Probability Theory and Related Fields (2019) 175:1177–1181
https://doi.org/10.1007/s00440-019-00930-4

CORRECTION

Correction to: Multivariate approximations in Wasserstein distance by Stein's method and Bismut's formula

Xiao Fang[1] · Qi-Man Shao[1] · Lihu Xu[2,3]

Published online: 11 July 2019

Correction to: Probability Theory and Related Fields https://doi.org/10.1007/s00440-018-0874-5

We write this note to correct [1, **(6.9), (6.13), (7.1), (7.2)**] because there was one term *missed* in [1, (6.9)]. To estimate this missed term, we need to add an extra condition to [1, Assumption 2.1]:

Assumption 2.1 $g \in \mathcal{C}^3(\mathbb{R}^d, \mathbb{R}^d)$, and there exist $\theta_0 > 0$ and $\theta_1, \theta_2, \theta_3, \theta_3' \geq 0$ such that the conditions (2.3) and (2.4) in [1] hold, i.e.,

$$\langle u, \nabla_u g(x) \rangle \leq -\theta_0 \left(1 + \theta_1 |x|^{\theta_2}\right) |u|^2, \qquad \forall\, u, x \in \mathbb{R}^d;$$
$$|\nabla_{u_1} \nabla_{u_2} g(x)| \leq \theta_3 (1 + \theta_1 |x|)^{\theta_2 - 1} |u_1||u_2|, \qquad \forall\, u_1, u_2, x \in \mathbb{R}^d.$$

and **additionally**,

$$|\nabla_{u_1} \nabla_{u_2} \nabla_{u_3} g(x)| \leq \theta_3' (1 + |x|)^{\theta_2 - 2} |u_1||u_2||u_3|, \qquad \forall\, u_1, u_2, u_3, x \in \mathbb{R}^d;$$

Under the above-strengthened Assumption 2.1, all the conclusions and examples in [1] still hold true, except that all the constants C_θ therein will depend on the constants in the new assumption.

The original article can be found online at https://doi.org/10.1007/s00440-018-0874-5.

✉ Lihu Xu
lihuxu@umac.mo

Xiao Fang
xfang@sta.cuhk.edu.hk

Qi-Man Shao
qmshao@sta.cuhk.edu.hk

1 Department of Statistics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

2 Department of Mathematics, Faculty of Science and Technology, University of Macau, Av. Padre Tomás Pereira, Taipa, Macau, China

3 Zhuhai UM Science and Technology Research Institute, Zhuhai, China

Before correcting [1, (6.9), (6.13), (7.1), (7.2)], let us recall some notations in [1], give the missed term, and prove an auxiliary lemma. Let $u, u_1, u_2 \in \mathbb{R}^d$, recall

$$\mathcal{I}_u^x(t) = \frac{1}{\sqrt{2}t}\int_0^t \langle \nabla_u X_s^x, \mathrm{d}B_s\rangle, \quad \mathcal{I}_{u_1,u_2}^x(t) = \mathcal{I}_{u_1}^x(t)\mathcal{I}_{u_2}^x(t) - D_{V_2}\mathcal{I}_{u_1}^x(t)$$

with $V_{i,t} = \int_0^t v_i(s)\mathrm{d}s$ and $v_i(s) = \frac{1}{\sqrt{2}t}\nabla_{u_i}X_s^x$ for $0 \le s \le t$ and $i = 1, 2$, see [1, (5.12),(5.13)]. The *missed* term is defined by

$$\mathcal{R}_{u_1,u_2}^x(t) := \nabla_{u_2}\nabla_{u_1}X_t^x - D_{V_2}(\nabla_{u_1}X_t^x).$$

Lemma 0.1 *We have*

$$\begin{aligned}
|\mathcal{R}_{u_1,u_2}^x(t)| &\le C_\theta |u_2||u_1|,\\
|\nabla_{u_3}\mathcal{R}_{u_1,u_2}^x(t)| &\le C_\theta |u_3||u_2||u_1|,\\
|D_{V_3}\mathcal{R}_{u_2,u_1}^x(t)| &\le C_\theta |u_3||u_2||u_1|,
\end{aligned}$$

for all $u_1, u_2, u_3, x \in \mathbb{R}^d$.

Proof The first bound follows immediately from [1, (5.7),(5.17)]. It is easy to check that $\nabla_{u_3}\nabla_{u_2}\nabla_{u_1}X_t$ satisfies the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla_{u_3}\nabla_{u_2}\nabla_{u_1}X_t = \nabla g(X_t)\nabla_{u_3}\nabla_{u_2}\nabla_{u_1}X_t + \nabla^2 g(X_t)\mathcal{R}_1(t) + \nabla^3 g(X_t)\mathcal{R}_2(t),$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are terms about first and second-order derivatives of X_t. By [1, (5.6), (5.7)], we have

$$|\mathcal{R}_1(t)| \le C_\theta |u_1||u_2||u_3|, \quad |\mathcal{R}_2(t)| \le C_\theta |u_1||u_2||u_3|.$$

Differentiating $|\nabla_{u_3}\nabla_{u_2}\nabla_{u_1}X_t|^2$ with respect to t and using the above two bounds, we can prove by the same argument as showing [1, (5.7)]

$$|\nabla_{u_3}\nabla_{u_2}\nabla_{u_1}X_t| \le C_\theta |u_1||u_2||u_3|.$$

Similarly, first finding the differential equations of $\nabla_{u_3}D_{V_2}\nabla_{u_1}X_t$, $D_{V_3}\nabla_{u_2}\nabla_{u_1}X_t$, $D_{V_3}D_{V_2}\nabla_{u_1}X_t$, and then using the same argument as above, we get

$$\begin{aligned}
|\nabla_{u_3}D_{V_2}\nabla_{u_1}X_t| &\le C_\theta |u_1||u_2||u_3|,\\
|D_{V_3}\nabla_{u_2}\nabla_{u_1}X_t| &\le C_\theta |u_1||u_2||u_3|,\\
|D_{V_3}D_{V_2}\nabla_{u_1}X_t| &\le C_\theta |u_1||u_2||u_3|.
\end{aligned}$$

Collecting the previous estimates, we immediately obtain the other two estimates in the lemma. □

Correction to [1, (6.9), (6.13)]: The original [1, (6.9), (6.13)] should be corrected as

$$\nabla_{u_2}\nabla_{u_1}\mathbb{E}[\phi(X_t^x)] = \mathbb{E}[\nabla_{u_1}\phi(X_t^x)\mathcal{I}_{u_2}^x(t)] + \mathbb{E}[\nabla\phi(X_t^x)\mathcal{R}_{u_1,u_2}^x(t)]$$

and

$$\nabla_{u_2}\nabla_{u_1} f(x) = \int_0^\infty e^{-t}\mathbb{E}\left\{\left[\nabla_{u_1} f(X_t^x) - \nabla_{u_1} h(X_t^x)\right]\mathcal{I}_{u_2}^x(t)\right\} \mathrm{d}t + \int_0^\infty e^{-t}\mathbb{E}\left\{\left[\nabla f(X_t^x) - \nabla h(X_t^x)\right]\mathcal{R}_{u_2,u_1}^x(t)\right\} \mathrm{d}t,$$

for $u_1, u_2, x \in \mathbb{R}^d$.

Proof We have

$$\nabla_{u_2}\nabla_{u_1}\mathbb{E}[\phi(X_t^x)] = \mathbb{E}\left[\nabla^2\phi(X_t^x)\nabla_{u_2}X_t^x\nabla_{u_1}X_t^x\right] + \mathbb{E}\left[\nabla\phi(X_t^x)\nabla_{u_2}\nabla_{u_1}X_t^x\right],$$

By [1, (5.14), (5.9), (5.11)],

$$\begin{aligned}\mathbb{E}\left[\nabla^2\phi(X_t^x)\nabla_{u_2}X_t^x\nabla_{u_1}X_t^x\right] &= \mathbb{E}\left[\nabla^2\phi(X_t^x)D_{V_2}X_t^x\nabla_{u_1}X_t^x\right]\\ &= \mathbb{E}\left[D_{V_2}(\nabla\phi(X_t^x))\nabla_{u_1}X_t^x\right]\\ &= \mathbb{E}\left[D_{V_2}(\nabla\phi(X_t^x)\nabla_{u_1}X_t^x)\right] - \mathbb{E}\left[\nabla\phi(X_t^x)D_{V_2}(\nabla_{u_1}X_t^x)\right]\\ &= \mathbb{E}\left[\nabla_{u_1}\phi(X_t^x)\mathcal{I}_{u_2}^x(t)\right] - \mathbb{E}\left[\nabla\phi(X_t^x)D_{V_2}(\nabla_{u_1}X_t^x)\right].\end{aligned}$$

Combining the above relations, we immediately obtain the first relation in the proposition. The second relation can immediately be obtained from the first one. □

Correction to the proofs of (7.1) and (7.2) in [1]: The conclusions of (7.1) and (7.2) still hold under the strengthened Assumption 2.1., but we need to estimate the extra terms related to $\mathcal{R}_{u_2,u_1}^x(t)$. From the second relation in the above proposition, we have

$$\left|\nabla_{u_2}\nabla_{u_1} f(x)\right| \le \int_0^\infty e^{-t}\left|\mathbb{E}\left\{\left[\nabla_{u_1} f(X_t^x) - \nabla_{u_1} h(X_t^x)\right]\mathcal{I}_{u_2}^x(t)\right\}\right| \mathrm{d}t + \int_0^\infty e^{-t}\left|\mathbb{E}\left\{\left[\nabla f(X_t^x) - \nabla h(X_t^x)\right]\mathcal{R}_{u_1,u_2}^x(t)\right\}\right| \mathrm{d}t.$$

Since we have shown in the original proof that

$$\int_0^\infty e^{-t}\left|\mathbb{E}\left\{\left[\nabla_{u_1} f(X_t^x) - \nabla_{u_1} h(X_t^x)\right]\mathcal{I}_{u_2}^x(t)\right\}\right| \mathrm{d}t \le C_\theta\|\nabla h\||u_1||u_2|,$$

it remains to bound the second integral. By [1, (5.7), (5.17)], we immediately obtain

$$\int_0^\infty e^{-t}\left|\mathbb{E}\left\{\left[\nabla f(X_t^x) - \nabla h(X_t^x)\right]\mathcal{R}_{u_1,u_2}^x(t)\right\}\right| \mathrm{d}t \le C_\theta\|\nabla h\||u_1||u_2|.$$

Combining the previous three inequalities, we conclude that [1, (7.1)] still holds true.

To prove [1, (7.2)], we have

$$\nabla_{u_2}\nabla_{u_1} f(x+\varepsilon u) - \nabla_{u_2}\nabla_{u_1} f(x) \;=\; \int_0^\infty e^{-t}\Psi \mathrm{d}t \;+\int_0^\infty e^{-t}\Phi \mathrm{d}t = J_1 + J_2,$$

where

$$\begin{aligned}\Psi &= \mathbb{E}\left\{\left[\nabla_{u_1} f(X_t^{x+\varepsilon u}) - \nabla_{u_1} h(X_t^{x+\varepsilon u})\right]\mathcal{I}_{u_2}^{x+\varepsilon u}(t)\right\}\\ &\quad - \mathbb{E}\left\{\left[\nabla_{u_1} f(X_t^{x}) - \nabla_{u_1} h(X_t^{x})\right]\mathcal{I}_{u_2}^{x}(t)\right\}.\\ \Phi &= \mathbb{E}\left\{\left[\nabla f(X_t^{x+\varepsilon u}) - \nabla h(X_t^{x+\varepsilon u})\right]\mathcal{R}_{u_1,u_2}^{x+\varepsilon u}(t)\right\}\\ &\quad - \mathbb{E}\left\{\left[\nabla f(X_t^{x}) - \nabla h(X_t^{x})\right]\mathcal{R}_{u_1,u_2}^{x}(t)\right\}.\end{aligned}$$

We have shown in the original proof that

$$|J_1| \le C_\theta ||\nabla h|||\varepsilon|\,(|\log|\varepsilon|| \vee 1)\,|u_1||u_2|.$$

We prove below that

$$|J_2| \le C_\theta ||\nabla h|||\varepsilon||u_1||u_2||u|.$$

Combining the estimates of J_1 and J_2, we immediately get that [1, (7.2)] still holds true.

Let us show the above bound about J_2. Write

$$J_2 = J_{2,1} + J_{2,2},$$

with

$$\begin{aligned}J_{2,1} &= \int_0^\infty e^{-t}\mathbb{E}\left\{\left[\nabla f(X_t^{x+\varepsilon u}) - \nabla h(X_t^{x+\varepsilon u})\right]\left[\mathcal{R}_{u_1,u_2}^{x+\varepsilon u}(t) - \mathcal{R}_{u_1,u_2}^{x}(t)\right]\right\}\mathrm{d}t,\\ J_{2,2} &= \int_0^\infty e^{-t}\mathbb{E}\left\{\left[\nabla f(X_t^{x+\varepsilon u}) - \nabla h(X_t^{x+\varepsilon u}) - \nabla f(X_t^{x}) + \nabla h(X_t^{x})\right]\mathcal{R}_{u_1,u_2}^{x}(t)\right\}\mathrm{d}t,\end{aligned}$$

For $J_{2,1}$, observe

$$J_{2,1} = \varepsilon\int_0^\infty e^{-t}\int_0^1 \mathbb{E}\left\{[\nabla f(X_t^{x+\varepsilon u}) - \nabla h(X_t^{x+\varepsilon u})]\nabla_u \mathcal{R}_{u_1,u_2}^{x+s\varepsilon u}(t)\right\}\mathrm{d}s\mathrm{d}t,$$

which, together with Lemma 0.1, immediately gives

$$|J_{2,1}| \le C_\theta|\varepsilon|(\|\nabla f\| + \|\nabla h\|) \le C_\theta|\varepsilon|\|\nabla h\||u||u_1||u_2|.$$

For $J_{2,2}$, we have

$$
\begin{aligned}
J_{2,2} &= \varepsilon \int_0^\infty e^{-t} \int_0^1 \mathbb{E}\left\{\nabla[\nabla f(X_t^{x+s\varepsilon u}) - \nabla h(X_t^{x+s\varepsilon u})]\nabla_u X_t^{x+s\varepsilon u} \mathcal{R}_{u_1,u_2}^x(t)\right\} \mathrm{d}s\mathrm{d}t \\
&= \varepsilon \int_0^\infty e^{-t} \int_0^1 \mathbb{E}\left\{\nabla[\nabla f(X_t^{x+s\varepsilon u}) - \nabla h(X_t^{x+s\varepsilon u})]D_V X_t^{x+s\varepsilon u} \mathcal{R}_{u_1,u_2}^x(t)\right\} \mathrm{d}s\mathrm{d}t \\
&= \varepsilon \int_0^\infty e^{-t} \int_0^1 \mathbb{E}\left\{D_V[\nabla f(X_t^{x+s\varepsilon u}) - \nabla h(X_t^{x+s\varepsilon u})]\mathcal{R}_{u_1,u_2}^x(t)\right\} \mathrm{d}s\mathrm{d}t \\
&= \varepsilon(J_{2,2,1} - J_{2,2,2})
\end{aligned}
$$

where the last equality is by [1, (5.14), (5.9), (5.11)] and

$$
\begin{aligned}
J_{2,2,1} &= \int_0^\infty e^{-t} \int_0^1 \mathbb{E}\left\{[\nabla f(X_t^{x+s\varepsilon u}) - \nabla h(X_t^{x+s\varepsilon u})]\mathcal{R}_{u_1,u_2}^x(t)\mathcal{I}_u^{x+s\varepsilon u}(t)\right\} \mathrm{d}s\mathrm{d}t, \\
J_{2,2,2} &= \int_0^\infty e^{-t} \int_0^1 \mathbb{E}\left\{\left[\nabla f(X_t^{x+s\varepsilon u}) - \nabla h(X_t^{x+s\varepsilon u})\right] D_V \mathcal{R}_{u_1,u_2}^x(t)\right\} \mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

By Lemma 0.1 and [1, (5.18)],

$$
\begin{aligned}
|J_{2,2,2}| &\le C_\theta(\|\nabla f\| + \|\nabla h\|)|u_1||u_2||u| \le C_\theta\|\nabla h\||u_1||u_2||u|, \\
|J_{2,2,1}| &\le C_\theta(\|\nabla f\| + \|\nabla h\|)|u_1||u_2| \int_0^\infty e^{-t} \int_0^1 \mathbb{E}\left[|\mathcal{I}_u^{x+s\varepsilon u}(t)|\right] \mathrm{d}s\mathrm{d}t \\
&\le C_\theta\|\nabla h\||u_1||u_2||u|.
\end{aligned}
$$

Combining the estimates above, we immediately obtain the bound of J_2. □

Due to the new assumption on g, [1, Remark 3.2] should be revised as

Remark 3.2 Gorham et. al. (see [18] in [1]) recently put forward a method to measure sample quality with diffusions by a Stein discrepancy, in which the same Stein equation as (3.1) has to be considered. Under the assumption that g is third-order differentiable, they used the Bismut–Elworthy–Li formula (see [16] in [1]), together with smooth convolution and interpolation techniques, to prove a bound on the first, second, and $(3 - \epsilon)$th derivative of f for $\epsilon > 0$. They can also obtain the bound (3.4) by their approach (personal communication (see [24] in [1]) after their reading our paper on ArXiv) together with a limiting argument.

Acknowledgements We thank Jim Dai and James Thompson for pointing out the errors.

Reference

1. Fang, X., Shao, Q.M., Xu, L.: Multivariate approximations in Wasserstein distance by Stein's method and Bismut's formula. Probab. Theory Relat. Fields (2019) **(to appear)**