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Multiple levels of symmetry breaking

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Abstract. In the previous paper in this volume we have studied the p-spin interaction model just below the critical temperature, and we have rigorously proved several aspects of the physicist's prediction that this model exhibits "one level of symmetry breaking". In the present paper we show how to construct systems that exhibit an arbitrarily large, but finite number of "levels of symmetry-breaking". As the temperature decreases, such systems exhibit many phase transitions, as the structure of the overlaps gains complexity. This phenomenon does not seem to have been described previously, even in the physics literature.

1. Introduction

In the previous paper in this volume [T] we studied the p-spins interaction model, that is the model with Hamiltonian

$$H(\boldsymbol{\sigma}) = -\Big(\frac{p!}{2N^{p-1}}\Big)^{1/2} \sum_{1\le i_1<\cdots<i_p\le N} g_{i_1\cdots i_p}\sigma_{i_1}\cdots\sigma_{i_p}. \tag{1.1}$$

There, the summation is over all possible choices of indices $1 \le i_1 < \cdots < i_p \le N$, and the $g_{i_1\cdots i_p}$ are realizations of independent standard normal r.v. The spins σ_i are Ising spins, $\sigma_i \in \{-1, 1\}$, $\boldsymbol{\sigma} = (\sigma_i)_{i\le N} \in \Sigma_N = \{-1, 1\}^N$. We will briefly repeat those of the conclusions from [T] that are needed to understand our present purpose. For more details about the (elementary) ideas of statistical mechanics we use, and for references, the reader should consult [T]. Our results are better described in terms of overlaps. The overlap of two configurations $\boldsymbol{\sigma}$, $\boldsymbol{\sigma}'$ is given by

$$R(\boldsymbol{\sigma}, \boldsymbol{\sigma}') =: N^{-1}\sum_{i\le N}\sigma_i\sigma_i'. \tag{1.2}$$

The useful way to think about the overlap is as a function on the square of the configuration space, provided with the probability $G \otimes G$, where G is Gibbs' measure at a given temperature. In other words one fixes the temperature and the disorder

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(i.e. the variables $g_{i_1 \cdots i_p}$) and one tries to understand the behavior of R as a function of $\boldsymbol{\sigma}, \boldsymbol{\sigma}'$, weighted by their Gibbs' weights.

For p large enough, the picture is as follows: There is a critical temperature β_p such that for $\beta < \beta_p$ "the overlap are essentially zero". This means simply that

$$\lim_{N \to \infty} E\langle |R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')|\rangle = 0 \tag{1.3}$$

where $\langle \ \rangle$ denotes integration with respect to $G \otimes G$, and E denotes average with respect to disorder. On the other hand, for $\beta > \beta_p$ the overlaps are no longer zero, and (1.3) fails. (This is our definition of β_p.) But, if β is not too large ($\beta \le \beta_p + 1/L$, where L is a number) something remarkable happens: the overlaps are almost certainly either small or large. More precisely

$$E(G \otimes G(\{\boldsymbol{\sigma}, \boldsymbol{\sigma}'; \frac{1}{10} \le |R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')| \le \frac{9}{10}\}) \le \exp(-N/L) \tag{1.4}$$

There is thus a phase transition concerning the behavior of the overlaps at $\beta = \beta_p$. The physicists predict that for $\beta > \beta_p$, (but not too large) the overlaps take essentially only two non random values, $q_0 = 0$ and q_1. The part $q_0 = 0$ is proved in [T] for $\beta \le \beta_p + 1/L$. Such a situation is described in physics as a "one level of symmetry breaking", and is the predicted low-temperature behavior of many systems. In contrast, the celebrated solution proposed by G. Parisi for the Sherrington-Kirkpatrick model (that is, the case $p = 2$ of (1.1)) exhibits "infinitely many levels of symmetry breaking" and is a much more complex situation, where the overlaps are expected to range over entire intervals. It is thus natural to investigate whether there are intermediate situations, with "a finite number of levels of symmetry breaking". This situation does not appear to have been described in the physics literature (except in the uninteresting case of the "Generalized Random Energy Models", where the desired structure is built in from the start in the model). Our purpose is to construct a "real" example of such a situation. We have not been able to do this using Ising model, and we will use the spherical model, where the configuration space is now

$$S_N = \{\boldsymbol{\sigma}; \sum_{i \le N} \sigma_i^2 = N\}. \tag{1.5}$$

The physicists believe that the spherical model is easier that the Ising model because they think that the one level of symmetry breaking picture remains valid at arbitrarily low temperature, while at very low temperatures for the Ising model it has to be replaced by Parisi-type solutions. We should point out that up to this point it seems harder to obtain mathematical results for spherical models. This is because one does not know a priori that only the values of spins of order 1 are relevant. (This difficulty will create considerable complications here). A technical key reason however in our case for using the spherical model is that the critical temperature goes to infinity with p (while it stays bounded for the Ising model). In fact, we have the following estimate that deserves to be stated separately.

Theorem 1.1. *There exists a constant L such that the critical temperature β_p of the spherical p-spins interaction model satisfies*

$$|\beta_p - (2\log(p\log p))^{1/2}| \le \frac{L}{\sqrt{\log p}}. \tag{1.6}$$

The critical temperature is defined as in the Ising case, that is the largest number β_p such that (1.3) holds for $\beta < \beta_p$.

Before we formally state our main result, let us explain in words the properties it has. At high temperature $\beta < \theta_1$, for a certain number θ_1, the absolute value $|R|$ of the overlaps is small, it belongs to a small interval I_0 around zero. As β increases from θ_1, to a certain number $\theta_2 > \theta_1$ the overlaps are no longer small; but $|R|$ belongs to $I_0 \cup I_1$ where I_1, is disjoint from I_0, and to its right. For some intermediate value $\theta_1 < \theta_1' < \theta_2$ we know that overlaps do appear in I_1 if $\theta_1' < \beta < \theta_2$; We are in a situation similar to the Ising case as proved in [T]. As β increases from θ_2 to a certain number $\theta_3 > \theta_2$, we know that $|R|$ remains in $I_0 \cup I_1 \cup I_2$, where I_2 is disjoint from I_1 and to its right at a certain intermediate value θ_2' we are guaranteed that overlaps do appear in each of I_0, I_1, I_2; we are (morally at least because we do not know how to guarantee that the overlaps asymptotically take only 3 values) in a situation with "two levels of symmetry breaking"'; and we can achieve this with any prescribed number of levels.

If we denote by $H_p(\boldsymbol{\sigma})$ the left-hand side of (1.2), we will use Hamiltonians of the type

$$\sum_{1\le \ell \le k} a_\ell H_{p_\ell}(\boldsymbol{\sigma}) \tag{1.7}$$

where a_ℓ are numbers and p_ℓ integers. It is understood that all the Gaussian r.v. involved in (1.7) are independent.

Theorem 1.2. *For each integer $k > 0$, we can find for $\ell \le k$ coefficients a_ℓ, integers p_ℓ, numbers $\epsilon_\ell > 0$, we can find numbers*

$$r_0 = 0 < m_0 < r_1 < m_1 < \cdots < r_k < m_k < 1$$

and numbers

$$0 < \theta_1 < \theta_1' < \theta_2 < \theta_2' < \cdots < \theta_{k-1}' < \theta_k \tag{1.8}$$

such that if we set $I_\ell = [r_\ell, m_\ell]$, then for $\beta < \theta_r$, $1 \le r \le k$, we have

$$EG^2(R \notin \bigcup_{0\le \ell\le r-1} I_\ell) \le \exp(-N/L) \tag{1.9}$$

while if $r \le k-1$ for $\theta_r' < \beta < \theta_k$ we have

$$\liminf_{N\to\infty} EG^2(|R| \in I_r) \ge \epsilon_r. \tag{1.10}$$

Let us now comment on the methods and the organization of the paper. Half of the proof of Theorem 1.1 is a pretty exercise on Gaussian processes, although readers who have not spent years thinking about these might find it instructive. The proof of Theorem 1.2 builds upon the ideas of [T], sections 2 and 3, and assumes that the reader is familiar with this material. The high technicity of this proof could disappoint some; but rather, I think we should look at the bright side; knowing (and understanding) as little as we currently do about the topic studied here, it is quite unexpected that a control as precise as what Theorem 1.2 provides can be achieved at all. The main idea is simple. The sequence (p_k) increases very fast. As β increases, each term of the sum (1.7) successively goes through a high temperature to low temperature behavior, creating a new phase transition. It is easy to control the terms in the "high temperature" behavior. The difficulty is of course to control the others.

2. Critical temperature for the spherical model

It will be useful to think to $H(\boldsymbol{\sigma})$ as a Gaussian process indexed by the sphere S_N, given by (1.5). A look at the proof of the inequality $\beta_p \le 2\sqrt{\log 2}$ of [T], Theorem 1.1 reveals the general fact that

$$\beta_p \le 2 \liminf_{N\to\infty} \frac{1}{N} E(\sup_{\boldsymbol{\sigma}\in S_N} H(\boldsymbol{\sigma})) \tag{2.1}$$

and our upper bound for β_p will rely upon the following.

Proposition 2.1. *We have*

$$E(\sup_{\boldsymbol{\sigma}\in S_N} H(\boldsymbol{\sigma})) \le N\left[\sqrt{\frac{1}{2}\log(p\log p)} + \frac{L}{\sqrt{\log p}}\right]. \tag{2.2}$$

An expert on Gaussian processes might consider the proof of this inequality as a somewhat standard exercise. The first step is to control the canonical distance associated to the process.

Lemma 2.2. *For $\boldsymbol{\sigma}$, $\boldsymbol{\rho}$ in S_N we have*

$$E(H(\boldsymbol{\sigma}) - H(\boldsymbol{\rho}))^2 \le \frac{p}{2}\|\boldsymbol{\sigma} - \boldsymbol{\rho}\|^2. \tag{2.3}$$

Proof. We have

$$\begin{aligned} E(H(\boldsymbol{\sigma}) - H(\boldsymbol{\rho}))^2 &= u^2 \sum_{i_1<\cdots<i_p} (\sigma_{i_1}\cdots\sigma_{i_p} - \rho_{i_1}\cdots\rho_{i_p})^2 \\ &= \frac{1}{2N^{p-1}} \sum_{i_1,\cdots,i_p} (\sigma_{i_1}\cdots\sigma_{i_p} - \rho_{i_1}\cdots\rho_{i_p})^2 \end{aligned}$$

where the second summation is over all choices of $i_1, \cdots, i_p \le N$, all different. Let us set

$$c^k_{i_1\cdots i_p} = \sigma_{i_1}\cdots\sigma_{i_{k-1}}(\sigma_{i_k} - \rho_{i_k})\rho_{i_{k+1}}\cdots\rho_{i_p}$$

so that

$$(\sigma_{i_1}\cdots\sigma_{i_p}-\rho_{i_1}\ldots\rho_{i_p})^2=(\sum_{1\le k\le p}c^k_{i_1\cdots i_p})^2$$
$$\le p\sum_{1\le k\le p}(c^k_{i_1\cdots i_p})^2.$$

Now,

$$\sum_{i_1,\cdots,i_p}c^p_{i_1\cdots i_p}\le\sum(\sigma_i-\rho_i)^2\sum\sigma^2_{i_1}\cdots\sigma^2_{i_{k-1}}\rho^2_{i_{k+1}}\cdots\rho^2_{i_p} \tag{2.4}$$

where the last summation is over *all* possible choices of $i_1,\cdots,i_{k-1},i_{k+1},\cdots,i_p$. The right hand side of (2.4) is at most $\sum_{i\le N}(\sigma_i-\rho_i)^2N^{p-1}$. The result follows. □

We now denote by $B(\boldsymbol{\sigma},r)$ the ball of $\mathbb{R}^N$ of center $\boldsymbol{\sigma}$ and radius r. The following is standard.

Lemma 2.3. *In $\mathbb{R}^N$, it is possible to cover $B(\boldsymbol{\sigma},1)$ by at most $(1+2/\epsilon)^N$ balls of radius ϵ.*

Lemma 2.4. *We have, for each $\boldsymbol{\rho},r$*

$$E\sup_{\boldsymbol{\sigma}\in B(\boldsymbol{\rho},r)}H(\boldsymbol{\sigma})\le Lr\sqrt{Np}.$$

Proof. This is a consequence of the bound known as "Dudley's entropy integral"

$$E\sup_{\boldsymbol{\sigma}\in B(\boldsymbol{\rho},r)\cap S_N}H(\boldsymbol{\sigma})\le L\int_0^\infty\sqrt{\log N(\epsilon)}d\epsilon \tag{2.5}$$

where $N(\epsilon)$ is the minimum number of balls of radius ϵ, for the distance

$$d(\boldsymbol{\sigma},\boldsymbol{\sigma}')^2=(E(H(\boldsymbol{\sigma})-H(\boldsymbol{\sigma}'))^2)^{1/2}$$

that are needed to cover $B(\boldsymbol{\rho},r)\cap S_N$. It follows from Lemmas 2.2 and 2.3 that

$$N(\epsilon)\le(1+\frac{r\sqrt{2p}}{\epsilon})^N$$

and that $N(\epsilon)=1$ for $\epsilon\ge r\sqrt{2p}$. Lemma 2.4 then follows from (2.5) by a routine computation.

Lemma 2.5. *For all $t>0$,*

$$P\left(\sup_{\boldsymbol{\sigma}\in B(\boldsymbol{\rho},r)\cap S_N}H(\boldsymbol{\sigma})\ge t+Lr\sqrt{Np}\right)\le\exp-\frac{t^2}{N}.$$

Proof. This is the same general principle as in [T], Proposition 2.5, using that

$$\forall \boldsymbol{\sigma} \in S_N,\, EH(\boldsymbol{\sigma})^2 \le \frac{N}{2}$$

and Lemma 2.4. □

Proof of Proposition 2.1. In Lemma 2.5 we choose $r = a\sqrt{N}$ where the parameter a will be determined later. We cover S_N by at most $(1+2/a)^N$ balls of the type $B(\boldsymbol{\rho}, r)$, and we get from Lemma 2.5 that

$$P(\sup_{\boldsymbol{\sigma}\in S_N} H(\boldsymbol{\sigma}) \ge t + LaN\sqrt{p}) \le (1+\frac{2}{a})^N \exp(-\frac{t^2}{N}) \tag{2.6}$$

so that, setting $t_0 = N\sqrt{\log(1+2/a)}$

$$\begin{aligned} E(\sup_{\boldsymbol{\sigma}\in S_N} H(\boldsymbol{\sigma})) &\le LaN\sqrt{p} + \int_0^\infty \min(1, (1+\frac{2}{a})^N \exp -\frac{t^2}{N})dt \\ &\le LaN\sqrt{p} + t_0 + \int_{t_0}^\infty (1+\frac{2}{a})^N \exp -\frac{t^2}{N} dt. \end{aligned}$$

Now,

$$\int_{t_0}^\infty \exp -\frac{t^2}{N} dt \le \frac{1}{t_0}\int_{t_0}^\infty t \exp -\frac{t^2}{N} dt = \frac{N}{2t_0}\exp -\frac{t_0^2}{N}$$

and thus

$$E(\sup_{\boldsymbol{\sigma}\in S_N} H(\boldsymbol{\sigma})) \le N\left[La\sqrt{p} + \sqrt{\log(1+\frac{2}{a})} + \frac{1}{2\sqrt{\log(1+\frac{2}{a})}}\right].$$

Using $\log(1+\frac{2}{a}) \le 1 + \log\frac{1}{a}$, we see that the choice $a = 1/\sqrt{p\log p}$ yields (2.2). □

Proposition 2.6. *We have*

$$\beta_p^2 \ge \inf_{0<t<1} (1+t^{-p}) \log \frac{1}{1-t^2}. \tag{2.7}$$

The proof of this proposition requires only small modifications from the proof of Proposition 2.6 of [T]. The main modification is the replacement of $\varphi(t)$ by $-\frac{1}{2}\log(1-t^2)$. □

Proposition 2.7. *We have*

$$\inf_{0<t<1} -(1+t^{-p})\log(1-t^2) \ge 2\log(p\log p) - L.$$

Proof. We set $x = 1 - t^2, q = p/2,$

$$f(x) = (1 + (1-x)^{-q}) \log(\frac{1}{x})$$

so algebra shows that $f'(x_0) = 0$ where

$$qx_0 \log \frac{1}{x_0} = 1 - x_0 + (1-x_0)^q.$$

Thus

$$x_0 \log \frac{1}{x_0} \leq \frac{2}{q}$$

and, since we can assume q large enough, we have $x_0 \leq 1/q$, so that $\log 1/x_0 \geq \log q$, and $x_0 \leq 2/q \log q$.

Now, $(1-x)^{-q} \geq 1$, so

$$f(x_0) \geq 2 \log \frac{1}{x_0} \geq 2 \log((q \log q)/2) \geq 2 \log p \log p - L. \qquad \square$$

Combining with Proposition 2.6, we have

$$\beta_p \geq \sqrt{2 \log(p \log p)} - \frac{L}{\sqrt{\log p}}$$

and together with Proposition 2.1 this finishes the proof of Theorem 1.1.

3. Multiple phase transitions

We will consider a sequence $p_1 \leq p_2 \leq \cdots \leq p_k$ of integers, and we set

$$H_\ell(\boldsymbol{\sigma}) = H_{p_\ell}(\boldsymbol{\sigma}) \tag{3.1}$$

where

$$H_p(\boldsymbol{\sigma}) = -\left(\frac{p!}{2N^{p-1}}\right)^{1/2} \sum_{i_1 < \cdots < i_p} g_{i_1 \cdots i_p} \sigma_{i_1} \cdots \sigma_{i_p}.$$

Each quantity $H_k(\boldsymbol{\sigma})$ represents a certain interaction between the spins. It will clarify matters to step slightly outside the usual framework of statistical mechanics, and to assume that to each H_ℓ corresponds an inverse temperature γ_ℓ. That is, we define

$$Z_N = Z_N(\gamma) = \int \exp(-\sum_{\ell \leq k} \gamma_\ell H_\ell(\boldsymbol{\sigma})) d\mu_N(\boldsymbol{\sigma})$$

where $\gamma = (\gamma_1, \cdots, \gamma_k)$, and we define the corresponding Gibbs measure accordingly. Once most of the work is done, we will revert to the traditional setting by setting $\gamma_\ell = \beta x_\ell$ for a suitable choice of x_ℓ.

The proof of Theorem 1.2 will build upon the ideas of [T], Sections 2 and 3. In order to provide motivation for the main construction, we mention two of the principles we will use. We will consider $F_N(\gamma) = \log Z_N(\gamma)$, so that, by Jensen's inequality

$$\frac{1}{N} EF_N(\gamma) \le \sum_{\ell \le k} \frac{\gamma_\ell^2}{4}$$

and we set

$$\Delta_N(\gamma) = \sum_{\ell \le k} \frac{\gamma_\ell^2}{4} - \frac{EF_N(\gamma)}{N} \tag{3.2}$$

Lemma 3.1. *Assume that for all $t > 0$ we have*

$$\sum_{\ell \le k} \gamma_\ell^2 \frac{t^{p_\ell}}{1 + t^{p_\ell}} < \log(1 - t^2).$$

Then we have $\lim_{N\to\infty} \Delta_N(\gamma) = 0$.

This is obtained by a rather direct adaptation of the arguments of [T], Section 2.

A basic ingredient will be a principle that ensures that the overlap essentially never belongs to certain sets.

Lemma 3.2. *Assume that for a certain number Δ, and certain numbers $\theta_\ell > 1$, we have, for $1 \le \ell \le k$,*

$$\gamma_\ell^2(\theta_\ell - 1)^2/4 \ge \Delta. \tag{3.3}$$

Then if $\Delta_N(\gamma) \le \Delta$, the set

$$U = \{t; \sum_{\ell \le k} f_\ell \le -\log(1 - t^2) - 2\Delta - 1\},$$

satisfies

$$EG^2(R \in U) \le \exp -\frac{N}{L},$$

where

$$f_\ell = f_\ell(t, \gamma_\ell) = \gamma_\ell^2 t^{p_\ell} \text{ for } 1 + t^{p_\ell} < \theta_\ell \tag{3.4}$$

$$f_\ell = f_\ell(t, \gamma_\ell) = 2\gamma_\ell^2(\theta_\ell - 1) + \gamma_\ell^2 \frac{t^{p_\ell}}{1 + t^{p_\ell}} \text{ for } 1 + t^{p_\ell} \ge \theta_\ell. \tag{3.5}$$

The proof of this statement requires only rather straightforward modifications to the proof of [T], Theorem 3.3, and these are left to the reader.

Lemma 3.2 has been stated in a way that should allow the interested reader to figure out a proof; Formula (3.4) corresponds to the case where we do not use truncation; and (3.5) to the case we do. It will always be used for $\theta_\ell = 1+2\sqrt{\Delta}/\gamma_\ell$, which satisfies (3.3). The definition of f_ℓ then becomes

$$f_\ell = f_\ell(t, \gamma_\ell, \Delta) = \gamma_\ell^2 t^{p_\ell} \quad \text{if } \gamma_\ell \mathrm{t}^{\mathrm{p}_\ell} < 2\sqrt{\Delta} \tag{3.6}$$

$$f_\ell = f_\ell(t, \gamma_\ell, \Delta) = 4\sqrt{\Delta}\gamma_\ell + \gamma_\ell^2 \frac{t^{p_\ell}}{1 + t^{p_\ell}} \quad \text{if } \gamma_\ell \mathrm{t}^{\mathrm{p}_\ell} \ge 2\sqrt{\Delta}. \tag{3.7}$$

These are the formulae that are going to be used in the rest of the section. The discontinuity of f_ℓ at $\gamma_\ell t^{p\ell} = 2\sqrt{\Delta}$ should not disturb the reader. It has no special meaning, and simply arises from the fact that our estimates were sharper when we did not use truncation than when we did. It should be obvious that $f_\ell(t, \gamma_\ell, \Delta)$ increases with γ_ℓ.

In Proposition 2.7 we have shown that for a certain number L_1 we have, if $p \in \mathbb{N}$ is large enough

$$0 < t < 1 \Rightarrow (2\log(p\log p) - L_1)\frac{t^p}{1+t^p} \leq -\log(1-t^2). \tag{3.8}$$

We consider now a number $L_0 \geq 31$, $L_0 \geq L_1$ that is fixed once and for all. We consider

$$\delta_\ell = (2 + L_0)^{2(\ell-1)} \tag{3.9}$$

so that $\delta_1 = 1$ and

$$\delta_{\ell+1} = (2 + L_0)^2 \delta_\ell. \tag{3.10}$$

We set $\delta = \delta_{k+2}$. The reason for these choices will be apparent in due time. The main part of the construction is to build the sequence of integers $(p_\ell)_{1\leq\ell\leq k}$. To simplify notation, we set

$$\beta_\ell = \sqrt{2\log(p_\ell \log p_\ell)} \tag{3.11}$$

$$b_\ell = \beta_\ell - L_0\sqrt{\delta_\ell} \tag{3.12}$$

$$c_\ell = \beta_\ell + L_0 \tag{3.13}$$

$$r_\ell = 1 - \frac{1}{p_\ell};\ m_\ell = 1 - \frac{1}{p_\ell^3} \tag{3.14}$$

Thus, all these quantities are in fact functions of p_ℓ. An important feature of the construction of the integers p_ℓ is that the only requirement is that this sequence increases fast enough; that is for certain functions ψ_ℓ

$$p_{\ell+1} \geq \psi_\ell(p_1, \ldots, p_\ell). \tag{3.15}$$

In particular we can require additional conditions of this type if we so wish.

Lemma 3.3. *Consider a number k, and the sequence δ_ℓ given by (3.9), $\delta = \delta_{k+2}$. Then we can construct a sequence $(p_\ell)_{1\leq\ell\leq k}$ and $m_0 < r_1$ with the following properties.*

$$\forall\, 1 \leq \ell \leq k, t \in [m_{\ell-1}, r_\ell] \Rightarrow \sum_{m\leq k} f_m(t, c_m, \delta) \leq -\log(1-t^2) - 3\delta \tag{3.16}$$

$$\forall\, \ell \leq k, t \in [r_\ell, m_\ell]$$
$$\Rightarrow \sum_{m\neq\ell, m\leq k} f_m(t, c_m, \delta_\ell) + f_\ell(t, b_\ell, \delta_\ell) \leq -\log(1-t^2) - 3\delta_\ell \tag{3.17}$$

Moreover, we can also assume that

$$0 < t \leq m_0 \Rightarrow \sum_{\ell\leq k} c_\ell^2 t^{p_\ell} < -\log(1-t^2). \tag{3.18}$$

Proof. First, we choose m_0 close enough to one that

$$t \geq m_0 \Rightarrow 5\delta \leq -\log(1-t^2).$$

Next, if we choose p_1 large enough, we will have $m_0 < r_1$, and (3.18) will hold because for $t < 1$, $c_\ell^2 t^{p_\ell} \to 0$ as $p_\ell \to \infty$.

To simplify the construction, a first observation is that if we arrange that, for each ℓ,

$$t \leq m_\ell \Rightarrow \forall\, x \leq \delta,\ f_{\ell+1}(t, c_{\ell+1}, x) \leq 2^{-\ell-1}, \tag{3.19}$$

then to prove (3.16) it suffices to prove that

$$\begin{aligned} &\forall\, 1 \leq \ell \leq k+1, t \in [m_{\ell-1}, r_\ell] \\ &\quad \Rightarrow \sum_{m \leq \ell} f_m(t, c_m, \delta) \leq -\log(1-t^2) - 4\delta \end{aligned} \tag{3.20}$$

and to prove (3.17) it suffices to prove that

$$\begin{aligned} &\forall\, \ell \leq k, t \in [r_\ell, m_\ell] \\ &\quad \Rightarrow \sum_{m < \ell} f_m(t, c_m, \delta_\ell) + f_\ell(t, b_\ell, \delta_\ell) \leq -\log(1-t^2) - 4\delta_\ell. \end{aligned} \tag{3.21}$$

Now, condition (3.19) is automatically satisfied if the sequence (p_ℓ) increases fast enough. This is the case because, given $t < 1$, $c_\ell^2 t^{p_\ell} \to 0$ as $p_\ell \to \infty$.

Thus, we can turn our attention to (3.20), (3.21). Consider the quantity

$$d_\ell = \sum_{m \leq \ell} 4\sqrt{\delta} c_m + \frac{c_m^2}{2}. \tag{3.22}$$

The motivation for this definition is simply that

$$0 \leq t < 1 \Rightarrow f_m(t, c_m, \delta_\ell) \leq 4\sqrt{\delta} c_m + \frac{c_m^2}{2}.$$

It will help during the construction to ensure that the following holds

$$t \geq m_\ell \Rightarrow d_\ell \leq -\log(1-t^2) - 5\delta. \tag{C_ℓ}$$

Our choice of m_0 ensures that (C_0) holds.

To perform the construction, assuming that $p_1, \dots, p_{\ell-1}$ have been constructed, and that $(C_{\ell-1})$ holds, we show that if p_ℓ is large enough, then (C_ℓ) will hold, as well as (3.20), (3.21) (for this value of ℓ). This will complete the proof. First, to prove (C_ℓ), we note that

$$t \geq m_\ell \Rightarrow t^2 \geq 1 - \frac{2}{p_\ell^3} \Rightarrow -\log(1-t^2) \geq 3\log p_\ell - 2. \tag{3.23}$$

Also, if p_ℓ is large enough,

$$c_\ell^2 \leq 2.5 \log p_\ell. \tag{3.24}$$

Since $4\sqrt{\delta}c_\ell + c_\ell^2/2 \le 8\delta + c_\ell^2$, we see that (C_ℓ) holds provided

$$d_{\ell-1} + 13\delta + 2 < \frac{1}{2}\log p_\ell.$$

To prove (3.21), it suffices to show that

$$t \ge r_\ell \Rightarrow d_{\ell-1} + f_\ell(t, b_\ell, \delta_\ell) \le -\log(1-t^2) - 4\delta_\ell. \tag{3.25}$$

By (3.8) we have

$$b_\ell^2 \frac{t^{p_\ell}}{1+t^{p_\ell}} + (\beta_\ell^2 - L_1 - b_\ell^2)\frac{t^{p_\ell}}{1+t^{p_\ell}} \le -\log(1-t^2). \tag{3.26}$$

For $t \ge r_\ell$, we have $t^{p_\ell} \ge 1/3$, so that (3.26) yields

$$b_\ell^2 \frac{t^{p_\ell}}{1+t^{p_\ell}} + \frac{1}{3}(\beta_\ell^2 - L_1 - b_\ell^2) \le -\log(1-t^2). \tag{3.27}$$

Thus, to ensure (3.25) it suffices that

$$d_{\ell-1} + 4\sqrt{\delta_\ell}b_\ell + 4\delta_\ell \le \frac{1}{3}(\beta_\ell^2 - L_1 - b_\ell^2). \tag{3.28}$$

Now, since $b_\ell \le \beta_\ell$, we have

$$\begin{aligned} \beta_\ell^2 - b_\ell^2 &\ge 2b_\ell(\beta_\ell - b_\ell) \\ &= 2L_0 b_\ell\sqrt{\delta_\ell}. \end{aligned}$$

Since $\delta_\ell, \beta_\ell \ge 1$ and $L_1 \le L_0$, (3.28) holds provided

$$d_{\ell-1} + 4\sqrt{\delta_\ell}b_\ell + 4\delta_\ell \le \frac{1}{3}L_0 b_\ell\sqrt{\delta_\ell}.$$

Since $L_0 \ge 30$, this holds provided

$$d_{\ell-1} + 4\delta_\ell \le b_\ell\sqrt{\delta_\ell}$$

which is true for p_ℓ large enough.

To prove (3.20), we will prove the stronger condition

$$t \in [m_{\ell-1}, r_\ell] \Rightarrow d_{\ell-1} + f_\ell(t, c_\ell, \delta) \le -\log(1-t^2) - 4\delta. \tag{3.29}$$

Case 1. $c_\ell^2 t^{p_\ell} \le \delta$

In that case if p_ℓ is large, $c_\ell t^{p_\ell} \le 2\sqrt{\delta}$, so that $f_\ell(t, \gamma_\ell, \delta) = c_\ell^2 t^{p_\ell} \le \delta$, and (3.29) follows from $(C_{\ell-1})$.

Case 2. $c_\ell^2 t^{p_\ell} \geq \delta, c_\ell t^{p_\ell} \leq 24\sqrt{\delta}$

In that case

$$f_\ell(t, c_\ell, \delta) \leq 28\sqrt{\delta} c_\ell$$

and all we need to show is that

$$t^{p_\ell} \geq \delta/c_\ell^2 \Rightarrow d_{\ell-1} + 28\sqrt{\delta} c_\ell \leq -\log(1-t^2) - 4\delta. \tag{3.30}$$

But

$$(\delta/c_\ell^2)^{1/p_\ell} \simeq 1 - \frac{1}{p_\ell} \log(c_\ell^2/\delta)$$

so that for p_ℓ large,

$$-\log(1-(\delta/c_\ell)^{2/p_\ell}) \geq \frac{1}{2} \log p_\ell$$

while c_ℓ is of order $\sqrt{\log p_\ell}$. Thus (3.30) holds if p_ℓ is large enough.

Case 3. $c_\ell t^{p_\ell} \geq 24\sqrt{\delta}, t \leq r_\ell$.

The proof of Proposition 3.8 shows that

$$t \leq r_\ell \Rightarrow -(1+t^{-p_\ell}) \log(1-t^2) \geq -(1+r_\ell^{-p_\ell}) \log(1-r_\ell^2)$$

so that

$$t \leq r_\ell \Rightarrow \frac{t^{p_\ell}}{1+t^{p_\ell}} V_\ell \leq -\log(1-t^2) \tag{3.31}$$

where

$$V_\ell = -(1+r_\ell^{-p_\ell}) \log(1-r_\ell^2) \geq 3 \log p_\ell$$

for p_ℓ large. Thus (3.31) implies

$$t \leq r_\ell \Rightarrow \frac{5}{4} c_\ell^2 \frac{t^{p_\ell}}{1+t^{p_\ell}} \leq -\log(1-t^2) \tag{3.32}$$

and (3.29) will hold provided

$$d_{\ell-1} + 2\sqrt{\delta} c_\ell \leq \frac{1}{8} c_\ell^2 t^{p_\ell}. \tag{3.33}$$

Since $c_\ell t^{p_\ell} \geq 24\sqrt{\delta}$, it suffices that $d_{\ell-1} \leq \sqrt{\delta} c_\ell$, which is true if p_ℓ is large enough. □

In order to use Lemma 3.2, we need bounds for $\Delta_N(\gamma)$, and we turn to the task of finding such bounds under the information of Lemma 3.3.

Lemma 3.4. *If $\gamma_\ell \leq b_\ell$ for each $\ell \leq k$, we have*

$$\lim_{N\to\infty} \Delta_N(\gamma) = 0.$$

Proof. Since for $x > 0$ we have

$$b_\ell^2 \frac{t^{p_\ell}}{1+t^{p_\ell}} \le f_\ell(t, b_\ell, x)$$

it follows from (3.16) to (3.18) that

$$\forall\, t > 0, \quad \sum_{\ell \le k} b_\ell^2 \frac{t^{p_\ell}}{1+t^{p_\ell}} < -\log(1-t^2)$$

and the conclusion follows from Lemma 3.1. □

Next, by induction over $m \le k+1$ we show how to bound $\Delta_N(\gamma)$ under the conditions

$$\forall\, \ell < m, \quad \gamma_\ell \le c_\ell; \quad \forall \ell \ge m, \gamma_\ell \le b_\ell. \tag{3.34}$$

We will prove the following.

Lemma 3.5. *Under (3.34) we have*

$$\lim_{N\to\infty} \Delta_N(\gamma) \le \delta_{m+1}. \tag{3.35}$$

Proof. It is by induction over $m \ge 0$. We have shown in Lemma 3.4 that (3.35) holds when $m = 0$, for $\delta_1 = 1$.

For the induction from $m-1$ to m, consider γ as in (3.34). We set

$$\gamma(t) = (\gamma_1, \cdots, \gamma_{m-1}, t, \gamma_{m+1}, \cdots, \gamma_k).$$

Thus, by induction hypothesis, we have

$$\forall t \le b_m, \quad \lim_{N\to\infty} \Delta_N(\gamma(t)) \le \delta_m.$$

In other words, if

$$a(t) = \frac{1}{4}\Big(\sum_{\ell \ne m} \gamma_\ell^2 + t^2\Big)$$

we have, for $t \le b_m$,

$$\lim_{N\to\infty} \frac{1}{N} EF_N(\gamma(t)) \ge a(t) - \delta_m. \tag{3.36}$$

Now, we simply use convexity to write

$$\frac{1}{N} EF_N(\gamma) \ge \frac{1}{N} EF_N(\gamma') + (\gamma_m - b_m)h_N \tag{3.37}$$

when $\gamma' = \gamma(b_m)$, and h_N is the derivative at $t = b_m$ of $EF_N(\gamma(t))$. Thus

$$\Delta_N(\gamma) \le \frac{1}{4}(\gamma_m^2 - b_m^2) - (\gamma_m - b_m)h_N + \Delta_N(\gamma'). \tag{3.38}$$

To avoid repetition, let us prove an elementary fact.

Lemma 3.6. *Consider a sequence of convex functions φ_N on $I = [0, t_0]$, such that for t in I we have $\varphi_N(t) \le \frac{t^2}{4} + a$, and assume that for numbers a, $\Delta \le t_0/\sqrt{2}$ we have*

$$\liminf \varphi_N(t_0) \ge \frac{t_0^2}{4} + a - \Delta .$$

Then

$$\liminf \varphi_N'(t_0) \ge \frac{t_0}{2} - \sqrt{2\Delta}.$$

Proof. We have, for $t < t_0$

$$(t - t_0)\varphi_N'(t_0) + \varphi_N(t_0) \le \varphi_N(t) \le \frac{t^2}{4} + a.$$

So

$$\liminf \varphi_N'(t_0) \ge \liminf_N \left(\frac{\varphi_N(t_0) - \frac{t^2}{4} - a}{t_0 - t} \right) \ge \frac{t_0 + t}{4} - \frac{\Delta}{t_0 - t}$$

and we take $t = t_0 - \sqrt{2}\Delta$. □

We go back to the proof of Lemma 3.5. There is nothing to prove for the induction step unless $\gamma_m \ge b_m$, so we can assume that this is the case. It follows from Lemma 3.6 and induction hypothesis that $\liminf h_N \ge \frac{1}{2} b_m - \sqrt{2\delta_m}$ and going back to (3.38) show that

$$\begin{aligned}\limsup_{N\to\infty} \Delta_N(\gamma) &\le \delta_m + \sqrt{2\delta_m}(\gamma_m - b_m) + \frac{1}{4}(\gamma_m - b_m)^2 \\ &\le \delta_m + \sqrt{2\delta_m}(c_m - b_m) + \frac{1}{4}(c_m - b_m)^2.\end{aligned}$$

Since $c_m - b_m \le 2L_0\sqrt{\delta_m}$, we get a bound

$$\delta_m[1+2\sqrt{2}L_0 + L_0^2] \le \delta_m(2 + L_0)^2 = \delta_{m+1}. \qquad \square$$

Remark. Lemma 3.5 holds for $m = k + 1$; that is, if $\gamma_\ell \le c_\ell$ for $\ell \le k$, then $\lim_{N\to\infty} \Delta_N(\gamma) \le \delta_{k+2} = \delta$.

If we combine Lemma 3.5 with Lemmas 3.2 and 3.3, we have shown the following.

Lemma 3.7. *a) If for each $\ell \le k$ we have $\gamma_\ell \le c_\ell$ then for each $1 \le \ell \le k+1$ we have*

$$E(G^2(|R| \in [m_{\ell-1}, r_\ell])) \le K \exp -\frac{N}{L}.$$

b) If $0 \le s \le k - 1$ and if for each $\ell < s$ we have $\gamma_\ell \le c_\ell$, while for $\ell \ge s$ we have $\gamma_\ell \le b_\ell$, then we have

$$E(G^2(|R| \in [r_{s+1}, m_{s+1}])) \le K \exp -\frac{N}{L}.$$

Now, we will show that if γ_ℓ is close to c_ℓ, the overlaps do take values in $[r_\ell, m_\ell]$, for $\ell \leq k-1$. (Although this is likely to hold also for $\ell = k$, we do not know how to show this.) The proof relies upon the following.

Lemma 3.8. *We have*

$$\frac{1}{N} E \frac{\partial F_N}{\partial \gamma_\ell} = \frac{\gamma_\ell p_\ell!}{2N^{p_\ell}} E\Bigg(\sum_{1 \leq i_1 < \cdots < i_{p_\ell} \leq N} < \sigma_{i_1}^2 \cdots \sigma_{i_{p_\ell}}^2 > - < \sigma_{i_1} \cdots \sigma_{i_{p_\ell}} >^2 \Bigg). \tag{3.39}$$

Proof. See [T], (2.30), (2.31) □

In the case of the Ising model, we knew that $\sigma_i^2 = 1$. Although it is likely to be true, we do not know how to show that

$$\lim_{N\to\infty} \frac{p_\ell!}{N^{p_\ell}} E\Bigg(\sum_{1 \leq i_1 < \cdots < i_{p_\ell} \leq N} < \sigma_{i_1}^2 \cdots \sigma_{i_{p_\ell}}^2 > \Bigg) = 1 \tag{3.40}$$

which makes it very difficult to use (3.39). In order to find a substitute for (3.40), we will use a “trick” (i.e. a somewhat unnatural argument).

Let us define (keeping γ implicit)

$$C(N, p) = E < \sigma_1^2 \cdots \sigma_p^2 > .$$

Our aim is to prove (3.44) below. It shows that $C(N, p)$ is nearly 1. By symmetry among the variables,

$$C(N, p) = E < \sigma_{i_1}^2 \cdots \sigma_{i_p}^2 >$$

whenever $i_1 < \cdots < i_p$. Thus

$$\begin{aligned} (N-p)C(N, p+1) &= \sum_{p < \ell \leq N} E < \sigma_1^1 \cdots \sigma_p^2 \sigma_\ell^2 > \\ &\leq \sum_{1 \leq \ell \leq N} E < \sigma_1^2 \cdots \sigma_p^2 \sigma_\ell^2 > \\ &= E < \sigma_1^2 \cdots \sigma_p^2 (\sum_{\ell \leq N} \sigma_\ell^2) > \\ &= N C(N, p) \end{aligned}$$

by the spherical constraint $\sum_{\ell \leq N} \sigma_\ell^2 = N$. Thus

$$C(N, p) \geq (1 - \frac{p}{N}) C(N, p+1). \tag{3.41}$$

This will allow us to control $C(N, p_\ell)$ from below if we control $C(N, p_k)$ from below. To do this, we deduce from (3.39) for $\ell = k$,

$$\frac{1}{N} E \frac{\partial F_N}{\partial \gamma_k} \leq \frac{1}{2N^{p_k}} \gamma_k N(N-1) \cdots (N-p_k) C(N, p_k)$$

so that

$$C(N, p_k) \geq \frac{2}{N}\frac{1}{\gamma_k} E \frac{\partial F_N}{\partial \gamma_k}. \tag{3.42}$$

Let us now set $\boldsymbol{\gamma}(t) = (\gamma_1, \ldots, \gamma_{k-1}, t)$. Thus, for $t \leq c_k$, using Lemma 3.5 for $m = k+1$, we have, for $a(t) = (\sum_{\ell \leq k-1} \gamma_\ell^2 + t^2)/4$

$$a(t) - \delta \leq \liminf \frac{1}{N} E F_N(\boldsymbol{\gamma}(t)) \leq a(t) \tag{3.43}$$

and Lemma 3.6 shows that

$$\liminf_{N\to\infty} \frac{1}{N} E \frac{\partial F_N}{\partial \gamma_k} \geq \frac{1}{2}\gamma_k - \sqrt{2\delta}$$

so that (3.42) yields

$$C(N, p_k) \geq 1 - \frac{3\sqrt{\delta}}{\gamma_k}.$$

Combining with (3.41) we see that for each ℓ, if N is large enough

$$C(N, p_\ell) \geq 1 - \frac{4\sqrt{\delta}}{\gamma_k}. \tag{3.44}$$

We now show why controlling $C(N, p_\ell)$ from below helps to use (3.39).

Lemma 3.9. *We have*

$$\frac{1}{N} E \frac{\partial F_N}{\partial \gamma_\ell} \leq \frac{\gamma_\ell}{2}(1 - E < R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^{p_\ell} >) \tag{3.45}$$

$$\frac{1}{N} E \frac{\partial F_N}{\partial \gamma_\ell} \geq \frac{\gamma_\ell}{2}(1 - E < R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^{p_\ell} >) - \gamma_\ell(1 - C(N, p_\ell)) - \gamma_\ell K(p_\ell)/N \tag{3.46}$$

Proof. To prove (3.45), it suffices to use the arguments of [T], Lemma 2.1, so we prove (3.46). Using the notations $\sum_d$ and $\sum_{nd}$ of [T], Lemma 2.1 and writing p for p_ℓ, we have

$$\begin{aligned} p! \sum_{i_1 < \cdots < i_p} < \sigma_{i_1}^2 \cdots \sigma_{i_p}^2 > &= \sum < \sigma_{i_1}^2 \cdots \sigma_{i_p}^2 > - \sum_{nd} < \sigma_{i_1}^2 \cdots \sigma_{i_p}^2 > \\ &= N^p - \sum_{nd} < \sigma_{i_1}^2 \cdots \sigma_{i_p}^2 > \end{aligned}$$

so that

$$N(N-1)\cdots(N-p)C(N, p) = N^p - E(\sum_{nd} < \sigma_{i_1}^2 \cdots \sigma_{i_p}^2 >) \tag{3.47}$$

and thus

$$E(\sum_{nd} < \sigma_{i_1}^2 \cdots \sigma_{i_p}^2 >) \leq N^p(1 - C(N, p)) + K(p)N^{p-1}. \tag{3.48}$$

Next, as used in the proof of [T], Lemma 2.1,

$$\begin{aligned}\sum_d <\sigma_{i_1}\cdots\sigma_{i_p}>^2 = <\sum_d \sigma_{i_1}\sigma'_{i_1}\cdots\sigma_{i_p}\sigma'_{i_p}>\\ = N^p < R(\boldsymbol{\sigma},\boldsymbol{\sigma}')^p > - < \sum_{nd}\sigma_{i_1}\sigma'_{i_1}\cdots\sigma_{i_p}\sigma'_{i_p}>\\ \leq N^p < R(\boldsymbol{\sigma},\boldsymbol{\sigma}')^p > + < \sum_{nd}\sigma^2_{i_1}\cdots\sigma^2_{i_p}>\end{aligned}$$

Combining with (3.48)

$$E(\sum_d <\sigma_{i_1}\cdots\sigma_{i_p}>^2) \leq N^p < R(\boldsymbol{\sigma},\boldsymbol{\sigma}')^p > + N^p(1-C(N,p)) + K(p)N^{p-1}.$$

The result follows from this and (3.39), (3.48). □

Corollary 3.10. *If $\gamma_k \geq c_k/2$, and N is large enough we have, for $\ell \leq k-1$*

$$\frac{1}{N}E\frac{\partial F_N}{\partial\gamma_\ell} \geq \frac{\gamma_\ell}{2}(1 - E < R(\boldsymbol{\sigma},\boldsymbol{\sigma}')^{p_\ell} >) - \frac{10\gamma_\ell}{c_k}\sqrt{\delta} \tag{3.49}$$

Proof. Combine (3.44) with (3.46). □

Lemma 3.11. *Assume that the following conditions hold:*

$$\forall\, \ell \leq k, m_{\ell-1}^{p_\ell} \leq \frac{1}{\beta_\ell} \tag{3.50}$$

$$\forall\, \ell \leq k-1, \frac{24\sqrt{\delta}}{c_{\ell+1}} \leq \frac{1}{\beta_\ell} \tag{3.51}$$

$$\forall\, \ell \leq k-1, \frac{1}{\delta}L_0\frac{\sqrt{\delta_\ell}}{c_\ell} \geq 20\frac{\sqrt{\delta}}{c_k} \tag{3.52}$$

Then, if for each $m \leq k$, we have $c_m/2 \leq \gamma_m \leq c_m$, we have, for $\ell < k-1$

$$\gamma_\ell \geq c_\ell - 1 \Rightarrow \liminf_{N\to\infty} EG^2(|R| \in [r_\ell, m_\ell]) \geq \frac{1}{\beta_\ell}.$$

Proof. Using (2.2) we have

$$\frac{1}{N}E\frac{\partial F_N}{\partial\gamma_\ell} \leq \frac{1}{N}E\sup_{\boldsymbol{\sigma}} H_\ell(\boldsymbol{\sigma}) \leq \frac{\beta_\ell}{2} + \frac{L}{\sqrt{\log p_\ell}} \leq \frac{\beta_\ell}{2} + 1.$$

Assuming p_1 large enough, we see from (3.49) that

$$E < R(\boldsymbol{\sigma},\boldsymbol{\sigma}')^{p_\ell} > \geq 1 - \frac{2}{\gamma_\ell}[\frac{\beta_\ell}{2}+1] - \frac{20}{c_k}\sqrt{\delta} \tag{3.53}$$

Now, for $\gamma_\ell \geq c_\ell - 1$, we have $\gamma_\ell \geq \beta_\ell + 30$ so that using (3.51) the right-hand side of (3.51) is at least

$$1 - \frac{\beta_\ell+2}{\beta_\ell+30} - \frac{20}{c_k}\sqrt{\delta} \geq \frac{16}{\beta_\ell}. \tag{3.54}$$

Let us observe that

$$E < R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^{p_\ell} > \leq m_{\ell-1}^{p_\ell} + EG^2(|R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')| \in [m_{\ell-1}, r_{\ell+1}]) \\ + EG^2(|R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')| \geq r_{\ell+1}) \tag{3.55}$$

For $t \geq r_{\ell+1}$, we have $t^{p_{\ell+1}} \geq \frac{1}{3}$, so that

$$EG^2(|R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')| \geq r_{\ell+1}) \leq 3E < R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^{p_{\ell+1}} > . \tag{3.56}$$

Now, from (3.45) we have

$$E < R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^{p_{\ell+1}} > \leq 1 - \frac{2}{\gamma_{\ell+1}} \frac{1}{N} E \frac{\partial F_N}{\partial \gamma_{\ell+1}}. \tag{3.57}$$

Using Lemma 3.5 for $m = k + 1$, and Lemma 3.6, we see that for N large

$$E < R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')^{p_{\ell+1}} > \leq \frac{2\sqrt{2\delta}}{\gamma_{\ell+1}} \leq \frac{8\sqrt{\delta}}{c_{\ell+1}}$$

since $\gamma_{\ell+1} \geq c_{\ell+1}/2$. Going back to (3.55), and using (3.56), we see that for N large

$$EG^2(|R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')| \in [m_{\ell-1}, r_{\ell+1}]) \geq E < (\boldsymbol{\sigma}, \boldsymbol{\sigma}')^{p_\ell} > -m_{\ell-1}^{p_\ell} - \frac{24\sqrt{\delta}}{c_{\ell+1}} \geq \frac{14}{\beta_\ell}$$

by using (3.53), (3.54), (3.50), (3.51).

But by Lemma 3.7 a),

$$EG^2(|R(\boldsymbol{\sigma}, \boldsymbol{\sigma}')| \in [m_{\ell-1}, r_\ell] \cup [m_\ell, r_{\ell+1}]) \leq \exp(-\frac{N}{K}).$$

The Lemma is proved. □

Proof of Theorem 1.2. We perform the previous construction, making sure that (3.50) to (3.52) hold.

We consider the Hamiltonian

$$H_N(\boldsymbol{\sigma}) = \sum_{\ell \leq k} c_\ell H_\ell(\boldsymbol{\sigma}).$$

We set $\theta_\ell = \beta_\ell / c_\ell, \theta'_\ell = (c_\ell - 1)/c_\ell$. If the sequence (p_ℓ) increases fast enough, (1.8) holds. Condition (1.9) follows from Lemma 3.7, b; and (1.10) from Lemma 3.10. □

Remark. It is possible to adapt the arguments of [T], Section 4 to show that the configuration space exhibits a tree structure. The depth of this tree increases by one as β increases from θ_ℓ to θ'_ℓ.

References

[T] Talagrand, M.: Rigorous low temperature results for the p-spins interaction model. Probab. Theory Relat. Fields, **117**, 303–360 (2000) (DOI: 10.1007/s004400000070)