



Hydrodynamic limit of simple exclusion processes in symmetric random environments via duality and homogenization

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Abstract

We consider continuous-time random walks on a random locally finite subset of \mathbb{R}^d with random symmetric jump probability rates. The jump range can be unbounded. We assume some second-moment conditions and that the above randomness is left invariant by the action of the group $\mathbb{G} = \mathbb{R}^d$ or $\mathbb{G} = \mathbb{Z}^d$. We then add a site-exclusion interaction, thus making the particle system a simple exclusion process. We show that, for almost all environments, under diffusive space–time rescaling the system exhibits a hydrodynamic limit in path space. The hydrodynamic equation is non-random and governed by the effective homogenized matrix D of the single random walk, which can be degenerate. The above result covers a very large family of models including e.g. simple exclusion processes built from random conductance models on \mathbb{Z}^d and on crystal lattices (possibly with long conductances), Mott variable range hopping, simple random walks on Delaunay triangulations, random walks on supercritical percolation clusters.

Keywords Simple point process · Palm distribution · Random walk in random environment · Stochastic homogenization · Hydrodynamic limit

Mathematics Subject Classification 60G55 · 60K35 · 60K37 · 35B27

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1 Introduction

The simple exclusion process is a fundamental interacting particle system obtained by adding a site-exclusion interaction to multiple random walks [26]. We assume here that particles lie on a random locally finite subset of \mathbb{R}^d (a simple point process) and allow the jump probability rates to be random as well, but symmetric (i.e. they do not depend on the orientation of the jump). We require that the law of the environment is stationary and ergodic w.r.t. the action of a group \mathbb{G} of \mathbb{R}^d -translations, \mathbb{G} being the full group of translations or a subgroup isomorphic to \mathbb{Z}^d . Under weak second moment assumptions on the jump rates and a percolation assumption assuring the existence of the process, we then prove for almost all environments that the simple exclusion process admits a hydrodynamic limit (HL) in path space with hydrodynamic equation $\partial_t \rho = \nabla \cdot (D \nabla \rho)$, D being the non random effective homogenized matrix associated to a single random walk (D can also be degenerate). The above result (stated in Theorem 4.1 in Sect. 4) covers a very large class of simple exclusion processes in symmetric random environments, e.g. those obtained by adding a site-exclusion interaction to random walks on \mathbb{Z}^d and on general crystal lattices with random (possibly arbitrarily long) conductances, to random walks performing a Mott variable range hopping, to simple random walks on Delaunay triangulations [18] or on supercritical percolation clusters (in Sect. 5 we discuss some examples). In Sect. 2 we provide a brief presentation of our class of models and our main result, without insisting on technicalities (faced in the subsequent sections). We discuss below how the present work relates with the existing literature, the strategy we have followed and the most original aspects of our contribution.

Given a realization of the environment the resulting simple exclusion process is non-gradient. The usual derivation of the HL for non-gradient interacting particle systems based on the method introduced by Varadhan and further developed by Quastel (cf. [26, 34, 39]) is very technical. It becomes even harder in the disordered case (cf. [14, 35]). On the other hand, for disordered simple exclusion processes with symmetric jump rates one can try to avoid the non-gradient machinery by exploiting some *duality property* between the particle system and the single random walk and some *averaging property* of the single random walk. This was first realized by Nagy in [33] for the simple exclusion process on \mathbb{Z} with symmetric random jump rates. Nagy's analysis had two main ingredients: a representation of the exclusion process in terms of compensated Poisson processes and the Markov semigroup of the random walk (see [33, Eq. (12), (13)] and a quenched CLT for the random walk uniformly in the starting point (see [33, Theorem 1]). Nagy's representation (coming from duality) has been further generalized in [9, 10] and in [10] we showed that Nagy's second ingredient can be replaced but a suitable homogenization result of the L^2 -Markov semigroup of the random walk. The advantage comes from the fact that homogenization requires much weaker assumptions than quenched CLT's (moreover, it is also more natural from a physical viewpoint: the light bulb turns on because of the motion of many electrons and not of a single one). One advantage of the approach based on Nagy's representation and homogenization is that one can prove the HL without proving the uniqueness of the weak solution of the Cauchy problem associated to the hydrodynamic limit. On

the other hand, one gets the HL at a fixed macroscopic time (in the form usually stated e.g. in [26]) but not in path space.

To gain the HL in path space, one has to prove the tightness of the empirical measure. This has been achieved in [22] by developing the method of corrected empirical measure (initially introduced in [24]). This method again relies on duality and on homogenization property of the resolvent of the random walk. Once proved the tightness one can proceed in two ways. If a uniqueness result for the Cauchy problem is available, one can try to push further the analysis of the corrected empirical measure and characterize all limit points of the empirical measures as in [22]. Otherwise, one can try to extend Nagy's representation and use homogenization (or some averaging, in general) to get the HL for a fixed time, avoiding results of uniqueness. This has revealed useful e.g. for the subdiffusive system considered in [16], where a quenched CLT for varying and converging initial points was used instead of homogenization.

Of course, the above strategies have been developed in specific contexts and not in full generality. The applications to other models require some work, already in the choice of the right function spaces and topologies. In our proof we used the corrected empirical measure and homogenization to prove tightness. To proceed we have presented the two independent routes: by proving uniqueness for the Cauchy problem in weak form we characterize the limit points of the empirical measure continuing to work with the corrected one; alternatively we prove in "Appendix C" Nagy's representation in our context and use homogenization to get the HL at a fixed time.

We comment now how our result differs from the previous contributions concerning the diffusive HL of simple exclusion processes in symmetric environments. The main novelty is the huge class of models for which the HL has been proved. In particular, (i) we go beyond the lattice (\mathbb{Z}^d or toroidal) structure and deal with a very broad range of random environments including geometrically amorphous ones (think e.g. to a simple exclusion process on a Poisson point process), (ii) our assumptions on the jump rates are minimal and given by 2nd moment assumptions plus a percolation assumption for Harris' percolation argument, (iii) we remove ellipticity conditions on the jump rates and treat also the case of degenerate effective homogenized matrix D , (iv) the jump range can be unbounded. Concerning Item (i) we point out that to gain such a generality we have used the theory of \mathbb{G} -stationary random measures, where $\mathbb{G} = \mathbb{R}^d, \mathbb{Z}^d$ (cf. [20, 21, 25]), in order to fix our general setting in Sect. 3. This also allows to describe the ergodic properties of the environment in terms of the Palm distribution. To achieve the HL in great generality we needed the same generality for the homogenization results. This part, which has also an independent interest, has been presented in the companion work [12], where our homogenization analysis is based on 2-scale convergence. Although [12] has been preliminary to the present work, here we have kept the presentation self-contained.

For completeness, we point out that Theorem 4.1 includes also as very special cases the HL in [10, 33, 36] (for the part concerning non-dynamical random environments in [36]). We recall that in [36] the authors prove the HL for the random conductance model on \mathbb{Z}^d with possibly time-dependent random conductances in a given interval $[a, b]$, with $0 < a < b < +\infty$. Finally, we point out that for reversible but not symmetric jump rates the homogenization results in [12] for a single random walk still hold, but the duality properties of the simple exclusion process fail. An explicit

example is given by the simple exclusion process with site disorder treated in [14, 35]. In general, for reversible but not symmetric jump rates, the hydrodynamic limit is expected to be described by the non-linear equation $\partial_t \rho = \nabla \cdot (D(\rho) \nabla \rho)$ with a density-dependent diffusion matrix $D(\rho)$. As rigorously proved in [35, Theorem 1] in the case of site-disorder, $D(0)$ is expected to coincide with the effective homogenized matrix D associated to a single random walk.

Outline of the paper In Sect. 2 we give a non-technical presentation of setting and results. In Sect. 3 we present more precisely our setting and basic assumptions for the single random walk. In Sect. 4 we state our HL (see Theorem 4.1). In Sect. 5 we discuss some examples. In Sect. 6 we recall the homogenization results from [12] used in the proof of Theorem 4.1. In Sect. 7 we present the graphical construction of the simple exclusion process and analyze its Markov semigroup. In Sect. 8 we collect some results concerning duality. In Sect. 9 we recall some properties of the space \mathcal{M} of Radon measures on \mathbb{R}^d and of the Skorohod space $D([0, T], \mathcal{M})$ and show the uniqueness of the weak solution of the Cauchy problem. In Sect. 10 we study the family of typical environments, for which the HL will be proved. In Sect. 11 we prove Theorem 4.1. In ‘‘Appendix A’’ we present a model satisfying all our assumptions for which the effective homogenized matrix D is nonzero but degenerate. ‘‘Appendix B’’ concerns the proof of Proposition 7.4. In ‘‘Appendix C’’ we give an independent proof of the HL for fixed times by proving Nagy’s representation in our context and by using homogenization.

2 Overview

In this section we give a brief presentation of our context and results postponing a detailed discussion to Sects. 3 and 4. Not surprisingly, this story starts with a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Here are the other characters: the group \mathbb{G} acting on the probability space and acting by translations on \mathbb{R}^d , a simple point process and a family of jump probability rates.

The group \mathbb{G} can be \mathbb{R}^d or \mathbb{Z}^d (the former endowed with the Euclidean distance, the latter with the discrete topology). \mathbb{G} is a measurable space endowed with the Borel σ -algebra and it acts on $(\Omega, \mathcal{F}, \mathcal{P})$ by a family of maps $(\theta_g)_{g \in \mathbb{G}}$, with $\theta_g : \Omega \rightarrow \Omega$, such that

$$\begin{cases} \theta_0 = \mathbb{1}, \\ \theta_g \circ \theta_{g'} = \theta_{g+g'} \text{ for all } g, g' \in \mathbb{G}, \\ \text{the map } \mathbb{G} \times \Omega \ni (g, \omega) \mapsto \theta_g \omega \in \Omega \text{ is measurable.} \end{cases} \quad (1)$$

The group \mathbb{G} acts also on the space \mathbb{R}^d by translations. We denote its action by $(\tau_g)_{g \in \mathbb{G}}$, where $\tau_g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is given by

$$\tau_g x = x + g_1 v_1 + \cdots + g_d v_d, \quad g = (g_1, \dots, g_d) \in \mathbb{G}, \quad (2)$$

for a fixed basis v_1, \dots, v_d of \mathbb{R}^d . For many applications, $\tau_g x = x + g$. When dealing with processes on general lattices (as e.g. the triangular or hexagonal lattice on \mathbb{R}^2), the general form (2) is more suited (see Sect. 5).

We assume to have a simple point process on \mathbb{R}^d defined on our probability space. In particular, to each $\omega \in \Omega$ we associate a locally finite subset $\hat{\omega} \subset \mathbb{R}^d$ by a measurable map $\Omega \ni \omega \rightarrow \hat{\omega} \in \mathcal{N}$. Above, \mathcal{N} is the measurable space of locally finite subsets of \mathbb{R}^d with σ -algebra generated by the sets $\{|\hat{\omega} \cap A| = n\}$, where $A \subset \mathbb{R}^d$ is Borel and $n \in \mathbb{N}$ (cf. [5]). As discussed in [5] one can introduce a metric d on \mathcal{N} such that the above σ -algebra equals the Borel σ -algebra.

Finally, we fix a measurable function

$$c : \Omega \times \mathbb{R}^d \times \mathbb{R}^d \ni (\omega, x, y) \mapsto c_{x,y}(\omega) \in [0, +\infty), \tag{3}$$

symmetric in x, y : $c_{x,y}(\omega) = c_{y,x}(\omega)$. As it will be clear below, only the value of $c_{x,y}(\omega)$ with $x \neq y$ in $\hat{\omega}$ will be relevant. Hence, without loss of generality, we take

$$c_{x,x}(\omega) \equiv 0 \text{ and } c_{x,y}(\omega) \equiv 0 \text{ if } \{x, y\} \not\subset \hat{\omega}. \tag{4}$$

All the above objects are related by \mathbb{G} -invariance. As detailed in Sect. 3, we assume that \mathcal{P} is stationary and ergodic for the action $(\theta_g)_{g \in \mathbb{G}}$. We recall that stationarity means that $\mathcal{P} \circ \theta_g^{-1} = \mathcal{P}$ for all $g \in \mathbb{G}$, while ergodicity means that $\mathcal{P}(A) = 1$ for all translation invariant sets $A \in \mathcal{F}$, i.e. such that $\theta_g A = A$ for all $g \in \mathbb{G}$ (we can identify \mathbb{G} with a subset of Euclidean translations by (2), thus motivating our terminology). We also assume that, for \mathcal{P} -a.a. $\omega \in \Omega$ and for all $g \in \mathbb{G}$, it holds

$$\begin{aligned} \widehat{\theta_g \omega} &= \tau_{-g}(\hat{\omega}), \\ c_{x,y}(\theta_g \omega) &= c_{\tau_g x, \tau_g y}(\omega) \quad \forall x, y \in \mathbb{R}^d. \end{aligned} \tag{5}$$

The minus sign in (5) could appear ugly, but indeed if one identifies $\hat{\omega}$ with the counting measure $\mu_\omega(A) := \sharp(\hat{\omega} \cap A)$, one would restate (5) as $\mu_{\theta_g \omega}(A) = \mu_\omega(\tau_g A)$ for all $A \subset \mathbb{R}^d$ Borel.

Given the environment ω , we will introduce by the standard graphical construction the simple exclusion process on $\hat{\omega}$ with probability rate $c_{x,y}(\omega)$ for a jump between x and y when the exclusion constraint is satisfied. As discussed in Sect. 7 this simple exclusion process is a Feller process whose Markov semigroup on $C(\{0, 1\}^{\hat{\omega}})$ has infinitesimal generator \mathcal{L}_ω acting on local functions as

$$\mathcal{L}_\omega f(\eta) = \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) \eta(x) (1 - \eta(y)) [f(\eta^{x,y}) - f(\eta)], \quad \eta \in \{0, 1\}^{\hat{\omega}}. \tag{7}$$

Above and in what follows, $\{0, 1\}^{\hat{\omega}}$ is endowed with the product topology and $C(\{0, 1\}^{\hat{\omega}})$ denotes the space of continuous functions on $\{0, 1\}^{\hat{\omega}}$ endowed with the uniform topology. We recall that a function f on $\{0, 1\}^{\hat{\omega}}$ is called local if $f(\eta)$ depends on η only through $\eta(x)$ with x varying among a finite set. The configuration $\eta^{x,y}$ is obtained from η by exchanging the occupation variables at x and y , i.e.

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \\ \eta(z) & \text{otherwise.} \end{cases} \quad (8)$$

The generator \mathcal{L}_ω given in (7) can be thought of as an exchange operator:

$$\mathcal{L}_\omega f(\eta) = \sum_{\{x,y\} \subset \hat{\omega}} c_{x,y}(\omega) [f(\eta^{x,y}) - f(\eta)]. \quad (9)$$

When the starting configuration is given by a single particle, the dynamics reduces to a random walk in random environment, denoted by X_t^ω . In Sects. 3 and 4 we will fix basic assumptions assuring the existence of the above processes for all times for \mathcal{P} -a.a. ω .

We can now present the content of our Theorem 4.1 (see Sect. 4), in which we show that, under suitable weak assumptions, for \mathcal{P} -a.a. environments ω the above simple exclusion process admits a hydrodynamic limit under diffusive rescaling. More precisely, for \mathcal{P} -a.a. ω the following holds. Fix an initial macroscopic profile given by a Borel function $\rho_0 : \mathbb{R}^d \rightarrow [0, 1]$. Suppose that for any $\varepsilon > 0$ the simple exclusion process starts with an initial distribution m_ε such that

$$\lim_{\varepsilon \downarrow 0} m_\varepsilon \left(\left| \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta(x) - \int_{\mathbb{R}^d} \varphi(x) \rho_0(x) dx \right| > \varepsilon \right) = 0 \quad \forall \varphi \in C_c(\mathbb{R}^d).$$

Call $\mathbb{P}_{\omega, m_\varepsilon}^\varepsilon$ the law of the exclusion process on $\hat{\omega}$ with initial distribution m_ε and generator $\varepsilon^{-2} \mathcal{L}_\omega$. Then for all $T > 0$ one has

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon \left(\sup_{0 \leq t \leq T} \left| \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta_t(x) - \int_{\mathbb{R}^d} \varphi(x) \rho(x, t) dx \right| > \delta \right) = 0 \quad \forall \varphi \in C_c(\mathbb{R}^d),$$

where $\rho : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$ is given by $\rho(x, t) := P_t \rho_0(x)$, $(P_t)_{t \geq 0}$ being the Markov semigroup on bounded measurable functions of the Brownian motion with diffusion matrix $2D$.

Above D is the so called *effective homogenized matrix*. D is a $d \times d$ symmetric non-negative matrix, admitting a variational characterization in terms of the Palm distribution \mathcal{P}_0 associated to \mathcal{P} (cf. Definition 3.3). D is related to the homogenization properties of the diffusively rescaled random walk $\varepsilon X_{\varepsilon^{-2}t}^\omega$ on $\varepsilon \hat{\omega}$ as discussed in [12]. Some of these properties are collected in Proposition 6.1.

3 Basic assumptions and homogenization

In this section we describe our setting and our basic assumptions for the single random walk X_t^ω (hence the site-exclusion interaction does not appear here). The context is

the same of [12] with the simplification that the jump rates are symmetric, hence the counting measure on $\hat{\omega}$ is reversible for X_t^ω .

We first fix some basic notation. We denote by e_1, \dots, e_d the canonical basis of \mathbb{R}^d , by $\ell(A)$ the Lebesgue measure of the Borel set $A \subset \mathbb{R}^d$, by $a \cdot b$ the standard scalar product of $a, b \in \mathbb{R}^d$. Given a topological space W , without further mention, W will be thought of as a measurable space endowed with the σ -algebra $\mathcal{B}(W)$ of its Borel subsets. \mathcal{N} is the space of locally finite subset $\{x_i\}$ of \mathbb{R}^d . \mathcal{N} is endowed with a metric such that the Borel σ -algebra $\mathcal{B}(\mathcal{N})$ is generated by the sets $\{|\hat{\omega} \cap A| = n\}$, where $A \in \mathcal{B}(\mathbb{R}^d)$ and $n \in \mathbb{N}$ (cf. [5]).

Recall that \mathbb{G} acts on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$ by $(\theta_g)_{g \in \mathbb{G}}$ [see (1)] and that \mathcal{P} is assumed to be stationary and ergodic for this action. Moreover, \mathbb{G} acts on \mathbb{R}^d by $(\tau_g)_{g \in \mathbb{G}}$, where [cf. (2)]

$$\tau_g x = x + Vg, \quad V := [v_1 | v_2 | \dots | v_d]. \tag{10}$$

Above, V is the matrix with columns given by the basis vectors v_1, v_2, \dots, v_d , fixed once and for all.

We set

$$\Delta := \{t_1 v_1 + \dots + t_d v_d : (t_1, \dots, t_d) \in [0, 1)^d\}. \tag{11}$$

Given $x \in \mathbb{R}^d$, the \mathbb{G} -orbit of x is defined as the set $\{\tau_g x : g \in \mathbb{G}\}$.

If $\mathbb{G} = \mathbb{R}^d$, then the \mathbb{G} -orbit of the origin of \mathbb{R}^d equals \mathbb{R}^d . In this case we introduce the function $g : \mathbb{R}^d \rightarrow \mathbb{G}$ as follows:

$$x = \tau_g 0 = Vg \implies g(x) := g. \tag{12}$$

Simply, for each $x \in \mathbb{R}^d$, $g(x) = V^{-1}x$. When $V = \mathbb{I}$ (as in many applications), we have $\tau_g x = x + g$ and therefore $g(x) = x$.

If $\mathbb{G} = \mathbb{Z}^d$, Δ is a set of \mathbb{G} -orbit representatives for the action $(\tau_g)_{g \in \mathbb{G}}$. We introduce the functions $\beta : \mathbb{R}^d \rightarrow \Delta$ and $g : \mathbb{R}^d \rightarrow \mathbb{G}$ as follows:

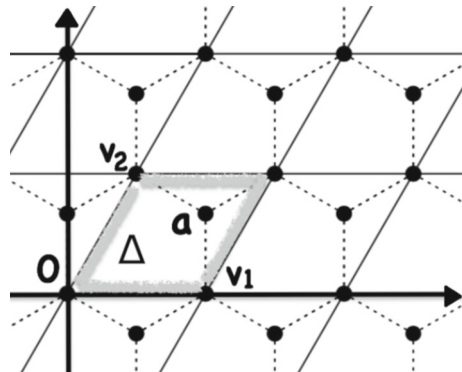
$$x = \tau_g \bar{x} \text{ and } \bar{x} \in \Delta \implies \beta(x) := \bar{x}, \quad g(x) := g. \tag{13}$$

Hence, given $x \in \mathbb{R}^d$, \bar{x} denotes the unique element of Δ such that x and \bar{x} are in the same \mathbb{G} -orbit, and $g(x)$ denotes the unique element in \mathbb{G} such that $x = \tau_{g(x)} \bar{x}$.

3.1 An example with $\mathbb{G} = \mathbb{Z}^d$ and $V \neq \mathbb{I}$

Although we will discuss several examples in Sect. 5, our mathematical objects for $\mathbb{G} = \mathbb{Z}^d$ and $V \neq \mathbb{I}$ could appear very abstract at a first sight. To have in mind something concrete to which refer below, we present an example related to the random walk and the simple exclusion process on the infinite cluster of the supercritical site Bernoulli percolation on the hexagonal lattice (see Sect. 5 for a further discussion). Consider the hexagonal lattice graph $\mathcal{L} = (\mathcal{V}, \mathcal{E})$ in \mathbb{R}^2 , partially drawn in Fig. 1. \mathcal{V} and \mathcal{E} denote respectively the vertex set and the edge set. The vectors v_1, v_2 in Fig. 1 form a fundamental basis for the hexagonal lattice.

Fig. 1 The parallelogram corresponds to the fundamental cell Δ , the vectors v_1, v_2 are the columns of V , $\{0, a\}$ equals $\mathcal{V} \cap \Delta$



We take $\Omega := \{0, 1\}^{\mathcal{V}}$ endowed with the product topology and with the Bernoulli product probability measure \mathcal{P} with supercritical parameter p . We set $\mathbb{G} := \mathbb{Z}^2$. The action $(\theta_g)_{g \in \mathbb{Z}^2}$ is given by $\theta_{(g_1, g_2)}\omega = (\omega_{x - g_1 v_1 - g_2 v_2})_{x \in \mathcal{V}}$ if $\omega = (\omega_x)_{x \in \mathcal{V}}$ (note that v_1, v_2 are 2d vectors and not coordinates, while $(g_1, g_2) \in \mathbb{Z}^2$). Trivially, \mathcal{P} is stationary and ergodic for this action. The action of \mathbb{Z}^2 on \mathbb{R}^2 is given by the translations $\tau_{(g_1, g_2)}x := x + g_1 v_1 + g_2 v_2$. Note that $V = [v_1 | v_2]$.

The cell Δ in (11) is here the fundamental cell of the lattice \mathcal{L} given by the parallelogram with ticked border in Fig. 1 (one has to remove the upper and right edges). Indeed, $\mathcal{V} = \cup_{g \in \mathbb{Z}^d} \tau_g \{0, a\}$ and $\{0, a\} = \mathcal{V} \cap \Delta$. Then the map $\beta : \mathbb{R}^2 \rightarrow \Delta$ in (13) is the map $\beta(x) := \bar{x}$ where \bar{x} is the unique element of Δ such that $x = \bar{x} \bmod \mathbb{Z}v_1 + \mathbb{Z}v_2$. Moreover, the map $g : \mathbb{R}^2 \rightarrow \mathbb{Z}^2$ in (13) assigns to x the only element $g = (g_1, g_2) \in \mathbb{Z}^2$ such that $x \in \tau_g \Delta = \Delta + g_1 v_1 + g_2 v_2$.

We now describe the simple point process $\hat{\omega}$. As p is supercritical, for \mathcal{P} -a.a. ω the set $\{x \in \mathcal{V} : \omega_x = 1\}$ has a unique infinite connected component $\mathcal{C}(\omega)$ inside the lattice \mathcal{L} . We set $\hat{\omega} := \mathcal{C}(\omega)$. To extend this definition to all ω , we set $\mathcal{C}(\omega) := \emptyset$ if ω does not have a unique infinite connected component.

3.2 Palm distribution

We recall that we have a simple point process on \mathbb{R}^d defined on our probability space $(\Omega, \mathcal{F}, \mathcal{P})$. This means that to each $\omega \in \Omega$ we associate a locally finite subset $\hat{\omega} \subset \mathbb{R}^d$ by a measurable map $\Omega \ni \omega \rightarrow \hat{\omega} \in \mathcal{N}$. We now recall the definition of Palm distribution \mathcal{P}_0 associated to our simple point process by distinguishing between two main cases and a special subcase. For a more detailed discussion we refer to [12] and references therein. We remark that our treatment reduces to the one in [5] when $\mathbb{G} = \mathbb{R}^d$, $\Omega = \mathcal{N}$, $\hat{\omega} = \omega$, $V = \mathbb{I}$ (i.e. $\tau_g x = x + g$) and $\theta_g \omega := \tau_{-g} \omega = \omega - g$. When $\mathbb{G} = \mathbb{R}^d$ and in the special discrete case treated below, the Palm distribution \mathcal{P}_0 can be thought of as the probability measure \mathcal{P} conditioned to the event $\{0 \in \hat{\omega}\}$. For the special discrete case see (18) below, while for $\mathbb{G} = \mathbb{R}^d$ some care is required

as the above event has zero \mathcal{P} -probability (see [5, 40] for more details). We will write $\mathbb{E}[\cdot]$ and $\mathbb{E}_0[\cdot]$ for the expectation w.r.t. \mathcal{P} and \mathcal{P}_0 , respectively.¹

- Case $\mathbb{G} = \mathbb{R}^d$. The intensity of the simple point process $\hat{\omega}$ is defined as

$$m := \mathbb{E} \left[\# \left(\hat{\omega} \cap [0, 1)^d \right) \right]. \tag{14}$$

We will assume that $m \in (0, +\infty)$. By the \mathbb{G} -stationarity of \mathcal{P} we have $m\ell(B) = \mathbb{E} \left[\# \left(\hat{\omega} \cap B \right) \right]$ for any $B \in \mathcal{B}(\mathbb{R}^d)$. Then the Palm distribution \mathcal{P}_0 is the probability measure on (Ω, \mathcal{F}) such that, for any $U \in \mathcal{B}(\mathbb{R}^d)$ with $0 < \ell(U) < \infty$ ($\ell(U)$ is the Lebesgue measure of U),

$$\mathcal{P}_0(A) := \frac{1}{m\ell(U)} \int_{\Omega} d\mathcal{P}(\omega) \sum_{x \in \hat{\omega} \cap U} \mathbb{1}_A(\theta_{g(x)}\omega), \quad \forall A \in \mathcal{F}. \tag{15}$$

One can check that \mathcal{P}_0 has support inside the set $\Omega_0 := \{\omega \in \Omega : 0 \in \hat{\omega}\}$.

- Case $\mathbb{G} = \mathbb{Z}^d$. The intensity of the simple point process $\hat{\omega}$ is defined as

$$m := \mathbb{E} \left[\# \left(\hat{\omega} \cap \Delta \right) \right] / \ell(\Delta). \tag{16}$$

By the \mathbb{G} -stationarity of \mathcal{P} , $m\ell(B) = \mathbb{E} \left[\hat{\omega}(B) \right]$ for any $B \in \mathcal{B}(\mathbb{R}^d)$ which is an overlap of translated cells $\tau_g\Delta$ with $g \in \mathbb{G}$. We will assume that $m \in (0, +\infty)$. Then the Palm distribution \mathcal{P}_0 is the probability measure on $(\Omega \times \Delta, \mathcal{F} \otimes \mathcal{B}(\Delta))$ such that

$$\mathcal{P}_0(A) := \frac{1}{m\ell(\Delta)} \int_{\Omega} d\mathcal{P}(\omega) \sum_{x \in \hat{\omega} \cap \Delta} \mathbb{1}_A(\omega, x), \quad \forall A \in \mathcal{F} \otimes \mathcal{B}(\Delta). \tag{17}$$

\mathcal{P}_0 has support inside $\Omega_0 := \{(\omega, x) \in \Omega \times \Delta : x \in \hat{\omega}\}$.

Note that in the Example of Sect. 3.1, the set $\hat{\omega} \cap \Delta$ equals $\{0, a\} \cap \mathcal{C}(\omega)$, a being as in Fig. 1. Moreover, $\Omega_0 = \{(\omega, 0) : \omega \in \Omega, 0 \in \mathcal{C}(\omega)\} \cup \{(\omega, a) : \omega \in \Omega, a \in \mathcal{C}(\omega)\}$.

- *Special discrete case:* $\mathbb{G} = \mathbb{Z}^d$, $V = \mathbb{I}$ and $\hat{\omega} \subset \mathbb{Z}^d \forall \omega \in \Omega$ [see (10)]. This is a subcase of the previous one and in what follows we will call it simply *special discrete case*. Due to its relevance in discrete probability, we discuss it apart pointing out some simplifications. As $V = \mathbb{I}$ we have $\Delta = [0, 1)^d$. In particular (see the case $\mathbb{G} = \mathbb{Z}^d$) \mathcal{P}_0 is concentrated on $\{\omega \in \Omega : 0 \in \hat{\omega}\} \times \{0\}$. Hence we can think of \mathcal{P}_0 simply as a probability measure concentrated on the set $\Omega_0 := \{\omega \in \Omega : 0 \in \hat{\omega}\}$. Formulas (16) and (17) then read

$$m := \mathcal{P}(0 \in \hat{\omega}), \quad \mathcal{P}_0(A) := \mathcal{P}(A \mid 0 \in \hat{\omega}) \quad \forall A \in \mathcal{F}. \tag{18}$$

In what follows, when treating the special discrete case, we will use the above identifications without explicit mention.

¹ With some abuse, when f has a complex form, we will write $\mathbb{E}[f(\omega)]$ instead of $\mathbb{E}[f]$, and similarly $\mathbb{E}_0[f(\omega)]$ instead of $\mathbb{E}_0[f]$.

3.3 Basic assumptions

Recall that the jump probability rates are given by the measurable function $c_{x,y}(\omega)$ in (3), which is symmetric in x, y (i.e. $c_{x,y}(\omega) = c_{y,x}(\omega)$) and recall our convention (4). We also define

$$c_x(\omega) := \sum_{y \in \hat{\omega}} c_{x,y}(\omega) \quad \forall x \in \hat{\omega}. \tag{19}$$

We define the functions $\lambda_k : \Omega_0 \rightarrow [0, +\infty]$ (for $k = 0, 2$) as follows:

$$\begin{cases} \lambda_k(\omega) := \sum_{x \in \hat{\omega}} c_{0,x}(\omega) |x|^k & \text{Case } \mathbb{G} = \mathbb{R}^d \text{ and} \\ \Omega_0 = \{\omega \in \hat{\omega} : 0 \in \hat{\omega}\} & \text{special discrete case,} \end{cases} \tag{20}$$

$$\begin{cases} \lambda_k(\omega, a) := \sum_{x \in \hat{\omega}} c_{a,x}(\omega) |x - a|^k & \text{Case } \mathbb{G} = \mathbb{Z}^d. \\ \Omega_0 = \{(\omega, x) \in \Omega \times \Delta : x \in \hat{\omega}\} \end{cases}$$

For $\mathbb{G} = \mathbb{R}^d$ and in the special discrete case, $\lambda_0(\omega) = c_0(\omega)$ for all $\omega \in \Omega_0$.

We collect below all our assumptions leading to homogenization of the massive Poisson equation of the diffusively rescaled random walk (some of them have already been mentioned in Sect. 2). We will not recall here the above homogenization results obtained in [12], as not necessary. On the other hand, we will collect some of their consequences in Proposition 6.1 in Sect. 6, since used in the proof of Theorem 4.1.

Assumptions for homogenization:

- (A1) \mathcal{P} is stationary and ergodic w.r.t. the action $(\theta_g)_{g \in \mathbb{G}}$ of the group \mathbb{G} ;
- (A2) the intensity m of the simple point process $\hat{\omega}$ is finite and positive [cf. (14), (16) and (18)];
- (A3) the ω 's in Ω such that $\theta_g \omega \neq \theta_{g'} \omega$ for all $g \neq g'$ in \mathbb{G} form a measurable set of \mathcal{P} -probability 1;
- (A4) the ω 's in Ω such that, for all $g \in \mathbb{G}$ and $x, y \in \mathbb{R}^d$,

$$\widehat{\theta_g \omega} = \tau_{-g}(\hat{\omega}), \tag{21}$$

$$c_{x,y}(\theta_g \omega) = c_{\tau_g x, \tau_g y}(\omega), \tag{22}$$

form a measurable set of \mathcal{P} -probability 1;

- (A5) for \mathcal{P} -a.a. $\omega \in \Omega$, for all $x, y \in \mathbb{R}^d$ it holds

$$c_{x,y}(\omega) = c_{y,x}(\omega); \tag{23}$$

- (A6) for \mathcal{P} -a.a. $\omega \in \Omega$, given any $x, y \in \hat{\omega}$ there exists a path $x = x_0, x_1, \dots, x_{n-1}, x_n = y$ such that $x_i \in \hat{\omega}$ and $c_{x_i, x_{i+1}}(\omega) > 0$ for all $i = 0, 1, \dots, n - 1$;
- (A7) $\lambda_0, \lambda_2 \in L^1(\mathcal{P}_0)$;
- (A8) $L^2(\mathcal{P}_0)$ is separable.

The above assumptions implies that \mathcal{P} -a.s. the random walk X_t^ω on $\hat{\omega}$ introduced in Sect. 2 is well defined for all times $t \geq 0$ (recall that a set $A \subset \Omega$ is called translation invariant if $\theta_g A = A$ for all $g \in \mathbb{G}$):

Lemma 3.1 [12, Lemma 3.5] *There exists a translation invariant measurable set $\mathcal{A} \subset \Omega$ with $\mathcal{P}(\mathcal{A}) = 1$ such that, for all $\omega \in \mathcal{A}$, (i) $c_x(\omega) \in (0, +\infty)$ for all $x \in \hat{\omega}$ [cf. (19)], (ii) the continuous-time Markov chain on $\hat{\omega}$ starting at any $x_0 \in \hat{\omega}$, with waiting time parameter $c_x(\omega)$ at $x \in \hat{\omega}$ and with probability $c_{x,y}(\omega)/c_x(\omega)$ for a jump from x to y , is non-explosive.*

In Sect. 4 we will make an additional assumption [called Assumption (SEP)] assuring that the simple exclusion process introduced via the universal graphical construction is well defined for all times [see (7) for its generator on local functions]. Hence, by thinking the random walk X_t^ω as a simple exclusion process with only one particle, also Assumption (SEP) guarantees the well-definedness of X_t^ω .

We now report some other comments on the above assumptions (A1), ..., (A8) taken from [12, Section 2.4] (where more details are provided). By Zero-Infinity Dichotomy (see [5, Proposition 10.1.IV]) and Assumptions (A1) and (A2), for \mathcal{P} -a.a. ω the set $\hat{\omega}$ is infinite. (A3) is a rather superfluous assumption as one can add some randomness by enlarging Ω to assure (A3). The assumption of measurability in (A3) and (A4) is always satisfied for $\mathbb{G} = \mathbb{Z}^d$ by (4) (as discussed in [12, Section 2.4], one can even weaken this requirement). Considering the random walk X_t^ω , (A5) and (A6) correspond \mathcal{P} -a.s. to reversibility of the counting measure and to irreducibility. Finally, we point out that, by [3, Theorem 4.13], (A8) is fulfilled if $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is a separable measure space where $\mathcal{F}_0 := \{A \cap \Omega_0 : A \in \mathcal{F}\}$ (i.e. there is a countable family $\mathcal{G} \subset \mathcal{F}_0$ such that the σ -algebra \mathcal{F}_0 is generated by \mathcal{G}). For example, if Ω_0 is a separable metric space and $\mathcal{F}_0 = \mathcal{B}(\Omega_0)$ [which is valid if Ω is a separable metric space and $\mathcal{F} = \mathcal{B}(\Omega)$] then (cf. [3, p. 98]) $(\Omega_0, \mathcal{F}_0, \mathcal{P}_0)$ is a separable measure space and (A8) is valid.

We now explain why the Palm distribution \mathcal{P}_0 will play a crucial role in the hydrodynamic limit of the simple exclusion process. \mathcal{P}_0 is indeed the natural object to express the ergodic property of the environment when dealing with observables keeping track also of the local microscopic details of the environment. This is formalized by the following result which will be frequently used below (cf. [11, Appendix B], [12, Proposition 3.1] and recall that \mathbb{E}_0 denotes the expectation w.r.t. \mathcal{P}_0):

Proposition 3.2 *Let $f : \Omega_0 \rightarrow \mathbb{R}$ be a measurable function with $\|f\|_{L^1(\mathcal{P}_0)} < \infty$. Then there exists a translation invariant measurable subset $\mathcal{A}[f] \subset \Omega$ such that $\mathcal{P}(\mathcal{A}[f]) = 1$ and such that, for any $\omega \in \mathcal{A}[f]$ and any $\varphi \in C_c(\mathbb{R}^d)$, it holds*

$$\lim_{\varepsilon \downarrow 0} \int d\mu_\omega^\varepsilon(x) \varphi(x) f(\theta_{g(x/\varepsilon)}\omega) = \int dx m\varphi(x) \cdot \mathbb{E}_0[f], \tag{24}$$

where $\mu_\omega^\varepsilon := \sum_{x \in \hat{\omega}} \varepsilon^d \delta_{\varepsilon x}$.

We point out that the above proposition implies that $m = \lim_{\ell \uparrow \infty} \#(\hat{\omega} \cap [-\ell, \ell]^d) / (2\ell)^d$ \mathcal{P} -a.s.

We can now also introduce the effective homogenized matrix D , defined in terms of the Palm distribution:

Definition 3.3 We define the *effective homogenized matrix* D as the unique $d \times d$ symmetric matrix such that:

- **Case $\mathbb{G} = \mathbb{R}^d$ and special discrete case**

$$a \cdot Da = \inf_{f \in L^\infty(\mathcal{P}_0)} \frac{1}{2} \int_{\Omega_0} d\mathcal{P}_0(\omega) \sum_{x \in \hat{\omega}} c_{0,x}(\omega) (a \cdot x - \nabla f(\omega, x))^2, \tag{25}$$

for any $a \in \mathbb{R}^d$, where $\nabla f(\omega, x) := f(\theta_{g(x)}\omega) - f(\omega)$.

- **Case $\mathbb{G} = \mathbb{Z}^d$**

$$\begin{aligned} a \cdot Da &= \inf_{f \in L^\infty(\mathcal{P}_0)} \frac{1}{2} \int_{\Omega \times \Delta} d\mathcal{P}_0(\omega, x) \sum_{y \in \hat{\omega}} c_{x,y}(\omega) (a \cdot (y - x) - \nabla f(\omega, x, y - x))^2, \end{aligned} \tag{26}$$

for any $a \in \mathbb{R}^d$, where $\nabla f(\omega, x, y - x) := f(\theta_{g(y)}\omega, \beta(y)) - f(\omega, x)$ [recall (13)].

We give some comments on the above definition of D . Firstly, it is well posed due to (A7). We also point out that the effective homogenized matrix D , which is defined by a variational formula, can be computed explicitly essentially only in dimension $d = 1$ with positive conductances $c_{x,y}(\omega)$ only between nearest neighboring points x, y of $\hat{\omega}$ (see e.g. [2] and [4, Eq. (4.22)]). On the other hand, in the last years numerical approximation methods for D have been developed in quantitative stochastic homogenization theory (see e.g. [7]).

Under Assumption (A1),..., (A8) the random walk X_t^ω satisfies a weak form of central limit theorem where $2D$ equals the asymptotic diffusion matrix (cf. [12, Theorem 4.4]). Since the position of the random walk can be thought of as an antisymmetric additive functional of the environment viewed from the particle, D has the same structure of a Green–Kubo formula (cf. [4, 27, 29, 30, 38] and references therein).

Finally we introduce an additional assumption assuring a weak form of convergence for the L^2 -Markov semigroup and the L^2 -resolvent associated to the random walk X_t^ω as discussed in Sect. 6 [recall definition (11) of Δ].

Additional assumption for semigroup and resolvent convergence:

(A9) At least one of the following conditions is satisfied:

- (i) for \mathcal{P} -a.a. $\omega \exists C(\omega) > 0$ such that

$$\sharp(\hat{\omega} \cap \tau_k \Delta) \leq C(\omega) \text{ for all } k \in \mathbb{Z}^d; \tag{27}$$

- (ii) at cost to enlarge the probability space Ω one can define random variables $(N_k)_{k \in \mathbb{Z}^d}$ with $\sharp(\hat{\omega} \cap \tau_k \Delta) \leq N_k$ and such that, for some $C_0 \geq 0$, it holds

$$\sup_{k \in \mathbb{Z}^d} \mathbb{E}[N_k] < +\infty, \quad \sup_{k \in \mathbb{Z}^d} \mathbb{E} \left[N_k^2 \right] < +\infty, \tag{28}$$

$$|\text{Cov}(N_k, N_{k'})| \leq C_0 |k - k'|^{-1} \quad \forall k \neq k' \text{ in } \mathbb{Z}^d. \tag{29}$$

Remark 3.4 If one set $N_k := \sharp(\hat{\omega} \cap \tau_k \Delta)$ for $k \in \mathbb{Z}^d$, then to check Condition (ii) in (A9) it is enough to check that $\mathbb{E}[N_0^2] < +\infty$ and (29) [due to (A1) and (A2)]. As discussed in [12, Remark 4.3], when $\mathbb{G} = \mathbb{R}^d$, in (A9) one can replace the cells $\{\tau_k \Delta\}_{k \in \mathbb{Z}^d}$ by the cells of any lattice partition of \mathbb{R}^d .

4 Hydrodynamic limit

Given $\omega \in \Omega$ we consider the simple exclusion process on $\hat{\omega}$ with particle exchange probability rate $c_{x,y}(\omega)$. To have a well defined process for all times $t \geq 0$, \mathcal{P} -a.s., we will use in Sect. 7 Harris’ percolation argument [6]. To this aim, we define

$$\mathcal{E}_\omega := \{ \{x, y\} : x, y \in \hat{\omega}, x \neq y \}. \tag{30}$$

Then, given ω , we associate to each unordered pair $\{x, y\} \in \mathcal{E}_\omega$ a Poisson process $(N_{x,y}(t))_{t \geq 0}$ with intensity $c_{x,y}(\omega)$, such that the $N_{x,y}(\cdot)$ ’s are independent processes when varying the pair $\{x, y\}$. The random object $(N_{x,y}(\cdot))_{\{x,y\} \in \mathcal{E}_\omega}$ takes value in the product space $D(\mathbb{R}_+, \mathbb{N})^{\mathcal{E}_\omega}$, $D(\mathbb{R}_+, \mathbb{N})$ being endowed with the standard Skorohod topology. In the rest, we will denote by $\mathcal{K} = (\mathcal{K}_{x,y}(\cdot))_{\{x,y\} \in \mathcal{E}_\omega}$ a generic element of $D(\mathbb{R}_+, \mathbb{N})^{\mathcal{E}_\omega}$. Moreover, we denote by \mathbb{P}_ω the law on $D(\mathbb{R}_+, \mathbb{N})^{\mathcal{E}_\omega}$ of $(N_{x,y}(\cdot))_{\{x,y\} \in \mathcal{E}_\omega}$.

In this section we add the following assumption (we call it “SEP” for “simple exclusion process”) as the assumption is introduced to assure the existence of the simple exclusion process):

Assumption (SEP). For \mathcal{P} -a.a. ω there exists $t_0 = t_0(\omega) > 0$ such that for \mathbb{P}_ω -a.a. \mathcal{K} the undirected graph $\mathcal{G}_{t_0}(\omega, \mathcal{K})$ with vertex set $\hat{\omega}$ and edges

$$\{ \{x, y\} \in \mathcal{E}_\omega : \mathcal{K}_{x,y}(t_0) \geq 1 \}$$

has only connected components with finite cardinality.

In Sect. 7 we discuss the universal graphical construction of the exclusion process on $\hat{\omega}$ under Assumption (SEP). For \mathcal{P} -a.a. ω the resulting process is a Feller process and the infinitesimal generator \mathcal{L}_ω acts on local functions as in (7) and (9) (see Proposition 7.4).

We denote by \mathcal{M} the space of Radon measures on \mathbb{R}^d endowed with the vague topology and we denote by $D([0, T], \mathcal{M})$ the Skorohod space of càdlàg paths from $[0, T]$ to \mathcal{M} endowed with the Skorohod metric (see Sect. 9 for details). For each $\varepsilon > 0$ we consider the map

$$\{0, 1\}^{\hat{\omega}} \ni \eta \mapsto \pi_\omega^\varepsilon[\eta] := \varepsilon^d \sum_{x \in \hat{\omega}} \eta(x) \delta_{\varepsilon x} \in \mathcal{M}.$$

Above $\pi_\omega^\varepsilon[\eta]$ is the so called *empirical measure* associate to η . Given a path $\eta = (\eta_s)_{0 \leq s \leq T}$ and given $t \in [0, T]$, we define $\pi_{\omega,t}^\varepsilon[\eta] := \pi_\omega^\varepsilon[\eta_t]$.

In what follows, given $\varepsilon > 0$ and a probability measure m on $\{0, 1\}^{\hat{\omega}}$, we denote by $\mathbb{P}_{\omega, m}^\varepsilon$ the law of the diffusively rescaled exclusion process on $\hat{\omega}$ with generator $\varepsilon^{-2}\mathcal{L}_\omega$ and initial distribution m . Note that the time T is fixed and does not appear in the notation.

We denote by $(B_t)_{t \geq 0}$ the Brownian motion on \mathbb{R}^d with diffusion matrix given by $2D$, D being the effective homogenized matrix (see Definition 3.3). As D is symmetric we can fix an orthonormal basis e_1, \dots, e_d of eigenvectors of D , such that e_1, \dots, e_{d_*} have positive eigenvalues, while the other basis vectors have zero eigenvalue. Then the Brownian motion $(B_t)_{t \geq 0}$ is not degenerate when projected on $\text{span}(e_1, \dots, e_{d_*})$, while no motion is present along $\text{span}(e_{d_*+1}, \dots, e_d)$. Given a bounded function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ we set $P_t f(x) := E[f(x + B_t)]$.

Theorem 4.1 *Suppose that Assumptions (A1), ..., (A9) and Assumption (SEP) are satisfied. Then there exists a translation invariant measurable set $\Omega_{\text{typ}} \subset \Omega$ with $\mathcal{P}(\Omega_{\text{typ}}) = 1$, such that for any $\omega \in \Omega_{\text{typ}}$ the simple exclusion process is well defined for any initial distribution and exhibits the following hydrodynamic behavior.*

Let $\rho_0 : \mathbb{R}^d \rightarrow [0, 1]$ be a measurable function and let $\rho : \mathbb{R}^d \times [0, \infty) \rightarrow [0, 1]$ be the function $\rho(x, t) := P_t \rho_0(x)$. Let $\{m_\varepsilon\}_{\varepsilon > 0}$ be an ε -parametrized family of probability measures on $\{0, 1\}^{\hat{\omega}}$ such that the random empirical measure $\pi_\omega^\varepsilon[\eta]$ in \mathcal{M} , with η sampled according to m_ε , converges in probability to $\rho_0(x)dx$ inside \mathcal{M} . In other words, we suppose that, for all $\delta > 0$ and $\varphi \in C_c(\mathbb{R}^d)$, it holds

$$\lim_{\varepsilon \downarrow 0} m_\varepsilon \left(\left| \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta(x) - \int_{\mathbb{R}^d} \varphi(x) \rho_0(x) dx \right| > \delta \right) = 0. \tag{31}$$

Then:

- (i) For all $T > 0$, as $\varepsilon \downarrow 0$ the random path $(\pi_{\omega, t}^\varepsilon[\eta])_{0 \leq t \leq T}$ in $D([0, T], \mathcal{M})$, with $\eta = (\eta_t)_{0 \leq t \leq T}$ sampled according to $\mathbb{P}_{\omega, m_\varepsilon}^\varepsilon$, converges in probability to the deterministic path $(\rho(x, t)dx)_{0 \leq t \leq T}$.
- (ii) For all $T > 0$, $\varphi \in C_c(\mathbb{R}^d)$ and $\delta > 0$, it holds

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon \left(\sup_{0 \leq t \leq T} \left| \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta_t(x) - \int_{\mathbb{R}^d} \varphi(x) \rho(x, t) dx \right| > \delta \right) = 0. \tag{32}$$

The proof of the above theorem is given in Sect. 11 (Sect. 11.2 can be replaced by ‘‘Appendix C’’, the two approaches are alternative). The function $\rho(x, t) = P_t \rho_0(x)$ is the unique weak solution of the Cauchy system

$$\begin{cases} \partial_t \rho = \nabla \cdot (D \nabla \rho) & \text{for } t > 0, \\ \rho(0, \cdot) = \rho_0, \end{cases} \tag{33}$$

in the sense specified by Lemma 9.3 in Sect. 11.2.

Remark 4.2 Theorem 4.1 remains valid if Assumption (SEP) is replaced by any other assumption leading to Proposition 7.4 below. Indeed, the latter contains all the properties used in the proof provided in Sect. 11. See also Remark 6.2 for what concerns modifications to Assumption (A9).

Remark 4.3 The assumptions of Theorem 4.1 do not include that the effective homogenized matrix D is strictly positive definite. Checking this property can be a non-trivial task (see the discussion on the non-degeneracy of D in [12, Introduction and Section 5]). For an example of degenerate and nonzero D see “Appendix A”.

5 Some applications

There are plenty of examples to which Theorem 4.1 can be applied. We discuss here four main classes. The application of Theorem 4.1 to the simple exclusion process with random jump rates on the Delaunay triangulation is discussed in [18].

5.1 Nearest-neighbor random conductance model on $\mathbb{Z}^d, d \geq 1$

We take $\mathbb{G} := \mathbb{Z}^d$ acting on \mathbb{R}^d by standard translations, i.e. $\tau_g x = x + g$. Let \mathbb{E}^d be the set of unoriented edges of \mathbb{Z}^d and endow $\Omega := (0, +\infty)^{\mathbb{E}^d}$ with the product topology. Given $\omega \in \Omega$, we write $\omega_{x,y}$ for the component of ω associated to the edge $\{x, y\} \in \mathbb{E}^d$. The action $(\theta_x)_{x \in \mathbb{Z}^d}$ is the standard one: $(\theta_x \omega)_{a,b} := \omega_{a+x, b+x}$. We set $\hat{\omega} := \mathbb{Z}^d$, hence the exclusion process lives on \mathbb{Z}^d . We define $c_{x,y}(\omega) := \omega_{x,y}$ if $\{x, y\} \in \mathbb{E}^d$ and $c_{x,y}(\omega) := 0$ otherwise. It is simple to check that Assumptions (A1), ..., (A9) are satisfied whenever \mathcal{P} is stationary and ergodic, \mathcal{P} satisfies (A3) (which is a rather superfluous assumption, as already commented) and $\mathbb{E}[\omega_{x,y}] < +\infty$ for all $\{x, y\} \in \mathbb{E}^d$. When $d = 1$, D can be explicitly computed and one gets $D = 1/\mathbb{E}[1/c_{0,1}(\omega)] \in [0, +\infty)$ (apply [2, Proposition 4.1 and Exercise 4.3] or use the characterization of D as a.s. limit (for $n \rightarrow +\infty$) of $2n$ times the effective conductivity under unit potential of the 1d resistor network with node set $[-n, n] \cap \mathbb{Z}$ and with nearest-neighbors conductances $c_{x,y}(\omega)$ [13]). For $d \geq 2$ the variational problem in (25) leading to D does not have an explicit solution.

Below, given $k > 0$, we say that the random conductances $\omega_{x,y}$ are k -dependent if, given $A, B \subset \mathbb{Z}^d$ with Euclidean distance between A and B larger than k , the random fields

$$\left(\omega_{x,y} : \{x, y\} \in \mathbb{E}^d, x, y \in A\right) \text{ and } \left(\omega_{x,y} : \{x, y\} \in \mathbb{E}^d, x, y \in B\right)$$

are independent (see [23, page 178] for a similar definition).

Proposition 5.1 *Assumption (SEP) is satisfied if at least one of the following conditions is satisfied:*

- (i) \mathcal{P} -a.s. there exists a constant $C(\omega)$ such that $\omega_{x,y} \leq C(\omega)$ for all $\{x, y\} \in \mathbb{E}^d$;
- (ii) under \mathcal{P} the random conductances $\omega_{x,y}$ are independent;

(iii) under \mathcal{P} the random conductances $\omega_{x,y}$ are k -dependent with $k > 0$.

We note that, by ergodicity, in Item (i) one could just restrict to a non-random upper bound C . Item (ii) is a special case of Item (iii).

Proof We start with Item (i). As $\mathbb{P}_\omega(\mathcal{K}_{x,y}(t_0) > 0) = 1 - e^{-\omega_{x,y}t_0}$, it is enough to take t_0 small to have $1 - e^{-C(\omega)t_0} < p_c$, $p_c > 0$ being the critical probability for the Bernoulli bond percolation on \mathbb{Z}^d .

Let us consider Items (ii) and (iii). We present an argument valid for all $d \geq 1$ (but for $d = 1$ one can give easily a more direct proof). By \mathbb{Z}^d -stationarity the distribution of $\omega_{x,y}$ depends only on the axis parallel to the edge $\{x, y\}$. To simplify the notation we suppose that the conductances are identically distributed with common distribution ν (otherwise one has just to deal with a finite family of distributions $\nu_1, \nu_2, \dots, \nu_d$ in the stochastic domination below). We observe that, for any $C_0 > 0$, the graph $\mathcal{G}_{t_0}(\omega, \mathcal{K})$ described in Assumption (SEP) is contained in the graph $\mathcal{G}'_{t_0}(\omega, \mathcal{K})$ with edges $\{x, y\} \in \mathbb{E}^d$ such that

$$\omega_{x,y} > C_0 \quad \text{or} \quad \begin{cases} \omega_{x,y} \leq C_0, \\ \mathcal{K}_{x,y}(t_0) > 0. \end{cases}$$

Given $e \in \mathbb{E}^d$ we set $Y_e(\omega, \mathcal{K}) := 1$ if e is present in $\mathcal{G}'_{t_0}(\omega, \mathcal{K})$, otherwise we set $Y_e = 0$. We define $\alpha(C_0) := \nu((C_0, +\infty))$. Then, under $\mathbb{P} := \int d\mathcal{P}(\omega)\mathbb{P}_\omega$, the random field $Y = (Y_e)_{e \in \mathbb{E}^d}$ is stationary, satisfies $\mathbb{P}(Y_e = 1) \leq \alpha(C_0) + (1 - \alpha(C_0))(1 - e^{-C_0t_0})$ and is given by independent r.v.'s under (ii) and by k -dependent r.v.'s under (iii). Hence, fixed $p_* \in (0, p_c)$, we can first choose C_0 large and afterwards t_0 small to have $\mathbb{P}(Y_e = 1) \leq p_*$. In particular, in case (ii) we conclude that \mathbb{P} -a.s. Y does not percolate. Similarly to [23, Theorem (7.65)] (invert the role between 0 and 1 there), by taking p_* small enough we get that the random field Y is stochastically dominated by a subcritical Bernoulli bond percolation (i.e. of parameter smaller than p_c) and therefore \mathbb{P} -a.s. Y does not percolate. Hence, in both cases (ii) and (iii), by suitably choosing C_0, t_0 , the graph $\mathcal{G}'_{t_0}(\omega, \mathcal{K})$ has only connected components with finite cardinality \mathbb{P} a.s. (i.e. for \mathcal{P} -a.a. ω and for \mathbb{P}_ω -a.a. \mathcal{K}). The same then must hold for $\mathcal{G}_{t_0}(\omega, \mathcal{K}) \subset \mathcal{G}'_{t_0}(\omega, \mathcal{K})$. □

5.2 Nearest-neighbor random conductance models on a generic crystal lattice

We consider a generic crystal lattice $\mathcal{L} = (\mathcal{V}, \mathcal{E})$ in \mathbb{R}^d , $d \geq 1$, as follows. We fix a basis v_1, \dots, v_d of \mathbb{R}^d , write V for the matrix with columns v_1, \dots, v_d and write Δ for the d -dimensional cell (11). Given $g \in \mathbb{G} := \mathbb{Z}^d$, we denote by τ_g the translation (10), i.e. $\tau_g x = x + Vg$. We fix a finite set $\mathcal{A} \subset \Delta$. Then the vertex set \mathcal{V} of the crystal lattice is given by $\sqcup_{g \in \mathbb{G}} (\tau_g \mathcal{A})$. The edge set \mathcal{E} has to be a family of unoriented pairs of vertices $\{x, y\}$ with $x \neq y$ in \mathcal{V} , such that $\tau_g \mathcal{E} = \mathcal{E}$ for all $g \in \mathbb{G}$. In particular, the crystal lattice $\mathcal{L} = (\mathcal{V}, \mathcal{E})$ is left invariant by the action $(\tau_g)_{g \in \mathbb{G}}$ on \mathbb{R}^d . As an example consider the hexagonal lattice $\mathcal{L} = (\mathcal{V}, \mathcal{E})$ in \mathbb{R}^2 (cf. Sect. 3.1). Then $\mathcal{A} = \{0, a\}$ (see Fig. 1).

We take $\Omega := (0, +\infty)^\mathcal{E}$ endowed with the product topology and set $\omega_{x,y} := \omega_{\{x,y\}}$. The action of $(\theta_g)_{g \in \mathbb{G}}$ on Ω is given by $\theta_g \omega := (\omega_{x-Vg,y-Vg} : \{x,y\} \in \mathcal{E})$ if $\omega = (\omega_{x,y} : \{x,y\} \in \mathcal{E})$. For any $\omega \in \Omega$, we set $\hat{\omega} := \mathcal{V}$, hence our simple exclusion process lives on \mathcal{V} . The set Ω_0 introduced after (17) equals $\Omega \times \mathcal{A}$ and, by (16), $m\ell(\Delta) = |\mathcal{A}|$. Hence [see (17)] $\mathcal{P}_0(d\omega, dx) = \mathcal{P}(d\omega) \otimes \text{Av}_{u \in \mathcal{A}} \delta_u(dx)$, where Av denotes the arithmetic average and δ_u is the Dirac measure at u .

We set $c_{x,y}(\omega) := \omega_{x,y}$ if $\{x,y\} \in \mathcal{E}$ and $c_{x,y}(\omega) := 0$ otherwise. If \mathcal{P} satisfies (A1), (A2), (A3) and the crystal lattice is connected, then all assumptions (A1), ..., (A9) are satisfied if $\sum_{y \in \mathcal{V}} \sum_{u \in \mathcal{A}} \mathbb{E}[\omega_{u,y}] |y - u|^2 < +\infty$. If the crystal lattice is locally finite (i.e. vertices have finite degree), then the above moment bound equal the bound $\mathbb{E}[\omega_{x,y}] < +\infty$ for $\{x,y\} \in \mathcal{E}$ (by \mathbb{G} -stationarity and local finiteness, we have just a finite family of bounds).

For locally finite crystal lattices, by reasoning as done for the lattice \mathbb{Z}^d , we get that Assumption (SEP) is satisfied if the conductances $\omega_{x,y}$ are uniformly bounded or if the conductances $\omega_{x,y}$ are independent or k -dependent under \mathcal{P} .

5.3 Simple exclusion processes on marked simple point processes

We take $\mathbb{G} := \mathbb{R}^d$ ($d \geq 1$) acting on \mathbb{R}^d by standard translations ($\tau_g x = x + g$). Ω is given by the space of marked counting measures with marks in \mathbb{R} [5], hence $(\Omega, \mathcal{F}, \mathcal{P})$ describes a marked simple point process [5]. By identifying ω with its support, we have $\omega = \{(x_i, E_i)\}$ where $E_i \in \mathbb{R}$ and the set $\{x_i\}$ is locally finite. The action θ_x on Ω is given by $\theta_x \omega := \{(x_i - x, E_i)\}$ if $\omega = \{(x_i, E_i)\}$. Our simple point process is obtained by setting $\hat{\omega} = \{x_i\}$ when $\omega = \{(x_i, E_i)\}$. We take

$$c_{x_i,x_j}(\omega) := \exp\{-|x_i - x_j| - u(E_{x_i}, E_{x_j})\} \quad x_i \neq x_j, \tag{34}$$

where $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a symmetric measurable function bounded from below. We point out that Mott random walk, used to model Mott variable range hopping in amorphous solids (see e.g. [15, 17] and references therein) is the random walk with jump rates $c_{x,y}(\omega)$ as above, with $u(a,b) = |E_a - E_b| + |E_a| + |E_b|$.

Suppose that \mathcal{P} satisfies (A1),(A2) and (A3). Then \mathcal{P}_0 is simply the standard Palm distribution associated to the marked simple point process with law \mathcal{P} [5]. Assumptions (A4), (A5), (A6) are automatically satisfied. As the above space Ω is Polish (see [5]) and $\Omega_0 = \{\omega : 0 \in \hat{\omega}\}$ is a Borel subset of Ω , Ω_0 is separable and therefore (A8) is satisfied. As proven in [12, Section 5.4], (A7) is implied by the bound $\mathbb{E}[|\hat{\omega} \cap [0, 1]^d|^2] < +\infty$. Assumption (A9) is verified in numerous examples of marked simple point processes, including the Poisson point process (PPP) with intensity $m \in (0, +\infty)$. Assumption (SEP) is of percolation nature. We show its validity for PPP's. Moreover, since one can consider as well other jump rates $c_{x,y}(\omega)$ for a random walk on a marked simple point process, we state our percolation result in a more general form.

Proposition 5.2 *Suppose that under \mathcal{P} the random set $\{x_i\}$ is a PPP with intensity $m \in (0, +\infty)$. Take jump rates $c_{x,y}(\omega)$ satisfying (22) in (A4) and (23) in (A5). Suppose that, for \mathcal{P} -a.a. ω , $c_{x,y}(\omega) \leq g(|x - y|)$ for any $x, y \in \hat{\omega}$, where $g(r)$ is*

a fixed bounded function such that the map $x \mapsto g(|x|)$ belongs to $L^1(\mathbb{R}^d, dx)$ [for example take $g(r) = Ce^{-r}$ for (34)]. Then Assumption (SEP) is satisfied.

Proof Note that $\mathbb{P}_\omega(\mathcal{K}_{x,y}(t) \geq 1) = 1 - e^{-c_{x,y}(\omega)t} \leq 1 - \exp\{-g(|x - y|)t\} \leq C_1 g(|x - y|)t$ for some fixed $C_1 > 0$ if we take $t \leq 1$ (since g is bounded). We restrict to t small enough such that $C_1 \|g\|_\infty t < 1$ and $t \leq 1$. Consider the random connection model [31] on a PPP with intensity m where an edge between $x \neq y$ is created with probability $C_1 g(|x - y|)t$. Due to the independence of the Poisson processes $N_{x,y}(\cdot)$'s given ω , one can couple the above random connection model with the field (ω, \mathcal{K}) with law $\mathbb{P} := \int d\mathcal{P}(\omega)\mathbb{P}_\omega$ in a way that the graph in the random connection model contains the graph $\mathcal{G}_t(\omega, \mathcal{K})$. We choose $t = t_0$ small enough to have $mC_1 t_0 \int_{\mathbb{R}^d} dx g(|x|) < 1$. The above bound and the branching process argument in the proof of [31, Theorem 6.1] [cf. (6.3) there] imply that a.s. the random connection model has only connected components with finite cardinality. Hence the same must hold for $\mathcal{G}_{t_0}(\omega, \mathcal{K})$ □

5.4 Simple exclusion processes on infinite clusters

For completeness we give an example associated to the random geometric structure introduced in Sect. 3.1. Recall that there $\mathcal{L} = (\mathcal{V}, \mathcal{E})$ is the hexagonal lattice, $\Omega = \{0, 1\}^\mathcal{V}$, \mathcal{P} is a Bernoulli site percolation, $\hat{\omega} = \mathcal{C}(\omega)$ is the unique infinite percolation cluster inside \mathcal{L} \mathcal{P} -a.s. We consider the simple exclusion process on $\mathcal{C}(\omega)$ with $c_{x,y}(\omega) = 1$ if $x, y \in \mathcal{C}(\omega)$ and $\{x, y\} \in \mathcal{E}$. The it is trivial to check that Assumptions (A1), . . . , (A9) and (SEP) are all satisfied.

We now explain how Theorem 4.1 improves the hydrodynamic result given by [10, Theorem 2.2]. We take $\mathbb{G} := \mathbb{Z}^d$ and $V := \mathbb{I}$ and define \mathbb{E}^d as in Example 5.1. We take $\Omega := [0, +\infty)^{\mathbb{E}^d}$ with the product topology. The action $(\theta_x)_{x \in \mathbb{Z}^d}$ is the standard one as in Example 5.1. Let \mathcal{P} be a probability measure on Ω stationary, ergodic and fulfilling (A3) for the above action. We assume that for \mathcal{P} -a.a. ω there exists a unique infinite connected component $\mathcal{C}(\omega) \subset \mathbb{Z}^d$ in the graph given by the edges $\{x, y\}$ in \mathbb{E}^d with positive $\omega_{x,y} := \omega_{\{x,y\}}$. We set $\hat{\omega} := \mathcal{C}(\omega)$, $c_{x,y}(\omega) := \omega_{x,y}$ if $\{x, y\}$ is an edge of $\mathcal{C}(\omega)$ and $c_{x,y}(\omega) := 0$ otherwise and assume that $\mathbb{E}[c_{0,e_i}] < +\infty$ for $i = 1, 2, \dots, d$. Then all Assumptions (A1), . . . , (A9) are satisfied. If at least one of the Items (i), (ii), (iii) in Proposition 5.1 is satisfied, then Assumption (SEP) is satisfied too (by the arguments in the proof of Proposition 5.1) and Theorem 4.1 applies, implying the hydrodynamic limit in path space. This result is stronger than [10, Theorem 2.2], since in [10] $c_{x,y}(\omega)$ has to be bounded uniformly in x, y and ω , D has to be strictly positive definite and the hydrodynamic limit is for a fixed time.

6 Random walk semigroup and resolvent convergence by homogenization

In this section we recall the main results from [12] which will be used in the proof of Theorem 4.1. As in Proposition 3.2 we introduce the atomic measure

$$\mu_\omega^\varepsilon := \sum_{x \in \hat{\omega}} \varepsilon^d \delta_{\varepsilon x}. \tag{35}$$

We also introduce the set [recall (19)]

$$\Omega_1 := \{\omega \in \Omega : c_x(\omega) < +\infty \ \forall x \in \hat{\omega}, \ c_{x,y}(\omega) = c_{y,x}(\omega) \ \forall x, y \in \hat{\omega}\}. \tag{36}$$

As explained in [12, Section 3.3], the set Ω_1 is translation invariant and satisfies $\mathcal{P}(\Omega_1) = 1$. Let us fix $\omega \in \Omega_1$. We call $C_{\text{loc}}(\varepsilon\hat{\omega})$ the space of local functions $f : \varepsilon\hat{\omega} \rightarrow \mathbb{R}$ (here local means that f has finite support, i.e. f is zero outside a finite set).

We define

$$\mathcal{D}_\omega^\varepsilon := \left\{ f \in L^2(\mu_\omega^\varepsilon) : \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) (f(\varepsilon y) - f(\varepsilon x))^2 < +\infty \right\}$$

and introduce the bilinear form

$$\mathcal{E}_\omega^\varepsilon(f, g) := \frac{\varepsilon^{d-2}}{2} \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) (f(\varepsilon y) - f(\varepsilon x))(g(\varepsilon y) - g(\varepsilon x))$$

with domain $\mathcal{D}_\omega^\varepsilon$. Since $\omega \in \Omega_1$ it holds $C_{\text{loc}}(\varepsilon\hat{\omega}) \subset \mathcal{D}_\omega^\varepsilon$, as explained in [12, Section 3.3]. We call $\mathcal{D}_{\omega,*}^\varepsilon$ the closure of $C_{\text{loc}}(\varepsilon\hat{\omega})$ w.r.t. the norm $\|f\|_{L^2(\mu_\omega^\varepsilon)} + \mathcal{E}_\omega^\varepsilon(f, f)^{1/2}$. Then, as stated in [19, Example 1.2.5], the bilinear form $\mathcal{E}_\omega^\varepsilon$ restricted to $\mathcal{D}_{\omega,*}^\varepsilon$ is a regular Dirichlet form. In particular, there exists a unique nonpositive self-adjoint operator $\mathbb{L}_\omega^\varepsilon$ in $L^2(\mu_\omega^\varepsilon)$ such that $\mathcal{D}_{\omega,*}^\varepsilon$ equals the domain of $\sqrt{-\mathbb{L}_\omega^\varepsilon}$ and $\mathcal{E}_\omega^\varepsilon(f, f) = \|\sqrt{-\mathbb{L}_\omega^\varepsilon} f\|_{L^2(\mu_\omega^\varepsilon)}^2$ for any $f \in \mathcal{D}_{\omega,*}^\varepsilon$ (see [19, Theorem 1.3.1]). Due to [19, Lemma 1.3.2 and Exercise 4.4.1], $\mathbb{L}_\omega^\varepsilon$ is the infinitesimal generator of the strongly continuous Markov semigroup $(P_{\omega,t}^\varepsilon)_{t \geq 0}$ on $L^2(\mu_\omega^\varepsilon)$ associated to the random walk $(\varepsilon X_{\varepsilon^{-2}t}^\omega)_{t \geq 0}$ on $\varepsilon\hat{\omega}$ defined in terms of holding times and jump probabilities (see Lemma 3.1). Hence, $P_{\omega,t}^\varepsilon f(x) = E_x[f(\varepsilon X_{\varepsilon^{-2}t}^\omega)]$ for $f \in L^2(\mu_\omega^\varepsilon)$ and $x \in \varepsilon\hat{\omega}$, E_x denoting the expectation when the random walk starts at x . For completeness, although not used below, we report that (using that $\omega \in \Omega_1$) one can check that $C_{\text{loc}}(\varepsilon\hat{\omega}) \subset \mathcal{D}(\mathbb{L}_\omega^\varepsilon)$ and that $\mathbb{L}_\omega^\varepsilon f(\varepsilon x) = \varepsilon^{-2} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) (f(\varepsilon y) - f(\varepsilon x))$ for all $x \in \hat{\omega}$, $\forall f \in C_{\text{loc}}(\varepsilon\hat{\omega})$ (the series in the r.h.s. is well defined being absolutely convergent).

We recall that we write $(P_t)_{t \geq 0}$ for the Markov semigroup associated to the Brownian motion $(B_t)_{t \geq 0}$ on \mathbb{R}^d with diffusion matrix $2D$ given in Definition 3.3 (strictly speaking it would be natural here to refer to the semigroup on $L^2(mdx)$ but P_t will be applied below to bounded functions, hence one can keep the same definition of P_t as for Theorem 4.1). Given $\lambda > 0$ we write $R_{\omega,\lambda}^\varepsilon : L^2(\mu_\omega^\varepsilon) \rightarrow L^2(\mu_\omega^\varepsilon)$ for the resolvent associated to the random walk $\varepsilon X_{\varepsilon^{-2}t}^\omega$, i.e. $R_{\omega,\lambda}^\varepsilon := (\lambda - \mathbb{L}_\omega^\varepsilon)^{-1} = \int_0^\infty e^{-\lambda s} P_{\omega,s}^\varepsilon ds$. We write $R_\lambda : L^2(mdx) \rightarrow L^2(mdx)$ for the resolvent associated to the above Brownian motion $(B_t)_{t \geq 0}$.

Proposition 6.1 [12, Theorem 4.4] *Let Assumptions (A1), ..., (A9) be satisfied. Then there exists a translation invariant measurable set $\Omega_{\sharp} \subset \Omega$ with $\mathcal{P}(\Omega_{\sharp}) = 1$ such that for any $\omega \in \Omega_{\sharp}$, any $f \in C_c(\mathbb{R}^d)$, $\lambda > 0$, $t \geq 0$ it holds:*

$$\lim_{\varepsilon \downarrow 0} \int |P_{\omega,t}^{\varepsilon} f(x) - P_t f(x)| d\mu_{\omega}^{\varepsilon}(x) = 0. \tag{37}$$

$$\lim_{\varepsilon \downarrow 0} \int |R_{\omega,\lambda}^{\varepsilon} f(x) - R_{\lambda} f(x)| d\mu_{\omega}^{\varepsilon}(x) = 0. \tag{38}$$

Remark 6.2 As stated in [12, Remark 4.5] Assumption (A9) is used in [12] only to prove for \mathcal{P} -a.a. ω that

$$\lim_{\ell \uparrow \infty} \overline{\lim}_{\varepsilon \downarrow 0} \int d\mu_{\omega}^{\varepsilon}(x) \psi(|x|) \mathbb{1}_{\{|x| \geq \ell\}} = 0, \quad \psi(r) := 1/(1 + r^{d+1}). \tag{39}$$

Hence, in Theorem 4.1 one could replace (A9) by any other condition leading to the above property (39) \mathcal{P} -a.s.

For later use, we also point out that the ω 's satisfying (39) form a translation invariant measurable set.

7 Graphical construction of the simple exclusion process

Let $t_0 = t_0(\omega)$ be as in Assumption (SEP) in Sect. 4. Recall definition (30) of \mathcal{E}_{ω} .

Definition 7.1 (*Property (P_r)*) Given $r \in \mathbb{N}$ we say that the pair $(\omega, \mathcal{K}) \in \Omega \times D(\mathbb{R}_+, \mathbb{N})^{\mathcal{E}_{\omega}}$ satisfies property (P_r) if the undirected graph $\mathcal{G}_{t_0}^r(\omega, \mathcal{K})$ with vertex set $\hat{\omega}$ and edge set $\{\{x, y\} \in \mathcal{E}_{\omega} : \mathcal{K}_{x,y}((r + 1)t_0) > \mathcal{K}_{x,y}(rt_0)\}$ has only connected components with finite cardinality.

Recall definition (19) of $c_x(\omega)$.

Definition 7.2 (*Set $\tilde{\Omega}$*) The set $\tilde{\Omega}$ is given by the elements $\omega \in \Omega$ such that $c_x(\omega) < +\infty \forall x \in \hat{\omega}$ and such that the properties in Assumptions (A4) and (A5) are fulfilled [namely, (21), (22), (23) hold for all x, y, g].

As already pointed out in Sect. 6, the set Ω_1 defined in (36) is a translation invariant set and $\mathcal{P}(\Omega_1) = 1$. It is trivial to check that the same holds for $\tilde{\Omega} \subset \Omega_1$.

Definition 7.3 (*Sets $\mathbb{K}_{\omega}, \Omega_*$*) Fixed $\omega \in \Omega$, \mathbb{K}_{ω} is the set given by the elements $\mathcal{K} \in D(\mathbb{R}_+, \mathbb{N})^{\mathcal{E}_{\omega}}$ such that

- (i) (ω, \mathcal{K}) satisfies property (P_r) for all $r \in \mathbb{N}$;
- (ii) the jump time sets $\{t > 0 : \mathcal{K}_{x,y}(t-) \neq \mathcal{K}_{x,y}(t)\}$ are disjoint as $\{x, y\}$ varies among \mathcal{E}_{ω} ;
- (iii) $\mathcal{K}_x(t) := \sum_{y:\{x,y\} \in \mathcal{E}_{\omega}} \mathcal{K}_{x,y}(t) < +\infty$ for all $x \in \hat{\omega}$ and $t \geq 0$.

We define Ω_* as the set of $\omega \in \tilde{\Omega}$ such that $\mathbb{P}_{\omega}(\mathbb{K}_{\omega}) = 1$.

Since $\mathcal{P}(\tilde{\Omega}) = 1$ and by the loss of memory of the Poisson point process, we have that $\mathcal{P}(\Omega_*) = 1$. It is simple to check that Ω_* is translation invariant.

Also for later use, we now recall the graphical construction of the simple exclusion process. To this aim it is convenient to think the simple exclusion process as an exchange process.

Let us fix $\omega \in \Omega_*$ and $\mathcal{K} \in \mathbb{K}_\omega$. Given a particle configuration $\xi \in \{0, 1\}^{\hat{\omega}}$ we now define a deterministic trajectory $(\eta_t^\xi[\mathcal{K}])_{t \geq 0}$ in $D(\mathbb{R}_+, \{0, 1\}^{\hat{\omega}})$ and starting at ξ by an iterative procedure. We set $\eta_0^\xi[\mathcal{K}] := \xi$. Suppose that the deterministic trajectory has been defined up to time $rt_0, r \in \mathbb{N}$ (note that for $r = 0$ this follows from our definition of $\eta_0^\xi[\mathcal{K}]$). As $\mathcal{K} \in \mathbb{K}_\omega$ all connected components of $\mathcal{G}_{t_0}^r(\omega, \mathcal{K})$ have finite cardinality. Let \mathcal{C} be such a connected component and let

$$\begin{aligned} & \{s_1 < s_2 < \dots < s_k\} \\ & = \{s : \mathcal{K}_{x,y}(s) = \mathcal{K}_{x,y}(s-) + 1, \{x, y\} \text{ bond in } \mathcal{C}, rt_0 < s \leq (r + 1)t_0\}. \end{aligned}$$

As $\mathcal{K} \in \mathbb{K}_\omega$, the l.h.s. is indeed a finite set. The local evolution $\eta_t^\xi[\mathcal{K}](z)$ with $z \in \mathcal{C}$ and $rt_0 < t \leq (r + 1)t_0$ is described as follows. Start with $\eta_{rt_0}^\xi[\mathcal{K}]$ as configuration at time rt_0 in \mathcal{C} . At time s_1 exchange the values between $\eta(x)$ and $\eta(y)$ if $\mathcal{K}_{x,y}(s_1) = \mathcal{K}_{x,y}(s_1-) + 1$ and $\{x, y\}$ is an edge in \mathcal{C} (there is exactly one such edge as $\mathcal{K} \in \mathbb{K}_\omega$). Repeat the same operation orderly for times s_2, s_3, \dots, s_k . Then move to another connected component of $\mathcal{G}_{t_0}^r(\omega, \mathcal{K})$ and repeat the above construction and so on. As the connected components are disjoint, the resulting path does not depend on the order by which we choose the connected components in the above algorithm. This procedure defines $(\eta_t^\xi[\mathcal{K}])_{rt_0 < t \leq (r+1)t_0}$. Starting with $r = 0$ and progressively increasing r by 1 we get the trajectory $(\eta_t^\xi[\mathcal{K}])_{t \geq 0}$.

We recall that $C(\{0, 1\}^{\hat{\omega}})$ is the space of continuous functions on $\{0, 1\}^{\hat{\omega}}$ endowed with the uniform topology. Given $\omega \in \Omega_*$ we consider the probability space $(\mathbb{K}_\omega, \mathbb{P}_\omega)$, and write \mathbb{E}_ω for the associated expectation. We set

$$S(t)f(\xi) := \mathbb{E}_\omega[f(\eta_t^\xi[\mathcal{K}])], \quad t \geq 0, \quad f \in C(\{0, 1\}^{\hat{\omega}}).$$

Proposition 7.4 *Take $\omega \in \Omega_*$ and fix $\xi \in \{0, 1\}^{\hat{\omega}}$. Then the random trajectory $(\eta_t^\xi[\mathcal{K}])_{t \geq 0}$ with \mathcal{K} sampled in the probability space $(\mathbb{K}_\omega, \mathbb{P}_\omega)$ belongs to the Skorohod space $D(\mathbb{R}_+, \{0, 1\}^{\hat{\omega}})$ and it starts at ξ . It describes a Feller process, called simple exclusion process. In particular, $(S(t))_{t \geq 0}$ is a Markov semigroup on $C(\{0, 1\}^{\hat{\omega}})$. Moreover, the domain of its infinitesimal generator \mathcal{L}_ω contains the family of local functions and for any local function f the function $\mathcal{L}_\omega f$ is given by the right hand sides of (7) and (9), which are absolutely convergent series in $C(\{0, 1\}^{\hat{\omega}})$.*

The above proposition can be derived by the standard arguments used for the graphical construction of the SEP usually presented under the assumption of finite range jumps (see e.g. [37, Section 2.1]). The only exception is given by the derivation of the identities (7) and (9) for local functions, due to possible unbounded jump range. We refer to ‘‘Appendix B’’ for the proof of (7) and (9).

8 Duality

In order to prove the tightness of the empirical measure by means of the corrected empirical one, we need to deal with non local functions on $\{0, 1\}^{\hat{\omega}}$. In what follows we collect the extended results concerning \mathcal{L}_ω and Dynkin martingales, which will be used in Sect. 11.1. Recall (19).

In all this section we restrict to $\omega \in \Omega_*$ (cf. Definition 7.3).

Definition 8.1 Given a function $u : \varepsilon\hat{\omega} \rightarrow \mathbb{R}$ such that $\sum_{x \in \hat{\omega}} c_x(\omega)|u(\varepsilon x)| < +\infty$, we define $\tilde{\mathbb{L}}_\omega^\varepsilon u(x) := \varepsilon^{-2} \sum_{y \in \hat{\omega}} c_{x,y}(\omega)(u(\varepsilon y) - u(\varepsilon x))$.

By symmetry of the jump rates we have

$$\sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega)(|u(\varepsilon x)| + |u(\varepsilon y)|) = 2 \sum_{x \in \hat{\omega}} c_x(\omega)|u(\varepsilon x)|. \tag{40}$$

Hence, if $\sum_{x \in \hat{\omega}} c_x(\omega)|u(\varepsilon x)| < +\infty$, the series defining $\tilde{\mathbb{L}}_\omega^\varepsilon u(x)$ is absolutely convergent for any $x \in \hat{\omega}$.

In what follows, to simplify the notation, we write $\pi_\omega^\varepsilon(u), \pi_{\omega,t}^\varepsilon(u)$ for the integral of u w.r.t. $\pi_\omega^\varepsilon[\eta], \pi_{\omega,t}^\varepsilon[\eta]$, respectively. Recall that $\mathbb{L}_\omega^\varepsilon$ is the Markov generator in $L^2(\mu_\omega^\varepsilon)$ of the random walk $(\varepsilon X_{\varepsilon^{-2}t}^\omega)_{t \geq 0}$ (see Sect. 6). Recall that \mathcal{L}_ω is the Markov generator of the simple exclusion process in the function space $C(\{0, 1\}^{\hat{\omega}})$ of continuous functions on $\{0, 1\}^{\hat{\omega}}$ endowed with the uniform topology (see Proposition 7.4). We now state two lemmas which will be crucial when dealing with the corrected empirical measure. We postpone their proofs to the end of the section.

Lemma 8.2 (Duality) *Suppose that $u : \varepsilon\hat{\omega} \rightarrow \mathbb{R}$ satisfies*

$$\sum_{x \in \hat{\omega}} |u(\varepsilon x)| < +\infty \text{ and } \sum_{x \in \hat{\omega}} c_x(\omega)|u(\varepsilon x)| < +\infty. \tag{41}$$

Then $\pi_\omega^\varepsilon(u) = \varepsilon^d \sum_{x \in \hat{\omega}} u(\varepsilon x)\eta(x)$ is an absolutely convergent series in $C(\{0, 1\}^{\hat{\omega}})$. It belongs to the domain $\mathcal{D}(\mathcal{L}_\omega) \subset C(\{0, 1\}^{\hat{\omega}})$ of \mathcal{L}_ω and

$$\mathcal{L}_\omega(\pi_\omega^\varepsilon(u)) = \varepsilon^{d+2} \sum_{x \in \hat{\omega}} \eta(x)\tilde{\mathbb{L}}_\omega^\varepsilon u(\varepsilon x), \tag{42}$$

the r.h.s. of (42) being an absolutely convergent series in $C(\{0, 1\}^{\hat{\omega}})$. If, in addition to (41), it holds $u \in \mathcal{D}(\mathbb{L}_\omega^\varepsilon) \subset L^2(\mu_\omega^\varepsilon)$, then $\mathbb{L}_\omega^\varepsilon u = \tilde{\mathbb{L}}_\omega^\varepsilon u$ and in particular we have the duality relation

$$\mathcal{L}_\omega(\pi_\omega^\varepsilon(u)) = \varepsilon^{d+2} \sum_{x \in \hat{\omega}} \eta(x)\mathbb{L}_\omega^\varepsilon u(\varepsilon x). \tag{43}$$

Let $u : \varepsilon\hat{\omega} \rightarrow \mathbb{R}$ be a function satisfying (41). As, by Lemma 8.2, $\pi_\omega^\varepsilon(u) \in \mathcal{D}(\mathcal{L}_\omega)$, we can introduce on the Skorohod space $D(\mathbb{R}_+, \{0, 1\}^{\hat{\omega}})$ the Dynkin martingale $(\mathcal{M}_{\omega,t}^\varepsilon)_{t \geq 0}$ given by (see e.g. [26, Appendix 1] or [37, Section 3.2])

$$\mathcal{M}_{\omega,t}^\varepsilon := \pi_{\omega,t}^\varepsilon(u) - \pi_{\omega,0}^\varepsilon(u) - \varepsilon^{-2} \int_0^t \mathcal{L}_\omega(\pi_\omega^\varepsilon(u)) (\eta_s) ds. \tag{44}$$

$(\mathcal{M}_{\omega,t}^\varepsilon)_{t \geq 0}$ is a square integrable martingale w.r.t. the filtered probability space $(D(\mathbb{R}_+, \{0, 1\}^{\hat{\omega}}), \mathbb{P}_{\omega, \mathbf{n}_\varepsilon}^\varepsilon, (\mathcal{F}_t)_{t \geq 0})$, \mathbf{n}_ε being an arbitrary initial distribution and \mathcal{F}_t being the σ -field generated $\{\eta_s : 0 \leq s \leq t\}$. Square integrability follows from the property that $\|\mathcal{M}_{\omega,t}^\varepsilon\|_\infty < +\infty$ as the same holds for all addenda in the r.h.s. of (44) (see Lemma 8.2).

Lemma 8.3 *Let $u : \varepsilon\hat{\omega} \rightarrow \mathbb{R}$ be a function satisfying (41). Suppose in addition that $\sum_{x \in \varepsilon\hat{\omega}} c_x(\omega) u(\varepsilon x)^2 < +\infty$. Then the sharp bracket process of $\mathcal{M}_{\omega,t}^\varepsilon$ is given by $\langle \mathcal{M}_{\omega,t}^\varepsilon \rangle_t = \int_0^t B_\omega^\varepsilon(\eta_s) ds$, where*

$$B_\omega^\varepsilon(\eta) = \varepsilon^{2d-2} \sum_{x \in \varepsilon\hat{\omega}} \sum_{y \in \varepsilon\hat{\omega}} c_{x,y}(\omega) [u(\varepsilon x) - u(\varepsilon y)]^2 \eta(x)(1 - \eta(y)). \tag{45}$$

Note that the bound $\sum_{x \in \varepsilon\hat{\omega}} c_x(\omega) u(\varepsilon x)^2 < +\infty$ implies that the r.h.s. of (45) is an absolutely convergent series of functions in $C(\{0, 1\}^{\hat{\omega}})$. For later use, we recall that $\langle \mathcal{M}_{\omega,t}^\varepsilon \rangle_t$ can be characterized as the unique predictable increasing process such that $(\mathcal{M}_{\omega,t}^\varepsilon)^2 - \langle \mathcal{M}_{\omega,t}^\varepsilon \rangle_t$ is a martingale [28, Theorem 8.24].

Remark 8.4 In the proof of Theorem 4.1 (see Sect. 11.1) we will apply the above Lemmas 8.2 and 8.3 just to functions u of the form $R_{\omega,\lambda}^\varepsilon \psi$ for suitable functions $\psi \in C_c(\mathbb{R}^d)$, where $R_{\omega,\lambda}^\varepsilon \psi$ is the resolvent introduced in Sect. 6.

Proof of Lemma 8.2 As $\sum_{x \in \varepsilon\hat{\omega}} |u(\varepsilon x)| < +\infty$, it is simple to check that the series defining $\pi_\omega^\varepsilon(u)$ is indeed an absolutely convergent series of continuous functions w.r.t. the uniform norm. The same holds for the series corresponding to the r.h.s. of (42). Indeed, by (40), $\sum_{x \in \varepsilon\hat{\omega}} \|\eta(x) \tilde{\mathbb{L}}_\omega^\varepsilon u(\varepsilon x)\|_\infty \leq 2\varepsilon^{-2} \sum_{x \in \varepsilon\hat{\omega}} c_x(\omega) |u(\varepsilon x)| < +\infty$.

When the function u is local, also the map $\eta \mapsto \pi_\omega^\varepsilon(u)$ is local. By locality and Proposition 7.4, this map belongs to $\mathcal{D}(\mathcal{L}_\omega)$. In the case of local u , (42) follows from easy computations by (9). We now treat the general case. Given $n \in \mathbb{N}$, we define $u_n(\varepsilon x) := u(\varepsilon x) \mathbb{1}(|\varepsilon x| \leq n)$. As observed above, $\pi_\omega^\varepsilon(u_n)$ is a local function on $\{0, 1\}^{\hat{\omega}}$ belonging to $\mathcal{D}(\mathcal{L}_\omega)$ and (42) holds with u_n instead of u . We claim that

$$\lim_{n \rightarrow \infty} \|\pi_\omega^\varepsilon(u_n) - \pi_\omega^\varepsilon(u)\|_\infty = 0, \tag{46}$$

$$\lim_{n \rightarrow \infty} \left\| \sum_{x \in \varepsilon\hat{\omega}} \eta(x) \tilde{\mathbb{L}}_\omega^\varepsilon u_n(\varepsilon x) - \sum_{x \in \varepsilon\hat{\omega}} \eta(x) \tilde{\mathbb{L}}_\omega^\varepsilon u(\varepsilon x) \right\|_\infty = 0. \tag{47}$$

As \mathcal{L}_ω is a closed operator being a Markov generator, (42) with u_n instead of u , (46) and (47) imply that $\pi_\omega^\varepsilon(u) \in \mathcal{D}(\mathcal{L}_\omega)$ and that (42) holds. To prove (46) and (47) it is enough

to bound the uniform norms appearing there by, respectively, $\varepsilon^d \sum_{x \in \hat{\omega}: |\varepsilon x| > n} |u(\varepsilon x)|$ and $2\varepsilon^{-2} \sum_{x \in \hat{\omega}: |\varepsilon x| > n} c_x(\omega) |u(\varepsilon x)|$ and use (41). This concludes the proof of (42).

It remains to show that $\mathbb{L}_\omega^\varepsilon u = \tilde{\mathbb{L}}_\omega^\varepsilon u$ if $u \in \mathcal{D}(\mathbb{L}_\omega^\varepsilon) \subset L^2(\mu_\omega^\varepsilon)$ in addition to (41). Given a function $f \in C(\{0, 1\}^{\hat{\omega}})$ we write $S(t)f(\eta) := \mathbb{E}_\eta[f(\eta_t)]$ for the Markov semigroup associated to the simple exclusion process (without any time rescaling). Then (42) can be read as

$$\lim_{t \downarrow 0} \sup_{\eta \in \{0, 1\}^{\hat{\omega}}} \left| \frac{(S(t)\pi_\omega^\varepsilon(u))(\eta) - \pi_\omega^\varepsilon[\eta](u)}{t} - \varepsilon^{d+2} \sum_{x \in \hat{\omega}} \eta(x) \tilde{\mathbb{L}}_\omega^\varepsilon u(\varepsilon x) \right| = 0. \tag{48}$$

Given $x_0 \in \hat{\omega}$ we take η corresponding to a single particle located at x_0 . Then $(S(t)\pi_\omega^\varepsilon(u))(\eta) = \varepsilon^d E_{x_0}[u(\varepsilon X_t^\omega)]$ and (48) implies that $\frac{d}{dt} E_{x_0}[u(\varepsilon X_{\varepsilon^{-2}t}^\omega)]|_{t=0} = \tilde{\mathbb{L}}_\omega^\varepsilon u(\varepsilon x_0)$. On the other hand, we know that $u \in \mathcal{D}(\mathbb{L}_\omega^\varepsilon)$. Hence

$$\lim_{t \downarrow 0} \sum_{x \in \hat{\omega}} \left| \frac{E_x[u(\varepsilon X_{\varepsilon^{-2}t}^\omega)] - u(x)}{t} - \mathbb{L}_\omega^\varepsilon u(\varepsilon x) \right|^2 = 0, \tag{49}$$

which implies that $\frac{d}{dt} E_{x_0}[u(\varepsilon X_{\varepsilon^{-2}t}^\omega)]|_{t=0} = \mathbb{L}_\omega^\varepsilon u(\varepsilon x_0)$. Then it must be $\mathbb{L}_\omega^\varepsilon u(\varepsilon x_0) = \tilde{\mathbb{L}}_\omega^\varepsilon u(\varepsilon x_0)$. □

Proof of Lemma 8.3 For u local both $\pi_\omega^\varepsilon(u)$ and its square belong to $\mathcal{D}(\mathcal{L}_\omega)$ being local functions of η . Then the statement in the lemma can be checked by simple computations due to (9), Lemma 8.2 and [26, Lemma 5.1, Appendix 1] (equivalently, [37, Exercise 3.1 and Lemma 8.3]). For the computation of the sharp bracket process we just comment that, by using the symmetry of $c_{x,y}(\omega)$, one easily gets

$$\begin{aligned} & \mathcal{L}_\omega(\pi_\omega^\varepsilon(u)^2) - 2\pi_\omega^\varepsilon(u)\mathcal{L}_\omega(\pi_\omega^\varepsilon(u)) \\ &= \varepsilon^{2d} \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) [u(\varepsilon x) - u(\varepsilon y)]^2 \eta(x)(1 - \eta(y)). \end{aligned}$$

We now move to the general case. For simplicity of notation we write $\mathcal{M}_t, B(\eta)$ instead of $\mathcal{M}_{\omega,t}^\varepsilon, B_\omega^\varepsilon(\eta)$. Similarly, we define $\mathcal{M}_{n,t}$ and $B_n(\eta)$ as in (44) and (45) with u replaced by $u_n, u_n(\varepsilon x) := u(\varepsilon x)\mathbb{1}(|\varepsilon x| \leq n)$. Note that $\lim_{n \rightarrow +\infty} \sup_{t \in [0, T]} \|\mathcal{M}_t - \mathcal{M}_{n,t}\|_\infty = 0$ for any $T > 0$ [see (42), (46) and (47)]. Hence, by the characterization of the sharp bracket process recalled after Lemma 8.3 and by our results for the local case (applied to u_n), to get (45) it is enough to show that $\lim_{n \rightarrow \infty} \|B_n(\eta) - B(\eta)\|_\infty = 0$. To this aim it is enough to show that $\sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) [u_n(\varepsilon x) - u_n(\varepsilon y)]^2$ converges, as $n \rightarrow \infty$, to the analogous expression with u instead of u_n . This follows from the dominated convergence theorem, by dominating $[u_n(\varepsilon x) - u_n(\varepsilon y)]^2$ with $2u(\varepsilon x)^2 + 2u(\varepsilon y)^2$ and by using that $\sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) [u(\varepsilon x)^2 + u(\varepsilon y)^2] = 2 \sum_{x \in \hat{\omega}} c_x(\omega) u(\varepsilon x)^2 < +\infty$. □

9 Space \mathcal{M} of Radon measures and Skorohod space $D([0, T], \mathcal{M})$

Given a measure μ on \mathbb{R}^d and a real function G on \mathbb{R}^d , we will denote by $\mu(G)$ the integral $\int d\mu(x)G(x)$. We denote by \mathcal{M} the space of Radon measures on \mathbb{R}^d , i.e. locally bounded Borel measures on \mathbb{R}^d . \mathcal{M} is endowed with the vague topology, for which $\mu_n \rightarrow \mu$ if and only if $\mu_n(f) \rightarrow \mu(f)$ for all $f \in C_c(\mathbb{R}^d)$. This topology can be defined through a metric, that we now recall also for later use (for more details, see e.g. [37, Appendix A.10]). To this aim we set $B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$. For each $\ell \in \mathbb{N}$ we choose a sequence of functions $(\varphi_{\ell,n})_{n \geq 0}$ such that²

- (i) $\varphi_{\ell,n} \in C_c^\infty(\mathbb{R}^d)$ and $\varphi_{\ell,n}$ is supported on $B_{\ell+1}$;
- (ii) the family $(\varphi_{\ell,n})_{n \geq 0}$ contains a function with values in $[0, 1]$, equal to 1 on B_ℓ and equal to 0 outside $B_{\ell+1}$;
- (iii) for each $\delta > 0$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ with support in B_ℓ there exists $n \geq 0$ such that $\|\varphi_{\ell,n} - \varphi\|_\infty \leq \delta$ and $\sup_{i,k=1}^d \|\partial_{x_i,x_k}^2 \varphi_{\ell,n} - \partial_{x_i,x_k}^2 \varphi\|_\infty \leq \delta$.

For the existence of such a set of functions $\varphi_{\ell,n}$ we refer [37, Appendix A] and discuss only Item (iii) which is in part new. To deal with Item (iii) we use an extended version of the classical Weierstrass approximation theorem (see [32, Theorem 1.6.2]) implying that, for any compact set K , the family \mathcal{P} of polynomial functions with rational coefficients is dense in $C^2(\mathbb{R}^d)$ w.r.t. to the semi-norm $\|f\|_K := \|f\|_{L^\infty(K)} + \sum_{i=1}^d \|\partial_{x_i} f\|_{L^\infty(K)} + \sum_{i,j=1}^d \|\partial_{x_i,x_j}^2 f\|_{L^\infty(K)}$. For each ℓ fix a function g_ℓ as in Item (ii). Given $\varphi \in C_c^\infty(\mathbb{R}^d)$ as in Item (iii), by applying Leibniz rule to $\varphi - g_\ell f = g_\ell(\varphi - f)$, one easily gets that $\|\varphi - g_\ell f\|_{B_{\ell+1}} \leq C(d)\|g_\ell\|_{B_{\ell+1}}\|\varphi - f\|_{B_{\ell+1}}$. Hence, to fulfill Item (iii), it is enough to include into $\{\varphi_{\ell,n}\}$ the countable family of functions $\{g_\ell f : f \in \mathcal{P}\}$.

Definition 9.1 By a relabeling, we write $(\varphi_j)_{j \in \mathbb{N}}$ for the family $(\varphi_{\ell,n})_{\ell,n \in \mathbb{N}}$.

On \mathcal{M} we define the metric $d_{\mathcal{M}}$ as $d_{\mathcal{M}}(\mu, \nu) := \sum_{j=0}^\infty 2^{-j} (1 \wedge |\mu(\varphi_j) - \nu(\varphi_j)|)$. It can be proved that $(\mathcal{M}, d_{\mathcal{M}})$ is a Polish space and that the topology induced by the metric $d_{\mathcal{M}}$ coincides with the vague topology (see e.g. [37, Appendix A.10], [5]).

We write $D([0, T], \mathcal{M})$ for the Skorohod space of \mathcal{M} -valued càdlàg paths $(\mu_t)_{0 \leq t \leq T}$. We recall (cf. [26, Section 4.1]) that $D([0, T], \mathcal{M})$ is a Polish space endowed with the metric

$$d(\mu_\cdot, \nu_\cdot) := \inf_{\lambda \in \Lambda} \max \left\{ \|\lambda\|, \sup_{0 \leq t \leq T} d_{\mathcal{M}}(\mu_t, \nu_t) \right\}, \tag{50}$$

where Λ is the set of strictly increasing continuous functions $\lambda : [0, T] \rightarrow [0, T]$ with $\lambda(0) = 0, \lambda(T) = T$, and $\|\lambda\| := \sup_{s \neq t} |\ln[(\lambda(t) - \lambda(s))/(t - s)]|$. As a subset $A \subset \mathcal{M}$ is relatively compact if and only if $\sup\{\mu(K) : \mu \in A\} < +\infty$ for any compact set $K \subset \mathbb{R}^d$ (cf. [37, Appendix A]), by the same arguments used in the proof of [26, prop 1.7, Chapter 4] one gets the following:

² Some of our requirements will be used to prove the hydrodynamic behavior for \mathcal{P} -a.a. ω and are not strictly necessary to define the metric on \mathcal{M} .

Lemma 9.2 *Given an index set \mathcal{A} , a family of probability measures $\{Q^\alpha\}_{\alpha \in \mathcal{A}}$ on $D([0, T], \mathcal{M})$ is relatively compact (w.r.t. weak convergence) if and only if for any $j \in \mathbb{N}$ the family of probability measures $\{Q^\alpha \circ \Phi_j^{-1}\}_{\alpha \in \mathcal{A}}$ on $D([0, T], \mathbb{R})$ is relatively compact, where*

$$\Phi_j : D([0, T], \mathcal{M}) \ni (\mu_t)_{0 \leq t \leq T} \mapsto (\mu_t(\varphi_j))_{0 \leq t \leq T} \in D([0, T], \mathbb{R}). \tag{51}$$

Recall that $B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$. The following fact can be obtained by suitably modifying and afterwards extend the proof of [37, Theorem A.28 in Appendix A]. W.r.t. the version in [37], we have removed the assumption of non-degenerate diffusion matrix and we have modified the mass bounds.

Lemma 9.3 *Let $v_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ be Borel and bounded. Let $\alpha : [0, T] \rightarrow \mathcal{M}$ be a map such that*

- (i) α is continuous when \mathcal{M} is endowed with the vague topology;
- (ii) $\alpha_0(dx) = v_0(x)dx$;
- (iii) for all $\varphi \in C_c^\infty(\mathbb{R}^d)$ and $t \in [0, T]$ it holds

$$\int_{\mathbb{R}^d} \varphi(x)\alpha_t(dx) = \int_{\mathbb{R}^d} \varphi(x)\alpha_0(dx) + \int_0^t ds \int_{\mathbb{R}^d} \nabla \cdot (D\nabla\varphi)(x)\alpha_s(dx); \tag{52}$$

- (iv) for some constants $C, r_0, \gamma > 0$ it holds $\alpha_s(B_r) \leq Cr^\gamma$ for all $s \in [0, T]$ and all $r \geq r_0$.

Then $\alpha_t(dx) = P_t v_0(x)dx$ for all $t \in [0, T]$.

Proof We distinguish two cases according to the non-degeneracy of D .

• *Case D non-degenerate.* The proof is the same of [37, Theorem A.28] apart of modifying [37, Eq. (A.40)]. To this aim, as in [37], let $f \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative, symmetric function with $\int_{\mathbb{R}^d} f(x)dx = 1$. Set $f^\epsilon(x) := \epsilon^{-d} f(x/\epsilon)$ and $v^\epsilon(x, t) := \int_{\mathbb{R}^d} f^\epsilon(x - y)\alpha_t(dy)$. Then [37, Eq. (A.40)] has to be replaced by the bound (for $0 < \epsilon \leq 1$)

$$|v^\epsilon(x, t)| \leq \|f^\epsilon\|_\infty \alpha_t(x + B_1) \leq \|f^\epsilon\|_\infty \alpha_t(B_{r_0+1+|x|}) \leq C \|f\|_\infty \epsilon^{-d} (r_0 + 1 + |x|)^\gamma,$$

which holds uniformly in $t \in [0, T]$ due to Item (iv). The above bound is enough to apply [37, Theorem A.30] (which is a byproduct of [8, Theorems 1 and 7, Section 2.3]). Then one can proceed and conclude as in [37].

• *Case D degenerate.* Without loss, at cost of a linear change of coordinates, we can assume that D is diagonal with strictly positive eigenvalues on e_1, e_2, \dots, e_{d_*} , and zero eigenvalue on e_{d_*+1}, \dots, e_d (e_1, \dots, e_d being the canonical basis). By writing $p_t(\cdot, \cdot)$ for the probability transition kernel of the Brownian motion on \mathbb{R}^{d_*} with non-degenerate diffusion matrix $2\tilde{D} := (2D_{i,j})_{1 \leq i, j \leq d_*}$, it holds

$$P_t v_0(x', x'') = \int_{\mathbb{R}^{d_*}} p_t(x', z') v_0(z', x'') dz' \quad (x', x'') \in \mathbb{R}^{d_*} \times \mathbb{R}^{d-d_*} = \mathbb{R}^d. \tag{53}$$

Given $\psi \in C_c^\infty(\mathbb{R}^{d-d_*})$ with $\psi \geq 0$, we define $\tilde{\alpha}_t(dx')$ as the measure on \mathbb{R}^{d_*} given by $\tilde{\alpha}_t(B) := \int_{\mathbb{R}^d} \mathbb{1}_B(x')\psi(x'')\alpha_t(dx', dx'')$ for all Borel $B \subset \mathbb{R}^{d_*}$. Above, and in what follows, $x' \in \mathbb{R}^{d_*}$ and $x'' \in \mathbb{R}^{d-d_*}$. Then $\tilde{\alpha}_t \in \mathcal{M}(\mathbb{R}^{d_*})$, where $\mathcal{M}(\mathbb{R}^{d_*})$ is defined as \mathcal{M} but with \mathbb{R}^{d_*} instead of \mathbb{R}^d . Due to Item (i) the path $\tilde{\alpha} : [0, T] \rightarrow \mathcal{M}(\mathbb{R}^{d_*})$ is continuous. Due to Item (ii) we have $\tilde{\alpha}_0(dx') = \tilde{v}_0(x')dx'$ where $\tilde{v}_0(x') := \int_{\mathbb{R}^{d-d_*}} v_0(x', x'')\psi(x'')dx''$ (trivially \tilde{v}_0 is bounded and Borel). Moreover, taking $\varphi(x', x'') := \tilde{\varphi}(x')\psi(x'')$ in Item (iii) with $\tilde{\varphi} \in C_c^\infty(\mathbb{R}^{d_*})$, we get that $\int_{\mathbb{R}^{d_*}} \tilde{\varphi}(x')\tilde{\alpha}_t(dx') = \int_{\mathbb{R}^{d_*}} \tilde{\varphi}(x')\tilde{\alpha}_0(dx') + \int_0^t ds \int_{\mathbb{R}^{d_*}} \nabla \cdot (\tilde{D}\nabla\tilde{\varphi})(x')\tilde{\alpha}_s(dx')$. We set $\tilde{B}_r := \{x' \in \mathbb{R}^{d_*} : |x'| \leq r\}$ and let r_ψ be the minimal radius such that ψ has support in the ball of \mathbb{R}^{d-d_*} centered at the origin with radius r_ψ . Then, due to Item (iv), it holds $\tilde{\alpha}_s(\tilde{B}_r) = \int_{\mathbb{R}^d} \mathbb{1}_{\tilde{B}_r}(x')\psi(x'')\alpha_s(dx', dx'') \leq C2^\gamma r^\gamma$ if $r \geq \tilde{r}_0 := \max\{r_0, r_\psi\}$. Hence, we have checked that the path $\tilde{\alpha}$ satisfies the same conditions appearing in Lemma 9.3, restated for \mathbb{R}^{d_*} with D replaced by \tilde{D} . By the non-degenerate case we conclude that $\tilde{\alpha}_t(dx') = [\int_{\mathbb{R}^{d_*}} dz' p_t(x', z')\tilde{v}_0(z')]dx'$. Hence, for all $\tilde{\varphi} \in C_c(\mathbb{R}^{d_*})$ and $\psi \in C_c^\infty(\mathbb{R}^{d-d_*})$ with $\psi \geq 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} \tilde{\varphi}(x')\psi(x'')\alpha_t(dx', dx'') &= \int_{\mathbb{R}^{d_*}} dx' \tilde{\varphi}(x') \int_{\mathbb{R}^{d-d_*}} dz' p_t(x', z')\tilde{v}_0(z') \\ &= \int_{\mathbb{R}^{d_*}} dx' \int_{\mathbb{R}^{d-d_*}} dx'' \tilde{\varphi}(x')\psi(x'') \int_{\mathbb{R}^{d_*}} dz' p_t(x', z')v_0(z', x'') \\ &= \int_{\mathbb{R}^{d_*}} dx' \int_{\mathbb{R}^{d-d_*}} dx'' \tilde{\varphi}(x')\psi(x'') P_t v_0(x', x''). \end{aligned} \tag{54}$$

By additivity and density we then get that $\alpha_t(dx) = P_t v_0(x)dx$. □

10 Set Ω_{typ} of typical environments

In this section we describe the set Ω_{typ} of typical environments ω for which the properties stated in Theorem 4.1 will hold. We denote by $p_{\omega,t}^\varepsilon(\cdot, \cdot)$ the transition probability kernel of $(\varepsilon X_{\varepsilon^{-2}t}^\omega)_{t \geq 0}$. Recall Definition 9.1.

Definition 10.1 (Set $\hat{\Omega}$) We define $\hat{\Omega}$ as the family of $\omega \in \Omega$ such that

$$\sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega} : \varepsilon y \in B_r} c_x(\omega) \int_0^\infty e^{-t} p_{\omega,t}^\varepsilon(\varepsilon x, \varepsilon y) dt < +\infty \tag{55}$$

for all $\varepsilon, r \in (0, +\infty)$, where $B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$.

Definition 10.2 (Set Ω_{typ}) The set Ω_{typ} is given by the environments $\omega \in \mathcal{A}[1] \cap \Omega_{\#} \cap \Omega_* \cap \hat{\Omega}$ satisfying (39) (see respectively Propositions 3.2 and 6.1, Definitions 7.3 and 10.1).

Remark 10.3 Due to Proposition 3.2, for any $\omega \in \Omega_{\text{typ}}$ we have $\lim_{\varepsilon \downarrow 0} \mu_\omega^\varepsilon(\varphi) = \int_{\mathbb{R}^d} \varphi(x) m dx$ for all $\varphi \in C_c(\mathbb{R}^d)$.

Given $G \in \{\varphi_j\}_{j \in \mathbb{N}}$ and $\lambda > 0$, we set $G^{(\lambda)} := \lambda G - \nabla \cdot D \nabla G$. Moreover, we denote by $G_{\omega, \lambda}^\varepsilon \in \mathcal{D}(\mathbb{L}_\omega^\varepsilon)$ the unique solution of $\lambda G_{\omega, \lambda}^\varepsilon - \mathbb{L}_\omega^\varepsilon G_{\omega, \lambda}^\varepsilon = G^{(\lambda)}$ in $\mathbb{L}^2(\mu_\omega^\varepsilon)$. As $G_{\omega, \lambda}^\varepsilon = \int_0^\infty e^{-\lambda t} P_{\omega, t}^\varepsilon G^{(\lambda)} dt$ (for the notation see Sect. 6), we have the integral representation

$$G_{\omega, \lambda}^\varepsilon(\varepsilon x) = \sum_{y \in \hat{\omega}} \int_0^\infty e^{-\lambda t} p_{\omega, t}^\varepsilon(\varepsilon x, \varepsilon y) G^{(\lambda)}(\varepsilon y) dt, \quad \forall x \in \hat{\omega}. \tag{56}$$

Remark 10.4 If $\omega \in \Omega_{\text{typ}}$, then for all $G \in \{\varphi_j\}_{j \in \mathbb{N}}$, $\varepsilon > 0$ and $\lambda = 1$ it holds

$$\sum_{x \in \hat{\omega}} c_x(\omega) |G_{\omega, \lambda}^\varepsilon(\varepsilon x)| < +\infty \quad \text{and} \quad \sum_{x \in \hat{\omega}} c_x(\omega) G_{\omega, \lambda}^\varepsilon(\varepsilon x)^2 < +\infty. \tag{57}$$

Indeed, by (56) we get that $\|G_{\omega, \lambda}^\varepsilon\|_\infty \leq \|G^{(\lambda)}\|_\infty$. Hence, one has just to check the first bound in (57), which follows from (55) and (56) as $G^{(\lambda)} \in C_c^\infty(\mathbb{R}^d)$.

Proposition 10.5 Ω_{typ} is measurable, translation invariant and $\mathcal{P}(\Omega_{\text{typ}}) = 1$.

Proof The sets $\mathcal{A}[1]$, $\Omega_\#$, Ω_* are translation invariant measurable sets of \mathcal{P} -probability one as stated in Propositions 3.2 and 6.1 and after Definition 7.3, respectively. The same holds for the set of ω 's satisfying (39) as stated after Proposition 6.1.

To conclude it is enough to show that $\hat{\Omega} \cap \Omega_*$ is a translation invariant measurable set with $\mathcal{P}(\hat{\Omega} \cap \Omega_*) = 1$. For what concerns measurability, it is enough to show that $\hat{\Omega}$ is measurable. Trivially, in Definition 10.1 one can restrict to $r \in \mathbb{Q} \cap (0, +\infty)$. It is also simple to check that one can restrict also to $\varepsilon \in \mathbb{Q} \cap (0, +\infty)$ by using that, given $0 < \varepsilon_* < \varepsilon$ with $\varepsilon_* \in \mathbb{Q}$, it holds

$$\begin{aligned} \int_0^\infty e^{-t} p_{\omega, t}^\varepsilon(\varepsilon x, \varepsilon y) dt &= \int_0^\infty e^{-t} p_{\omega, \varepsilon^{-2}t}^1(x, y) dt = \varepsilon^2 \int_0^\infty e^{-\varepsilon^2 s} p_{\omega, s}^1(x, y) ds \\ &\leq \varepsilon^2 \int_0^\infty e^{-\varepsilon_*^2 s} p_{\omega, s}^1(x, y) ds \\ &= (\varepsilon/\varepsilon_*)^2 \int_0^\infty e^{-t} p_{\omega, t}^{\varepsilon_*}(\varepsilon_* x, \varepsilon_* y) dt. \end{aligned} \tag{58}$$

Since in in Definition 10.1 one can restrict to $r, \varepsilon \in \mathbb{Q} \cap (0, +\infty)$ (hence to a countable set of parameters), we conclude that $\hat{\Omega}$ is measurable.

To prove that $\mathcal{P}(\hat{\Omega} \cap \Omega_*) = 1$, it is enough to prove that, given $r, \varepsilon \in \mathbb{Q} \cap (0, +\infty)$, it holds $H(\omega) < +\infty$ for \mathcal{P} -almost all $\omega \in \Omega_*$ (recall that $\mathcal{P}(\Omega_*) = 1$), where H denotes the l.h.s. of (55).

We treat the case $\mathbb{G} = \mathbb{R}^d$ (the case $\mathbb{G} = \mathbb{Z}^d$ can be reduced to the present one by the transformation described in [12, Section 6]). We also assume that $V = \mathbb{I}$ in (10), w.l.o.g. at cost to apply an affine transformation. This implies that $\tau_g x = x + g$ and $g(x) = x$ [see (10) and (12)]. Given $\omega \in \Omega_*$ and given $x, y \in \hat{\omega}$ (see Definitions 7.2 and 7.3), we then have $p_{\omega, t}^\varepsilon(\varepsilon x, \varepsilon y) = p_{\omega, t}^\varepsilon(\varepsilon y, \varepsilon x) = p_{\theta_y \omega, t}^\varepsilon(0, \varepsilon(x - y))$, $c_x(\omega) = c_{x-y}(\theta_y \omega)$

and $\widehat{\theta_y} = \tau_{-y}\hat{\omega} = \hat{\omega} - y$. We set $F(\xi) := \sum_{z \in \hat{\xi}} c_z(\xi) \int_0^\infty e^{-t} p_{\xi,t}^\varepsilon(0, \varepsilon z) dt$ for $\xi \in \Omega_0 = \{\xi \in \Omega : 0 \in \hat{\xi}\}$. By the above observations, given $\omega \in \Omega_*$ we get

$$H(\omega) = \sum_{y \in \hat{\omega}: \varepsilon y \in B_r} \sum_{z \in \widehat{\theta_y \omega}} c_z(\theta_y \omega) \int_0^\infty e^{-t} p_{\theta_y \omega, t}^\varepsilon(0, \varepsilon z) dt = \sum_{y \in \hat{\omega}: \varepsilon y \in B_r} F(\theta_y \omega). \tag{59}$$

Hence, to prove that $\mathcal{P}(\hat{\Omega} \cap \Omega_*) = 1$, we just need to show that the last expression is finite \mathcal{P} -a.s. To this aim we apply Campbell’s identity (see [12, Appendix B]): for any nonnegative measurable function f on $\mathbb{R}^d \times \Omega_0$ it holds

$$\int_{\mathbb{R}^d} dx \int_{\Omega_0} d\mathcal{P}_0(\omega) f(x, \omega) = \frac{1}{m} \int_{\Omega} d\mathcal{P}(\omega) \sum_{x \in \hat{\omega}} f(x, \theta_x \omega) \tag{60}$$

(we recall that \mathcal{P}_0 denotes the Palm distribution associated to \mathcal{P}). Taking $f(x, \omega) := \mathbb{1}_{B_r}(\varepsilon x) F(\omega)$ we get $\varepsilon^{-d} \ell(B_r) \mathbb{E}_0[F] = m^{-1} \mathbb{E}[\sum_{x \in \hat{\omega}: \varepsilon x \in B_r} F(\theta_x \omega)]$, where \mathbb{E}_0, \mathbb{E} denote the expectation w.r.t. $\mathcal{P}_0, \mathcal{P}$ respectively and $\ell(B_r)$ denotes the Lebesgue measure of the ball B_r . Hence, to conclude it is enough to show that $\mathbb{E}_0[F] < +\infty$. As it can be easily deduced from the proof of [12, Lemma 3.5], the Palm distribution \mathcal{P}_0 is a reversible and ergodic (w.r.t. time shifts) distribution for the environment viewed from the random walk, i.e. for the process $(\theta_{X_t^\omega} \omega)_{t \geq 0}$. Indeed, in the proof of [12, Lemma 3.5] we considered the jump chain associated to the environment viewed from the random walk and proved that $\mathbb{E}[c_0]^{-1} c_0(\omega) d\mathcal{P}_0(\omega)$ is reversible and ergodic for the jump chain. As accelerating time does not change the class of reversible and ergodic distributions, we get that \mathcal{P}_0 is a reversible and ergodic distribution also for the process $(\bar{\omega}_t)_{t \geq 0}$, $\bar{\omega}_t := \theta_{X_{\varepsilon^{-2}t}^\omega} \omega$. On the other hand, by (A7), $\mathbb{E}_0[c_0] = \mathbb{E}_0[\lambda_0] < +\infty$. Hence, by the L^1 -Birkhoff ergodic theorem, we get $\lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t c_0(\bar{\omega}_s) ds = \mathbb{E}_0[c_0]$ in $L^1(\mathbb{P}_{\mathcal{P}_0})$, where $\mathbb{P}_{\mathcal{P}_0}$ is the law of the random path $(\bar{\omega}_t)_{t \geq 0}$ when the starting configuration ω is sampled with distribution \mathcal{P}_0 . As the above limit implies the limit of expectations and $c_0(\theta_z \omega) = c_z(\omega)$, we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \sum_{z \in \hat{\omega}} \int \mathcal{P}_0(d\omega) \int_0^t p_{\omega, s}^\varepsilon(0, \varepsilon z) c_z(\omega) ds = \mathbb{E}_0[c_0]. \tag{61}$$

Now note that, for some positive constant C , it holds

$$F(\omega) \leq C \sum_{n=0}^\infty \frac{e^{-n/2}}{n+1} \sum_{z \in \hat{\omega}} \int_n^{n+1} p_{\omega, t}^\varepsilon(0, \varepsilon z) c_z(\omega) dt. \tag{62}$$

Hence, setting $a_n := \frac{1}{n+1} \int \mathcal{P}_0(d\omega) \sum_{z \in \hat{\omega}} \int_0^{n+1} p_{\omega, t}^\varepsilon(0, \varepsilon z) c_z(\omega) dt$, we get $\mathbb{E}_0[F] \leq C \sum_{n=0}^\infty e^{-n/2} a_n$. By (61) we have $\lim_{n \rightarrow \infty} a_n = \mathbb{E}_0[c_0] < +\infty$, hence the series $\sum_{n=0}^\infty e^{-n/2} a_n$ is finite, thus implying that $\mathbb{E}_0[F] < +\infty$. This concludes the proof that $\mathcal{P}(\hat{\Omega} \cap \Omega_*) = 1$

We now show that $\hat{\Omega} \cap \Omega_*$ is translation invariant (always restricting to $\mathbb{G} = \mathbb{R}^d$ and $V = \mathbb{I}$). Take $\varepsilon, r > 0, g \in \mathbb{G}$ and $\omega \in \hat{\Omega} \cap \Omega_*$. Then we have

$$\begin{aligned} H(\theta_g \omega) &= \sum_{x \in \widehat{\theta_g \omega}} \sum_{y \in \widehat{\theta_g \omega}: \varepsilon y \in B_r} c_x(\theta_g \omega) \int_0^\infty e^{-t} p_{\theta_g \omega, t}^\varepsilon(\varepsilon x, \varepsilon y) dt \\ &\leq \sum_{a \in \hat{\omega}} \sum_{b \in \hat{\omega}: \varepsilon b \in B_{r+\varepsilon|g|}} c_a(\omega) \int_0^\infty e^{-t} p_{\omega, t}^\varepsilon(\varepsilon a, \varepsilon b) dt < +\infty. \end{aligned}$$

This proves that $\theta_g(\hat{\Omega} \cap \Omega_*) \subset \hat{\Omega}$ for all $g \in \mathbb{G}$. Using that Ω_* is translation invariant, it is then trivial to conclude that $\theta_g(\hat{\Omega} \cap \Omega_*) \subset \hat{\Omega} \cap \Omega_*$ for all $g \in \mathbb{G}$, which implies the translation invariance of $\hat{\Omega} \cap \Omega_*$. □

11 Proof of Theorem 4.1

In Proposition 7.4 we discussed the existence of the simple exclusion process for $\omega \in \Omega_* \supset \Omega_{\text{typ}}$. In Proposition 10.5 we showed that Ω_{typ} is a translation invariant measurable set of \mathcal{P} -probability one. To get the hydrodynamic behavior we will proceed as follows. We fix $\omega \in \Omega_{\text{typ}}$. We consider the random path $(\pi_{\omega, t}^\varepsilon[\eta.\!])_{0 \leq t \leq T}$ with $\eta.$ sampled according to $\mathbb{P}_{\omega, m_\varepsilon}^\varepsilon$. We call Q^ε its law, which is a probability measure on $D([0, T], \mathcal{M})$. Note that, to simplify the notation, ω is understood in Q^ε . We call Q^* the law of the deterministic path $(\rho(x, t)dx)_{0 \leq t \leq T}$ in $D([0, T], \mathcal{M})$ where $\rho(x, t) = P_t \rho_0(x)$. To get Theorem 4.1 it is enough to prove that, for $\omega \in \Omega_{\text{typ}}$, Q^ε weakly converges to Q^* . Indeed, this implies the convergence in probability of the random path $(\pi_{\omega, t}^\varepsilon[\eta.\!])_{0 \leq t \leq T}$ towards $(\rho(x, t)dx)_{0 \leq t \leq T}$. As $(\rho(x, t)dx)_{0 \leq t \leq T} \in C([0, T], \mathcal{M})$, the above convergence in probability implies (32) (cf. [1, page 124]).

By adapting the method of the corrected empirical process of [22] to the L^2 -context and the unbounded domain \mathbb{R}^d , we prove the tightness of $\{Q^\varepsilon\}$ in Sect. 11.1 (since $D([0, T], \mathcal{M})$ is a Polish space, tightness is here equivalent to relative compactness, cf. [1, Theorems 5.1 and 5.2]). The homogenization result used in this part is given by (38).

After getting tightness, one can proceed in two ways. A first route is to show that all limit points of $\{Q^\varepsilon\}$ must equal Q^* since concentrated on continuous \mathcal{M} -valued paths solving in a weak sense the hydrodynamic equation with initial value $\rho_0(x)dx$ and satisfying suitable mass bounds on balls. Then one can invoke the uniqueness result for these weak solutions given by Lemma 9.3. This is the route followed in Sect. 11.2 in the same spirit of [22]. Again, the homogenization result used here is given by (38).

We now describe the second route. Due to tightness and by [1, Theorem 13.1], to prove that $Q^\varepsilon \Rightarrow Q^*$ it is enough to show the finite dimensional distribution convergence and, by a union bound, the convergence for the distribution at a fixed time, i.e. that for any $t \geq 0, \delta > 0$ and $\varphi \in C_c(\mathbb{R}^d)$ it holds

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon \left(\left| \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta_t(x) - \int_{\mathbb{R}^d} \varphi(x) \rho(x, t) dx \right| > \delta \right) = 0. \tag{63}$$

This can be obtained by a completely autonomous analysis with two main ingredients: an extension to our context of the Nagy’s representation of the simple exclusion process (based on duality with the random walk) and the homogenization limit (37). Note that here one does not need Lemma 9.3. We have discussed this second route in “Appendix C”.

11.1 Relative compactness of the empirical measure

To simplify the notation, we fix once and for all a sequence $\{\varepsilon_n\}$ of positive numbers with $\varepsilon_n \downarrow 0$. In what follows all limits $\varepsilon \downarrow 0$ have to be thought along the above sequence $\{\varepsilon_n\}$. By Lemma 9.2, to prove that the family $\{Q^\varepsilon\}$ is relatively compact as $\varepsilon \downarrow 0$ it is enough to prove that, given $G \in \{\varphi_j\}_{j \in \mathbb{N}}$ (cf. Definition 9.1), the ε -parameterized laws of the random paths $(\pi_{\omega, t}^\varepsilon(G))_{0 \leq t \leq T}$ form a relatively compact family of probability measures on $D([0, T], \mathbb{R})$ as $\varepsilon \downarrow 0$. Note that we have dropped from the notation the dependence on the path η . By [26, Theorem 1.3, Chapter 4] and Aldous’ criterion given in [26, Proposition 1.6, Chapter 4], it is enough to prove that

- (i) for every $t \in [0, T]$ and every $\beta > 0$, there exists $\ell > 0$ such that $\overline{\lim}_{\varepsilon \downarrow 0} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon (|\pi_{\omega, t}^\varepsilon(G)| > \ell) \leq \beta$;
- (ii) calling \mathcal{J}_T the family of stopping times bounded by T w.r.t. to the filtration $(\mathcal{F}_t)_{t \geq 0}$, with $\mathcal{F}_t := \sigma\{\eta_s : 0 \leq s \leq t\}$, for any $\beta > 0$ it holds

$$\lim_{\gamma \downarrow 0} \overline{\lim}_{\varepsilon \downarrow 0} \sup_{\substack{\tau \in \mathcal{J}_T \\ \theta \leq \gamma}} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon \left(\left| \pi_{\omega, \tau}^\varepsilon(G) - \pi_{\omega, (\tau+\theta) \wedge T}^\varepsilon(G) \right| > \beta \right) = 0. \tag{64}$$

Item (i) gives no problem. Indeed, $|\pi_{\omega, t}^\varepsilon(G)| \leq \mu_\omega^\varepsilon(|G|) \rightarrow \int dx m|G(x)| dx$ as $\omega \in \Omega_{\text{typ}}$ (cf. Remark 10.3). Item (ii) is more delicate and can be treated by the corrected empirical measure. To this aim we fix $\lambda > 0$ (let us take $\lambda = 1$ as in Remark 10.4) and define $G^{(\lambda)} \in C_c^\infty(\mathbb{R}^d)$ as

$$G^{(\lambda)} := \lambda G - \nabla \cdot D \nabla G, \tag{65}$$

where D is the effective homogenized matrix (see Definition 3.3). As in Sect. 10, we define $G_{\omega, \lambda}^\varepsilon$ as the unique element of $\mathcal{D}(\mathbb{L}_\omega^\varepsilon) \subset L^2(\mu_\omega^\varepsilon)$ such that

$$\lambda G_{\omega, \lambda}^\varepsilon - \mathbb{L}_\omega^\varepsilon G_{\omega, \lambda}^\varepsilon = G^{(\lambda)} \quad \text{in } L^2(\mu_\omega^\varepsilon). \tag{66}$$

By using the resolvent operators R_λ and $R_{\omega, \lambda}^\varepsilon$ defined in Sect. 6, we can restate the above definitions as

$$G = R_\lambda G^{(\lambda)}, \quad G_{\omega, \lambda}^\varepsilon = R_{\omega, \lambda}^\varepsilon G^{(\lambda)}. \tag{67}$$

We point out some standard bounds which we will be useful below. By taking the scalar product with $G_{\omega,\lambda}^\varepsilon$ in the massive Poisson equation (66) and by applying Schwarz inequality, we get that

$$\lambda \|G_{\omega,\lambda}^\varepsilon\|_{L^2(\mu_\omega^\varepsilon)} \leq \|G^{(\lambda)}\|_{L^2(\mu_\omega^\varepsilon)}, \tag{68}$$

$$\langle G_{\omega,\lambda}^\varepsilon, -\mathbb{L}_\omega^\varepsilon G_{\omega,\lambda}^\varepsilon \rangle_{L^2(\mu_\omega^\varepsilon)} \leq \langle G_{\omega,\lambda}^\varepsilon, G^{(\lambda)} \rangle_{L^2(\mu_\omega^\varepsilon)} \leq \lambda^{-1} \|G^{(\lambda)}\|_{L^2(\mu_\omega^\varepsilon)}^2. \tag{69}$$

We also note that

$$\langle G_{\omega,\lambda}^\varepsilon, -\mathbb{L}_\omega^\varepsilon G_{\omega,\lambda}^\varepsilon \rangle_{L^2(\mu_\omega^\varepsilon)} = \frac{\varepsilon^{d-2}}{2} \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) [G_{\omega,\lambda}^\varepsilon(\varepsilon x) - G_{\omega,\lambda}^\varepsilon(\varepsilon y)]^2. \tag{70}$$

To justify (70) we proceed as follows. For any $f \in \mathcal{D}(\mathbb{L}_\omega^\varepsilon) \subset \mathcal{D}(\sqrt{-\mathbb{L}_\omega^\varepsilon})$ we have $\langle f, -\mathbb{L}_\omega^\varepsilon f \rangle_{L^2(\mu_\omega^\varepsilon)} = \|\sqrt{-\mathbb{L}_\omega^\varepsilon} f\|_{L^2(\mu_\omega^\varepsilon)}^2 = \mathcal{E}_\omega^\varepsilon(f, f)$, the last identity being discussed in Sect. 6. By taking $f = G_{\omega,\lambda}^\varepsilon$, we then get (70).

We now use our homogenization result for the resolvent convergence. Indeed, $\Omega_{\text{typ}} \subset \Omega_\#$ and (67) and (38) imply that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^d \sum_{x \in \hat{\omega}} |G(\varepsilon x) - G_{\omega,\lambda}^\varepsilon(\varepsilon x)| = 0. \tag{71}$$

As $|\pi_{\omega,t}^\varepsilon(G) - \pi_{\omega,t}^\varepsilon(G_{\omega,\lambda}^\varepsilon)| \leq \varepsilon^d \sum_{x \in \hat{\omega}} |G(\varepsilon x) - G_{\omega,\lambda}^\varepsilon(\varepsilon x)|$, we get

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon \left(\sup_{0 \leq t \leq T} |\pi_{\omega,t}^\varepsilon(G) - \pi_{\omega,t}^\varepsilon(G_{\omega,\lambda}^\varepsilon)| > \delta \right) = 0. \tag{72}$$

By (72), to get Item (ii), it is enough to prove the same result with G replaced by $G_{\omega,\lambda}^\varepsilon$, i.e. that for any $\beta > 0$ it holds

$$\lim_{\gamma \downarrow 0} \overline{\lim}_{\varepsilon \downarrow 0} \sup_{\substack{\tau \in \mathcal{I}_T \\ \theta \leq \gamma}} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon \left(\left| \pi_{\omega,\tau}^\varepsilon(G_{\omega,\lambda}^\varepsilon) - \pi_{\omega,(\tau+\theta) \wedge T}^\varepsilon(G_{\omega,\lambda}^\varepsilon) \right| > \beta \right) = 0. \tag{73}$$

We have now to deal with the Dynkin martingale associated to $\pi_\omega^\varepsilon(G_{\omega,\lambda}^\varepsilon)$. We will use below Lemmas 8.2 and 8.3. Let us check their hypotheses. Trivially, $G_{\omega,\lambda}^\varepsilon \in \mathcal{D}(\mathbb{L}_\omega^\varepsilon)$ (by definition). We claim that

$$G_{\omega,\lambda}^\varepsilon \in L^1(\mu_\omega^\varepsilon), \quad \sum_{x \in \hat{\omega}} c_x(\omega) |G_{\omega,\lambda}^\varepsilon(\varepsilon x)| < +\infty, \quad \sum_{x \in \hat{\omega}} c_x(\omega) G_{\omega,\lambda}^\varepsilon(\varepsilon x)^2 < +\infty.$$

The last two bounds follow from Remark 10.4. To derive the first one we observe that, by the integral representation (56) and the symmetry of $p_{\omega,t}^\varepsilon(\cdot, \cdot)$, it holds $\|G_{\omega,\lambda}^\varepsilon\|_{L^1(\mu_\omega^\varepsilon)} \leq \|G^{(\lambda)}\|_{L^1(\mu_\omega^\varepsilon)}/\lambda < +\infty$.

By applying Lemma 8.2, we get that $\pi_\omega^\varepsilon(G_{\omega,\lambda}^\varepsilon)$ corresponds to an absolutely convergent series in $C(\{0, 1\}^{\hat{\omega}})$ and, as function of η , belongs to the domain of \mathcal{L}_ω . This observation allows us to introduce the Dynkin martingale

$$M_{\omega,t}^\varepsilon := \pi_{\omega,t}^\varepsilon(G_{\omega,\lambda}^\varepsilon) - \pi_{\omega,0}^\varepsilon(G_{\omega,\lambda}^\varepsilon) - \varepsilon^{-2} \int_0^t \mathcal{L}_\omega(\pi_\omega^\varepsilon(G_{\omega,\lambda}^\varepsilon))(\eta_s) ds. \tag{74}$$

By (43) and (66), we can rewrite $M_{\omega,t}^\varepsilon$ as

$$M_{\omega,t}^\varepsilon = \pi_{\omega,t}^\varepsilon(G_{\omega,\lambda}^\varepsilon) - \pi_{\omega,0}^\varepsilon(G_{\omega,\lambda}^\varepsilon) - \varepsilon^d \sum_{x \in \hat{\omega}} \int_0^t \eta_s(x) (\lambda G_{\omega,\lambda}^\varepsilon - G^{(\lambda)})(\varepsilon x) ds. \tag{75}$$

We can now prove (73). Due to (75) it is enough to prove that

$$\lim_{\gamma \downarrow 0} \overline{\lim}_{\varepsilon \downarrow 0} \sup_{\substack{\tau \in \mathcal{T}_T \\ \theta \leq \gamma}} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon \left(\varepsilon^d \theta \sum_{x \in \hat{\omega}} \left| \lambda G_{\omega,\lambda}^\varepsilon - G^{(\lambda)} \right|(\varepsilon x) > \beta/2 \right) = 0, \tag{76}$$

$$\lim_{\gamma \downarrow 0} \overline{\lim}_{\varepsilon \downarrow 0} \sup_{\substack{\tau \in \mathcal{T}_T \\ \theta \leq \gamma}} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon \left(\left| M_{\omega,(\tau+\theta) \wedge T}^\varepsilon - M_{\omega,\tau}^\varepsilon \right| > \beta/2 \right) = 0. \tag{77}$$

• *Proof of (76).* The inequality inside (76) is indeed deterministic. As $\theta \leq \gamma \downarrow 0$, to prove (76) it is enough to prove that

$$\overline{\lim}_{\varepsilon \downarrow 0} \varepsilon^d \sum_{x \in \hat{\omega}} \left| \lambda G_{\omega,\lambda}^\varepsilon - G^{(\lambda)} \right|(\varepsilon x) < +\infty. \tag{78}$$

We have already observed that $\|G_{\omega,\lambda}^\varepsilon\|_{L^1(\mu_\omega^\varepsilon)} \leq \|G^{(\lambda)}\|_{L^1(\mu_\omega^\varepsilon)}/\lambda$. Then, to get (78) it is enough to apply Remark 10.3.

• *Proof of (77).* We write $\mathbb{E}_{\omega, m_\varepsilon}^\varepsilon$ for the expectation w.r.t. $\mathbb{P}_{\omega, m_\varepsilon}^\varepsilon$. We bound the probability in (77) by $(2/\beta)^2 \mathbb{E}_{\omega, m_\varepsilon}^\varepsilon \left[(M_{\omega,(\tau+\theta) \wedge T}^\varepsilon - M_{\omega,\tau}^\varepsilon)^2 \right]$. Using that τ is a stopping time and the form of the sharp bracket process in Lemma 8.3, we get [cf. (69) and (70)]

$$\begin{aligned} \mathbb{E}_{\omega, m_\varepsilon}^\varepsilon \left[(M_{\omega,(\tau+\theta) \wedge T}^\varepsilon - M_{\omega,\tau}^\varepsilon)^2 \right] &\leq \theta \varepsilon^{2d-2} \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) [G_{\omega,\lambda}^\varepsilon(\varepsilon x) - G_{\omega,\lambda}^\varepsilon(\varepsilon y)]^2 \\ &= 2\theta \varepsilon^d \langle G_{\omega,\lambda}^\varepsilon, -\mathbb{L}_\omega^\varepsilon G_{\omega,\lambda}^\varepsilon \rangle_{L^2(\mu_\omega^\varepsilon)} \\ &\leq 2\theta \varepsilon^d \lambda^{-1} \|G^{(\lambda)}\|_{L^2(\mu_\omega^\varepsilon)}^2. \end{aligned} \tag{79}$$

As $\omega \in \Omega_{\text{typ}}$ (see Remark 10.3), as $\varepsilon \downarrow 0$ we have $\|G^{(\lambda)}\|_{L^2(\mu_\omega^\varepsilon)}^2 \rightarrow C_0 := \int dx m G^{(\lambda)}(x)^2$. In conclusion we have proved that the probability in (77) is bounded from above by $(2/\beta)^2 2\theta \varepsilon^d \lambda^{-1} (C_0 + o(1))$ as $\varepsilon \downarrow 0$. This implies (77).

11.2 Characterization of the limit points

Recall that $\omega \in \Omega_{\text{typ}}$ is fixed. Let Q be any limit point $\{Q^\varepsilon\}$ as $\varepsilon \downarrow 0$. We claim that Q is concentrated on paths $\alpha \in D([0, T], \mathcal{M})$ satisfying the conditions of Lemma 9.3 with $v_0 = \rho_0$. Then, by applying Lemma 9.3, we can conclude that $Q = \delta_{(\rho(x,t)dx)_{0 \leq t \leq T}}$, thus completing the proof of Theorem 4.1.

Let us prove our claim. Item (ii) in Lemma 9.3 follows from condition (31). We move to Item (iv). We recall that, for any integer $\ell \geq 0$, there is some $[0, 1]$ -valued function $\varphi_{j_0} \in \{\varphi_j\}_{j \in \mathbb{N}}$ equal to 1 on B_ℓ and equal to zero outside $B_{\ell+1}$. Then, by Remark 10.3, we have for all $t \in [0, T]$ and for a suitable constant $C(d)$ depending only on the dimension d that

$$\pi_{\omega,t}^\varepsilon(\varphi_{j_0}) \leq \mu_\omega^\varepsilon(\varphi_{j_0}) \xrightarrow{\varepsilon \downarrow 0} m \int \varphi_{j_0}(x)dx \leq C(d)m\ell^d. \tag{80}$$

Setting $H := \{\alpha : \sup_{0 \leq t \leq T} \alpha_t(\varphi_{j_0}) \leq 2C(d)m\ell^d\}$, we get $\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega,m_\varepsilon}^\varepsilon(\pi_{\omega,\cdot}^\varepsilon \in H) = 1$. As H is closed in $D([0, T], \mathcal{M})$, we conclude that $Q(H) = 1$. By varying ℓ in \mathbb{N} , this implies Item (iv) in Lemma 9.3 with $\gamma = d$.

We move to Item (iii). By Doob’s inequality and reasoning as in (79) we get

$$\mathbb{P}_{\omega,m_\varepsilon}^\varepsilon \left(\sup_{t \in [0,T]} |M_{\omega,t}^\varepsilon| \geq \delta \right) \leq \delta^{-2} \mathbb{E}_{\omega,m_\varepsilon}^\varepsilon \left((M_{\omega,T}^\varepsilon)^2 \right) \leq 2\delta^{-2} T \varepsilon^d \lambda^{-1} \|G^{(\lambda)}\|_{L^2(\mu_{\omega_0}^\varepsilon)}^2 \xrightarrow{\varepsilon \downarrow 0} 0. \tag{81}$$

By (65) and (71) (the latter is due to (38) in Proposition 6.1), we have

$$\sup_{0 \leq t \leq T} \left| \int_0^t \pi_{\omega,s}^\varepsilon \left(\lambda G_{\omega,\lambda}^\varepsilon - G^{(\lambda)} - \nabla \cdot D\nabla G \right) ds \right| \leq T \varepsilon^d \sum_{x \in \hat{\omega}} |\lambda G_{\omega,\lambda}^\varepsilon - \lambda G|(\varepsilon x) \xrightarrow{\varepsilon \downarrow 0} 0. \tag{82}$$

At this point, by combining (72), (75), (81) and (82) we get that

$$\mathbb{P}_{\omega,m_\varepsilon}^\varepsilon \left(\sup_{0 \leq t \leq T} \left| \pi_{\omega,t}^\varepsilon(G) - \pi_{\omega,0}^\varepsilon(G) - \int_0^t \pi_{\omega,s}^\varepsilon(\nabla \cdot D\nabla G) ds \right| \leq \delta \right) = 1. \tag{83}$$

As a consequence, given $G \in \{\varphi_j\}_{j \in \mathbb{N}}$, Q -a.s. it holds $\alpha_t(G) - \alpha_0(G) - \int_0^t \alpha_s(\nabla \cdot D\nabla G)ds = 0$ for all $0 \leq t \leq T$ (adapt the proof of [37, Lemma 8.7] to show that $\{\alpha : \sup_{0 \leq t \leq T} |\alpha_t(G) - \alpha_0(G) - \int_0^t \alpha_s(\nabla \cdot D\nabla G)ds| \leq \delta\}$ is closed in $D([0, T], \mathcal{M})$). By the construction of $\{\varphi_j\}_{j \in \mathbb{N}}$ in Sect. 9, given a generic $\varphi \in C_c^\infty(\mathbb{R}^d)$ with support in some B_ℓ , we know that for each $\delta > 0$ there exists $G \in \{\varphi_j\}_{j \in \mathbb{N}}$ with support in $B_{\ell+1}$ such that $\|G - \varphi\|_\infty \leq \delta$ and $\sup_{1 \leq i,k \leq d} \|\partial_{x_i,x_k}^2 G - \partial_{x_i,x_k}^2 \varphi\|_\infty \leq \delta$. Hence both $\sup_{0 \leq t \leq T} |\alpha_t(G) - \alpha_t(\varphi)|$ and $\sup_{0 \leq t \leq T} |\alpha_t(\nabla \cdot D\nabla G) - \alpha_t(\nabla \cdot D\nabla \varphi)|$ can be bounded by $C\delta \sup_{0 \leq t \leq T} \alpha_t(B_{\ell+1})$, where $C = C(D)$. Due to Item (iv) (already proved) and by density, we conclude that Q -a.s. it holds $\alpha_t(\varphi) - \alpha_0(\varphi) - \int_0^t \alpha_s(\nabla \cdot D\nabla \varphi)ds = 0$ for all $0 \leq t \leq T$ and all $\varphi \in C_c^\infty(\mathbb{R}^d)$. Hence Item (iii) in Lemma 9.3 is verified.

We now check Item (i) in Lemma 9.3. By Remark 10.3, given $G \in \{\varphi_j\}_{j \in \mathbb{N}}$ we get

$$\sup_{\substack{0 \leq s \leq t \leq T \\ |t-s| < \beta}} \left| \int_s^t \pi_{\omega, u}^\varepsilon (\nabla \cdot D \nabla G) du \right| \leq \beta \varepsilon^d \sum_{x \in \hat{\omega}} |\nabla \cdot D \nabla G(\varepsilon x)| \xrightarrow{\varepsilon \downarrow 0} C(G) \beta. \tag{84}$$

We set $\tilde{H} := \{\alpha : |\alpha_t(G) - \alpha_s(G)| \leq 2C(G)\beta \text{ for all } 0 \leq s \leq t \leq T \text{ with } |t - s| < \beta\}$. By combining (83) and (84) we get $\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega, m_\varepsilon}^\varepsilon (\pi_{\omega, \cdot}^\varepsilon \in \tilde{H}) = 1$. As \tilde{H} is closed in $D([0, T], \mathcal{M})$, we conclude that $Q(\tilde{H}) = 1$. By varying G among $\{\varphi_j\}_{j \in \mathbb{N}}$ and by taking $\beta \downarrow 0$ along a sequence, we get that $Q(C([0, T], \mathcal{M})) = 1$.

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Appendix A. An example of degenerate nonzero effective homogenized matrix D

In this appendix we present a model satisfying Assumptions (A1),...,(A9) and (SEP) for which the effective homogenized matrix D is nonzero but degenerate.

All product spaces appearing below are endowed with the product topology. We take $\Omega := (0, 2)^{\mathbb{Z}^2} \times (0, 2)^{\mathbb{Z}}$. We denote a generic element of Ω as $\omega = ((u_x)_{x \in \mathbb{Z}^2}, (a_s)_{s \in \mathbb{Z}})$. The probability measure \mathcal{P} on Ω is such that, under \mathcal{P} , all coordinates are independent random variables, all u_x ’s are uniformly distributed on $(1, 2)$ and all a_s ’s are identically distributed with $\mathbb{E}[a_s^{-1}] = +\infty$. We take $\hat{\omega} := \mathbb{Z}^2$ and $\mathbb{G} = \mathbb{Z}^2$. The action of \mathbb{G} on Ω is the following:

$$\theta_g \omega := ((u_{x-g})_{x \in \mathbb{Z}^2}, (a_{s-g_2})_{s \in \mathbb{Z}}) \text{ if } g = (g_1, g_2) \in \mathbb{Z}^2, \quad \omega = ((u_x)_{x \in \mathbb{Z}^2}, (a_s)_{s \in \mathbb{Z}}).$$

\mathbb{G} acts on \mathbb{Z} by standard translations: $\tau_g x := x + g, g \in \mathbb{G}$. The random conductance field is defined as

$$c_{x,y}(\omega) := \begin{cases} u_z & \text{if } \{x, y\} = \{z, z + e_1\}, \quad z \in \mathbb{Z}^2, \\ a_{z_2} & \text{if } \{x, y\} = \{z, z + e_2\}, \quad z = (z_1, z_2) \in \mathbb{Z}^2, \\ 0 & \text{otherwise.} \end{cases}$$

The geometric idea behind the definition of ω and the action $(\theta_g)_{g \in \mathbb{Z}^2}$ is that we attach to each point $x = (x_1, x_2)$ the two numbers u_x and a_{x_2} and we think of u_x as the conductance of the edge $\{x, x + e_1\}$ and of a_{x_2} as the conductance of the edge $\{x, x + e_2\}$.

We claim that all Assumptions (A1),..., (A9) and (SEP) are satisfied. Indeed, trivially \mathcal{P} is stationary. To check the ergodicity of \mathcal{P} we introduce the bijection $\Phi : \Omega \rightarrow \Gamma^{\mathbb{Z}}$ where $\Gamma = (0, 2)^{\mathbb{Z} \cup \{*\}}$ as

$$\Phi(\omega) = (\gamma_s)_{s \in \mathbb{Z}} \quad (\gamma_s)_t := \begin{cases} u_{(t,s)} & \text{if } t \in \mathbb{Z}, \\ a_s & \text{if } t = *, \end{cases}$$

whenever $\omega = ((u_x)_{x \in \mathbb{Z}^2}, (a_s)_{s \in \mathbb{Z}})$. Simply, we organize the elements of ω in rows having in mind the above geometric idea of ω . We write \mathcal{Q} for the probability measure on $\Gamma^{\mathbb{Z}}$ such that $\mathcal{Q}(\mathcal{B}) := \mathcal{P}(\Phi^{-1}(\mathcal{B}))$ for all Borel sets $\mathcal{B} \subset \Gamma^{\mathbb{Z}}$. Then under \mathcal{Q} the coordinates γ_s are i.i.d. random variables. As a consequence \mathcal{Q} is ergodic w.r.t. standard shifts of $\Gamma^{\mathbb{Z}}$. Take now a translation invariant measurable set $\mathcal{A} \subset \Omega$. Then $\theta_{te_2}\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{Z}$ and therefore $\Phi(\mathcal{A})$ is left invariant by the standard shifts of $\Gamma^{\mathbb{Z}}$. Due to the ergodicity of \mathcal{Q} we obtain that $\mathcal{P}(\mathcal{A}) = \mathcal{Q}(\Phi(\mathcal{A})) \in \{0, 1\}$. This concludes the proof that \mathcal{P} is ergodic. All other Assumptions (A2),..., (A9) are trivially satisfied. To check (SEP) it is enough to argue as in the proof of Proposition 5.1 in case (i).

Let us now prove that $D_{1,1} > 0$ and $D_{i,j} = 0$ for $(i, j) \neq (1, 1)$.

As in the derivation of [2, Prop. 4.1] one can lower bound the scalar product $a \cdot Da$ by $C \sum_{x=e_1, e_2} (a \cdot x)^2 / \mathbb{E}[1/c_{0,x}(\omega)]$ with $C > 0$. Taking $a = e_1$ and using that $c_{0,e_1}(\omega) = u_0 \geq 1$ \mathcal{P} -a.s., we get that $D_{1,1} > 0$.

We now show that if $D_{2,2} = 0$ then $D_{1,2} = D_{2,1} = 0$. Recall that in general D is a symmetric and positive semidefinite matrix. Having nonnegative eigenvalues, the determinant of D is nonnegative, i.e. $D_{1,1}D_{2,2} - D_{1,2}D_{2,1} \geq 0$. Since $D_{2,2} = 0$ and $D_{1,2} = D_{2,1}$, we then get that $-D_{1,2}^2 \geq 0$, thus implying that $D_{1,2} = D_{2,1} = 0$.

It remains to prove that $D_{2,2} = 0$. Note that, by (18) and since $\hat{\omega} = \mathbb{Z}^2$, it holds $\mathcal{P} = \mathcal{P}_0$ and $\Omega = \Omega_0$. Then (25) implies that

$$D_{2,2} = \inf_{f \in L^\infty(\mathcal{P})} \frac{1}{2} \int_{\Omega} d\mathcal{P}(\omega) \sum_{x=\pm e_1, \pm e_2} c_{0,x}(\omega) (x_2 - \nabla f(\omega, x))^2, \quad (85)$$

where $\nabla f(\omega, x) := f(\theta_x \omega) - f(\omega)$. Let \mathcal{W} be the family of bounded measurable functions on Ω depending only on the coordinates $(a_s)_{s \in \mathbb{Z}}$. Then $\nabla f(\omega, \pm e_1) = f(\theta_{e_1} \omega) - f(\omega) = 0$. Using also that $c_{x,y}(\omega) \leq 2$, due to (85) we get

$$\begin{aligned}
 D_{2,2} &\leq 2 \inf_{f \in \mathcal{W}} \frac{1}{2} \int_{\Omega} d\mathcal{P}(\omega) \sum_{x=\pm e_2} (\pm 1 - \nabla f(\omega, x))^2 \\
 &= 2 \inf_{f \in L^\infty((0,2)^{\mathbb{Z}})} \frac{1}{2} \int_{(0,2)^{\mathbb{Z}}} d\bar{\mathcal{P}}(\bar{a}) \sum_{x=\pm 1} (x - \bar{\nabla} f(\bar{a}, x))^2, \tag{86}
 \end{aligned}$$

where $\bar{a} \in (0, 2)^{\mathbb{Z}}$, $\bar{\mathcal{P}}$ is the probability measure on $(0, 2)^{\mathbb{Z}}$ making the coordinates into i.i.d. random variables with the same distribution of a_0 under \mathcal{P} , $\bar{\theta}_x$ is the standard shift operator on $(0, 2)^{\mathbb{Z}}$ and $\bar{\nabla} f(\bar{a}, x) = f(\bar{\theta}_x \bar{a}) - f(\bar{a})$. Then the last expression in (86) is twice the effective homogenized coefficient for the nearest-neighbor random conductance model on \mathbb{Z} with i.i.d. conductances given by $(\bar{a}_s)_{s \in \mathbb{Z}}$. Since \bar{a}_0^{-1} has infinite expectation under $\bar{\mathcal{P}}$, we get that this coefficient is zero (see the discussion in Sect. 5.1 for $d = 1$).

Appendix B. Proof of (7) and (9) for local functions f

Recall the notation of Sect. 7. We take $\omega \in \Omega_*$ and $\xi \in \{0, 1\}^{\hat{\omega}}$. Below \mathcal{K} will always vary in \mathbb{K}_ω , without further mention. Given $t \in (0, t_0]$ we denote by $\mathcal{G}_t(\omega, \mathcal{K})$ the undirected graph with vertex set $\hat{\omega}$ and edge set $\{\{x, y\} \in \mathcal{E}_\omega : \mathcal{K}_{x,y}(t) > 0\}$. We recall that $\mathcal{E}_\omega = \{\{x, y\} : x, y \in \hat{\omega}, x \neq y\}$. As $\mathcal{G}_t(\omega, \mathcal{K})$ is a subgraph of $\mathcal{G}_{t_0}(\omega, \mathcal{K})$, the graph $\mathcal{G}_t(\omega, \mathcal{K})$ has only connected components of finite cardinality. Moreover, as $t \leq t_0$, one can check that $\eta_t^\xi[\mathcal{K}]$ can be obtained by the graphical construction detailed in Sect. 7 but working with the graph $\mathcal{G}_t(\omega, \mathcal{K})$ instead of $\mathcal{G}_{t_0}(\omega, \mathcal{K})$.

Let $f : \{0, 1\}^{\hat{\omega}} \rightarrow \mathbb{R}$ be a local function. Let $A \subset \hat{\omega}$ be a finite set such that $f(\eta)$ is defined in terms only of $\eta(x)$ with $x \in A$. We set $\mathcal{E}_A := \{\{x, y\} \in \mathcal{E}_\omega : \{x, y\} \cap A \neq \emptyset\}$ (as ω is fixed, in the notation we do not stress the dependence of \mathcal{E}_A from ω). As $\omega \in \Omega_* \subset \Omega_1$ [see (36) and Definition 7.3], we have

$$c_A(\omega) := \sum_{\{x,y\} \in \mathcal{E}_A} c_{x,y}(\omega) \leq \sum_{x \in A} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) = \sum_{x \in A} c_x(\omega) < +\infty. \tag{87}$$

Due to the above bound, it is simple to check that the r.h.s.'s of (7) and (9) are absolutely convergent series in $C(\{0, 1\}^{\hat{\omega}})$ defining the same function, that we denote by $\hat{\mathcal{L}}_\omega f$. Hence we just need to prove that $\mathcal{L}_\omega f = \hat{\mathcal{L}}_\omega f$.

We note that $\mathcal{K}_A(t) := \sum_{\{x,y\} \in \mathcal{E}_A} \mathcal{K}_{x,y}(t)$ is a Poisson random variable (cf. also Item (iii) in Definition 7.3) with finite parameter $c_A(\omega)$. In particular, it holds

$$\mathbb{P}_\omega(\mathcal{K}_A(t) \geq 2) = 1 - e^{-c_A(\omega)t} (1 + c_A(\omega)t) \leq C(\omega)t^2. \tag{88}$$

When $\mathcal{K}_A(t) = 1$, we define the pair $\{x_0, y_0\}$ as the only edge in \mathcal{E}_A such that $\mathcal{K}_{x_0,y_0}(t) = 1$. To have a univocally defined labelling, we fix a total order $<$ of $\hat{\omega}$. If the pair has only one point in A , then we call this point x_0 and the other one y_0 . Otherwise, we call x_0 the minimal point inside the pair w.r.t. the order $<$.

Claim B.1 Let F be the event that (i) $\mathcal{K}_A(t) = 1$ and (ii) $\{x_0, y_0\}$ is not a connected component of $\mathcal{G}_t(\omega, \mathcal{K})$. Then $\mathbb{P}_\omega(F) = o(t)$.

Proof of Claim B.1 We first show that $F \subset G$, where

$$G = \{ \mathcal{K}_A(t) = 1, x_0 \in A, y_0 \notin A, \exists z \in \hat{\omega} \setminus (A \cup \{y_0\}) \text{ with } \mathcal{K}_{y_0,z}(t) \geq 1 \}.$$

To prove the above inclusion suppose first that $\mathcal{K}_A(t) = 1$ and $x_0, y_0 \in A$. Then $\{x_0, y_0\}$ must be a connected component in $\mathcal{G}_t(\omega, \mathcal{K})$ otherwise we would contradict $\mathcal{K}_A(t) = 1$. Hence, the event F implies that $x_0 \in A$ and $y_0 \notin A$. By F , $\{x_0, y_0\}$ is not a connected component of $\mathcal{G}_t(\omega, \mathcal{K})$, and therefore there exists a point $z \in \hat{\omega} \setminus \{x_0, y_0\}$ such that $\mathcal{K}_{x_0,z}(t) \geq 1$ or $\mathcal{K}_{y_0,z}(t) \geq 1$. The first case cannot indeed occur as $\mathcal{K}_A(t) = 1$. By the same reason, in the second case it must be $z \notin A$. Hence, we conclude that there exists $z \in \hat{\omega} \setminus (A \cup \{y_0\})$ such that $\mathcal{K}_{y_0,z}(t) \geq 1$. This concludes the proof that $F \subset G$.

We have

$$\begin{aligned} \mathbb{P}_\omega(G) &\leq \sum_{x \in A} \sum_{y \in \hat{\omega} \setminus A} \mathbb{P}_\omega \left(\mathcal{K}_{x,y}(t) = 1, \sum_{z \in \hat{\omega} \setminus (A \cup \{y\})} \mathcal{K}_{y,z}(t) \geq 1 \right) \\ &\leq t \sum_{x \in A} \sum_{y \in \hat{\omega}} c_{x,y}(\omega) e^{-c_{x,y}(\omega)t} (1 - e^{-c_y(\omega)t}). \end{aligned} \tag{89}$$

By (87) and the dominated convergence theorem applied to the last expression in (89), we get $\lim_{t \downarrow 0} \mathbb{P}_\omega(G)/t = 0$. As $F \subset G$, the same holds for F . \square

Now let H be the event that (i) $\mathcal{K}_A(t) = 1$ and (ii) $\{x_0, y_0\}$ is a connected component of $\mathcal{G}_t(\omega, \mathcal{K})$. Moreover, given $\{x, y\} \in \mathcal{E}_A$, we set $H_{x,y} := H \cap \{\{x_0, y_0\} = \{x, y\}\}$. Due to (88) and Claim B.1 we get

$$\mathbb{P}_\omega(\{\mathcal{K}_A(t) = 0\} \cup H) = 1 - o(t). \tag{90}$$

Hence we have $S(t)f(\xi) - f(\xi) = \sum_{\{x,y\} \in \mathcal{E}_A} [f(\xi^{x,y}) - f(\xi)] \mathbb{P}_\omega(H_{x,y}) + \|f\|_\infty o(t)$. As $\mathbb{P}_\omega(F) = o(t)$, we can rewrite the r.h.s. as

$$\begin{aligned} &\sum_{\{x,y\} \in \mathcal{E}_A} [f(\xi^{x,y}) - f(\xi)] \mathbb{P}_\omega(\{\mathcal{K}_A(t) = 1\} \cap \{\{x_0, y_0\} = \{x, y\}\}) + \|f\|_\infty o(t) \\ &= t \sum_{\{x,y\} \in \mathcal{E}_A} [f(\xi^{x,y}) - f(\xi)] c_{x,y}(\omega) e^{-c_A(\omega)t} + \|f\|_\infty o(t). \end{aligned}$$

As $\lim_{t \downarrow 0} o(t)/t = 0$ uniformly in ξ , by the dominated convergence theorem we can conclude that $\hat{\mathcal{L}}_\omega f = \mathcal{L}_\omega f$.

Appendix C. Convergence at a fixed time

In this appendix we prove (63) for any $\omega \in \Omega_{\text{typ}}, t > 0, \delta > 0$ and $\varphi \in C_c(\mathbb{R}^d)$ [the case $t = 0$ follows from (31)]. To this aim recall the semigroups P_t and $P_{\omega,t}^\varepsilon$

discussed before Proposition 6.1 in Sect. 6. Recall also the sets Ω_{\sharp} , $\tilde{\Omega}$, Ω_* and Ω_{typ} (cf. respectively Proposition 6.1, Definitions 7.2, 7.3 and 10.2). One main tool to get (63) will be the following fact, that we will prove at the end:

Lemma C.1 Fix $\omega \in \Omega_{\text{typ}}$, $\delta > 0$, $t > 0$, $\varphi \in C_c(\mathbb{R}^d)$ and let \mathfrak{m}_ε be an ε -parametrized family of probability measures on $\{0, 1\}^{\hat{\omega}}$. Then it holds

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega, \mathfrak{m}_\varepsilon}^\varepsilon \left(\left| \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta_t(x) - \varepsilon^d \sum_{x \in \hat{\omega}} \eta_0(x) P_{\omega, t}^\varepsilon \varphi(\varepsilon x) \right| > \delta \right) = 0. \tag{91}$$

Remark C.2 The second sum in (91) can be an infinite series. It is anyway absolutely convergent as it can be bounded by $\sum_{x \in \hat{\omega}} P_{\omega, t}^\varepsilon |\varphi|(\varepsilon x) = \sum_{x \in \hat{\omega}} |\varphi(\varepsilon x)|$ (by using the symmetry of the rates for $\omega \in \Omega_{\text{typ}} \subset \Omega_* \subset \tilde{\Omega}$).

Let us first prove (63). As $\int \varphi(x) \rho(x, t) dx = \int \rho_0(x) P_t \varphi(x) dx$, we can bound

$$\begin{aligned} & \left| \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta_t(x) - \int_{\mathbb{R}^d} \varphi(x) \rho(x, t) dx \right| \\ & \leq \left| \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta_t(x) - \varepsilon^d \sum_{x \in \hat{\omega}} \eta_0(x) P_{\omega, t}^\varepsilon \varphi(\varepsilon x) \right| \\ & \quad + \left| \varepsilon^d \sum_{x \in \hat{\omega}} \eta_0(x) P_{\omega, t}^\varepsilon \varphi(\varepsilon x) - \varepsilon^d \sum_{x \in \hat{\omega}} \eta_0(x) P_t \varphi(\varepsilon x) \right| \\ & \quad + \left| \varepsilon^d \sum_{x \in \hat{\omega}} \eta_0(x) P_t \varphi(\varepsilon x) - \int_{\mathbb{R}^d} \rho_0(x) P_t \varphi(x) dx \right| =: I_1 + I_2 + I_3. \end{aligned} \tag{92}$$

Trivially we can bound $I_2 \leq \varepsilon^d \sum_{x \in \hat{\omega}} |P_{\omega, t}^\varepsilon \varphi(\varepsilon x) - P_t \varphi(\varepsilon x)|$. The r.h.s. goes to zero as $\varepsilon \downarrow 0$ as $\Omega_{\text{typ}} \subset \Omega_{\sharp}$ [cf. (37)]. By combining this limit with Lemma C.1, to prove (63) we only need to show that $\lim_{\varepsilon \downarrow 0} \mathfrak{m}_\varepsilon(I_3 > \delta) = 0$. The continuous function $P_t \varphi$ decays fast to infinity. In particular, for some $C > 0$ we have $|P_t \varphi|(z) \leq C \psi(|z|)$ for all $z \in \mathbb{R}^d$, where $\psi(r) := 1/(1+r^{d+1})$. Due to (39) and since $\omega \in \Omega_{\text{typ}}$, we can fix ℓ such that $\int_{\mathbb{R}^d} \psi(|z|) \mathbb{1}_{\{|z| \geq \ell\}} dz < \delta/(5C)$ and $\overline{\lim}_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} d\mu_\omega^\varepsilon(z) \psi(|z|) \mathbb{1}_{\{|z| \geq \ell\}} \leq \delta/(5C)$. This implies for ε small that

$$\varepsilon^d \sum_{x \in \hat{\omega}} |P_t \varphi|(\varepsilon x) \mathbb{1}_{\{|\varepsilon x| \geq \ell\}} \leq \delta/5, \quad \int_{\mathbb{R}^d} |P_t \varphi|(z) \mathbb{1}_{\{|z| \geq \ell\}} dz \leq \delta/5. \tag{93}$$

Then we fix a function $\tilde{\varphi} \in C_c(\mathbb{R}^d)$ such that $|\tilde{\varphi}| \leq |P_t \varphi|$ and $\tilde{\varphi}(x) = P_t \varphi(x)$ if $|x| \leq \ell$. Hence, due to (93), to prove that $\lim_{\varepsilon \downarrow 0} \mathfrak{m}_\varepsilon(I_3 > \delta) = 0$ it is enough to show that

$$\lim_{\varepsilon \downarrow 0} \mathfrak{m}_\varepsilon \left(\left| \varepsilon^d \sum_{x \in \hat{\omega}} \eta_0(x) \tilde{\varphi}(\varepsilon x) - \int_{\mathbb{R}^d} \rho_0(x) \tilde{\varphi}(x) dx \right| > \delta/5 \right) = 0. \tag{94}$$

The above limit follows from our assumption on \mathfrak{m}_ε [cf. (31) in Theorem 4.1]. This concludes the derivation of (63) assuming Lemma C.1.

We now give the proof of Lemma C.1:

Proof of Lemma C.1 It is convenient here to work with the non speeded-up exclusion process with formal generator \mathcal{L}_ω [cf. (7) and (9)]. We write $\mathbb{P}_{\omega, n_\varepsilon}$ for its law on the path space $D(\mathbb{R}_+, \{0, 1\}^{\hat{\omega}})$, when starting with distribution n_ε . Then we can restate (91) as

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}_{\omega, n_\varepsilon} \left(\left| \varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta_{\varepsilon^{-2}t}(x) - \varepsilon^d \sum_{x \in \hat{\omega}} \eta_0(x) P_{\omega, t}^\varepsilon \varphi(\varepsilon x) \right| > \delta \right) = 0. \tag{95}$$

We divide the proof of (95) in some main steps.

- *Step 1: Reduction to distributions \bar{n}_ε concentrating on configurations having a finite number of particles.* We think of the exclusion process as built according to the graphical construction described in Sect. 7, after sampling η_0 with distribution n_ε . As $\omega \in \Omega_{\text{typ}} \subset \Omega_*$ we have $\mathbb{P}_\omega(\mathbb{K}_\omega) = 1$. Given $x \in \hat{\omega}$, $r \in \mathbb{N}$ and $\mathcal{K} \in \mathbb{K}_\omega$, we denote by $\mathcal{C}_r(x)$ the connected component of x in the graph $\mathcal{G}'_0(\omega, \mathcal{K})$. Fix $s \in (rt_0, (r + 1)t_0]$. Due to the graphical construction, if we know \mathcal{K} , then to determine $\eta_s^\xi[\mathcal{K}](x)$ we only need to know $\eta_{rt_0}^\xi[\mathcal{K}](z)$ with $z \in \mathcal{C}_r(x)$ (and this holds for any $\xi \in \{0, 1\}^{\hat{\omega}}$). By iterating the above argument we conclude that, knowing \mathcal{K} , the value of $\eta_s^\xi[\mathcal{K}](x)$ is determined by $\xi(z)$ as z varies in the finite set

$$Q_r(x) := \cup_{z_r \in \mathcal{C}_r(x)} \cup_{z_{r-1} \in \mathcal{C}_{r-1}(z_r)} \cdots \cup_{z_1 \in \mathcal{C}_1(z_2)} \mathcal{C}_0(z_1).$$

The above set $Q_r(x)$ is finite as $\mathcal{K} \in \mathbb{K}_\omega$. As φ has compact support, we can take $\ell > 0$ such that φ has support in the ball B_ℓ of radius ℓ centered at the origin. Then, by the above considerations, given $t > 0$ for $\ell_* = \ell_*(\omega, \varepsilon, t)$ large enough we have $\mathbb{P}_\omega(A_{\omega, \varepsilon, t}^c) \leq \varepsilon$, where

$$A_{\omega, \varepsilon, t} := \{ \mathcal{K} \in \mathbb{K}_\omega : \cup_{\substack{x \in \hat{\omega}: \\ \varepsilon|x| \leq \ell}} Q_r(\varepsilon^{-2}t)(x) \subset B_{\ell_*} \}$$

and $r(\varepsilon^{-2}t)$ is the unique integer $r \geq 0$ such that $\varepsilon^{-2}t \in (rt_0, (r + 1)t_0]$. Note that, when the event $A_{\omega, \varepsilon, t}$ takes place, the value $\varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \eta_{\varepsilon^{-2}t}^\xi[\mathcal{K}](x)$ depends on ξ only through $\xi(z)$ with $z \in \hat{\omega} \cap B_{\ell_*}$.

As $\omega \in \Omega_{\text{typ}}$ [see (39)] and since $P_t \varphi$ decays fast to infinity, we have $\overline{\lim}_{L \uparrow \infty} \overline{\lim}_{\varepsilon \downarrow 0} \int d\mu_\omega^\varepsilon(z) |P_t \varphi|(z) \mathbb{1}_{\{|z| > L\}} = 0$. In particular, we can fix $L_* = L_*(\varphi, \omega)$, such that $\overline{\lim}_{\varepsilon \downarrow 0} \int d\mu_\omega^\varepsilon(z) |P_t \varphi|(z) \mathbb{1}_{\{|z| > L_*\}} \leq \delta/4$. On the other hand, as $\omega \in \Omega_{\text{typ}} \subset \Omega_\sharp$ [cf. (37) in Proposition 6.1] for ε small enough we have $\int d\mu_\omega^\varepsilon(z) |P_{\omega, t}^\varepsilon \varphi(z) - P_t \varphi(z)| \leq \delta/4$. Due to the above observations, for ε small it holds $\varepsilon^d \sum_{x \in \hat{\omega}: |x| \geq L_*/\varepsilon} \xi(x) |P_{\omega, t}^\varepsilon \varphi|(\varepsilon x) \leq \delta/2$ for all $\xi \in \{0, 1\}^{\hat{\omega}}$.

Call \bar{n}_ε the law of the following random configuration in $\{0, 1\}^{\hat{\omega}}$: sample ξ with law n_ε , then set the particle number of ξ equal to zero at any site $x \in \hat{\omega}$ with $|x| > \ell_* \vee (L_*/\varepsilon)$. By the above considerations, to get (95) it is enough to prove the same limit with n_ε replaced by \bar{n}_ε and with δ replaced by $\delta/2$. The fact that the constant

$\ell_* \vee (L_*/\varepsilon)$ depends on $\varepsilon, \omega, \varphi, t$ does not interfere with the arguments presented below (moreover, ω, φ, t can be thought as fixed).

• *Step 2: special pathwise representation of $\eta_t^\xi[\mathcal{K}](x)$.* We fix $\xi \in \{0, 1\}^{\hat{\omega}}$ with a finite number of particles. On the probability space $(\mathbb{K}_\omega, \mathbb{P}_\omega)$ (cf. Definition 7.3) we introduce the martingales $(M_y^\xi(t))_{t \geq 0}$, with y varying among $\hat{\omega}$, by setting $M_y^\xi(0) := 0$ and

$$dM_y^\xi(t) := \sum_{z \in \hat{\omega}} (\eta_{t-}^\xi[\mathcal{K}](z) - \eta_{t-}^\xi[\mathcal{K}](y)) dA_{y,z}(t), \quad A_{y,z}(t) := \mathcal{K}_{y,z}(t) - c_{y,z}(\omega)t. \tag{96}$$

The key observation now, going back to [33] and proved below, is that the symmetry of the jump rates implies the following pathwise representation for all $x \in \hat{\omega}$ and $\mathcal{K} \in \mathbb{K}_\omega$:

$$\eta_t^\xi[\mathcal{K}](x) = \sum_{y \in \hat{\omega}} p_\omega(t, x, y) \xi(y) + \sum_{y \in \hat{\omega}} \int_0^t p_\omega(t-s, x, y) dM_y^\xi(s). \tag{97}$$

Above $p_\omega(t, x, y)$ is the probability to be at y for the random walk X^ω starting at x (before we used the notation $p_{\omega,t}^1(x, y)$, which would not be very readable in the rest). We first show that the r.h.s. of (97) is well posed and afterwards we check (97) itself.

⊙ *Step 2.a: the r.h.s. of (97) is well posed.* As $\omega \in \Omega_{\text{typ}}$ it holds $c_y(\omega) := \sum_{z \in \hat{\omega}} c_{y,z}(\omega) < +\infty$ for all $y \in \hat{\omega}$. As $\mathcal{K} \in \mathbb{K}_\omega$, by Definition 7.3 we also have $\mathcal{K}_y(t) < +\infty$ for all $y \in \hat{\omega}$ and $t \geq 0$.

As ξ has a finite number of particles, the first sum in the r.h.s. of (97) is trivially finite. We now show that the second sum in the r.h.s. is absolutely convergent, thus implying that the r.h.s. of (97) is well posed. To this aim call $D = D(\mathcal{K}, \xi)$ the set of points $y \in \hat{\omega}$ such that $\eta_s^\xi[\mathcal{K}](y) = 1$ for some $s \in [0, t]$. By the graphical construction and since ξ has a finite number of particles, D is a finite set. We also note that, if $|\eta_{s-}^\xi[\mathcal{K}](z) - \eta_{s-}^\xi[\mathcal{K}](y)|$ is nonzero, then y or z must belong to D . Hence we can bound

$$\begin{aligned} & \sum_{y \in \hat{\omega}} \sum_{z \in \hat{\omega}} \int_0^t p_\omega(t-s, x, y) |\eta_{s-}^\xi[\mathcal{K}](z) - \eta_{s-}^\xi[\mathcal{K}](y)| c_{y,z}(\omega) ds \\ & \leq t \sum_{y \in D} \sum_{z \in \hat{\omega}} c_{y,z}(\omega) + t \sum_{y \in \hat{\omega}} \sum_{z \in D} c_{y,z}(\omega) = 2t \sum_{y \in D} c_y(\omega) < +\infty \end{aligned} \tag{98}$$

and (using also that $\mathcal{K}_{y,z}(s) = \mathcal{K}_{z,y}(s)$)

$$\begin{aligned} & \sum_{y \in \hat{\omega}} \sum_{z \in \hat{\omega}} \int_0^t p_\omega(t-s, x, y) |\eta_{s-}^\xi[\mathcal{K}](z) - \eta_{s-}^\xi[\mathcal{K}](y)| d\mathcal{K}_{y,z}(s) \\ & \leq \sum_{y \in \hat{\omega}} \sum_{z \in \hat{\omega}} \int_0^t \mathbb{1}(y \in D \text{ or } z \in D) d\mathcal{K}_{y,z}(s) \leq 2 \sum_{v \in D} \mathcal{K}_v(t) < +\infty. \end{aligned} \tag{99}$$

As a byproduct of (98) and (99) the second series in the r.h.s. of (97) is absolutely convergent.

⊙ *Step 2.b: proof of (97).* We now verify (97) (the proof is different from the one in [33], which does not adapt well to our setting). To this aim we fix $\mathcal{K} \in \mathbb{K}_\omega$. Recall the finite set D introduced in Step 2.a. Let $t_1 < t_2 < \dots < t_n$ be the jump times of the Poisson processes $\mathcal{K}_v(\cdot)$ up to time t , as v varies among D . Let $a_i, b_i \in \hat{\omega}$ be such that $\mathcal{K}_{a_i, b_i}(t_i) = \mathcal{K}_{a_i, b_i}(t_i-) + 1$ (the pair $\{a_i, b_i\}$ is univocally determined, the way we label its elements will be irrelevant). We set $t_0 := 0, t_{n+1} := t$. As (see Step 2.a) the series in the r.h.s. of (97) are absolutely convergent, we have

$$\begin{aligned} & \sum_{y \in \hat{\omega}} \int_0^t p_\omega(t-s, x, y) dM_y^\xi(s) = A_1 - A_2, \\ A_1 & := \sum_{i=0}^n \sum_{y \in \hat{\omega}} \sum_{z \in \hat{\omega}} (\eta_{t_i}^\xi[\mathcal{K}](z) - \eta_{t_i}^\xi[\mathcal{K}](y)) \int_{(t_i, t_{i+1})} p_\omega(t-s, x, y) d\mathcal{K}_{y,z}(s), \\ A_2 & := \sum_{i=0}^n \sum_{y \in \hat{\omega}} \sum_{z \in \hat{\omega}} c_{y,z}(\omega) (\eta_{t_i}^\xi[\mathcal{K}](z) - \eta_{t_i}^\xi[\mathcal{K}](y)) \int_{t_i}^{t_{i+1}} p_\omega(t-s, x, y) ds. \end{aligned} \tag{100}$$

Consider the expression $(\eta_{t_i}^\xi[\mathcal{K}](z) - \eta_{t_i}^\xi[\mathcal{K}](y)) d\mathcal{K}_{y,z}(s)$. If it is nonzero, then $\{y, z\}$ intersects D and s is a jump time of $\mathcal{K}_{y,z}(\cdot) = \mathcal{K}_{z,y}(\cdot)$. In particular, it must be $s \in \{t_1, t_2, \dots, t_n\}$ and $\{y, z\} = \{a_i, b_i\}$ if $s = t_i$. The above considerations imply that $A_1 = \sum_{i=0}^{n-1} C_i$, where

$$C_i := (\eta_{t_i}^\xi[\mathcal{K}](b_{i+1}) - \eta_{t_i}^\xi[\mathcal{K}](a_{i+1})) [p_\omega(t-t_{i+1}, x, a_{i+1}) - p_\omega(t-t_{i+1}, x, b_{i+1})].$$

We write E_x for the expectation w.r.t. the random walk $(X_t^\omega)_{t \geq 0}$ on $\hat{\omega}$ starting at x . Fixed $i \in \{1, \dots, n\}$, we consider the function $f_i : \hat{\omega} \rightarrow \mathbb{R}$ given by $f_i(a) := \eta_{t_i}^\xi[\mathcal{K}](a)$. Note that f_i has finite support. Since $\tilde{\mathbb{L}}_\omega^1 f_i(y) = \sum_{z \in \hat{\omega}} c_{y,z}(\omega) (\eta_{t_i}^\xi[\mathcal{K}](z) - \eta_{t_i}^\xi[\mathcal{K}](y))$ (cf. Definition 8.1), we have

$$\begin{aligned} A_2 & = \sum_{i=0}^n \sum_{y \in \hat{\omega}} \tilde{\mathbb{L}}_\omega^1 f_i(y) \int_{t_i}^{t_{i+1}} p_\omega(t-s, x, y) ds \\ & = \sum_{i=0}^n \sum_{y \in \hat{\omega}} \int_{t-t_{i+1}}^{t-t_i} p_\omega(s, x, y) \tilde{\mathbb{L}}_\omega^1 f_i(y) ds = \sum_{i=0}^n \int_{t-t_{i+1}}^{t-t_i} \frac{d}{ds} E_x[\eta_{t_i}^\xi[\mathcal{K}](X_s^\omega)] ds \\ & = \sum_{i=0}^n \left(E_x[\eta_{t_i}^\xi[\mathcal{K}](X_{t-t_i}^\omega)] - E_x[\eta_{t_i}^\xi[\mathcal{K}](X_{t-t_{i+1}}^\omega)] \right). \end{aligned}$$

Note that the third identity can be derived from Proposition 7.4 (recall that $\Omega_{\text{typ}} \subset \Omega_*$) as the random walk can be thought of as a simple exclusion process with just one particle (having (9) on local functions of η , it is enough to compute $\mathcal{L}_\omega F$ with

$F(\eta) := \sum_{a \in \Delta_i} f(a)\eta(a)$, Δ_i being the finite support of f_i , and evaluate $\mathcal{L}_\omega F$ on configurations with just one particle). Since, for $1 \leq i \leq n$ and $u \in \hat{\omega}$, it holds

$$\eta_i^\xi[\mathcal{K}](u) = \eta_{i-1}^\xi[\mathcal{K}](u) + (\delta_{u,a_i} - \delta_{u,b_i})[\eta_{i-1}^\xi[\mathcal{K}](b_i) - \eta_{i-1}^\xi[\mathcal{K}](a_i)],$$

we have $E_x[\eta_i^\xi[\mathcal{K}](X_{t-t_i}^\omega)] = E_x[\eta_{i-1}^\xi[\mathcal{K}](X_{t-t_i}^\omega)] + C_{i-1}$. Hence we can write

$$\begin{aligned} A_2 &= E_x[\eta_0^\xi[\mathcal{K}](X_t^\omega)] + \sum_{i=1}^n (E_x[\eta_{i-1}^\xi[\mathcal{K}](X_{t-t_i}^\omega)] + C_{i-1}) - \sum_{i=0}^n E_x[\eta_i^\xi[\mathcal{K}](X_{t-t_{i+1}}^\omega)] \\ &= E_x[\xi(X_t^\omega)] - E_x[\eta_n^\xi[\mathcal{K}](X_0^\omega)] + A_1 = \sum_{y \in \hat{\omega}} p_\omega(t, x, y)\xi(y) - \eta_i^\xi[\mathcal{K}](x) + A_1. \end{aligned}$$

The above identity and (100) imply (97).

• *Step 3: Conclusion.* Recall that, due to Step 1, to prove (95) it is enough to prove the same limit with n_ε replaced by \bar{n}_ε and with δ replaced by $\delta/2$. We denote by \mathbb{E}_ω the expectation w.r.t. \mathbb{P}_ω . By the symmetry $p_\omega(t, x, y) = p_\omega(t, y, x)$, we have

$$\varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \sum_{y \in \hat{\omega}} p_\omega(t, x, y)\xi(y) = \varepsilon^d \sum_{x \in \hat{\omega}} \xi(x) P_{\omega,t}^\varepsilon \varphi(\varepsilon x), \quad \forall \xi \in \{0, 1\}^{\hat{\omega}}.$$

Hence, due to (97), in order to conclude the proof of (95) it is enough to show that

$$\lim_{\varepsilon \downarrow 0} \int d\bar{n}_\varepsilon(\xi) \mathbb{E}_\omega \left[\left(\varepsilon^d \sum_{x \in \hat{\omega}} \varphi(\varepsilon x) \sum_{y \in \hat{\omega}} \int_0^{\varepsilon^{-2}t} p_\omega(\varepsilon^{-2}t - s, x, y) dM_y^\xi(s) \right)^2 \right] = 0. \tag{101}$$

Due to (96), we can rewrite the expression inside the (\cdot) -brackets as

$$\begin{aligned} \mathcal{R}_\varepsilon^\xi[\mathcal{K}] &:= \frac{\varepsilon^d}{2} \sum_{x \in \hat{\omega}} \sum_{y \in \hat{\omega}} \sum_{z \in \hat{\omega}} \varphi(\varepsilon x) \cdot \\ &\int_0^{\varepsilon^{-2}t} [\eta_{s-}^\xi(z) - \eta_{s-}^\xi(y)] (p_\omega(\varepsilon^{-2}t - s, x, y) - p_\omega(\varepsilon^{-2}t - s, x, z)) dA_{y,z}(s) \\ &= \frac{\varepsilon^d}{2} \sum_{y \in \hat{\omega}} \sum_{z \in \hat{\omega}} \int_0^{\varepsilon^{-2}t} [\eta_{s-}^\xi(z) - \eta_{s-}^\xi(y)] \left(P_{\omega, \frac{t}{2}-s}^1 \varphi(\varepsilon y) - P_{\omega, \frac{t}{2}-s}^1 \varphi(\varepsilon z) \right) dA_{y,z}(s) \end{aligned}$$

where $\eta^\varepsilon = \eta^\varepsilon[\mathcal{K}]$. As the $A_{y,z}(\cdot)$'s are orthogonal martingales by varying $\{y, z\}$ (while $A_{y,z}(\cdot) = A_{z,y}(\cdot)$), similarly to [33] we get (using the symmetry of $p_\omega(s, \cdot, \cdot)$)

$$\begin{aligned} \int d\bar{n}_\varepsilon(\xi) \mathbb{E}_\omega[(\mathcal{R}_\varepsilon^\xi)^2] &\leq \frac{\varepsilon^{2d}}{2} \sum_{y \in \hat{\omega}} \sum_{z \in \hat{\omega}} \int_0^{\varepsilon^{-2t}} c_{y,z}(\omega) (P_{\omega,s}^1 \varphi(\varepsilon y) - P_{\omega,s}^1 \varphi(\varepsilon z))^2 ds \\ &= \varepsilon^d \int_0^t \langle P_{\omega,s}^\varepsilon \varphi, -\mathbb{L}_\omega^\varepsilon P_{\omega,s}^\varepsilon \varphi \rangle_{L^2(\mu_\omega^\varepsilon)} = -\frac{\varepsilon^d}{2} \int_0^t \frac{d}{ds} \|P_{\omega,s}^\varepsilon \varphi\|_{L^2(\mu_\omega^\varepsilon)}^2 ds \\ &= \frac{\varepsilon^d}{2} \|P_{\omega,0}^\varepsilon \varphi\|_{L^2(\mu_\omega^\varepsilon)}^2 - \frac{\varepsilon^d}{2} \|P_{\omega,t}^\varepsilon \varphi\|_{L^2(\mu_\omega^\varepsilon)}^2 \leq \frac{\varepsilon^d}{2} \|\varphi\|_{L^2(\mu_\omega^\varepsilon)}^2 \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

This concludes the proof of (101). \square

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