Poisson limit of bumping routes in the Robinson–Schensted correspondence

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Received: 16 June 2020 / Revised: 12 April 2021 / Accepted: 15 July 2021 / Published online: 17 August 2021
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Abstract

We consider the Robinson–Schensted–Knuth algorithm applied to a random input and investigate the shape of the bumping route (in the vicinity of the y -axis) when a specified number is inserted into a large Plancherel-distributed random tableau. We show that after a projective change of the coordinate system the bumping route converges in distribution to the Poisson process.

Keywords Robinson–Schensted–Knuth algorithm · RSK · Plancherel growth process · Bumping route · Limit shape · Poisson process

Mathematics Subject Classification 60C05 · 05E10 · 60F05 · 60K35

1 Introduction

1.1 Notations

The set of Young diagrams will be denoted by $\mathbb{Y}$; the set of Young diagrams with n boxes will be denoted by $\mathbb{Y}_n$. The set $\mathbb{Y}$ has a structure of an oriented graph, called *Young graph*; a pair $\mu \nearrow \lambda$ forms an oriented edge in this graph if the Young diagram λ can be created from the Young diagram μ by addition of a single box.

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Fig. 1 **a** The original tableau T . **b** We consider the Schensted row insertion of the number 18 to the tableau T . The highlighted boxes form the corresponding bumping route. The small numbers on the left (next to the arrows) indicate the inserted/bumped numbers. **c** The output $T \leftarrow 18$ of the Schensted insertion

We will draw Young diagrams and tableaux in the French convention with the Cartesian coordinate system Oxy , cf. Figs. 1 and 2a. We index the rows and the columns of tableaux by *non-negative* integers from $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. In particular, if $\square$ is a box of a tableau, we identify it with the Cartesian coordinates of its *lower-left corner*: $\square = (x, y) \in \mathbb{N}_0 \times \mathbb{N}_0$. For a tableau T we denote by $T_{x,y}$ its entry which lies in the intersection of the row $y \in \mathbb{N}_0$ and the column $x \in \mathbb{N}_0$. The position of the box s in the tableau T will be denoted by $\text{Pos}_s(T) \in \mathbb{N}_0 \times \mathbb{N}_0$.

Also the rows of any Young diagram $\lambda = (\lambda_0, \lambda_1, \dots)$ are indexed by the elements of $\mathbb{N}_0$; in particular the length of the bottom row of λ is denoted by λ_0 .

1.2 Schensted row insertion

The *Schensted row insertion* is an algorithm which takes as an input a tableau T and some number a . The number a is inserted into the first row (that is, the bottom row, the row with the index 0) of T to the leftmost box which contains an entry which is strictly bigger than a .

In the case when the row contains no entries which are bigger than a , the number a is inserted into the leftmost empty box in this row and the algorithm terminates.

If, however, the number a is inserted into a box which was not empty, the previous content a' of the box is *bumped* into the second row (that is, the row with the index 1). This means that the algorithm is iterated but this time the number a' is inserted into the second row to the leftmost box which contains a number bigger than a' . We repeat these steps of row insertion and bumping until some number is inserted into a previously empty box. This process is illustrated on Fig. 1b and c. The outcome of the Schensted insertion is defined as the result of the aforementioned procedure; it will be denoted by $T \leftarrow a$.

Note that this procedure is well defined also in the setup when T is an *infinite* tableau (see Fig. 2a for an example), even if the above procedure does not terminate after a finite number of steps.

1.3 Robinson–Schensted–Knuth algorithm

For the purposes of this article we consider a simplified version of *the Robinson–Schensted–Knuth algorithm*; for this reason we should rather call it *the Robinson–Schensted algorithm*. Nevertheless, we use the first name because of its well-known acronym RSK. The RSK algorithm associates to a finite sequence $w = (w_1, \dots, w_\ell)$ a pair of tableaux: *the insertion tableau* $P(w)$ and *the recording tableau* $Q(w)$.

The insertion tableau

$$P(w) = \left(((\emptyset \leftarrow w_1) \leftarrow w_2) \leftarrow \dots \right) \leftarrow w_\ell \quad (1)$$

is defined as the result of the iterative Schensted row insertion applied to the entries of the sequence w , starting from *the empty tableau* $\emptyset$.

The recording tableau $Q(w)$ is defined as the standard Young tableau of the same shape as $P(w)$ in which each entry is equal to the number of the iteration of (1) in which the given box of $P(w)$ stopped being empty; in other words the entries of $Q(w)$ give the order in which the entries of the insertion tableau were filled.

The tableaux $P(w)$ and $Q(w)$ have the same shape; we will denote this common shape by $\text{RSK}(w)$ and call it *the RSK shape associated to* w .

The RSK algorithm is of great importance in algebraic combinatorics, especially in the context of the representation theory [5].

1.4 Plancherel measure, Plancherel growth process

Let $\mathfrak{S}_n$ denote the symmetric group of order n . We will view each permutation $\pi \in \mathfrak{S}_n$ as a sequence $\pi = (\pi_1, \dots, \pi_n)$ which has no repeated entries, and such that $\pi_1, \dots, \pi_n \in \{1, \dots, n\}$. The restriction of RSK to the symmetric group is a bijection which to a given permutation from $\mathfrak{S}_n$ associates a pair (P, Q) of standard Young tableaux of the same shape, consisting of n boxes. A fruitful area of study concerns the RSK algorithm applied to a uniformly random permutation from $\mathfrak{S}_n$, especially asymptotically in the limit $n \rightarrow \infty$, see [12] and the references therein.

The Plancherel measure on $\mathbb{Y}_n$, denoted Plan_n , is defined as the probability distribution of the random Young diagram $\text{RSK}(w)$ for a uniformly random permutation $w \in \mathfrak{S}_n$.

An *infinite standard Young tableau* [7, Section 2.2] is a filling of the boxes in a subset of the upper-right quarterplane with positive integers, such that each row and each column is increasing, and each positive integer is used exactly once. There is a natural bijection between the set of infinite standard Young tableaux and the set of infinite paths in the Young graph

$$\lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \dots \quad \text{with } \lambda^{(0)} = \emptyset; \quad (2)$$

this bijection is given by setting $\lambda^{(n)}$ to be the set of boxes of a given infinite standard Young tableau which are $\leq n$.

If $w = (w_1, w_2, \dots)$ is an *infinite* sequence, the recording tableau $Q(w)$ is defined as the infinite standard Young tableau in which each non-empty entry is equal to the number of the iteration in the infinite sequence of Schensted row insertions

$$((\emptyset \leftarrow w_1) \leftarrow w_2) \leftarrow \dots$$

in which the corresponding box stopped being empty, see [13, Section 1.2.4]. Under the aforementioned bijection, the recording tableau $Q(w)$ corresponds to the sequence (2) with

$$\lambda^{(n)} = \text{RSK}(w_1, \dots, w_n).$$

Let $\xi = (\xi_1, \xi_2, \dots)$ be an infinite sequence of independent, identically distributed random variables with the uniform distribution $U(0, 1)$ on the unit interval $[0, 1]$. The *Plancherel measure on the set of infinite standard Young tableaux* is defined as the probability distribution of $Q(\xi)$. Any sequence with the same probability distribution as (2) with

$$\lambda^{(n)} = \text{RSK}(\xi_1, \dots, \xi_n) \quad (3)$$

will be called the *Plancherel growth process* [7]. It turns out that the Plancherel growth process is a Markov chain [7, Sections 2.2 and 2.4]. For a more systematic introduction to this topic we recommend the monograph [12, Section 1.19].

1.5 Bumping route

The *bumping route* consists of the boxes the entries of which were changed by the action of Schensted insertion, including the last, newly created box, see Fig. 1b and c. The bumping route will be denoted by $\mathcal{T} \leftarrow a$ or by $\mathcal{T} \leftarrow_a$ depending on current typographic needs. In any row $y \in \mathbb{N}_0$ there is at most one box from the bumping route $\mathcal{T} \leftarrow a$; we denote by $\mathcal{T} \leftarrow_a(y)$ its x -coordinate. We leave $\mathcal{T} \leftarrow_a(y)$ undefined if such a box does not exist. In this way

$$\mathcal{T} \leftarrow a = \left\{ (\mathcal{T} \leftarrow_a(y), y) : y \in \mathbb{N}_0 \right\}. \quad (4)$$

For example, for the tableau $\mathcal{T}$ from Fig. 1a and $a = 18$ we have

$$\mathcal{T} \leftarrow_a(y) = \begin{cases} 1 & \text{for } y \in \{0, 1\}, \\ 0 & \text{for } y \in \{2, 3\}. \end{cases}$$

The bumping route $\mathcal{T} \leftarrow a$ can be visualized either as a collection of its boxes or as a plot of the function

$$x(y) = \mathcal{T} \leftarrow_a(\lfloor y \rfloor), \quad y \in \mathbb{R}_+,$$

cf. the thick red line on Fig. 2a.

Fig. 2 **a** The French convention for drawing tableaux. An example of an infinite standard Young tableau $\mathcal{T}$ sampled with the Plancherel distribution. The highlighted boxes form a bumping route obtained by adding the entry $m + 1/2$ for $m = 3$. The thick red line is the corresponding plot of the function $x(y) = \mathcal{T}_{\leftarrow m+1/2}(\lfloor y \rfloor)$. **b** The same tableau shown in the projective coordinates Oxz with $z = \frac{2m}{y}$. The thick red line is the plot of the function $x(z) = \mathcal{T}_{\leftarrow m+1/2}(\lfloor \frac{2m}{z} \rfloor)$

1.6 Bumping routes for infinite tableaux

Any bumping route which corresponds to an insertion to a *finite* tableau is, well, also finite. This is disadvantageous when one aims at the asymptotics of such a bumping route in a row of index y in the limit $y \rightarrow \infty$. For such problems it would be preferable to work in a setup in which the bumping routes are infinite; we present the details in the following.

Let us fix the value of an integer $m \in \mathbb{N}_0$. Now, for an integer $n \geq m$ we consider a real number $0 < \alpha_n < 1$ and a finite sequence $\xi = (\xi_1, \dots, \xi_n)$ of independent, identically distributed random variables with the uniform distribution $U(0, 1)$ on the unit interval $[0, 1]$. In order to remove some randomness from this picture we will condition the choice of ξ in such a way that there are exactly m entries of ξ which are smaller than α_n ; heuristically this situation is similar to a scenario without conditioning, for the choice of

$$\alpha_n = \frac{m}{n}. \quad (5)$$

We will study the bumping route

$$P(\xi_1, \dots, \xi_n) \leftarrow \alpha_n \quad (6)$$

in the limit as $n \rightarrow \infty$ and m is fixed.

Without loss of generality we may assume that the entries of the sequence ξ are all different. Let $\pi \in \mathfrak{S}_n$ be the unique permutation which encodes the relative order of the entries in the sequence ξ , that is

$$(\pi_i < \pi_j) \iff (X_i < X_j)$$

for any $1 \leq i, j \leq n$. Since the algorithm behind the Robinson–Schensted–Knuth correspondence depends only on the relative order of the involved numbers and not their exact values, it follows that the bumping route (6) coincides with the bumping route

$$P(\pi_1, \dots, \pi_n) \leftarrow m + 1/2. \quad (7)$$

The probability distribution of π is the uniform measure on $\mathfrak{S}_n$; it follows that the probability distribution of the tableau $P(\pi_1, \dots, \pi_n)$ which appears in (7) is the Plancherel measure Plan_n on the set of standard Young tableaux with n boxes. Since such a Plancherel-distributed random tableau with n boxes can be viewed as a truncation of an infinite standard Young tableau $\mathcal{T}$ with the Plancherel distribution, the bumping routes (6) and (7) can be viewed as truncations of the infinite bumping route

$$\mathcal{T} \leftarrow m + 1/2, \quad (8)$$

see Fig. 2a for an example.

1.7 The main problem: asymptotics of infinite bumping routes

The aim of the current paper is to investigate the asymptotics of the *infinite* bumping route (8) in the limit $m \rightarrow \infty$.

Heuristically, this corresponds to investigation of the asymptotics of the *finite* bumping routes (6) in the simplified setting when we do not condition over some additional properties of ξ , in the scaling in which α_n does not tend to zero too fast [so that $\lim_{n \rightarrow \infty} \alpha_n n = \infty$, cf. (5)], but on the other hand α_n should tend to zero fast enough so that the bumping route is long enough that our asymptotic questions are well defined. We will not pursue in this direction and we will stick to the investigation of the *infinite* bumping route (8).

Even though Romik and the last named author [14] considered the asymptotics of *finite* bumping routes, their discussion is nevertheless applicable in our context. It shows that in the *balanced scaling* when we focus on the part of the bumping route with

Fig. 3 Four sample bumping routes corresponding to an insertion $\mathcal{T} \leftarrow m + 1/2$ for $m = 100$ and a random infinite standard Young tableau $\mathcal{T}$ which was sampled according to the Plancherel measure. In order to improve visibility, each bumping route is visualized as the plot of the corresponding function $y \mapsto T_{\leftarrow m+1/2}(\lfloor y \rfloor)$, cf. Fig. 2a, and not as a collection of boxes. Colour and thickness were added in order to help identify the routes. The solid line is the (rescaled) *limit bumping curve*. The dashed line is the hyperbola $xy = 2m$, cf. Eq. (9)

the Cartesian coordinates (x, y) of magnitude $x, y = O(\sqrt{m})$, the shape of the bumping route (scaled down by the factor $\frac{1}{\sqrt{m}}$) converges in probability towards an explicit curve, which we refer to as *the limit bumping curve*, see Fig. 3 for an illustration.

In the current paper we go beyond the scaling used by Romik and the last named author and investigate the part of the bumping route with the Cartesian coordinates of order $x = O(1)$ and $y \gg \sqrt{m}$. This part of the bumping curves was not visible on Fig. 3; in order to reveal it one can use the semi-logarithmic plot, cf. Figs. 4 and 5.

1.8 The Naive hyperbola

The first step in this direction would be to stretch the validity of the results of Romik and the last named author [14] beyond their limitations and to expect that the limit bumping

Fig. 4 The content of Fig. 3 shown with the axis y in the logarithmic scale

curve describes the asymptotics of the bumping routes also in this new scaling. This would correspond to the investigation of the asymptotics of the (non-rescaled) limit bumping curve $(x(y), y)$ in the regime $y \rightarrow \infty$. The latter analysis was performed by the first named author [9]; one of his results is that

$$\lim_{y \rightarrow \infty} x(y)y = 2;$$

in other words, for $y \rightarrow \infty$ the *non-rescaled* limit bumping curve can be approximated by the hyperbola $xy = 2$ while its *rescaled* counterpart which we consider in the current paper by the hyperbola

$$xy = 2m \tag{9}$$

which is shown on Fig. 3 as the dashed line. At the very end of Sect. 1.10 we will discuss the extent to which this naive approach manages to confront the reality.

1.9 In which row a bumping route reaches a given column?

Let us fix some (preferably infinite) standard Young tableau $\mathcal{T}$. The bumping route in each step jumps to the next row, directly up or to the left to the original column; in other words

Fig. 5 Zoom on a part of Fig. 4. In this kind of scaling when $x = O(1)$ and $y \gg \sqrt{m}$ the results of Romik and the last named author [14] are *not* applicable and the rescaled limit bumping curve (the solid line) is shown for illustration purposes only

$$\mathcal{T}_{\leftarrow m+1/2}(0) \geq \mathcal{T}_{\leftarrow m+1/2}(1) \geq \dots$$

is a weakly decreasing sequence of non-negative integers.

For $x, m \in \mathbb{N}_0$ we denote by

$$Y_x^{[m]} = Y_x = \min \{y \in \mathbb{N}_0 : \mathcal{T}_{\leftarrow m+1/2}(y) \leq x\} \quad (10)$$

the index of the first row in which the bumping route $\mathcal{T}_{\leftarrow m+1/2}$ reaches the column with the index x (or less, if the bumping route skips the column x completely). For example, for the tableau $\mathcal{T}$ from Fig. 2a we have

$$Y_0^{[3]} = 4, \quad Y_1^{[3]} = 2, \quad Y_2^{[3]} = 1, \quad Y_3^{[3]} = Y_4^{[3]} = \dots = 0.$$

If such a row does not exist we set $Y_x = \infty$; the following result shows that we do not have to worry about such a scenario.

Proposition 1.1 *For a random infinite standard Young tableau $\mathcal{T}$ with the Plancherel distribution*

$$Y_x^{[m]} < \infty \quad \text{for all } x, m \in \mathbb{N}_0$$

holds true almost surely.

The proof is postponed to Sect. 3.8. For a sketch of the proof of an equivalent result see the work of Vershik [17, Предложение] who uses different methods.

Theorem 1.2 (The main result) *Assume that $\mathcal{T}$ is an infinite standard Young tableau with the Plancherel distribution. With the above notations, the random set*

$$\left(\frac{2m}{Y_0^{[m]}}, \frac{2m}{Y_1^{[m]}}, \dots \right)$$

converges in distribution, as $m \rightarrow \infty$, to the Poisson point process on $\mathbb{R}_+$ with the unit intensity.

The proof is postponed to Sect. 5.3.

Remark 1.3 The Poisson point process [8, Section 4]

$$(0 < \xi_0 < \xi_1 < \dots) \quad (11)$$

on $\mathbb{R}_+$ can be viewed concretely as the sequence of partial sums

$$\begin{aligned} \xi_0 &= \psi_0, \\ \xi_1 &= \psi_0 + \psi_1, \\ \xi_2 &= \psi_0 + \psi_1 + \psi_2, \\ &\vdots \end{aligned}$$

for a sequence (ψ_i) of independent, identically distributed random variables with the exponential distribution $\text{Exp}(1)$. Thus a concrete way to express the convergence in Theorem 1.2 is to say that for each $l \in \mathbb{N}_0$ the joint distribution of the *finite* tuple of random variables

$$\left(\frac{2m}{Y_0^{[m]}}, \dots, \frac{2m}{Y_l^{[m]}} \right)$$

converges, as $m \rightarrow \infty$, to the joint distribution of the sequence of partial sums

$$(\psi_0, \quad \psi_0 + \psi_1, \quad \dots, \quad \psi_0 + \psi_1 + \dots + \psi_l).$$

Corollary 1.4 *For each $x \in \mathbb{N}_0$ the random variable $\frac{Y_x^{[m]}}{2m}$ converges in distribution, as $m \rightarrow \infty$, to the reciprocal of the Erlang distribution $\text{Erlang}(x+1, 1)$.*

In particular, for $x = 0$ it follows that the random variable $\frac{Y_0^{[m]}}{2m}$ which measures the (rescaled) number of steps of the bumping route to reach the leftmost column converges in distribution, as $m \rightarrow \infty$, to the Fréchet distribution of shape parameter $\alpha = 1$:

$$\lim_{m \rightarrow \infty} \mathbb{P} \left(\frac{Y_0^{[m]}}{2m} \leq u \right) = e^{-\frac{1}{u}} \quad \text{for any } u \in \mathbb{R}_+. \quad (12)$$

Fig. 6 Monte Carlo simulations of the cumulative probability distribution function for the random variable $e^{-\frac{2m}{Y_0}}$. The thick red line corresponds to $m = 1$ (sample size is equal to 10,000); the blue line corresponds to $m = 6$ (sample size 3000); the thin green line corresponds to $m = 25$ (sample size 2500). The dashed line corresponds to the cumulative probability distribution function of the uniform distribution $U(0, 1)$ on the unit interval. Due to constraints on computation time it was not possible to get Monte Carlo data for all values of u . The staircase feature of the plots is due to the discrete nature of the probability distribution of Y_0 (color figure online)

The Fréchet distribution has a heavy tail; in particular its first moment is infinite which provides a theoretical explanation for a bad time complexity of some of our Monte Carlo simulations.

Equivalently, the random variable $e^{-\frac{2m}{Y_0}}$ converges in distribution, as $m \rightarrow \infty$, to the uniform distribution $U(0, 1)$ on the unit interval. Figure 6 presents the results of Monte Carlo simulations which illustrate this result.

1.10 Projective convention for drawing Young diagrams

Usually in order to draw a Young diagram we use the French convention and the Oxy coordinate system, cf. Fig. 2a. For our purposes it will be more convenient to change the parametrization of the coordinate y by setting

$$z = z(y) = \frac{2m}{y}.$$

This convention allows us to show an infinite number of rows of a given tableau on a finite piece of paper, cf. Fig. 2b. We will refer to this way of drawing Young tableaux as *the projective convention*; it is somewhat reminiscent of the *English convention*

Fig. 7 The bumping routes from Fig. 3 shown in the projective convention (see also Fig. 2b). The dashed line $x = z$ corresponds to the hyperbola (9); it is tangent in 0 to the rescaled limit bumping curve (the solid line); it is also the plot of the mean value of the Poisson process $z \mapsto \mathbb{E}N(z)$

in the sense that the numbers in the tableau increase along the columns *from top to bottom*.

In the projective convention the bumping route can be seen as the plot of the function

$$x_{T,m}^{\text{proj}}(z) = \mathcal{T}_{\leftarrow m+1/2} \left(\left\lfloor \frac{2m}{z} \right\rfloor \right) \quad \text{for } z \in \mathbb{R}_+ \quad (13)$$

shown on Fig. 2b as the thick red line.

With these notations Theorem 1.2 allows the following convenient reformulation.

Theorem 1.5 (The main result, reformulated) *Let $\mathcal{T}$ be a random infinite standard Young tableau with the Plancherel distribution. For $m \rightarrow \infty$ the stochastic process*

$$\left\{ x_{T,m}^{\text{proj}}(z), \quad z > 0 \right\} \quad (14)$$

converges in distribution to the standard Poisson counting process $\{N(z), \quad z > 0\}$ with the unit intensity.

For an illustration see Fig. 7.

Remark 1.6 In Theorem 1.5 above, the convergence in distribution for stochastic processes is understood as follows: for any *finite* collection $z_1, \dots, z_l > 0$ we claim that

the joint distribution of the tuple of random variables

$$(x_{T,m}^{\text{proj}}(z_1), \dots, x_{T,m}^{\text{proj}}(z_l))$$

converges in the weak topology of probability measures, as $m \rightarrow \infty$, to the joint distribution of the tuple of random variables

$$(N(z_1), \dots, N(z_l)).$$

Proof of Theorem 1.5 The process (14) is a counting process. By the definition (13), the time of its k -th jump (for an integer $k \geq 1$)

$$\inf \left\{ z > 0 : x_{T,m}^{\text{proj}}(z) = k \right\} = \frac{2m}{Y_{k-1}^{[m]}}$$

is directly related to the number of the row in which the bumping route reaches the column with the index $k - 1$. By Theorem 1.2 the joint distribution of the times of the jumps converges to the Poisson point process; it follows therefore that (14) converges to the Poisson counting process, as required. $\square$

The plot of the mean value of the standard Poisson process $z \mapsto \mathbb{E}N(z)$ is the straight line $x = z$ which is shown on Fig. 7 as the dashed line. Somewhat surprisingly it coincides with the hyperbola (9) shown in the projective coordinate system; a posteriori this gives some justification to the naive discussion from Sect. 1.8.

1.11 The main result with the right-to-left approach

Theorem 1.2 was formulated in a compact way which may obscure the true nature of this result. Our criticism is focused on the *left-to-right approach* from Remark 1.3 which might give a false impression that the underlying mechanism for generating the random variable $\frac{2m}{Y_{x+1}^{[m]}}$ describing the ‘time of arrival’ of the bumping route to the column number $x + 1$ is based on generating first the random variable $\frac{2m}{Y_x^{[m]}}$ related to the previous column (that is the column directly to the left), and adding some ‘waiting time’ for the transition. In fact, such a mechanism is not possible without the time travel because the chronological order of the events is opposite: the bumping route first visits the column $x + 1$ and *then* lands in the column x . In the following we shall present an alternative, *right-to-left* viewpoint which explains better the true nature of Theorem 1.2.

For the Poisson point process (11) and an integer $l \geq 1$ we consider the collection of random variables

$$\xi_l, R_0, R_1, \dots, R_{l-1} \quad (15)$$

which consists of ξ_l and the ratios

$$R_i := \frac{\xi_{i+1}}{\xi_i}$$

of consecutive entries of (ξ_i) . Then (15) are independent random variables with the distributions that can be found easily. This observation can be used to define $\xi_0, \dots, \xi_l$ from the Poisson point process by setting

$$\xi_i = \xi_l \frac{1}{R_{l-1}} \frac{1}{R_{l-2}} \cdots \frac{1}{R_i} \quad \text{for } 0 \leq i \leq l.$$

With this in mind we may reformulate Theorem 1.2 as follows.

Theorem 1.7 (The main result, reformulated) *For any integer $l \geq 0$ the joint distribution of the tuple of random variables*

$$\left(\frac{Y_l^{[m]}}{2m}, \frac{Y_{l-1}^{[m]}}{2m}, \frac{Y_{l-2}^{[m]}}{2m}, \dots, \frac{Y_0^{[m]}}{2m} \right) \quad (16)$$

converges, as $m \rightarrow \infty$, to the joint distribution of the random variables

$$\left(\frac{1}{\xi_l}, \frac{1}{\xi_l} R_{l-1}, \frac{1}{\xi_l} R_{l-1} R_{l-2}, \dots, \frac{1}{\xi_l} R_{l-1} \cdots R_0 \right), \quad (17)$$

where $\xi_l, R_{l-1}, \dots, R_0$ are independent random variables, the distribution of ξ_l is equal to $\text{Erlang}(l+1, 1)$, and for each $i \geq 0$ the distribution of the ratio R_i is supported on $[1, \infty)$ with the power law

$$\mathbb{P}(R_i > u) = \frac{1}{u^{i+1}} \quad \text{for } u \geq 1. \quad (18)$$

The order of the random variables in (16) reflects the chronological order of the events, from left to right. Heuristically, (17) states that the transition of the bumping route from the column $x+1$ to the column x gives a *multiplicative* factor R_x to the total waiting time, with the factors $R_0, R_1, \dots$ independent.

It is more common in mathematical and physical models that the total waiting time for some event arises as a *sum* of some independent summands, so the *multiplicative* structure in Theorem 1.7 comes as a small surprise. We believe that this phenomenon can be explained heuristically as follows: when we study the transition of the bumping route from row y to the next row $y+1$, the probability of the transition from column $x+1$ to column x seems asymptotically to be equal to

$$\frac{x+1}{y} + o\left(\frac{1}{y}\right) \quad \text{for fixed value of } x, \text{ and for } y \rightarrow \infty.$$

This kind of decay would explain both the multiplicative structure ('if a bumping route arrives to a given column very late, it will stay in this column even longer') as well as

the power law (18). We are tempted therefore to state the following conjecture which might explain the aforementioned transition probabilities of the bumping routes.

Conjecture 1.8 *For a Plancherel-distributed random infinite standard Young tableau $\mathcal{T}$*

$$\begin{aligned}\mathbb{P}\{\mathcal{T}_{x-1,y+1} < \mathcal{T}_{x,y}\} &= \frac{x}{y} + o\left(\frac{1}{y}\right) \text{ for fixed } x \geq 1 \text{ and } y \rightarrow \infty, \\ \mathbb{P}\{\mathcal{T}_{x-2,y+1} < \mathcal{T}_{x,y}\} &= o\left(\frac{1}{y}\right) \text{ for fixed } x \geq 2 \text{ and } y \rightarrow \infty.\end{aligned}$$

Furthermore, for each $x \in \{1, 2, \dots\}$ the set of points

$$\left\{ \log \frac{y}{c} : y \in \{c, c+1, \dots\} \text{ and } \mathcal{T}_{x-1,y+1} < \mathcal{T}_{x,y} \right\} \quad (19)$$

converges, as $c \rightarrow \infty$, to Poisson point process on $\mathbb{R}_+$ with the constant intensity equal to x .

Numerical experiments are not conclusive and indicate interesting clustering phenomena for the random set (19).

1.12 Asymptotics of fixed m

The previous results concerned the bumping routes $\mathcal{T} \leftarrow m + \frac{1}{2}$ in the limit $m \rightarrow \infty$ as the inserted number tends to infinity. In the following we concentrate on another class of asymptotic problems which concern the fixed value of m .

The following result shows that (12) gives asymptotically a very good approximation for the distribution tail of $Y_0^{[m]}$ in the scaling when m is fixed and the number of the row $y \rightarrow \infty$ tends to infinity.

Proposition 1.9 *For each integer $m \geq 1$*

$$\lim_{y \rightarrow \infty} y \mathbb{P}\{Y_0^{[m]} \geq y\} = 2m.$$

This result is illustrated on Fig. 6 in the behavior of each of the cumulative distribution functions in the neighborhood of $u = 1$. The proof is postponed to Sect. 5.1.

Question 1.10 *What can we say about the other columns, that is the tail asymptotics of $\mathbb{P}\{Y_x^{[m]} \geq y\}$ for fixed values of $x \in \mathbb{N}_0$ and $m \geq 1$, in the limit $y \rightarrow \infty$?*

1.13 More open problems

Let $\mathcal{T}$ be a random Plancherel-distributed infinite standard Young tableau. We consider the *bumping tree* [3] which is defined as the collection of all possible bumping routes

Fig. 8 All possible bumping routes (“the bumping tree”) for a given Plancherel-distributed random infinite standard Young tableau. The y axis is shown using the logarithmic scale. In order to improve visibility, each bumping route was drawn as a piecewise-affine function connecting the points (4) and not as a jump function as in Fig. 2a

for this tableau

$$(\mathcal{T} \leftarrow m + 1/2 : m \in \mathbb{N}_0),$$

which can be visualized, for example, as on Fig. 8. Computer simulations suggest that the set of boxes which can be reached by *some* bumping route for a given tableau $\mathcal{T}$ is relatively ‘small’. It would be interesting to state this vague observation in a meaningful way. We conjecture that the pictures such as Fig. 8 which use the logarithmic scale for the y coordinate converge (in the scaling when $x = O(1)$ is bounded and $y \rightarrow \infty$) to some meaningful particle jump-and-coalescence process.

1.14 Overview of the paper. Sketch of the proof of Theorem 1.2

As we already mentioned, the detailed proof of Theorem 1.2 is postponed to Sect. 5.3. In the following we present an overview of the paper and a rough sketch of the proof.

1.14.1 Trajectory of infinity. Lazy parametrization of the bumping route

Without loss of generality we may assume that the Plancherel-distributed infinite tableau $\mathcal{T}$ from the statement of Theorem 1.2 is of the form $\mathcal{T} = Q(\xi_1, \xi_2, \dots)$ for a

sequence $\xi_1, \xi_2, \dots$ of independent, identically distributed random variables with the uniform distribution $U(0, 1)$.

We will iteratively apply Schensted row insertion to the entries of the infinite sequence

$$\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \xi_{m+2}, \dots \quad (20)$$

which is the initial sequence ξ with our favorite symbol ∞ inserted at the position $m+1$. At step $m+1$ the symbol ∞ is inserted at the end of the bottom row; as further elements of the sequence (20) are inserted, the symbol ∞ stays put or is being bumped to the next row, higher and higher.

In Proposition 3.1 we will show that the trajectory of ∞ in this infinite sequence of Schensted row insertions

$$\left((P(\xi_1, \dots, \xi_m, \infty) \leftarrow \xi_{m+1}) \leftarrow \xi_{m+2} \right) \leftarrow \dots \quad (21)$$

coincides with the bumping route $\mathcal{T} \leftarrow \dots \leftarrow m+1/2$. Thus our main problem is equivalent to studying the time evolution of the position of ∞ in the infinite sequence of row insertions (21). This time evolution also provides a convenient alternative parametrization of the bumping route, called *lazy parametrization*.

1.14.2 Augmented Young diagrams

For $t \geq m$ we consider the insertion tableau

$$\mathcal{T}^{(t)} = P(\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \dots, \xi_t) \quad (22)$$

which appears at an intermediate step in (21) after some finite number of row insertions was performed. By removing the information about the entries of the tableau $\mathcal{T}^{(t)}$ we obtain the *shape* of $\mathcal{T}^{(t)}$, denoted by $\text{sh } \mathcal{T}^{(t)}$, which is a Young diagram with $t+1$ boxes. In the following we will explain how to modify the notion of the *shape of a tableau* so that it better fits our needs.

Let us remove from the tableau $\mathcal{T}^{(t)}$ the numbers $\xi_1, \dots, \xi_t$ and let us keep only the information about the position of the box which contains the symbol ∞ . The resulting object, called *augmented Young diagram* (see Fig. 9 for an illustration), can be regarded as a pair $\Lambda^{(t)} = (\lambda, \square)$ which consists of:

- the Young diagram λ with t boxes which keeps track of the positions of the boxes with the entries ξ_i , $i \in \{1, \dots, t\}$, in $\mathcal{T}^{(t)}$;
- the outer corner $\square$ of λ which is the position of the box with ∞ in $\mathcal{T}^{(t)}$.

We will say that $\text{sh}^* \mathcal{T}^{(t)} = \Lambda^{(t)}$ is the *augmented shape* of $\mathcal{T}^{(t)}$.

The set of augmented Young diagrams, denoted $\mathbb{Y}^*$, has a structure of an oriented graph which is directly related to Schensted row insertion, as follows. For a pair of augmented Young diagrams $\Lambda, \tilde{\Lambda} \in \mathbb{Y}^*$ we say that $\Lambda \nearrow \tilde{\Lambda}$ if there exists a tableau $\mathcal{T}$ (which contains exactly one entry equal to ∞) such that $\Lambda = \text{sh}^* \mathcal{T}$ and there exists some number x such that $\tilde{\Lambda} = \text{sh}^*(\mathcal{T} \leftarrow x)$, see Fig. 10 and Sect. 3.4 for more details.

With these notations the time evolution of the position of ∞ in the sequence of row insertions (21) can be extracted from the sequence of the corresponding augmented shapes

$$\Lambda^{(m)} \nearrow \Lambda^{(m+1)} \nearrow \dots \quad (23)$$

1.14.3 Augmented Plancherel growth processes

The random sequence (23) is called *the augmented Plancherel growth process initiated at time m* ; in Sect. 3.6 we will show that it is a Markov chain with dynamics closely related to the usual (i.e., non-augmented) Plancherel growth process. Since we have a freedom of choosing the value of the integer $m \in \mathbb{N}_0$, we get a whole family of augmented Plancherel growth processes. It turns out that the transition probabilities for these Markov chains do not depend on the value of m .

Our strategy is to use the Markov property of augmented Plancherel growth processes combined with the following two pieces of information.

- *Probability distribution at a given time t .* In Proposition 3.9 we give an asymptotic description of the probability distribution of $\Lambda^{(t)}$ in the scaling when $m, t \rightarrow \infty$ in such a way that $t = \Theta(m^2)$.
- *Averaged transition probabilities.* In Proposition 4.2 we give an asymptotic description of the transition probabilities for the augmented Plancherel growth processes between two moments of time n and n' (with $n < n'$) in the scaling when $n, n' \rightarrow \infty$.

Thanks to these results we will prove Theorem 4.3 which gives an asymptotic description of the probability distribution of the trajectory of the symbol ∞ or, equivalently, the bumping route in the lazy parametrization.

Finally, in Sect. 5 we explain how to translate this result to the non-lazy parametrization of the bumping route in which the boxes of the bumping route are parametrized by the index of the row; this completes the proof of Theorem 1.2.

The main difficulty lies in the proofs of the aforementioned Propositions 3.9 and 4.2; in the following we sketch their proofs.

1.14.4 Probability distribution of the augmented Plancherel growth process at a given time

In order to prove the aforementioned Proposition 3.9 we need to understand the probability distribution of the augmented shape of the insertion tableau $\mathcal{T}^{(t)}$ given by (22) in the scaling when $m = O(\sqrt{t})$. Thanks to some symmetries of the RSK algorithm, the tableau $\mathcal{T}^{(t)}$ is equal to the transpose of the insertion tableau

$$P(\underbrace{\xi_t, \xi_{t-1}, \dots, \xi_{m+1}}_{t-m \text{ entries}}, \infty, \xi_m, \dots, \xi_1) \quad (24)$$

which corresponds to the original sequence read backwards. Since the probability distribution of the sequence ξ is invariant under permutations, the augmented shape

of the tableau (24) can be viewed as the value at time t of the augmented Plancherel growth process initiated at time $m' := t - m$.

The remaining difficulty is therefore to understand the probability distribution of the augmented Plancherel growth process initiated at time m' , after additional m steps of Schensted row insertion were performed. We are interested in the asymptotic setting when $m' \rightarrow \infty$ and the number of additional steps $m = O(\sqrt{m'})$ is relatively small. This is precisely the setting which was considered in our recent paper about the Poisson limit theorem for the Plancherel growth process [10]. We summarize these results in Sect. 2; based on them we prove in Proposition 3.6 that the index of the row of the symbol ∞ in the tableau (24) is asymptotically given by the Poisson distribution.

By taking the transpose of the augmented Young diagrams we recover Proposition 3.9, as desired.

1.14.5 Averaged transition probabilities

We will sketch the proof of the aforementioned Proposition 4.2 which concerns an augmented Plancherel growth process

$$\Lambda^{(n)} \nearrow \Lambda^{(n+1)} \nearrow \dots \quad (25)$$

for which the initial probability distribution at time n is given by $\Lambda^{(n)} = (\lambda^{(n)}, \square^{(n)})$, where $\lambda^{(n)}$ is a random Young diagram with n boxes distributed (approximately) according to the Plancherel measure and $\square^{(n)}$ is its outer corner located in the column with the fixed index k . Our goal is to describe the probability distribution of this augmented Plancherel growth process at some later time n' , asymptotically as $n, n' \rightarrow \infty$.

Our first step in this direction is to approximate the probability distribution of the Markov process (25) by a certain linear combination (with real, positive and negative, coefficients) of the probability distributions of augmented Plancherel growth processes initiated at time m . This linear combination is taken over the values of m which are of order $O(\sqrt{n})$. Finding such a linear combination required the results which we discussed above in Sect. 1.14.4, namely a good understanding of the probability distribution at time n of the augmented Plancherel growth process initiated at some specified time $m = O(\sqrt{n})$.

The probability distribution of $\Lambda^{(n')}$ is then approximately equal to the aforementioned linear combination of the laws (this time evaluated at time n') of the augmented Plancherel growth processes initiated at some specific times m . This linear combination is straightforward to analyze because for each individual summand the results from Sect. 1.14.4 are applicable. This completes the sketch of the proof of Proposition 4.2.

2 Growth of the bottom rows

In the current section we will gather some results and some notations from our recent paper [10, Section 2] which will be necessary for the purposes of the current work.

2.1 Total variation distance

Suppose that μ and ν are (signed) measures on the same discrete set S . Such measures can be identified with real-valued functions on S . We define the *total variation distance* between the measures μ and ν

$$\delta(\mu, \nu) := \frac{1}{2} \|\mu - \nu\|_{\ell^1} \quad (26)$$

as half of their ℓ^1 distance as functions. If X and Y are two random variables with values in the same discrete set S , we define their total variation distance $\delta(X, Y)$ as the total variation distance between their probability distributions (which are probability measures on S).

Usually in the literature the total variation distance is defined only for probability measures. In such a setup the total variation distance can be expressed as

$$\delta(\mu, \nu) = \max_{X \subset S} |\mu(X) - \nu(X)| = \|(\mu - \nu)^+\|_{\ell^1}. \quad (27)$$

In the current paper we will occasionally use the notion of the total variation distance also for signed measures for which (26) and (27) are *not* equivalent.

2.2 Growth of rows in Plancherel growth process

Let $\lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \dots$ be the Plancherel growth process. For integers $n \geq 1$ and $r \in \mathbb{N}_0$ we denote by $E_r^{(n)}$ the random event which occurs if the unique box of the skew diagram $\lambda^{(n)}/\lambda^{(n-1)}$ is located in the row with the index r .

The following result was proved by Okounkov [11, Proposition 2], see also [10, Proposition 2.7] for an alternative proof.

Proposition 2.1 *For each $r \in \mathbb{N}_0$*

$$\lim_{n \rightarrow \infty} \sqrt{n} \mathbb{P}\left(E_r^{(n)}\right) = 1.$$

Let us fix an integer $k \in \mathbb{N}_0$. We define $\mathcal{N} = \{0, 1, \dots, k, \infty\}$. For $n \geq 1$ we define the random variable $R^{(n)} \in \mathcal{N}$ which is given by

$$R^{(n)} = \begin{cases} r & \text{if the event } E_r^{(n)} \text{ occurs for } 0 \leq r \leq k, \\ \infty & \text{if the event } E_r^{(n)} \text{ occurs for some } r > k. \end{cases}$$

Let $\ell = \ell(n)$ be a sequence of non-negative integers such that

$$\ell = O(\sqrt{n}).$$

For a given integer $n \geq (k+1)^2$ we focus on the specific part of the Plancherel growth process

$$\lambda^{(n)} \nearrow \dots \nearrow \lambda^{(n+\ell)}. \quad (28)$$

We will encode some partial information about the growths of the rows as well as about the final Young diagram in (28) by the random vector

$$V^{(n)} = \left(R^{(n+1)}, \dots, R^{(n+\ell)}, \lambda^{(n+\ell)} \right) \in \mathcal{N}^\ell \times \mathbb{Y}. \quad (29)$$

We also consider the random vector

$$\overline{V}^{(n)} = \left(\overline{R}^{(n+1)}, \dots, \overline{R}^{(n+\ell)}, \overline{\lambda}^{(n+\ell)} \right) \in \mathcal{N}^\ell \times \mathbb{Y} \quad (30)$$

which is defined as a sequence of independent random variables; the random variables $\overline{R}^{(n+1)}, \dots, \overline{R}^{(n+\ell)}$ have the same distribution given by

$$\begin{aligned} \mathbb{P} \left\{ \overline{R}^{(n+i)} = r \right\} &= \frac{1}{\sqrt{n}} \quad \text{for } r \in \{0, \dots, k\}, \quad 1 \leq i \leq \ell, \\ \mathbb{P} \left\{ \overline{R}^{(n+i)} = \infty \right\} &= 1 - \frac{k+1}{\sqrt{n}} \end{aligned}$$

and $\overline{\lambda}^{(n+\ell)}$ is distributed according to the Plancherel measure $\text{Plan}_{n+\ell}$; in particular the random variables $\lambda^{(n+\ell)}$ and $\overline{\lambda}^{(n+\ell)}$ have the same distribution.

Heuristically, the following result states that when the Plancherel growth process is in an advanced stage and we observe a relatively small number of its additional steps, the growths of the bottom rows occur approximately like independent random variables. Additionally, these growths do not affect too much the final shape of the Young diagram.

Theorem 2.2 [10, Theorem 2.2] *With the above notations, for each fixed $k \in \mathbb{N}_0$ the total variation distance between $V^{(n)}$ and $\overline{V}^{(n)}$ converges to zero, as $n \rightarrow \infty$; more specifically*

$$\delta \left(V^{(n)}, \overline{V}^{(n)} \right) = o \left(\frac{\ell}{\sqrt{n}} \right).$$

3 Augmented Plancherel growth process

In this section we will introduce our main tool: *the augmented Plancherel growth process* which keeps track of the position of a very large number in the insertion tableau when new random numbers are inserted.

3.1 Lazy parametrization of bumping routes

Our first step towards the proof of Theorem 1.2 is to introduce a more convenient parametrization of the bumping routes. In (4) we used y , the number of the row, as the variable which parametrizes the bumping route. In the current section we will introduce the *lazy parametrization*.

Let us fix a (finite or infinite) standard Young tableau $\mathcal{T}$ and an integer $m \in \mathbb{N}_0$. For a given integer $t \geq m$ we denote by

$$\square_{\mathcal{T},m}^{\text{lazy}}(t) = (x_{\mathcal{T},m}^{\text{lazy}}(t), y_{\mathcal{T},m}^{\text{lazy}}(t))$$

the coordinates of the first box in the bumping route $\mathcal{T} \leftarrow m + 1/2$ which contains an entry of $\mathcal{T}$ which is bigger than t . If such a box does not exist, this means that the bumping route is finite, and all boxes of the tableau $\mathcal{T}$ which belong to the bumping route are $\leq t$. If this is the case we define $\square_{\mathcal{T},m}^{\text{lazy}}(t)$ to be the last box of the bumping route, i.e. the box of the bumping route which lies outside of $\mathcal{T}$. We will refer to

$$t \mapsto (x_{\mathcal{T},m}^{\text{lazy}}(t), y_{\mathcal{T},m}^{\text{lazy}}(t)) \quad (31)$$

as the *lazy parametrization of the bumping route*.

For example, for the infinite tableau $\mathcal{T}$ from Fig. 2a and $m = 3$ the usual parametrization of the bumping route is given by

$$\mathcal{T} \leftarrow m + 1/2(y) = \begin{cases} 3 & \text{for } y = 0, \\ 2 & \text{for } y = 1, \\ 1 & \text{for } y = 2, \\ 1 & \text{for } y = 3, \\ 0 & \text{for } y \geq 4, \end{cases}$$

while its lazy counterpart is given by

$$\square_{\mathcal{T},m}^{\text{lazy}}(t) = (x_{\mathcal{T},m}^{\text{lazy}}(t), y_{\mathcal{T},m}^{\text{lazy}}(t)) = \begin{cases} (3, 0) & \text{for } t \in \{3, 4, 5\}, \\ (2, 1) & \text{for } t \in \{6, 7, 8\}, \\ (1, 2) & \text{for } t = 9, \\ (1, 3) & \text{for } t \in \{10, 11\}, \\ (0, 4) & \text{for } t = 12, \\ (0, 5) & \text{for } t = 13, \\ (0, 6) & \text{for } t \in \{14, 15, 16\}, \\ \vdots & \end{cases}$$

Clearly, the set of values of the function (31) coincides with the bumping route understood in the traditional way (4).

We denote by $\mathcal{T}|_{\leq t}$ the outcome of keeping only these boxes of $\mathcal{T}$ which are at most t . Note that the element of the bumping route

$$\square_{\mathcal{T},m}^{\text{lazy}}(t) = \text{sh}(\mathcal{T}|_{\leq t} \leftarrow m + 1/2) / \text{sh}(\mathcal{T}|_{\leq t}) \quad (32)$$

is the unique box of the difference of two Young diagrams on the right-hand side.

3.2 Trajectory of ∞

Let $\xi = (\xi_1, \xi_2, \dots)$ be a sequence of independent, identically distributed random variables with the uniform distribution $U(0, 1)$ on the unit interval $[0, 1]$ and let $m \geq 0$ be a fixed integer. We will iteratively apply Schensted row insertion to the entries of the infinite sequence

$$\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \xi_{m+2}, \dots$$

which is the initial sequence ξ with our favorite symbol ∞ inserted at position $m + 1$. (The Readers who are afraid of infinity may replace it by any number which is strictly bigger than all of the entries of the sequence ξ .) Our goal is to investigate the position of the box containing ∞ as a function of the number of iterations. More specifically, for an integer $t \geq m$ we define

$$\square_m^{\text{traj}}(t) = \text{Pos}_{\infty}(P(\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \dots, \xi_t)) \quad (33)$$

to be the position of the box containing ∞ in the appropriate insertion tableau. This problem was formulated by Duzhin [3]; the first asymptotic results in the scaling in which $m \rightarrow \infty$ and $t = O(m)$ were found by the first named author [9]. In the current paper we go beyond this scaling and consider $m \rightarrow \infty$ and $t = O(m^2)$; the answer for this problem is essentially contained in Theorem 4.3.

The following result shows a direct link between the above problem and the asymptotics of bumping routes. This result also shows an interesting link between the papers [9, 14].

Proposition 3.1 *Let $\xi_1, \xi_2, \dots$ be a (non-random or random) sequence and $\mathcal{T} = Q(\xi_1, \xi_2, \dots)$ be the corresponding recording tableau. Then for each $m \in \mathbb{N}$ the bumping route in the lazy parametrization coincides with the trajectory of ∞ as defined in (33):*

$$\square_{\mathcal{T},m}^{\text{lazy}}(t) = \square_m^{\text{traj}}(t) \quad \text{for each integer } t \geq m. \quad (34)$$

We will provide two proofs of Proposition 3.1. The first one is based on the following classic result of Schützengerger.

Fact 3.2 [15] *For any permutation σ the insertion tableau $P(\sigma)$ and the recording tableau $Q(\sigma^{-1})$, which corresponds to the inverse of σ , are equal.*

The first proof of Proposition 3.1 Let $\pi = (\pi_1, \dots, \pi_t) \in \mathfrak{S}_t$ be the permutation generated by the sequence $(\xi_1, \dots, \xi_t)$, that is the unique permutation such that for any choice of indices $i < j$ the condition $\pi_i < \pi_j$ holds true if and only if $\xi_i \leq \xi_j$. Let $\pi^{-1} = (\pi_1^{-1}, \dots, \pi_t^{-1})$ be the inverse of π . Since RSK depends only on the relative order of entries, the restricted tableau $\mathcal{T}|_{\leq t}$ is equal to

$$\mathcal{T}|_{\leq t} = Q(\xi_1, \dots, \xi_t) = Q(\pi) = P(\pi^{-1}). \quad (35)$$

By (32), (35) and Fact 3.2 it follows that

$$\begin{aligned} \square_{\mathcal{T}, m}^{\text{lazy}}(t) &= \text{sh} \left(P(\pi^{-1}) \leftarrow m + 1/2 \right) / \text{sh} P(\pi^{-1}) \\ &= \text{sh} P \left(\pi_1^{-1}, \dots, \pi_t^{-1}, m + 1/2 \right) / \text{sh} P \left(\pi_1^{-1}, \dots, \pi_t^{-1} \right) \\ &= \text{Pos}_{t+1} \left(Q \left(\pi_1^{-1}, \dots, \pi_t^{-1}, m + 1/2 \right) \right) \\ &\stackrel{\text{Fact 3.2}}{=} \text{Pos}_{t+1} (P(\pi_1, \dots, \pi_m, t + 1, \pi_{m+1}, \dots, \pi_t)) \\ &= \text{Pos}_{\infty} (P(\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \dots, \xi_t)) \\ &= \square_m^{\text{traj}}(t) \end{aligned}$$

since the permutation $(\pi_1, \dots, \pi_m, t + 1, \pi_{m+1}, \dots, \pi_t)$ is the inverse of the permutation generated by the sequence $(\pi_1^{-1}, \dots, \pi_t^{-1}, m + 1/2)$. $\square$

The above proof has an advantage of being short and abstract. The following alternative proof highlights the ‘dynamic’ aspects of the bumping routes and the trajectory of infinity.

The second proof of Proposition 3.1 We use induction over the variable t .

The induction base $t = m$ is quite easy: $\square_{\mathcal{T}, m}^{\text{lazy}}(m)$ is the leftmost box in the bottom row of $\mathcal{T}$ which contains a number which is bigger than m . This box is the first to the right of the last box in the bottom row in the tableau $Q(\xi_1, \dots, \xi_m)$. On the other hand, since this recording tableau has the same shape as the insertion tableau $P(\xi_1, \dots, \xi_m)$, it follows that $\square_m^{\text{traj}}(m) = \square_{\mathcal{T}, m}^{\text{lazy}}(m)$ and the proof of the induction base is completed.

We start with an observation that ∞ is bumped in the process of calculating the row insertion

$$P(\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \dots, \xi_t) \leftarrow \xi_{t+1} \quad (36)$$

if and only if the position of ∞ at time t , that is $\square_m^{\text{traj}}(t)$, is the unique box which belongs to the skew diagram

$$\text{RSK}(\xi_1, \dots, \xi_{t+1}) / \text{RSK}(\xi_1, \dots, \xi_t).$$

The latter condition holds true if and only if the entry of $\mathcal{T}$ located in the box $\square_m^{\text{traj}}(t)$ fulfills

$$\mathcal{T}_{\square_m^{\text{traj}}(t)} = t + 1.$$

In order to make the induction step we assume that the equality (34) holds true for some $t \geq m$. There are the following two cases.

Case 1. Assume that the entry of $\mathcal{T}$ located in the box $\square_{\mathcal{T},m}^{\text{lazy}}(t)$ is strictly bigger than $t + 1$. In this case the lazy bumping route stays put and

$$\square_{\mathcal{T},m}^{\text{lazy}}(t + 1) = \square_{\mathcal{T},m}^{\text{lazy}}(t).$$

By the induction hypothesis, the entry of $\mathcal{T}$ located in the box $\square_m^{\text{traj}}(t) = \square_{\mathcal{T},m}^{\text{lazy}}(t)$ is bigger than $t + 1$. By the previous discussion, ∞ is not bumped in the process of calculating the row insertion (36) hence

$$\square_m^{\text{traj}}(t + 1) = \square_m^{\text{traj}}(t)$$

and the inductive step holds true.

Case 2. Assume that the entry of $\mathcal{T}$ located in the box $\square_{\mathcal{T},m}^{\text{lazy}}(t)$ is equal to $t + 1$. In this case the lazy bumping route moves to the next row. It follows that $\square_{\mathcal{T},m}^{\text{lazy}}(t + 1)$ is the leftmost box of $\mathcal{T}$ in the row above $\square_{\mathcal{T},m}^{\text{lazy}}(t)$ which contains a number which is bigger than $\mathcal{T}_{\square_{\mathcal{T},m}^{\text{lazy}}(t)} = t + 1$.

By the induction hypothesis, $\mathcal{T}_{\square_m^{\text{traj}}(t)} = \mathcal{T}_{\square_{\mathcal{T},m}^{\text{lazy}}(t)} = t + 1$, so ∞ is bumped in the process of calculating the row insertion (36) to the next row r . The box $\square_m^{\text{traj}}(t + 1)$ is the first to the right of the last box in the row r in $\text{RSK}(\xi_1, \dots, \xi_t, \xi_{t+1})$. Clearly, this is the box in the row r of $\mathcal{T}$ which has the least entry among those which are bigger than $t + 1$, so it is the same as $\square_{\mathcal{T},m}^{\text{lazy}}(t + 1)$. $\square$

3.3 Augmented Young diagrams. Augmented shape of a tableau

For the motivations and heuristics behind the notion of augmented Young diagrams see Sect. 1.14.2.

A pair $\Lambda = (\lambda, \square)$ will be called an *augmented Young diagram* if λ is a Young diagram and $\square$ is one of its outer corners, see Fig. 9b. We will say that λ is *the regular part of Λ* and that $\square$ is *the special box of Λ* .

The set of augmented Young diagrams will be denoted by $\mathbb{Y}^*$ and for $n \in \mathbb{N}_0$ we will denote by $\mathbb{Y}_n^*$ the set of augmented Young diagrams $(\lambda, \square)$ with the additional property that λ has n boxes (which we will shortly denote by $|\lambda| = n$).

Suppose $\mathcal{T}$ is a tableau with the property that exactly one of its entries is equal to ∞ . We define the *augmented shape of $\mathcal{T}$*

$$\text{sh}^* \mathcal{T} = (\text{sh}(\mathcal{T} \setminus \{\infty\}), \text{Pos}_\infty \mathcal{T}) \in \mathbb{Y}^*,$$

Fig. 9 **a** Example of a tableau $\mathcal{T}$ which has exactly one entry equal to ∞ . **b** The augmented shape $\text{sh}^* \mathcal{T} = (\lambda, \square)$ of the tableau $\mathcal{T}$. The regular part $\lambda = (4, 2, 1)$ is shown as the Young diagram drawn with solid lines, the position of the special box $\square = (x_\square, y_\square) = (2, 1)$ is marked as the decorated box drawn with the dotted lines. With the notations of Sect. 4.2 this augmented Young diagram corresponds to $(x_\square, \lambda) = (2, \lambda) \in \mathbb{N}_0 \times \mathbb{Y}$

as the pair which consists of (a) the shape of $\mathcal{T}$ after removal of the box with ∞ , and (b) the location of the box with ∞ in $\mathcal{T}$, see Fig. 9.

3.4 Augmented Young graph

The set $\mathbb{Y}^*$ can be equipped with a structure of an oriented graph, called *augmented Young graph*. We declare that a pair $\Lambda \nearrow \tilde{\Lambda}$ forms an oriented edge (with $\Lambda = (\lambda, \square) \in \mathbb{Y}^*$ and $\tilde{\Lambda} = (\tilde{\lambda}, \tilde{\square}) \in \mathbb{Y}^*$) if the following two conditions hold true:

$$\lambda \nearrow \tilde{\lambda} \text{ and } \tilde{\square} = \begin{cases} \text{the outer corner of } \tilde{\lambda} \\ \text{which is in the row above } \square & \text{if } \tilde{\lambda}/\lambda = \{\square\}, \\ \square & \text{otherwise,} \end{cases} \quad (37)$$

see Fig. 9 for an illustration. If $\Lambda \nearrow \tilde{\Lambda}$ (with $\Lambda = (\lambda, \square) \in \mathbb{Y}^*$ and $\tilde{\Lambda} = (\tilde{\lambda}, \tilde{\square}) \in \mathbb{Y}^*$) are such that $\square \neq \tilde{\square}$ (which corresponds to the first case on the right-hand side of (37)), we will say that *the edge $\Lambda \nearrow \tilde{\Lambda}$ is a bump*.

The above definition was specifically tailored so that the following simple lemma holds true.

Lemma 3.3 *Assume that $\mathcal{T}$ is a tableau which has exactly one entry equal to ∞ and let x be some finite number. Then*

$$(\text{sh}^* \mathcal{T}) \nearrow (\text{sh}^*(\mathcal{T} \leftarrow x)).$$

Proof Let $\mathcal{T}' := \mathcal{T}/\{\infty\}$ be the tableau $\mathcal{T}$ with the box containing ∞ removed. Denote $(\lambda, \square) = \text{sh}^* \mathcal{T}$ and $(\tilde{\lambda}, \tilde{\square}) = \text{sh}^*(\mathcal{T} \leftarrow x)$; their regular parts

$$\begin{aligned} \lambda &= \text{sh } \mathcal{T}', \\ \tilde{\lambda} &= \text{sh } (\mathcal{T}' \leftarrow x) \end{aligned}$$

clearly fulfill $\lambda \nearrow \tilde{\lambda}$.

The position $\tilde{\square}$ of ∞ in $\mathcal{T} \leftarrow x$ is either:

- in the row immediately above the position $\square$ of ∞ in $\mathcal{T}$ (this happens exactly if ∞ was bumped in the insertion $\mathcal{T} \leftarrow x$; equivalently if $\tilde{\lambda}/\lambda = \{\square\}$), or
- the same as the position $\square$ of ∞ in $\mathcal{T}$ (this happens exactly when ∞ was not bumped; equivalently if $\tilde{\lambda}/\lambda \neq \{\square\}$).

Clearly these two cases correspond to the second condition in (37) which completes the proof. $\square$

3.5 Lifting of paths

We consider the ‘covering map’ $p : \mathbb{Y}^* \rightarrow \mathbb{Y}$ given by taking the regular part

$$\mathbb{Y}^* \ni (\lambda, \square) \xrightarrow{p} \lambda \in \mathbb{Y}.$$

Lemma 3.4 *For any $\Lambda^{(m)} \in \mathbb{Y}^*$ and any path in the Young graph*

$$\lambda^{(m)} \nearrow \lambda^{(m+1)} \nearrow \dots \in \mathbb{Y} \quad (38)$$

with a specified initial element $\lambda^{(m)} = p(\Lambda^{(m)})$ there exists the unique lifted path

$$\Lambda^{(m)} \nearrow \Lambda^{(m+1)} \nearrow \dots \in \mathbb{Y}^*$$

in the augmented Young graph with the specified initial element $\Lambda^{(m)}$, and such that $\lambda^{(t)} = p(\Lambda^{(t)})$ holds true for each $t \in \{m, m+1, \dots\}$.

Proof From (37) it follows that for each $(\lambda, \square) \in \mathbb{Y}^*$ and each $\tilde{\lambda}$ such that $\lambda \nearrow \tilde{\lambda}$ there exists a unique $\tilde{\square}$ such that $(\lambda, \square) \nearrow (\tilde{\lambda}, \tilde{\square})$. This shows that, given $\Lambda^{(i)}$, the value of $\Lambda^{(i+1)}$ is determined uniquely. This observation implies that the lemma can be proved by a straightforward induction. $\square$

3.6 Augmented Plancherel growth process

We keep the notations from the beginning of Sect. 3.2, i.e., we assume that $\xi = (\xi_1, \xi_2, \dots)$ is a sequence of independent, identically distributed random variables with the uniform distribution $U(0, 1)$ on the unit interval $[0, 1]$ and $m \geq 0$ is a fixed integer. We consider a path in the augmented Young graph

$$\Lambda_m^{(m)} \nearrow \Lambda_m^{(m+1)} \nearrow \dots \quad (39)$$

given by

$$\Lambda_m^{(t)} = \text{sh}^* P(\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \dots, \xi_t) \quad \text{for any integer } t \geq m$$

Fig. 10 A part of the augmented Young graph. For each vertex $(\lambda, \square) \in \mathbb{Y}^*$ the regular part λ is drawn with the solid line and the special box $\square$ is indicated as a decorated dotted square. For clarity this figure shows only the direct neighborhood of the augmented Young diagram $(\lambda, \square)$ with $\lambda = (4, 2, 1)$ and $\square = (2, 1)$. The double thick arrows indicate the edges which are *bumps*

(Lemma 3.3 shows that (39) is indeed a path in $\mathbb{Y}^*$). We will call (39) *the augmented Plancherel growth process initiated at time m* . The coordinates of the special box of $\Lambda_m^{(t)} = (\lambda^{(t)}, \square_m^{(t)})$ will be denoted by

$$\square_m^{(t)} = (x_m^{(t)}, y_m^{(t)}).$$

Theorem 3.5 *The augmented Plancherel growth process initiated at time m is a Markov chain with the transition probabilities given for any $t \geq m$ by*

$$\begin{aligned} \mathbb{P} \left(\Lambda_m^{(t+1)} = \tilde{\Lambda} \mid \Lambda_m^{(t)} = \Lambda \right) &= \begin{cases} \mathbb{P} (\lambda^{(t+1)} = \tilde{\lambda} \mid \lambda^{(t)} = \lambda) & \text{if } \Lambda \nearrow \tilde{\Lambda}, \\ 0 & \text{otherwise} \end{cases} \end{aligned} \quad (40)$$

for any $\Lambda, \tilde{\Lambda} \in \mathbb{Y}^*$, where λ is the regular part of Λ and $\tilde{\lambda}$ is the regular part of $\tilde{\Lambda}$. These transition probabilities do not depend on the choice of m . The conditional probability on the right-hand side is the transition probability for the Plancherel growth process $\lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \dots$.

Proof The path (39) is the unique lifting (cf. Lemma 3.4) of the sequence of the regular parts

$$\lambda^{(m)} \nearrow \lambda^{(m+1)} \nearrow \dots \quad (41)$$

with the initial condition that the special box $\square_m^{(m)}$ is the outer corner of $\lambda^{(m)}$ which is located in the bottom row. It follows that for any augmented Young diagrams $\Sigma_m, \dots, \Sigma_{t+1} \in \mathbb{Y}^*$ with the regular parts $\sigma_m, \dots, \sigma_{t+1} \in \mathbb{Y}$

$$\mathbb{P} \left(\Lambda_m^{(m)} = \Sigma_m, \dots, \Lambda_m^{(t+1)} = \Sigma_{t+1} \right) \quad (42)$$

$$= \begin{cases} \mathbb{P} \left(\lambda^{(m)} = \sigma_m, \dots, \lambda^{(t+1)} = \sigma_{t+1} \right) & \text{if } \Sigma_m \nearrow \dots \nearrow \Sigma_{t+1} \text{ and} \\ & \text{the special box of } \Sigma_m \\ & \text{is in the bottom row,} \\ 0 & \text{otherwise.} \end{cases}$$

The sequence of the regular parts (41) forms the usual Plancherel growth process (with the first m entries truncated) hence it is a Markov chain (the proof that the usual Plancherel growth process is a Markov chain can be found in [7, Sections 2.2 and 2.4]). It follows that the probability on the top of the right-hand side of (42) can be written in the product form in terms of the probability distribution of $\lambda^{(m)}$ and the transition probabilities for the Plancherel growth process.

We compare (42) with its counterpart for $t := t - 1$; this shows that the conditional probability

$$\mathbb{P} \left(\Lambda_m^{(t+1)} = \Sigma_{t+1} \mid \Lambda_m^{(m)} = \Sigma_m, \dots, \Lambda_m^{(t)} = \Sigma_t \right)$$

is equal to the right-hand side of (40) for $\tilde{\Lambda} := \Sigma_{t+1}$ and $\Lambda := \Sigma_t$. In particular, this conditional probability does not depend on the values of $\Sigma_m, \dots, \Sigma_{t-1}$ and the Markov property follows. $\square$

The special box in the augmented Plancherel growth process can be thought of as a *test particle* which provides some information about the local behavior of the usual Plancherel growth process. From this point of view it is reminiscent of the *second class particle* in the theory of interacting particle systems or *jeu de taquin trajectory* for infinite tableaux [13].

3.7 Probability distribution of the augmented Plancherel growth process

Propositions 3.6 and 3.9 below provide information about the probability distribution of the augmented Plancherel growth process at time t for $t \rightarrow \infty$ in two distinct asymptotic regimes: very soon after the augmented Plancherel process was initiated (that is when $t = m + O(\sqrt{m})$, cf. Proposition 3.6) and after a very long time after the augmented Plancherel process was initiated (that is when $t = \Theta(m^2) \gg m$, cf. Proposition 3.9).

Proposition 3.6 *Let $z > 0$ be a fixed positive number and let $t = t(m)$ be a sequence of positive integers such that $t(m) \geq m$ and with the property that*

$$\lim_{m \rightarrow \infty} \frac{t - m}{\sqrt{t}} = z.$$

Let $\Lambda_m^{(m)} \nearrow \Lambda_m^{(m+1)} \nearrow \dots$ be the augmented Plancherel growth process initiated at time m . We denote $\Lambda_m^{(t)} = (\lambda^{(t)}, \square_m^{(t)})$; let $\square_m^{(t)} = (x_m^{(t)}, y_m^{(t)})$ be the coordinates of the special box of $\Lambda_m^{(t)}$.

- (a) The probability distribution of $y_m^{(t)}$ converges, as $m \rightarrow \infty$, to the Poisson distribution $\text{Pois}(z)$ with parameter z .
- (b) For each $k \in \mathbb{N}_0$ the total variation distance between
- the conditional probability distribution of $\lambda^{(t)}$ under the condition that $y_m^{(t)} = k$, and
 - the Plancherel measure Plan_t

converges to 0, as $m \rightarrow \infty$.

- (c) The total variation distance between
- the probability distribution of the random vector

$$(\lambda^{(t)}, y_m^{(t)}) \in \mathbb{Y} \times \mathbb{N}_0 \quad (43)$$

and

- the product measure

$$\text{Plan}_t \times \text{Pois}(z)$$

converges to 0, as $m \rightarrow \infty$.

Let us fix an integer $k \geq 0$. We use the notations from Sect. 2.2 for $n := m$ and $\ell = t - m$ so that $n + \ell = t$; we assume that m is big enough so that $m \geq (k + 1)^2$. Our general strategy is to read the required information from the vector $V^{(n)}$ given by (29) and to apply Theorem 2.2. Before the proof of Proposition 3.6 we start with the following auxiliary result.

For $s \geq m$ we define the random variable $y_m^{(s)} \downarrow_{\mathcal{N}} \in \mathcal{N}$ by

$$y_m^{(s)} \downarrow_{\mathcal{N}} = \begin{cases} y_m^{(s)} & \text{if } y_m^{(s)} \in \{0, 1, \dots, k\}, \\ \infty & \text{otherwise.} \end{cases}$$

We also define the random variable $F_s \in \{0, \dots, k\}$ by

$$F_s = \begin{cases} y_m^{(s)} \downarrow_{\mathcal{N}} & \text{if } y_m^{(s)} \downarrow_{\mathcal{N}} \neq \infty, \\ \text{arbitrary element of } \{0, \dots, k\} & \text{otherwise.} \end{cases}$$

Lemma 3.7 *For each $s \geq m$ the value of F_s can be expressed as an explicit function of the entries of the sequence R related to the past, that is*

$$R^{(m+1)}, \dots, R^{(s)}.$$

For any integer $p \in \{0, \dots, k\}$ the equality $y_m^{(s)} = p$ holds true if and only if there are exactly p values of the index $u \in \{m+1, \dots, s\}$ with the property that

$$R^{(u)} = F_{u-1}. \quad (44)$$

The inequality $y_m^{(s)} > k$ holds if and only if there are at least $k+1$ values of the index $u \in \{m+1, \dots, s\}$ with this property.

Proof There are exactly $y_m^{(s)}$ edges which are bumps in the path

$$\Lambda^{(m)} \nearrow \dots \nearrow \Lambda^{(s)} \quad (45)$$

because each bump increases the y -coordinate of the special box by 1. Note that an edge $\Lambda^{(u-1)} \nearrow \Lambda^{(u)}$ in this path is a bump if and only if

$$\text{the event } E_r^{(u)} \text{ occurs for the row } r = y_m^{(u-1)}. \quad (46)$$

If $y_m^{(s)} \leq k$ then for any $u \in \{m+1, \dots, s\}$ the equality $F_u = y_m^{(u)}$ holds true; furthermore the event (46) occurs if and only if $R^{(u)} = y_m^{(u-1)}$. It follows that there are exactly $y_m^{(s)}$ values of the index $u \in \{m+1, \dots, s\}$ such that (44) holds true.

On the other hand, if $y_m^{(s)} > k$ we can apply the above reasoning to the truncation of the path (45) until after the $(k+1)$ -st bump occurs. It follows that in this case there are at least $k+1$ values of the index $u \in \{m+1, \dots, s\}$ with the property (44). In this way we proved the second part of the lemma.

By the second part of the lemma, the value of $y_m^{(s)} \downarrow_{\mathcal{N}}$ can be expressed as an explicit function of both (i) the previous values

$$y_m^{(m)} \downarrow_{\mathcal{N}}, \dots, y_m^{(s-1)} \downarrow_{\mathcal{N}}, \quad (47)$$

and (ii) the entries of the sequence R related to the past, that is

$$R^{(m+1)}, \dots, R^{(s)}. \quad (48)$$

By iteratively applying this observation to the previous values (47) it is possible to express the value of $y_m^{(s)} \downarrow_{\mathcal{N}}$ purely in terms of (48). Also the value of

$$F_s = F_s(R^{(m+1)}, \dots, R^{(s)})$$

can be expressed as a function of the entries of the sequence R related to the past, as required. $\square$

Proof of Proposition 3.6 Lemma 3.7 shows that the event $y_m^{(t)} = k$ can be expressed in terms of the vector $V^{(n)}$ given by (29). We apply Theorem 2.2; it follows that the probability $\mathbb{P}\{y_m^{(t)} = k\}$ is equal, up to an additive error term $o(1)$, to the probability that

there are exactly k values of the index $u \in \{m+1, \dots, t\}$ with the property that

$$\overline{R}^{(u)} = F_{u-1} \left(\overline{R}^{(m+1)}, \dots, \overline{R}^{(u-1)} \right). \quad (49)$$

We denote by A_u the random event that the equality (49) holds true.

Let $i_1 < \dots < i_l$ be an increasing sequence of integers from the set $\{m+1, \dots, t\}$ for $l \geq 1$. We will show that

$$\mathbb{P}(A_{i_1} \cap \dots \cap A_{i_l}) = \frac{1}{\sqrt{m}} \mathbb{P}(A_{i_1} \cap \dots \cap A_{i_{l-1}}). \quad (50)$$

Indeed, by Lemma 3.7, the event $A_{i_1} \cap \dots \cap A_{i_{l-1}}$ is a disjoint finite union of some random events of the form

$$B_{r_{m+1}, \dots, r_j} = \left\{ \overline{R}^{(m+1)} = r_{m+1}, \overline{R}^{(m+2)} = r_{m+2}, \dots, \overline{R}^{(j)} = r_j \right\}$$

over some choices of $r_{m+1}, r_{m+2}, \dots, r_j \in \mathcal{N}$, where $j := i_l - 1$. Since the random variables $(\overline{R}^{(i)})$ are independent, it follows that

$$\mathbb{P}(B_{r_{m+1}, \dots, r_j} \cap A_{i_l}) = \frac{1}{\sqrt{m}} \mathbb{P}(B_{r_{m+1}, \dots, r_j}). \quad (51)$$

By summing over the appropriate values of $r_{m+1}, \dots, r_j \in \mathcal{N}$ the equality (50) follows.

By iterating (50) it follows that the events $A_{m+1}, \dots, A_t$ are independent and each has equal probability $\frac{1}{\sqrt{m}}$.

By the Poisson limit theorem [2, Theorem 3.6.1] the probability of k successes in ℓ Bernoulli trials as above converges to the probability of the atom k in the Poisson distribution with the intensity parameter equal to

$$\lim_{m \rightarrow \infty} \frac{\ell}{\sqrt{m}} = \lim_{m \rightarrow \infty} \frac{t-m}{\sqrt{t}} = z$$

which concludes the proof of part (a).

The above discussion also shows that the conditional probability distribution considered in point (b) is equal to the conditional probability distribution of the last coordinate $\lambda^{(t)}$ of the vector $V^{(n)}$ under certain condition which is expressed in terms of the coordinates $R^{(m+1)}, \dots, R^{(t)}$. By Theorem 2.2 this conditional probability distribution is in the distance $o(1)$ (with respect to the total variation distance) to its counterpart for the random vector $\overline{V}^{(n)}$. The latter conditional probability distribution, due to the independence of the coordinates of $\overline{V}^{(n)}$, is equal to the Plancherel measure Plan_t , which concludes the proof of (b).

Part (c) is a direct consequence of parts (a) and (b). $\square$

For an augmented Young diagram $\Lambda = (\lambda, (x, y))$ we define its *transpose* $\Lambda^T = (\lambda^T, (y, x))$.

Lemma 3.8 *For any integers $m, m' \geq 0$ the probability distributions at time $t = m + m'$ of the augmented Plancherel growth processes initiated at times m and m' respectively are related by*

$$\Lambda_m^{(t)} \stackrel{d}{=} \left[\Lambda_{m'}^{(t)} \right]^T.$$

Proof Without loss of generality we may assume that the random variables $\xi_1, \dots, \xi_t$ are distinct real numbers. An application of Greene's theorem [6, Theorem 3.1] shows that the insertion tableaux which correspond to a given sequence of distinct numbers and this sequence read backwards

$$P(\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \dots, \xi_t) = \left[P(\xi_t, \xi_{t-1}, \dots, \xi_{m+1}, \infty, \xi_m, \xi_{m-1}, \dots, \xi_1) \right]^T$$

are transposes of one another. It follows that also the augmented shapes are transposes of one another:

$$\begin{aligned} \Lambda_m^{(t)} &= \text{sh}^* P(\xi_1, \dots, \xi_m, \infty, \xi_{m+1}, \dots, \xi_t) \\ &= \left[\text{sh}^* P(\underbrace{\xi_t, \xi_{t-1}, \dots, \xi_{m+1}}_{m' \text{ entries}}, \infty, \xi_m, \xi_{m-1}, \dots, \xi_1) \right]^T. \end{aligned}$$

Since the sequence (ξ_i) and its any permutation $(\xi_{\sigma(i)})$ have the same distributions, the right-hand side has the same probability distribution as $\left[\Lambda_{m'}^{(t)} \right]^T$, as required. $\square$

Proposition 3.9 *Let $z > 0$ be a fixed real number. Let $t = t(m)$ be a sequence of positive integers such that $t(m) \geq m$ and with the property that*

$$\lim_{m \rightarrow \infty} \frac{m}{\sqrt{t}} = z.$$

Let $\Lambda_m^{(m)} \nearrow \Lambda_m^{(m+1)} \nearrow \dots$ be the augmented Plancherel growth process initiated at time m . We denote $\Lambda_m^{(t)} = (\lambda^{(t)}, \square_m^{(t)})$; let $\square_m^{(t)} = (x_m^{(t)}, y_m^{(t)})$ be the coordinates of the special box at time t .

The total variation distance between

- *the probability distribution of the random vector*

$$\left(x_m^{(t)}, \lambda^{(t)} \right) \in \mathbb{N}_0 \times \mathbb{Y} \quad (52)$$

and

- *the product measure*

$$\text{Pois}(z) \times \text{Plan}_t$$

converges to 0, as $m \rightarrow \infty$.

Proof By Lemma 3.8 the probability distribution of (52) coincides with the probability distribution of

$$\left(y_{m'}^{(t)}, [\lambda^{(t)}]^T\right) \quad (53)$$

for $m' := t - m$. The random vector (53) can be viewed as the image of the vector $(y_{m'}^{(t)}, \lambda^{(t)})$ under the bijection

$$\text{id} \times T : (y, \lambda) \mapsto (y, \lambda^T).$$

By Proposition 3.6 it follows that the total variation distance between (53) and the push-forward measure

$$(\text{id} \times T)(\text{Pois}(z) \times \text{Plan}_t) = \text{Pois}(z) \times \text{Plan}_t$$

converges to zero as $m \rightarrow \infty$; the last equality holds since the Plancherel measure is invariant under transposition. $\square$

3.8 Lazy version of Proposition 1.9. Proof of Proposition 1.1

In Sect. 1.9 we parametrized the shape of the bumping route by the sequence $Y_0, Y_1, \dots$ which gives the number of the row in which the bumping route reaches a specified column, cf. (10). With the help of Proposition 3.1 we can define the lazy counterpart of these quantities: for $x, m \in \mathbb{N}_0$ we denote by

$$T_x^{[m]} = T_x = \min \left\{ t : x_m^{(t)} \leq x \right\}$$

the time it takes for the bumping route (in the lazy parametrization) to reach the specified column.

The following result is the lazy version of Proposition 1.9.

Lemma 3.10 For each integer $m \geq 1$

$$\lim_{u \rightarrow \infty} \sqrt{u} \mathbb{P} \left\{ T_0^{[m]} > u \right\} = m.$$

Proof By Lemma 3.8, for any $u \in \mathbb{N}_0$

$$\mathbb{P} \left\{ T_0^{[m]} > u \right\} = \mathbb{P} \left\{ x_m^{(u)} \geq 1 \right\} = \mathbb{P} \left\{ y_{u-m}^{(u)} \geq 1 \right\}.$$

In the special case $m = 1$ the proof is particularly easy: the right-hand side is equal to $\mathbb{P} \left(E_0^{(u)} \right)$ and Proposition 2.1 provides the necessary asymptotics.

For the general case $m \geq 1$ we use the notations from Sect. 2.2 for $k = 0$, and $n = u - m$, and $\ell = m$. The event $y_{u-m}^{(u)} \geq 1$ occurs if and only if at least one of the numbers

$R^{(n+1)}, \dots, R^{(n+\ell)}$ is equal to 0. We apply Theorem 2.2; it follows that the probability of the latter event is equal, up to an additive error term of the order $o\left(\frac{m}{\sqrt{u-m}}\right) = o\left(\frac{1}{\sqrt{u}}\right)$, to the probability that in m Bernoulli trials with success probability $\frac{1}{\sqrt{n}}$ there is at least one success. In this way we proved that

$$\mathbb{P}\left\{y_{u-m}^{(u)} \geq 1\right\} = \frac{m}{\sqrt{u}} + o\left(\frac{1}{\sqrt{u}}\right),$$

as desired. $\square$

Proof of Proposition 1.1 Since $Y_0^{[m]} \geq Y_1^{[m]} \geq \dots$ is a weakly decreasing sequence, it is enough to consider the case $x = 0$. We apply Lemma 3.10 in the limit $u \rightarrow \infty$. It follows that the probability that the bumping route $\mathcal{T} \leftarrow m + 1/2$ does not reach the column with the index 0 is equal to

$$\lim_{u \rightarrow \infty} \mathbb{P}\left\{T_0^{[m]} \geq u\right\} = 0,$$

as required. $\square$

4 Transition probabilities for the augmented Plancherel growth process

Our main result in this section is Theorem 4.3. It will be the key tool for proving the main results of the current paper.

4.1 Approximating Bernoulli distributions by linear combinations of Poisson distributions

The following Lemma 4.1 is a technical result which will be necessary later in the proof of Proposition 4.2. Roughly speaking, it gives a positive answer to the following question: *for a given value of $k \in \mathbb{N}_0$, can the point measure δ_k be approximated by a linear combination of the Poisson distributions in some explicit, constructive way?* A naive approach to this problem would be to consider a scalar multiple of the Poisson distribution $e^z \text{Pois}(z)$ which corresponds to the sequence of weights

$$\mathbb{N}_0 \ni m \mapsto \frac{1}{m!} z^m$$

and then to consider its k -th derivative with respect to the parameter z for $z = 0$. This is not exactly a solution to the original question (the derivative is not a linear combination), but since the derivative can be approximated by the forward difference operator, this naive approach gives a hint that an expression such as (54) in the special case $p = 1$ might be, in fact, a good answer.

Lemma 4.1 *Let us fix an integer $k \geq 0$ and a real number $0 \leq p \leq 1$. For each $h > 0$ the linear combination of the Poisson distributions*

$$\nu_{k,p,h} := \frac{1}{(e^h - 1)^k} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j} e^{jh} \text{Pois}(pjh) \quad (54)$$

is a probability measure on $\mathbb{N}_0$.

As $h \rightarrow 0$, the measure $\nu_{k,p,h}$ converges (in the sense of total variation distance) to the binomial distribution $\text{Binom}(k, p)$.

Proof *The special case $p = 1$. For a function f on the real line we consider its forward difference function $\Delta[f]$ given by*

$$\Delta[f](x) = f(x+1) - f(x).$$

It follows that the iterated forward difference is given by

$$\Delta^k[f](x) = \sum_{0 \leq j \leq k} (-1)^j \binom{k}{j} f(x+k-j).$$

A priori, $\nu_{k,1,h}$ is a signed measure with the total mass equal to

$$\frac{1}{(e^h - 1)^k} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j} e^{jh} = \frac{1}{(e^h - 1)^k} \Delta^k [e^{hx}] (0). \quad (55)$$

The right-hand side of (55) is equal to 1, since the forward difference of an exponential function is again an exponential:

$$\Delta [e^{hx}] = (e^h - 1) e^{hx}.$$

The atom of $\nu_{k,1,h}$ at an integer $m \geq 0$ is equal to

$$\begin{aligned} \nu_{k,1,h}(m) &= \frac{1}{(e^h - 1)^k m!} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j} (jh)^m \\ &= \frac{h^m}{(e^h - 1)^k m!} \Delta^k [x^m] (0). \end{aligned}$$

Note that the monomial x^m can be expressed in terms of the falling factorials $x^{\underline{p}}$ with the coefficients given by the Stirling numbers of the second kind:

$$x^m = \sum_{0 \leq p \leq m} \left\{ \begin{matrix} m \\ p \end{matrix} \right\} x^{\underline{p}},$$

hence

$$\Delta^k [x^m] = \sum_p \left\{ \begin{matrix} m \\ p \end{matrix} \right\} \Delta^k [x^p] = \sum_{p \geq k} \left\{ \begin{matrix} m \\ p \end{matrix} \right\} p^k x^{p-k}.$$

When we evaluate the above expression at $x = 0$, there is only one non-zero summand

$$\Delta^k [x^m] (0) = \left\{ \begin{matrix} m \\ k \end{matrix} \right\} k!.$$

Thus

$$v_{k,1,h}(m) = \frac{h^m k!}{(e^h - 1)^k m!} \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \geq 0,$$

and the above expression is non-zero only for $m \geq k$. All in all, $v_{k,1,h}$ is a probability measure on $\mathbb{N}_0$, as required.

It follows that the total variation distance between $\text{Binom}(k, 1) = \delta_k$ and $v_{k,1,h}$ is equal to

$$\sum_{i=0}^{\infty} [\delta_k(i) - v_{k,1,h}(i)]^+ = 1 - v_{k,1,h}(k) = 1 - \frac{h^k}{(e^h - 1)^k} \xrightarrow{h \rightarrow 0} 0,$$

as required.

The general case. For a signed measure μ which is supported on $\mathbb{N}_0$ and $0 \leq p \leq 1$ we define the signed measure $C_p[\mu]$ on $\mathbb{N}_0$ by

$$C_p[\mu](k) = \sum_{j \geq k} \mu(j) \binom{j}{k} p^k (1-p)^{j-k}.$$

In the case when μ is a probability measure, $C_p[\mu]$ has a natural interpretation as the probability distribution of a compound binomial random variable $\text{Binom}(M, p)$, where M is a random variable with the probability distribution given by μ .

It is easy to check that for any $0 \leq q \leq 1$ the image of a binomial distribution

$$C_p[\text{Binom}(n, q)] = \text{Binom}(n, pq)$$

is again a binomial distribution, and for any $\lambda \geq 0$ the image of a Poisson distribution

$$C_p[\text{Pois}(\lambda)] = \text{Pois}(p\lambda)$$

is again a Poisson distribution. Since C_p is a linear map, by the very definition (54) it follows that

$$C_p[v_{k,1,h}] = v_{k,p,h}; \quad (56)$$

in particular the latter is a probability measure, as required. By considering the limit $h \rightarrow 0$ of (56) we get

$$\lim_{h \rightarrow 0} v_{k,p,h} = C_p [\text{Binom}(k, 1)] = \text{Binom}(k, p)$$

in the sense of total variation distance, as required. $\square$

4.2 The inclusion $\mathbb{Y}^* \subset \mathbb{N}_0 \times \mathbb{Y}$

We will extend the meaning of the notations from Sect. 3.3 to a larger set. The map

$$\mathbb{Y}^* \ni (\lambda, \square) \mapsto (x_\square, \lambda) \in \mathbb{N}_0 \times \mathbb{Y}, \quad (57)$$

where $\square = (x_\square, y_\square)$, allows us to identify $\mathbb{Y}^*$ with a subset of $\mathbb{N}_0 \times \mathbb{Y}$. For a pair $(x, \lambda) \in \mathbb{N}_0 \times \mathbb{Y}$ we will say that λ is its *regular part*.

We define the edges in this larger set $\mathbb{N}_0 \times \mathbb{Y} \supset \mathbb{Y}^*$ as follows: we declare that $(x, \lambda) \nearrow (\tilde{x}, \tilde{\lambda})$ if the following two conditions hold true:

$$\lambda \nearrow \tilde{\lambda} \quad \text{and} \quad \tilde{x} = \begin{cases} \max \{ \tilde{\lambda}_i : \tilde{\lambda}_i \leq x \} & \text{if the unique box of } \tilde{\lambda}/\lambda \\ & \text{is located in the column } x, \\ x & \text{otherwise.} \end{cases} \quad (58)$$

In this way the oriented graph $\mathbb{Y}^*$ is a subgraph of $\mathbb{N}_0 \times \mathbb{Y}$.

An analogous lifting property as in Lemma 3.4 remains valid if we assume that the initial element $\Lambda^{(m)} \in \mathbb{N}_0 \times \mathbb{Y}$ and the elements of the lifted path

$$\Lambda^{(m)} \nearrow \Lambda^{(m+1)} \nearrow \dots \in \mathbb{N}_0 \times \mathbb{Y}$$

are allowed to be taken from this larger oriented graph.

With these definitions the transition probabilities (40) also make sense if $\Lambda, \tilde{\Lambda} \in \mathbb{N}_0 \times \mathbb{Y}$ are taken from this larger oriented graph and can be used to define Markov chains valued in $\mathbb{N}_0 \times \mathbb{Y}$.

4.3 Transition probabilities for augmented Plancherel growth processes

For the purposes of the current section we will view $\mathbb{Y}^*$ as a subset of $\mathbb{N}_0 \times \mathbb{Y}$, cf. (57). In this way the augmented Plancherel growth process initiated at time m , cf. (39), can be viewed as the aforementioned Markov chain

$$\left((x_m^{(t)}, \lambda^{(t)}) \right)_{t \geq m} \quad (59)$$

valued in $\mathbb{N}_0 \times \mathbb{Y}$.

Let us fix some integer $n \in \mathbb{N}_0$. For each integer $m \in \{0, \dots, n\}$ we may remove some initial entries of the sequence (59) and consider the Markov chain

$$\left((x_m^{(t)}, \lambda^{(t)}) \right)_{t \geq n} \quad (60)$$

which is indexed by the time parameter $t \geq n$. In this way we obtain a whole family of Markov chains (60) indexed by an integer $m \in \{0, \dots, n\}$ which have the same transition probabilities (40).

The latter encourages us to consider a general class of Markov chains

$$\left((x^{(t)}, \lambda^{(t)}) \right)_{t \geq n} \quad (61)$$

valued in $\mathbb{N}_0 \times \mathbb{Y} \supset \mathbb{Y}^*$, for which the transition probabilities are given by (40) and for which the initial probability distribution of $(x^{(n)}, \lambda^{(n)})$ can be arbitrary. We will refer to each such a Markov chain as *augmented Plancherel growth process*.

Proposition 4.2 *Let an integer $k \in \mathbb{N}_0$ and a real number $0 < p < 1$ be fixed, and let $n' = n'(n)$ be a sequence of integers such that $n' \geq n$ and*

$$\lim_{n \rightarrow \infty} \sqrt{\frac{n}{n'}} = p.$$

For a given integer $n \geq 0$ let (61) be an augmented Plancherel growth process with the initial probability distribution at time n given by

$$\delta_k \times \text{Plan}_n.$$

Then the total variation distance

$$\delta \left\{ \left(x^{(n')}, \lambda^{(n')} \right), \text{Binom}(k, p) \times \text{Plan}_{n'} \right\} \quad (62)$$

converges to 0, as $n \rightarrow \infty$.

Proof Let $\epsilon > 0$ be given. By Lemma 4.1 there exists some $h > 0$ with the property that for each $q \in \{1, p\}$ the total variation distance between the measure $\nu_{k,q,h}$ defined in (54) and the binomial distribution $\text{Binom}(k, q)$ is bounded from above by ϵ .

Let T be a map defined on the set of probability measures on $\mathbb{N}_0 \times \mathbb{Y}_n$ in the following way. For a probability measure μ on $\mathbb{N}_0 \times \mathbb{Y}_n$ consider the augmented Plancherel growth process (61) with the initial probability distribution at time n given by μ and define $T\mu$ to be the probability measure on $\mathbb{N}_0 \times \mathbb{Y}_{n'}$ which gives the probability distribution of $(x^{(n')}, \lambda^{(n')})$ at time n' .

It is easy to extend the map T so that it becomes a linear map between the vector space of *signed* measures on $\mathbb{N}_0 \times \mathbb{Y}_n$ and the vector space of *signed* measures on $\mathbb{N}_0 \times \mathbb{Y}_{n'}$. We equip both vector spaces with a metric which corresponds to the total variation distance. Then T is a contraction because of Markovianity of the augmented Plancherel growth process.

For $m \in \{0, \dots, n\}$ and $t \geq n$ we denote by $\mu_m(t)$ the probability measure on $\mathbb{N}_0 \times \mathbb{Y}$, defined by the probability distribution at time t of the augmented Plancherel growth process $(x_m^{(t)}, \lambda^{(t)})$ initiated at time m . For the aforementioned value of $h > 0$ we consider the signed measure on $\mathbb{N}_0 \times \mathbb{Y}_t$ given by the linear combination

$$\mathbb{P}(t) := \frac{1}{(e^h - 1)^k} \sum_{0 \leq j \leq k} (-1)^{k-j} \binom{k}{j} e^{jh} \mu_{\lfloor jh\sqrt{n} \rfloor}(t)$$

(which is well-defined for sufficiently big values of n which assure that $kh\sqrt{n} < n \leq t$).

We apply Proposition 3.9; it follows that for any $j \in \{0, \dots, k\}$ the total variation distance between $\mu_{\lfloor jh\sqrt{n} \rfloor}(n)$ and the product measure

$$\text{Pois}(jh) \times \text{Plan}_n$$

converges to 0, as $n \rightarrow \infty$; it follows that the total variation distance between $\mathbb{P}(n)$ and the product measure

$$\nu_{k,1,h} \times \text{Plan}_n \quad (63)$$

converges to 0, as $n \rightarrow \infty$. On the other hand, the value of $h > 0$ was selected in such a way that the total variation distance between the probability measure (63) and the product measure

$$\delta_k \times \text{Plan}_n \quad (64)$$

is smaller than ϵ . In this way we proved that

$$\limsup_{n \rightarrow \infty} \delta \left\{ \mathbb{P}(n), \delta_k \times \text{Plan}_n \right\} \leq \epsilon.$$

An analogous reasoning shows that

$$\limsup_{n \rightarrow \infty} \delta \left\{ \mathbb{P}(n'), \text{Binom}(k, p) \times \text{Plan}_{n'} \right\} \leq \epsilon.$$

The image of $\mathbb{P}(n)$ under the map T can be calculated by linearity of T :

$$\mathbb{P}(n') = T\mathbb{P}(n).$$

By the triangle inequality and the observation that the map T is a contraction,

$$\begin{aligned} (62) &\leq \delta \left\{ (x^{(n')}, \lambda^{(n')}), \mathbb{P}(n') \right\} + \epsilon \\ &\leq \delta \left\{ (x^{(n)}, \lambda^{(n)}), \mathbb{P}(n) \right\} + \epsilon \leq 2\epsilon \end{aligned}$$

holds true for sufficiently big values of n , as required. $\square$

4.4 Bumping route in the lazy parametrization converges to the Poisson process

Let $(N(t) : t \geq 0)$ denote the Poisson counting process which is independent from the Plancherel growth process $\lambda^{(0)} \nearrow \lambda^{(1)} \nearrow \dots$. The following result is the lazy version of Theorem 1.5.

Theorem 4.3 *Let $l \geq 1$ be a fixed integer, and $z_1 > \dots > z_l$ be a fixed sequence of positive real numbers.*

Let $\Lambda_m^{(m)} \nearrow \Lambda_m^{(m+1)} \nearrow \dots$ be the augmented Plancherel growth process initiated at time m . We denote $\Lambda_m^{(t)} = (\lambda^{(t)}, \square_m^{(t)})$; let $\square_m^{(t)} = (x_m^{(t)}, y_m^{(t)})$ be the coordinates of the special box at time t .

For each $1 \leq i \leq l$ let $t_i = t_i(m)$ be a sequence of positive integers such that

$$\lim_{m \rightarrow \infty} \frac{m}{\sqrt{t_i}} = z_i.$$

We assume that $t_1 \leq \dots \leq t_l$. Then the total variation distance between

- *the probability distribution of the vector*

$$(x_m^{(t_1)}, \dots, x_m^{(t_l)}, \lambda^{(t_l)}), \quad (65)$$

and

- *the probability distribution of the vector*

$$(N(z_1), \dots, N(z_l), \lambda^{(t_l)}) \quad (66)$$

converges to 0, as $m \rightarrow \infty$.

Proof We will perform the proof by induction over l . Its main idea is that the collection of the random vectors (65) over $l \in \{1, 2, \dots\}$ forms a Markov chain; the same holds true for the analogous collection of the random vectors (66). We will compare their initial probability distributions (thanks to Proposition 3.9) and — in a very specific sense — we will compare the kernels of these Markov chains (with Proposition 4.2). We present the details below.

The induction base $l = 1$ coincides with Proposition 3.9.

We will prove now the induction step. We start with the probability distribution of the vector (65) (with the substitution $l := l + 1$). Markovianity of the augmented Plancherel growth process implies that this probability distribution is given by

$$\begin{aligned} & \mathbb{P} \left\{ \left(x_m^{(t_1)}, \dots, x_m^{(t_{l+1})}, \lambda^{(t_{l+1})} \right) = (x_1, \dots, x_{l+1}, \lambda) \right\} \\ &= \sum_{\mu \in \mathbb{Y}_{t_l}} \mathbb{P} \left\{ \left(x_m^{(t_1)}, \dots, x_m^{(t_l)}, \lambda^{(t_l)} \right) = (x_1, \dots, x_l, \mu) \right\} \\ & \quad \times \mathbb{P} \left\{ \left(x_m^{(t_{l+1})}, \lambda^{(t_{l+1})} \right) = (x_{l+1}, \lambda) \mid \left(x_m^{(t_l)}, \lambda^{(t_l)} \right) = (x_l, \mu) \right\} \end{aligned} \quad (67)$$

for any $x_1, \dots, x_{l+1} \in \mathbb{N}_0$ and $\lambda \in \mathbb{Y}$. We define the probability measure $\mathbb{Q}$ on $\mathbb{N}_0^{l+1} \times \mathbb{Y}$ which to a tuple $(x_1, \dots, x_{l+1}, \lambda)$ assigns the probability

$$\begin{aligned} \mathbb{Q}(x_1, \dots, x_{l+1}, \lambda) &:= \sum_{\mu \in \mathbb{Y}_{t_l}} \mathbb{P}\{(N(z_1), \dots, N(z_l)) = (x_1, \dots, x_l)\} \times \text{Plan}_{t_l}(\mu) \\ &\times \mathbb{P}\left\{\left(x_m^{(t_{l+1})}, \lambda^{(t_{l+1})}\right) = (x_{l+1}, \lambda) \mid \left(x_m^{(t_l)}, \lambda^{(t_l)}\right) = (x_l, \mu)\right\}. \end{aligned} \quad (68)$$

In the light of the general definition (61) of the augmented Plancherel growth process, the measures (67) and (68) on $\mathbb{N}_0^{l+1} \times \mathbb{Y}$ can be viewed as applications of the same Markov kernel (which correspond to the last factors on the right-hand side of (67) and (68))

$$\mathbb{P}\left\{\left(x_m^{(t_{l+1})}, \lambda^{(t_{l+1})}\right) = (x_{l+1}, \lambda) \mid \left(x_m^{(t_l)}, \lambda^{(t_l)}\right) = (x_l, \mu)\right\}, \quad \mu \in \mathbb{Y}_{t_l},$$

to two specific initial probability distributions. Since such an application of a Markov kernel is a contraction (with respect to the total variation distance), we proved in this way that the total variation distance between (67) and (68) is bounded from above by the total variation distance between the initial distributions, that is the random vectors (65) and (66). By the inductive hypothesis the total variation distance between the measures $\mathbb{P}$ and $\mathbb{Q}$ converges to zero as $m \rightarrow \infty$. The remaining difficulty is to understand the asymptotic behavior of the measure $\mathbb{Q}$.

Observe that the sum on the right hand side of (68)

$$\begin{aligned} \sum_{\mu \in \mathbb{Y}_{t_l}} \text{Plan}_{t_l}(\mu) \times \mathbb{P}\left\{\left(x_m^{(t_{l+1})}, \lambda^{(t_{l+1})}\right) = (x_{l+1}, \lambda) \mid \left(x_m^{(t_l)}, \lambda^{(t_l)}\right) = (x_l, \mu)\right\} \\ = \mathbb{P}\left\{\left(x^{(n')}, \lambda^{(n')}\right) = (x_{l+1}, \lambda) \mid \left(x^{(n)}, \lambda^{(n)}\right) \stackrel{d}{=} \delta_k \times \text{Plan}_n\right\} \end{aligned} \quad (69)$$

is the probability distribution of the random vector $(x^{(n')}, \lambda^{(n')})$ which appears in Proposition 4.2 with $n' = t_{l+1}$, and $n = t_l$, and $p = \frac{z_{l+1}}{z_l}$, and $k = x_l$. Therefore we proved that the measure $\mathbb{Q}$ is in an $o(1)$ -neighborhood of the following probability measure

$$\begin{aligned} \mathbb{Q}'(x_1, \dots, x_{l+1}, \lambda) &:= \mathbb{P}\{(N(z_1), \dots, N(z_l)) = (x_1, \dots, x_l)\} \\ &\times \text{Plan}_{n'}(\lambda) \text{Binom}\left(x_l, \frac{z_{l+1}}{z_l}\right)(x_{l+1}). \end{aligned} \quad (70)$$

It is easy to check that

$$\mathbb{P}(N(z_{l+1}) = x_{l+1} \mid N(z_l) = x_l) = \text{Binom}\left(x_l, \frac{z_{l+1}}{z_l}\right)(x_{l+1}).$$

Hence the probability of the binomial distribution which appears as the last factor on the right-hand side of (70) can be interpreted as the conditional probability distribution of the Poisson process in the past, given its value in the future.

We show that the Poisson counting process with the reversed time is also a Markov process. Since the Poisson counting process has independent increments, the probability of the event

$$(N(z_1), \dots, N(z_l)) = (x_1, \dots, x_l)$$

can be written as a product; an analogous observation is valid for $l := l + 1$. Due to cancellations of the factors which contribute to the numerator and the denominator, the following conditional probability can be simplified:

$$\begin{aligned} & \mathbb{P}(N(z_{l+1}) = x_{l+1} \mid (N(z_1), \dots, N(z_l)) = (x_1, \dots, x_l)) \\ &= \frac{\mathbb{P}\{(N(z_1), \dots, N(z_{l+1})) = (x_1, \dots, x_{l+1})\}}{\mathbb{P}\{(N(z_1), \dots, N(z_l)) = (x_1, \dots, x_l)\}} \\ &= \frac{\mathbb{P}(N(z_l) = x_l \wedge N(z_{l+1}) = x_{l+1})}{\mathbb{P}(N(z_l) = x_l)} \\ &= \mathbb{P}(N(z_{l+1}) = x_{l+1} \mid N(z_l) = x_l). \end{aligned}$$

By combining the above observations with (70) it follows that

$$\begin{aligned} & \mathbb{Q}'(x_1, \dots, x_{l+1}, \lambda) \\ &= \mathbb{P}\{(N(z_1), \dots, N(z_{l+1})) = (x_1, \dots, x_{l+1})\} \text{Plan}_{t_{l+1}}(\lambda) \end{aligned}$$

is the probability distribution of (66) (with the obvious substitution $l := l + 1$) which completes the inductive step. $\square$

4.5 Lazy version of Remark 1.3

The special case $l = 0$ of the following result seems to be closely related to a very recent work of Azangulov and Ovechkin [1] who used different methods.

Proposition 4.4 *Let (ψ_i) be a sequence of independent, identically distributed random variables with the exponential distribution $\text{Exp}(1)$.*

For each $l \in \mathbb{N}_0$ the joint distribution of the finite tuple of random variables

$$\left(\frac{m}{\sqrt{T_0^{[m]}}, \dots, \frac{m}{\sqrt{T_l^{[m]}}} \right) \tag{71}$$

converges, as $m \rightarrow \infty$, to the joint distribution of the sequence of partial sums

$$(\psi_0, \psi_0 + \psi_1, \dots, \psi_0 + \psi_1 + \dots + \psi_l).$$

Proof For any $s_0, \dots, s_l > 0$ the cumulative distribution function of the random vector (71)

$$\mathbb{P}\left(\frac{m}{\sqrt{T_0^{[m]}}} < s_0, \dots, \frac{m}{\sqrt{T_l^{[m]}}} < s_l\right) = \mathbb{P}\left(x_m^{(t_0)} > 0, x_m^{(t_1)} > 1, \dots, x_m^{(t_l)} > l\right) \quad (72)$$

can be expressed directly in terms of the cumulative distribution of the random vector $(x_m^{(t_0)}, \dots, x_m^{(t_l)})$ with

$$t_i = t_i(m) = \left\lfloor \left(\frac{m}{s_i}\right)^2 \right\rfloor.$$

Theorem 4.3 shows that the right-hand side of (72) converges to

$$\begin{aligned} &\mathbb{P}\left(N(s_0) > 0, N(s_1) > 1, \dots, N(s_l) > l\right) \\ &= \mathbb{P}\left(\psi_0 \leq s_0, \psi_0 + \psi_1 \leq s_1, \dots, \psi_0 + \dots + \psi_l \leq s_l\right), \end{aligned}$$

where

$$\psi_i = \inf \left\{ t : N(t) \geq i + 1 \right\} - \inf \left\{ t : N(t) \geq i \right\}, \quad i \in \mathbb{N}_0,$$

denote the time between the jumps of the Poisson process. Since $(\psi_0, \psi_1, \dots)$ form a sequence of independent random variables with the exponential distribution, this concludes the proof. $\square$

4.6 Conjectural generalization

We revisit Sect. 3.2 with some changes. This time let

$$\xi = (\dots, \xi_{-2}, \xi_{-1}, \xi_0, \xi_1, \dots)$$

be a *doubly* infinite sequence of independent, identically distributed random variables with the uniform distribution $U(0, 1)$ on the unit interval $[0, 1]$. Let us fix $m \in \mathbb{R}_+$. For $s, t \in \mathbb{R}_+$ we define

$$\begin{aligned} &(x_m(s, t), y_m(s, t)) \\ &= \text{Pos}_\infty \left(P \left(\xi_{-\lfloor ms \rfloor}, \dots, \xi_{-2}, \xi_{-1}, \infty, \xi_1, \xi_2, \dots, \xi_{\lfloor \frac{m^2}{t^2} \rfloor} \right) \right). \end{aligned}$$

Fig. 11 Computer simulation of the level curves of the function (73) for $m = 100$. A part of the plot which corresponds to small values of t was not shown due to restrictions on the computation time

Let $\mathcal{N}$ denote the Poisson point process with the uniform unit intensity on $\mathbb{R}_+^2$. For $s, t \in \mathbb{R}_+$ we denote by

$$\mathcal{N}_{s,t} = \mathcal{N}([0, s] \times [0, t])$$

the number of sampled points in the specified rectangle.

Conjecture 4.5 *The random function*

$$\mathbb{R}_+^2 \ni (s, t) \mapsto x_m(s, t) \quad (73)$$

converges in distribution to Poisson point process

$$\mathbb{R}_+^2 \ni (s, t) \mapsto \mathcal{N}_{s,t} \quad (74)$$

in the limit as $m \rightarrow \infty$.

Note that the results of the current paper show the convergence of the marginals which correspond to (a) fixed value of s and all values of $t > 0$ (cf. Theorem 4.3), or (b) fixed value of t and all values of $s > 0$ (this is a corollary from the proof of Proposition 3.9).

It is a bit discouraging that the contour curves obtained in computer experiments (see Fig. 11) *do not seem* to be counting the number of points from some set which belong to a specified rectangle, see Fig. 12 for comparison. On the other hand, maybe the value of m used in our experiments was not big enough to reveal the asymptotic behavior of these curves.

Fig. 12 Computer simulation of the level curves of the function (74) for Poisson point process

5 Removing laziness

Most of the considerations above concerned the lazy parametrization of the bumping routes. In this section we will show how to pass to the parametrization by the row number and, in this way, to prove the remaining claims from Sect. 1 (that is Theorem 1.2 and Proposition 1.9).

5.1 Proof of Proposition 1.9

Our general strategy in this proof is to use Lemma 3.10 and to use the observation that a Plancherel-distributed random Young diagram with n boxes has approximately $2\sqrt{n}$ columns in the scaling when $n \rightarrow \infty$.

Proof of Proposition 1.9 We denote by $c^{(n)}$ the number of rows (or, equivalently, the length of the leftmost column) of the Young diagram $\lambda^{(n)}$. Our proof will be based on an observation (recall Proposition 3.1) that

$$Y_0^{[m]} = c \left(T_0^{[m]} \right).$$

Let $\epsilon > 0$ be fixed. Since $c^{(n)}$ has the same distribution as the length of the bottom row of a Plancherel-distributed random Young diagram with n boxes, the large deviation results [4, 16] show that there exists a constant $C_\epsilon > 0$ such that

$$\begin{aligned} \mathbb{P} \left(\sup_{n \geq n_0} \left| \frac{c^{(n)}}{\sqrt{n}} - 2 \right| > \epsilon \right) &\leq \sum_{n \geq n_0} \mathbb{P} \left(\left| \frac{c^{(n)}}{\sqrt{n}} - 2 \right| > \epsilon \right) \\ &\leq \sum_{n \geq n_0} e^{-C_\epsilon \sqrt{n}} = O(e^{-C_\epsilon \sqrt{n_0}}) \leq o \left(\frac{1}{n_0} \right) \end{aligned} \quad (75)$$

in the limit as $n_0 \rightarrow \infty$.

Consider an arbitrary integer $y \geq 1$. Assume that (i) the event on the left-hand side of (75) *does not* hold true for $n_0 := y$, and (ii) $Y_0^{[m]} \geq y$. Since $T_0^{[m]} \geq Y_0^{[m]} \geq y$ it follows that

$$\left| \frac{Y_0^{[m]}}{\sqrt{T_0^{[m]}}} - 2 \right| = \left| \frac{c(T_0^{[m]})}{\sqrt{T_0^{[m]}}} - 2 \right| \leq \epsilon$$

hence

$$T_0^{[m]} \geq \left(\frac{y}{2 + \epsilon} \right)^2. \quad (76)$$

By considering two possibilities: either the event on the left-hand side of (75) holds true for $n_0 := y$ or not, it follows that

$$\mathbb{P} \left\{ Y_0^{[m]} \geq y \right\} \leq o \left(\frac{1}{y} \right) + \mathbb{P} \left\{ T_0^{[m]} \geq \left(\frac{y}{2 + \epsilon} \right)^2 \right\}.$$

Lemma 3.10 implies therefore that

$$\mathbb{P} \left\{ Y_0^{[m]} \geq y \right\} \leq \frac{(2 + \epsilon)m}{y} + o \left(\frac{1}{y} \right)$$

which completes the proof of the upper bound.

For the lower bound, assume that (i) the event on the left-hand side of (75) *does not* hold true for

$$n_0 := \left\lceil \left(\frac{y}{2 - \epsilon} \right)^2 \right\rceil$$

and (ii) $T_0^{[m]} \geq n_0$. In an analogous way as in the proof of (76) it follows that

$$Y_0^{[m]} \geq (2 - \epsilon) \sqrt{T_0^{[m]}} \geq (2 - \epsilon) \sqrt{n_0} \geq y.$$

By considering two possibilities: either the event on the left-hand side of (75) holds true or not, it follows that that

$$\mathbb{P} \left\{ T_0^{[m]} \geq n_0 \right\} \leq o \left(\frac{1}{y} \right) + \mathbb{P} \left\{ Y_0^{[m]} \geq y \right\}.$$

Lemma 3.10 implies therefore that

$$\mathbb{P} \left\{ Y_0^{[m]} \geq y \right\} \geq \frac{(2 - \epsilon)m}{y} + o \left(\frac{1}{y} \right)$$

which completes the proof of the lower bound. $\square$

5.2 Lazy parametrization versus row parametrization

Proposition 5.1 *For each $x \in \mathbb{N}_0$*

$$\lim_{m \rightarrow \infty} \frac{Y_x^{[m]}}{\sqrt{T_x^{[m]}}} = 2$$

holds true in probability.

Regretfully, the ideas used in the proof of Proposition 1.9 (cf. Sect. 5.1 above) are not directly applicable for the proof of Proposition 5.1 when $x \geq 1$ because we are not aware of suitable large deviation results for the lower tail of the distribution of a specific row a Plancherel-distributed Young diagram, other than the bottom row.

Our general strategy in this proof is to study the length $\mu_x^{(t)}$ of the column with the fixed index x in the Plancherel growth process $\lambda^{(t)}$, as $t \rightarrow \infty$. Since we are unable to get asymptotic *uniform* bounds for

$$\left| \frac{\mu_x^{(t)}}{\sqrt{t}} - 2 \right| \tag{77}$$

over *all* integers t such that $\frac{t}{m^2}$ belongs to some compact subset of $(0, \infty)$ in the limit $m \rightarrow \infty$, as a substitute we consider a *finite* subset of $(0, \infty)$ of the form

$$\left\{ c(1 + \epsilon), \dots, c(1 + \epsilon)^l \right\}$$

for arbitrarily small values of $c, \epsilon > 0$ and arbitrarily large integer $l \geq 0$ and prove the appropriate bounds for the integers $t_i(m)$ for which $\frac{t_i}{m^2}$ are approximately elements of this finite set. We will use monotonicity in order to get some information about (77) also for the integers t which are between the numbers $\{t_i(m)\}$.

Proof Let $\epsilon > 0$ be fixed. Let $\delta > 0$ be arbitrary. By Proposition 4.4 the law of the random variable $\frac{m}{\sqrt{T_x^{[m]}}}$ converges to the Erlang distribution which is supported on $\mathbb{R}_+$ and has no atom in 0. Let W be a random variable with the latter probability distribution; in this way the law of $\frac{T_x^{[m]}}{m^2}$ converges to the law of W^{-2} . Let $c > 0$ be a sufficiently small number such that

$$\mathbb{P}(W^{-2} < c) < \delta.$$

Now, let $l \in \mathbb{N}_0$ be a sufficiently big integer so that

$$\mathbb{P}(c(1 + \epsilon)^l < W^{-2}) < \delta.$$

We define

$$t_i = t_i(m) = \lfloor m^2 c(1 + \epsilon)^i \rfloor \quad \text{for } i \in \{0, \dots, l\}.$$

With these notations there exists some m_1 with the property that for each $m \geq m_1$

$$\mathbb{P}(t_0 < T_x^{[m]} \leq t_l) > 1 - 2\delta. \quad (78)$$

Let $\mu^{(n)} = [\lambda^{(n)}]^T$ be the transpose of $\lambda^{(n)}$; in this way $\mu_x^{(n)}$ is the number of the boxes of $\mathcal{T}$ which are in the column x and contain an entry $\leq n$. The probability distribution of $\mu^{(n)}$ is also given by the Plancherel measure. The monograph of Romik [12, Theorem 1.22] contains that proof that

$$\frac{\mu_x^{(t_i(m))}}{\sqrt{t_i(m)}} \xrightarrow{\mathbb{P}} 2$$

holds true in the special case of the bottom row $i = 0$; it is quite straightforward to check that this proof is also valid for each $i \in \{0, \dots, l\}$, for the details see [10, proof of Lemma 2.5]. Hence there exists some m_2 with the property that for each $m \geq m_2$ the probability of the event

$$\left| \frac{\mu_x^{(t_i(m))}}{\sqrt{t_i(m)}} - 2 \right| < \epsilon \quad \text{holds for each } i \in \{0, \dots, l\} \quad (79)$$

is at least $1 - \delta$.

Let us consider an elementary event $\mathcal{T}$ with the property that the event considered in (78) occurred, that is $t_0 < T_x^{[m]} \leq t_l$, and the event (79) occurred. Since $t_0 \leq \dots \leq t_l$ form a weakly increasing sequence, there exists an index $j = j(\mathcal{T}) \in \{0, \dots, l-1\}$ such that

$$t_j < T_x^{[m]} \leq t_{j+1}.$$

It follows that

$$\mu_x^{(t_j)} < Y_x^{[m]} \leq \mu_x^{(t_{j+1})}$$

hence

$$\begin{aligned} (2 - \epsilon) \frac{1}{\sqrt{1 + \epsilon + o(1)}} &< \frac{\mu_x^{(t_j)}}{\sqrt{t_{j+1}}} \leq \frac{Y_x^{[m]}}{\sqrt{T_x^{[m]}}} \\ &\leq \frac{\mu_x^{(t_{j+1})}}{\sqrt{t_j}} < (2 + \epsilon) \left(\sqrt{1 + \epsilon + o(1)} \right). \end{aligned} \quad (80)$$

In this way we proved that for each $m \geq \max(m_1, m_2)$ the probability of the event (80) is at least $1 - 3\delta$, as required. $\square$

5.3 Proof of Theorem 1.2

Proof For each integer $l \geq 0$ Proposition 4.4 gives the asymptotics of the joint probability distribution of the random variables $T_0^{[m]}, \dots, T_l^{[m]}$ which concern the shape of the bumping route in the lazy parametrization. On the other hand, Proposition 5.1 allows us to express asymptotically these random variables by their non-lazy counterparts $Y_0^{[m]}, \dots, Y_l^{[m]}$. The discussion from Remark 1.3 completes the proof. $\square$

Acknowledgements We thank Iskander Azangulov, Maciej Dołęga, Vadim Gorin, Piet Groeneboom, Adam Jakubowski, Grigory Ovechkin, Timo Seppäläinen, and Anatoly Vershik for discussions and bibliographic suggestions.

Funding Research supported by Narodowe Centrum Nauki, Grant Number 2017/26/A/ST1/00189. Mikołaj Marciniak was additionally supported by Narodowe Centrum Badań i Rozwoju, grant number POWR.03.05.00-00-Z302/17-00.

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