Probability Theory and Related Fields (2020) 176:1439–1444
https://doi.org/10.1007/s00440-020-00965-y

CORRECTION

Correction to: Tightness for the cover time of the two dimensional sphere

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Published online: 6 March 2020

Correction to: Probability Theory and Related Fields https://doi.org/10.1007/s00440-019-00940-2

Lemma 9.1 in the paper is incorrect as stated. The proof given there yields the following corrected statement.

Lemma 9.1 *For any* $L - 2k \geq k' > k + 10 \geq 11$ *and* $n > 1$, *let* $\mathcal{A}_{k'}$ *denote an event, measurable on the excursions of the Brownian motion from* $\partial B_d(y, h_{k'})$ *to* $\partial B_d(y, h_{k'-1})$ *during the first n excursions from* $\partial B_d(y, h_k) \to \partial B_d(y, h_{k-1})$. *Then,*

$$\begin{aligned} &\mathbb{P}(\mathcal{A}_{k'} \mid T^{y,k \overset{n}{\to} k-1}_{y,k'-1 \to k'} = m_{k'},\, \mathcal{G}^y_k) \\ &= \left(1 + O\left((k'-k)\frac{h_{k'-1}}{h_k}\right)\right)^{m_{k'}} \mathbb{P}(\mathcal{A}_{k'} \mid T^{y,k \overset{n}{\to} k-1}_{y,k'-1 \to k'} = m_{k'})\,. \end{aligned} \tag{9.1}$$

In particular, for all $m_l; l = k', \dots L$, *and all* $y \in \mathbf{S}^2$,

$$\mathbb{P}(T^{y,k \overset{n}{\to} k-1}_{y,l-1 \to l} = m_l; l = k'+1, \dots L \mid T^{y,k \overset{n}{\to} k-1}_{y,k'-1 \to k'} = m_{k'},\, \mathcal{G}^y_k)$$

The original article can be found online at https://doi.org/10.1007/s00440-019-00940-2.

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$$= \left(1 + O\left((k' - k)\frac{h_{k'-1}}{h_k}\right)\right)^{m_{k'}}$$

$$\mathbb{P}(T^{y,k \xrightarrow{n} k-1}_{y,l-1\to l} = m_l; l = k'+1, \dots L \mid T^{y,k \xrightarrow{n} k-1}_{y,k'-1\to k'} = m_{k'}). \tag{9.2}$$

In Lemma 9.1 we have $\left|O\left((k'-k)\frac{h_{k'-1}}{h_k}\right)\right| \leq 100\left((k'-k)\frac{h_{k'-1}}{h_k}\right)$.

This necessitates numerous corrections, mostly in the order in which estimates are applied. The changes occur in Section **4**, as sketched below. Full details are given in the companion posting [2]. Here and in what follows, boldfaced numbers refer to the original numbering in [1].

1. Section 4.1 (first moment estimate) The estimate **(4.56)** is incorrect. Instead, one needs to use the decoupling estimates from Section **4.5**. This necessitates changes starting from the paragraph following **(4.47)** through the end of the subsection, as follows.

Let

$$\mathcal{W}^{\in x}_{y,k}(n) = \left\{ d^1_{\mathrm{Wa}}\left(\frac{1}{n}\sum_{i=1}^{n} \delta_{\theta_{k,i}}, \nu_k\right) \in \frac{c_0}{2\sqrt{n}} I_x \right\}, \tag{1.1}$$

so that

$$\mathcal{W}^{c}_{y,k}(N_{k,a}) \subseteq \cup_{m=\log k}^{\infty} \mathcal{W}^{\in m}_{y,k}(N_{k,a}), \tag{1.2}$$

and consequently, setting

$$\mathcal{L}_{k,m,p,a} = \mathcal{K}_{k,p,a} \cap \mathcal{W}^{\in m}_{y,k}(N_{k,a}), \tag{1.3}$$

we have

$$\mathbb{P}\left(\mathcal{K}_{k,p,a}\right) \leq \sum_{m=\log k}^{\infty} \mathbb{P}\left(\mathcal{L}_{k,m,p,a}\right). \tag{1.4}$$

Thus to prove **(4.47)** it suffices to show that for all $m \geq \log k$, all $d^* \leq k \leq L_-$, and all $0 \leq z \leq (\log L)^{1/4}$,

$$\sum_{a=[k^{1/4}]}^{L^{3/4}} \sum_{p=[(k-3)^{1/4}]}^{L^{3/4}} \mathbb{P}\left(\mathcal{L}_{k,m,p,a}\right) \leq c(1+z)e^{-2L}e^{-2z}e^{-c' m^2}. \tag{1.5}$$

Write

$$\mathcal{L}'_{k,m,p,a} = \mathcal{H}_{k-3,p} \bigcap \mathcal{H}_{k,a} \bigcap \mathcal{W}^{\in m}_{y,k}(N_{k,a}). \tag{1.6}$$

Since the $\theta_{k,i}$ are i.i.d. ν_k-distributed random variables, as explained in the paragraph before **(4.19)**, it follows from **(4.17)** that

$$\mathbb{P}\left(\mathcal{W}^{\in m}_{y,k}(N_{k,a}) \,\middle|\, T^{y,t_z}_{k-3}\right) \leq 2e^{-m^2/4}. \tag{1.7}$$

The following lemma is immediate, details in [2, Lemma 4.6].

Lemma 1.1 *There exist constants $c, c' > 0$ so that*

$$\mathbb{P}\left(\mathcal{L}'_{k,m,p,a}\right) \leq c\, e^{-2k-2(z-p)-(z-p)^2/4k} e^{-m^2/8} e^{-c'(a-p)^2}. \tag{1.8}$$

Let

$$k^+ = k + \lceil 10^{10} \log L \rceil, \quad k^{++} = k + 2\lceil 10^{10} \log L \rceil, \tag{1.9}$$

$$\mathbb{B}^{j'}_{y,k+3,k^+} = \Big\{ \gamma(l) \leq \sqrt{2T_l^{y,t_z}} \text{ for } l = k+3, \ldots, k^+, \sqrt{2T_{k^+}^{y,t_z}} \in I_{\rho_L(L-k^+)+j'} \Big\}, \tag{1.10}$$

and

$$\widehat{\mathfrak{B}}_{y,k^{++}+1,L} = \Big\{ \rho_L(L-l) \leq \sqrt{2T_l^{y,t_z}} \text{ for } l = k^{++}+1, \ldots, L-1, \sqrt{2T_L^{y,t_z}} = 0 \Big\}. \tag{1.11}$$

We have that,

$$\mathbb{P}\left(\mathcal{L}_{k,m,p,a}\right) = \mathbb{P}\left(\mathcal{L}'_{k,m,p,a} \bigcap \mathcal{B}_{y,k+1,L}\right) \leq \sum_{j'=(k^+)^{1/4}}^{L^{3/4}} \sum_{j''=(k^{++})^{1/4}}^{L^{3/4}} \mathbb{P}\left(\mathcal{L}'_{k,m,p,a} \cap \mathbb{B}^{j'}_{y,k+3,k^+} \cap \widehat{\mathfrak{B}}_{y,k^{++}+1,L}; \sqrt{2T_{k^{++}}^{y,t_z}} \in I_{\rho_L(L-k^{++})+j''}\right), \tag{1.12}$$

up to an error due to the restriction of $j', j'' \leq L^{3/4}$, which is little o of the right-hand side of [1, (4.43)]. Let

$$\mathcal{G}^y_{k^+} = \sigma\text{-algebra generated by the excursions from } \partial B_d(y, h_{k^+-1}) \text{ to } \partial B_d(y, h_{k^+}) \tag{1.13}$$

(In the definition of $\mathcal{G}^y_{k^+}$, if we start outside $\partial B_d(y, h_{k^+-1})$ we include the initial excursion to $\partial B_d(y, h_{k^+-1})$. Do not confuse with **(3.38)**.) Note that

$$\mathcal{A}_{j'} := \mathcal{L}'_{k,m,p,a} \cap \mathbb{B}^{j'}_{y,k+3,k^+} \in \mathcal{G}^y_{k^+}. \tag{1.14}$$

Using (1.14) we have

$$\mathbb{P}\left(\mathcal{A}_{j'}; \sqrt{2T_{k^{++}}^{y,t_z}} \in I_{\rho_L(L-k^{++})+j''}; \widehat{\mathfrak{B}}_{y,k^{++}+1,L}\right)$$

$$\leq \sup_{s\in I_{\rho(L-k^+)+j'},\ v\in I_{\rho(L-k^{++})+j''}} \mathbb{P}\left(\mathcal{A}_{j'};\ \sqrt{2T^{y,t_z}_{k^{++}}} \in I_{\rho(L-k^{++})+j''};\right.$$
$$\left.\mathbb{P}\left(\widehat{\mathfrak{B}}_{y,k^{++}+1,L} \mid T^{y,k^+\overset{s^2/2}{\to}k^+-1}_{y,k^{++}-1\to k^{++}} = v^2/2,\ \mathcal{G}^y_{k^+}\right)\right). \tag{1.15}$$

By Lemma 9.1 with the k, k' there replaced by k^+, k^{++} and using that $(1+10(k^{++}-k^+)h_{k^{++}}/h_{k^+})^{4L^2}$ is bounded above uniformly, we have that for some universal $c<\infty$

$$\mathbb{P}\left(\widehat{\mathfrak{B}}_{y,k^{++}+1,L} \mid T^{y,k^+\overset{s^2/2}{\to}k^+-1}_{y,k^{++}-1\to k^{++}} = v^2/2,\ \mathcal{G}^y_{k^+}\right)$$
$$\leq c\mathbb{P}\left(\widehat{\mathfrak{B}}_{y,k^{++}+1,L} \mid T^{y,k^+\overset{s^2/2}{\to}k^+-1}_{y,k^{++}-1\to k^{++}} = v^2/2\right). \tag{1.16}$$

By the barrier estimate of Lemma **8.4** in Appendix I, we see that for j'' in the range of summation in (1.12), uniformly in $s \in I_{\rho(L-k^+)+j'}$ and $v \in I_{\rho(L-k^{++})+j''}$,

$$\mathbb{P}\left(\widehat{\mathfrak{B}}_{y,k^{++}+1,L} \mid T^{y,k^+\overset{s^2/2}{\to}k^+-1}_{y,k^{++}-1\to k^{++}} = v^2/2\right) \leq c(1+j'')e^{-2(L-k^{++})-2j''}. \tag{1.17}$$

Thus, uniformly in $[k^{1/4}] \leq a \leq L^{3/4}$,

$$\mathbb{P}\left(\mathcal{L}_{k,m,p,a}\right) = \mathbb{P}\left(\mathcal{L}'_{k,m,p,a} \cap \mathcal{B}_{y,k+3,L}\right) \leq ce^{-2(L-k^{++})}$$
$$\times \sum_{j'=(k^+)^{1/4}}^{L^{3/4}} \sum_{j''=(k^{++})^{1/4}}^{L^{3/4}} j''e^{-2j''}\mathbb{P}\left(\mathcal{L}'_{k,m,p,a} \cap \mathbb{B}^{j',j''}_{y,k+3,k^+,k^{++}}\right), \tag{1.18}$$

where

$$\mathbb{B}^{j',j''}_{y,k+3,k^+,k^{++}} = \left\{\gamma(l) \leq \sqrt{2T^{y,t_z}_l} \text{ for } l = k+3,\ldots,k^+,k^{++},\right.$$
$$\left.\sqrt{2T^{y,t_z}_{k^+}} \in I_{\rho_L(L-k^+)+j'},\ \sqrt{2T^{y,t_z}_{k^{++}}} \in I_{\rho_L(L-k^{++})+j''}\right\}. \tag{1.19}$$

From this point, the estimates needed are similar to the argument in [1, Section 4.1], using the barrier estimates of Section **8** according to different values of m. The full details and the division to different cases appear in [2, Section 4.1].

Section 4.4 (Second moment estimate: early branching) The last inequality in **(4.87)** is incorrect. In order to fix that, one needs to change the events to which one applies decoupling, starting from below **(4.83)**, as follows.

Recall the events $\mathbb{B}^{j'}_{y,k+3,k^+}$ and $\widehat{\mathfrak{B}}_{y,k^{++}+1,L}$, and the σ-algebra $\mathcal{G}^y_{k^+}$. We have that

$$\mathbb{P}\left(\mathcal{W}_{y,k}(N_{k,a}) \cap \mathcal{B}_{y,k+3,L} \cap \mathcal{H}_{k,a} \cap \mathcal{I}_{y',z}\right) \le \sum_{j'=(k^+)^{1/4}}^{L^{3/4}} \sum_{j''=(k^{++})^{1/4}}^{L^{3/4}} \mathbb{P}\Bigg(\mathcal{W}_{y,k}(N_{k,a}) \cap \mathbb{B}^{j'}_{y,k+3,k^+} \cap \widehat{\mathfrak{B}}_{y,k^{++}+1,L} \cap \mathcal{H}_{k,a} \cap \widehat{\mathcal{I}}_{y',k,z}, \sqrt{2T^{y,t_z}_{k^{++}}} \in I_{\rho_L(L-k^{++})+j''}\Bigg) + o(e^{-a}E(k)), \tag{1.20}$$

where the error term is coming from the restriction $j', j'' \le L^{3/4}$. Next, set

$$\mathcal{A}_{j'} := \mathcal{W}_{y,k}(N_{k,a}) \bigcap \mathcal{H}_{k,a} \bigcap \widehat{\mathcal{I}}_{y',k,z} \cap \mathbb{B}^{j'}_{y,k+3,k^+} \in \mathcal{G}^y_{k^+}. \tag{1.21}$$

Using (1.21) we have

$$\mathbb{P}\left(\mathcal{A}_{j'};\ \sqrt{2T^{y,t_z}_{k^{++}}} \in I_{\rho_L(L-k^{++})+j''};\ \widehat{\mathfrak{B}}_{y,k^{++}+1,L}\right) \le \sup_{s\in I_{\rho(L-k^+)+j'},\ v\in I_{\rho(L-k^{++})+j''}} \mathbb{P}\Bigg(\mathcal{A}_{j'};\ \sqrt{2T^{y,t_z}_{k^{++}}} \in I_{\rho(L-k^{++})+j''};\ \mathbb{P}\left(\widehat{\mathfrak{B}}_{y,k^{++}+1,L} \mid T^{y,k^+ \overset{s^2/2}{\to} k^+-1}_{y,k^{++}-1\to k^{++}} = v^2/2,\ \mathcal{G}^y_{k^+}\right)\Bigg). \tag{1.22}$$

Recall the estimates (1.16) and (1.17). We then have that uniformly in $[k^{1/4}] \le a \le L^{3/4}$,

$$\mathbb{P}\left(\mathcal{W}_{y,k}(N_{k,a}) \cap \mathcal{B}_{y,k+3,L} \cap \mathcal{H}_{k,a} \cap \mathcal{I}_{y',z}\right) \le ce^{-2(L-k^{++})} \sum_{j'=(k^+)^{1/4}}^{L^{3/4}} \sum_{j''=(k^{++})^{1/4}}^{L^{3/4}} (1+j'')e^{-2j''} \times\mathbb{P}\left(\mathcal{W}_{y,k}(N_{k,a}) \cap \mathbb{B}^{j',j''}_{y,k+3,k^+,k^{++}} \cap \mathcal{H}_{k,a} \cap \widehat{\mathcal{I}}_{y',k,z}\right), \tag{1.23}$$

where $\mathbb{B}^{j',j''}_{y,k+3,k^+,k^{++}}$ is as in (1.19).

The next lemma handles the last term in (1.23) and replaces Lemma **4.7** in [1]. The proof, which is given in detail in [2, Lemma 4.9], is very similar to the proof of Lemma **4.7**; it uses the decoupling lemma (Lemma **4.11**), barrier estimates and the control on Wasserstein distance contained in **(4.17)**.

Lemma 1.2 *For some $M_0 < \infty$ and k, a, j' in the ranges specified above,*

$$
\begin{aligned}
&\mathbb{P}(\mathcal{W}_{y,k}(N_{k,a}) \bigcap \mathbb{B}^{j',j''}_{y,k+3,k^+,k^{++}} \bigcap \mathcal{H}_{k,a} \bigcap \widehat{\mathcal{I}}_{y',k,z}) \\
&\quad \leq F_{k,a,j',j''}\ \mathbb{P}(\widehat{\mathcal{I}}_{y',k,z}) + e^{-2L-\sqrt{L}},
\end{aligned} \tag{1.24}
$$

where

$$
\begin{aligned}
F_{k,a,j',j''} = cae^{-2a} e^{-2(k^{++}-k)} &\sum_{\substack{\{\bar{j}':|\bar{j}'-j'|\leq 2M_0 \log k\} \\ \{\bar{j}'':|\bar{j}''-j''|\leq 2M_0 \log k\}}} \\
&\bar{j}' \frac{e^{-(\bar{j}'-a)^2/(4(k^+-k))}}{(k^+-k)} e^{2\bar{j}''} \frac{e^{-(\bar{j}'-\bar{j}'')^2/2(k^+-k)}}{(k^+-k)}.
\end{aligned} \tag{1.25}
$$

Assuming Lemma 1.2, the rest of the proof is similar to [1, Section 4.4]. Full details appear in [2, Section 4.4].

References

1. Belius, D., Rosen, J., Zeitouni, O.: Tightness for the cover time of the two dimensional sphere. Probab. Theory. Relat. Fields **176**, 1357–1437 (2020)
2. Belius, D., Rosen, J., Zeitouni, O.: Tightness for the cover time of the two dimensional sphere. (2020). arXiv:1711.02845v6