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Exponential loss of memory for the 2-dimensional Allen–Cahn equation with small noise

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Abstract

We prove an asymptotic coupling theorem for the 2-dimensional Allen–Cahn equation perturbed by a small space-time white noise. We show that with overwhelming probability two profiles that start close to the minimisers of the potential of the deterministic system contract exponentially fast in a suitable topology. In the 1-dimensional case a similar result was shown in Martinelli et al. (Commun Math Phys 120(1):25–69, 1988; J Stat Phys 55(3–4):477–504, 1989). It is well-known that in two or more dimensions solutions of this equation are distribution-valued, and the equation has to be interpreted in a renormalised sense. Formally, this renormalisation corresponds to moving the minima of the potential infinitely far apart and making them infinitely deep. We show that despite this renormalisation, solutions behave like perturbations of the deterministic system without renormalisation: they spend large stretches of time close to the minimisers of the (un-renormalised) potential and the exponential contraction rate of different profiles is given by the second derivative of the potential in these points. As an application we prove an Eyring–Kramers law for the transition times between the stable solutions of the deterministic system for fixed initial conditions.

Keywords Singular SPDEs · Metastability · Asymptotic coupling · Eyring–Kramers law

Mathematics Subject Classification 60H15 · 35K57

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1 Introduction

We are interested in the behaviour of solutions to the Allen–Cahn equation, perturbed by a small noise term. The deterministic equation is given by

$$(\partial_t - \Delta)X = -X^3 + X, \tag{1.1}$$

and it is well-known that (1.1) is a gradient flow with respect to the potential

$$V(X) := \int \left(\frac{1}{2}|\nabla X(z)|^2 - \frac{1}{2}X(z)^2 + \frac{1}{4}X(z)^4 \right) \mathrm{d}z. \tag{1.2}$$

The fluctuation-dissipation theorem suggests an additive Gaussian space-time white noise ξ as a natural random perturbation of (1.1); so we consider

$$(\partial_t - \Delta)X = -X^3 + X + \sqrt{2\varepsilon}\xi, \tag{1.3}$$

for a small parameter $\varepsilon > 0$.

In the 1-dimensional case, i.e. the case where the solution X depends on time and a 1-dimensional spatial argument, the behaviour of solutions to (1.3) is well-understood. Solutions exhibit the phenomenon of metastability, i.e. they typically spend large stretches of time close to the minimisers of the potential (1.2) with rare and relatively quick noise-induced transitions between them. Early contributions go back to the 80s where Farris and Jona–Lasinio [9] studied the system on the level of large deviations.

We are particularly interested in the "exponential loss of memory property" first observed by Martinelli, Olivieri and Scoppola in [17,18]. They studied the flow map induced by (1.3), i.e. the random map $x \mapsto X(t; x)$ which associates to any initial condition the corresponding solution at time t, and showed that for large t the map becomes essentially constant. They also showed that with overwhelming probability, solutions that start within the basin of attraction of the same minimiser of V contract exponentially fast, with exponential rate given by the smallest eigenvalue of the linearisation of V in this minimiser. This implies for example that the law of such solutions at large times is essentially insensitive to the precise location at which they are started.

It is very natural to consider higher dimensional analogues of (1.3), but unfortunately for space dimension $d \geq 2$, Eq. (1.3) is ill-posed. In fact, for $d \geq 2$ the space-time white noise becomes so irregular, that solutions have to be interpreted in the sense of Schwartz distributions, and the interpretation of the nonlinear term is a priori unclear. These kind of singular stochastic partial differential equations (SPDEs) have received a lot of attention recently (see e.g. [6,10,12]). The solution proposed in these works is to *renormalise* the equation, by removing some infinite terms, formally leading to the equation

$$(\partial_t - \Delta)X = -X^3 + (1 + 3\varepsilon\infty)X + \sqrt{2\varepsilon}\xi. \tag{1.4}$$

Note that formally, this renormalisation corresponds to moving the minima of the double-well potential out to $\pm\infty$ and making them infinitely deep at the same time. So at first glance, it seems unclear why these renormalised distribution-valued solutions should exhibit similar behaviour to the 1-dimensional function-valued solutions of (1.3).

In [14] Hairer and the second named author studied the small ε asymptotics for (1.4) for space dimension $d = 2$ and 3 on the level of Freidlin–Wentzell type large deviations. They obtained a large deviation principle with rate function $\mathcal{I}$ given by

$$\mathcal{I}(X) := \frac{1}{4}\int_0^T \int \Big(\partial_t X(t,z) - \Big(\Delta X(t,z) - \Big(X(t,z)^3 - X(t,z)\Big)\Big)\Big)^2 \,\mathrm{d}z\,\mathrm{d}t. \tag{1.5}$$

In fact, a result in a similar spirit had already appeared in the 90s [15]. The striking fact is that this rate function is exactly the 2-dimensional version of the rate function obtained in the 1-dimensional case [9]. The infinite renormalisation constant does not affect the rate functional. This result implies that for small ε solutions of the renormalised SPDE (1.4) stay close to solutions of the deterministic PDE (1.1) suggesting that (1.4) may indeed be the natural small noise perturbation of (1.1).

In this article we consider (1.4) over a 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/L\mathbb{Z}^2$ for $L < 2\pi$. It is known that under this assumption on the torus size L, the deterministic equation (1.1) has exactly three stationary solutions, namely the constant profiles $-1, 0, 1$ (see [16, Appendix B.1]). Here ± 1 are stable minimisers of V and 0 is unstable. We prove that in the small noise regime with overwhelming probability solutions that start close to the same stable minimiser ± 1 contract exponentially fast. The exponential contraction rate is arbitrarily close to 2, the second derivative of the double-well $x \mapsto \frac{1}{4}x^4 - \frac{1}{2}x^2$ in ± 1. This is precisely the 2-dimensional version of [17, Corollary 3.1].

On a technical level we work with the Da Prato–Debussche decomposition (see Sect. 2 for more details). An immediate observation is that differences of any two profiles have much better regularity than the solutions themselves. We split the time axis into random “good” and “bad” intervals depending on whether a reference profile is close to ± 1 or not. The key idea is that on “good” intervals solutions should contract exponentially, while they should not diverge too fast on “bad” intervals. Furthermore, “good” intervals should be much longer than “bad” intervals.

The control on the “good” intervals is relatively straightforward: the exponential contraction follows by linearising the equation and the fact that these intervals are typically long follows from exponential moment bounds on the explicit stochastic objects appearing in the Da Prato–Debussche approach. The control on the “bad” intervals is much more involved: in the 1-dimensional case two profiles cannot diverge too fast, because the second derivative of the double-well potential is bounded from below. But in the 2-dimensional case, where solutions are distribution-valued, there is no obvious counterpart of this property. Instead we use a strong a priori estimate obtained in our previous work [24] and the local Lipschitz continuity of the non-linearity. Ultimately this yields an exponential growth bound where the exponential rate is given by a polynomial in the explicit stochastic objects. We use a large deviation estimate to prove that these intervals cannot be too long. In the final step we show that the exponential contraction holds for all t if a certain random walk with positive drift stays positive for all times. This random walk is then analysed using techniques developed for the classical Cramér–Lunberg model in risk theory.

The original motivation for our work was to prove an Eyring–Kramers law for the transition times of X. In [2] Berglund, Di Gesú and the second named author studied spectral Galerkin approximations X_N of (1.4) and obtained explicit estimates on the expected first transition times from a neighbourhood of -1 to a neighbourhood of 1. These estimates give a precise asymptotic as $\varepsilon \to 0$ and hold uniformly in the discretisation parameter N. Their method was based on the potential theoretic approach developed in the finite-dimensional context by Bovier et al. in [3]. This approach relies heavily on the reversibility of the dynamics and provides explicit formulas for the expected transition times in terms of certain integrals of the reversible measure. The key observation in [2] was that in the context of (1.3) these integrals can be analysed uniformly in the parameter N using the classical Nelson’s estimate [22] from constructive Quantum Field Theory. However, the result in [2] was not optimal for the following two reasons: First, it does not allow to pass to the limit as $N \to \infty$ to retrieve the estimate for the transition times of X. Second, and more important, the bounds could only be obtained for a certain N-dependent choice of initial distribution on the neighbourhood of -1. This problem is inherent to the potential theoretic approach, which only yields an exact formula for the diffusion started in this so-called normalised equilibrium measure. In fact, a large part of the original work [3] was dedicated to removing this problem using regularity theory for the finite-dimensional transition probabilities.

In this paper we overcome these two barriers. We first justify the passage to the limit $N \to \infty$ based on our previous work [24]: we use the strong a priori estimates on the level of the approximation X_N and the support theorem obtained there to prove uniform integrability of the transition times of X_N. The only difficulty here comes from the action of the Galerkin projection on the non-linearity which does not allow to test the equation with powers greater than 1. To remove the unnatural assumption on the initial distribution we make use of our main result, the exponential contraction estimate. This estimate allows us to couple the solution started with an arbitrary but fixed initial condition with the solution started in the normalised equilibrium measure.

1.1 Outline

In Sect. 2 we briefly review the solution theory of (1.4). In Sect. 3 we state our main results, that is, the exponential loss of memory, Theorem 3.1, and the Eyring–Kramers law, Theorem 3.5. In Sect. 4 we prove Theorem 3.1 based on some auxiliary propositions. These propositions are proved in Sects. 5 and 6. Finally, in Sect. 7 we prove the Eyring–Kramers law, Theorem 3.5, generalising [2, Theorem 2.3]. Several known results that are used throughout this article as well as some additional technical statements can be found in the Appendix.

1.2 Notation

We fix a torus $\mathbb{T}^2 = \mathbb{R}^2/L\mathbb{Z}^2$ of size $0 < L < 2\pi$. All function spaces are defined over $\mathbb{T}^2$. We write $\mathcal{C}^\infty$ for the space of smooth functions and L^p, $p \in [1, \infty]$, for the space of p-integrable periodic functions endowed with the usual norm $\| \cdot \|_{L^p}$ and the usual interpretation if $p = \infty$.

We denote by $\mathcal{B}^\alpha_{p,q}$ the (inhomogeneous) Besov space of regularity α and exponents $p, q \in [1, \infty]$ with norm $\| \cdot \|_{\mathcal{B}^\alpha_{p,q}}$ (see Definition A.1). We write $\mathcal{C}^\alpha$ and $\| \cdot \|_{\mathcal{C}^\alpha}$ to denote the space $\mathcal{B}^\alpha_{\infty,\infty}$ and the corresponding norm. Many useful results about Besov spaces that we repeatedly use throughout the article can be found in Appendix A.

For any Banach space $(V, \| \cdot \|_V)$ we denote by $B_V(x_0; \delta)$ the open ball $\{x \in V : \|x - x_0\|_V < \delta\}$ and by $\bar{B}_V(x_0; \delta)$ its closure.

Throughout this article we write C for a positive constant which might change from line to line. In proofs we sometimes write $\lesssim$ instead of $\le C$. We also write $a \vee b$ and $a \wedge b$ to denote the maximum and the minimum of a and b.

In several statements we write ± 1 to signify that the statement holds true for either choice of $+1$ or -1.

2 Preliminaries

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let ξ be a space-time white noise defined over Ω. More precisely, ξ is a family $\{\xi(\phi)\}_{\phi \in L^2((0,\infty)\times\mathbb{T}^2)}$ of centred Gaussian random variables such that

$$\mathbb{E}\xi(\phi)\xi(\psi) = \langle \phi, \psi \rangle_{L^2((0,\infty)\times\mathbb{T}^2)}.$$

A natural filtration $\{\mathcal{F}_t\}_{t\ge 0}$ is given by the usual augmentation (as in [23, Chapter 1.4]) of

$$\tilde{\mathcal{F}}_t = \sigma\left(\{\xi(\phi) : \phi|_{(t,\infty)\times\mathbb{T}^2} \equiv 0\}\right), \quad t \geq 0.$$

We interpret solutions of (1.4) following [6] and [20]. We write $X(\cdot; x)$ for the solution started in x and use the decomposition $X(\cdot; x) = v(\cdot; x) + \varepsilon^{\frac{1}{2}}\<1>(\cdot)$ where $\<1>$ solves the stochastic heat equation

$$\begin{aligned} \left(\partial_t - (\Delta - 1)\right)\<1> &= \sqrt{2}\xi \\ \<1>\big|_{t=0} &= 0. \end{aligned} \tag{2.1}$$

The remainder term v solves

$$\begin{aligned} (\partial_t - \Delta)\, v &= -v^3 + v - \left(3v^2\varepsilon^{\frac{1}{2}}\<1> + 3v\varepsilon\<2> + \varepsilon^{\frac{3}{2}}\<3> - 2\varepsilon^{\frac{1}{2}}\<1>\right) \\ v\big|_{t=0} &= x \end{aligned} \tag{2.2}$$

where $\<2>, \<3>$ are the 2nd and 3rd Wick powers of the solution to the stochastic heat equation $\<1>$. The random distributions $\<2>$ and $\<3>$ can be constructed as limits of $\<1>_N^2 - \mathfrak{R}_N$ and $\<1>_N^3 - 3\mathfrak{R}_N\<1>_N$, where $\<1>_N$ is a spatial Galerkin approximation of $\<1>$, and $\mathfrak{R}_N$ is a renormalisation constant which diverges logarithmically in the regularisation parameter N. The value of $\mathfrak{R}_N$ is given by

$$\mathfrak{R}_N := \lim_{t\to\infty} \mathbb{E}\<1>_N(t,z)^2 = \frac{1}{L^2}\sum_{|k|\leq N} \frac{1}{(2\pi|k|/L)^2 + 1}. \tag{2.3}$$

Note that $\<1>_N(t,z)$ is stationary in the space variable z, hence the expectation is independent of z. We refer the reader to [6, Lemma 3.2], [24, Section 2] for more details on the construction of the Wick powers. We recall that $\<1>, \<2>$ and $\<3>$ can be realised as continuous (in time) processes taking values in $\mathcal{C}^{-\alpha}$ for $\alpha > 0$ and that $\mathbb{P}$-almost surely for every $T > 0$, and $\alpha' > 0$

$$\max\left\{\sup_{t\leq T}\|\<1>(t)\|_{\mathcal{C}^{-\alpha}},\ \sup_{t\leq T}(t\wedge 1)^{\alpha'}\|\<2>(t)\|_{\mathcal{C}^{-\alpha}},\ \sup_{t\leq T}(t\wedge 1)^{2\alpha'}\|\<3>(t)\|_{\mathcal{C}^{-\alpha}}\right\} < \infty. \tag{2.4}$$

The blow-up of $\|\<2>(t)\|_{\mathcal{C}^{-\alpha}}$ and $\|\<3>(t)\|_{\mathcal{C}^{-\alpha}}$ for t close to 0 is due to the fact that we define the stochastic objects $\<2>$ and $\<3>$ with zero initial condition, but we work with a time-independent renormalisation constant $\mathfrak{R}_N$ (see (2.3)). We define the stochastic heat equation with a Laplacian with mass 1 because this allows us to prove exponential moment bounds of $\<1>, \<2>$ and $\<3>$ which hold uniformly in time (see Proposition D.1). Throughout the paper we use $\<n>$ to refer to all the stochastic objects $\<1>, \<2>$ and $\<3>$ simultaneously. In this notation (2.4) turns into

$$\sup_{t\leq T}(t\wedge 1)^{(n-1)\alpha'}\|\<n>(t)\|_{\mathcal{C}^{-\alpha}} < \infty.$$

We fix $\alpha_0 \in (0, \frac{1}{3})$ (to measure the regularity of the initial condition x in $\mathcal{C}^{-\alpha_0}$), $\beta > 0$ (to measure the regularity of v in $\mathcal{C}^{\beta}$) and $\gamma > 0$ (to measure the blow-up of $\|v(t;x)\|_{\mathcal{C}^{\beta}}$ for t close to 0) such that

$$\gamma < \frac{1}{3}, \quad \frac{\alpha_0 + \beta}{2} < \gamma. \tag{2.5}$$

We also assume that $\alpha' > 0$ and $\alpha > 0$ in (2.4) satisfy

$$\alpha' < \gamma, \quad \alpha < \alpha_0, \quad \frac{\alpha + \beta}{2} + 2\gamma < 1. \tag{2.6}$$

In [24, Theorems 3.3 and 3.9]) it was shown that for every $x \in \mathcal{C}^{-\alpha_0}$ there exist a unique solution $v \in C\left((0,\infty); \mathcal{C}^{\beta}\right)$ of (2.2) such that for every $T > 0$

$$\sup_{t \le T} (t \wedge 1)^{\gamma} \|v(t;x)\|_{\mathcal{C}^{\beta}} < \infty.$$

Remark 2.1 In Condition (2.5) β has to be strictly less than $\frac{2}{3}$. This is necessary if one wants to treat all of the terms arising in a fixed point problem for (2.2) with the same norm for v. A simple post-processing of [24, Theorems 3.3 and 3.9] shows that in fact v is continuous in time taking values in $\mathcal{C}^{2-\lambda}$ for any $\lambda > \alpha$.

Equations (2.1), (2.2) suggest that indeed X can be seen as a perturbation of the Allen–Cahn equation (1.1), because the terms $\<1>$, $\<2>$ and $\<3>$ in (2.2) all appear with a positive power of ε. It is important to note that v is much more regular than X. The irregular part of $X(\cdot;x)$ is $\varepsilon^{\frac{1}{2}}\<1>$. Therefore differences of solutions are much more regular than solutions themselves.

We repeatedly work with restarted stochastic terms: we define $\<1>_s$ as the solution of

$$\begin{aligned} \left(\partial_t - (\Delta - 1)\right)\<1>_s &= \sqrt{2}\xi, \qquad t > s, \\ \<1>_s\big|_{t=s} &= 0, \end{aligned}$$

and let $\<2>_s$ and $\<3>_s$ be its Wick powers. By [24, Proposition 2.3] for every $s > 0$, $\<n>_s(s+\cdot)$ are independent of $\mathcal{F}_s$ and equal in law to $\<n>(\cdot)$. For $t \ge s$ we can define a restarted remainder $v_s(t; X(s;x))$ through the identity $X(t;x) = v_s(t; X(s;x)) + \varepsilon^{\frac{1}{2}}\<1>_s(t)$. Rearranging (2.2) and using the pathwise identities in [24, Corollary 2.4] one can see that v_s solves

$$\begin{aligned} (\partial_t - \Delta)\, v_s &= -v_s^3 + v_s - \left(3v_s^2\varepsilon^{\frac{1}{2}}\<1>_s + 3v_s\varepsilon\<2>_s + \varepsilon^{\frac{3}{2}}\<3>_s - 2\varepsilon^{\frac{1}{2}}\<1>_s\right) \\ v_s\big|_{t=s} &= X(s;x). \end{aligned} \tag{2.7}$$

In [24, Theorem 4.2] this is used to prove the Markov property for $X(\cdot;x)$.

3 Main results

In this article we prove the following main theorem.

Theorem 3.1 *For every $\kappa > 0$ there exist $\delta_0, a_0, C > 0$ and $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}\left(\sup_{\|y-x\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \frac{\|X(t;y) - X(t;x)\|_{\mathcal{C}^\beta}}{\|y-x\|_{\mathcal{C}^{-\alpha_0}}} \leq C\mathrm{e}^{-(2-\kappa)t} \textit{ for every } t \geq 1 \right) \geq 1 - \mathrm{e}^{-a_0/\varepsilon}.$$

Proof See Sect. 4.2. □

This theorem is a variant of [17, Corollary 3.1] in space dimension $d = 2$, but in that work the supremum is taken over both x and y inside the probability measure. We also obtain this version of the theorem as a corollary.

Corollary 3.2 *For every $\kappa > 0$ there exist $\delta_0, a_0, C > 0$ and $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\mathbb{P}\left(\sup_{x,y \in \bar{B}_{\mathcal{C}^{-\alpha_0}}(\pm 1; \delta_0)} \frac{\|X(t;y) - X(t;x)\|_{\mathcal{C}^\beta}}{\|y-x\|_{\mathcal{C}^{-\alpha_0}}} \leq C\mathrm{e}^{-(2-\kappa)t} \textit{ for every } t \geq 1 \right) \geq 1 - \mathrm{e}^{-a_0/\varepsilon}.$$

Proof See Sect. 4.2. □

Remark 3.3 The restriction $t \geq 1$ in Theorem 3.1 appears only because we measure $y - x$ in a lower regularity norm than $X(t;y) - X(t;x)$. To prove the theorem we first prove Theorem 4.9 were we assume that $y - x \in \mathcal{C}^\beta$ and in this case we prove a bound which holds for every $t > 0$.

Remark 3.4 Theorem 3.1 is an asymptotic coupling of solutions that start close to the same minimiser. In [17, Proposition 3.4] it was shown that in the 1-dimensional case, solutions which start with initial conditions x and y close to different minimisers also contract exponentially fast, but only after time $T_\varepsilon \propto \mathrm{e}^{[(V(0)-V(\pm 1))+\eta]/\varepsilon}$ for any $\eta > 0$. This is the "typical" time needed for one of the two profiles to jump close to the other minimiser. We expect that Theorem 3.1 and the large deviation theory developed in [14] could be combined to prove a similar result in the case $d = 2$.

As an application of Theorem 3.1 we prove an Eyring–Kramers law for the transition times of X. Before we state our main result in this direction let us briefly introduce some extra notation.

For $\delta \in (0, 1/2)$ and $\alpha > 0$ we define the symmetric subsets $A(\alpha;\delta)$ and $B(\alpha;\delta)$ of $\mathcal{C}^{-\alpha}$ by

$$A(\alpha;\delta) := \left\{ f \in \mathcal{C}^{-\alpha} : \bar{f} \in [-1-\delta, -1+\delta],\ f - \bar{f} \in D_\perp \right\} \tag{3.1}$$

$$B(\alpha;\delta) := \left\{ f \in \mathcal{C}^{-\alpha} : \bar{f} \in [1-\delta, 1+\delta],\ f - \bar{f} \in D_\perp \right\} \tag{3.2}$$

where $D_\perp$ is the closed ball of radius δ in $\mathcal{C}^{-\alpha}$ and $\bar{f} = L^{-2}\langle f, 1\rangle$. For $x \in A(\alpha;\delta)$ we define the transition time

$$\tau_{B(\alpha;\delta)}(X(\cdot;x)) := \inf\{t > 0 : X(t;x) \in B(\alpha;\delta)\}\,.$$

Last, for $k \in \mathbb{Z}^2$ we let

$$\lambda_k := \left(\frac{2\pi|k|}{L}\right)^2 - 1 \quad \text{and} \quad \nu_k := \left(\frac{2\pi|k|}{L}\right)^2 + 2 = \lambda_k + 3. \tag{3.3}$$

The sequences $\{\lambda_k\}_{k\in\mathbb{Z}^2}$ and $\{\nu_k\}_{k\in\mathbb{Z}^2}$ are the eigenvalues of the operators $-\Delta - 1$ and $-\Delta + 2$ endowed with periodic boundary conditions.

With this notation at hand the Eyring–Kramers law can be expressed as follows. Notice that by symmetry the same result holds if we swap the neighbourhoods of -1 and 1 below.

Theorem 3.5 *There exist $\delta_0 > 0$ such that the following holds. For every $\alpha \in (0, \alpha_0)$ and $\delta \in (0, \delta_0)$ there exist $c_+, c_- > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\begin{aligned}
&\sup_{x\in A(\alpha_0;\delta)} \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x)) \\
&\quad\leq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}\left(1 + c_+\sqrt{\varepsilon}\right) \\
&\inf_{x\in A(\alpha_0;\delta)} \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x)) \\
&\quad\geq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}\left(1 - c_-\varepsilon\right).
\end{aligned} \tag{3.4}$$

Proof See Sect. 7.3. □

4 Proof of the exponential loss of memory

In this section we prove the exponential loss of memory. In Sect. 4.1 we present the basic ingredients needed in the proof and in Sect. 4.2 we give the proofs of Theorem 3.1 and Corollary 3.2.

4.1 Methodology

We define two sequences $\{\nu_i(x)\}_{i\geq1}$ and $\{\rho_i(x)\}_{i\geq1}$ of stopping times which partition our time axis and allow us to keep track of the time spent close to and away from the minimisers ±1 (see Fig. 1 for a sketch). On the "good" intervals $[\rho_{i-1}(x), \nu_i(x)]$ we require both the restarted diagrams  $_{\rho_{i-1}(x)}$ to be small and the restarted remainder

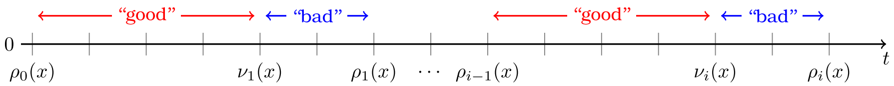

Fig. 1 A partition of the time axis with respect to the times $\nu_i(x)$ and $\rho_i(x)$. The "good" intervals are "typically" much larger than the "bad" intervals

$v_{\rho_{i-1}(x)}$ to be close to ± 1. The "bad" intervals $[\nu_i(x), \rho_i(x)]$ end when $X(\cdot; x)$ re-enters a small neighbourhood of the minimisers. The stopping times $\rho_i(x)$ are defined in terms of the $\mathcal{C}^{-\alpha_0}$ norm for $X(\cdot; x)$, while we define good intervals in terms of the stronger $\mathcal{C}^{\beta}$ topology for $v_{\rho_{i-1}(x)}$. To connect the two, we need to allow for a blow-up close to the starting point of the "good" intervals.

Definition 4.1 For $x \in \mathcal{C}^{-\alpha_0}$ we define the sequence of stopping times $\{\rho_i(x)\}_{i\geq 0}$, $\{\nu_i(x)\}_{i\geq 1}$ recursively by $\rho_0(x) = 0$ and

$$\begin{aligned}\nu_i(x) := \inf\Big\{t > \rho_{i-1}(x) : \min_{x_* \in\{-1,1\}} &((t-\rho_{i-1}(x)) \wedge 1)^{\gamma} \|v_{\rho_{i-1}(x)}(t; X(\rho_i(x); x))\\ &- x_*\|_{\mathcal{C}^{\beta}} \geq \delta_1 \\ &\text{or } ((t-\rho_{i-1}(x)) \wedge 1)^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \boldsymbol{\vee}^{n}_{\rho_{i-1}(x)}(t)\|_{\mathcal{C}^{-\alpha}} \geq \delta_2^n\Big\}\\ \rho_i(x) := \inf\{t > \nu_i(x) : \min_{x_*\in\{-1,1\}} &\|X(t;x) - x_*\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0\}.\end{aligned}$$

We now define the time increments

$$\begin{aligned}\tau_i(x) &= \nu_i(x) - \rho_{i-1}(x)\\ \sigma_i(x) &= \rho_i(x) - \nu_i(x).\end{aligned} \tag{4.1}$$

The process $X(\cdot; x)$ is expected to spend long time intervals close to the minimisers ± 1, which corresponds to large values of $\tau_i(x)$. Large values of $\sigma_i(x)$ are "atypical". This behaviour is established Propositions 6.3 and 6.6.

The following proposition shows contraction on the "good" intervals. We distinguish between the cases (4.2) and (4.3) for $y - x$ that lie in $\mathcal{C}^{\beta}$ and $\mathcal{C}^{-\alpha_0}$ respectively. The Da Prato–Debussche decomposition shows that differences of any two profiles lie in $\mathcal{C}^{\beta}$ for any $t > 0$ but at $t = 0$ they maintain the irregularity of the initial conditions. Hence we only use (4.3) on the first "good" interval.

Proposition 4.2 *For every $\kappa > 0$ there exist $\delta_0, \delta_1, \delta_2 > 0$ and $C > 0$ such that if $\|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$ and $y - x \in \mathcal{C}^{\beta}$, $\|y - x\|_{\mathcal{C}^{\beta}} \leq \delta_0$ then*

$$\|X(t; y) - X(t; x)\|_{\mathcal{C}^{\beta}} \leq C \exp\left\{-\left(2 - \frac{\kappa}{2}\right) t\right\} \|y - x\|_{\mathcal{C}^{\beta}}, \tag{4.2}$$

for every $t \leq \tau_1(x)$ defined with respect to δ_1 and δ_2. If we only assume that $\|y - x\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$ then

$$(t \wedge 1)^{\gamma} \|X(t; y) - X(t; x)\|_{\mathcal{C}^{\beta}} \leq C \exp\left\{-\left(2 - \frac{\kappa}{2}\right) t\right\} \|y - x\|_{\mathcal{C}^{-\alpha_0}}, \tag{4.3}$$

for every $t \le \tau_1(x)$.

Proof See Sect. 5.1. □

Our next aim is to control the growth of the differences on the "bad" intervals in terms of the stochastic objects $\overset{n}{\vee}$. This is done by partitioning the intervals $[\nu_i(x), \rho_i(x)]$ into tiles of length one. To achieve independence we restart the stochastic objects at the starting point of each tile.

Definition 4.3 For $k \ge 0$ and $\rho \ge \nu \ge 0$ let $t_k = \nu + k$. For $k \ge 1$ we define a random variable $L_k(\nu, \rho)$ by

$$L_k(\nu, \rho) := \left(\sup_{t \in [t_{k-1}, t_k \wedge \rho]} (t - t_{k-1})^{(n-1)\alpha'} \| \varepsilon^{\frac{n}{2}} \overset{n}{\vee}_{t_{k-1}}(t) \|_{\mathcal{C}^{-\alpha}} \right)^{\frac{2}{n}}. \quad (4.4)$$

In our analysis we use a second tiling defined by setting $s_k = t_k + \frac{1}{2}$, i.e. the tiles $[t_k, t_{k+1}]$ and $[s_k, s_{k+1}]$ overlap. In order to bound $X(t; y) - X(t; x)$ on a time interval $[t_k, s_k]$ we restart the stochastic objects at s_{k-1} and write $X(t; y) - X(t; x) = v_{s_{k-1}}(t; X(s_{k-1}; y)) - v_{s_{k-1}}(t; X(s_{k-1}; x))$. In Lemma 5.1 we upgrade the strong a priori bound obtained in [24, Proposition 3.7] to get a control on the $\mathcal{C}^\beta$ norm of both remainders. This bound holds uniformly over all possible values of $X(s_{k-1}; y)$ and $X(s_{k-1}; x)$ and while the bound allows for a blow-up for times t close to s_{k-1} it holds uniformly over all times in $[t_k, s_k]$. Ultimately, the bound only depends on $L_k(\nu + \frac{1}{2}, \rho)$ in a polynomial way as shown in Fig. 2. Then we can use the local Lipschitz property of the non-linearity in (2.2) to bound the exponential growth rate of $X(t; y) - X(t; x)$. For the first interval $[t_0, t_1]$ we do not use this trick, because we want to avoid bounds that depend on the realisation of the white noise outside of $[\nu, \rho]$. On this interval, we make use of an a priori assumption that we have some control on $\|X(\nu; y)\|_{\mathcal{C}^{-\alpha_0}}$ and $\|X(\nu; x)\|_{\mathcal{C}^{-\alpha_0}}$.

Proposition 4.4 *Let* $R > 0$. *Then there exists a constant* $C \equiv C(R) > 0$ *such that for every* $\|X(\nu; x)\|_{\mathcal{C}^{-\alpha_0}}, \|X(\nu; y)\|_{\mathcal{C}^{-\alpha_0}} \le R$, $\rho > \nu \ge 0$ *and* $t \in [\nu, \rho]$

$$\|X(t; y) - X(t; x)\|_{\mathcal{C}^\beta} \le C \exp\{L(\nu, \rho; t - \nu)\} \|X(\nu; y) - X(\nu; x)\|_{\mathcal{C}^\beta}, \quad (4.5)$$

where

$$L(\nu, \rho; t - \nu) = \frac{c_0}{2} \sum_{k=1}^{\lfloor t-\nu \rfloor} \sum_{l=0,\frac{1}{2}} (1 \vee L_k(\nu + l, \rho))^{p_0} + L_0(t - \nu), \quad (4.6)$$

for L_k *as in* (4.4), *and for some constants* $p_0 \ge 1$ *and* $c_0 \equiv c_0(R)$, $L_0 \equiv L_0(R) \ge 0$.

Proof See Sect. 5.2. □

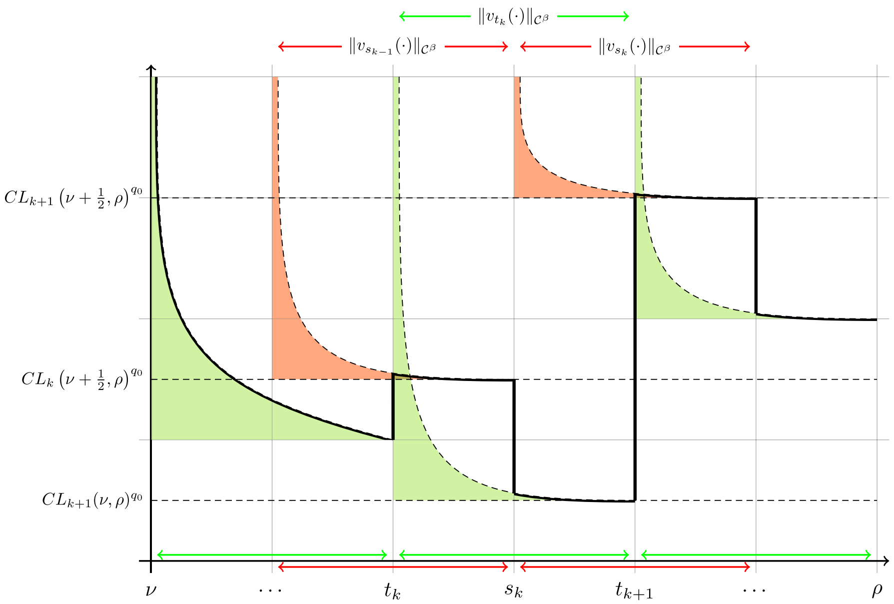

Fig. 2 Bounds on the $\mathcal{C}^\beta$ norm of the restarted remainder v on the overlapping tiles of the partition of $[\nu, \rho]$. On a time interval $[t_k, s_k]$ we restart the stochastic objects at time s_{k-1} and bound $v_{s_{k-1}}$ by a polynomial function of $L_k\left(\nu+\frac{1}{2},\rho\right)$. On a time interval $[s_k, t_{k+1}]$ we restart the stochastic objects at time t_k and bound v_{t_k} by a polynomial function of $L_k(\nu,\rho)$

If we assume that $y - x \in \mathcal{C}^\beta$, combining the estimates in Propositions 4.2 and 4.4 suggest the bound

$$\|X(\rho_N(x); y) - X(\rho_N(x); x)\|_{\mathcal{C}^\beta} \\ \leq \exp\left\{\sum_{i\leq N}\left[-\left(2-\frac{\kappa}{2}\right)\tau_i + L(\nu_i(x), \rho_i(x); \sigma_i(x)) + 2\log C\right]\right\}\|y-x\|_{\mathcal{C}^\beta}, \tag{4.7}$$

for any $N \geq 1$. If we can show that the exponents satisfy

$$\sum_{i\leq N}\left[-\left(2-\frac{\kappa}{2}\right)\tau_i + L(\nu_i(x), \rho_i(x); \sigma_i(x)) + 2\log C\right] \leq -(2-\kappa)\rho_N(x),$$

then (4.7) yields exponential contraction at time $\rho_N(x)$ with rate $2-\kappa$. The difference of the right hand side and the left hand side of the last inequality is given by the random walk $S_N(x)$ in the next definition.

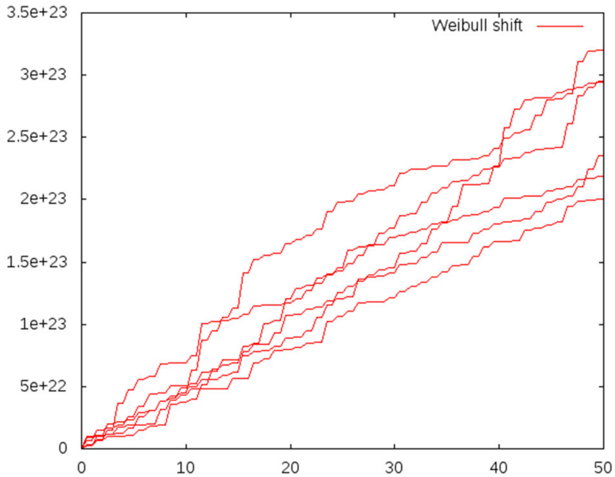

Fig. 3 "Typical" realisations of a random walk $S_N = \sum_{i \le N}(f_i - g_i)$ for $f_i \sim e^{0.5/\varepsilon}\exp(1)$, $g_i \sim e^{0.1/\varepsilon}\text{Weibull}(0.5, 1)$, $N = 50$ and $\varepsilon = 0.01$. The choice of a Weibull distribution here captures the fact that the random variables $L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2 - \kappa)\sigma_i(x) + M_0$ in Definition 4.5 have stretched exponential tails as shown in Proposition 6.7

Definition 4.5 Let $\|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$. We define the random walk $(S_N(x))_{N \ge 1}$ by

$$S_N(x) := \sum_{i \le N} \left[\frac{\kappa}{2}\tau_i(x) - \big(L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2 - \kappa)\sigma_i(x) + M_0\big)\right],$$

where $M_0 = 2\log C$ for $C > 0$ as in Propositions 4.2 and 4.4 .

The next proposition shows that the random walk $S_N(x)$ stays positive for every $N \ge 1$ with overwhelming probability (see Fig. 3 for an illustration). The proof is based on a variant of the classical Cramér–Lundberg model in risk theory (see [8, Chapter 1.2]). In this classical model a random walk $S_N = \sum_{i \le N}(f_i - g_i)$ with i.i.d. exponential random variables f_i and i.i.d. non-negative random variables g_i is considered. The probability for S_N to stay positive for every $N \ge 1$ can be calculated explicitly in terms of the expectations of f_i and g_i using a renewal equation. In our case we use the Markov property and Propositions 6.3 and Proposition 6.7 to compare the random walk $S_N(x)$ in Definition 4.5 to this classical case.

Remark 4.6 If the family $\{L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2-\kappa)\sigma_i(x) + M_0\}_{i \ge 1}$ had exponential moments, a simple exponential Chebyshev argument would imply the following proposition without any reference to the Cramér–Lundberg model. However, by (4.4) and (4.6) one sees that $L(\nu_i(x), \rho_i(x); \sigma_i(x))$ is a polynomial of potentially high degree in the explicit stochastic objects (which are themselves polynomials of the Gaussian noise ξ). Hence, we cannot expect more than stretched exponential moments, and indeed, such bounds are established in Proposition 6.7. In the proof of the next proposition we also use an exponential Chebyshev argument, but only to compare $\frac{\kappa}{2}\tau_i(x)$ with a suitable exponential random variable which does not depend on x.

Proposition 4.7 *For every* $\kappa > 0$ *there exist* $a_0 > 0$ *and* $\varepsilon_0 \in (0,1)$ *such that for every* $\varepsilon \leq \varepsilon_0$

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}(S_N(x) \geq 0 \text{ for every } N \geq 1) \geq 1 - \mathrm{e}^{-a_0/\varepsilon}. \tag{4.8}$$

Proof See Sect. 6.3. □

4.2 Proofs of Theorem 3.1 and Corollary 3.2

We first treat the case where $y - x \in \mathcal{C}^\beta$: let $x \in \mathcal{C}^{-\alpha_0}$ such that $\|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$ and let y be such that $y - x \in \mathcal{C}^\beta$ and $\|y - x\|_{\mathcal{C}^\beta} \leq \delta_0$. We also write $Y(t) = X(t; y) - X(t; x)$. We consider the event

$$\mathcal{S}(x) = \{S_N(x) \geq 0 \text{ for every } N \geq 1\}, \tag{4.9}$$

for $S_N(x)$ as in Definition 4.5.

We first prove the following proposition which provides explicit estimates on the differences at the stopping times $\nu_N(x)$ and $\rho_N(x)$ for every $N \geq 1$ and $\omega \in \mathcal{S}(x)$ by iterating Propositions 4.2 and 4.4. To shorten the notation we drop the explicit dependence on the starting point x in the stopping times ν_N and ρ_N and the random walk S_N. We also drop the dependence on the realisation ω but we assume throughout that $\omega \in \mathcal{S}(x)$.

Proposition 4.8 *For any* $\kappa > 0$ *let* $C > 0$ *be as in Proposition* 4.2*. Then for every* $\omega \in \mathcal{S}(x)$ *and* $N \geq 1$

$$\|Y(\nu_N)\|_{\mathcal{C}^\beta} \leq C \exp\{-S_{N-1}\} \exp\left\{-\frac{\kappa}{2}\tau_N\right\} \exp\{-(2-\kappa)\nu_N\} \|Y(0)\|_{\mathcal{C}^\beta} \tag{4.10}$$

$$\|Y(\rho_N)\|_{\mathcal{C}^\beta} \leq \exp\{-S_N\} \exp\{-(2-\kappa)\rho_N\} \|Y(0)\|_{\mathcal{C}^\beta}. \tag{4.11}$$

Proof We prove our claim by induction on $N \geq 1$, observing that it is obvious for $N = 0$.

To prove (4.10) for $N+1$ we first notice that by the definition of ρ_N we have that $\|X^\varepsilon(\rho_N; x) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$ and since $\omega \in \mathcal{S}(x)$ (4.11) implies that $\|Y(\rho_N)\|_{\mathcal{C}^\beta} \leq \delta_0$. Hence we can use (4.2) to get

$$\|Y(\nu_{N+1})\|_{\mathcal{C}^\beta} \lesssim \exp\left\{-\frac{\kappa}{2}\tau_{N+1}\right\} \exp\{-(2-\kappa)\tau_{N+1}\} \|Y(\rho_N)\|_{\mathcal{C}^\beta}.$$

Combining with the estimate on $\|Y(\rho_N)\|_{\mathcal{C}^\beta}$ the above implies (4.10) for $N+1$.

To prove (4.11) for $N+1$ we first notice that by Proposition 6.2 $\|X(\nu_{N+1}; x)\|_{\mathcal{C}^{-\alpha_0}} \leq 2\delta_1 + 1$. This bound, (4.10) for $N+1$ and the triangle inequality imply that $\|X(\nu_{N+1}; y)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0 + 2\delta_1 + 1$. Hence we can use Proposition 4.4 for $\nu = \nu_{N+1}$, $\rho = \rho_{N+1}$ and $R = \delta_0 + 2\delta_1 + 1$ to obtain

$$\|Y(\rho_{N+1})\|_{\mathcal{C}^\beta} \lesssim \exp\{L(\nu_{N+1}, \rho_{N+1}; \sigma_{N+1})\} \|Y(\nu_{N+1})\|_{\mathcal{C}^\beta}.$$

If we combine with (4.10) for $N+1$ we have that

$$\|Y(\rho_{N+1})\|_{\mathcal{C}^\beta} \leq \exp\{L(\nu_{N+1},\rho_{N+1};\sigma_{N+1})+M_0\}\exp\{-S_N\} \\ \times\exp\left\{-\frac{\kappa}{2}\tau_{N+1}\right\} \\ \exp\{-(2-\kappa)\nu_{N+1}\}\,\|Y(0)\|_{\mathcal{C}^\beta}.$$

We then rearrange the terms to obtain (4.11), which completes the proof. □

We are ready to prove the following version of Theorem 3.1 for sufficiently smooth initial conditions.

Theorem 4.9 *For every $\kappa>0$ there exist $\delta_0, a_0, C>0$ and $\varepsilon_0\in(0,1)$ such that for every $\varepsilon\leq\varepsilon_0$*

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}\leq\delta_0} \\ \mathbb{P}\left(\sup_{\substack{y-x\in\mathcal{C}^\beta\\ \|y-x\|_{\mathcal{C}^\beta}\leq\delta_0}}\frac{\|X(t;y)-X(t;x)\|_{\mathcal{C}^\beta}}{\|y-x\|_{\mathcal{C}^\beta}}\leq C\mathrm{e}^{-(2-\kappa)t}\ \textit{for every } t\geq 0\right) \\ \geq 1-\mathrm{e}^{-a_0/\varepsilon}.$$

Proof Let $\omega\in\mathcal{S}(x)$ as in (4.9). For any $t>0$ there exists $N\equiv N(\omega)\geq 0$ such that $t\in[\rho_N,\nu_{N+1})$ or $t\in[\nu_{N+1},\rho_{N+1})$.

If $t\in[\rho_N,\nu_{N+1})$ then

$$\|X(t;y)-X(t;x)\|_{\mathcal{C}^\beta} \\ \overset{(4.2),(4.11)}{\lesssim} \exp\left\{-\left(2-\frac{\kappa}{2}\right)(t-\rho_N)\right\}\|X(\rho_N;y))-X(\rho_N;x)\|_{\mathcal{C}^\beta} \\ =\exp\left\{-\frac{\kappa}{2}(t-\rho_N)\right\}\exp\{-(2-\kappa)(t-\rho_N)\}\,\|X(\rho_N;y))-X(\rho_N;y)\|_{\mathcal{C}^\beta} \\ \overset{(4.11)}{\lesssim} \exp\{-(2-\kappa)t\}\,\|y-x\|_{\mathcal{C}^\beta}.$$

If $t\in[\nu_{N+1},\rho_{N+1})$ then

$$\|X(t;y)-X(t;x)\|_{\mathcal{C}^\beta} \\ \overset{(4.5)}{\lesssim} \exp\{L(\nu_{N+1},\rho_{N+1};t-\nu_{N+1})\}\|X(\nu_{N+1};y)-X(\nu_{N+1};x)\|_{\mathcal{C}^\beta} \\ =\exp\{L(\nu_{N+1},\rho_{N+1};t-\nu_{N+1})+(2-\kappa)(t-\nu_{N+1})\} \\ \times\exp\{-(2-\kappa)(t-\nu_{N+1})\}\|X(\nu_{N+1};y)-X(\nu_{N+1};x)\|_{\mathcal{C}^\beta} \\ \overset{(4.10),\omega\in\mathcal{S}(x)}{\lesssim} \exp\{-(2-\kappa)t\}\|y-x\|_{\mathcal{C}^\beta}.$$

By Proposition 4.7 there exist $a_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}(\mathcal{S}(x)) \ge 1 - \mathrm{e}^{-a_0/\varepsilon},$$

which completes the proof. □

We are now ready to prove Theorem 3.1 and Corollary 3.2.

Proof of Theorem 3.1 This is a consequence of (4.3), Proposition 6.3 and Theorem 4.9. Let $\delta_1, \delta_2 > 0$ sufficiently small such that $\delta_1 + \delta_2 < \delta_0$ and assume that $\tau_1(x) \ge 1$. By the definition of $\tau_1(x)$

$$\|X(1;x) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \|v(1;x) - (\pm 1)\|_{\mathcal{C}^{\beta}} + \|\varepsilon^{\frac{1}{2}} \mathbf{1}(1)\|_{\mathcal{C}^{-\alpha_0}} < \delta_1 + \delta_2 < \delta_0.$$

If we also choose $\delta_0' < \delta_0$ by (4.3) we have that for every $\|y - x\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0'$

$$\|X(1;y) - X(1;x)\|_{\mathcal{C}^{\beta}} \lesssim \|y - x\|_{\mathcal{C}^{-\alpha_0}}.$$

The probability of the event $\{\tau_1(x) \ge 1\}$ can be estimated from below by Proposition 6.3 uniformly in $\|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0'$. Combining with Theorem 4.9 completes the proof. □

Proof of Corollary 3.2 We only prove the case where initial conditions are close to the minimiser 1. We fix $\delta_0', \delta_1' > 0$ such that $2\delta_0' < \delta_0$ and $\delta_0' + \delta_1' < \delta_1$. By Proposition 6.2 if we chose δ_2 sufficiently small then

- $\sup_{t \le 1} t^{(n-1)\alpha'} \|\mathbf{\Psi}^{n}(t)\|_{\mathcal{C}^{-\alpha}} \le \delta_2 \Rightarrow \sup_{t \le 1} t^{\gamma} \|v(t;y) - 1\|_{\mathcal{C}^{\beta}} \le \delta_1$ uniformly for $\|y - 1\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0'$.

This together with (4.3) implies that for every $x, y \in B_{\mathcal{C}^{-\alpha_0}}(1; \delta_0')$

$$\|X(1;y) - X(1;x)\|_{\mathcal{C}^{\beta}} \lesssim \|y - x\|_{\mathcal{C}^{-\alpha_0}} \lesssim \delta_0'.$$

Let

$$\omega \in \mathcal{S}' := \left\{ \sup_{\|y-1\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0'} \frac{\|X(t;y) - X(t;1)\|_{\mathcal{C}^{\beta}}}{\|y - 1\|_{\mathcal{C}^{-\alpha_0}}} \le C\mathrm{e}^{-(2-\kappa)t} \text{ for every } t \ge 1 \right\},$$

$t \ge 1$ and $y \in B_{\mathcal{C}^{-\alpha_0}}(-1; \delta_0')$. Then

- $\sup_{s \le t \le T} (t-s)^{\gamma} \|v_s(t; X(s;1)) - (\pm 1)\|_{\mathcal{C}^{\beta}} \le \delta_1' \Rightarrow \sup_{s \le t \le T} (t-s)^{\gamma} \| v_s(t; X(s;y)) - (\pm 1)\|_{\mathcal{C}^{\beta}} \le \delta_1$ for $T, s \ge 1$.
- $\|X(t;1) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0' \Rightarrow \|X(t;y) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$.

This implies that if we consider the process $X(t;y)$ for $t \ge 1$, the times $\nu_i(X(1;y))$ and $\rho_i(X(1;y))$ of Definition 4.1 for δ_0, δ_1 and δ_2 can be replaced by the times $\nu_i(X(1;1))$ and $\rho_i(X(1;1))$ for δ_0', δ_1' and the same δ_2. Hence the corresponding random walk $S_N(X(1;y))$ in Definition 4.5 can be replaced by $S_N(X(1;1))$.

We can now repeat the proof of Theorem 4.9 for the difference $X(t; y) - X(t; x)$, $t \geq 1$, step by step, replacing the event in (4.9) by

$$\mathcal{S}' \cap \left\{ \sup_{t \leq 1} t^{(n-1)\alpha'} \|\vee^{n}(t)\|_{\mathcal{C}^{-\alpha}} \leq \delta_2,\ S_N(X(1; 1)) \geq 0 \text{ for every } N \geq 1 \right\}. \quad (4.12)$$

This allows us to prove that

$$\begin{aligned} \|X(t; y) - X(t; x)\|_{\mathcal{C}^\beta} &\leq C\mathrm{e}^{-(2-\kappa)(t-1)} \|X(1; y) - X(1; x)\|_{\mathcal{C}^\beta} \\ &\leq C\mathrm{e}^{-(2-\kappa)t} \|y - x\|_{\mathcal{C}^{-\alpha_0}}, \end{aligned}$$

uniformly in $y, x \in B_{\mathcal{C}^{-\alpha_0}}(1; \delta_0')$.

To estimate the event in (4.12) we use Theorem 3.1 and Propositions 6.1 and 4.7. This completes the proof. □

5 Pathwise estimates on the difference of two profiles

In this section we prove Propositions 4.2 and Propositions 4.4. Our analysis here is pathwise and uses no probabilistic tools.

5.1 Proof of Proposition 4.2

Proof of Proposition 4.2 We only prove (4.3). To prove (4.2) we follow the same strategy as below. However in this case we do not need to encounter the blow-up of $\|Y(t)\|_{\mathcal{C}^\beta}$ close to 0 and hence we omit the proof since it poses no extra difficulties.

Let $Y(t) = X(t; y) - X(t; x)$ and notice that from (2.2) we get

$$(\partial_t - \Delta)Y = -\left(v(\cdot; y)^3 - v(\cdot; x)^3\right) + Y - 3(v(\cdot; y) + v(\cdot; x))\varepsilon^{\frac{1}{2}} \uparrow Y - 3\varepsilon \vee Y.$$

We use the identity $v(\cdot; y) = v(\cdot; x) + Y$ to rewrite this equation in the form

$$\begin{aligned} (\partial_t - (\Delta - 2))\, Y = &-3\left(v(\cdot; x)^2 - 1\right) Y + \texttt{Error}(v(\cdot; x); Y) \\ &-3(Y + 2v(\cdot; x))\varepsilon^{\frac{1}{2}} \uparrow Y - 3\varepsilon \vee Y \end{aligned}$$

where $\texttt{Error}(v(\cdot; x); Y) = -Y^3 - 3v(\cdot; x)Y^2$ collects all the terms which are higher order in Y. Then

$$\begin{aligned} Y(t) = \mathrm{e}^{-2t}\mathrm{e}^{\Delta t} Y(0) + \int_0^t &\mathrm{e}^{-2(t-s)}\mathrm{e}^{\Delta(t-s)} \\ &\Big[-3\left(v(s; x)^2 - 1\right) Y(s) + \texttt{Error}(v(s; x); Y(s)) \\ &- 3(Y(s) + 2v(s; x))\varepsilon^{\frac{1}{2}} \uparrow(s) Y(s) - 3\varepsilon \vee(s) Y(s) \Big]\, \mathrm{d}s. \end{aligned} \quad (5.1)$$

We set

$$\tilde{\kappa} = \sup_{t \le \tau_1(x)} (t \wedge 1)^{2\gamma} \| - 3 \left(v(t;x)^2 - 1 \right) \|_{\mathcal{C}^\beta}.$$

Let $\iota = \inf\{t > 0 : (t \wedge 1)^\gamma \|Y(t)\|_{\mathcal{C}^\beta} > \zeta\}$ for $1 \ge \zeta > \delta_0$ and notice that for $t \le \tau_1(x) \wedge \iota$ using (5.1) we get

$$\begin{aligned} \|Y(t)\|_{\mathcal{C}^\beta} &\overset{(A.6),(A.7),(A.8)}{\le} \mathrm{e}^{-2t} C (t \wedge 1)^{-\frac{\alpha_0+\beta}{2}} \|Y(0)\|_{\mathcal{C}^{-\alpha_0}} \\ &+ \tilde{\kappa} \int_0^t \mathrm{e}^{-2(t-s)} (s \wedge 1)^{-2\gamma} \|Y(s)\|_{\mathcal{C}^\beta} \, \mathrm{d}s \\ &+ \zeta\, C_1 \int_0^t \mathrm{e}^{-2(t-s)} (s \wedge 1)^{-2\gamma} \|Y(s)\|_{\mathcal{C}^\beta} \, \mathrm{d}s \\ &+ \delta_2\, C_2 \int_0^t \mathrm{e}^{-2(t-s)} (t-s)^{-\frac{\alpha+\beta}{2}} (s \wedge 1)^{-\gamma} \|Y(s)\|_{\mathcal{C}^\beta} \, \mathrm{d}s \\ &+ \delta_2\, C_3 \int_0^t \mathrm{e}^{-2(t-s)} (t-s)^{-\frac{\alpha+\beta}{2}} (s \wedge 1)^{-\alpha'} \|Y(s)\|_{\mathcal{C}^\beta} \, \mathrm{d}s, \end{aligned}$$

where we also use that for $s \le t$

$$\|\mathtt{Error}(v(s;x); Y(s))\|_{\mathcal{C}^\beta} \lesssim \zeta s^{-2\gamma} \|Y(s)\|_{\mathcal{C}^\beta}.$$

Choosing $\zeta \le \tilde{\kappa}/C_1$ and $\delta_2 \le \tilde{\kappa}/C_2 \vee C_3$ we have

$$\begin{aligned} \|Y(t)\|_{\mathcal{C}^\beta} \le \mathrm{e}^{-2t} C (t \wedge 1)^{-\frac{\alpha+\beta}{2}} \|Y(0)\|_{\mathcal{C}^{-\alpha_0}} \\ + \tilde{\kappa} \int_0^t \mathrm{e}^{-2(t-s)} (t-s)^{-\frac{\alpha+\beta}{2}} (s \wedge 1)^{-2\gamma} \|Y(s)\|_{\mathcal{C}^\beta} \, \mathrm{d}s. \end{aligned}$$

Then, for $t \le \tau_1(x) \wedge \iota$, by the generalised Gronwall inequality, Lemma B.1, on $f(t) = (t \wedge 1)^\gamma \|Y(t)\|_{\mathcal{C}^\beta}$ there exist $c > 0$ such that

$$(t \wedge 1)^\gamma \|Y(t)\|_{\mathcal{C}^\beta} \le C \exp\left\{ -2t + c\tilde{\kappa}^{\overline{\frac{1}{1-\frac{\alpha+\beta}{2}-3\gamma}}} t + M \right\} \|Y(0)\|_{\mathcal{C}^{-\alpha_0}}.$$

We now fix $\delta_1 > 0$ such that $c\tilde{\kappa}^{\overline{\frac{1}{1-\frac{\alpha+\beta}{2}-3\gamma}}} \le \frac{\kappa}{2}$. This implies that for $t \le \tau_1(x) \wedge \iota$

$$(t \wedge 1)^\gamma \|Y(t)\|_{\mathcal{C}^\beta} \le C \exp\left\{ -\left(2 - \frac{\kappa}{2}\right) t \right\} \|Y(0)\|_{\mathcal{C}^{-\alpha_0}}.$$

Finally choosing δ_0 sufficiently small we furthermore notice that $\tau_1(x) \wedge \iota = \tau_1(x)$, which completes the proof of (4.3). ◻

5.2 Proof of Proposition 4.4

Before we proceed to the proof of Proposition 4.4 we need the following lemma which upgrades the a priori estimates in [24, Proposition 3.7]. Here and below we let $S(t) = \mathrm{e}^{\Delta t}$.

Lemma 5.1 *There exist* $\alpha, \gamma', C > 0$ *and* $p_0 \geq 1$ *such that if* $\sup_{t\leq 1} t^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \overset{n}{\vee}(t)\|_{\mathcal{C}^{-\alpha}} \leq L^n$ *then*

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \sup_{t\leq 1} t^{\gamma'} \|v(t;x)\|_{\mathcal{C}^{\beta}} \leq C(1\vee L)^{p_0}.$$

Proof Throughout this proof we simply write $v(t)$ to denote $v(t;x)$. We first need bounds on $\|v(t)\|_{L^p}$, for p sufficiently large, and $\int_s^t \|\nabla v(r)\|_{L^2}^2\,\mathrm{d}r$. These bounds can be obtained by classical energy estimates. A bound on $\|v(t)\|_{L^p}$ has already been obtained in [24, Proposition 3.7], which states that for every $p \geq 2$ even

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \sup_{t\leq 1} t^{\frac{1}{2}} \|v(t)\|_{L^p} \leq C \left(1 \vee \sup_{t\leq 1} t^{(n-1)\alpha' p_n} \|\varepsilon^{\frac{n}{2}} \overset{n}{\vee}(t)\|_{\mathcal{C}^{-\alpha}}^{p_n} \right), \tag{5.2}$$

for some exponents $p_n \geq 1$. To bound $\int_s^t \|\nabla v(r)\|_{L^2}^2\,\mathrm{d}r$ we need to slightly modify the strategy used in the proof of [24, Proposition 3.7]. In particular, combining [24, Equations (3.13) and (3.22)] and integrating from s to t we obtain the energy inequality,

$$\|v(t)\|_{L^2}^2 - \|v(s)\|_{L^2}^2 + \int_s^t \|\nabla v(r)\|_{L^2}^2\,\mathrm{d}r \leq C \int_s^t \Big(1 + \sum_{n\leq 3} \|\varepsilon^{\frac{n}{2}} \overset{n}{\vee}(r)\|_{\mathcal{C}^{-\alpha}}^{p_n}\Big)\,\mathrm{d}r,$$

which implies that

$$\int_s^t \|\nabla v(r)\|_{L^2}^2\,\mathrm{d}r \leq C \int_s^t \Big(1 + \sum_{n\leq 3} \|\varepsilon^{\frac{n}{2}} \overset{n}{\vee}(r)\|_{\mathcal{C}^{-\alpha}}^{p_n}\Big)\,\mathrm{d}r + \|v(s)\|_{L^2}^2. \tag{5.3}$$

We now upgrade these bounds to bounds on $\|v(t)\|_{\mathcal{C}^{\beta}}$. Using the mild form of (2.2) we have for $1 \geq t > s > 0$

$$\begin{aligned}
\|v(t)\|_{\mathcal{C}^{\beta}} \lesssim{}& \underbrace{\|S(t-s)v(s)\|_{\mathcal{C}^{\beta}}}_{=:I_1} + \underbrace{\int_s^t \|S(t-r)v(r)^3\|_{\mathcal{C}^{\beta}}\,\mathrm{d}r}_{=:I_2} \\
&+ \underbrace{\int_s^t \|S(t-r)\left(v(r)^2\varepsilon^{\frac{1}{2}}\overset{1}{\vee}(r)\right)\|_{\mathcal{C}^{\beta}}\,\mathrm{d}r}_{=:I_3} \\
&+ \underbrace{\int_s^t \|S(t-r)\left(v(r)\varepsilon\overset{2}{\vee}(r)\right)\|_{\mathcal{C}^{\beta}}\,\mathrm{d}r}_{=:I_4} + \underbrace{\int_s^t \|S(t-r)\varepsilon^{\frac{3}{2}}\overset{3}{\vee}(r)\|_{\mathcal{C}^{\beta}}\,\mathrm{d}r}_{=:I_5}
\end{aligned}$$

$$+\underbrace{\int_s^t \|S(t-r)\varepsilon^{\frac{1}{2}}\mathbf{\uparrow}(r)\|_{\mathcal{C}^\beta}\,\mathrm{d}r}_{=:I_6}+\underbrace{\int_s^t \|S(t-r)v(r)\|_{\mathcal{C}^\beta}\,\mathrm{d}r}_{=:I_7}. \tag{5.4}$$

To estimate $\|v(t)\|_{\mathcal{C}^\beta}$ we use the L^p bound (5.2), the energy inequality (5.3) and the embedding $\mathcal{B}^1_{2,\infty}$ to bound the terms appearing on the right hand side of the last inequality as shown below.

We treat each term in (5.4) separately. Below p may change from term to term and α, λ can be taken arbitrarily small. We write p_1 and p_2 for conjugate exponents of p, i.e. $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}$. We also denote by $(1\vee L)^{p_0}$ a polynomial of degree $p_0\geq 1$ in the variable $1\vee L$ where the value of p_0 may change from line to line.

Term I_1:

$$I_1 \overset{(A.5),(A.6)}{\lesssim} (t-s)^{-\frac{\beta+\frac{2}{p}}{2}}\|v(s)\|_{L^p} \overset{(5.2)}{\lesssim} (t-s)^{-\frac{\beta+\frac{2}{p}}{2}} s^{-\frac{1}{2}}(1\vee L)^{p_0}.$$

Term I_2:

$$\begin{aligned} I_2 &\overset{(A.5),(A.6)}{\lesssim} \int_s^t (t-r)^{-\frac{\beta+\frac{2}{p}}{2}}\|v(r)^3\|_{L^p}\,\mathrm{d}r \overset{(5.2)}{\lesssim} (1\vee L)^{p_0}\int_s^t (t-r)^{-\frac{\beta+\frac{2}{p}}{2}} r^{-\frac{3}{2}}\,\mathrm{d}r \\ &\lesssim (1\vee L)^{p_0} s^{-\frac{3}{2}}\int_s^t (t-r)^{-\frac{\beta+\frac{2}{p}}{2}}\,\mathrm{d}r. \end{aligned}$$

Term I_3:

$$\begin{aligned} I_3 &\overset{(A.5),(A.6),(A.8),\lambda>0}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}}\|v(r)^2\|_{\mathcal{B}^{\alpha+\lambda}_{p,\infty}}\|\varepsilon^{\frac{1}{2}}\mathbf{\uparrow}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \\ &\overset{(A.9)}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}}\|v(r)\|_{L^{p_1}}\|v(r)\|_{\mathcal{B}^{\alpha+\lambda}_{p_2,\infty}}\|\varepsilon^{\frac{1}{2}}\mathbf{\uparrow}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \\ &\overset{(A.5),(5.2)}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}} r^{-\frac{1}{2}}\|v(r)\|_{\mathcal{B}^{\alpha+\lambda+1-\frac{2}{p_2}}_{2,\infty}}\|\varepsilon^{\frac{1}{2}}\mathbf{\uparrow}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \\ &\overset{(5.2),\,\frac{2}{p_2}=\alpha+\lambda}{\lesssim} (1\vee L)^{p_0} s^{-\frac{1}{2}}\int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}}\|v(r)\|_{\mathcal{B}^1_{2,\infty}}\,\mathrm{d}r \\ &\overset{\text{Cauchy–Schwarz}}{\lesssim} (1\vee L)^{p_0} s^{-\frac{1}{2}}\left(\int_s^t (t-r)^{-\left(2\alpha+\lambda+\frac{2}{p}\right)}\,\mathrm{d}r\right)^{\frac{1}{2}}\left(\int_s^t \|v(r)\|^2_{\mathcal{B}^1_{2,\infty}}\,\mathrm{d}r\right)^{\frac{1}{2}}. \end{aligned}$$

Term I_4:

$$\begin{aligned} I_4 &\overset{(A.5),(A.6),(A.8),\lambda>0}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}}\|v(r)\|_{\mathcal{B}^{\alpha+\lambda}_{p,\infty}}\|\varepsilon\mathbf{V}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \\ &\overset{(A.5)}{\lesssim} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}}\|v(r)\|_{\mathcal{B}^{\alpha+\lambda+1-\frac{2}{p}}_{2,\infty}}\|\varepsilon\mathbf{V}(r)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}r \end{aligned}$$

$$
\overset{\frac{2}{p}=\alpha+\lambda}{\lesssim} (1 \vee L)^{p_0} s^{-\alpha'} \int_s^t (t-r)^{-\frac{2\alpha+\lambda+\frac{2}{p}}{2}} \|v(r)\|_{\mathcal{B}^1_{2,\infty}} \, \mathrm{d}r
$$

$$
\overset{\text{Cauchy–Schwarz}}{\lesssim} (1 \vee L)^{p_0} s^{-\alpha'} \left(\int_s^t (t-r)^{-\left(2\alpha+\lambda+\frac{2}{p}\right)} \, \mathrm{d}r \right)^{\frac{1}{2}}
$$

$$
\left(\int_s^t \|v(r)\|^2_{\mathcal{B}^1_{2,\infty}} \, \mathrm{d}r \right)^{\frac{1}{2}}.
$$

Term I_5:

$$
I_5 \overset{(A.6)}{\lesssim} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}} \|\varepsilon^{\frac{3}{2}} \mathbf{\Psi}(r)\|_{\mathcal{C}^{-\alpha}} \, \mathrm{d}r \lesssim (1 \vee L)^{p_0} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}} r^{-2\alpha'} \, \mathrm{d}r
$$

$$
\lesssim (1 \vee L)^{p_0} s^{-2\alpha'} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}} \, \mathrm{d}r.
$$

Term I_6:

$$
I_6 \overset{(A.6)}{\lesssim} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}} \|\varepsilon^{\frac{1}{2}} \mathbf{\Upsilon}(r)\|_{\mathcal{C}^{-\alpha}} \, \mathrm{d}r \lesssim (1 \vee L)^{p_0} \int_s^t (t-r)^{-\frac{\alpha+\beta}{2}} \, \mathrm{d}r.
$$

Term I_7:

$$
I_7 \overset{(A.5),(A.6)}{\lesssim} \int_s^t (t-r)^{-\frac{\beta+\frac{2}{p}}{2}} \|v(r)\|_{L^p} \, \mathrm{d}r \overset{(5.2)}{\lesssim} (1 \vee L)^{p_0} \int_s^t (t-r)^{-\frac{\beta+\frac{2}{p}}{2}} r^{-\frac{1}{2}} \, \mathrm{d}r
$$

$$
\lesssim (1 \vee L)^{p_0} s^{-\frac{1}{2}} \int_s^t (t-r)^{-\frac{\beta+\frac{2}{p}}{2}} \, \mathrm{d}r.
$$

Using Proposition A.9, (5.2) and (5.3) we notice that

$$
\left(\int_s^t \|v(r)\|^2_{\mathcal{B}^1_{2,\infty}} \, \mathrm{d}r \right)^{\frac{1}{2}} \lesssim \left(\int_s^t \|\nabla v(r)\|^2_{L^2} \, \mathrm{d}r \right)^{\frac{1}{2}} + \left(\int_s^t \|v(r)\|^2_{L^2} \, \mathrm{d}r \right)^{\frac{1}{2}}
$$

$$
\lesssim (1 \vee L)^{p_0} s^{-\frac{1}{2}}.
$$

Combining the above and choosing $s = t/2$ we find $\gamma' > 0$ such that

$$
t^{\gamma'} \|v(t)\|_{\mathcal{C}^\beta} \lesssim (1 \vee L)^{p_0}
$$

which completes the proof. □

Proof of Proposition 4.4 We denote by $(1 \vee L)^{p_0}$ a polynomial of degree $p_0 \geq 1$ in the variable $1 \vee L$ where the value of p_0 may change from line to line.

For $k \geq 0$ recall that $t_k = \nu + k$ and $s_k = t_k + \frac{1}{2}$. As before, we write $Y(t) = X(t; y) - X(t; x)$.

Let $t \in (t_k, s_k]$, $k \geq 1$. We restart the stochastic terms at time s_{k-1} and write $Y(t) = v_{s_{k-1}}(t; \tilde{y}) - v_{s_{k-1}}(t; \tilde{x})$ where for simplicity $\tilde{y} = X(s_{k-1}; y)$ and $\tilde{x} = X(s_{k-1}; x)$. Together with (2.7), this implies that

$$
\begin{aligned}
(\partial_t - \Delta) Y = &- \left(v_{s_{k-1}}(\cdot; \tilde{y})^3 - v_{s_{k-1}}(\cdot; \tilde{x})^3 \right) \\
&+ Y - 3(v_{s_{k-1}}(\cdot; \tilde{y}) + v_{s_{k-1}}(\cdot; \tilde{x})) \varepsilon^{\frac{1}{2}} \Psi_{s_{k-1}} Y - 3\varepsilon \curlyvee_{s_{k-1}} Y.
\end{aligned}
$$

Using the mild form of the above equation, now starting at $t_k = s_{k-1} + \frac{1}{2}$, we get

$$
\begin{aligned}
\|Y(t)\|_{\mathcal{C}^\beta} \overset{(A.6),(A.7),(A.8)}{\lesssim} &\; \|Y(t_k)\|_{\mathcal{C}^\beta} + \int_{t_k}^{t} \|v_{s_{k-1}}(r; \tilde{y})^3 - v_{s_{k-1}}(r; \tilde{x})^3\|_{\mathcal{C}^\beta} \, \mathrm{d}r \\
&+ \int_{t_k}^{t} (t-r)^{-\frac{\alpha+\beta}{2}} \|v_{s_{k-1}}(r; \tilde{y})^2 - v_{s_{k-1}}(r; \tilde{x})^2\|_{\mathcal{C}^\beta} \|\varepsilon^{\frac{1}{2}} \Psi_{s_{k-1}}(r)\|_{\mathcal{C}^{-\alpha}} \, \mathrm{d}r \\
&+ \int_{t_k}^{t} (t-r)^{-\frac{\alpha+\beta}{2}} \|Y(r)\|_{\mathcal{C}^\beta} \|\varepsilon \curlyvee_{s_{k-1}}(r)\|_{\mathcal{C}^{-\alpha}} \, \mathrm{d}r + \int_{t_k}^{t} \|Y(r)\|_{\mathcal{C}^\beta} \, \mathrm{d}r.
\end{aligned}
$$

By Lemma 5.1 there exist $\gamma' > 0$ such that

$$
\sup_{x \in \mathcal{C}^{-\alpha_0}} \sup_{t \in [s_{k-1}, s_k]} (t - s_{k-1})^{\gamma'} \|v_{s_{k-1}}(t; x)\|_{\mathcal{C}^\beta} \lesssim \left(1 \vee L_k \left(\nu + \frac{1}{2}, \rho \right) \right)^{p_0}.
$$

Combining the above we get

$$
\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \|Y(t_k)\|_{\mathcal{C}^\beta} + \left(1 \vee L_k \left(\nu + \frac{1}{2}, \rho \right) \right)^{p_0} \int_{t_k}^{t} (t-r)^{-\frac{\alpha+\beta}{2}} \|Y(r)\|_{\mathcal{C}^\beta} \, \mathrm{d}r.
$$

By the generalised Gronwall inequality, Lemma B.1, there exists $c_0 > 0$ such that

$$
\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \exp\left\{ c_0 \left(1 \vee L_k \left(\nu + \frac{1}{2}, \rho \right) \right)^{p_0} (t - s) \right\} \|Y(t_k)\|_{\mathcal{C}^\beta}. \tag{5.5}
$$

Following the same strategy we prove that for $t \in [s_k, t_{k+1}]$, $k \geq 1$,

$$
\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \exp\left\{ c_0 \left(1 \vee L_{k+1}(\nu, \rho) \right)^{p_0} (t - s) \right\} \|Y(s_k)\|_{\mathcal{C}^\beta}. \tag{5.6}
$$

Finally, we also need a bound for $t \in [t_0, t_1]$. To obtain an estimate which does not depend on any information before time t_0 we use local solution theory. By [24, Theorem 3.3] there exists $t_* \in (t_0, t_1)$ such that

$$
\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \sup_{r \in [t_0, t_*]} (r - t_0)^{\gamma} \|v_{t_0}(r; x)\|_{\mathcal{C}^\beta} \leq 1
$$

and furthermore we can take

$$t_* = \left(\frac{1}{C(R \vee L_1(\nu, \rho))} \right)^{p_0}.$$

By Lemma 5.1 we also have that

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \sup_{r \in (t_0, t_1]} (r - t_0)^{\gamma'} \| v_{t_0}(r; x) \|_{\mathcal{C}^\beta} \lesssim (1 \vee L_1(\nu, \rho))^{p_0}.$$

Combining these two bounds we get

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} \sup_{r \in [t_0, t_1]} (r - t_0)^{\gamma} \| v_{t_0}(r; x) \|_{\mathcal{C}^\beta} \lesssim (1 \vee L_1(\nu, \rho))^{p_0}, \tag{5.7}$$

where the implicit constant depends on R. Note that $\gamma < \frac{1}{3}$ whereas γ' is much larger. We write $Y(t) = v_{t_0}(t; y) - v_{t_0}(t; x)$ and use the mild form starting at t_0. We then use (5.7) to bound $\|v_{t_0}(t; \cdot)\|_{\mathcal{C}^\beta}$ on $[t_0, t_1]$ which implies the estimate

$$\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \|Y(t_0)\|_{\mathcal{C}^\beta} + (1 \vee L_1(\nu, \rho))^{p_0} \int_{t_0}^{t} (t - r)^{-\frac{\alpha+\beta}{2}} (r - t_0)^{-2\gamma} \|Y(r)\|_{\mathcal{C}^\beta} \, \mathrm{d}r.$$

The extra term $(r - t_0)^{-2\gamma}$ in the last inequality appears because of the blow-up of $v_{t_0}(t; \cdot)$ and $\vee^{n}_{t_0}(t)$ for t close to t_0. By the generalised Gronwall inequality, Lemma B.1, we obtain that

$$\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \exp\left\{ c_0 \, (1 \vee L_1(\nu, \rho))^{p_0} \, (t - s) \right\} \|Y(s)\|_{\mathcal{C}^\beta}. \tag{5.8}$$

For arbitrary $t \in [\nu, \rho]$ we glue together (5.5), (5.6) and (5.8) to get

$$\|Y(t)\|_{\mathcal{C}^\beta} \lesssim \exp\left\{ \frac{c_0}{2} \sum_{k=1}^{\lfloor t-\nu \rfloor} \sum_{l=0,\frac{1}{2}} (1 \vee L_k(\nu + l, \rho))^{p_0} + L_0(t - \nu) \right\} \|Y(\nu)\|_{\mathcal{C}^\beta},$$

for some $L_0 > 0$ which collects the implicit constants in the inequalities. □

6 Random walk estimates

In this section we prove Proposition 4.7 based mainly on probabilistic arguments. In Sects. 6.1 and 6.2 we provide estimates on $\frac{\kappa}{2}\tau_i(x)$ and $L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2 - \kappa)\sigma_i(x) + M_0$ from Definition 4.5. In Sect. 6.3 we use these estimates to prove Proposition 4.7.

6.1 Estimates on the exit times

Proposition 6.1 *Let* $\delta > 0$ *and* $\tau_{\text{tree}} = \inf\{t > 0 : (t \wedge 1)^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \overset{n}{\vee}(t)\|_{\mathcal{C}^{-\alpha}} \geq \delta^n\}$. *Then there exist* $a_0 > 0$ *and* $\varepsilon_0 \in (0, 1)$ *such that for every* $\varepsilon \leq \varepsilon_0$

$$\mathbb{P}\left(\tau_{\text{tree}} \leq \mathrm{e}^{3a_0/\varepsilon}\right) \leq \mathrm{e}^{-3a_0/\varepsilon}.$$

Proof First notice that for $N \geq 1$

$$\begin{aligned}\mathbb{P}(\tau_{\text{tree}} \leq N) &\leq \sum_{k=0}^{N-1} \mathbb{P}(\tau_{\text{tree}} \in (k, k+1)) \\ &\leq \sum_{k=0}^{N-1} \mathbb{P}\left(\sup_{t\in(k,k+1]} (t \wedge 1)^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \overset{n}{\vee}(t)\|_{\mathcal{C}^{-\alpha}} \geq \delta^n\right).\end{aligned}$$

By Proposition D.1 and the exponential Chebyshev inequality there exists $a_0 > 0$ such that for every $k \geq 0$

$$\mathbb{P}\left(\sup_{t\in(k,k+1]} (t \wedge 1)^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \overset{n}{\vee}(t)\|_{\mathcal{C}^{-\alpha}} \geq \delta^n\right) \leq \mathrm{e}^{-6a_0/\varepsilon}.$$

Hence

$$\mathbb{P}(\tau_{\text{tree}} \leq N) \leq N\mathrm{e}^{-6a_0/\varepsilon}$$

and choosing $N = \mathrm{e}^{3a_0/\varepsilon}$ completes the proof. □

Proposition 6.2 *For* $\delta_1 > 0$ *sufficiently small there exist* $\delta_0, \delta_2 > 0$ *such that if*

$$\sup_{t\leq T} (t \wedge 1)^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \overset{n}{\vee}(t)\|_{\mathcal{C}^{-\alpha}} < \delta_2^n, \tag{6.1}$$

then for every $\|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0$

$$\sup_{t\leq T} (t \wedge 1)^{\gamma} \|v(t; x) - (\pm 1)\|_{\mathcal{C}^{\beta}} < \delta_1$$

and

$$\sup_{t\leq T} \|X(t; x) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq 2\delta_1.$$

Proof Let $u(t) = v(t; x) - (\pm 1)$. By an exact expansion of $-v^3 + v$ around ± 1 we have that

$$(\partial_t - (\Delta - 2))u = \mathtt{Error}(u) - \left(3v^2 \varepsilon^{\frac{1}{2}} \uparrow + 3v\varepsilon \vee + \varepsilon^{\frac{3}{2}} \Psi\right) + 2\varepsilon^{\frac{1}{2}} \uparrow, \tag{6.2}$$

where $\texttt{Error}(u) = -u^3 \pm 3u^2$ and $\|\texttt{Error}(u)\|_{\mathcal{C}^\beta} \lesssim \|u\|^3_{\mathcal{C}^\beta} + \|u\|^2_{\mathcal{C}^\beta}$. Let $T > 0$ and $\iota = \inf\{t > 0 : (t \wedge 1)^\gamma \|u(t)\|_{\mathcal{C}^\beta} \geq \delta_1\}$ for some $\delta_1 > 0$ which we fix below. Using the mild form of (6.2) we get

$$
\begin{aligned}
(t \wedge 1)^\gamma \|u(t)\|_{\mathcal{C}^\beta} \overset{(A.6),(A.7),(A.8)}{\lesssim} & \; \mathrm{e}^{-2t} \|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \\
& + \int_0^t \mathrm{e}^{-2(t-s)} \left(\|u(s)\|^3_{\mathcal{C}^\beta} + \|u(s)\|^2_{\mathcal{C}^\beta} \right) \mathrm{d}s \\
& + \int_0^t \mathrm{e}^{-2(t-s)} (t-s)^{-\frac{\alpha+\beta}{2}} \Big(\|v(s)\|^2_{\mathcal{C}^\beta} \|\varepsilon^{\frac{1}{2}} \uparrow(s)\|_{\mathcal{C}^{-\alpha}} \\
& + \|v(s)\|_{\mathcal{C}^\beta} \|\varepsilon \curlyvee(s)\|_{\mathcal{C}^{-\alpha}} \\
& + \|\varepsilon^{\frac{3}{2}} \Psi(s)\|_{\mathcal{C}^{-\alpha}} + \|\varepsilon^{\frac{1}{2}} \uparrow(s)\|_{\mathcal{C}^{-\alpha}} \Big) \mathrm{d}s.
\end{aligned}
$$

If we furthermore assume (6.1) for $t \leq T \wedge \iota$ we obtain that

$$
\begin{aligned}
& (t \wedge 1)^\gamma \|u(t)\|_{\mathcal{C}^\beta} \\
& \lesssim \delta_0 \mathrm{e}^{-2t} + \delta_1^3 \int_0^t \mathrm{e}^{-2(t-s)} (s \wedge 1)^{-3\gamma} \, \mathrm{d}s + \delta_1^2 \int_0^t \mathrm{e}^{-2(t-s)} (s \wedge 1)^{-2\gamma} \, \mathrm{d}s \\
& \quad + \delta_2 \int_0^t \mathrm{e}^{-2(t-s)} (t-s)^{-\frac{\alpha+\beta}{2}} \Big((s \wedge 1)^{-2\gamma} + (s \wedge 1)^{-\gamma} (s \wedge 1)^{-\alpha'} \\
& \quad + (s \wedge 1)^{-2\alpha'} + 1 \Big) \, \mathrm{d}s.
\end{aligned}
$$

Then Lemma B.2 implies the bound

$$
\sup_{t \leq T \wedge \iota} (t \wedge 1)^\gamma \|u(t)\|_{\mathcal{C}^\beta} \lesssim \delta_0 + \delta_1^3 + \delta_1^2 + \delta_2.
$$

Choosing $\delta_0 < \frac{\delta_1}{4C}, \delta_1 < \frac{1}{4C}$ and $\delta_2 < \frac{\delta_1}{4C}$ this implies that $\sup_{t \leq T \wedge \iota} (t \wedge 1)^\gamma \|u(t)\|_{\mathcal{C}^\beta} < \delta_1$ which in turn implies that $\iota \leq T$ and proves the first bound.

To prove the second bound we notice that for every $t \leq T$

$$
\|X(t; x) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \|u(t)\|_{\mathcal{C}^{-\alpha_0}} + \|\uparrow(t)\|_{\mathcal{C}^{-\alpha_0}} \leq \|u(t)\|_{\mathcal{C}^{-\alpha_0}} + \delta_2.
$$

Hence it suffices to prove that $\sup_{t \leq T} \|u(t)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_1$. Using again the mild form of (6.2) we get

$$
\begin{aligned}
\|u(t)\|_{\mathcal{C}^{-\alpha_0}} \overset{(A.6),(A.2),(A.7),(A.8)}{\lesssim} & \; \mathrm{e}^{-2t} \|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \\
& + \int_0^t \mathrm{e}^{-2(t-s)} \left(\|u(s)\|^3_{\mathcal{C}^\beta} + \|u(s)\|^2_{\mathcal{C}^\beta} \right) \mathrm{d}s \\
& + \int_0^t \mathrm{e}^{-2(t-s)} \Big(\|v(s)\|^2_{\mathcal{C}^\beta} \|\varepsilon^{\frac{1}{2}} \uparrow(s)\|_{\mathcal{C}^{-\alpha}} + \|v(s)\|_{\mathcal{C}^\beta} \|\varepsilon \curlyvee(s)\|_{\mathcal{C}^{-\alpha}}
\end{aligned}
$$

$$+ \|\varepsilon^{\frac{3}{2}} \Psi(s)\|_{\mathcal{C}^{-\alpha}} + \|\varepsilon^{\frac{1}{2}} \mathsf{I}(s)\|_{\mathcal{C}^{-\alpha}} \Big) \, \mathrm{d}s,$$

for every $t \le T$. Plugging in (6.1) and the bound $\sup_{t \le T} (t \wedge 1)^{\gamma} \|u(t)\|_{\mathcal{C}^{\beta}} \le \delta_1$ the last inequality implies

$$\begin{aligned} \|u(t)\|_{\mathcal{C}^{\beta}} &\lesssim \delta_0 \mathrm{e}^{-2t} + \delta_1^3 \int_0^t \mathrm{e}^{-2(t-s)} (s \wedge 1)^{-3\gamma} \, \mathrm{d}s + \delta_1^2 \int_0^t \mathrm{e}^{-2(t-s)} (s \wedge 1)^{-2\gamma} \, \mathrm{d}s \\ &\quad + \delta_2 \int_0^t \mathrm{e}^{-2(t-s)} \Big((s \wedge 1)^{-2\gamma} + (s \wedge 1)^{-\gamma} (s \wedge 1)^{-\alpha'} \\ &\quad + (s \wedge 1)^{-2\alpha'} + 1 \Big) \, \mathrm{d}s. \end{aligned}$$

Using again Lemma B.2 we obtain that $\sup_{t \le T} \|u(t)\|_{\mathcal{C}^{-\alpha_0}} < \delta_1$, which completes the proof.

Proposition 6.3 *For every $\kappa > 0$ and $\delta_1 > 0$ sufficiently small there exist $a_0, \delta_0, \delta_2 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$*

$$\sup_{\|x - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P} \left(\frac{\kappa}{2} \tau_1(x) \le \mathrm{e}^{2a_0/\varepsilon} \right) \le \mathrm{e}^{-3a_0/\varepsilon},$$

where $\tau_1(x)$ is given by (4.1).

Proof We first notice that there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$

$$\mathbb{P} \left(\frac{\kappa}{2} \tau_1(x) \le \mathrm{e}^{2a_0/\varepsilon} \right) \le \mathbb{P} \left(\tau_1(x) \le \mathrm{e}^{3a_0/\varepsilon} \right).$$

The last probability can be estimated by Propositions 6.2 and 6.1 for $\delta = \delta_2$. □

6.2 Estimates on the entry times

In this section we use large deviation theory and in particular a lower bound of the form

$$\begin{aligned} &\liminf_{\varepsilon \searrow 0} \log \varepsilon \inf_{x \in \aleph} \mathbb{P}(X(\cdot; x) \in \mathcal{A}(T; x)) \\ &\quad \ge - \sup_{x \in \aleph} \inf_{\substack{f \in \mathcal{A}(T;x) \\ f(0) = x}} \left\{ \underbrace{\frac{1}{4} \int_0^T \|(\partial_t - \Delta) f(t) + f(t)^3 - f(t)\|_{L^2}^2 \, \mathrm{d}t}_{=: I(f)} \right\}, \end{aligned} \tag{6.3}$$

where $\aleph$ is a compact subset of $\mathcal{C}^{-\alpha}$ and $\mathcal{A}(T; x) \subset \{f : (0, T) \to \mathcal{C}^{-\alpha}\}$ is open. This bound is an immediate consequence of [14] and the remark that the solution map

$$\mathcal{C}^{-\alpha_0} \times \left(\mathcal{C}^{-\alpha}\right)^3 \ni \left(x, \left\{\varepsilon^{\frac{n}{2}} \vee^{n}\right\}_{n\leq 3}\right) \mapsto X(\cdot;x) \in \mathcal{C}^{-\alpha}$$

is jointly continuous on compact time intervals. This estimate implies a “nice” lower bound for the probabilities $\mathbb{P}(X(\cdot;x) \in \mathcal{A}(T;x))$ if a suitable path $f \in \mathcal{A}(T;x)$ is chosen.

In the next proposition we use the lower bound (6.3) for suitable sets $\aleph$ and $\mathcal{A}(T;x)$ to estimate probabilities of the entry time of X in a neighbourhood of ± 1. In particular, we construct a path $f(\cdot;x)$ and obtain bounds on $I(f(\cdot;x))$ uniformly in $x \in \aleph$. This construction is similar in spirit to the one used in [9, proof of Theorem 9.1] for the 1-dimensional analogue of (1.4), although here we consider a slightly different event and the initial conditions are not regular functions since they lie in $\mathcal{C}^{-\alpha_0}$. For this reason we make use of the smoothing properties of the deterministic flow given by Proposition C.3.

Proposition 6.4 *Let* $\delta_0 > 0$ *and* $\sigma(x) = \inf\left\{t > 0 : \min_{x_* \in\{-1,1\}} \|X(t;x) - x_*\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0\right\}$. *For every* $R, b > 0$ *there exists* $T_0 > 0$ *such that*

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\leq R} \mathbb{P}(\sigma(x) \geq T_0) \leq 1 - \mathrm{e}^{-b/\varepsilon}.$$

Proof First notice that

$$\mathbb{P}(\sigma(x) \leq T_0) = \mathbb{P}(\underbrace{\|X(T_*;x) - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} < \delta_0 \text{ for some } T_* \leq T_0}_{=:\mathcal{A}(T_0;x)}).$$

By the large deviation estimate (6.3) it suffices to bound

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\leq R} \inf_{\substack{f\in\mathcal{A}(T_0;x)\\ f(0)=x}} I(f(\cdot;x)).$$

We construct a suitable path $g \in \mathcal{A}(T_0;x)$ and we use the trivial inequality

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\leq R} \inf_{\substack{f\in\mathcal{A}(T_0;x)\\ f(0)=x}} I(f(\cdot;x)) \leq \sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\leq R} I(g(\cdot;x)).$$

We now give the construction of g which involves five different steps. In Steps 1, 3 and 5, g follows the deterministic flow. The contribution of these steps to the energy functional I is zero. In Steps 2 and 4, g is constructed by linear interpolation. The contribution of these steps is estimated by Lemma 6.5. Below we write $X_{det}(\cdot;x)$ to denote the solution of (1.1) with initial condition x. We also pass through the space $\mathcal{B}^1_{2,2}$ to use convergence results for $X_{det}(\cdot;x)$ which hold in this topology (see Propositions C.1 and C.2).

Step 1 (Smoothness of initial condition via the deterministic flow):

Let $\tau_1 = 1$. For $t \in [0,\tau_1]$ we set $g(t;x) = X_{det}(t;x)$. By Proposition C.3 there exist $C \equiv C(r) > 0$ and $\lambda > 0$ such that

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\leq R} \|X_{det}(1;x)\|_{\mathcal{C}^{2+\lambda}} \leq C.$$

Step 2 (Reach points that lead to a stationary solution):

By Step 1 $g(\tau_1;x) \in B_{\mathcal{C}^{2+\lambda}}(0;C)$ uniformly for $\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R$. Let $\delta > 0$ to be fixed below. By compactness there exists $\{y_i\}_{1\leq i\leq N}$ such that $B_{\mathcal{C}^{2+\lambda}}(0;C)$ is covered by $\cup_{1\leq i\leq N} B_{\mathcal{B}^1_{2,2}}(y_i;\delta)$. Here we use that $\mathcal{C}^{2+\lambda}$ is compactly embedded in $\mathcal{B}^1_{2,2}$ (see Proposition A.8).

Without loss of generality we assume that $\{y_i\}_{1\leq i\leq N}$ is such that $y_i \in \mathcal{C}^\infty$ and $X_{det}(t;y_i)$ converges to a stationary solution $-1, 0, 1$ in $\mathcal{B}^1_{2,2}$. Otherwise we choose $\{y_i^*\}_{1\leq i\leq N} \in B_{\mathcal{B}^1_{2,2}}(y_i;\delta)$ such that $y_i^* \in \mathcal{C}^\infty$ and relabel them. This is possible because of Proposition C.1.

Let $\tau_2 = \tau_1 + \tau$, for $\tau > 0$ which we fix below. For $t \in [\tau_1, \tau_2]$ we set $g(t;x) = g(\tau_1;x) + \frac{t-\tau_1}{\tau_2-\tau_1}(y_i - g(\tau_1;x))$, where y_i is such that $g(\tau_1;x) \in B_{\mathcal{B}^1_{2,2}}(y_i;\delta)$.

Step 3 (Follow the deterministic flow to reach a stationary solution):

Let T_i^* be such that $X_{det}(t;y_i) \in B_{\mathcal{B}^1_{2,2}}(x_*;\delta)$ for every $t \geq T_i^*$, where $x_* \in \{-1,0,1\}$ is the limit of $X_{det}(t;y_i)$ in $\mathcal{B}^1_{2,2}$, for $\{y_i\}_{1\leq i\leq N}$ as in Step 2. Let $\tau_3 = \tau_2 + \max_{1\leq i\leq N} T_i^* \vee 1$. For $t \in [\tau_2,\tau_3]$ we set $g(t;x) = X_{det}(t-\tau_2;y_i)$. If $X_{det}(\tau_3 - \tau_2;y_i) \in B_{\mathcal{B}^1_{2,2}}(\pm 1;\delta)$ we stop here. Otherwise $X_{det}(\tau_3-\tau_2;y_i) \in B_{\mathcal{B}^1_{2,2}}(0;\delta) \cap B_{\mathcal{C}^{2+\lambda}}(0;C)$ (here we use again Proposition C.3 to ensure that $X_{det}(\tau_3-\tau_2;y_i) \in B_{\mathcal{C}^{2+\lambda}}(0;C)$) and we proceed to Steps 4 and 5.

Step 4 (If an unstable solution is reached move to a point nearby which leads to a stable solution):

We choose $y_0 \in B_{\mathcal{B}^1_{2,2}}(0;\delta)$ such that $y_0 \in \mathcal{C}^\infty$ and $X_{det}(t;y_0)$ converges to either 1 or -1 in $\mathcal{B}^1_{2,2}$. This is possible because of Proposition C.2.

Let $\tau_4 = \tau_3 + \tau$ for $\tau > 0$ as in Step 2 which we fix below. For $t \in [\tau_3,\tau_4]$ we set $g(t;x) = g(\tau_3;x) + \frac{t-\tau_3}{\tau_4-\tau_3}(y_0 - g(\tau_3;x))$.

Step 5 (Follow the deterministic flow again to finally reach a stable solution):

Let T_0^* be such that $X_{det}(t;y_0) \in B_{\mathcal{B}^1_{2,2}}(\pm 1;\delta)$ for every $t \geq T_0^*$, where y_0 is as in Step 4. Let $\tau_5 = \tau_4 + T_0^* \vee 1$. For $t \in [\tau_4,\tau_5]$ we set $g(t;x) = X_{det}(t-\tau_4;y_0)$.

For the path $g(\cdot;x)$ constructed above we see that after time $t \geq \tau_5$, $g(t;x) \in B_{\mathcal{B}^1_{2,2}}(\pm 1;\delta)$ for every $\|x\|-\mathcal{C}^{-\alpha_0} \leq R$. This implies that $\|g(t;x)-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} < C\delta$ since by (A.5), $\mathcal{B}^1_{2,2} \subset \mathcal{C}^{-\alpha_0}$. We now choose $\delta > 0$ such that $C\delta < \delta_0$ and let $T_0 = \tau_5 + 1$. Then $g \in \mathcal{A}(T_0;x)$.

To bound $I(g(\cdot;x))$ we split our time interval based on the construction of g i.e. $I_k = [\tau_{k-1},\tau_k]$ for $k = 1,\ldots,4$ and $I_5 = [\tau_5, T_0]$. We first notice that for $k = 1,3,5$

$$\frac{1}{4}\int_{I_k} \|(\partial_t - \Delta)g(t;x) + g(t;x)^3 - g(t;x)\|^2_{L^2}\,\mathrm{d}t = 0$$

since on these intervals we follow the deterministic flow. For the remaining two intervals, i.e. $k = 2,4$, we first notice that by construction $\|g(\tau_{k-1};x)\|_{\mathcal{C}^{2+\lambda}}, \|g(\tau_k;x)\|_{\mathcal{C}^{2+\lambda}} \leq C$. By (A.3), $\mathcal{C}^{2+\lambda} \subset \mathcal{B}^2_{\infty,2}$ for every $\lambda > 0$, hence we also have that $\|g(\tau_{k-1};x)\|_{\mathcal{B}^2_{\infty,2}}, \|g(\tau_k;x)\|_{\mathcal{B}^2_{\infty,2}} \leq C$. We can now choose τ in Steps 2 and 4 according to Lemma 6.5, which implies that

$$\frac{1}{4}\int_{I_k} \|(\partial_t - \Delta)g(t;x) + g(t;x)^3 - g(t;x)\|_{L^2}^2 \,\mathrm{d}t \le C\delta.$$

Hence

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\le R} \frac{1}{4}\int_0^{T_0} \|(\partial_t - \Delta)g(t;x) + g(t;x)^3 - g(t;x))\|_{L^2}^2 \,\mathrm{d}t \le C\delta.$$

For $b > 0$ we choose δ even smaller to ensure that $C\delta < b$. Finally, by (6.3) there exists $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\inf_{\|x\|_{\mathcal{C}^{-\alpha_0}}\le R} \mathbb{P}(\sigma(x) \le T_0) \ge \mathrm{e}^{-b/\varepsilon},$$

which completes the proof. □

Lemma 6.5 ([9, Lemma 9.2]) *Let $f(t) = x + \frac{t}{\tau}(y - x)$ such that $\|x\|_{\mathcal{B}^2_{2,2}}, \|y\|_{\mathcal{B}^2_{2,2}} \le R$ and $\|x - y\|_{L^2} \le \delta$. There exist $\tau > 0$ and $C \equiv C(R)$ such that*

$$\frac{1}{4}\int_0^{\tau} \|(\partial_t - \Delta)f(t) + f(t)^3 - f(t)\|_{L^2}^2 \,\mathrm{d}t \le C\delta.$$

Proof We first notice that $\partial_t f(t) = \frac{1}{\tau}(y - x)$, hence $\|\partial_t f(t)\|_{L^2} \le \frac{1}{\tau}\delta$. For the term $\Delta f(t)$ we have

$$\|\Delta f(t)\|_{L^2} \le \|\Delta x\|_{L^2} + \|\Delta y\|_{L^2} \lesssim \|x\|_{\mathcal{B}^2_{2,2}} + \|y\|_{\mathcal{B}^2_{2,2}} \lesssim R,$$

where we use that the Besov space $\mathcal{B}^2_{2,2}$ is equivalent with the Sobolev space H^1. This is immediate from Definition A.1 for $p = q = 2$ if we write $\|f * \eta_k\|_{L^2}$ using Plancherel's identity. For the term $f(t)^3 - f(t)$ we have

$$\begin{aligned}\|f(t)^3 - f(t)\|_{L^2} &\lesssim \|f(t)\|_{L^6}^3 + \|f(t)\|_{L^2} \overset{(A.4)}{\lesssim} \|f(t)\|^3_{\mathcal{B}^0_{6,1}} + \|f(t)\|_{\mathcal{B}^0_{2,1}} \\ &\overset{(A.5),\lambda>0}{\lesssim} \|f(t)\|^3_{\mathcal{B}^{\frac{2}{3}+\lambda}_{2,2}} + \|f(t)\|_{\mathcal{B}^{\lambda}_{2,2}} \\ &\overset{(A.2),\lambda<\frac{1}{3}}{\lesssim} \|f(t)\|^3_{\mathcal{B}^2_{2,2}} + \|f(t)\|_{\mathcal{B}^2_{2,2}}.\end{aligned}$$

Hence for $C \equiv C(R)$

$$\frac{1}{2}\int_0^{\tau} \|(\partial_t - \Delta)f(t) + f(t)^3 - f(t)\|_{L^2}^2 \,\mathrm{d}t \le \frac{1}{\tau}\delta^2 + C\tau.$$

Choosing $\tau = \delta$ completes the proof. □

In the next proposition we estimate the tails of the entry time of X in a neighbourhood of ± 1 uniformly in the initial condition x. This is achieved by Proposition 6.4 and the Markov property combined with [24, Corollary 3.10] which implies that after time $t = 1$ the process $X(\cdot; x)$ enters a compact subset of the state space with positive probability uniformly in x.

Proposition 6.6 *Let $\delta_0 > 0$ and $\sigma(x) = \inf\big\{t > 0 : \min_{x_* \in \{-1,1\}} \|X(t;x) - x_*\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0\big\}$. For every $b > 0$ there exist $T_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\sigma(x) \geq m T_0) \leq \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m ,$$

for every $m \geq 1$.

Proof By [24, Corollary 3.10] we know that $\sup_{x \in \mathcal{C}^{-\alpha_0}} \sup_{t \in (0,1]} t^{\frac{p}{2}} \mathbb{E}\|X(t;x)\|_{L^p}^p < \infty$, for every $p \geq 2$, and the bound is uniform in $\varepsilon \in (0,1]$ since it only depends polynomially on $\sqrt{\varepsilon}$. Hence by a simple application of Markov's inequality there exist $R_0 > 0$ such that

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \sup_{\varepsilon \in (0,1]} \mathbb{P}(\|X(1;x)\|_{\mathcal{C}^{-\alpha}} > R_0) \leq \frac{1}{2}. \tag{6.4}$$

By Proposition 6.4 for every $b > 0$ there exists $T_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R_0} \mathbb{P}(\sigma(x) \geq T_0) \leq 1 - \mathrm{e}^{-b/\varepsilon}. \tag{6.5}$$

Then for every $x \in \mathcal{C}^{-\alpha_0}$ and $\varepsilon \leq \varepsilon_0$

$$\begin{aligned} \mathbb{P}(\sigma(x) \geq T_0 + 1) &\leq \mathbb{E}\left(\mathbf{1}_{\{\|X(1;x)\|_{\mathcal{C}^{-\alpha_0}} \leq R_0\}} \mathbb{P}(\sigma(X(1;x)) \geq T_0)\right) \\ &\quad + \mathbb{P}(\|X(1;x)\|_{\mathcal{C}^{-\alpha_0}} > R_0) \\ &\overset{(6.4),(6.5)}{\leq} 1 - \frac{1}{2}\mathrm{e}^{-b/\varepsilon}. \end{aligned} \tag{6.6}$$

Using the Markov property successively implies for every $m \geq 1$ and $x \in \mathcal{C}^{-\alpha_0}$

$$\mathbb{P}(\sigma(x) \geq m(T_0 + 1)) \leq \sup_{y \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\sigma(y) \geq (T_0 + 1))\, \mathbb{P}(\sigma(x) \geq (m-1)(T_0 + 1)). \tag{6.7}$$

Combining (6.6) and (6.7) we obtain that

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\sigma(x) \geq m(T_0 + 1)) \leq \left(1 - \frac{1}{2}\mathrm{e}^{-b/\varepsilon}\right)^m .$$

The last inequality completes the proof if we relabel b and T_0. □

Proposition 6.7 *Let $\delta_0 > 0$, $\nu_1(x), \rho_1(x)$ as in Definition 4.1, $\sigma_1(x)$ as in (4.1) and $L(\nu_1(x), \rho_1(x); \sigma_1(x))$ as in (4.6). For every $\kappa, M_0, b > 0$ there exist $T_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$*

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}\left([L(\nu_1(x), \rho_1(x); \sigma_1(x)) + (2-\kappa)\sigma_1(x) + M_0]^{\frac{1}{p_0}} \ge mT_0 \right)$$
$$\le \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m,$$

for every $m \ge 1$ and $p_0 \ge 1$ as in (4.6).

Proof We first condition on $\nu_1(x)$ to obtain the bound

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}\left([L(\nu_1(x), \rho_1(x); \sigma_1(x)) + (2-\kappa)\sigma_1(x) + M_0]^{\frac{1}{p_0}} \ge mT_0 \right)$$
$$\le \sup_{x \in \mathcal{C}^{-\alpha_0}} \underbrace{\mathbb{P}\left([L(0, \sigma(x); \sigma(x)) + (2-\kappa)\sigma(x) + M_0]^{\frac{1}{p_0}} \ge mT_0 \right)}_{=:\mathbb{P}\left(g(\sigma(x))^{\frac{1}{p_0}} \ge mT_0\right)},$$

where $\sigma(x) = \inf\left\{t > 0 : \min_{x_* \in \{-1,1\}} \|X(t;x) - x_*\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0\right\}$. Let $T_0 \ge 1$ to be fixed below and notice that for any $T_1 > 0$

$$\mathbb{P}\left(g(\sigma(x))^{\frac{1}{p_0}} \ge mT_0\right) \le \mathbb{P}\left(g(\sigma(x))^{\frac{1}{p_0}} \ge mT_0,\ \sigma(x) \le mT_1\right) + \mathbb{P}(\sigma(x) \ge mT_1)$$
$$\le \mathbb{P}\left(\sum_{k=1}^{\lfloor mT_1 \rfloor} \sum_{l=0,\frac{1}{2}} L_k(l, mT_1) \ge m(T_0 - C) \right)$$
$$+ \mathbb{P}(\sigma(x) \ge mT_1),$$

for some $C > 0$, where in the second inequality we use convexity of the mapping $g \mapsto g^{\frac{1}{p_0}}$ and the fact that $L_k(l, \sigma)$ is increasing in σ by Definition 4.3. By Proposition 6.6 we can choose $T_1 > 0$ and $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\sigma(x) \ge mT_1) \le \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m.$$

We also notice that

$$\mathbb{P}\left(\sum_{k=1}^{\lfloor mT_1 \rfloor} \sum_{l=0,\frac{1}{2}} L_k(l, mT_1) \ge m(T_0 - C) \right)$$
$$\le \sum_{l=0,\frac{1}{2}} \mathbb{P}\left(\sum_{k=1}^{\lfloor mT_1 \rfloor} L_k(l, l+k) \ge m\left(\frac{T_0 - C}{2}\right) \right)$$
$$\le \sum_{l=0,\frac{1}{2}} \exp\left\{-cm\left(\frac{T_0 - C}{2\varepsilon}\right)\right\} \left(\mathbb{E}\mathrm{e}^{cL_1(l,1)/\varepsilon}\right)^{mT_1},$$

where in the first inequality we use that $L_k(l, mT_1) \leq L_k(l, l+k)$, for every $1 \leq k \leq \lfloor mT_1 \rfloor$, and in the second we use an exponential Chebyshev inequality, independence and equality in law of the $L_k(l, l+k)$'s. For any $T > 0$ we choose $c \equiv c(n) > 0$ according to Proposition D.1, T_0 sufficiently large and $\varepsilon_0 \in (0, 1)$ sufficiently small such that for every $\varepsilon \leq \varepsilon_0$

$$\sum_{l=0,\frac{1}{2}} \exp\left\{-cm\left(\frac{T_0 - C}{2\varepsilon}\right)\right\}\left(\mathbb{E}\mathrm{e}^{cL_1(l,1)/\varepsilon}\right)^{mT_1} \leq \mathrm{e}^{-mT/\varepsilon}.$$

Combining all the previous inequalities imply that

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}\left([L(\nu_1(x), \rho_1(x); \sigma_1(x)) + (2-\kappa)\sigma_1(x) + M_0]^{\frac{1}{p_0}} \geq mT_0\right)$$
$$\leq \mathrm{e}^{-mT/\varepsilon} + \left(1 - \mathrm{e}^{-b/\varepsilon}\right)^m .$$

This completes the proof if we relabel b since T is arbitrary. □

6.3 Proof of Proposition 4.7

In this section we set

$$f_i(x) := \frac{\kappa}{2}\tau_i(x)$$
$$g_i(x) := L(\nu_i(x), \rho_i(x); \sigma_i(x)) + (2-\kappa)\sigma_i(x) + M_0.$$

In this notation the random walk $S_N(x)$ in Definition 4.5 is given by $\sum_{i\leq N}(f_i(x) - g_i(x))$.

To prove Proposition 4.7 we first consider a sequence of i.i.d. random variables $\{\tilde{f}_i\}_{i\geq 1}$ such that $\tilde{f}_1 \sim \exp(1)$. We furthermore assume that the family $\{\tilde{f}_i\}_{i\geq 1}$ is independent from both $\{f_i(x)\}_{i\geq 1}$ and $\{g_i(x)\}_{i\geq 1}$. For $\lambda > 0$ which we fix later on, we set

$$\tilde{S}_N(x) := \lambda \sum_{i\leq N} \tilde{f}_i - \sum_{i\leq N} g_i(x).$$

In the proof of Proposition 4.7 below we compare the random walk $S_N(x)$ with $\tilde{S}_N(x)$. The idea is that $\sum_{i\leq N} f_i(x)$ behaves like $\lambda \sum_{i\leq N} \tilde{f}_i$ for suitable $\lambda > 0$.

In the next proposition we estimate the new random walk $\tilde{S}_N(x)$ using stochastic dominance. In particular we assume that the family of random variables $\{g_i(x)\}_{i\geq 1}$ is stochastically dominated by a family of i.i.d. random variables $\{\tilde{g}_i\}_{i\geq 1}$ which does not depend on x and obtain a lower bound on $\mathbb{P}(-\tilde{S}_N(x) \leq u$ for every $N \geq 1)$.

From now on we denote by μ_Z the law of a random variable Z.

Proposition 6.8 *Assume that there exists a family of i.i.d. random variables* $\{\tilde{g}_i\}_{i\geq 1}$*, independent from both* $\{g_i(x)\}_{i\geq 1}$ *and* $\{\tilde{f}_i\}_{i\geq 1}$*, such that*

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}(g_i(x) \ge g) \le \mathbb{P}(\tilde{g}_i \ge g),$$

for every $g \ge 0$. *Let* $\tilde{S}_N = \lambda \sum_{i \le N} \tilde{f}_i - \sum_{i \le N} \tilde{g}_i$. *Then*

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}(-\tilde{S}_N(x) \le u \textit{ for every } N \ge 1) \ge \mathbb{P}(-\tilde{S}_N \le u \textit{ for every } N \ge 1).$$

Proof Let

$$\begin{aligned} G_N(x,u) &= \mathbb{P}(-\tilde{S}_M(x) \le u \text{ for every } N \ge M \ge 1) \\ G_N(u) &= \mathbb{P}(-\tilde{S}_M \le u \text{ for every } N \ge M \ge 1). \end{aligned}$$

We first prove that for every $N \ge 1$ and every x

$$G_N(x,u) \ge G_N(u). \tag{6.8}$$

For $N = 1$ we have that

$$\begin{aligned} G_1(x,u) = \mathbb{P}(-\lambda \tilde{f}_1 + g_1(x) \le u) &= \int_0^\infty \mathbb{P}(g_1(x) \le u + \lambda f)\, \mu_{\tilde{f}_1}(\mathrm{d}f) \\ &\ge \int_0^\infty \mathbb{P}(\tilde{g}_1 \le u + \lambda f)\, \mu_{\tilde{f}_1}(\mathrm{d}f) = \mathbb{P}(-\lambda \tilde{f}_1 + \tilde{g}_1 \le u) = G_1(u). \end{aligned}$$

Let us assume that (6.8) holds for N. Let $\partial B_0 = \{y : \|y - (\pm 1)\|_{\mathcal{C}^{-\alpha_0}} = \delta_0\}$. Conditioning on $\left(\tilde{f}_1, g_1(x), X(\nu_2(x); x)\right)$ and using independence of $\tilde{f}_1$ from the joint law of $(g_1(x), X(\nu_2(x); x))$ we notice that

$$\begin{aligned} &G_{N+1}(x,u) \\ &\quad = \int_0^\infty \int_{[0,u+\lambda f] \times \partial B_0} G_N(y, u + \lambda f - g)\, \mu_{(g_1(x), X(\nu_2(x);x))}(\mathrm{d}g, \mathrm{d}y)\, \mu_{\tilde{f}_1}(\mathrm{d}f) \\ &\quad \overset{(6.8)}{\ge} \int_0^\infty \int_{[0,u+\lambda f] \times \partial B_0} G_N(u + \lambda f - g)\, \mu_{(g_1(x), X(\nu_2(x);x))}(\mathrm{d}g, \mathrm{d}y)\, \mu_{\tilde{f}_1}(\mathrm{d}f) \\ &\quad = \int_0^\infty \int_{[0,u+\lambda f]} G_N(u + \lambda f - g)\, \mu_{g_1(x)}(\mathrm{d}g)\, \mu_{\tilde{f}_1}(\mathrm{d}f). \end{aligned} \tag{6.9}$$

In the last equality above we use that $G_N(u + \lambda f - g)$ does not depend on y, hence we can drop the integral with respect to y. Let

$$H(g) = \mathbf{1}_{\{g \le u + \lambda f\}} G_N(u + \lambda f - g).$$

Then for fixed $u, f \ge 0$, H is decreasing with respect to g. By Lemma E.1

$$\int_{[0,u+\lambda f]} G_N(u+\lambda f-g)\,\mu_{g_1(x)}(\mathrm{d}g) = \int H(g)\mu_{g_1(x)}(\mathrm{d}g) \geq \int H(g)\mu_{\tilde{g}_1}(\mathrm{d}g)$$
$$= \int_{[0,u+\lambda f]} G_N(u+\lambda f-g)\,\mu_{\tilde{g}_1}(\mathrm{d}g).$$

Integrating the last inequality with respect to f with $\mu_{\tilde{f}_1}$ and combining with (6.9) we obtain

$$G_{N+1}(x,u) \geq \int_0^\infty \int_{[0,u+\lambda f]} G_N(u+\lambda f-g)\,\mu_{\tilde{g}_1}(\mathrm{d}g)\,\mu_{\tilde{f}_1}(\mathrm{d}f) = G_{N+1}(u),$$

which proves (6.8). If we now take $N \to \infty$ in (6.8) we get for arbitrary x

$$G(x,u) \geq G(u),$$

which completes the proof. □

In the next proposition we prove existence of a family of random variables $\{\tilde{g}_i\}_{i\geq 1}$ that satisfy the assumption of Proposition 6.8 and estimate their first moment.

Proposition 6.9 *There exists a family of i.i.d. random variables $\{\tilde{g}_i\}_{i\geq 1}$, independent from both $\{g_i(x)\}_{i\geq 1}$ and $\{\tilde{f}_i\}_{i\geq 1}$, such that*

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}\leq \delta_0} \mathbb{P}(g_i(x)\geq g) \leq \mathbb{P}(\tilde{g}_i \geq g),$$

and furthermore for every $b>0$ there exist $\varepsilon_0 \in (0,1)$ and $C>0$ such that for every $\varepsilon \leq \varepsilon_0$

$$\mathbb{E}\tilde{g}_1 \leq C\mathrm{e}^{b/\varepsilon}.$$

Proof We first notice that by the Markov property

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}\leq \delta_0} \mathbb{P}(g_i(x)\geq g) \leq \sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}}\leq \delta_0} \mathbb{P}(g_1(x)\geq g).$$

Let $F(g)$ be the right continuous version of the increasing function $1-\sup_{x\in\mathcal{C}^{-\alpha_0}} \mathbb{P}(g_1(x)\geq g)$. We consider a family of i.i.d. random variables such $\{\tilde{g}_i\}_{i\geq 1}$ independent from both $\{g_i(x)\}_{i\geq 1}$ and $\{\tilde{f}_i\}_{i\geq 1}$ such that $\mathbb{P}(\tilde{g}_i\leq g)=F(g)$. To estimate $\mathbb{E}\tilde{g}_1$ let $c_\varepsilon>0$ to be fixed below. We notice that

$$\mathbb{E}\tilde{g}_1 \leq \sup_{g\geq 0} g\mathrm{e}^{-c_\varepsilon g^{\frac{1}{p_0}}}\,\mathbb{E}\exp\left\{c_\varepsilon \tilde{g}_1^{\frac{1}{p_0}}\right\} \leq \left(\frac{p_0\mathrm{e}^{-1}}{c_\varepsilon}\right)^{p_0} \mathbb{E}\exp\left\{c_\varepsilon \tilde{g}_1^{\frac{1}{p_0}}\right\}. \tag{6.10}$$

For $b>0$ we choose $T_0>0$ and $\varepsilon_0\in(0,1)$ as in Proposition 6.7. Then for every $\varepsilon\leq\varepsilon_0$

$$\mathbb{E}\exp\left\{c_\varepsilon \tilde{g}_1^{\frac{1}{p_0}}\right\} = 1+\int_0^\infty c_\varepsilon \mathrm{e}^{c_\varepsilon g}\mathbb{P}\left(\tilde{g}_1^{\frac{1}{p_0}}\geq g\right)\mathrm{d}g$$

$$
\begin{aligned}
&\le 1 + \sum_{m\ge 0} \mathbb{P}\left(\tilde{g}_1^{\frac{1}{p_0}} \ge mT_0\right) \int_{mT_0}^{(m+1)T_0} c_\varepsilon e^{c_\varepsilon g}\, dg \\
&= 1 + \sum_{m\ge 0} \sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}\left(g_1(x)^{\frac{1}{p_0}} \ge mT_0\right) \int_{mT_0}^{(m+1)T_0} c_\varepsilon e^{c_\varepsilon g}\, dg \\
&\le 1 + e^{c_\varepsilon T_0} \sum_{m\ge 0} e^{mc_\varepsilon T_0} \left(1 - e^{-b/\varepsilon}\right)^m,
\end{aligned}
$$

where in the last inequality we use Proposition 6.7 to estimate $\mathbb{P}\left(g_1(x)^{\frac{1}{p_0}} \ge mT_0\right)$. We now choose $c_\varepsilon > 0$ such that $c_\varepsilon T_0 = \log\left(1 + e^{-b/\varepsilon}\right)$. Then

$$
\begin{aligned}
\mathbb{E}\exp\left\{c_\varepsilon \tilde{g}_1^{\frac{1}{p_0}}\right\} &\le 1 + \left(1 + e^{-b/\varepsilon}\right) \sum_{m\ge 0} \left(1 + e^{-b/\varepsilon}\right)^m \left(1 - e^{-b/\varepsilon}\right)^m \\
&\le 1 + 2\sum_{m\ge 0} \left(1 - e^{-2b/\varepsilon}\right)^m \\
&= 1 + 2e^{2b/\varepsilon}.
\end{aligned}
$$

Finally, by (6.10) we obtain that

$$
\mathbb{E}\tilde{g}_1 \le \left(\frac{p_0 e^{-1} T_0}{\log\left(1 + e^{-b/\varepsilon}\right)}\right)^{p_0} \left(1 + 2e^{2b/\varepsilon}\right),
$$

which completes the proof if we relabel b. □

Remark 6.10 In the proof of Proposition 6.9 we use stretched exponential moments of $\tilde{g}_1$, although we only need 1st moments (see Lemma 6.12 below). This simplifies our calculations.

From now on we let $\tilde{S}_N = \lambda \sum_{i\le N} \tilde{f}_i - \sum_{i\le N} \tilde{g}_i$ for $\{\tilde{g}_i\}_{i\ge 1}$ as in Proposition 6.9.

In the next proposition we explicitly compute the probability $\mathbb{P}(-\tilde{S}_N \le 0 \text{ for every } N \ge 1)$. The proof is essentially the same as the classical Cramér–Lundberg estimate (see [8, Chapter 1.2]). We present it here for the reader's convenience.

Proposition 6.11 *For the random walk $\tilde{S}_N$ the following estimate holds,*

$$
\mathbb{P}(-\tilde{S}_N \le 0 \textit{ for every } N \ge 1) = 1 - \frac{1}{\lambda}\mathbb{E}\tilde{g}_1.
$$

Proof Let $G(u) = \mathbb{P}(-\tilde{S}_N \le u \text{ for every } N \ge 1)$. Conditioning on $(\tilde{f}_1, \tilde{g}_1)$ and using independence we notice that

$$
G(u) = \mathbb{P}\left(-\lambda \sum_{i=2}^{N} \tilde{f}_i + \sum_{i=2}^{N} \tilde{g}_i \le u + \lambda \tilde{f}_1 - \tilde{g}_1 \text{ for every } N \ge 2,\ -\lambda \tilde{f}_1 + \tilde{g}_1 \le u\right)
$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^{u+\lambda f} G(u+\lambda f - g)\, \mu_{\tilde{g}_1}(\mathrm{d}g)\, \mu_{\tilde{f}_1}(\mathrm{d}f) \\
&= \frac{1}{\lambda} \mathrm{e}^{u/\lambda} \int_u^\infty \mathrm{e}^{-\bar{f}/\lambda} \int_0^{\bar{f}} G(\bar{f} - g)\, \mu_{\tilde{g}_1}(\mathrm{d}g)\, \mathrm{d}\bar{f}, \qquad (6.11)
\end{aligned}
$$

where in the last equality we use that $\tilde{f}_1 \sim \exp(1)$ and we also make the change of variables $\bar{f} = u + \lambda f$. This implies that $G(u)$ is differentiable with respect to u and in particular

$$
\partial_{\bar{u}} G(\bar{u}) = \frac{1}{\lambda} G(\bar{u}) - \frac{1}{\lambda} \int_0^{\bar{u}} G(\bar{u} - g)\, \mu_{\tilde{g}_1}(\mathrm{d}g).
$$

Integrating the last equation form 0 to u we obtain that

$$
G(u) = G(x,0) + \frac{1}{\lambda} \int_0^u G(u - \bar{u})\, \mathrm{d}\bar{u} - \frac{1}{\lambda} \int_0^u \int_0^{\bar{u}} G(\bar{u} - g)\, \mu_{\tilde{g}_1}(\mathrm{d}g)\, \mathrm{d}\bar{u}. \quad (6.12)
$$

Let $F(g) := \mu_{\tilde{g}_1}([0,g])$. A simple integration by parts implies

$$
\begin{aligned}
&\int_0^u \int_0^{\bar{u}} G(\bar{u} - g)\, \mu_{\tilde{g}_1}(\mathrm{d}g)\, \mathrm{d}\bar{u} \\
&\quad = \int_0^u \left([G(\bar{u} - g)]_{g=0}^{\bar{u}} + \int_0^{\bar{u}} \partial_g G(\bar{u} - g) F(g)\, \mathrm{d}g \right) \mathrm{d}\bar{u} \\
&\quad = \int_0^u G(0) F(\bar{u})\, \mathrm{d}\bar{u} + \int_0^u \int_g^u \partial_g G(\bar{u} - g)\, \mathrm{d}\bar{u}\, F(g)\, \mathrm{d}g \\
&\quad = \int_0^u G(0) F(\bar{u})\, \mathrm{d}\bar{u} - \int_0^u [-G(\bar{u} - g)]_g^u\, F(g)\, \mathrm{d}g \\
&\quad = \int_0^u G(u - g) F(g)\, \mathrm{d}g. \qquad (6.13)
\end{aligned}
$$

Combining (6.12) and (6.13) we get

$$
G(u) = G(0) + \frac{1}{\lambda} \int_0^u G(u - \bar{u})\, \mathrm{d}\bar{u} - \frac{1}{\lambda} \int_0^u G(u - \bar{u}) F(\bar{u})\, \mathrm{d}\bar{u}.
$$

By taking $u \to \infty$ in the last equation and using the dominated convergence theorem and the law of large numbers we finally obtain

$$
1 = G(0) + \frac{1}{\lambda} \mathbb{E} \tilde{g}_1,
$$

which completes the proof. □

Combining Propositions 6.8, 6.9 and 6.11 we obtain the following lemma.

Lemma 6.12 *For any $b > 0$ there exist $\varepsilon_0 \in (0,1)$ and $C > 0$ such that for every $\varepsilon \le \varepsilon_0$*

$$\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}(-\tilde{S}_N(x) \le 0 \textit{ for every } N \ge 1) \ge 1 - C\frac{e^{b/\varepsilon}}{\lambda}.$$

Proof By Propositions 6.8, 6.9 and 6.11 and

$$\begin{aligned}\inf_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} &\mathbb{P}(-\tilde{S}_N(x) \le 0 \text{ for every } N \ge 1) \\ &\ge \mathbb{P}(-\tilde{S}_N \le 0 \text{ for every } N \ge 1) = 1 - \frac{1}{\lambda}\mathbb{E}\tilde{g}_1.\end{aligned}$$

Moreover, by Proposition 6.9 for every $b > 0$ there exist $\varepsilon_0 \in (0,1)$ and $C > 0$ such that for every $\varepsilon \le \varepsilon_0$, $\mathbb{E}\tilde{g}_1 \le Ce^{b/\varepsilon}$ which completes the proof. □

We are now ready to prove Proposition 4.7 which is the main goal of this section.

Proof of Proposition 4.7 We estimate $\mathbb{P}(S_N(x) \le 0 \text{ for some } N \ge 1)$ in the following way,

$$\begin{aligned}&\mathbb{P}(-S_N(x) \ge 0 \text{ for some } N \ge 1) \\ &\quad \le \mathbb{P}\left(-\sum_{i\le N} f_i(x) + \lambda \sum_{i \le N} \tilde{f}_i \ge 0 \text{ for some } N \ge 1\right) \\ &\quad\quad + \mathbb{P}(-\tilde{S}_N(x) \ge 0 \text{ for some } N \ge 1). \end{aligned} \tag{6.14}$$

The second term on the right hand side can be estimated by Lemma 6.12 which provides a bound of the form

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}(-\tilde{S}_N(x) \ge 0 \text{ for some } N \ge 1) \le C\frac{e^{b/\varepsilon}}{\lambda}. \tag{6.15}$$

For the first term we notice that

$$\begin{aligned}&\mathbb{P}\left(-\sum_{i\le N} f_i(x) + \lambda \sum_{i \le N} \tilde{f}_i \ge 0 \text{ for some } N \ge 1\right) \\ &\quad \le \sum_{N \ge 1} \mathbb{P}\left(-\sum_{i\le N} f_i(x) + \lambda \sum_{i \le N} \tilde{f}_i \ge 0\right) \\ &\quad \le \sum_{N \ge 1} \mathbb{P}\left(\exp\left\{-\frac{1}{2\lambda}\sum_{i\le N} f_i(x) + \frac{1}{2}\sum_{i \le N} \tilde{f}_i\right\} \ge 1\right).\end{aligned}$$

By Markov's inequality, independence of $\{f_i(x)\}_{i\geq 1}$ and $\{\tilde{f}_i\}_{i\geq 1}$ and equality in law of the $\tilde{f}_i$'s the last inequality implies that

$$\mathbb{P}\left(-\sum_{i\leq N} f_i(x) + \lambda \sum_{i\leq N} \tilde{f}_i \geq 0 \text{ for some } N \geq 1\right) \leq \sum_{N\geq 1} \underbrace{\mathbb{E}\exp\left\{-\frac{1}{2\lambda}\sum_{i\leq N} f_i(x)\right\}}_{=:I_N(x)} \underbrace{\left(\mathbb{E}\exp\left\{\frac{\tilde{f}_1}{2}\right\}\right)^N}_{\leq 2^N \text{ since } \tilde{f}_1 \sim \exp(1)}. \tag{6.16}$$

Let $\varepsilon_0 \in (0,1)$ as in Proposition 6.3. For the term $I_N(x)$ we notice that for every $\varepsilon \leq \varepsilon_0$

$$\begin{aligned} I_N(x) &\leq \left(\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{E}\exp\left\{-\frac{1}{2\lambda} f_1(x)\right\}\right)^N \\ &\leq \left(\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \left[\mathbb{E}\exp\left\{-\frac{1}{2\lambda} f_1(x)\right\} \mathbf{1}_{\{f_1(x)\geq e^{2a_0/\varepsilon}\}} \right.\right. \\ &\quad \left.\left. + \mathbb{P}\left(f_1(x) \leq e^{2a_0/\varepsilon}\right)\right]\right)^N \\ &\leq \left(e^{-e^{2a_0/\varepsilon}/2\lambda} + e^{-3a_0/\varepsilon}\right)^N, \end{aligned}$$

where in the first inequality we use the Markov property and in the last we use Proposition 6.3. If we choose $\frac{1}{2\lambda} = e^{-(2a_0-b)/\varepsilon}$ and choose $\varepsilon_0 \in (0,1)$ even smaller the last inequality implies that for every $\varepsilon \leq \varepsilon_0$

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} I_N(x) \leq \left(e^{-e^{b/\varepsilon}} + e^{-3a_0/\varepsilon}\right)^N \leq e^{-5a_0 N/2\varepsilon}.$$

Combining with (6.16) we find $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \leq \varepsilon_0$

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}\left(-\sum_{i\leq N} f_i(x) + \lambda \sum_{i\leq N} \tilde{f}_i \geq 0 \text{ for some } N \geq 1\right) \leq \sum_{N\geq 1} e^{-5a_0 N/2\varepsilon} 2^N \leq \sum_{N\geq 1} e^{-2a_0 N/\varepsilon} = \frac{e^{-2a_0/\varepsilon}}{1-e^{-2a_0/\varepsilon}}. \tag{6.17}$$

Finally (6.14), (6.15) and (6.17) imply that

$$\sup_{\|x-(\pm 1)\|_{\mathcal{C}^{-\alpha_0}} \leq \delta_0} \mathbb{P}(-S_N(x) \geq 0 \text{ for some } N \geq 1) \leq C\frac{e^{b/\varepsilon}}{e^{(2a_0-b)/\varepsilon}} + \frac{e^{-2a_0/\varepsilon}}{1-e^{-2a_0/\varepsilon}},$$

which completes the proof since b is arbitrary. □

7 Proof of the Eyring–Kramers law

In this section we prove the Eyring–Kramers law Theorem 3.5. We first need to introduce some additional tools.

We consider the spatial Galerkin approximation $X_N(\cdot; x)$ of $X(\cdot; x)$ given by

$$\begin{aligned} (\partial_t - \Delta) X_N &= -\Pi_N \left(X_N^3 - X_N - 3\varepsilon \mathfrak{R}_N X_N \right) + \sqrt{2\varepsilon}\,\xi_N \\ X_N\big|_{t=0} &= x_N \end{aligned} \tag{7.1}$$

where Π_N is the projection on $\{f \in L^2 : f(z) = \sum_{|k| \le N} \hat{f}(k) L^{-2} \mathrm{e}^{2\pi \mathrm{i} k \cdot z / L}\}$, $\xi_N = \Pi_N \xi$, $x_N = \Pi_N x$ and $\mathfrak{R}_N$ is as in (2.3). Here for $k \in \mathbb{Z}^2$ we set $|k| = |k_1| \vee |k_2|$. In this notation we have that $\Pi_N f = f * D_N$, where D_N is the 2-dimensional square Dirichlet kernel given by $D_N(z) = \sum_{|k| \le N} L^{-2} \mathrm{e}^{2\pi \mathrm{i} k \cdot z / L}$.

To treat (7.1) we write $X_N(\cdot; x) = v_N(\cdot; x) + \varepsilon^{\frac{1}{2}} \text{⫯}_N(\cdot; x)$ for

$$\begin{aligned} (\partial_t - (\Delta - 1)) \text{⫯}_N &= \sqrt{2}\,\xi_N \\ \text{⫯}_N(0) &= 0. \end{aligned}$$

Then $v_N(\cdot; x)$ solves

$$\begin{aligned} (\partial_t - \Delta)\, v_N &= -\Pi_N v_N^3 + v_N - \Pi_N \left(3 v_N^2 \varepsilon^{\frac{1}{2}} \text{⫯}_N + 3 v_N \varepsilon \mathbf{V}_N + \varepsilon^{\frac{3}{2}} \mathbf{W}_N \right) + 2\varepsilon^{\frac{1}{2}} \text{⫯}_N \\ v_N\big|_{t=0} &= x_N \end{aligned} \tag{7.2}$$

where $\mathbf{V}_N = \text{⫯}_N^2 - \mathfrak{R}_N$ and $\mathbf{W}_N = \text{⫯}_N^3 - 3\mathfrak{R}_N \text{⫯}_N$.

As in (3.1) and (3.2), for $\delta \in (0, 1/2)$ and $\alpha > 0$ we define the symmetric subsets A and B of $\mathcal{C}^{-\alpha}$ by

$$A := \left\{ f \in \mathcal{C}^{-\alpha} : \bar{f} \in [-1-\delta, -1+\delta],\ f - \bar{f} \in D_\perp \right\} \tag{7.3}$$

$$B := \left\{ f \in \mathcal{C}^{-\alpha} : \bar{f} \in [1-\delta, 1+\delta],\ f - \bar{f} \in D_\perp \right\} \tag{7.4}$$

where $D_\perp$ is the closed ball of radius δ in $\mathcal{C}^{-\alpha}$ and $\bar{f} = L^{-2}\langle f, 1\rangle$. To simplify the notation in this section, we have dropped the dependence of A and B on the parameters α and δ. We will only write $A(\alpha; \delta)$ and $B(\alpha; \delta)$ if we need to specify the values of these parameters. For $x \in A$ we define

$$\tau_B(X_N(\cdot; x)) := \inf\{t > 0 : X_N(t; x) \in B\}$$

and

$$\tau_B(X(\cdot; x)) := \inf\{t > 0 : X(t; x) \in B\}.$$

Last, recall that for $k \in \mathbb{Z}^2$ (see (3.3)),

$$\lambda_k = \left(\frac{2\pi|k|}{L}\right)^2 - 1 \text{ and } \quad \nu_k = \left(\frac{2\pi|k|}{L}\right)^2 + 2 = \lambda_k + 3.$$

The next theorem is essentially [2, Theorem 2.3].

Theorem 7.1 ([2, Theorem 2.3]) *Let* $0 < L < 2\pi$. *For every* $\alpha > 0$, $\delta \in (0, 1/2)$ *and* $\varepsilon \in (0, 1)$ *there exists a sequence* $\{\mu_{\varepsilon,N}\}_{N\geq 1}$ *of probability measures concentrated on* ∂A *such that*

$$\begin{aligned}
&\limsup_{N\to\infty} \int \mathbb{E}\tau_B(X_N(\cdot;x))\, \mu_{\varepsilon,N}(\mathrm{d}x) \\
&\quad \leq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{\lambda_k + 2}\right\}} \mathrm{e}^{(V(0)-V(-1))/\varepsilon} \left(1 + c_+\sqrt{\varepsilon}\right) \\
&\liminf_{N\to\infty} \int \mathbb{E}\tau_B(X_N(\cdot;x))\, \mu_{\varepsilon,N}(\mathrm{d}x) \\
&\quad \geq \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{\lambda_k + 2}\right\}} \mathrm{e}^{(V(0)-V(-1))/\varepsilon} \left(1 - c_-\varepsilon\right) \qquad (7.5)
\end{aligned}$$

where the constants c_+ *and* c_- *are uniform in* ε.

Proof The proof of (7.5) is given in [2, Sections 4 and 5], but the following should be modified.

- In [2], the sets A and B are defined as in (7.3) and (7.4) with $D_\perp$ replaced by a ball in H^s for $s < 0$. The explicit form of $D_\perp$ is only used in [2, Lemma 5.9]. There the authors consider the 0-mean Gaussian measure $\gamma_0^\perp$ with quadratic form $\frac{1}{2\varepsilon}\left(\|\nabla f\|_{L^2}^2 - \|f - \bar{f}\|_{L^2}^2\right)$, and prove that $D_\perp$ has probability bounded from below by $1 - c\varepsilon^2$. Here we assume that $D_\perp$ is a ball in $\mathcal{C}^{-\alpha}$. To obtain the same estimate for this set, we first notice that the random field f associated with the measure $\gamma_0^\perp$ satisfies

$$\mathbb{E}\langle f, L^{-2}\mathrm{e}^{2\mathrm{i}\pi k\cdot/L}\rangle \lesssim \frac{\varepsilon \log \varepsilon^{-1} \log \lambda_k}{1 + \lambda_k},$$

  for every $k \in \mathbb{Z}^2$, where the explicit constant depends on L. This decay of the Fourier modes of f and [21, Proposition 3.6] imply that the measure $\gamma_0^\perp$ is concentrated in $\mathcal{C}^{-\alpha}$, for every $\alpha > 0$, which in turn implies [2, Lemma 5.9] for the set $D_\perp$ considered here.
- In [2], the authors consider (7.1) with $\mathfrak{R}_N$ replaced by

$$C_N = \frac{1}{L^2}\sum_{|k|\leq N} \frac{1}{|\lambda_k|}$$

and obtain (7.5) with the pre-factor given by

$$\frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k}\right\}} = \lim_{N\to\infty}\frac{2\pi}{|\lambda_0|}\sqrt{\prod_{|k|\le N}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{3L^2C_N}{2}\right\}}.$$

In our case one can check by (2.3) that $\mathfrak{R}_N$ is given by

$$\mathfrak{R}_N = \frac{1}{L^2}\sum_{|k|\le N}\frac{1}{|\lambda_k+2|}.$$

According to [2, Remark 2.5] this choice of renormalisation constant modifies [2, Theorem 2.3] by multiplying the pre-factor there with

$$\exp\left\{-3L^2\lim_{N\to\infty}(\mathfrak{R}_N - C_N)/2\lambda_0\right\}.$$

□

Remark 7.2 The finite dimensional measure $\mu_{\varepsilon,N}$ in (7.5) is given by

$$\mu_{\varepsilon,N}(\mathrm{d}f) = \frac{1}{\mathrm{cap}_A(B)}\mathrm{e}^{-V(\Pi_N f)/\varepsilon}\rho_{A,B}(\mathrm{d}f),$$

where $\rho_{A,B}$ is a probability measure concentrated on ∂A, called the equilibrium measure, and $\mathrm{cap}_A(B)$ is a normalisation constant. Under this measure and the assumption that the sets A and B are symmetric, the integrals appearing in (7.5) can be rewritten using potential theory as

$$\int \mathbb{E}\tau_B(X_N(\cdot;x))\,\mu_{\varepsilon,N}(\mathrm{d}x) = \frac{1}{2\mathrm{cap}_A(B)}\int_{\mathbb{R}^{(2N+1)^2}}\mathrm{e}^{-V(\Pi_N f)/\varepsilon}\,\mathrm{d}f.$$

This formula is derived in [2, Section 3] and it is then analysed to obtain (7.5).

Theorem 3.5 generalises (7.5) for the limiting process $X(\cdot;x)$ for fixed initial condition x in a suitable neighbourhood of -1. To prove this theorem, we first fix $\alpha\in(0,\alpha_0)$ and pass to the limit as $N\to\infty$ in (7.5) to prove a version of (3.4) where the initial condition x is averaged with respect to a measure μ_ε concentrated on a closed ball with respect to the weaker topology $\mathcal{C}^{-\alpha_0}$ (see Proposition 7.7). This measure is the weak limit, up to a subsequence, of the measures $\mu_{\varepsilon,N}$ in Theorem 7.1. We then use the exponential loss of memory, Theorem 3.1, to pass from averages of initial conditions with respect to the limiting measure μ_ε to fixed initial conditions.

The rest of this section is structured as follows. In Sect. 7.1 we prove convergence of the Galerkin approximations $X_N(\cdot;x_N)$ and obtain estimates uniform in the initial condition x and the regularisation parameter N. In Sect. 7.2 we prove uniform integrability of the stopping times $\tau_B(X(\cdot;x))$ and pass to the limit as $N\to\infty$ in (7.5). Finally in Sect. 7.3 we prove Theorem 3.5.

7.1 Convergence of the Galerkin scheme and a priori estimates

In the next proposition we prove convergence of $X_N(\cdot;x)$ to $X(\cdot;x_N)$ in $C([0,T];\mathcal{C}^{-\alpha})$ using convergence of the stochastic objects $\curlyvee^{n}_N$ which is proven in [24, Proposition 2.3]. This is a technical result and the proof is given in the Appendix.

Proposition 7.3 *Let $\aleph \subset \mathcal{C}^{-\alpha_0}$ be bounded and assume that for every $x \in \aleph$, there exists a sequence $\{x_N\}_{N\geq 1}$ such that $x_N \to x$ uniformly in x. Then for every $\alpha \in (0,\alpha_0)$ and $0 < s < T$*

$$\lim_{N\to\infty} \sup_{x\in\aleph} \sup_{t\in[s,T]} \|X_N(t;x_N) - X(t;x)\|_{\mathcal{C}^{-\alpha}} = 0$$

in probability.

Proof See Appendix F. □

The next proposition provides a bound for $X_N(\cdot;x)$ uniformly in the initial condition x and the regularisation parameter N in the $\mathcal{B}^{-\alpha}_{2,2}$ norm, for $0<\alpha<\alpha_0$. This result has been already established in [24, Corollary 3.10] for the limiting process $X(\cdot;x)$ in the $\mathcal{C}^{-\alpha}$ norm. There (2.2) is tested with $v(\cdot;x)^{p-1}$, for $p \geq 2$ even, to bound $\|v(\cdot;x)\|_{L^p}$ by using the "good" sign of the non-linear term $-v^3$. In the case of (7.2) this argument allows us to bound $\|v_N(\cdot;x)\|_{L^p}$ for $p=2$ only, because of the projection Π_N in front of the non-linearity.

Proposition 7.4 *For every $\alpha \in (0,\alpha_0]$ and $p \geq 1$ we have that*

$$\sup_{N\geq 1} \sup_{x\in\mathcal{C}^{-\alpha_0}} \sup_{t\leq 1} t^{\frac{p}{2}} \mathbb{E}\|X_N(s;x)\|^p_{\mathcal{B}^{-\alpha}_{2,2}} < \infty. \tag{7.6}$$

Proof Proceeding exactly as in the proof of [24, Proposition 3.7] we first show that there exist $\alpha \in (0,1)$ and $p_n \geq 1$ such that for every $t \in (0,1)$

$$\|v_N(t;x)\|^2_{L^2} \lesssim t^{-1} \vee \left(\sum_{n=1}^{3} t^{-\alpha'(n-1)p_n} \sup_{s\leq t} s^{\alpha'(n-1)p_n} \|\varepsilon^{\frac{n}{2}} \curlyvee^{n}_N(s)\|^{p_n}_{\mathcal{C}^{-\alpha}} \right)^{\frac{1}{2}}, \tag{7.7}$$

for every $\alpha' \in (0,1)$, uniformly in $x \in \mathcal{C}^{-\alpha_0}$. We then proceed as in the proof of [24, Corollary 3.10] and use (7.7) to prove (7.6). The only difference is that here we use the norm $\|\cdot\|_{\mathcal{B}^{-\alpha}_{2,2}}$ and the embedding $L^2 \hookrightarrow \mathcal{B}^{-\alpha}_{2,2}$ on the level of $v_N(\cdot;x)$ together with the fact that

$$\sup_{N\geq 1} \mathbb{E} \left(\sup_{t\leq 1} t^{(n-1)\alpha'} \|\curlyvee^{n}_N(t)\|_{\mathcal{C}^{-\alpha}} \right)^p < \infty,$$

for every $\alpha,\alpha' > 0$ and $p \geq 1$, which is immediate from [24, Proposition 2.2, Proposition 2.3]. □

7.2 Passing to the limit

In this section we pass to the limit as $N \to \infty$ in (7.5) using uniform integrability of the stopping time $\tau_B(X_N(\cdot; x))$. To obtain uniform integrability we prove exponential moment bounds for $\tau_B(X_N(\cdot; x))$ uniformly in the initial condition $x \in \mathcal{C}^{-\alpha_0}$ and the regularisation parameter N. We first bound $\mathbb{P}(\tau_B(X_N(\cdot; x)) \geq 1)$ using a support theorem and a strong a priori bound for $X_N(\cdot; x)$ in $\mathcal{C}^{-\alpha}$. A support theorem for the limiting process $X(\cdot; x)$ has been already established in [24, Corollary 6.4]. To use it for $X_N(\cdot; x)$ we combine it with the convergence result in Proposition 7.3. To obtain a strong a priori bound for $X_N(\cdot; x)$ in $\mathcal{C}^{-\alpha}$ we first use Proposition 7.4 which implies the bound in $\mathcal{B}^{-\alpha}_{2,2}$ and then use Proposition G.2 to pass from the $\mathcal{B}^{-\alpha}_{2,2}$ norm to the $\mathcal{C}^{-\alpha}$ norm.

Proposition 7.5 *For every $\alpha \in (0, \alpha_0)$, $\delta \in (0, 1/2)$ and $\varepsilon \in (0, 1)$ there exist $c_0 \in (0, 1)$ and $N_0 \geq 1$ such that for every $N \geq N_0$*

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\tau_B(X_N(\cdot; x)) \geq 1) \leq c_0.$$

Proof Let $\alpha \in (0, \alpha_0)$ and let $\aleph$ be a compact subset of $\mathcal{C}^{-\alpha_0}$ which we fix below. Using the Markov property

$$\mathbb{P}(\tau_B(X_N(\cdot; x)) \geq 1) \leq \sup_{y \in \aleph} \mathbb{P}(\tau_B(X_N(\cdot; y)) \geq 1/2)\, \mathbb{P}(X_N(1/2; x) \in \aleph) \\ + \mathbb{P}(X_N(1/2; x) \notin \aleph).$$

The proof is complete if for every $N \geq N_0$

$$\sup_{y \in \aleph} \mathbb{P}(\tau_B(X_N(\cdot; y)) \geq 1/2) < 1, \quad \sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(X_N(1/2; x) \notin \aleph) < 1. \tag{7.8}$$

We notice that there exists $\delta' > 0$ such that for any $y \in \aleph$

$$\mathbb{P}(\tau_B(X_N(\cdot; y)) \leq 1/2) \geq \mathbb{P}(X_N(1/2; y) \in B) \geq \mathbb{P}\left(X(1/2; y) \in B_{\mathcal{C}^{-\alpha}}(1; \delta')\right) \\ - \mathbb{P}\left(\|X_N(1/2; y) - X(1/2; y)\|_{\mathcal{C}^{-\alpha}} \geq \delta'\right). \tag{7.9}$$

Here we use that if $\|X(1/2; y) - 1\|_{\mathcal{C}^{-\alpha}}, \|X_N(1/2; y) - X(1/2; y)\|_{\mathcal{C}^{-\alpha}} \leq \delta'$, then $X_N(1/2; y) \in B$ for δ' sufficiently small. By the support theorem [24, Corollary 6.4] there exists $c_1 \equiv c_1(\delta, \varepsilon) > 0$ such that

$$\inf_{y \in \aleph} \mathbb{P}\left(X(1/2; y) \in B_{\mathcal{C}^{-\alpha}}(1; \delta')\right) \geq c_1. \tag{7.10}$$

On the other hand Proposition 7.3 implies convergence in probability of $X_N(1/2; y)$ to $X(1/2; y)$ in $\mathcal{C}^{-\alpha}$ uniformly in $y \in \aleph$. Hence there exists $N_0 \geq 1$ such that for every $N \geq N_0$

$$\sup_{y \in \aleph} \mathbb{P}\left(\|X_N(1/2; y) - X(1/2; y)\|_{\mathcal{C}^{-\alpha}} \geq \delta/2\right) \leq c_1/2. \tag{7.11}$$

Plugging (7.10) and (7.11) in (7.9) implies the first bound in (7.8).

We now prove the second bound in (7.8). By the Markov inequality for every $R > 0$

$$\mathbb{P}\left(\|X_N(1/4; x)\|_{\mathcal{B}^{-\alpha}_{2,2}} \geq R\right) \leq \frac{1}{R} \mathbb{E}\|X_N(1/4; x)\|_{\mathcal{B}^{-\alpha}_{2,2}}.$$

By (7.6) the expectation on the right hand side of the last inequality is uniformly bounded over $x \in \mathcal{C}^{-\alpha_0}$ and $N \geq 1$. Thus choosing $R > 0$ large enough

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}\left(\|X_N(1/4; x)\|_{\mathcal{B}^{-\alpha}_{2,2}} \geq R\right) \leq \frac{1}{2}. \tag{7.12}$$

By Proposition G.2 for every $K, R > 0$ there exist $C \equiv C(K, R)$ such that

$$\sup_{\|y\|_{\mathcal{B}^{-\alpha}_{2,2}} \leq R} \mathbb{P}\left(\|X_N(1/4; y)\|_{\mathcal{C}^{-\alpha}} \geq C\right) \leq \mathbb{P}\left(\sup_{t \leq 1} t^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \vee^{n}_N(t)\|_{\mathcal{C}^{-\alpha}} \geq K\right).$$

Choosing K sufficiently large, combining the last inequality with [24, Propositions 2.2 and 2.3] and using the Markov inequality imply that

$$\sup_{\|y\|_{\mathcal{B}^{-\alpha}_{2,2}} \leq R} \mathbb{P}\left(\|X_N(1/4; y)\|_{\mathcal{C}^{-\alpha}} \geq C\right) \leq \frac{1}{2}. \tag{7.13}$$

Using the Markov property and (7.12) and (7.13) we get for arbitrary $x \in \mathcal{C}^{-\alpha_0}$

$$\begin{aligned} &\mathbb{P}\left(\|X_N(1/2; x)\|_{\mathcal{C}^{-\alpha}} \geq C\right) \\ &\quad \leq \mathbb{P}\left(\|X_N(1/4; x)\|_{\mathcal{B}^{-\alpha}_{2,2}} \leq R\right) \sup_{y \in \mathcal{B}^{-\alpha}_{2,2}} \mathbb{P}(\|X_N(1/4; y)\|_{\mathcal{C}^{-\alpha}} \geq C) \\ &\qquad + \mathbb{P}\left(\|X_N(1/4; x)\|_{\mathcal{B}^{-\alpha}_{2,2}} \geq R\right) \leq \frac{3}{4}. \end{aligned}$$

We finally notice that for every $\alpha < \alpha_0$ the set $\aleph = \{f \in \mathcal{C}^{-\alpha_0} : \|f\|_{\mathcal{C}^{-\alpha}} \leq C\}$ is compact in $\mathcal{C}^{-\alpha_0}$ which implies the second bound in (7.8). □

In the next corollary we use Proposition 7.5 to prove exponential moments for the stopping time $\tau_B(X_N(\cdot; x))$.

Corollary 7.6 *For every $\delta > 0$ and $\varepsilon \in (0, 1)$ there exist $\eta_0 > 0$ and $N_0 \geq 1$ such that*

$$\sup_{N \geq N_0} \sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{E} \exp\{\eta_0 \tau_B(X_N(\cdot; x))\} < \infty.$$

Proof By the Markov property we have that

$$\mathbb{P}(\tau_B(X_N(\cdot;x)) \geq k+1) \leq \sup_{y \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\tau_B(X_N(\cdot;y)) \geq 1)\, \mathbb{P}(\tau_B(X_N(\cdot;x)) \geq k).$$

Iterating this inequality and using Proposition 7.5 we obtain that

$$\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\tau_B(X_N(\cdot;x)) \geq k+1) \leq c_0^{k+1}.$$

Then

$$\begin{aligned}
\mathbb{E}\exp\{\eta_0 \tau_B(X_N(\cdot;x))\} &= 1 + \int_0^\infty \eta_0 \mathrm{e}^{\eta_0 t} \mathbb{P}(\tau_B(X_N(\cdot;x)) \geq t)\, \mathrm{d}t \\
&\leq 1 + \sum_{k=0}^\infty \mathbb{P}(\tau_B(X_N(\cdot;x)) \geq k) \int_k^{k+1} \eta_0 \mathrm{e}^{\eta_0 t}\, \mathrm{d}t \\
&\leq 1 + \mathrm{e}^{\eta_0} \sum_{k=0}^\infty \mathrm{e}^{\eta_0 k} c_0^k
\end{aligned}$$

and the proof is complete if we choose $\eta_0 < \log c_0^{-1}$. □

In the next proposition we pass to the limit as $N \to \infty$ in (7.5). Here we use Corollary 7.6, which implies uniform integrability of $\tau_B(X_N(\cdot;x))$, and the weak convergence of the measures $\mu_{\varepsilon,N}$.

Proposition 7.7 *For every $\alpha \in (0,\alpha_0)$, $\delta \in (0,1/2)$ except possibly a countable subset, and $\varepsilon \in (0,1)$ there exists a probability measure $\mu_\varepsilon \in \mathcal{M}_1(A(\alpha_0;\delta))$ such that*

$$\begin{aligned}
&\int \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x))\, \mu_\varepsilon(\mathrm{d}x) \\
&\quad \leq \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{\lambda_k + 2}\right\}} \mathrm{e}^{(V(0)-V(-1))/\varepsilon} \left(1 + c_+\sqrt{\varepsilon}\right) \\
&\int \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x))\, \mu_\varepsilon(\mathrm{d}x) \\
&\quad \geq \frac{2\pi}{|\lambda_0|} \sqrt{\prod_{k \in \mathbb{Z}^2} \frac{|\lambda_k|}{\nu_k} \exp\left\{\frac{\nu_k - \lambda_k}{\lambda_k + 2}\right\}} \mathrm{e}^{(V(0)-V(-1))/\varepsilon} (1 - c_-\varepsilon)
\end{aligned} \tag{7.14}$$

where the constants c_+ and c_- are uniform in ε.

Proof We only prove the upper bound in 7.14. The lower bound follows similarly.

Let $\alpha \in (0,\alpha_0)$ and $\delta \in (0,1/2)$. Using the compact embedding $\mathcal{C}^{-\alpha} \hookrightarrow \mathcal{C}^{-\alpha_0}$ (see Proposition A.8), for any $\alpha < \alpha_0$, we have that $A(\alpha;\delta) \subset A(\alpha_0;\delta)$. Let $\{\mu_{\varepsilon,N}\}_{N \geq 1}$

be the family of probability measures in (7.5). Using again the compact embedding $\mathcal{C}^{-\alpha} \hookrightarrow \mathcal{C}^{-\alpha_0}$, for any $\alpha < \alpha_0$, this family is trivially tight since it is concentrated on $\partial A(\alpha;\delta)$. Hence there exists $\mu_\varepsilon \in \mathcal{M}_1(A(\alpha_0;\delta))$ such that $\mu_{\varepsilon,N} \overset{\text{weak}}{\to} \mu_\varepsilon$ up to a subsequence.

By Skorokhod's represantation theorem (see [7, Theorem 2.4]) there exist a probability space $(\Omega_\mu, \mathcal{F}_\mu, \mathbb{P}_\mu)$ and random variables $\{x_N\}_{N\geq 1}$ and x taking values in $A(\alpha_0;\delta)$ such that $x_N \overset{\text{law}}{=} \mu_N$, $x \overset{\text{law}}{=} \mu_\varepsilon$ and $x_N \to x$ $\mathbb{P}_{\mu_\varepsilon}$-almost surely in $\mathcal{C}^{-\alpha_0}$. If we denote by $\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}}$ the expectation of the probability measure $\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}$, we have that

$$\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}} \tau_{B(\alpha;\delta)}(X_N(\cdot;x_N)) = \int \mathbb{E}\tau_{B(\alpha;\delta)}(X_N(\cdot;x))\,\mu_N(\mathrm{d}x)$$
$$\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}} \tau_{B(\alpha;\delta)}(X(\cdot;x)) = \int \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x))\,\mu_\varepsilon(\mathrm{d}x). \tag{7.15}$$

By Proposition 7.3 $X_N(\cdot;x_N)$ converges to $X(\cdot;x)$ $\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}$-almost surely on compact time intervals of $(0,\infty)$ up to a subsequence. Let

$$\mathtt{L} = \left\{\delta \in (0,1/2) : \mathbb{P}\left(\tau_{B(\alpha;\delta)}(\cdot) \text{ is discontinuous on } X(\cdot;x)\right) > 0\right\}$$

and notice that for $x(t) = L^{-2}\langle X(t;x), 1\rangle$

$$\mathtt{L} \subset \{\delta \in (0,1/2) : \mathbb{P}\,(t \mapsto |x(t)-1| \vee \|X(t;x)-x(t)\|_{\mathcal{C}^{-\alpha}} \text{ has a local minimum at height } \delta) > 0\}.$$

As in [20, Proof of Theorem 6.1] the last set is at most countable, hence $\tau_{B(\alpha;\delta)}(X_N(\cdot;x_N)) \to \tau_{B(\alpha;\delta)}(X(\cdot;x))$ $\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}$-almost surely up to a subsequence, except possibly a countable number of $\delta \in (0,1/2)$.

By Corollary 7.6 the family $\{\tau_{B(\alpha;\delta)}(X_N(\cdot;x))\}_{N\geq N_0}$ is uniformly integrable. Hence by Vitali's convergence theorem (see [5, Theorem 4.5.4]) we obtain that

$$\mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}} \tau_{B(\alpha;\delta)}(X_N(\cdot;x_N)) \to \mathbb{E}_{\mathbb{P}\otimes\mathbb{P}_{\mu_\varepsilon}} \tau_{B(\alpha;\delta)}(X(\cdot;x)).$$

Combining with (7.5) and (7.15) the proof of the upper bound is complete. □

7.3 Proof of Theorem 3.5

In this section we combine Proposition 7.7 and Theorem 3.1 to prove Theorem 3.5. The idea we use here was first implemented in the 1-dimensional case in [4]. Generally speaking, if we restrict ourselves on the event where the first transition from a neighbourhood of -1 to a neighbourhood of 1 happens after the exponential loss of memory, $\tau_{B(\alpha;\delta)}(X(\cdot;x))$ behaves like $\int \tau_{B(\alpha;\delta)}(X(\cdot;x))\,\mu_\varepsilon(\mathrm{d}x)$ for $x \in A(\alpha_0;\delta)$. The probability of this event is quantified by Theorem 3.1 and Proposition 7.8. On the complement of this event the transition time $\tau_{B(\alpha;\delta)}(X(\cdot;x))$ is estimated using Proposition 7.9.

In the next proposition we prove that the first transition from a neighbourhood of -1 to a neighbourhood of 1 happens only after some time $T_0 > 0$ with overwhelming probability. This is a large deviation event which can be estimated using continuity of X with respect to the initial condition x and the stochastic objects $\left\{\varepsilon^{\frac{n}{2}} \vee^{n}\right\}_{n\le 3}$. We sketch the proof for completeness.

Proposition 7.8 *For every $\alpha \in (0, \alpha_0)$ and $\delta \in (0, 1/2)$ there exist $a_0, \delta_0, T_0 > 0$ and $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \le \varepsilon_0$*

$$\sup_{\|x-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0} \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \le T_0) \le \mathrm{e}^{-a_0/\varepsilon}.$$

Proof We first notice that for $\|x-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$

$$\mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \ge T_0) \ge \mathbb{P}\left(\sup_{t\le T_0} \|X(t;x)-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_1\right),$$

for some $\delta_1 > 0$. Using continuity of X with respect to x and the stochastic objects $\left\{\varepsilon^{\frac{n}{2}} \vee^{n}\right\}_{n\le 3}$, the last probability can be estimated from below uniformly in $\|x - (-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$, for δ_0 sufficiently small, by

$$\begin{aligned}\mathbb{P}&\left(\sup_{t\le T_0} \|X(t;x)-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_1\right)\\ &\ge \mathbb{P}\left(\sup_{t\le T_0} (t\wedge 1)^{-(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \vee^{n}(t)\|_{\mathcal{C}^{-\alpha}} \le \delta_2\right),\end{aligned}$$

for some $\delta_2 > 0$. Last by Proposition D.1 we find $a_0 > 0$ and $\varepsilon_0 \in (0, 1)$ such that for ever $\varepsilon \le \varepsilon_0$

$$\mathbb{P}\left(\sup_{t\le T_0} (t\wedge 1)^{-(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \vee^{n}(t)\|_{\mathcal{C}^{-\alpha}} \le \delta_2\right) \ge 1-\mathrm{e}^{-a_0/\varepsilon},$$

which completes the proof. □

In the next proposition we estimate the second moment of the transition time $\tau_{B(\alpha;\delta)}(X(\cdot;x))$ using the large deviation estimate (6.3). The proof combines the ideas in Propositions 6.4 and 6.6. However here we construct a path g which is different from the one in the proof of Proposition 6.4 to ensure that the process $X(\cdot;x)$ returns to a neighbourhood of -1. The same proof implies exponential moments of the transition time $\tau_{B(\alpha;\delta)}(X(\cdot;x))$, but we only need to estimate the second moment in the proof of Theorem 3.5.

Proposition 7.9 *Let $\alpha \in (0, \alpha_0)$ and $\delta \in (0, 1/2)$. For every $\eta > 0$ there exists $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \le \varepsilon_0$*

$$\sup_{x\in\mathcal{C}^{-\alpha_0}} \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;x))^2 \le C\mathrm{e}^{2[(V(0)-V(-1))+\eta]/\varepsilon},$$

for some $C > 0$ *independent of* η *and* ε.

Proof We first prove that for every $R, \eta > 0$ there exists $T_0 > 0$ and $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\le R} \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \ge T_0) \le 1 - \mathrm{e}^{-[(V(0)-V(-1))+\eta]/\varepsilon}.$$

We notice that there exists $\delta' > 0$ such that

$$\mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot;x)) \le T_0) \ge \mathbb{P}(\underbrace{\|X(T_*;x)-1\|_{\mathcal{C}^{-\alpha}} \le \delta' \text{ for some } T_* \le T_0}_{=:\mathcal{A}(T_0;x)}).$$

Here we use that if $\|X(T_*;x)-1\|_{\mathcal{C}^{-\alpha}} \le \delta'$, for δ' sufficiently small then $X(T_*;x) \in B(\alpha;\delta)$. By the large deviation estimate (6.3) we need to bound

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}}\le R} \sup_{\substack{f\in\mathcal{A}(T_0;x)\\ f(0)=x}} I(f(\cdot;x)).$$

To do so we proceed as in the proof of Proposition 6.4 by constructing a suitable path $g \in \mathcal{A}(T_0;x)$. The construction here is similar but some of the steps differ since we need to ensure that g returns to a neighbourhood of 1. To avoid repeating ourselves we give a sketch of the proof highlighting the different steps of the construction.

Steps 1, 2 and 3 are exactly as in the proof of Proposition 6.4. However we need to distinguish the value of δ there from the value of δ in the statement of the proposition. If $g(\tau_3;x) \in B_{\mathcal{B}^1_{2,2}}(1;\delta) \cap B_{\mathcal{C}^{2+\lambda}}(0;C)$ we stop at Step 3. If not then $g(\tau_3;x) \in B_{\mathcal{B}^1_{2,2}}(-1;\delta) \cap B_{\mathcal{C}^{2+\lambda}}(0;C)$ or $B_{\mathcal{B}^1_{2,2}}(0;\delta) \cap B_{\mathcal{C}^{2+\lambda}}(0;C)$. We only explain how to proceed in the first case since it also covers the other.

Before we describe the remaining steps we recall that by Proposition C.2 there exist $y_{0,-}, y_{0,+} \in B_{\mathcal{B}^1_{2,2}}(0;\delta)$ such that $y_{0,-}, y_{0,+} \in \mathcal{C}^\infty$ and $X_{det}(t;y_{0,\pm}) \to \pm 1$ in $\mathcal{B}^1_{2,2}$. In particular there exists $T_0^* > 0$ such that $X_{det}(T_0^*;y_{0,\pm}) \in B_{\mathcal{B}^1_{2,2}}(\pm 1;\delta) \cap B_{\mathcal{C}^{2+\lambda}}(0;C)$.

Step 4 (Jump to $X_{det}(T_0^*;y_{0,-})$):

Let $\tau_4 = \tau_3 + \tau$, for $\tau > 0$ as in Step 2 which we fix below according to Lemma 6.5. For $t \in [\tau_3, \tau_4]$ we set $g(t;x) = g(\tau_3;x) + \frac{t-\tau_3}{\tau_4-\tau_3}(X_{det}(T_0^*;y_{0,-}) - g(\tau_3;x))$.

Step 5 (Follow the deterministic flow backward to reach 0):

Let $\tau_5 = \tau_4 + T_0^*$. For $t \in [\tau_4, \tau_5]$ we set $g(t;x) = X_{det}(\tau_5 - t;y_{0,-})$.

Step 6 (Jump to $y_{0,+}$):

Let $\tau_6 = \tau_5 + \tau$, for τ as in Step 4. For $t \in [\tau_5, \tau_6]$ we set $g(t;x) = g(\tau_5;x) + \frac{t-\tau_5}{\tau_6-\tau_5}(y_{0,+} - g(\tau_5;x))$.

Step 7 (Follow the deterministic flow forward to reach 1):

Let $\tau_7 = \tau_6 + T_0^*$. For $t \in [\tau_6, \tau_7]$ we set $g(t; x) = X_{det}(t - \tau_6; y_{0,+})$.

For the path g constructed above we notice that for every $\|x\|_{\mathcal{C}^{-\alpha_0}} \le R$, if $t \ge \tau_7$ then $g(t; x) \in B_{\mathcal{B}^1_{2,2}}(1; \delta)$. By (A.5), $\mathcal{B}^1_{2,2} \subset \mathcal{C}^{-\alpha}$, for every $\alpha > 0$, hence if we choose δ sufficiently small and set $T_0 = \tau_7 + 1$ then $g \in \mathcal{A}(T_0; x)$.

To bound $I(g(\cdot; x))$ we proceed exactly as in the proof of Proposition 6.4 using Lemma 6.5. But when considering the contribution from Step 5 we get

$$
\begin{aligned}
&\frac{1}{4}\int_{\tau_4}^{\tau_5} \|(\partial_t - \Delta)g(t; x) + g(t; x)^3 - g(t; x)\|_{L^2}^2 \,\mathrm{d}t \\
&\quad = 2\int_0^{T_0^*} \Big\langle \partial_t X_{det}(t; y_{0,+}), \Delta X_{det}(t; y_{0,+}) - X_{det}(t; y_{0,+})^3 + X_{det}(t; y_{0,+})\Big\rangle \,\mathrm{d}t \\
&\quad = -2\left(V(X_{det}(T_0^*; y_0)) - V(y_{0,+})\right) \\
&\quad \le 2\left(V(0) - V(-1)\right).
\end{aligned}
$$

In total we obtain the bound

$$
\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} I(g(\cdot; x)) \le 2\left(V(0) - V(-1)\right) + C\delta.
$$

For $\eta > 0$ we choose δ even smaller to ensure that $C\delta < \eta$. Then by (6.3) we find $\varepsilon_0 \in (0, 1)$ such that for every $\varepsilon \le \varepsilon_0$

$$
\inf_{\|x\|_{\mathcal{C}^{-\alpha_0}} \le R} \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot; x)) \le T_0) \ge \mathrm{e}^{-[(V(0)-V(-1))+\eta]/\varepsilon}.
$$

The next step is to use the this estimate to show that for any $\eta > 0$ there exists $\varepsilon_0 \in (0, 1)$ and possibly a different $T_0 > 0$ such that for every $\varepsilon \le \varepsilon_0$

$$
\sup_{x \in \mathcal{C}^{-\alpha_0}} \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot; x)) \ge mT_0) \le \left(1 - \mathrm{e}^{-[(V(0)-V(-1))+\eta]/\varepsilon}\right)^m.
$$

We omit the proof since it is the same as the one of Proposition 6.6.

Finally we notice that

$$
\begin{aligned}
\mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot; x))^2 &= \int_0^\infty 2t\,\mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot; x)) \ge t)\,\mathrm{d}t \\
&\le \sum_{m=0}^\infty \mathbb{P}(\tau_{B(\alpha;\delta)}(X(\cdot; x)) \ge mT_0)\int_{mT_0}^{(m+1)T_0} 2t\,\mathrm{d}t \\
&\le 2T_0^2 \sum_{m=0}^\infty (m+1)\left(1 - \mathrm{e}^{-[(V(0)-V(-1))+\eta]/\varepsilon}\right)^m \\
&= 2T_0^2 \mathrm{e}^{2[(V(0)-V(-1))+\eta]/\varepsilon},
\end{aligned}
$$

which completes the proof. □

Proof of Theorem 3.5 Let

$$\mathtt{Pr}(\varepsilon) = \frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\mathrm{e}^{(V(0)-V(-1))/\varepsilon}$$

and $\delta \in (0,\delta_0)$, for $\delta_0 \in (0,1/2)$ which we fix below.

To prove the upper bound in (3.4) let $\delta_- < \delta$ and $T > 0$ which we also fix below. For $x \in A(\alpha_0;\delta_-)$ we define the set

$$A_T(x) = \left\{\tau_{B(\alpha;\delta_-)}(X(\cdot;x)) > T,\ \sup_{\|\bar{y}-x\|_{\mathcal{C}^{-\alpha_0}}\le\delta_0}\frac{\|X(t;\bar{y})-X(t;x)\|_{\mathcal{C}^\beta}}{\|\bar{y}-x\|_{\mathcal{C}^{-\alpha_0}}}\right.$$
$$\left.\le C\mathrm{e}^{-(2-\kappa)t}\text{ for every } t\ge T\right\},$$

where δ_0 and C are as in Theorem 3.1. For $y \in A(\alpha_0;\delta)$ and $x \in A(\alpha_0;\delta_-)$ we have that $\|y-x\|_{\mathcal{C}^{-\alpha_0}}, \|x-(-1)\|_{\mathcal{C}^{-\alpha_0}} \le \delta_0$, if we choose δ_0 sufficiently small. Furthermore for $y \in A(\alpha_0;\delta)$, $x \in A(\alpha_0;\delta_-)$ and $\omega \in A_T(x)$

$$\tau_{B(\alpha;\delta)}(X(\cdot;y)) \le \tau_{B(\alpha;\delta_-)}(X(\cdot;x)),$$

if we choose T sufficiently large. By Proposition 7.8 and Theorem 3.1 there exist $a_1 > 0$ and $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\sup_{x\in A(\alpha_0;\delta_-)}\mathbb{P}(A_T(x)^c) \le \sup_{\|x-(-1)\|_{\mathcal{C}^{-\alpha_0}}\le\delta_0}\mathbb{P}(A_T(x)^c) \le \mathrm{e}^{-a_1/\varepsilon}.$$

Then for every $y \in A(\alpha_0;\delta)$, $x \in A(\alpha_0;\delta_-)$ and $\eta > 0$, which we fix below, there exists $\varepsilon_0 \in (0,1)$ such that for every $\varepsilon \le \varepsilon_0$

$$\begin{aligned}
&\mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y))\\
&\quad\le \mathbb{E}\tau_{B(\alpha;\delta_-)}(X(\cdot;x)) + \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y))\mathbf{1}_{A_T(x)^c}\\
&\quad\overset{\text{Cauchy–Schwarz}}{\le} \mathbb{E}\tau_{B(\alpha;\delta_-)}(X(\cdot;x)) + \left(\mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y))^2\right)^{\frac{1}{2}}\mathbb{P}(A_T(x)^c)^{\frac{1}{2}}\\
&\quad\overset{\text{Prop. 7.9}}{\le} \mathbb{E}\tau_{B(\alpha;\delta_-)}(X(\cdot;x)) + C\mathrm{e}^{((V(0)-V(-1))+\eta-\frac{a_1}{2})/\varepsilon},
\end{aligned} \tag{7.16}$$

for some $C > 0$ independent of ε. By Proposition 7.7 there exist $\delta_- \in (0,\delta)$, $c_+ > 0$ and $\mu_\varepsilon \in \mathcal{M}_1(A(\alpha_0;\delta_-))$ such that for every $\varepsilon \in (0,1)$

$$\int \mathbb{E}\tau_{B(\alpha;\delta_-)}(X(\cdot;x))\,\mu_\varepsilon(\mathrm{d}x) \le \mathtt{Pr}(\varepsilon)(1+c_+\sqrt{\varepsilon}).$$

Integrating (7.16) over x with respect to μ_ε implies that

$$\sup_{y\in A(\alpha_0;\delta)} \mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y))$$
$$\leq \mathtt{Pr}(\varepsilon)\left((1+c_+\sqrt{\varepsilon})+\mathrm{e}^{(\eta-\frac{a_1}{2})/\varepsilon}C\left(\frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\right)^{-1}\right).$$

Let $\zeta>0$. Choosing $\eta<\frac{a_1}{2}$ we can find $\varepsilon_0\in(0,1)$ such that for every $\varepsilon\leq\varepsilon_0$

$$\mathrm{e}^{(\eta-\frac{a_1}{2})/\varepsilon}C\left(\frac{2\pi}{|\lambda_0|}\sqrt{\prod_{k\in\mathbb{Z}^2}\frac{|\lambda_k|}{\nu_k}\exp\left\{\frac{\nu_k-\lambda_k}{\lambda_k+2}\right\}}\right)^{-1}\leq\zeta\sqrt{\varepsilon},$$

which in turn implies that

$$\sup_{y\in A(\alpha_0;\delta)}\mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y))\leq\mathtt{Pr}(\varepsilon)\left(1+(c_++\zeta)\sqrt{\varepsilon}\right)$$

and proves the upper bound in (3.4).

To prove the lower bound, we let $\delta_+\in(\delta,\delta_0)$ which we fix below and for $y\in A(\alpha_0;\delta)$ and $x\in A(\alpha_0;\delta_+)$ we define the set

$$B_T(y,x)=\Bigg\{\tau_{B(\alpha;\delta)}(X(\cdot;y))\geq T,\ \sup_{\|\bar{y}-x\|_{\mathcal{C}^{-\alpha_0}}\leq\delta_0}\frac{\|X(t;\bar{y})-X(t;x)\|_{\mathcal{C}^\beta}}{\|\bar{y}-x\|_{\mathcal{C}^{-\alpha_0}}}$$
$$\leq C\mathrm{e}^{-(2-\kappa)t}\text{ for every }t\geq T\Bigg\}.$$

For $y\in A(\alpha_0;\delta)$ and $x\in A(\alpha_0;\delta_+)$ we have that $\|y-x\|_{\mathcal{C}^{-\alpha_0}}$, $\|y-(-1)\|_{\mathcal{C}^{-\alpha_0}}$, $\|x-(-1)\|_{\mathcal{C}^{-\alpha_0}}\leq\delta_0$, if we choose δ_0 sufficiently small. We also notice that for $y\in A(\alpha_0;\delta)$, $x\in A(\alpha_0;\delta_+)$ and $\omega\in B_T(y,x)$

$$\tau_{B(\alpha;\delta_+)}(X(\cdot;x))\leq\tau_{B(\alpha;\delta)}(X(\cdot;y)),$$

if we choose T sufficiently large. By Proposition 7.8 and Theorem 3.1 there exists $a_1>0$ and $\varepsilon_0\in(0,1)$ such that for every $\varepsilon\leq\varepsilon_0$

$$\sup_{\substack{y\in A(\alpha_0;\delta)\\ x\in A(\alpha_0;\delta_+)}}\mathbb{P}(B_T(y,x)^c)\leq\sup_{\substack{\|y-(-1)\|_{\mathcal{C}^{-\alpha_0}}\leq\delta_0\\ \|x-(-1)\|_{\mathcal{C}^{-\alpha_0}}\leq\delta_0}}\mathbb{P}(B_T(y,x)^c)\leq\mathrm{e}^{-a_1/\varepsilon}.$$

Then for every $y\in A(\alpha_0;\delta)$, $x\in A(\alpha_0;\delta_+)$ and $\varepsilon\leq\varepsilon_0$

$$\begin{aligned}\mathbb{E}\tau_{B(\alpha;\delta)}(X(\cdot;y))&\geq\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))\mathbf{1}_{B_T(y,x)}\\&=\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))-\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))\mathbf{1}_{B_T(y,x)^c}\\&\overset{\text{Cauchy–Schwarz}}{\geq}\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))\end{aligned}$$

$$- \left(\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))^2\right)^{\frac{1}{2}} \mathbb{P}\left(B_T(y,x)^c\right)^{\frac{1}{2}}$$
$$\geq \mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x)) - \left(\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))^2\right)^{\frac{1}{2}} \mathrm{e}^{-a_1/2\varepsilon}$$

and we proceed as in the case of the upper bound, using Proposition 7.9 for $\mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))^2$ and Proposition 7.7 to find $\delta_+ \in (\delta,\delta_0)$, $c_- > 0$ and $\mu_\varepsilon \in \mathcal{M}_1(A(\alpha_0;\delta_+))$ such that for every $\varepsilon \in (0,1)$

$$\int \mathbb{E}\tau_{B(\alpha;\delta_+)}(X(\cdot;x))\,\mu_\varepsilon(\mathrm{d}x) \geq \mathtt{Pr}(\varepsilon)(1-c_-\varepsilon).$$

□

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A Besov spaces

Definition A.1 Let $\alpha \in \mathbb{R}$ and $p,q \in [1,\infty]$. The Besov norm $\|\cdot\|_{\mathcal{B}^\alpha_{p,q}}$ is defined as

$$\|f\|_{\mathcal{B}^\alpha_{p,q}} := \left\| \left(2^{\alpha\kappa}\|f * \eta_\kappa\|_{L^p}\right)_{\kappa\geq -1}\right\|_{\ell^q}. \tag{A.1}$$

Here the family of functions $\{\eta_\kappa\}_{\kappa\geq-1}$ is given by $\hat{\eta}_\kappa = \chi_\kappa$ in Fourier space for $\{\chi_\kappa\}_{\kappa\geq-1}$ a suitable dyadic partition of unity as in [1, Proposition 2.10]. The Besov space space $\mathcal{B}^\alpha_{p,q}$ is defined as the completion of $\mathcal{C}^\infty$ with respect to (A.1).

In this appendix we present several useful results from [19,20] about Besov spaces that we repeatedly use in this article. For a complete survey of the full-space analogues of these results we refer the reader to [1]. A discussion on the validity of these results in the periodic case can be found in [20, Section 4.2].

The following estimate is immediate from the definition of the Besov norm (A.1),

$$\|f\|_{\mathcal{B}^\alpha_{p,q}} \leq C\|f\|_{\mathcal{B}^\beta_{p,q}}, \text{ if } \beta > \alpha. \tag{A.2}$$

Proposition A.2 ([20, Remark 9]) *Let $\alpha \in \mathbb{R}$ and $p, q_1, q_2 \in [1,\infty]$ such that $q_2 > q_1$. For every $\lambda > 0$*

$$\|f\|_{\mathcal{B}^\alpha_{p,q_2}} \leq C\|f\|_{\mathcal{B}^{\alpha+\lambda}_{p,q_1}}. \tag{A.3}$$

Proposition A.3 ([20, Remarks 10 and 11]) *For every* $p \in [1, \infty]$

$$C^{-1}\|f\|_{\mathcal{B}^0_{p,\infty}} \le \|f\|_{L^p} \le C\|f\|_{\mathcal{B}^0_{p,1}}. \tag{A.4}$$

Proposition A.4 ([20, Proposition 2]) *Let* $\alpha, \beta \in \mathbb{R}$ *and* $p, q \ge 1$ *such that* $p \ge q$ *and* $\beta = \alpha + d\left(\frac{1}{q} - \frac{1}{p}\right)$. *Then*

$$\|f\|_{\mathcal{B}^\alpha_{p,\infty}} \le C\|f\|_{\mathcal{B}^\beta_{q,\infty}}. \tag{A.5}$$

Proposition A.5 ([20, Proposition 5]) *For every* $\beta \ge \alpha$

$$\|e^{t\Delta} f\|_{\mathcal{B}^\beta_{p,q}} \le C(t \wedge 1)^{\frac{\alpha-\beta}{2}}\|f\|_{\mathcal{B}^\alpha_{p,q}}. \tag{A.6}$$

Proposition A.6 ([20, Corollary 1]) *Let* $\alpha \ge 0$ *and* $p, q \in [1, \infty]$. *Then*

$$\|fg\|_{\mathcal{B}^\alpha_{p,q}} \le C\|f\|_{\mathcal{B}^\alpha_{p_1,q_1}}\|g\|_{\mathcal{B}^\alpha_{p_2,q_2}}, \tag{A.7}$$

where $p = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $p = \frac{1}{q_1} + \frac{1}{q_2}$.

Proposition A.7 ([20, Corollary 2]) *Let* $\alpha < 0, \beta > 0$ *such that* $\alpha + \beta > 0$ *and* $p, q \in [1, \infty]$. *Then*

$$\|fg\|_{\mathcal{B}^\alpha_{p,q}} \le C\|f\|_{\mathcal{B}^\alpha_{p_1,q_1}}\|g\|_{\mathcal{B}^\beta_{p_2,q_2}}, \tag{A.8}$$

where $p = \frac{1}{p_1} + \frac{1}{p_2}$ *and* $p = \frac{1}{q_1} + \frac{1}{q_2}$.

Proposition A.8 ([20, Proposition 10]) *For every* $\alpha < \alpha'$ *the embedding* $\mathcal{C}^{\alpha'} \hookrightarrow \mathcal{B}^\alpha_{\infty,1}$ *is compact.*

Proposition A.9 ([19, Proposition A.6]) *For every* $p \in [1, \infty)$

$$\|f\|_{\mathcal{B}^1_{p,\infty}} \le C(\|\nabla f\|_{L^p} + \|f\|_{L^p}).$$

Proposition A.10 ([19, Corollary A.8]) *Let* $\alpha > 0$ *and* $p, q \in [1, \infty]$. *Then*

$$\|f^2\|_{\mathcal{B}^\alpha_{p,q}} \le C\|f\|_{L^{p_1}}\|f\|_{\mathcal{B}^\alpha_{p_2,q}}, \tag{A.9}$$

where $p = \frac{1}{p_1} + \frac{1}{p_2}$.

In the next proposition we prove convergence of the Galerkin approximations $\Pi_N f$ to f in Besov spaces. Here we use that the projection $\Pi_N f$ is defined as the convolution of f with the 2-dimensional square Dirichlet kernel, which satisfies a logarithmic growth bound in the L^1 norm.

Proposition A.11 *Let $\Pi_N : L^2 \to L^2$ be the projection on $\{f \in L^2 : f(z) = \sum_{|k|\leq N} \hat{f}(k) L^{-2} \mathrm{e}^{2\mathrm{i}\pi k\cdot z/L}\}$. Then for every $\alpha \in \mathbb{R}$, $p, q \in [1,\infty]$ and $\lambda > 0$*

$$\|\Pi_N f - f\|_{\mathcal{B}^{\alpha}_{p,q}} \leq \frac{C(\log N)^2}{N^{\lambda}} \|f\|_{\mathcal{B}^{\alpha+\lambda}_{p,q}} \tag{A.10}$$

$$\|\Pi_N f\|_{\mathcal{B}^{\alpha}_{p,q}} \leq C\|f\|_{\mathcal{B}^{\alpha+\lambda}_{p,q}}. \tag{A.11}$$

If we furthermore assume that $p = 2$ then

$$\|\Pi_N f - f\|_{\mathcal{B}^{\alpha}_{2,q}} \leq \frac{C}{N^{\lambda}} \|f\|_{\mathcal{B}^{\alpha+\lambda}_{2,q}} \tag{A.12}$$

$$\|\Pi_N f\|_{\mathcal{B}^{\alpha}_{2,q}} \leq \|f\|_{\mathcal{B}^{\alpha}_{2,q}}. \tag{A.13}$$

Proof We first notice that for $c_2 > c_1 > 0$

$$\delta_\kappa (\Pi_N f - f) = \begin{cases} 0, & \text{if } 2^\kappa \leq c_1 N \\ \delta_\kappa f, & \text{if } 2^\kappa > c_2 N \end{cases}.$$

Let $D_N(z) = \sum_{|k|\leq N} L^{-2} \mathrm{e}^{-2\mathrm{i}\pi k\cdot z/L}$ be the square Dirichlet kernel. Then $\Pi_N f = f * D_N$. Using the triangle inequality and Young's inequality for convolution we have that

$$\|\delta_\kappa (\Pi_N f - f)\|_{L^p} \leq (\|D_N\|_{L^1} + 1)\|\delta_\kappa f\|_{L^p}.$$

Thus

$$\|\delta_\kappa(\Pi_N f - f)\|_{L^p} \leq \begin{cases} 0, & \text{if } 2^\kappa \leq c_1 N \\ C(\log N)^2 \|\delta_\kappa f\|_{L^p}, & \text{if } c_1 N \leq 2^\kappa < c_2 N, \\ \|\delta_\kappa f\|_{L^p}, & \text{if } 2^\kappa > c_2 N \end{cases}$$

where in the second case we use that $\|D_N\|_{L^1} \lesssim (\log N)^2$. This bound immediate form the fact that the 2-dimensional square Dirichlet kernel is the product of two 1-dimensional Dirichlet kernels (see [11, Section 3.1.3]). The last implies (A.10) and (A.11). For $p = 2$ we notice that

$$\|\delta_\kappa \Pi_N f\|_{L^2} \leq \|\delta_\kappa f\|_{L^2},$$

which implies (A.12) and (A.13). □

B Generalised Gronwall inequality

Lemma B.1 (Generalised Gronwall inequality) *Let $f : [0,T] \to \mathbb{R}$ be a measurable function and $\sigma_1 + \sigma_2 < 1$ such that*

$$f(t) \leq \mathrm{e}^{-c_0 t} a + b \int_0^t \mathrm{e}^{-c_0(t-s)} (t-s)^{-\sigma_1} s^{-\sigma_2} f(s)\, \mathrm{d}s.$$

Then there exists $c, C > 0$ *such that*

$$f(t) \le C \exp\left\{-c_0 t + c b^{\frac{1}{1-\sigma_1-\sigma_2}} t\right\} a.$$

Proof The lemma is essentially [13, Lemma 5.7] if we set $x(t) = \mathrm{e}^{c_0 t} f(t)$ with their notation. □

Lemma B.2 *Let* $\alpha + \beta < 1$ *and* $c > 0$. *Then*

$$\sup_{t \ge 0} \int_0^t (t-s)^{-\alpha} (s \wedge 1)^{-\beta} \mathrm{e}^{-c(t-s)} \, \mathrm{d}s < \infty.$$

Proof Assume $t \ge 1$. Then

$$\int_0^1 (t-s)^{-\alpha} (s \wedge 1)^{-\beta} \mathrm{e}^{-c(t-s)} \, \mathrm{d}s \lesssim \mathrm{e}^{-ct} \int_0^t (t-s)^{-\alpha} (s \wedge 1)^{-\beta} \, \mathrm{d}s \lesssim t^{1-\alpha-\beta} \mathrm{e}^{-ct}$$

and

$$\begin{aligned} \int_1^t (t-s)^{-\alpha} (s \wedge 1)^{-\beta} \mathrm{e}^{-c(t-s)} \, \mathrm{d}s &\le \int_0^t s^{-\alpha} \mathrm{e}^{-cs} \, \mathrm{d}s \lesssim 1 + \int_1^t s^{-\alpha} \mathrm{e}^{-cs} \, \mathrm{d}s \\ &\lesssim 1 + \int_1^t \mathrm{e}^{-cs} \, \mathrm{d}s. \end{aligned}$$

The above implies that

$$\sup_{t \ge 1} \int_0^t (t-s)^{-\alpha} (s \wedge 1)^{-\beta} \mathrm{e}^{-c(t-s)} \, \mathrm{d}s < \infty.$$

The bound for $t \le 1$ follows easily. □

C Deterministic dynamics

Propositions C.1 and C.2 are a consequence of [9, Section 8] and [16, Appendix B.1]. Although the results in [9, Section 8] concern 1 space-dimension they can be easily generalised in 2 space-dimensions. For consistency we have also replaced the space H^1 appearing in [9, Section 8] by $\mathcal{B}^1_{2,2}$. The fact that these spaces coincide is immediate from Definition A.1 for $p = q = 2$ if we rewrite $\| f * \eta_k \|_{L^2}$ using Plancherel's identity.

Proposition C.1 *For every* $x \in \mathcal{B}^1_{2,2}$ *there exists* $x_* \in \{-1, 0, 1\}$ *such that* $X_{det}(t; x) \overset{\mathcal{B}^1_{2,2}}{\to} x_*$.

Proposition C.2 *For every* $\delta > 0$ *there exists* $x_\pm \in B_{\mathcal{B}^1_{2,2}}(0; \delta)$ *such that* $X_{det}(t; x_\pm) \overset{\mathcal{B}^1_{2,2}}{\to} \pm 1$.

Proposition C.3 *Let $R > 0$. Then there exists $C \equiv C(R) > 0$ such that for every $\lambda > 0$ sufficiently small*

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \|X_{det}(1; x)\|_{\mathcal{C}^{2+\lambda}} \leq C.$$

Proof By [24, Theorem 3.3, Theorem 3.9] there exists $C \equiv C(R) > 0$ such that

$$\sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \sup_{t \leq 1} t^{\gamma} \|X_{\mathrm{det}}(t; x)\|_{\mathcal{C}^{\beta}} \leq C.$$

Let $S(t) = \mathrm{e}^{\Delta t}$. Using the mild form we write

$$X_{det}(1; x) = S(1/2) X_{det}(1/2; x) - \int_{1/2}^{1} S(1-s) \left(X_{det}(s; x)^3 + X_{det}(s; x) \right) \mathrm{d}s.$$

Then

$$\begin{aligned} &\|X_{det}(1; x)\|_{\mathcal{C}^{2+\lambda}} \\ &\quad \lesssim \|X_{det}(1/2; x)\|_{\mathcal{C}^{\beta}} + \int_{1/2}^{1} (1-s)^{-\frac{2+\lambda-\beta}{2}} \left(\|X_{det}(s; x)\|_{\mathcal{C}^{\beta}}^3 + \|X_{det}(s; x)\|_{\mathcal{C}^{\beta}} \right) \end{aligned}$$

and if we choose $\lambda < \beta$ the above implies that

$$\begin{aligned} \sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \|X_{det}(1; x)\|_{\mathcal{C}^{2+\lambda}} &\lesssim \sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \sup_{t \leq 1} t^{3\gamma} \|X_{\mathrm{det}}(t; x)\|_{\mathcal{C}^{\beta}}^3 \\ &\quad + \sup_{\|x\|_{\mathcal{C}^{-\alpha_0}} \leq R} \sup_{t \leq 1} t^{\gamma} \|X_{\mathrm{det}}(t; x)\|_{\mathcal{C}^{\beta}}. \end{aligned}$$

□

D Stretched exponential moments for the stochastic objects

Proposition D.1 *For every $n \geq 1$ there exists $c \equiv c(n) > 0$ such that*

$$\sup_{k \geq 0} \mathbb{E} \exp \left\{ c \left(\sup_{t \in [k, k+1]} (t \wedge 1)^{(n-1)\alpha'} \|\overset{n}{\curlyvee}(t)\|_{\mathcal{C}^{-\alpha}} \right)^{\frac{2}{n}} \right\} < \infty.$$

Proof Following step by step the proof of [24, Theorem 2.1] but using the explicit bound in Nelson's estimate [24, Equation (B.3)] (see also [5, Section 1.6]), we have that for every $p \geq 1$

$$\sup_{k \geq 0} \mathbb{E} \left(\sup_{t \in [k, k+1]} (t \wedge 1)^{(n-1)\alpha'} \|\overset{n}{\curlyvee}(t)\|_{\mathcal{C}^{-\alpha}} \right)^{p} \leq (p-1)^{\frac{n}{2} p} C_n^{\frac{p}{2}},$$

for some $C_n > 0$. Then for any $c > 0$

$$\begin{aligned}
&\mathbb{E}\exp\left\{c\left(\sup_{t\in[k,k+1]}(t\wedge 1)^{(n-1)\alpha'}\|\overset{n}{\vee}(t)\|_{\mathcal{C}^{-\alpha}}\right)^{\frac{2}{n}}\right\}\\
&\quad=\sum_{k\geq 0}\frac{c^p\mathbb{E}\left(\sup_{t\in[k,k+1]}(t\wedge 1)^{(n-1)\alpha'}\|\overset{n}{\vee}_{-\infty}(t)\|_{\mathcal{C}^{-\alpha}}\right)^{\frac{2}{n}p}}{p!}\\
&\quad\leq\sum_{p\geq 0}\frac{c^p(p-1)^p(C_n)^{\frac{p}{n}}}{p!}
\end{aligned}$$

and by choosing $c\equiv c(n)>0$ sufficiently small the series converges. □

E An estimate for stochastically dominated random variables

Lemma E.1 *Let $g_1, \tilde{g}_1$ be positive random variables such that*

$$\mathbb{P}(g_1\geq g)\leq\mathbb{P}(\tilde{g}_1\geq g),$$

for every $g\geq 0$ and let F be a positive decreasing measurable function on $[0,\infty)$. Then

$$\int_0^\infty F(g)\,\mu_{g_1}(\mathrm{d}g)\geq\int_0^\infty F(g)\,\mu_{\tilde{g}_1}(\mathrm{d}g),$$

where μ_{g_1} and $\mu_{\tilde{g}_1}$ is the law of g_1 and $\tilde{g}_1$.

Proof We first assume that F is smooth. Then $\frac{\mathrm{d}}{\mathrm{d}g}F(g)\leq 0$ for every $g\geq 0$. Hence

$$\begin{aligned}
\int_0^\infty F(g)\,\mu_{g_1}(\mathrm{d}g) &= F(0)+\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}g}F(g)\,\mathbb{P}(g_1\geq g)\,\mathrm{d}g\geq F(0)\\
&\quad+\int_0^\infty\frac{\mathrm{d}}{\mathrm{d}g}F(g)\,\mathbb{P}(\tilde{g}_1\geq g)\,\mathrm{d}g\\
&=\int_0^\infty F(g)\,\mu_{\tilde{g}_1}(\mathrm{d}g),
\end{aligned}$$

which proves the estimate for F differentiable. To prove the estimate for a general decreasing function F we define $F_\delta = F*\eta_\delta$ for some positive mollifier η_δ to preserve monotonicity and use the last estimate together with the dominated convergence theorem. □

F Proof of Proposition 7.3

Proof of Proposition 7.3 By [24, Proposition 2.3] for every $\alpha > 0$, $p \geq 1$ and $T > 0$

$$\lim_{N\to\infty} \mathbb{E}\left(\sup_{t\leq T}(t \wedge 1)^{(n-1)\alpha'} \|\vee^{n}_N(t) - \vee^{n}(t)\|_{\mathcal{C}^{-\alpha}}\right)^p = 0.$$

Hence $\sup_{t\leq T}(t \wedge 1)^{(n-1)\alpha'} \|\vee^{n}_N(t) - \vee^{n}(t)\|_{\mathcal{C}^{-\alpha}}$ convergences to 0 in probability.

It is enough to prove that

$$\lim_{N\to\infty} \sup_{x\in\aleph} \sup_{t\leq T}(t \wedge 1)^{\gamma} \|v_N(t; x_N) - v(t; x)\|_{\mathcal{C}^{\beta}} = 0.$$

This, convergence in probability of $\sup_{t\leq T} \|\uparrow_N(t) - \uparrow(t)\|_{\mathcal{C}^{-\alpha}}$ to 0 and the embedding $\mathcal{C}^{\beta} \subset \mathcal{C}^{-\alpha}$ (see (A.2)) imply the result.

Let $S(t) = \mathrm{e}^{\Delta t}$. For simplicity we write $v_N(t)$ and $v(t)$ to denote $v_N(t; x_N)$ and $v(t; x)$. Using the mild forms of (7.2) and (2.2) we get

$$\begin{aligned}
\|v_N(t) - v(t)\|_{\mathcal{C}^{\beta}} \leq{}& \underbrace{\|S(t)(x_N - x)\|_{\mathcal{C}^{\beta}}}_{=:I_1} + \underbrace{\int_0^t \|S(t-s)[\Pi_N(v_N(s)^3) - v(s)^3]\|_{\mathcal{C}^{\beta}}\,\mathrm{d}s}_{=:I_2} \\
&+ \underbrace{3\int_0^t \|S(t-s)\left[\Pi_N\left(v_N(s)^2\varepsilon^{\frac{1}{2}}\uparrow_N(s)\right) - v(s)^2\varepsilon^{\frac{1}{2}}\uparrow(s)\right]\|_{\mathcal{C}^{\beta}}\,\mathrm{d}s}_{=:I_3} \\
&+ \underbrace{3\int_0^t \|S(t-s)[\Pi_N(v_N(s)\varepsilon\mathbf{V}_N(s)) - v(s)\varepsilon\mathbf{V}(s)]\|_{\mathcal{C}^{\beta}}\,\mathrm{d}s}_{=:I_4} \\
&+ \underbrace{\int_0^t \|S(t-s)\left(\Pi_N\varepsilon^{\frac{3}{2}}\Psi_N(s) - \varepsilon^{\frac{3}{2}}\Psi(s)\right)\|_{\mathcal{C}^{\beta}}\,\mathrm{d}s}_{=:I_5} \\
&+ \underbrace{2\int_0^t \|S(t-s)\left(\varepsilon^{\frac{1}{2}}\uparrow_N(s) - \varepsilon^{\frac{1}{2}}\uparrow(s)\right)\|_{\mathcal{C}^{\beta}}\,\mathrm{d}s}_{=:I_6} \\
&+ \underbrace{\int_0^t \|S(t-s)(v_N(s) - v(s))\|_{\mathcal{C}^{\beta}}\,\mathrm{d}s}_{=:I_7}.
\end{aligned} \tag{F.1}$$

Let $\iota = \inf\{t > 0 : (t \wedge 1)^{\gamma}\|v_N(t) - v(t)\|_{\mathcal{C}^{\beta}} \geq 1\}$ and $t \leq T \wedge \iota$. We treat each of the terms in (F.1) separately. Below the parameters α and λ can be taken arbitrarily small and all the implicit constants depend on $\sup_{t\leq T}(t \wedge 1)^{(n-1)\alpha'}\|\vee^{n}(t)\|_{\mathcal{C}^{-\alpha}}$, and $\sup_{x\in\aleph}\sup_{t\leq T}(t \wedge 1)^{\gamma}\|v(t)\|_{\mathcal{C}^{\beta}}$.

Term I_1:

$$I_1 \overset{(A.6)}{\lesssim} (t\wedge 1)^{-\frac{\alpha_0+\beta}{2}} \sup_{x\in\aleph} \|x_N - x\|_{\mathcal{C}^{-\alpha_0}}$$

Term I_2:

$$\begin{aligned}
I_2 &\overset{(A.6)}{\lesssim} \int_0^t \Big((t-s)^{-\frac{\lambda}{2}} \|\Pi_N(v_N(s)^3) - v_N(s)^3\|_{\mathcal{C}^{\beta-\lambda}} + \|v_N(s)^3 - v(s)^3\|_{\mathcal{C}^\beta}\Big)\, \mathrm{d}s \\
&\overset{(A.10)}{\lesssim} \int_0^t (t-s)^{-\frac{\lambda}{2}} \left(\frac{(\log N)^2}{N^\lambda} \|v_N(s)^3\|_{\mathcal{C}^\beta} + \|v_N(s)^3 - v(s)^3\|_{\mathcal{C}^{\beta-\lambda}}\right) \mathrm{d}s \\
&\overset{(A.7)}{\lesssim} \int_0^t \Big[(t-s)^{-\frac{\lambda}{2}} \frac{(\log N)^2}{N^\lambda} \|v_N(s)\|^3_{\mathcal{C}^\beta} + \|v_N(s) - v(s)\|_{\mathcal{C}^\beta} \\
&\quad \times \Big(\|v_N(s)\|^2_{\mathcal{C}^\beta} + \|v_N(s)\|_{\mathcal{C}^\beta}\|v(s)\|_{\mathcal{C}^\beta} + \|v(s)\|^2_{\mathcal{C}^\beta}\Big)\Big]\, \mathrm{d}s \\
&\lesssim \int_0^t \left((t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}} \frac{(\log N)^2}{N^\lambda} (s\wedge 1)^{-3\gamma} + (s\wedge 1)^{-2\gamma} \|v_N(s) - v(s)\|_{\mathcal{C}^\beta}\right) \mathrm{d}s.
\end{aligned}$$

Term I_3:

$$\begin{aligned}
I_3 &\overset{(A.6)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \|\Pi_N(v_N(s)^2 \cirmid_N(s)) - v(s)^2 \cirmid(s)\|_{\mathcal{C}^{-\alpha-\lambda}}\, \mathrm{d}s \\
&\overset{(A.10)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \left(\frac{(\log N)^2}{N^\lambda} \|v_N(s)^2 \cirmid_N(s)\|_{\mathcal{C}^{-\alpha}} + \|v_N(s)^2(\cirmid_N(s) - \cirmid(s))\|_{\mathcal{C}^{-\alpha}} \right. \\
&\quad \left. + \|\cirmid(s)(v_N(s)^2 - v(s)^2)\|_{\mathcal{C}^{-\alpha}}\right) \mathrm{d}s \\
&\overset{(A.8),(A.7)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \Big[\frac{(\log N)^2}{N^\lambda} \|v_N(s)\|^2_{\mathcal{C}^\beta} \|\cirmid_N(s)\|_{\mathcal{C}^{-\alpha}} + \|v_N(s)\|^2_{\mathcal{C}^\beta} \|\cirmid_N(s) \\
&\quad - \cirmid(s)\|_{\mathcal{C}^{-\alpha}} + \big(\|v_N(s)\|_{\mathcal{C}^\beta} + \|v(s)\|_{\mathcal{C}^\beta}\big) \|v_N(s) - v(s)\|_{\mathcal{C}^\beta} \|\cirmid(s)\|_{\mathcal{C}^{-\alpha}} \Big]\, \mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \left(\frac{(\log N)^2}{N^\lambda} (s\wedge 1)^{-2\gamma} + (s\wedge 1)^{-2\gamma} \|\cirmid_N(s) - \cirmid(s)\|_{\mathcal{C}^{-\alpha}} \right. \\
&\quad \left. + (s\wedge 1)^{-\gamma} \|v_N(s) - v(s)\|_{\mathcal{C}^\beta}\right) \mathrm{d}s.
\end{aligned}$$

Term I_4: Similarly to I_3,

$$\begin{aligned}
I_4 &\lesssim \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \left(\frac{(\log N)^2}{N^\lambda} (s\wedge 1)^{-\gamma-\alpha'} + (s\wedge 1)^{-\gamma} \|\Psi_N(s) - \Psi(s)\|_{\mathcal{C}^{-\alpha}} \right. \\
&\quad \left. + (s\wedge 1)^{-\alpha'} \|v_N(s) - v(s)\|_{\mathcal{C}^\beta}\right) \mathrm{d}s.
\end{aligned}$$

Term I_5:

$$
\begin{aligned}
I_5 &\overset{(A.6)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \|\Pi_N \Psi_N(s) - \Psi(s)\|_{\mathcal{C}^{-\alpha-\lambda}}\, \mathrm{d}s \\
&\overset{(A.10)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} \left(\frac{(\log N)^2}{N^\lambda} (s\wedge 1)^{-2\alpha'} + \|\Psi_N(s) - \Psi(s)\|_{\mathcal{C}^{-\alpha}} \right) \mathrm{d}s.
\end{aligned}
$$

Terms I_6, I_7:

$$
\begin{aligned}
I_6 &\overset{(A.6)}{\lesssim} \int_0^t (t-s)^{-\frac{\alpha+\beta}{2}} \|\uparrow_N(s) - \uparrow(s)\|_{\mathcal{C}^{-\alpha}}\, \mathrm{d}s. \\
I_7 &\overset{(A.6)}{\lesssim} \int_0^t \|v_N(s) - v(s)\|_{\mathcal{C}^{\beta}}\, \mathrm{d}s.
\end{aligned}
$$

Combining the above estimates we obtain that for $t \le T \wedge \iota$

$$
\begin{aligned}
\|v_N(t) - v(t)\|_{\mathcal{C}^\beta} &\lesssim (t\wedge 1)^{-\frac{\alpha_0+\beta}{2}} \sup_{x\in\aleph} \|x_N - x\|_{\mathcal{C}^{-\alpha_0}} \\
&+ T^{1-\frac{\alpha+\beta+\lambda}{2}-3\gamma} \left(\frac{(\log N)^2}{N} + \sup_{t\le T} (t\wedge 1)^{(n-1)\alpha'} \|\vee^{n}_N(t) - \vee^{n}(t)\|_{\mathcal{C}^{-\alpha}} \right) \\
&+ \int_0^t (t-s)^{-\frac{\alpha+\beta+\lambda}{2}} (s\wedge 1)^{-2\gamma} \|v_N(s) - v(s)\|_{\mathcal{C}^\beta}\, \mathrm{d}s.
\end{aligned}
$$

By the generalised Gronwall inequality, Lemma B.1, on $f(t) = (t\wedge 1)^\gamma \|v_N(t) - v(t)\|_{\mathcal{C}^\beta}$ we find $C \equiv C(T) > 0$ such that

$$
\begin{aligned}
&\sup_{t\le T\wedge\iota} (t\wedge 1)^\gamma \|v_N(t) - v(t)\|_{\mathcal{C}^\beta} \\
&\quad\le C \Bigg(\sup_{x\in\aleph} \|x_N - x\|_{\mathcal{C}^{-\alpha_0}} + \frac{(\log N)^2}{N} \\
&\qquad + \sup_{t\le T} (t\wedge 1)^{(n-1)\alpha'} \|\vee^{n}_N(t) - \vee^{n}(t)\|_{\mathcal{C}^{-\alpha}} \Bigg).
\end{aligned}
$$

This and convergence of $\sup_{t\le T} \|\vee^{n}_N(t) - \vee^{n}(t)\|_{\mathcal{C}^{-\alpha}}$ to 0 in probability imply the result. □

G Local existence in L^2-based Besov spaces

In this section we fix $\beta \in \left(\frac{1}{3}, \frac{2}{3}\right)$, $\gamma \in \left(\frac{\beta}{2}, \frac{1}{3}\right)$ and $p \in (1,2)$ such that

$$
1 - \frac{2}{3p} < \beta \text{ and } 1 - \frac{\beta + \frac{2}{p} - 1}{2} - 2\gamma > 0.
$$

The next proposition provides local existence of (7.2) in $\mathcal{B}^{\beta}_{2,2}$ up to some time $T_* > 0$ which is uniform in the regularisation parameter N.

Proposition G.1 *Let* $K, R, T > 0$ *such that* $\|x\|_{\mathcal{B}^{-\alpha_0}_{2,2}} \leq R$ *and* $\sup_{t\leq T}(t \wedge 1)^{(n-1)\alpha'} \|\overset{n}{\mathbf{V}}_N(t)\|_{\mathcal{B}^{-\alpha}_{\infty,2}} \leq K$*. Then there exist* $T_* \equiv T_*(K,R) \leq T$ *and* $C \equiv C(K,R) > 0$ *such that* (7.2) *has a unique solution* $v \in C((0,T_*];\mathcal{B}^{\beta}_{2,2})$ *satisfying*

$$\sup_{t\leq T_*}(t\wedge 1)^{\gamma}\|v_N(t;x)\|_{\mathcal{B}^{\beta}_{2,2}} \leq C.$$

Proof Let $S(t) = \mathrm{e}^{\Delta t}$. We define

$$\begin{aligned}\mathscr{T}(v)(t) &:= S(t)x \\ &\quad - \int_0^t S(t-s)\Pi_N\left(v(s)^3 + 3v(s)^2\varepsilon^{\frac{1}{2}}\uparrow_N(s) + 3v(s)\varepsilon\mathbf{V}_N(s) + \varepsilon^{\frac{3}{2}}\mathbf{\Psi}_N(s)\right)\mathrm{d}s \\ &\quad + 2\int_0^t S(t-s)\left(\varepsilon^{\frac{1}{2}}\uparrow_N(s) + v(s)\right)\mathrm{d}s.\end{aligned}$$

It is enough to prove that there exists $T_* > 0$ such that $\mathscr{T}$ is a contraction on

$$\mathscr{B}_{T_*} := \left\{ v : \sup_{t\leq T_*}(t\wedge 1)^{\gamma}\|v(t;x)\|_{\mathcal{B}^{\beta}_{2,2}} \leq 1 \right\}.$$

We first prove that for $T_* > 0$ sufficiently small $\mathscr{T}$ maps $\mathscr{B}_{T_*}$ to itself. To do so we notice that

$$\begin{aligned}\|\mathscr{T}(v)(t)\|_{\mathcal{B}^{\beta}_{2,2}} &\lesssim \underbrace{\|S(t)x\|_{\mathcal{B}^{\beta}_{2,2}}}_{=:I_1} + \underbrace{\int_0^t \|S(t-s)v(s)^3\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s}_{=:I_2} \\ &\quad + \underbrace{\int_0^t \|S(t-s)(v(s)^2\uparrow_N(s))\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s}_{=:I_3} \\ &\quad + \underbrace{\int_0^t \|S(t-s)(v(s)\mathbf{V}_N(s))\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s}_{=:I_4} + \underbrace{\int_0^t \|S(t-s)\mathbf{\Psi}_N(s)\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s}_{=:I_5} \\ &\quad + \underbrace{\int_0^t \|S(t-s)\uparrow_N(s)\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s}_{=:I_6} + \underbrace{\int_0^t \|S(t-s)v(s)\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s}_{=:I_7},\end{aligned}$$

where we use (A.13) together with the relation $S(\cdot)\Pi_N = \Pi_N S(\cdot)$ to drop Π_N. We treat each term separately.

Term I_1:

$$I_1 \overset{(A.6)}{\lesssim} (t\wedge 1)^{-\frac{\alpha_0+\beta}{2}} \|x\|_{\mathcal{B}^{-\alpha_0}_{2,2}} \lesssim (t\wedge 1)^{-\frac{\alpha_0+\beta}{2}} R.$$

Term I_2:

$$\begin{aligned}
I_2 &\overset{(A.5)}{\lesssim} \int_0^t \|S(t-s)v(s)^3\|_{\mathcal{B}^{\beta+\frac{2}{p}-1}_{p,2}}\,\mathrm{d}s \overset{(A.6)}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}} \|v(s)^3\|_{\mathcal{B}^0_{p,2}}\,\mathrm{d}s\\
&\overset{(A.7)}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}} \|v(s)\|^3_{\mathcal{B}^0_{3p,2}}\,\mathrm{d}s \overset{(A.5)}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}} \|v(s)\|^3_{\mathcal{B}^{1-\frac{2}{3p}}_{2,2}}\,\mathrm{d}s\\
&\overset{1-\frac{2}{3p}<\beta}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}} \|v(s)\|^3_{\mathcal{B}^\beta_{2,2}}\,\mathrm{d}s \lesssim \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1}{2}} (s\wedge 1)^{-3\gamma}\,\mathrm{d}s.
\end{aligned}$$

Term I_3:

$$\begin{aligned}
I_3 &\overset{(A.5),(A.6)}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} \|v(s)^2 \mathsf{I}_N(s)\|_{\mathcal{B}^{-\alpha}_{p,2}}\,\mathrm{d}s\\
&\overset{(A.8),(A.7)}{\lesssim} K\int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} \|v(s)\|^2_{\mathcal{B}^{\alpha+\lambda}_{2p,2}}\,\mathrm{d}s\\
&\overset{(A.5)}{\lesssim} K\int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} \|v(s)\|^2_{\mathcal{B}^{\alpha+\lambda+1-\frac{1}{p}}_{2,2}}\,\mathrm{d}s\\
&\overset{1-\frac{2}{3p}<\beta}{\lesssim} K\int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} \|v(s)\|^2_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s\\
&\lesssim K\int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} (s\wedge 1)^{-2\gamma}\,\mathrm{d}s.
\end{aligned}$$

Term I_4:

$$\begin{aligned}
I_4 &\overset{(A.5),(A.6)}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} \|v(s)\mathsf{V}_N(s)\|_{\mathcal{B}^{-\alpha}_{p,2}}\,\mathrm{d}s\\
&\overset{(A.8),(A.5)}{\lesssim} K\int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} (s\wedge 1)^{-\alpha'} \|v(s)\|_{\mathcal{B}^{\alpha+\lambda+1-\frac{2}{p}}_{2,2}}\,\mathrm{d}s\\
&\overset{1-\frac{2}{3p}<\beta}{\lesssim} K\int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} (s\wedge 1)^{-\alpha'} \|v(s)\|_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s\\
&\lesssim K\int_0^t (t-s)^{-\frac{\beta+\frac{2}{p}-1+\alpha}{2}} (s\wedge 1)^{-\gamma-\alpha'}\,\mathrm{d}s.
\end{aligned}$$

Terms I_5, I_6, I_7:

$$I_5 \overset{(A.6)}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\alpha}{2}} \|\mathsf{W}_N(s)\|_{\mathcal{B}^{-\alpha}_{2,2}}\,\mathrm{d}s \lesssim K\int_0^t (t-s)^{-\frac{\beta+\alpha}{2}} (s\wedge 1)^{-2\alpha'}\,\mathrm{d}s.$$

$$I_6 \overset{(A.6)}{\lesssim} \int_0^t (t-s)^{-\frac{\beta+\alpha}{2}} \|\stackrel{\bullet}{|}_N(s)\|_{\mathcal{B}^{-\alpha}_{2,2}} \,\mathrm{d}s \lesssim K \int_0^t (t-s)^{-\frac{\beta+\alpha}{2}} \,\mathrm{d}s.$$

$$I_7 \overset{(A.6)}{\lesssim} \int_0^t \|v(s)\|_{\mathcal{B}^{\beta}_{2,2}} \,\mathrm{d}s \lesssim \int_0^t (s\wedge 1)^{-\gamma} \,\mathrm{d}s.$$

Combining all the above we find $C \equiv C(K, R) > 0$ such that

$$\sup_{t\leq T_*} (t\wedge 1)^{\gamma} \|\mathscr{T}(v)(t)\|_{\mathcal{B}^{\beta}_{2,2}} \leq C T_*^{\theta},$$

for some $\theta \equiv \theta(\alpha, \alpha', \alpha_0, \beta, \gamma) \in (0,1)$. Choosing $T_* > 0$ sufficiently small the above implies that

$$\sup_{t\leq T_*} (t\wedge 1)^{\gamma} \|\mathscr{T}(v)(t)\|_{\mathcal{B}^{\beta}_{2,2}} \leq 1.$$

Hence for this choice of T_*, $\mathscr{T}$ maps $\mathscr{B}_{T_*}$ to itself. In a similar way, but by possibly choosing a smaller value of T_*, we prove that $\mathscr{T}$ is a contraction on $\mathscr{B}_{T_*}$. For simplicity we omit the proof. That way we obtain a unique solution $v \in C((0, T_*]; \mathcal{B}^{\beta}_{2,2})$. We can furthermore assume that T_* is maximal in the sense that either $T_* = T$ or $\lim_{t\nearrow T_*} \|v(t;x)\|_{\mathcal{B}^{\beta}_{2,2}} = \infty$. ☐

Proposition G.2 *For every $t_0 \in (0,1)$, and $K, R > 0$ there exists $C \equiv C(t_0, K, R) > 0$ such that if $\|x\|_{\mathcal{B}^{-\alpha}_{2,2}} \leq R$ and $\sup_{t\leq 1} t^{(n-1)\alpha'} \|\varepsilon^{\frac{n}{2}} \stackrel{n}{\bigvee}_N(t)\|_{\mathcal{C}^{-\alpha}} \leq K$ then*

$$\sup_{\|x\|_{\mathcal{B}^{-\alpha}_{2,2}}\leq R} \|X_N(t_0;x)\|_{\mathcal{C}^{-\alpha}} \leq C.$$

Proof Using the a priori estimate in Proposition 7.4 we can assume that $T_* = 1$ in Proposition G.1. This implies that

$$\sup_{\|x\|_{\mathcal{B}^{-\alpha}_{2,2}}\leq R} \sup_{t\leq 1} t^{\gamma} \|v_N(t;x)\|_{\mathcal{B}^{\beta}_{2,2}} \leq C. \tag{G.1}$$

For simplicity we assume that $t_0 = 1$. Let $S(t) = \mathrm{e}^{\Delta t}$. Using the mild form of (7.2) we obtain that

$$\begin{aligned}
&\|v_N(1)\|_{\mathcal{C}^{-\alpha}} \\
&\lesssim \underbrace{\|S(1/2)v_N(1/2)\|_{\mathcal{C}^{-\alpha}}}_{=:I_1} + \underbrace{\int_{1/2}^1 \|S(1-s)\Pi_N(v_N(s))^3\|_{\mathcal{C}^{-\alpha}} \,\mathrm{d}s}_{=:I_2} \\
&\quad + \underbrace{\int_{1/2}^1 \|S(1-s)\Pi_N\left(v_N(s)^2 \varepsilon^{\frac{1}{2}} \stackrel{\bullet}{|}_N(s)\right)\|_{\mathcal{C}^{-\alpha}} \,\mathrm{d}s}_{=:I_3}
\end{aligned}$$

$$+ \underbrace{\int_{1/2}^{1} \|S(1-s)\Pi_N \left(v_N(s)\varepsilon \vee_N(s) \right)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s}_{=:I_4}$$

$$+ \underbrace{\int_{1/2}^{1} \|S(1-s)\Pi_N \varepsilon^{\frac{3}{2}} \Psi_N(s)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s}_{=:I_5} + \underbrace{\int_{1/2}^{1} \|S(1-s)\varepsilon^{\frac{1}{2}} \uparrow_N(s)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s}_{=:I_6}$$

$$+ \underbrace{\int_{1/2}^{1} \|S(1-s)v_N(s)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s}_{=:I_7}.$$

We treat each term separately.
Term I_1:

$$I_1 \overset{(A.5)}{\lesssim} \|S(1/2)v_N(1/2)\|_{\mathcal{B}^{-\alpha+1}_{2,\infty}} \overset{(A.6)}{\lesssim} \|v_N(1/2)\|_{\mathcal{B}^{-\alpha}_{2,\infty}} \lesssim \|v_N(1/2)\|_{\mathcal{B}^{-\alpha}_{2,2}}$$

Term I_2:

$$\begin{aligned}
I_2 &\overset{(A.5)}{\lesssim} \int_{1/2}^{1} \|S(1-s)\Pi_N(v_N(s)^3)\|_{\mathcal{B}^{-\alpha+\frac{2}{p}}_{p,\infty}}\,\mathrm{d}s \\
&\overset{(A.6)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+\frac{2}{p}+\lambda}{2}} \|\Pi_N(v_N(s)^3)\|_{\mathcal{B}^{-\lambda}_{p,\infty}}\,\mathrm{d}s \\
&\overset{(A.11),(A.7)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+\frac{2}{p}+\lambda}{2}} \|v_N(s)\|^3_{\mathcal{B}^{0}_{3p,\infty}}\,\mathrm{d}s \\
&\overset{(A.5)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+\frac{2}{p}+\lambda}{2}} \|v_N(s)\|^3_{\mathcal{B}^{1-\frac{2}{3p}}_{2,\infty}}\,\mathrm{d}s \\
&\overset{1-\frac{2}{3p}<\beta}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+\frac{2}{p}+\lambda}{2}} \|v_N(s)\|^3_{\mathcal{B}^{\beta}_{2,2}}\,\mathrm{d}s.
\end{aligned}$$

Term I_3:

$$\begin{aligned}
I_3 &\overset{(A.5)}{\lesssim} \int_{1/2}^{1} \|S(1-s)\Pi_N \left(v_N(s)^2 \varepsilon^{\frac{1}{2}} \uparrow_N(s) \right)\|_{\mathcal{B}^{-\alpha+\frac{2}{p}}_{p,\infty}}\,\mathrm{d}s \\
&\overset{(A.6)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \|\Pi_N \left(v_N(s)^2 \varepsilon^{\frac{1}{2}} \uparrow_N(s) \right)\|_{\mathcal{B}^{-\alpha-\lambda}_{p,\infty}}\,\mathrm{d}s \\
&\overset{(A.11),(A.8),(A.7)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \|v_N(s)\|^2_{\mathcal{B}^{\alpha+\lambda}_{2p,\infty}} \|\varepsilon^{\frac{1}{2}} \uparrow_N(s)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s \\
&\overset{(A.5)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \|v_N(s)\|^2_{\mathcal{B}^{\alpha+\lambda+1-\frac{1}{p}}_{2,\infty}} \|\varepsilon^{\frac{1}{2}} \uparrow_N(s)\|_{\mathcal{C}^{-\alpha}}\,\mathrm{d}s
\end{aligned}$$

$$\overset{1-\frac{2}{3p}<\beta}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \|v_N(s)\|^2_{\mathcal{B}^{\beta}_{2,2}} \|\varepsilon^{\frac{1}{2}} \dot{\mathbf{I}}_N(s)\|_{\mathcal{C}^{-\alpha}}\, \mathrm{d}s.$$

Term I_4: Similarly to I_3,

$$I_4 \lesssim \int_{1/2}^{1} (1-s)^{-\frac{\frac{2}{p}+\lambda}{2}} \|v_N(s)\|_{\mathcal{B}^{\beta}_{2,2}} \|\varepsilon \mathbf{\Upsilon}_N(s)\|_{\mathcal{C}^{-\alpha}}\, \mathrm{d}s.$$

Terms I_5, I_6, I_7:

$$\begin{aligned}
I_5 &\overset{(A.6)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\lambda}{2}} \|\Pi_N \varepsilon^{\frac{3}{2}} \Psi_N(s)\|_{\mathcal{C}^{-\alpha-\lambda}}\, \mathrm{d}s \\
&\overset{(A.11)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\lambda}{2}} \|\varepsilon^{\frac{3}{2}} \Psi_N(s)\|_{\mathcal{C}^{-\alpha-\lambda}}\, \mathrm{d}s. \\
I_6 &\overset{(A.6)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{\lambda}{2}} \|\varepsilon^{\frac{1}{2}} \dot{\mathbf{I}}_N(s)\|_{\mathcal{C}^{-\alpha-\lambda}}\, \mathrm{d}s. \\
I_7 &\overset{(A.5)}{\lesssim} \int_{1/2}^{1} \|S(1-s) v_N(s)\|_{\mathcal{B}^{-\alpha+1}_{2,2}}\, \mathrm{d}s \overset{(A.6)}{\lesssim} \int_{1/2}^{1} (1-s)^{-\frac{-\alpha+1-\beta}{2}} \|v_N(s)\|_{\mathcal{B}^{\beta}_{2,2}}\, \mathrm{d}s.
\end{aligned}$$

The proof is complete if we combine these estimates with (G.1). □

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