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CORRECTION

Correction to: Random graph asymptotics on high-dimensional tori II: volume, diameter and mixing time

Markus Heydenreich[1] · **Remco van der Hofstad**[2]

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Abstract
In [3, Theorem 1.2], we claim that the maximal cluster for critical percolation on the high-dimensional torus is non-concentrated. This proof contains an error. In this note, we replace this statement by a *conditional* statement instead.

1 Correction of [3, Theorem 1.2]

Recall from [3] that $\mathcal{C}_{(i)}$ denotes the ith largest cluster for percolation on the d-dimensional torus $\mathbb{T}_{r,d}$, so that $\mathcal{C}_{(1)} = \mathcal{C}_{\max}$ is the largest component and $|\mathcal{C}_{(2)}| \leq |\mathcal{C}_{(1)}|$ is the size of the second largest component, etc. Then, the statement of [3, Theorem 1.2] should be replaced by the following (shortened) statement:

Theorem 1.2 (Random graph asymptotics of the ordered cluster sizes) *Fix $d > 6$ and L sufficiently large in the spread-out case, or d sufficiently large for nearest-neighbor percolation. For every $m = 1, 2, \ldots$ there exist constants $b_1, \ldots, b_m > 0$, such that for all $\omega \geq 1$, $r \geq 1$, and all $i = 1, \ldots, m$,*

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✉ Remco van der Hofstad
r.w.v.d.hofstad@tue.nl

Markus Heydenreich
MO.Heydenreich@few.vu.nl; m.heydenreich@lmu.de

1 Department of Mathematics, Vrije Universiteit Amsterdam, De Boelelaan 1081a, 1081 HV Amsterdam, The Netherlands

2 Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands

$$\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(\omega^{-1}V^{2/3} \le |\mathcal{C}_{(i)}| \le \omega V^{2/3}\Big) \ge 1 - \frac{b_i}{\omega}. \tag{1.1}$$

Consequently, the expected cluster sizes satisfy $\mathbb{E}_{\mathbb{T},p_c(\mathbb{Z}^d)}|\mathcal{C}_{(i)}| \ge b_i' V^{2/3}$ *for certain constants* $b_i' > 0$.

In [3, Theorem 1.2], an additional non-concentration result was claimed for $V^{-2/3}|\mathcal{C}_{\max}|$. The proof of this result is incorrect. Below we will explain why, and replace this statement by a *conditional* version. Unfortunately, we are not able to prove the required condition.

2 Last paragraph of discussion in [3, Section 1.3]

In the last paragraph of [3, Section 1.3], we discuss the non-concentration of $V^{-2/3}|\mathcal{C}_{\max}|$, a feature that is highly indicative of the critical behavior. This paragraph needs to be removed.

3 Corrections to the proof of Theorem 1.2

The proof of [3, Theorem 1.2] still applies, except for [3, Proposition 3.1], where the non-concentration of $|\mathcal{C}_{\max}|V^{-2/3}$ is proved. This statement can be replaced by the following conditional statement:

Proposition 3.1 ($|\mathcal{C}_{\max}|V^{-2/3}$ is not concentrated) *Under the conditions of* [3, Theorem 1.1], *and assuming that there exists* $\omega > 6^{2/3}$ *such that*

$$\liminf_{V\to\infty} V^{1/3}\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\Big(|\mathcal{C}| > \omega V^{2/3}\Big) > 0, \tag{3.2}$$

the random sequence $|\mathcal{C}_{\max}|V^{-2/3}$ *is non-concentrated.*

The proof of [3, Proposition 3.1] can be followed verbatim, except for the discussion right after [3, (3.19)]. Indeed, [3, (3.19)] reads

$$\begin{aligned}\mathrm{Var}_{p_c(\mathbb{Z}^d)}(Z_{>\omega V^{2/3}}V^{-2/3}) &\ge V^{-1/3}\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}\big(|\mathcal{C}| > \omega V^{2/3}\big)\\ &\quad\times\big[\omega V^{2/3} - V\,\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}(|\mathcal{C}| > \omega V^{2/3})\big]\\ &\ge V^{1/3}\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}(|\mathcal{C}| > \omega V^{2/3})\big[\omega - C_{\mathcal{C}}\omega^{-1/2}\big],\end{aligned} \tag{3.3}$$

and below it, we claim that this remains uniformly positive for $\omega \ge 1$ sufficiently large, by [3, (2.4)]. The problem is that [3, (2.4)] applies only to ω that are not too large, while to keep the second factor in (3.3) positive, we need to take $\omega > 0$ sufficiently large, which we cannot satisfy simultaneously.

An inspection of the proof of the upper bound in [3, (2.4)] (which is originally [1, Theorem 1.3]) shows that $C_{\mathcal{C}} = 6$ suffices. Indeed, by [2, Proposition 2.1], $\mathbb{P}_{\mathbb{T},p_c(\mathbb{Z}^d)}(|\mathcal{C}| \ge k) \le \mathbb{P}_{\mathbb{Z},p_c(\mathbb{Z}^d)}(|\mathcal{C}| \ge k)$. Further, by [4, (9.2.6)], $\mathbb{P}_{\mathbb{Z},p_c(\mathbb{Z}^d)}(|\mathcal{C}| \ge$

$k) \leq \frac{e}{e-1} M(p_c(\mathbb{Z}^d), 1/k)$, while [4, Lemma 9.3] proves that $M(p_c(\mathbb{Z}^d), \gamma) \leq \sqrt{12\gamma}$. Thus, we need that $\omega > 6^{2/3}$ to keep the second term in (3.3) strictly positive. For this choice, also [3, (3.19)] is satisfied. As a result, the proof can be repaired when we assume (3.2) for $\omega > 6^{2/3}$. □

Mind that [3, (2.4)] implies (3.2) for $\omega < b_C$ for a positive constant b_C (which is the same as b_1 in [1, Theorem 1.3]). The actual value of b_C depends of the position of $p_c(\mathbb{Z}^d)$ within the critical window of $p_c(\mathbb{T}_{r,d})$. While [3, Theorem 2.1] guarantees that $p_c(\mathbb{Z}^d)$ does lie within the critical window, it gives us no control on the precise position.

We believe that (3.2) is correct, in fact, even for *all* $\omega > 0$. For example, for the Erdős–Rényi random graph model, which is the corresponding mean-field model, a corresponding statement is true for all $\omega > 0$, cf. [5, Lemma 2.2], where even a local limit version of (3.2) is proved. However, we have not been able to show this for percolation on the high-dimensional torus.

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