

Mixing rates for intermittent maps of high exponent

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Abstract We obtain higher order theory for the long term behavior of the transfer operator associated with the unit interval map $f(x) = x(1 + 2^\alpha x^\alpha)$ if $0 < x < \frac{1}{2}$, $f(x) = 2x - 1$ if $\frac{1}{2} < x < 1$ for the whole range $\alpha > 1$, which corresponds to the infinite measure preserving case. Higher order theory for $\alpha \geq 2$ is more challenging and requires new techniques. Along the way, we provide higher order theory for scalar and operator renewal sequences with infinite measure and regular variation. Although the present work considers the unit interval map mentioned above as a toy model, our interest focuses on finding sufficient conditions under which the asymptotic behavior of the transfer operator associated to dynamical systems preserving an infinite measure is 'almost like' the asymptotic behavior of scalar renewal sequences associated to null recurrent Markov chains characterized by regular variation.

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1 Introduction and main results

Understanding the long term behaviour of the transfer operator $L : L^1(X) \rightarrow L^1(X)$ associated with infinite measure preserving transformations (X, f, μ) is still a challenging problem. To provide a summary of results in the infinite measure case we recall the general set up of scalar and operator renewal sequences. Let (X, μ) be a measure space (finite or infinite), and $f : X \rightarrow X$ a conservative measure preserving map. Fix $Y \subset X$ with $\mu(Y) \in (0, \infty)$. Let $\varphi : Y \rightarrow \mathbb{Z}_+$ be the first return time $\varphi(y) = \inf\{n \geq 1 : f^n(y) \in Y\}$ (finite almost everywhere by conservativity). Let

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$L : L^1(X) \rightarrow L^1(X)$ denote the transfer operator for f and

$$T_n v = 1_Y L^n(1_Y v), \quad n \geq 0, \quad R_n v = 1_Y L^n(1_{\{\varphi=n\}} v), \quad n \geq 1. \quad (1.1)$$

Thus T_n corresponds to general returns to Y and R_n corresponds to first returns to Y . The relationship $T_n = \sum_{j=1}^n T_{n-j} R_j$ generalizes the notion of scalar renewal sequences (see [5,9] and references therein).

Throughout, we assume that the return time function $\varphi : Y \rightarrow \mathbb{Z}_+$ satisfies $\int_Y \varphi d\mu = \infty$, which implies $\mu(X) = \infty$. Also, we assume φ is regularly varying with index $\beta \in (0, 1)$ satisfying a certain asymptotic expansion (see assumption (H) in Sect. 1.1). Under these assumptions on φ , for all $\beta \in (0, 1)$, Theorem 1.1 provides higher order asymptotics for scalar renewal sequences (as explicitly recalled in Sect. 1.1). Under the same assumption on φ , for all $\beta \in (0, 1)$, Theorem 1.3 provides higher order asymptotics for operator renewal sequences T_n associated to *non* independent (dynamical) systems (we refer to Sects. 1.1 and 1.2 for the precise use of terminology). Most of the body of this work is devoted to the proof of Theorem 1.3.

1.1 Higher order asymptotics for scalar renewal sequences with infinite mean

In this section, we recall some basic background on scalar renewal theory, focusing on the infinite mean case. For more details we refer the reader to [5,9]. Let $(Z_i)_{i \geq 0}$ be a sequence of positive integer-valued independent identically distributed random variables with probabilities $P(Z_i = j) = r_j$. Define the partial sums $S_n = \sum_{j=1}^n Z_j$, set $u_0 = 1$ and define $u_n = \sum_{j=1}^n r_j u_{n-j}$, $n \geq 1$. Then it is easy to see that $u_n = \sum_{j=1}^n P(S_j = n)$. The sequences $(u_n)_{n \geq 0}$ are called *scalar renewal sequences*.

To relate to the notions of the previous section, let $F = f^\varphi : Y \rightarrow Y$ be the first return map to $Y \subset X$ and rescale such that $\mu(Y) = 1$. For $n \geq 0$, let $Z_n = \varphi \circ F^n$. If $\{Z_n; n \geq 0\}$ are independent with respect to μ , one can reduce the study of the dynamics $f : X \rightarrow X$ to the setting of scalar renewal theory. To see this, let $r_j = \mu(Z_i = j)$ and reduce the action of the operator R_j defined in (1.1) to $R_j := r_j$. In this case, the operator T_n defined in (1.1) coincides with the scalar sequence $u_n = \sum_{j=1}^n r_j u_{n-j}$. For general dynamical systems, $\{\varphi \circ F^n; n \geq 0\}$ are *not* independent and thus, one *cannot* understand the dynamics by reducing to the scalar case. Instead, the study of the asymptotic behavior of the operator sequences $(T_n)_{n \geq 0}$ defined in (1.1) is helped by the study of scalar renewal sequences $(u_n)_{n \geq 0}$. In what follows, we let $r_j = \mu(Z_i = j)$ and set $u_0 = 1$, $u_n = \sum_{j=1}^n r_j u_{n-j}$, $n \geq 1$. For early use of scalar renewal sequences with infinite mean in the context of dynamical systems we refer to [1–3].

The analysis of scalar renewal sequences with infinite mean relies crucially on the assumption of regularly varying tails:

$$\mu(y \in Y : \varphi(y) > n) = \sum_{j>n} r_j = \ell(n)n^{-\beta},$$

where ℓ is slowly varying and $\beta \in [0, 1]$ (see [5,9] and references therein).

For $z \in \mathcal{S}^1$ define

$$\Psi(z) := \sum_{j=1}^{\infty} r_j z^j = \int_Y z^\varphi d\mu.$$

The asymptotics of scalar renewal sequences u_n can be obtained by estimating the Fourier coefficient $[(1 - \Psi)^{-1}]_n$ of $(1 - \Psi(z))^{-1}$, $z = e^{i\theta} \in \mathcal{S}^1$, that is

$$u_n = [(1 - \Psi)^{-1}]_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 - \Psi(e^{i\theta}))^{-1} e^{-in\theta} d\theta.$$

We refer to Garsia and Lamperti [11] and Erickson [8] for further details.

Throughout we let $\beta \in (0, 1)$. Define $k = \min\{j \geq 2 : \beta > \frac{1}{j}\}$ and assume that

- (H) $\mu(y \in Y : \varphi(y) > n) = \sum_{j=1}^{k-1} c_j n^{-j\beta} + A(n) + B(n)$, where c_j are real constants with $c_1 > 0$ and the functions $A(n)$, $B(n)$ are such that:
- (a) A is a finite sum $A(n) = \sum_j \ell_j(n) n^{-\xi_j}$, where for all j , $\xi_j \geq k\beta$ with $\ell_j(x) = C_j \log x + C'_j$, for C_j, C'_j real constants. We further assume that if $\xi_j = 2\beta$ then $C_j = 0$, so $\ell_j(x) = C'_j$.
 - (b) B is such that $n^2 B(n)$ is of bounded variation¹ and $B(n) = O(n^{-\gamma})$, for some $\gamma > 2$.²

Throughout, we set $q = \max\{j \geq 0 : (j + 1)\beta - j > 0\}$ and let d_0, \dots, d_q be nonnegative real constants that depend only the quantities defined in (H). For a precise definition of these constants we refer to Sect. 5. Here, we only mention that $d_0 = c_1^{-1}(\Gamma(1 - \beta)\Gamma(1 + \beta))^{-1}$ and note that d_1, \dots, d_q are nonzero only when $\beta > 1/2$.

With these specified, we can state our result on higher order expansions for the coefficients $u_n = [(1 - \Psi)^{-1}]_n$ of $(1 - \Psi(z))^{-1}$, $z \in \mathcal{S}^1$.

Theorem 1.1 *Assume that (H) holds and that $\text{g.c.d.}\{\varphi(y) : y \in Y\} = 1$. Let $\beta \in (0, 1)$. Let $r = 1$ if $\beta \neq 1/2$ and $r = 2$ if $\beta = 1/2$. Then*

$$u_n = [(1 - \Psi)^{-1}]_n = d_0 n^{\beta-1} + d_1 n^{2\beta-2} + d_2 n^{3\beta-3} + \dots + d_q n^{(q+1)(\beta-1)} + O((\log n)^r / n).$$

Remark 1.2 We note that if $\beta \leq 1/2$ then $q = 0$ and Theorem 1.1 says that $u_n = d_0 n^{\beta-1} + O((\log n)^r / n)$, so u_n is given by precisely one exact term plus the error term. We believe that it is unlikely to obtain more exact terms in the asymptotic expression of u_n when $\beta \leq 1/2$ (for this purpose, a stronger assumption (H) will not make any difference).

We already mentioned that d_1, \dots, d_q are nonzero only when $\beta > 1/2$. The definition of these constants in Sect. 5 says that the number of nonzero constants among

¹ Recall that a sequence a_n is of bounded variation if $\sum_{n=1}^{\infty} |a_{n+1} - a_n| < \infty$.

² We note that the function B includes all terms that are $O(n^{-\rho})$ with $\rho > 3$.

d_1, \dots, d_q increases as β gets larger (closer to 1). Thus, when $\beta > 1/2$, the number of exact terms in the asymptotic expression of u_n increases as β gets larger.

The error term $O((\log n)^r/n)$ in the expansion of u_n in Theorem 1.1 could, most probably, be considerably improved using the technique introduced in [23] (also used in the current work). However, we do not do this here since (except for the value $\beta = 1/2$) a better error term in Theorem 1.1 does not help us to improve the error term in Theorem 1.3 below.

To our knowledge, Theorem 1.1 is the first result on first order asymptotic with rates for *scalar renewal sequences* u_n with the error term $O((\log n)^r/n)$ for all $\beta \in (0, 1)$. The only other previous results on higher order asymptotic for scalar renewal sequences are contained in [19, 23] and do not address the regime $\beta \in (0, 1/2]$. Also, we note that Theorem 1.1 improves the error terms in [19, 23] for the range $\beta \in (1/2, 1)$.

1.2 Higher order asymptotics of operator renewal sequences for infinite measure preserving systems

Operator renewal sequences were introduced by Sarig [22] to study lower bounds for mixing rates associated with finite measure preserving systems, and this technique was substantially extended and refined by Gouëzel [13, 16]. In [19], Melbourne and Terhesiu developed a theory of operator renewal sequences for dynamical systems with infinite measure, generalizing the results of [8, 11] to the operator case. Under suitable assumptions on the first return map f^φ , [19] shows that for a ('sufficiently regular') function v supported on Y and a constant $d_0 = \frac{1}{\pi} \sin \beta\pi$, the following hold: (i) when $\beta \in (\frac{1}{2}, 1)$ then $\lim_{n \rightarrow \infty} \ell(n)n^{1-\beta}T_nv = d_0 \int_Y v d\mu$, uniformly on Y ; (ii) if $\beta \in (0, \frac{1}{2}]$ and $v \geq 0$ then $\liminf_{n \rightarrow \infty} \ell(n)n^{1-\beta}T_nv = d_0 \int_Y v d\mu$, pointwise on Y and (iii) if $\beta \in (0, \frac{1}{2})$ then $T_nv = O(\ell(n)n^{-\beta})$. In [19], the results summarized above are referred to as first order asymptotics of T_n . In the same work, the authors also obtain an optimal version of item i) above for the case $\beta = 1$. Since the case $\beta = 1$ has completely treated in [19] (also in the sense of higher order theory as explained below), we do not consider this case in the present work. For a different technique for operator renewal sequences satisfying the general assumption $\mu(\varphi > n) = \ell(n)n^{-\beta}$ (and implicitly, for scalar renewal sequences) we refer to [23].

As shown in [19], the above results on T_n extend to similar results on L^n associated with a large class of systems preserving an infinite measure. We recall that previous to the results in [19] via operator renewal techniques, Thaler [27] obtained first order asymptotics of L^n for a rather restrictive class of dynamical systems, which applies to reasonably large classes of systems (similar to the family of maps (1.2) recalled in Sect. 1.3) just in the case $\beta = 1$. Prior to the works [15, 19], the result [27] was the only success on this problem. Previous to the result [27] for the asymptotic of L^n , the works [25, 30] obtained first order asymptotic of the average operator $\sum_{j=1}^n L^j$ (a more tractable problem) for large classes of infinite measure preserving interval maps; in particular, [25] obtained first order asymptotic of the average operator for the class of Markov maps considered in [24], while [30] for the class of non Markov maps introduced in [29].

The apparently weaker results for the case $\beta < 1/2$ are in fact optimal under the general assumption $\mu(\varphi > n) = \ell(n)n^{-\beta}$ (see [11]). Under the *additional assumption* $\mu(\varphi = n) = O(\ell(n)n^{-(\beta+1)})$, Gouëzel [15] obtains first order asymptotics for $L^n v$ for all $\beta \in (0, 1)$. This additional assumption is satisfied in the setting of Pomeau–Manneville maps (see Sect. 1.3 below).

In this work we obtain higher order asymptotics of T_n for all $\beta \in (0, 1)$ with excellent error terms. The meaning of higher order asymptotics for T_n will become clear from the main result below. Comparisons with previous results in this direction are discussed after the statement of this result.

Theorem 1.3 *Assume (H) and assumptions (H1) and (H2) stated in Sect. 2. Let \mathcal{B} be an appropriate function space (defined by (H1) and (H2)), with norm $\|\cdot\|$. Let $r = 1$ if $\beta \neq 1/2$ and $r = 2$ if $\beta = 1/2$. Then for all $\beta \in (0, 1)$ and for all $v \in \mathcal{B}$*

$$T_n v = \left(d_0 n^{\beta-1} + d_1 n^{2\beta-2} + d_2 n^{3\beta-3} + \dots + d_q n^{(q+1)(\beta-1)} \right) \int v \, d\mu + D_n v,$$

where $D_n : \mathcal{B} \rightarrow \mathcal{B}$ is sequence of operators satisfying $\|D_n\| = O((\log n)^r/n)$.

As in [19] we let the notion of mixing rates refer to the case in which there exists an upper bound for $\|n^{1-\beta} T_n v - d_0 \int v \, d\mu\|$. If a lower bound exists and it is of the same order as the upper bound, we say that the mixing rates are sharp. The work [19] provides sharp mixing rates for $\beta \in (3/4, 1)$. The work [15] obtains first order asymptotics for L^n (but not mixing rates) for all $\beta \in (0, 1)$, and [23] provides sharp mixing rates for $\beta \in (2/3, 1)$.

Theorem 1.3 deals with the remaining cases. On the one hand, we obtain sharp mixing rates for all $\beta \in (1/2, 1)$ and improve the error terms (in the implied convergence) obtained in [19, 23]. More importantly, Theorem 1.3, for the first time, provides first order asymptotics of T_n along with mixing rates for the whole range $\beta \in (0, 1)$, so also for the small values of β that were the main obstacle so far. To deal with these problems, we need to exploit the full strength of (H), a much stronger assumption than the ones needed for first order theory [15, 19].

The new ingredients of the proof are a decomposition of the operator $\tilde{T}(z) - (1 - \Psi(z))^{-1} P$ given in (6.2) and the use of derivatives of various operator-valued power series, for which we need to work on an open set \mathbb{U} near 1 in the unit disk \mathbb{D} rather than on the unit circle \mathcal{S}^1 . This allows us to recognize the coefficients of these derivatives as convolutions integrated over a well chosen contour (see e.g. the proof of Proposition 6.6) and thus exploit the assumptions on the small tail $\mu(\varphi = n)$ (implicitly written in assumption (H)). We give a more detailed strategy in Sect. 3.

1.3 Application to Pomeau–Manneville maps

The family of Pomeau–Manneville intermittency maps [21] are interval maps with indifferent fixed points; that is, they are uniformly expanding except for an indifferent fixed point at 0. To fix notation, we focus on the version studied by Liverani et al.

[18]:

$$f(x) = \begin{cases} x(1 + 2^\alpha x^\alpha), & 0 < x < \frac{1}{2} \\ 2x - 1, & \frac{1}{2} < x < 1. \end{cases} \tag{1.2}$$

It is well known that for $\alpha \geq 1$ (equivalently $\beta := 1/\alpha \leq 1$), we are in the situation of infinite ergodic theory: there exists a unique (up to scaling) infinite, σ -finite, absolutely continuous invariant measure μ . Our main result in the setting of (1.2) reads as follows.

Theorem 1.4 *Let f be given as in (1.2). Let the observable $v : [0, 1] \rightarrow \mathbb{R}$ be Hölder or of bounded variation, and supported on a compact subset of $(0, 1)$.*

Let $q = \max\{j \geq 0 : (j + 1)\beta - j > 0\}$. Let $r = 1$ if $\beta \neq 1/2$ and $r = 2$ if $\beta = 1/2$. Then for all $\beta \in (0, 1)$, there exist real constants d_0, \dots, d_q (depending only on f) such that

$$L^n v = \left(d_0 n^{\beta-1} + d_1 n^{2\beta-2} + d_2 n^{3\beta-3} + \dots + d_q n^{(q+1)(\beta-1)} \right) \int v d\mu + O((\log n)^r / n),$$

uniformly on compact subsets of $(0, 1)$.

Proof Let $x_0 = 1/2$ and $x_{p+1} < x_p = f(x_{p+1})$ for each $p \geq 0$. Set $Y = [x_p, 1]$. Let the observable $v : [0, 1] \rightarrow \mathbb{R}$ be Hölder or of bounded variation, and supported on Y .

Let φ be the first return to Y . By Proposition 8.7 (see Appendix 2 for the corresponding proof)³ the sequence $\mu(\varphi > n)$ satisfies the assumption (H) with $A(n) = \sum_{j=k}^{k+N'} c_j n^{-j\beta} + \sum_{j=1}^{k+N} (\tilde{c}_j^1 \log n + \tilde{c}_j^2) n^{-(j\beta+1)}$, $B(n) = \sum_{j=1}^k (\hat{c}_j^1 \frac{(\log n)^2}{n^{j\beta+2}} + \hat{c}_j^2 \frac{\log n}{n^{j\beta+2}} + \hat{c}_j^3 \frac{1}{n^{j\beta+2}}) + O((\log n)^2 / n^{\beta+3})$ where $N' = \min\{\ell \geq 2 : \beta > \frac{3}{k+\ell}\}$, $N = \min\{\ell \geq 2 : \beta > \frac{2}{k+\ell}\}$ and $c_j, \tilde{c}_j^1, \tilde{c}_j^2, \hat{c}_j^1, \hat{c}_j^2, \hat{c}_j^3$ are real constants that depend only on f .

Next, Theorem 1.3 applies to this setting since the Banach space \mathcal{B} of Hölder or of bounded variation functions supported on Y is embedded in $L^\infty(Y)$. In particular, it is well-known that hypotheses (H1) and (H2) are satisfied on such sets Y (see for example [19, Section 11]). Putting these together, we obtain almost sure convergence at a uniform rate on Y . Redefining sequences on a set of measure zero, we obtain uniform convergence on Y . □

Remark 1.5 Using Theorem 1.3, the statement of Theorem 1.4 can be generalized to suitable functions supported on the whole of $[0, 1]$ as in, for instance, [19, Theorem 11.14].

As in [19], a result of the type of Theorem 1.4 implies convergence rates in the Dynkin–Lamperti arcsine law for waiting times. Corollary 1.6 below improves the convergence obtained in [19, Corollary 9.10] and [23, Corollary 3.5]. It is known that

³ This is an improved version of [20, Proposition C1] and [23, Proposition B.1].

the arcsine law holds for a large class of interval maps with indifferent fixed points for all $\beta \in (0, 1)$ [30]. See also [26,28] for more general transformations.

To state our next result we need to recall the following. Let $Y = [x_p, 1]$ as defined in the proof of Theorem 1.4. For $x \in \bigcup_{j=0}^n f^{-j}Y$, $n \geq 1$, let $Z_n(x) = \max\{0 \leq j \leq n : f^j(x) \in Y\}$ denote the time of the last visit of the orbit of x to Y during the time interval $[0, n]$. Let ζ_β denote a random variable distributed according to the $B(1-\beta, \beta)$ distribution: $\mathbb{P}(\zeta_\beta \leq t) = d_0 \int_0^t \frac{1}{u^{1-\beta}} \frac{1}{(1-u)^\beta} du$, for $t \in [0, 1]$ and $d_0 = \frac{1}{\pi} \sin \beta\pi$.

Corollary 1.6 *Assume the setting of (1.2) with $\beta = 1/\alpha \in (0, 1)$. Let ν be an absolutely continuous probability measure on Y with density g . Let \mathcal{B} be the space of Hölder or of bounded variation functions with norm $\|\cdot\|$.*

Assume that $g \in \mathcal{B}$ and let r, q and d_1, \dots, d_q be as defined in Theorem 1.4. Then, for $t \in [0, 1]$,

$$\begin{aligned} \left| \nu\left(\frac{1}{n}Z_n \leq t\right) - \mathbb{P}(\zeta_\beta \leq t) \right| &= \sum_{j=1}^q \frac{d_j}{n^{j(1-\beta)}} \int_0^t u^{-(j+1)(1-\beta)} (1-u)^{-\beta} du \\ &\quad + O(\|g\|(\log n)^r n^{-\beta}). \end{aligned}$$

Proof The proof goes exactly as the proof of [19, Corollary 9.10], except for the use of Theorem 1.3 instead of [19, Theorem 11.4]. □

Remark 1.7 Corollary 1.6 provides optimal convergence rates for $\beta \in (1/2, 1)$. Corollary [19, Corollary 9.10] and [23, Corollary 3.5] provide optimal convergence rates for: $\beta > 3/4$ in [19] and $\beta > 2/3$ in [23]. Here, by optimal convergence rates we mean that there exists a lower bound of the same order as the upper bound. When $\beta \leq 1/2$, the involved error rate is only an upper bound. Corollary 1.6 is new even in the setting of null recurrent Markov chains satisfying (H).

The rest of this paper is organized as follows. In Sect. 2, we describe the general framework and main assumptions required for our results on L^n . In Sect. 3 we describe the strategy for the proofs of the main results Theorems 1.1 and 1.3; in particular, we state Proposition 3.5 which is the key ingredient for proving Theorem 1.3 via Theorem 1.1. Sections 4, 5 and 6 are devoted to the proofs of Theorem 1.1 and Proposition 3.5. More concisely, Appendix 1 contains the proofs of several technical results used in Sect. 4 for the proof of Theorem 1.1, while Sects. 7 and 7.4 contain the proofs of some technical results used in Sect. 6 for the proof of Proposition 3.5. In Appendix 2 we improve the estimate on the tail sequence $\mu(\varphi > n)$ associated with (1.2) obtained in [20, Proposition C2], [23, Proposition B.1]. This result is required in the proof of Theorem 1.4.

Notation We use “big O” and \ll notation interchangeably, writing $a_n = O(b_n)$ or $a_n \ll b_n$ as $n \rightarrow \infty$ if there is a constant $C > 0$ such that $a_n \leq Cb_n$ for all $n \geq 1$.

2 Main assumptions and general setup

Let (X, f, μ) be a conservative measure preserving transformation, $\mu(X) = \infty$. Fix $Y \subset X$, $\mu(Y) \in (0, \infty)$ and scale such that $\mu(Y) = 1$. Let $\varphi : Y \rightarrow \mathbb{Z}_+$ be the first return time $\varphi(y) = \inf\{n \geq 1 : f^n(y) \in Y\}$ and define the first return map $F = f^\varphi : Y \rightarrow Y$. Throughout we assume that (H) holds. Recall that the transfer operator $R : L^1(Y) \rightarrow L^1(Y)$ for the first return map $F : Y \rightarrow Y$ is defined via the formula $\int_Y Rv w d\mu = \int_Y v w \circ F d\mu$, $w \in L^\infty(Y)$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\bar{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$. Given $z \in \bar{\mathbb{D}}$, we define $R(z) : L^1(Y) \rightarrow L^1(Y)$ to be the operator $R(z)v = R(z^\varphi v)$. Also, for each $n \geq 1$, we define $R_n : L^1(Y) \rightarrow L^1(Y)$, $R_n v = R(1_{\{\varphi=n\}}v)$. It is easily verified that $R(z) = \sum_{n=1}^\infty R_n z^n$.

We need some functional-analytic assumptions on the first return map $F : Y \rightarrow Y$. Our assumption (H1) below is stronger than assumption (H1) in [19, 20, 23]; it is of the same strength as the one in [15]. We assume that there is a function space $\mathcal{B} \subset L^\infty(Y)$ containing constant functions, with norm $\|\cdot\|$ satisfying $|v|_\infty \leq \|v\|$ for $v \in \mathcal{B}$, such that:

(H1) For all $n \geq 1$, $R_n : \mathcal{B} \rightarrow \mathcal{B}$ is a bounded linear operator with $\|R_n\| = O(n^{-(\beta+1)})$.

We notice that $z \mapsto R(z)$ is a continuous family of bounded linear operators on \mathcal{B} for $z \in \bar{\mathbb{D}}$. Since $R(1) = R$ and \mathcal{B} contains constant functions, 1 is an eigenvalue of $R(1)$. Throughout we assume:

- (H2) (i) The eigenvalue 1 is simple and isolated in the spectrum of $R(1)$.
- (ii) For $z \in \bar{\mathbb{D}} \setminus \{1\}$, the spectrum of $R(z)$ does not contain 1.

In particular, $z \mapsto (I - R(z))^{-1}$ is an analytic family of bounded linear operators on \mathcal{B} for $z \in \mathbb{D}$. Define $T_n : L^1(Y) \rightarrow L^1(Y)$ for $n \geq 0$ and $T(z) : L^1(Y) \rightarrow L^1(Y)$ for $z \in \bar{\mathbb{D}}$ by setting

$$T_n v = 1_Y L^n(1_Y v), \quad T(z) = \sum_{n=0}^\infty T_n z^n.$$

(Here, $T_0 = I$.) We have the usual relation $T_n = \sum_{j=1}^n T_{n-j} R_j$ for $n \geq 1$. An induction argument on n together with the boundedness of R_j (see (H1) above) shows that $\|T_n\|$ grows at most exponentially. Hence, $T(z)$ is well defined for z in a small disk around 0. Furthermore, $T(z) = I + T(z)R(z)$ on \mathbb{D} and thus, the renewal equation $T(z) = (I - R(z))^{-1}$ holds for $z \in \mathbb{D}$. It follows that $T(z) = \sum_{n=0}^\infty T_n z^n$ can be analytically extended to the whole of $\bar{\mathbb{D}}$.

By (H1) and (H2), there exist $\epsilon > 0$ and a continuous family of simple eigenvalues of $R(z)$, namely $\lambda(z)$ for $z \in \bar{\mathbb{D}} \cap B_\epsilon(1)$ with $\lambda(1) = 1$. Let $P(z) : \mathcal{B} \rightarrow \mathcal{B}$ denote the corresponding family of spectral projections with $P(1) = P$ and complementary projections $Q(z) = I - P(z)$. Also, let $v(z) \in \mathcal{B}$ denote the corresponding family of eigenfunctions normalized so that $\int_Y v(z) d\mu = 1$ for all z . In particular, $v(1) \equiv 1$.

Then we can write

$$T(z) = (1 - \lambda(z))^{-1}P(z) + (I - R(z))^{-1}Q(z), \tag{2.1}$$

for $z \in \mathbb{D} \cap B_\epsilon(1), z \neq 1$.

As shown in [20], much weaker versions of (H), (H1) and (H2) above are enough for first order expansion of $(1 - \lambda(z))^{-1}$, and consequently of $T(z)$, for $z \in \mathbb{D}$, as $z \rightarrow 1$. We recall this result as relevant to our setting.

Lemma 2.1 [20, Lemma 2.4] *Suppose $\mu(\varphi > n) = \ell(n)n^{-\beta}$, where ℓ is a slowly varying function. Assume (H1) and (H2). Then, writing $z = e^{-u+i\theta}$, $u > 0$, $\theta \in [-\pi, \pi)$, the following hold as $z \rightarrow 1$*

$$\begin{cases} \Gamma(1 - \beta)(1 - \lambda(z))^{-1} \sim \ell(1/|u - i\theta|)^{-1}(u - i\theta)^{-\beta}, \\ \Gamma(1 - \beta)T(z) \sim \ell(1/|u - i\theta|)^{-1}(u - i\theta)^{-\beta}P. \end{cases}$$

Higher order expansions for $1 - \lambda(z)$, $z \in \mathbb{D}$ (and thus for $T(z)$, $z \in \mathbb{D}$) were obtained in [20,23]. The assumptions in [20,23] are much more modest than the ones used in this work. For higher order expansions of $1 - \lambda(e^{i\theta})$ under very mild assumptions, we refer the reader to [19]. For first order expansions of $1 - \lambda(e^{i\theta})$ we also refer to [4].

3 Strategy of the proofs of Theorems 1.1 and 1.3

3.1 Strategy of the proof of Theorem 1.1

Theorem 1.1 is proved using the main idea of [23]. Since the Fourier coefficients of $(1 - \Psi(z))^{-1}$, $z \in \mathcal{S}^1$ coincide with the Taylor coefficients of $(1 - \Psi(z))^{-1}$, $z \in \mathbb{D}$ (see Corollary 3.2 below), and $(1 - \Psi(z))^{-1}$ is analytic on \mathbb{D} , we estimate the latter by understanding the asymptotics of the first derivative $\frac{d}{d\theta}(1 - \Psi(z))^{-1}$, $z \in \mathbb{D}$ (see Sect. 5).

The asymptotics of $\Psi(z)$ is entirely determined by the expansion of $\mu(\varphi > n)$; for higher order expansion of $1 - \Psi(z)$, $z \in \mathbb{D}$ under the assumption (H) stated below we refer to Proposition 4.2. Under more mild assumptions, the asymptotics of $1 - \Psi(z)$, $z \in \mathbb{D}$ was (implicitly) obtained in [20,23] in the process of understanding the asymptotics of $1 - \lambda(z)$, $z \in \mathbb{D}$. First order expansion of $1 - \Psi(e^{i\theta})$ under the assumption $\mu(\varphi > n) = \ell(n)n^{-\beta}$ was obtained in several other works (see, for instance, [11]).

The next two results justify that the Taylor coefficients of $(1 - \Psi(z))^{-1}$, $T(z)$, $z \in \mathbb{D}$ coincide with the Fourier coefficients of $(1 - \Psi(z))^{-1}$, $T(z)$, $z \in \mathcal{S}^1$.

By, for instance, the argument of [19, Corollary 4.2],

Lemma 3.1 *Let $A(z)$ be a function from \mathbb{D} to some Banach space \mathcal{B} , continuous on $\mathbb{D} \setminus \{1\}$ and analytic on \mathbb{D} . For $u \geq 0$, $\theta \in [-\pi, \pi)$, write $z = e^{-u+i\theta}$. Assume that*

$$|A(e^{-u+i\theta})| \ll |A(e^{i\theta})| \ll |\theta|^{-\gamma},$$

for some $\gamma \in (0, 1)$ as $z \rightarrow 1$. Then the Fourier coefficients A_n coincide with the Taylor coefficients \hat{A}_n , that is

$$A_n = \hat{A}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(e^{i\theta})e^{-in\theta} d\theta.$$

Corollary 3.2 *Suppose $\mu(\varphi > n) = \ell(n)n^{-\beta}$ for $\beta \in (0, 1)$ and ℓ a slowly varying function. Then,*

- (a) *The Taylor coefficients of $(1 - \Psi(z))^{-1}$, $z \in \mathbb{D}$ coincide with the Fourier coefficients of $(1 - \Psi(z))^{-1}$, $z \in \mathcal{S}^1$.*
- (b) *The Taylor coefficients of $T(z)$, $z \in \mathbb{D}$ coincide with the Fourier coefficients of $T(z)$, $z \in \mathcal{S}^1$.*

Proof As shown in [20], $1 - \Psi(z) \sim \ell(1/|u - i\theta|)(u - i\theta)^\beta$, as $z \rightarrow 1$. Item (a) follows from Lemma 3.1.

Item (b) follows from Lemmas 2.1 and 3.1. □

3.2 Strategy of the proof of Theorem 1.3

Roughly, Theorem 1.3 says that the coefficients of $T(z)$, $z \in \bar{\mathbb{D}}$, behave 'almost' like the coefficients of $(1 - \Psi(z))^{-1}$, $z \in \bar{\mathbb{D}}$. A key result used in the proof of this theorem is Proposition 3.5 below, which gives the asymptotic behaviour of the Fourier coefficients of the function $\tilde{T}(z) = (I - \tilde{R}(z))^{-1}$, $z \in \mathcal{S}^1$. Here, $\tilde{R}(z)$ denotes an operator with several good properties mentioned below. To provide a rough idea of the use of $\tilde{R}(z)$ in the proof of Theorem 1.3, we mention that its leading eigenvalue $\tilde{\lambda}(z)$ coincides with $\lambda(z)$ on a neighborhood of 1 and it is different from 1 on $\mathcal{S}^1 \setminus \{1\}$. As a consequence, the corresponding eigenprojection $\tilde{P}(z)$ and eigenfunction $\tilde{v}(z)$ are functions that are well defined on the whole of \mathcal{S}^1 and one can speak of the Fourier coefficients of $\tilde{P}(z)$ and $\tilde{v}(z)$.

In what follows we recall all the properties of the function $\tilde{R}(z)$ constructed in [13, Step 3 of proof of Lemma 3.1] which we will use in the sequel. Throughout this section, we assume that (H1) and (H2) hold.

Proposition 3.3 [13, Step 3 of proof of Lemma 3.1] *For any $\delta > 0$, there exists $\epsilon > 0$, a continuous function $\tilde{R}(z) : \mathcal{S}^1 \rightarrow \mathcal{B}$ and a compact set $K \subset \mathbb{C} \setminus \{1\}$ such that*

- (i) *There exists a continuous family $\tilde{\lambda}(z)$ of simple isolated eigenvalues for $\tilde{R}(z)$ with $\tilde{\lambda}(1) = 1$ and $\tilde{\lambda}(z) \neq 1$ for $z \in \mathcal{S}^1 \setminus \{1\}$.*
- (ii) *The spectrum of $\tilde{R}(z)$ is a subset of $\{\tilde{\lambda}(z)\} \cup K$ for all $z \in \mathcal{S}^1$.*
- (iii) *$\|\tilde{R}(z) - R(1)\| < \delta$ for all $z \in \mathcal{S}^1$.*
- (iv) *$\tilde{R}(z) = R(z)$ for all $z \in B_\epsilon(1)$.*
- (v) *$\|\tilde{R}_n\| \ll |n|^{-(\beta+1)}$, for all n .*

Proposition 3.4 [13,15] *Let $\tilde{R}(z)$ be an operator that satisfies the conclusions of Proposition 3.3. Let $\tilde{\lambda}(z)$ and $\tilde{P}(z)$ be the associated eigenvalue and corresponding spectral projection.*

Suppose that (H1) and (H2) hold. Then $\tilde{\lambda}(z), \tilde{P}(z)$ are continuous functions on S^1 , whose Fourier coefficients satisfy $|\tilde{\lambda}_n| \ll |n|^{-(\beta+1)}$ and $\|\tilde{P}_n\| \ll |n|^{-(\beta+1)}$, for all n .

We can now state our result on the asymptotics of \tilde{T}_n .

Proposition 3.5 *Assume the setting of Proposition 3.4. Suppose that (H) holds. Define $\tilde{T}(z) = (I - \tilde{R}(z))^{-1}$, $z \in S^1$ and let \tilde{T}_n be its n -th Fourier coefficient. Then,*

$$\tilde{T}_n = [(1 - \Psi)^{-1}]_n P + D_n$$

where $\|D_n\| = O((\log |n|)/|n|)$.

To conclude we need to show that the general case reduces to the case where $\lambda(z)$ is well defined and close to 1 for all $z \in S^1$. This follows by the partition of unity argument in [13, 15].

Proof of Theorem 1.3 By Proposition 3.5, the n -th Fourier coefficient of the function $\tilde{T}(z) = (I - \tilde{R}(z))^{-1}$, $z \in S^1$ satisfies $\tilde{T}_n = [(1 - \Psi)^{-1}]_n P + O((\log n)/n)$. By the argument in [13, 15], the n -th Fourier coefficient of $T(z)$, $z \in S^1$ satisfies $T_n = \tilde{T}_n + O(n^{-(\beta+1)})$. These facts together with Theorem 1.1 imply that the Fourier coefficients of $T(z)$, $z \in S^1$, have the desired asymptotics. This together with Corollary 3.2 implies the same asymptotics for the coefficients of $T(z)$, $z \in \mathbb{D}$. □

So far, we have reduced the proof of Theorem 1.3 to the proofs of Theorem 1.1 and Proposition 3.5. As already mentioned at the beginning of this section, Theorem 1.1 is proved using the main idea of [23] (see Sects. 4 and 5). The proof of Proposition 3.5 is the most difficult part of this paper. Roughly, our idea is to estimate the Fourier coefficients of each term/function in the expression of $\tilde{T}(z) - (1 - \Psi(z))^{-1}P$ (see Eq. (6.2)). As explained in Sect. 6 (see the paragraph after Eq. (6.2)), this comes down to estimating the Fourier coefficients of the functions of the form $(1 - \Psi(z))^{-1}(\tilde{R}(z) - R(z))$, $(1 - \Psi(z))^{-1}(R(z) - R(1))$ and variants of them.

To estimate the coefficients of $(1 - \Psi(z))^{-1}(\tilde{R}(z) - R(z))$ we use the fact that $\tilde{R}(z) = R(z)$ on a small neighborhood of 1 (see Proposition 6.3 and its proof). In this part, we need to exploit the full force of (H).

To estimate the coefficients of $(1 - \Psi(z))^{-1}(R(z) - R(1))$ (and variants of them) we use the fact that this function is analytic on \mathbb{D} , so we can exploit the use of the derivatives. In the process, we recognize the coefficients of some derivatives as convolutions integrated over a well chosen contour. This allows us to exploit the strength of (H) and (H1). For details we refer to the statements and proofs of Propositions 6.6 and 6.5.

4 Higher order expansion of the scalar part $1 - \Psi(z)$

In this section we obtain higher order expansions for $1 - \Psi(z) = 1 - \int_Y e^{(-u+i\theta)\varphi} d\mu$, $z \in \mathbb{D}$, using the full strength of (H).

We first fix some notation that will be used throughout the rest of this work.

Notation Recall that $\mu(y \in Y : \varphi(y) > n) = \sum_{j=1}^{k-1} c_j n^{-j\beta} + A(n) + B(n)$, where c_j and $A(n), B(n)$ are the constants and the functions defined in (H).

Recall that $k = \min\{j \geq 2 : \beta > \frac{1}{j}\}$. For $j = 1, \dots, k - 1$, define $\Delta_j(x) = \lfloor x \rfloor^{-j\beta} - x^{-j\beta}$. Define $H_1(x) = \sum_{j=1}^{k-1} c_j \Delta_j(x) + A(\lfloor x \rfloor) + B(\lfloor x \rfloor)$. With the convention $A(0) = B(0) = 0$ and $0^{-\beta} = 0$, the functions $A(x), B(x)$ and $\Delta_j(x), j = 1, \dots, k - 1$ are well defined on $[0, \infty)$. We set $c_H = \int_0^\infty H_1(x) dx$ if $\beta > 1/2$ and $c_H = 0$ otherwise.

First, we state a simple form of the expansion of $1 - \Psi(z)$ that will be used throughout the paper (mainly in Sect. 5).

Proposition 4.1 Assume (H). Write $z = e^{-u+i\theta}$, $u > 0$ and $\theta \in (-\pi, \pi)$. Then, as $z \rightarrow 1$,

$$1 - \Psi(z) = c_1 \Gamma(1 - \beta)(u - i\theta)^\beta + c_H(u - i\theta) + D(z),$$

where

- (i) For $\beta \neq 1/2$, $|D(z)| \ll |u - i\theta|^{2\beta}$. Also, $|\frac{d}{d\theta} D(z)| \ll |u - i\theta|^{2\beta-1}$ and $|\frac{d^2}{d\theta^2} D(z)| \ll |u - i\theta|^{2\beta-2} + u^{\gamma_1-1}$ for some $\gamma_1 \in (0, 1)$.
- (ii) For $\beta = 1/2$, $|D(z)| \ll |u - i\theta| \log(1/|u - i\theta|)$, $|\frac{d}{d\theta} D(z)| \ll \log(1/|u - i\theta|)$ and $|\frac{d^2}{d\theta^2} D(z)| \ll |u - i\theta|^{-1} \log(1/|u - i\theta|)$.

Proposition 4.1 is an immediate consequence of Proposition 4.2 below, which gives a more precise (but more complicated) expansion of $1 - \Psi(z)$.

Proposition 4.2 Assume (H). Write $z = e^{-u+i\theta}$, $u > 0$ and $\theta \in (-\pi, \pi)$. Set $\tilde{c}_H = \int_0^\infty H_1(x) dx$. Then, as $z \rightarrow 1$,

$$1 - \Psi(z) = \sum_{j=1}^K c_j \Gamma(1 - j\beta)(u - i\theta)^{j\beta} + \tilde{c}_H(u - i\theta) + D(z),$$

where for all $k \geq 2$, $K = k - 1$ if $\beta^{-1} \notin \mathbb{Z}_+$ and $K = k - 2$ if $\beta^{-1} \in \mathbb{Z}_+$ and $D(z)$ satisfies the following estimates:

- (i) For $\beta^{-1} \notin \mathbb{Z}_+$, there exists $\gamma_0 \in (1, 2)$ with $\gamma_0 \geq 2\beta$ such that $|D(z)| \ll |u - i\theta|^{\gamma_0}$. Also, $|\frac{d}{d\theta} D(z)| \ll |u - i\theta|^{\gamma_0-1}$ and $|\frac{d^2}{d\theta^2} D(z)| \ll u^{\gamma_0-2} |\log u|$.
- (ii) For $\beta^{-1} \in \mathbb{Z}_+$, $|D(z)| \ll |u - i\theta| \log(1/|u - i\theta|)$, $|\frac{d}{d\theta} D(z)| \ll \log(1/|u - i\theta|)$ and $|\frac{d^2}{d\theta^2} D(z)| \ll |u - i\theta|^{-1} \log(1/|u - i\theta|)$.

Proof Define the distribution function $G(x) = \mu(\varphi \leq x)$. Then $1 - \Psi(z) = \int_0^\infty (1 - e^{(-u+i\theta)x}) dG(x)$, where $1 - G(x) = \sum_{j=1}^{k-1} c_j x^{-j\beta} + H_1(x)$. Integration by parts gives

$$\int_0^\infty (1 - e^{(-u+i\theta)x}) dG(x) = - \int_0^\infty (1 - e^{(-u+i\theta)x}) d(1 - G(x))$$

$$\begin{aligned}
 &= (u - i\theta) \int_0^\infty (1 - e^{-(u+i\theta)x})(1 - G(x)) dx \\
 &= \sum_{j=1}^{k-1} c_j (u - i\theta)^{j\beta} \int_0^\infty \frac{e^{-(u-i\theta)x}}{((u - i\theta)x)^{j\beta}} (u - i\theta) dx \\
 &\quad + (u - i\theta) \int_0^\infty e^{-(u-i\theta)x} H_1(x) dx.
 \end{aligned}$$

By [20, Proposition B1], $I_j := \int_0^\infty e^{-(u-i\theta)x} ((u - i\theta)x)^{-j\beta} (u - i\theta) dx = \Gamma(1 - j\beta)$, for all $j < 1/\beta$. In particular this is the case when $j \leq K$ where K is as in the statement of the proposition. The remainder of the proof is divided in two cases $\beta^{-1} \notin \mathbb{Z}_+$ and $\beta^{-1} \in \mathbb{Z}_+$.

Proof of (i). The case $\beta^{-1} \notin \mathbb{Z}_+$. First, we note that in this case, $I_j = \Gamma(1 - j\beta)$ for all $j = 1, \dots, k - 1$ and put

$$D(z) = (u - i\theta) \int_0^\infty e^{-(u-i\theta)x} H_1(x) dx - \tilde{c}_H(u - i\theta).$$

Recall that A is a finite sum $A(n) = \sum_j \ell_j(n)n^{-\xi_j}$, where for all j , $\xi_j \geq k\beta > 1$ and $\ell_j(x) = C_j \log x + C'_j$ for real constants C_j, C'_j with $C_j = 0$ if $\xi_j = 2\beta$. In the case $\beta > 1/2$ (so $k = 2$), we choose $\gamma_0 = 2\beta$. For $\beta < 1/2$ (so $k \geq 3$), we choose $\gamma_0 \in (1, k\beta)$. With this choice of γ_0 we have $A(\lfloor x \rfloor) = O(x^{-\gamma_0})$.

Since $B(n) = O(n^{-\gamma})$, $\gamma > 2 > \gamma_0$, we have $B(\lfloor x \rfloor) = O(x^{-\gamma_0})$. Clearly, $\Delta_j(x) = O(x^{-(\beta+1)}) = O(x^{-\gamma_0})$. Thus, $H_1(x) = O(x^{-\gamma_0})$. By Proposition 8.1(a), $(u - i\theta) \int_0^\infty e^{-(u-i\theta)x} H_1(x) dx = \tilde{c}_H(u - i\theta) + O(|u - i\theta|^{\gamma_0})$ and thus,

$$|D(z)| \ll |u - i\theta|^{\gamma_0}.$$

We continue with the asymptotics of the first and second derivative (in θ) of $D(z)$. Recall $H_1(x) = \sum_{j=1}^{k-1} c_j \Delta_j(x) + A(\lfloor x \rfloor) + B(\lfloor x \rfloor)$. Hence,

$$\begin{aligned}
 \int_0^\infty e^{-(u-i\theta)x} H_1(x) dx &= \sum_{j=1}^{k-1} c_j \int_0^\infty e^{-(u-i\theta)x} \Delta_j(x) dx + \int_0^\infty e^{-(u-i\theta)x} A(\lfloor x \rfloor) dx \\
 &\quad + \int_0^\infty e^{-(u-i\theta)x} B(\lfloor x \rfloor) dx.
 \end{aligned}$$

For $j = 1, \dots, k - 1$, set

$$W_j(u, \theta) = \int_0^\infty e^{-(u-i\theta)x} \Delta_j(x) dx.$$

Put $c_{\Delta_j} = \int_0^\infty \Delta_j(x) dx$. By Proposition 8.6(b), (c),

$$\begin{cases} \left| \frac{d}{d\theta} \left((u - i\theta)W_j(u, \theta) - c_{\Delta_j}(u - i\theta) \right) \right| \ll |u - i\theta|^{j\beta} \\ \left| \frac{d^2}{d\theta^2} \left((u - i\theta)W_j(u, \theta) \right) \right| \ll u^{j\beta-1}. \end{cases}$$

Let $\hat{A}(u, \theta) = \int_0^\infty e^{-(u-i\theta)x} A(\lfloor x \rfloor) dx$ and $\hat{B}(u, \theta) = \int_0^\infty e^{-(u-i\theta)x} B(\lfloor x \rfloor) dx$. Set $c_{A+B} = \int_0^\infty (A(\lfloor x \rfloor) + B(\lfloor x \rfloor)) dx$. Recall that $A(\lfloor x \rfloor) + B(\lfloor x \rfloor) = O(x^{-\gamma_0})$, where $\gamma_0 \in (1, 2)$. By Proposition 8.1(b),

$$\left| \frac{d}{d\theta} \left((u - i\theta)(\hat{A}(u, \theta) + \hat{B}(u, \theta)) - c_{A+B}(u - i\theta) \right) \right| \ll |u - i\theta|^{\gamma_0-1}.$$

Next, we estimate the second derivative of terms associated with A and B . Recall that $x A(\lfloor x \rfloor)$ is of bounded variation. By Proposition 8.4,

$$\left| \frac{d^2}{d\theta^2} \left((u - i\theta)(\hat{A}(u, \theta)) \right) \right| \ll |\log u| u^{\gamma_0-2}.$$

Recall that $n^2 B(n)$ is of bounded variation and that $B(n) = O(n^{-\gamma})$, where $\gamma > 2$. So, $x^2 B(\lfloor x \rfloor)$ is of bounded variation and $B(\lfloor x \rfloor) = O(x^{-\gamma})$, $\gamma > 2$. By Proposition 8.1(c),

$$\left| \frac{d^2}{d\theta^2} \left((u - i\theta)\hat{B}(u, \theta) \right) \right| \ll 1.$$

Recall $\tilde{c}_H = \int_0^\infty H_1(x) dx$ and note that $\tilde{c}_H = \sum_{j=1}^{k-1} c_j c_{\Delta_j} + c_{A+B}$. Putting the above together, we have that

$$\begin{aligned} \left| \frac{d}{d\theta} \left((u - i\theta) \int_0^\infty e^{-(u-i\theta)x} H_1(x) dx - c_H(u - i\theta) \right) \right| &\ll |u - i\theta|^\beta + |u - i\theta|^{\gamma_0-1} \\ &\ll |u - i\theta|^{\gamma_0-1} \end{aligned}$$

and that

$$\begin{aligned} \left| \frac{d^2}{d\theta^2} \left((u - i\theta) \int_0^\infty e^{-(u-i\theta)x} H_1(x) dx \right) \right| &\ll u^{\beta-1} + u^{\gamma_0-2} |\log u| \\ &\ll u^{\gamma_0-2} |\log u|. \end{aligned}$$

Altogether,

$$\begin{cases} \left| \frac{d}{d\theta} D(z) \right| \ll |u - i\theta|^{\gamma_0-1} \\ \left| \frac{d^2}{d\theta^2} D(z) \right| \ll u^{\gamma_0-2} |\log u|, \end{cases}$$

which ends the proof of (i).

Proof of (ii). The case $\beta^{-1} \in \mathbb{Z}_+$. This is identical to case 9i) except for the term

$$I(u, \theta) = \int_0^\infty e^{-(u-i\theta)x} x^{-1} dx.$$

By Proposition 8.5, we have $|(u - i\theta)I(u, \theta)| \ll |u - i\theta| \log(1/|u - i\theta|)$, $|\frac{d}{d\theta}(u - i\theta)I(u, \theta)| \ll \log(1/|u - i\theta|)$ and $|\frac{d^2}{d\theta^2}(u - i\theta)I(u, \theta)| \ll |u - i\theta|^{-1} \log(1/|u - i\theta|)$. This together with the estimates obtained in case (i) completes the proof. \square

5 Proof of Theorem 1.1

The notation below provides the exact formulas for the constants d_0, \dots, d_q in Theorem 1.1.

Notation Recall $\beta \in (0, 1)$ and $q = \max\{j \geq 0 : (j + 1)\beta - j > 0\}$. Recall $c_H = \int_0^\infty H_1(x) dx$ if $\beta > 1/2$ and $c_H = 0$ otherwise. Set $C_H = -c_H c_1^{-1} \Gamma(1 - \beta)^{-1}$.

With the convention $(C_H)^0 = 1$, define $C_p = (C_H)^p ((p + 1)\beta - p)$ for $p = 0, \dots, q$. Set $d_p = C_p (c_1 \Gamma(1 - \beta))^{-1} \Gamma((p + 1)\beta - p + 1)^{-1}$. We note that when $\beta \leq 1/2, q = 0$ and the only non zero constant is $d_0 = (c_1 \Gamma(1 - \beta))^{-1} \Gamma(\beta + 1)^{-1}$.

The first result below is instrumental in the proof of Theorem 1.1.

Lemma 5.1 *Assume the setting of Proposition 4.2. Write $z = e^{-u+i\theta}$. Then, the following holds for all $\beta \in (0, 1)$ as $z \rightarrow 1$:*

$$c_1 \Gamma(1 - \beta) \frac{d}{d\theta} ((1 - \Psi(z))^{-1}) = i \sum_{p=0}^q C_p (u - i\theta)^{(p-1)-(p+1)\beta} + E(z),$$

where

$$|E(z)| \ll \begin{cases} |u - i\theta|^{-1}, & \text{if } \beta \neq 1/2, \\ |u - i\theta|^{-1} \log(1/|u - i\theta|), & \text{if } \beta = 1/2. \end{cases}$$

Proof Note that

$$\frac{d}{d\theta} (1 - \Psi(z))^{-1} = (1 - \Psi(z))^{-2} \frac{d}{d\theta} \Psi(z). \tag{5.1}$$

By Proposition 4.1 and the definition of C_H ,

$$1 - \Psi(z) = c_1 \Gamma(1 - \beta) (u - i\theta)^\beta (1 - C_H (u - i\theta)^{1-\beta} + D(z)),$$

where

$$|D(z)| \ll \begin{cases} |u - i\theta|^\beta, & \text{if } \beta \neq 1/2, \\ |u - i\theta|^{1/2} \log(1/|u - i\theta|), & \text{if } \beta = 1/2. \end{cases}$$

We recall that $q = \max\{j \geq 0 : (j + 1)\beta - j > 0\}$ and compute that

$$c_1 \Gamma(1 - \beta)(1 - \Psi(z))^{-1} = \sum_{p=0}^q (C_H)^p (u - i\theta)^{p - (p+1)\beta} + F(z),$$

where

$$|F(z)| \ll \begin{cases} 1, & \text{if } \beta \neq 1/2, \\ \log(1/|u - i\theta|), & \text{if } \beta = 1/2. \end{cases}$$

Based on the asymptotic expansion of $(1 - \Psi(z))^{-1}$ above we compute that

$$(c_1 \Gamma(1 - \beta))^2 (1 - \Psi(z))^{-2} = \left(\sum_{p=0}^q (C_H)^p (u - i\theta)^{p - (p+1)\beta} \right)^2 + G(z), \quad (5.2)$$

where

$$|G(z)| \ll \begin{cases} |u - i\theta|^{-\beta}, & \text{if } \beta \neq 1/2, \\ |u - i\theta|^{-1/2} \log(1/|u - i\theta|), & \text{if } \beta = 1/2. \end{cases}$$

Next, by Proposition 4.1 and the definition of C_H , we obtain that

$$(c_1 \Gamma(1 - \beta))^{-1} \frac{d}{d\theta} \Psi(z) = i\beta(u - i\theta)^{\beta-1} - iC_H + H(z), \quad (5.3)$$

where

$$|H(z)| \ll \begin{cases} |u - i\theta|^{2\beta-1}, & \text{if } \beta \neq 1/2, \\ \log(1/|u - i\theta|), & \text{if } \beta = 1/2. \end{cases}$$

By (5.1), (5.2) and (5.3), we compute that

$$c_1 \Gamma(1 - \beta) \frac{d}{d\theta} ((1 - \Psi(z))^{-1}) = i \sum_{p=0}^q C_p (u - i\theta)^{(p-1) - (p+1)\beta} + E(z),$$

where

$$|E(z)| \ll |u - i\theta|^{\beta-1} G(z) + |u - i\theta|^{-2\beta} H(z).$$

This ends the proof since

$$|E(z)| \ll \begin{cases} |u - i\theta|^{-1}, & \text{if } \beta \neq 1/2, \\ |u - i\theta|^{-1} \log(1/|u - i\theta|), & \text{if } \beta = 1/2. \end{cases}$$

□

Remark 5.2 For use below (in the proof of Proposition 6.5) we note that differentiating in (5.3) once more and using the information on the second derivative (in θ) of $D(z)$ provided by Proposition 4.1, one can easily show that for all $u > 0$ and $\theta \in (-\pi, \pi)$, $|\frac{d^2}{d\theta^2}((1 - \Psi(z))^{-1})| \ll |u - i\theta|^{-(\beta+2)} + |u - i\theta|^{-2\beta}u^{\gamma_1-1}$ for some $\gamma_1 \in (0, 1)$.

Remark 5.3 For use below (in the proof of Proposition 6.6) we note the following. Since $|\frac{d}{d\theta}((1 - \Psi(z))^{-1/2})| \ll |(1 - \Psi(z))^{-3/2}\frac{d}{d\theta}\Psi(z)|$, one can easily check that Proposition 4.1 (using just the information on the first derivative (in θ) of $D(z)$) implies that $|\frac{d}{d\theta}((1 - \Psi(z))^{-1/2})| \ll |u - i\theta|^{-(\beta/2+1)}$. Moreover, since

$$\left| \frac{d^2}{d\theta^2}((1 - \Psi(z))^{-1}) \right| \ll \left| (1 - \Psi(z))^{-5/2} \left(\frac{d}{d\theta} \Psi(z) \right)^2 \right| + \left| (1 - \Psi(z))^{-3/2} \frac{d^2}{d\theta^2} \Psi(z) \right|$$

one can easily check that using the information on the first and second derivative (in θ) of $D(z)$, $|\frac{d^2}{d\theta^2}((1 - \Psi(z))^{-1/2})| \ll |u - i\theta|^{-(\beta/2+2)} + |u - i\theta|^{-3\beta/2}u^{\gamma_1-1}$ for some $\gamma_1 \in (0, 1)$.

We can now proceed to the

Proof of Theorem 1.1 By Corollary 3.2 the Taylor coefficients of $(1 - \Psi(z))^{-1}, z \in \mathbb{D}$, coincide with the Fourier coefficients of $(1 - \Psi(z))^{-1}, z \in \mathcal{S}^1$.

We estimate the Taylor coefficients of $(1 - \Psi(z))^{-1}, z \in \mathbb{D}$, on the circle $\Gamma = \{e^{-u}e^{i\theta} : -\pi \leq \theta < \pi\}$ with $e^{-u} = e^{-1/n}$, where $n \geq 1$. Write

$$[(1 - \Psi)^{-1}]_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{(1 - \Psi(z))^{-1}}{z^{n+1}} dz = \frac{e}{2\pi} \int_{-\pi}^{\pi} (1 - \Psi(e^{-1/n}e^{i\theta}))^{-1} e^{-in\theta} d\theta.$$

Integration by parts gives

$$\frac{2\pi}{e} [(1 - \Psi)^{-1}]_n = -\frac{i}{n} \int_{-\pi}^{\pi} \frac{d}{d\theta} ((1 - \Psi(e^{-1/n}e^{i\theta}))^{-1}) e^{-in\theta} d\theta. \tag{5.4}$$

By Lemma 5.1 and (5.4),

$$\begin{aligned} \frac{2\pi}{e} [(c_1\Gamma(1 - \beta))[(1 - \Psi)^{-1}]_n &= \frac{1}{n} \sum_{p=0}^q C_p \int_{-\pi}^{\pi} \left(\frac{1}{n} - i\theta\right)^{(p-1)-(p+1)\beta} e^{-in\theta} d\theta \\ &+ \frac{i}{n} \int_{-\pi}^{\pi} E(e^{-1/n}e^{i\theta}) e^{-in\theta} d\theta \end{aligned}$$

$$= \frac{1}{n} \sum_{p=0}^q C_p \int_{-\pi}^{\pi} \frac{e^{-in\theta}}{\left(\frac{1}{n} - i\theta\right)^{(p+1)\beta - p + 1}} d\theta + \frac{1}{n} J.$$

If $\beta \neq 1/2$, using the asymptotics of $E(z)$ provided in Lemma 5.1, we compute that

$$\frac{1}{n} |J| \ll \frac{1}{n} \left(\int_0^{1/n} + \int_{1/n}^{\pi} \right) \left| \frac{1}{n} - i\theta \right|^{-1} d\theta \ll \frac{1}{n} \left(1 + \int_{1/n}^{\pi} \theta^{-1} d\theta \right) \ll \frac{\log n}{n}.$$

If $\beta = 1/2$, using again Lemma 5.1 we have

$$\begin{aligned} \frac{1}{n} |J| &\ll \frac{1}{n} \int_0^{\pi} \log \left(\left| \frac{1}{n} - i\theta \right|^{-1} \right) \left| \frac{1}{n} - i\theta \right|^{-1} d\theta \\ &\ll \frac{\log n}{n} \left(\int_0^{1/n} n d\theta + \log n \int_{1/n}^{\pi} |\theta|^{-1} d\theta \right) \ll \frac{(\log n)^2}{n}. \end{aligned}$$

By [20, Corollary B.3] with $\rho = (p + 1)\beta - p$, for $p = 0, \dots, q$, we have

$$\begin{aligned} \frac{1}{n} \sum_{p=0}^q C_p \int_{-\pi}^{\pi} \frac{e^{-in\theta}}{\left(\frac{1}{n} - i\theta\right)^{(p+1)\beta - p + 1}} d\theta &= \frac{2\pi}{e} \sum_{p=0}^q \frac{C_p}{\Gamma((p + 1)\beta - p + 1)} n^{(p+1)(\beta-1)} \\ &\quad + O\left(\frac{1}{n}\right). \end{aligned}$$

The result follows putting the above together and using the definition of d_p . □

Remark 5.4 For use below (in the proof of Proposition 6.5) we note that the coefficients of $(1 - \Psi(z))^{-1/2}$, $z \in \mathbb{D}$ satisfy $[(1 - \Psi)^{-1/2}]_n \ll n^{\beta/2-1}$. To see this recall from Remark 5.3 that $\left| \frac{d}{d\theta} (1 - \Psi(z))^{-1/2} \right| \ll |u - i\theta|^{-(1+\beta/2)}$. Hence, the result follows by the argument used in the proof of Theorem 1.1.

6 Main steps in estimating the Fourier coefficients of $\tilde{T}(z) - (1 - \Psi(z))^{-1}P$, $z \in \mathcal{S}^1$

6.1 Preliminaries on the use of Wiener’s lemma

An important ingredient in our proofs in the next sections are the versions of Wiener’s lemma for commutative and non-commutative Banach algebras recalled below. We first recall the standard Wiener lemma: Let $f : \mathcal{S}^1 \rightarrow \mathbb{C}$ be a continuous function, everywhere non-zero with absolutely summable Fourier coefficients. Then the Fourier coefficients of f^{-1} are also absolutely summable (see, for instance [17]).

To formulate the versions of Wiener’s lemma used here we introduce some notation. Let \mathcal{A} be the Banach algebra of continuous functions $f : \mathcal{S}^1 \rightarrow \mathbb{C}$ such that their Fourier coefficients \hat{f}_n are absolutely summable, with norm $\|f\|_{\mathcal{A}} = \sum_{n \in \mathbb{Z}} |\hat{f}_n|$.

Given $\gamma > 1$, define the commutative Banach algebra $\mathcal{A}_\gamma = \{f \in \mathcal{A} : \sup_{n \in \mathbb{Z}} |n|^\gamma |\hat{f}_n| < \infty\}$ with norm $\|f\|_{\mathcal{A}_\gamma} = \sum_{n \in \mathbb{Z}} |\hat{f}_n| + \sup_{n \in \mathbb{Z}} |n|^\gamma |\hat{f}_n|$. We can now state a Wiener lemma for commutative Banach algebras; for further details and proof we refer to, for instance, [10, Chapter 2].

Lemma 6.1 *Suppose that $f : \mathcal{S}^1 \rightarrow \mathbb{C}$ is a continuous function, everywhere non-zero and that f belongs to \mathcal{A}_γ . Then the function f^{-1} belongs to \mathcal{A}_γ .*

A similar version holds for operator-valued functions $F : \mathcal{S}^1 \rightarrow \mathcal{B}$, where \mathcal{B} is a Banach space with norm $\|\cdot\|$. In this case, let \hat{A} be the non-commutative Banach algebra of continuous functions $F : \mathcal{S}^1 \rightarrow \mathcal{B}$ such that their Fourier coefficients \hat{F}_n are absolutely summable, with norm $\|F\|_{\hat{A}} = \sum_{n \in \mathbb{Z}} \|\hat{F}_n\|$. Given $\gamma > 1$, define the non-commutative Banach algebra $\hat{A}_\gamma = \{F \in \hat{A} : \sup_{n \in \mathbb{Z}} |n|^\gamma \|\hat{F}_n\| < \infty\}$ with norm $\|F\|_{\hat{A}_\gamma} = \sum_{n \in \mathbb{Z}} \|\hat{F}_n\| + \sup_{n \in \mathbb{Z}} |n|^\gamma \|\hat{F}_n\|$. The result below can be obtained from [6]; see also [12, Chapter 2] (in particular [12, Theorem 2.2.16]) for a concise exposition.

Lemma 6.2 *Suppose that $F : \mathcal{S}^1 \rightarrow \mathcal{B}$ is a continuous function, everywhere non-zero and that F belongs to \hat{A}_γ . Then the function F^{-1} belongs to \hat{A}_γ .*

6.2 Main terms of $\tilde{T}(z) - (1 - \Psi(z))^{-1}P$

We first recall that $P(z) : \mathcal{B} \rightarrow \mathcal{B}$ is the family of spectral projections associated with the eigenvalue $\lambda(z)$, and $P(1) = P$. By (H2)(i), we can choose a closed loop $\Gamma \subset \mathbb{C} \setminus \text{spec } R(1)$ separating 1 from the remainder of the spectrum of $R(1)$; that is, there exists $\epsilon > 0$ such that the spectrum of $R(z)$ does not intersect Γ for $z \in \mathbb{D} \cap B_\epsilon(1)$. For $z \in B_\epsilon(1)$ we can define the spectral projection

$$P(z) = \frac{1}{2\pi i} \int_\Gamma (\psi I - R(z))^{-1} d\psi. \tag{6.1}$$

Also, we recall that one main property from Proposition 3.5 is: the eigenvalue $\tilde{\lambda}(z)$ of the new operator $\tilde{R}(z)$ is well defined and close to 1 for all $z \in \mathcal{S}^1$ (hence, $\tilde{P}(z)$ is well defined and close to P for all $z \in \mathcal{S}^1$). Since $\tilde{R}(1) = R(1) = R$, Eq. (6.1) with \tilde{P}, \tilde{R} instead of P, R , holds for all $z \in \mathcal{S}^1$.

Let $v(z) = P(z)1 / \int P(z)1$ and $\tilde{v}(z) = \tilde{P}(z)1 / \int \tilde{P}(z)1$ be the normalised eigenfunctions associated with $\lambda(z)$ and $\tilde{\lambda}(z)$ respectively.

Recall that $1 - \Psi(z) = \int_Y (1 - z^\varphi) d\mu$ (the function dealt with in the previous sections). Using the formalism in [14], a simplification of [4], we write $1 - \lambda(z) = 1 - \Psi(z) - \int_Y (R(z) - R(1))(v(z) - v(1)) d\mu$. Proceeding similarly we compute that

$$1 - \tilde{\lambda}(z) = 1 - \Psi(z) - \int_Y (R(z) - R)(\tilde{v}(z) - \tilde{v}(1)) d\mu - \int_Y (\tilde{R}(z) - R(z)) \tilde{v}(z) d\mu.$$

Put $\tilde{V}(z) = - \int_Y (R(z) - R(1))(\tilde{v}(z) - \tilde{v}(1))d\mu$ and define $\tilde{W}(z) = (1 - \Psi(z))^{-1} \tilde{V}(z)$. Also, let $\tilde{A}(z) = -(1 - \Psi(z))^{-1} \int_Y (\tilde{R}(z) - R(z))\tilde{v}(z) d\mu$. Hence,

$$(1 - \tilde{\lambda}(z))^{-1} = (1 - \Psi(z))^{-1}(1 + \tilde{W}(z) + \tilde{A}(z))^{-1} \\ = (1 - \Psi(z))^{-1} - (1 - \Psi(z))^{-1}(\tilde{W}(z) + \tilde{A}(z))(1 + \tilde{W}(z) + \tilde{A}(z))^{-1}.$$

Recall that $Q(z) = I - P(z)$ denotes the complementary spectral projection of $P(z)$. Let $\tilde{Q}(z) = I - \tilde{P}(z)$ be the complementary spectral projection of $\tilde{P}(z)$. The previous displayed equation together with Eq. (2.1) (with tilde everywhere) implies that

$$\tilde{T}(z) - (1 - \Psi(z))^{-1}P = (1 - \Psi(z))^{-1}(\tilde{P}(z) - P) \\ - (1 - \Psi(z))^{-1}(\tilde{W}(z) + \tilde{A}(z))(1 + \tilde{W}(z) + \tilde{A}(z))^{-1}\tilde{P}(z) \\ + (I - \tilde{R}(z))^{-1}\tilde{Q}(z). \tag{6.2}$$

Under (H1), the Fourier coefficients of $R(z)$, $z \in \mathbb{D}$, and $\tilde{R}(z)$, $z \in S^1$, satisfy $\|R_n\|, \|\tilde{R}_n\| = O(|n|^{-(\beta+1)})$; the latter estimate is given by Proposition 3.3(v). This property along with (H) and the decomposition (6.2) will be exploited in the next sections.

To begin we summarize the estimates for the Fourier coefficients of all the terms in (6.2) obtained in the next sections and as such provide

Proof of Proposition 3.5 By the argument in [13, 15] (based on Wiener Lemma 6.2), $\|[(I - \tilde{R})^{-1}\tilde{Q}]_n\| = O(|n|^{-(\beta+1)})$. Also, the coefficients of the first term $(1 - \Psi)^{-1}(\tilde{P} - P)$ are $O(1/|n|)$ by Proposition 6.9.

It remains to estimate the Fourier coefficients of the second term in (6.2), which we split in three factors. First, the Fourier coefficients of the third factor $\tilde{P}(z)$ are $O(|n|^{-(\beta+1)})$ by Proposition 3.4.

Next, by Corollary 6.4 (with $m = 1$), the Fourier coefficients of $\tilde{A}(z)$ are $O(|n|^{-(\beta+1)})$. By Corollary 6.8, the coefficients of $\tilde{W}(z)$ are $O(|n|^{-(1+\tau)})$ for some $\tau > 0$. Since $1 + \tilde{W}(z) + \tilde{A}(z)$ is continuous and non vanishing on S^1 , Wiener Lemma 6.1 applies. Hence, the coefficients of $(1 + \tilde{W}(z) + \tilde{A}(z))^{-1}$ are $O(|n|^{-(1+\tau)})$. This takes care of the middle factor.

By Corollary 6.4 (with $m = 2$), the coefficients of $(1 - \Psi(z))^{-1}\tilde{A}(z)$ are $O(|n|^{-(\beta+1)})$. By Corollary 6.10, the coefficients of $(1 - \Psi)^{-1}\tilde{W}(z)$ are $O(\log |n|/|n|)$. This takes care of the first factor.

Convolving the coefficients of the above three factors deals with the second term and hence completes the proof. □

6.3 Estimating the coefficients of $\tilde{A}(z)$ and $(1 - \Psi(z))^{-1}\tilde{A}(z)$

Proposition 6.3 *Let $m \in \mathbb{Z}_+$. The Fourier coefficients of the operator-valued function $(1 - \Psi(z))^{-m}(\tilde{R}(z) - R(z))$, $z \in S^1$ are $O(|n|^{-(1+\beta)})$ (in norm $\|\cdot\|$).*

Proof By Proposition 3.3(i), there exists $\epsilon > 0$ such that $\tilde{R}(e^{i\theta}) = R(e^{i\theta})$ for all $e^{i\theta} \in B_\epsilon(1)$. By (H1) and Proposition 3.3(v), $\|R_n\|, \|\tilde{R}_n\| \ll |n|^{-(\beta+1)}$. Consider a

C^∞ partition of unity on \mathcal{S}^1 given by ϕ and $1 - \phi$ with $\phi : \mathcal{S}^1 \rightarrow [0, 1]$ such that for $\epsilon > 0$ as above, $\phi(z) = 1$, for all $z \in B_{\epsilon/2}(1)$ and $\phi(z) = 0$, for all $z \in \mathcal{S}^1 \setminus B_\epsilon(1)$.

Define $\Phi = \phi + (1 - \phi)(1 - \Psi)^m$, $m \in \mathbb{Z}_+$. By construction, $(1 - \Psi)^{-m}(\tilde{R} - R) = \Phi^{-1}(\tilde{R} - R)$. Recall that the coefficients of $1 - \Psi$ are $O(n^{-(\beta+1)})$. Hence, the coefficients of $(1 - \Psi)^m$, and thus of Φ , are $O(n^{-(\beta+1)})$.

Next, note that Φ is continuous and nonvanishing on \mathcal{S}^1 . To see that it is nonvanishing, suppose the contrary. Splitting into real and imaginary parts, it is easy to see that Φ vanishes only if $\phi = 0$. But that means that $1 - \Psi = 0$ which is impossible.

Putting the above together, the coefficients of Φ^{-1} are $O(|n|^{-(\beta+1)})$, by Wiener Lemma 6.1. Thus, the coefficients of $\Phi(z)^{-1}(\tilde{R}(z) - R(z))$ are $O(|n|^{-(\beta+1)})$, as required. \square

Corollary 6.4 *Let $m \in \mathbb{Z}_+$. The Fourier coefficients of the operator-valued function $(1 - \Psi(z))^{-m}(\tilde{R}(z) - R(z))\tilde{v}(z)$ are $O(|n|^{-(1+\beta)})$ (in norm $\|\cdot\|$).*

Proof By Proposition 3.4, the Fourier coefficients of $\tilde{P}(z)$ satisfy $\|\tilde{P}_n\| = O(|n|^{-(1+\beta)})$. Since $\tilde{v}(z) = \tilde{P}(z)1 / \int \tilde{P}(z)1$, the coefficients of $\tilde{v}(z)$ are $O(|n|^{-(1+\beta)})$. The conclusion follows from this together with Proposition 6.3. \square

To justify the title of this subsection note that the estimates on the coefficients of $\tilde{A}(z)$ and $(1 - \Psi(z))^{-1}\tilde{A}(z)$ follow by Corollary 6.4 with $m = 1$ and $m = 2$, respectively.

6.4 Some abstract results

In this subsection we state some general results from which all the required estimates on the coefficients of the remaining terms in (6.2) are obtained. The corresponding proofs are postponed to Sect. 7.

Proposition 6.5 *Suppose that $B(z)$ is an operator-valued function (on some Banach space \mathcal{B} with norm $\|\cdot\|$) continuous on \mathcal{S}^1 with $B(1) = 0$. Assume that its Fourier coefficients satisfy $\|B_n\| = O(|n|^{-(\beta+1)})$.*

Define $C(z) = (1 - \Psi(z))^{-1}B(z)$. Then the Fourier coefficients of $C(z)$ satisfy $\|C_n\| = O(|n|^{-1})$.

Proposition 6.6 *Suppose that $B(z)$ is an operator-valued function (on some Banach space \mathcal{B} with norm $\|\cdot\|$) continuous on \mathcal{S}^1 with $B(1) = 0$. Assume that its Fourier coefficients satisfy $\|B_n\| = O(|n|^{-(\beta+1)})$.*

Define $C(z) = (1 - \Psi(z))^{-1/2}B(z)$. Then the Fourier coefficients of $C(z)$ satisfy $\|C_n\| = O(|n|^{-(\tau+1)})$, for some $\tau > 0$.

6.5 Estimating the Fourier coefficients of $\tilde{W}(z)$

Recall that $\tilde{V}(z) = -\int_Y (R(z) - R(1))(\tilde{v}(z) - \tilde{v}(1))d\mu$ and $\tilde{W}(z) = (1 - \Psi(z))^{-1}\tilde{V}(z)$. Clearly, the desired estimate for $|\tilde{W}_n|$ cannot be obtained by convolving the coefficients of $(1 - \Psi(z))^{-1}$ and $\tilde{V}(z)$. Moreover, using any other information about

the function $(1 - \Psi(z))^{-1}\tilde{V}(z)$ as a whole is bound to fail. For instance, the upper bound $O(|\mu - i\theta|^\beta)$ is useless by itself and the function is not analytic on \mathbb{D} . In fact, the knowledge about analyticity on \mathbb{D} would not be sufficient either; estimating the coefficients of the somewhat nicer (analytic) function $(1 - \Psi(z))^{-1} \int_Y (R(z) - R(1))(v(z) - v(1))d\mu$ without decomposing it into appropriate factors provides an unsatisfactory result for the present purpose.

To deal with the difficulties mentioned above, we write

$$\tilde{W}(z) = - \int_Y ((R(z) - R(1))(1 - \Psi(z))^{-1/2})(\tilde{v}(z) - \tilde{v}(1))(1 - \Psi(z))^{-1/2})d\mu. \tag{6.3}$$

The above decomposition of $\tilde{W}(z)$ allows us to exploit some immediate consequences of Proposition 6.6.

Proposition 6.7 *The Fourier coefficients (in the norm $\|\cdot\|$) of the operator valued functions $(1 - \Psi(z))^{-1/2}(R(z) - R(1))$ and $(1 - \Psi(z))^{-1/2}(\tilde{v}(z) - \tilde{v}(1))$ are $O(|n|^{-(1+\tau)})$, for some $\tau > 0$.*

Proof Let $B(z) = R(z) - R(1)$ and $C(z) = (1 - \Psi(z))^{-1/2}B(z)$. Since $\|R_n\| = O(n^{-(\beta+1)})$, we have $\|B_n\| = O(|n|^{-(\beta+1)})$. Also, we know that $B(z)$ is continuous on S^1 and $B(1) = 0$. Hence, the assumptions of Proposition 6.6 on the function B hold and the statement on the Fourier coefficients of $C(z)$ follows. The other part of the statement follows similarly by taking $B(z) = \tilde{v}(z) - \tilde{v}(1)$, $C(z) = (1 - \Psi(z))^{-1/2}B(z)$ and noticing that the assumptions of Proposition 6.6 on the function B are again satisfied (using the formula $\tilde{v}(z) = \tilde{P}(z)1 / \int \tilde{P}(z)1$ and Proposition 3.4). \square

We can now deal with the Fourier coefficients of $\tilde{W}(z)$.

Corollary 6.8 *The Fourier coefficients of the function $\tilde{W}(z)$ are $O(|n|^{-(1+\tau)})$ for some $\tau > 0$.*

Proof This follows from Eq. (6.3) together with Proposition 6.7. \square

6.6 Estimating the Fourier coefficients of the functions $(1 - \Psi(z))^{-1}(\tilde{P}(z) - P)$ and $(1 - \Psi(z))^{-1}\tilde{W}(z)$

Proposition 6.9 *The Fourier coefficients (in the norm $\|\cdot\|$) of the operator valued functions $(1 - \Psi(z))^{-1}(R(z) - R(1))$, $(1 - \Psi(z))^{-1}(\tilde{P}(z) - P(1))$ and $(1 - \Psi(z))^{-1}(\tilde{v}(z) - \tilde{v}(1))$ are $O(|n|^{-1})$.*

Proof The proof goes exactly as the proof of Proposition 6.7, except that this time we use Proposition 6.5 (instead of Proposition 6.6) to estimate the Fourier coefficients of the function $C(z) = (1 - \Psi(z))^{-1}B(z)$ with $B(z) = R(z) - R(1)$, $B(z) = \tilde{P}(z) - P(1)$ and $B(z) = \tilde{v}(z) - \tilde{v}(1)$, respectively. \square

Corollary 6.10 *The Fourier coefficients of the function $(1 - \Psi(z))^{-1}\tilde{W}(z)$ are $O(\log |n|/|n|)$.*

Proof Note that $(1 - \Psi(z))^{-1}\tilde{W}(z) = (1 - \Psi(z))^{-2}\tilde{V}(z)$, so we can write

$$(1 - \Psi(z))^{-1}\tilde{W}(z) = - \int_Y ((R(z) - R(1))(1 - \Psi(z))^{-1}) \times ((\tilde{v}(z) - \tilde{v}(1))(1 - \Psi(z))^{-1})d\mu.$$

By Proposition 6.9 we know that the Fourier coefficients (in the norm $\|\cdot\|$) of $(1 - \Psi(z))^{-1}(R(z) - R(1))$ and $(1 - \Psi(z))^{-1}(\tilde{v}(z) - \tilde{v}(1))$ are $O(|n|^{-1})$.

By a convolution argument (see, for instance, [13, Lemma 4.4]) the n -th coefficient of the function $((1 - \Psi(z))^{-1}(R(z) - R(1))((1 - \Psi(z))^{-1}(\tilde{v}(z) - \tilde{v}(1)))$ is $O(\log |n|/|n|)$, ending the proof. \square

7 Proofs of Propositions 6.5 and 6.6

7.1 Some preliminary results assuming analyticity

In this subsection we assume that $B(z)$ is an operator-valued function (on some Banach space \mathcal{B} with norm $\|\cdot\|$) continuous on $\bar{\mathbb{D}}$, analytic on \mathbb{D} with $B(1) = 0$. Moreover, we assume that the coefficients B_n of $B(z)$ satisfy $\|B_n\| \ll |n|^{-(\beta+1)}$.

The next result is an immediate consequence of our assumptions. We recall the standard argument only for completeness (see, for instance, [19, Proposition 2.7]).

Proposition 7.1 *For all $u \geq 0, \theta \in [-\pi, \pi), \|B(e^{-u+i\theta})\| \ll |u - i\theta|^\beta$.*

Proof Using $\|B_n\| \leq C|n|^{-(\beta+1)}$ for some $C > 0$, compute that

$$\|B(e^{-u+i\theta})\| = \|B(e^{-u+i\theta}) - B(1)\| \leq C|u - i\theta| \sum_{n < M} |n|^{-\beta} + 2C \sum_{n \geq M} |n|^{-(\beta+1)}.$$

The conclusion follows by taking M to be the integer part of $|u - i\theta|^{-1}$. \square

The analyticity of $B(z), z \in \mathbb{D}$ together with $\|B_n\| \ll n^{-(\beta+1)}$ imply that

Lemma 7.2 *Write $z = e^{-u+i\theta}$. Then, for all $u > 0, \|\frac{d}{d\theta}B(e^{-u+i\theta})\| \ll u^{\beta-1}$.*

Proof The result follows by standard computations. We provide the argument for completeness (see also [23, Proposition 4.6] for a more general statement). Compute that

$$\left\| \frac{d}{d\theta}B(z) \right\| \ll \sum_{j=0}^\infty j \|B_j\| e^{-(j-1)u} \ll \sum_j j^{-\beta} e^{-uj} \ll u^{\beta-1} \int_0^\infty \sigma^{-\beta} e^{-\sigma} d\sigma \ll u^{\beta-1}.$$

\square

Lemma 7.3 For $k = 1$ and $k = 1/2$, define $C(z) = (1 - \Psi(z))^{-k} B(z)$. Then the coefficients C_n of $C(z)$, $z \in \mathbb{D}$ satisfy

$$C_n = \frac{ei}{2\pi} \frac{1}{n} J_n + D_n,$$

where

$$J_n = \int_{-\pi}^{\pi} B(e^{-1/n} e^{i\theta}) \frac{d}{d\theta} ((1 - \Psi(e^{-1/n} e^{i\theta}))^{-k}) e^{-in\theta} d\theta$$

and D_n is a sequence of operators such that

$$\|D_n\| = \begin{cases} O(n^{-1}), & \text{if } k = 1; \\ O(n^{-(1+\tau)}), & \text{if } k = 1/2, \text{ for some } \tau > 0. \end{cases}$$

Proof We estimate the coefficients C_n of the function $C(z)$, $z \in \mathbb{D}$, on the circle $\Gamma = \{e^{-u+i\theta} : -\pi \leq \theta < \pi\}$ with $e^{-u} = e^{-1/n}$, where $n \geq 1$. Write

$$C_n = \frac{1}{2\pi i} \int_{\Gamma} \frac{C(z)}{z^{n+1}} dz = \frac{e}{2\pi} \int_{-\pi}^{\pi} C(z) e^{-in\theta} d\theta = \frac{e}{2\pi} \frac{i}{n} \int_{-\pi}^{\pi} \frac{d}{d\theta} C(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta.$$

Compute that

$$\begin{aligned} \frac{d}{d\theta} C(e^{-1/n} e^{i\theta}) &= (1 - \Psi(e^{-1/n} e^{i\theta}))^{-k} \frac{d}{d\theta} B(e^{-1/n} e^{i\theta}) \\ &\quad + B(e^{-1/n} e^{i\theta}) \frac{d}{d\theta} ((1 - \Psi(e^{-1/n} e^{i\theta}))^{-k}). \end{aligned}$$

Put $B^*(e^{-1/n} e^{i\theta}) = \frac{d}{d\theta} B(e^{-1/n} e^{i\theta})$. Thus,

$$C_n = \frac{i}{n} \frac{e}{2\pi} \int_{-\pi}^{\pi} (1 - \Psi(e^{1/n} e^{i\theta}))^{-k} B^*(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta + \frac{i}{n} \frac{e}{2\pi} J_n.$$

Note that $B^*(e^{-1/n} e^{i\theta}) = ie^{-1/n} e^{i\theta} B'(e^{-1/n} e^{i\theta})$ where $B'(z) := \frac{d}{dz} B(z)$. Hence,

$$\begin{aligned} &\frac{e}{2\pi} \int_{-\pi}^{\pi} (1 - \Psi(e^{1/n} e^{i\theta}))^{-k} B^*(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta \\ &= \frac{ei}{2\pi} \int_{-\pi}^{\pi} e^{-1/n} e^{i\theta} (1 - \Psi(z))^{-k} B'(z) e^{-in\theta} d\theta \\ &= \frac{1}{2\pi} \int_{\Gamma} \frac{z(1 - \Psi(z))^{-k} B'(z)}{z^{n+1}} dz. \end{aligned}$$

But $\frac{1}{2\pi i} \int_{\Gamma} \frac{z(1 - \Psi(z))^{-k} B'(z)}{z^{n+1}} dz$ is precisely the n -th coefficient of the function $z(1 - \Psi(z))^{-k} B'(z)$, $z \in \mathbb{D}$.

We claim that the n -th coefficient of $z(1 - \Psi(z))^{-k} B'(z)$ satisfies

$$\| [z(1 - \Psi(z))^{-1} B'(z)]_n \| = O(1), \quad \| [z(1 - \Psi(z))^{-1/2} B'(z)]_n \| = O(n^{-\beta/2}).$$

The conclusion follows. It remains to prove the claim.

By assumption, $\| B_n \| \ll n^{-(\beta+1)}$. Thus, $\| B'_n \| \ll n^{-\beta}$. Also, the function $z(1 - \Psi(z))^{-k}$, $k = 1/2, 1$ is analytic on \mathbb{D} .

By Theorem 1.1 we know that $[(1 - \Psi(z))^{-1}]_n \ll n^{\beta-1}$. Hence, $[z(1 - \Psi(z))^{-1}]_n \ll n^{\beta-1}$. The claim for the case $k = 1$ follows by a convolution argument applied to B'_n and $[z(1 - \Psi(z))^{-1}]_n$ (see, for instance, [13, Lemma 4.3]).

By Remark 5.4, $\| [z(1 - \Psi(z))^{-1/2}]_n \| \ll n^{\beta/2-1}$. The claim for the case $k = 1/2$ follows by a convolution argument applied to B'_n and $[z(1 - \Psi(z))^{-1/2}]_n$ (see, for instance, [13, Lemma 4.3]). □

7.2 Reducing the proofs of Propositions 6.5 and 6.6 to the analytic case

Recall that in the statements of Propositions 6.5 and 6.6 we only require that $B(z)$ is an operator-valued function (on some Banach space \mathcal{B} with norm $\| \cdot \|$) continuous on \mathcal{S}^1 with $B(1) = 0$. Moreover, we assume that the Fourier coefficients B_n of $B(z)$ satisfy $\| B_n \| \ll |n|^{-(\beta+1)}$. In this paragraph, we argue that without loss of generality, during the proofs of these results we can restrict to the case where B is a one sided Fourier series, that is $B(e^{i\theta}) = \sum_{n=0}^{\infty} B_n e^{in\theta}$.

If $B(e^{i\theta})$ also contains negative index coefficients, we write $B(e^{i\theta}) = \sum_{n=-\infty}^{-1} B_n e^{in\theta} + \sum_{n=0}^{\infty} B_n e^{in\theta} := B_-(e^{i\theta}) + B_+(e^{i\theta})$. Note that $B_+(1) + B_-(1) = 0$. Hence if we define $\tilde{B}_+(e^{i\theta}) = B_+(e^{i\theta}) - B_+(1)$ and $\tilde{B}_-(e^{i\theta}) = B_-(e^{i\theta}) - B_-(1)$ then we still have $B = \tilde{B}_+ + \tilde{B}_- = B_+ + B_-$ and moreover $\tilde{B}_+(1) = 0, \tilde{B}_-(1) = 0$.

Note that $\hat{B}_-(e^{i\theta}) := \tilde{B}_-(e^{-i\theta}) = \sum_{n=-\infty}^{-1} B_n e^{-in\theta} - B_+(1) = \sum_{n=1}^{\infty} B_{-n} e^{in\theta} - B_+(1)$. Since we assume that $\| B_{\pm n} \| = O(|n|^{-(\beta+1)})$ (hence, the coefficients of \hat{B}_- are summable) we can analytically extend \hat{B}_- to the unit disk \mathbb{D} . Moreover, \tilde{B}_+ is clearly analytic on the unit disk \mathbb{D} . Therefore, we can work with \tilde{B}_+ and \hat{B}_- separately and the proof for both cases goes similarly.

7.3 Proof of Proposition 6.5

Proof of Proposition 6.5 By Sect. 7.2, it suffices to deal with the case $B(e^{i\theta}) = \sum_{n=0}^{\infty} B_n e^{in\theta}$. That is, during the proof we can work as if B was also analytic on \mathbb{D} . By Lemma 7.3 with $k = 1$,

$$C_n = \frac{ei}{2\pi} \frac{1}{n} J_n + D_n,$$

where $\| D_n \| = O(n^{-1})$ and

$$J_n = \int_{-\pi}^{\pi} B(e^{-1/n} e^{i\theta}) \frac{d}{d\theta} ((1 - \Psi(e^{-1/n} e^{i\theta}))^{-1}) e^{-in\theta} d\theta.$$

It remains to show that $\|J_n\| = O(1)$. Write $J_n = \int_{-\pi}^0 + \int_0^\pi = J^- + J^+$. We estimate J^+ . The estimate for J^- follows by a similar argument.

Write $J^+ = \int_0^{1/n} + \int_{1/n}^\pi = J_1 + J_2$. By Proposition 7.1, $\|B(e^{-1/n}e^{i\theta})\| \ll |\frac{1}{n} - i\theta|^\beta$. By Lemma 5.1, $|\frac{d}{d\theta}((1 - \Psi(e^{-1/n}e^{i\theta}))^{-1})| \ll |\frac{1}{n} - i\theta|^{-(\beta+1)}$. Thus,

$$\|J_1\| \ll \int_0^{1/n} \left| \frac{1}{n} - i\theta \right|^{-1} d\theta \ll 1.$$

Next, put $M(e^{-1/n}e^{i\theta}) := \frac{d}{d\theta}((1 - \Psi(e^{-1/n}e^{i\theta}))^{-1})$. We already know that $|M(e^{-1/n}e^{i\theta})| \ll |\frac{1}{n} - i\theta|^{-(\beta+1)}$. By Proposition 7.2, $\|\frac{d}{d\theta}B(e^{-1/n}e^{i\theta})\| \ll n^{1-\beta}$. Compute that

$$\begin{aligned} J_2 &= \frac{i}{n} \int_{1/n}^\pi B(e^{-1/n}e^{i\theta})M(e^{-1/n}e^{i\theta})\frac{d}{d\theta}(e^{-in\theta})d\theta \\ &= -\frac{i}{n} \int_{1/n}^\pi M(e^{-1/n}e^{i\theta})\frac{d}{d\theta}B(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta \\ &\quad -\frac{i}{n} \int_{1/n}^\pi B(e^{-1/n}e^{i\theta})\frac{d}{d\theta}M(e^{-1/n}e^{i\theta})e^{-in\theta}d\theta + O(1) \\ &= -\frac{i}{n}J_2^1 - \frac{i}{n}J_2^2 + O(1). \end{aligned}$$

To justify the boundary term recall that $\|B(e^{-1/n}e^{i\theta})\| \ll |\frac{1}{n} - i\theta|^\beta$ and that $|M(e^{-1/n}e^{i\theta})| \ll |\frac{1}{n} - i\theta|^{-(\beta+1)}$. Hence, for $\theta = \frac{1}{n}$ and $\theta = \pi$ we have $\frac{1}{n}\|M(e^{-1/n}e^{i\theta})B(e^{-1/n}e^{i\theta})\| \ll 1$.

Next, using the estimates recalled above on $\|\frac{d}{d\theta}B(e^{-1/n}e^{i\theta})\|$ and $|M(e^{-1/n}e^{i\theta})|$,

$$\|J_2^1\| \ll n^{1-\beta} \int_{1/n}^\pi \left| \frac{1}{n} - i\theta \right|^{-(\beta+1)} d\theta \ll n.$$

By Remark 5.2,

$$\begin{aligned} \left| \frac{d}{d\theta} \left(M(e^{-1/n}e^{i\theta}) \right) \right| &= \left| \frac{d^2}{d\theta^2} ((1 - \Psi(e^{-1/n}e^{i\theta}))^{-1}) \right| \\ &\ll \left| \frac{1}{n} - i\theta \right|^{-(\beta+2)} + \left| \frac{1}{n} - i\theta \right|^{-2\beta} n^{1-\gamma_1} (\log n), \end{aligned}$$

for some $\gamma_1 \in (0, 1)$. Using the estimates above on $\|B(e^{-1/n}e^{i\theta})\|$ and $|\frac{d}{d\theta}(M(e^{-1/n}e^{i\theta}))|$,

$$\|J_2^2\| \ll \int_{1/n}^\pi \left| \frac{1}{n} - i\theta \right|^\beta \left(\left| \frac{1}{n} - i\theta \right|^{-(\beta+2)} + n^{1-\gamma_1} (\log n) \left| \frac{1}{n} - i\theta \right|^{-2\beta} \right) d\theta$$

$$\ll \int_{1/n}^{\pi} \theta^{-2} d\theta + n^{1-\gamma_1} (\log n) \ll n.$$

Putting these together, we obtain $\|J^+\| \ll \|J_1\| + \frac{1}{n}\|J_2^1\| + \frac{1}{n}\|J_2^2\| + 1 \ll 1$. Similarly, $\|J^-\| \ll 1$. Thus, $\|J_n\| \ll 1$, which ends the proof. \square

7.4 Proof of Proposition 6.6

Proof of Proposition 6.6 By Sect. 7.2, it suffices to deal with the case $B(e^{i\theta}) = \sum_{n=0}^{\infty} B_n e^{in\theta}$. That is, during the proof we can work as if B was also analytic on \mathbb{D} . By Lemma 7.3 with $k = 1/2$,

$$C_n = \frac{ei}{2\pi} \frac{1}{n} J_n + D_n,$$

where $\|D_n\| = O(n^{-(1+\tau)})$ for some $\tau > 0$, and

$$J_n = \int_{-\pi}^{\pi} B(e^{-1/n} e^{i\theta}) \frac{d}{d\theta} ((1 - \Psi(e^{-1/n} e^{i\theta}))^{-1/2}) e^{-in\theta} d\theta.$$

Thus, to complete the proof of Proposition 6.6 we need show that $\|J_n\| = O(n^{-\tau})$ for some $\tau > 0$. Write $J_n = \int_{-\pi}^0 + \int_0^{\pi} = J^- + J^+$. We estimate J^+ . The estimate for J^- follows by a similar argument. Write $J^+ = \int_0^{1/n} + \int_{1/n}^{\pi} = J_1 + J_2$.

By Proposition 7.1, $\|B(e^{-1/n} e^{i\theta})\| \ll |\frac{1}{n} - i\theta|^\beta$. Put $F(z) = \frac{d}{dz} ((1 - \Psi(z))^{-1/2})$. By Remark 5.3, $|F(e^{-1/n} e^{i\theta})| \ll |\frac{1}{n} - i\theta|^{-(\beta/2+1)}$. Hence, $\|B(e^{-1/n} e^{i\theta})F(e^{-1/n} e^{i\theta})\| \ll |\frac{1}{n} - i\theta|^{-(1-\beta/2)}$. Since $0 < 1 - \beta/2$,

$$\|J_1\| \ll \int_0^{1/n} \left| \frac{1}{n} - i\theta \right|^{-(1-\beta/2)} d\theta \ll n^{-\beta/2}.$$

It remains to estimate $\|J_2\|$. Compute that

$$\begin{aligned} J_2 &= \frac{i}{n} \int_{1/n}^{\pi} B(e^{-1/n} e^{i\theta}) F(e^{-1/n} e^{i\theta}) \frac{d}{d\theta} (e^{-in\theta}) d\theta \\ &= -\frac{i}{n} \int_{1/n}^{\pi} F(e^{-1/n} e^{i\theta}) \frac{d}{d\theta} B(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta \\ &\quad - \frac{i}{n} \int_{1/n}^{\pi} B(e^{-1/n} e^{i\theta}) \frac{d}{d\theta} F(e^{-1/n} e^{i\theta}) e^{-in\theta} d\theta + O(n^{-\beta/2}) \\ &= -\frac{i}{n} J_2^1 - \frac{i}{n} J_2^2 + O(n^{-\beta/2}). \end{aligned}$$

To justify the boundary term recall that $\|B(e^{-1/n}e^{i\theta})\| \ll |\frac{1}{n} - i\theta|^\beta$ and that $|F(e^{-1/n}e^{i\theta})| \ll |\frac{1}{n} - i\theta|^{-(\beta/2+1)}$. Hence, for $\theta = \frac{1}{n}$ and $\theta = \pi$ we have $\frac{1}{n}\|F(e^{-1/n}e^{i\theta})B(e^{-1/n}e^{i\theta})\| \ll n^{-\beta/2}$.

By Proposition 7.2, $\|\frac{d}{d\theta}B(e^{-1/n}e^{i\theta})\| \ll n^{1-\beta}$. This together with the estimate on $|F(e^{-1/n}e^{i\theta})|$ gives

$$\|J_2^1\| \ll n^{1-\beta} \int_{1/n}^\pi \left| \frac{1}{n} - i\theta \right|^{-(\beta/2+1)} d\theta \ll n^{1-\beta} \int_{1/n}^\pi \theta^{-(\beta/2+1)} d\theta \ll n^{1-\beta/2}.$$

By Remark 5.3,

$$\left| \frac{d}{d\theta}F(e^{-1/n}e^{i\theta}) \right| \ll \left| \frac{1}{n} - i\theta \right|^{-(\beta/2+2)} + n^{1-\gamma_1} \left| \frac{1}{n} - i\theta \right|^{-3\beta/2},$$

for some $\gamma_1 \in (0, 1)$. This together with the estimate on $\|B(e^{-1/n}e^{i\theta})\|$ gives

$$\begin{aligned} \|J_2^1\| &\ll \int_{1/n}^\pi |u - i\theta|^{\beta/2-2} d\theta + n^{1-\gamma_1} \int_{1/n}^\pi |u - i\theta|^{-\beta/2} d\theta \\ &\ll \int_{1/n}^\pi \theta^{\beta/2-2} d\theta + n^{1-\gamma_1} \int_{1/n}^\pi \theta^{-\beta/2} d\theta \ll n^{1-\beta/2} + n^{1-\gamma_1}. \end{aligned}$$

Putting these together, we obtain $\|J^+\| \ll \|J_1\| + \frac{1}{n}\|J_2^1\| + \frac{1}{n}\|J_2^2\| + n^{-\beta/2} \ll n^{-\tau}$ for $\tau = \min\{\beta/2, \gamma_1\}$. Similarly, $\|J^-\| \ll n^{-\tau}$. Thus, $\|J_n\| \ll n^{-\tau}$, which ends the proof. □

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Appendix 1: Proof of several results used in the proof of Proposition 4.2

Propositions 8.1 and 8.5 provide similar results for different regimes of $\rho \geq 1$.

Proposition 8.1 *Let $M : [0, \infty] \rightarrow \mathbb{R}$ be such that $M(x) = M \in \mathbb{R}$ for $x \in [0, 1)$ and $M(x) = O(x^{-\rho})$, for all $x \geq 1$ and some $\rho > 1$. For $u > 0, \theta \in (-\pi, \pi)$, define*

$$J(u, \theta) := \int_0^\infty e^{-(u-i\theta)x} M(x) dx. \tag{8.1}$$

Put $c_M = \int_0^\infty M(x) dx$. The following hold for all $u > 0, \theta \in (-\pi, \pi)$.

- (a) If $\rho \in (1, 2)$ then $(u - i\theta)J(u, \theta) = c_M(u - i\theta) + O(|u - i\theta|^\rho)$.
- (b) If $\rho \in (1, 2)$ and $xM(x)$ has bounded variation then $\frac{d}{d\theta}((u - i\theta)J(u, \theta) - c_M(u - i\theta)) = O(|u - i\theta|^{\rho-1})$.
- (c) If $\rho > 2$ and $x^2M(x)$ has bounded variation, then

$$\left| \frac{d^2}{d\theta^2}((u - i\theta)J(u, \theta)) \right| = O(1).$$

Remark 8.2 We notice that since $u > 0$, $\frac{d^k}{d\theta^k}$ Integrand($J(u, \theta)$) is bounded for any $k \geq 1$, hence we can move derivatives. This type of argument will be used in the proofs of several results below without further explanation.

Proof Items (a) and (b) are covered by [23, Proposition A.1]. Item (c) follows by the argument used in the proof of [23, Proposition A.1(b)]. □

The following technical result will be used in the proofs of Propositions 8.4 and 8.6 below.

Lemma 8.3 *Let $\gamma \in (0, 1)$. Then for all $u > 0$ and $\theta \in (-\pi, \pi)$.*

- (a) For $r = 0, 1$, set $I_r(u, \theta) = \int_1^\infty e^{-(u-i\theta)x} \{x\} (\log(\lfloor x \rfloor))^r x^{-\gamma} dx$. Then $|(u - i\theta)I_r(u, \theta)| = O(1)$.
- (b) Let $I(u, \theta) = \int_1^\infty e^{-(u-i\theta)x} \{x\} x^{1-\gamma} dx$. Then $|(u - i\theta)I(u, \theta)| \ll u^{\gamma-1}$.

Proof (a) Changing coordinates $x \rightarrow x - 1$,

$$I_r(u, \theta) = \int_0^\infty e^{-(u-i\theta)(x+1)} \{x\} (\log(\lfloor x + 1 \rfloor))^r (x + 1)^{-\gamma} dx.$$

But,

$$\begin{aligned} (1 - e^{-(u-i\theta)})I_r(u, \theta) &= \int_1^\infty e^{-(u-i\theta)(x+1)} \{x\} (\log(\lfloor x + 1 \rfloor))^r (x + 1)^{-\gamma} \\ &\quad - \log(\lfloor x \rfloor)^r x^{-\gamma} dx \\ &\quad + \int_0^1 e^{-(u-i\theta)(x+1)} \{x\} \log(\lfloor x + 1 \rfloor)^r (x + 1)^{-\gamma} dx. \end{aligned}$$

When $r = 1$,

$$\begin{aligned} |(1 - e^{-(u-i\theta)})I_1(u, \theta)| &\ll \int_1^\infty (\log x)x^{-(\gamma+1)} dx + \int_1^\infty (\log(x+1) - \log x)x^{-\gamma} dx \\ &\quad + \int_0^1 \log(x+1)(x+1)^{-\gamma} dx < \infty. \end{aligned}$$

When $r = 0$,

$$|(1 - e^{-(u-i\theta)})I_0(u, \theta)| \ll \int_1^\infty x^{-(\gamma+1)} dx + \int_0^1 1 dx < \infty.$$

Altogether, $|(u - i\theta)I_r(u, \theta)| \ll 1$, as required.

(b) Proceeding as in the proof of (a) above, we compute that

$$\begin{aligned} |(1 - e^{-(u-i\theta)})I(u, \theta)| &= \left| \int_1^\infty e^{-(u-i\theta)(x+1)} \{x\}((x+1)^{1-\gamma} - x^{1-\gamma}) dx \right. \\ &\quad \left. + \int_0^1 e^{-(u-i\theta)(x+1)} \{x\}x^{1-\gamma} dx \right| \\ &\ll \int_0^\infty e^{-ux} x^{-\gamma} dx \ll u^{\gamma-1}. \end{aligned}$$

Item (b) follows. □

The next result provides the second derivative of a function similar to the one considered in Proposition 8.1 for the range $\rho \in (1, 2)$. In this sense, our assumption are stronger than the ones in Proposition 8.1.

Proposition 8.4 *Let $M : [0, \infty] \rightarrow \mathbb{R}$ be such that $M(x) = M \in \mathbb{R}$ for $x \in [0, 1)$ and for all $x \geq 1$, $M(x) = \ell(\lfloor x \rfloor)x^{-\rho}$, where $\ell(x) = C_1 \log x + C_2$ for C_1, C_2 real constants and $\rho \in (1, 2)$.*

With the function M defined as above, let $J(u, \theta)$ be defined by Eq. (8.1). Then for all $u > 0$ and $\theta \in (-\pi, \pi)$,

$$\left| \frac{d^2}{d\theta^2}((u - i\theta)J(u, \theta)) \right| \ll |\log u|u^{\rho-2}.$$

Proof Compute that

$$\begin{aligned} \frac{d^2}{d\theta^2}((u - i\theta)J(u, \theta)) &= -2i \frac{d}{d\theta} J(u, \theta) - (u - i\theta) \int_0^\infty e^{-(u-i\theta)x} x^2 M(x) dx \\ &= -2i \frac{d}{d\theta} J(u, \theta) - (u - i\theta)J_1(u, \theta). \end{aligned}$$

By Proposition 8.1(b), $(u - i\theta) \frac{d}{d\theta} J(u, \theta) - iJ(u, \theta) + ic_M = O(|u - i\theta|^{\rho-1})$. Together with Proposition 8.1(a) this implies that $|\frac{d}{d\theta} J(u, \theta)| = O(|u - i\theta|^{\rho-2})$. It remains to estimate $|(u - i\theta)J_1(u, \theta)|$.

Write $(u - i\theta)J_1(u, \theta) = (u - i\theta) \int_0^1 + (u - i\theta) \int_1^\infty = O(|u - i\theta|) + (u - i\theta)J_2(u, \theta)$. Let $\{x\}$ denote the fractional part of x and compute that

$$x^2 \frac{\ell(\lfloor x \rfloor)}{\lfloor x \rfloor^\rho} = \ell(\lfloor x \rfloor)x^{2-\rho} + \rho x^2 \{x\} \frac{\ell(\lfloor x \rfloor)}{x^{\rho+1}} + O\left(\frac{\ell(\lfloor x \rfloor)}{x^\rho}\right).$$

Since $\ell(\lfloor x \rfloor) - \ell(x) = C_1(\log \lfloor x \rfloor - \log x) = -C_1 \{x\}x^{-1} + O(1/x^2)$,

$$J_2(u, \theta) = \int_1^\infty e^{-(u-i\theta)x} \ell(x)x^{2-\rho} dx - C_1 \int_1^\infty e^{-(u-i\theta)x} \{x\}x^{-(\rho-1)} dx$$

$$+ \rho \int_1^\infty e^{-(u-i\theta)x} \ell(\lfloor x \rfloor) \{x\} x^{-(\rho-1)} dx + \int_1^\infty e^{-(u-i\theta)x} g(x) dx,$$

where $|g(x)| = O(x^{-(\rho-\delta)})$, for any $\delta > 0$. Since $\rho > 1$, $|(u - i\theta) \int_1^\infty e^{-(u-i\theta)x} g(x) dx| = O(|u - i\theta|)$.

Next, by Lemma 8.3(a) (with $r = 0$ and $\tau = \rho - 1$)⁴

$$|(u - i\theta) \int_1^\infty e^{-(u-i\theta)x} \{x\} x^{-(\rho-1)} dx| = O(1).$$

Also, using the definition of ℓ and Lemma 8.3(a) (with $r = 1$ and $\tau = \rho - 1$),

$$\left| (u - i\theta) \int_1^\infty e^{-(u-i\theta)x} \ell(\lfloor x \rfloor) \{x\} x^{-(\rho-1)} dx \right| = O(1).$$

To conclude, we need to estimate $(u - i\theta) J_3(u, \theta)$, where $J_3(u, \theta) := \int_1^\infty e^{-(u-i\theta)x} \ell(x) x^{2-\rho} dx$.

First we consider the case $|\theta| \leq u$. Substituting $ux = \sigma$,

$$\begin{aligned} |(u - i\theta) J_3(u, \theta)| &\ll u \int_1^\infty e^{-ux} \ell(x) x^{2-\rho} dx \\ &= u^{\rho-2} \int_u^\infty e^{-\sigma} \sigma^{2-\rho} (C_1(\log \sigma - \log u) + C_2) d\sigma \\ &\ll |\log u| u^{\rho-2} \int_u^\infty e^{-\sigma} \sigma^{2-\rho} d\sigma + u^{\rho-2} \int_u^\infty e^{-\sigma} \sigma^{2-\rho} \log \sigma d\sigma \\ &\quad + u^{\rho-2} \int_u^\infty e^{-\sigma} \sigma^{2-\rho} d\sigma \ll |\log u| u^{\rho-2}. \end{aligned}$$

In the last inequality we have used that all integrands are independent of u and well behaved at 0 and at infinity.

It remains to consider the case $u \leq |\theta|$. Substituting $ux = \sigma$ and recalling that $\ell(x) = C_1 \log x + C_2$ (so, differentiable on $[1, \infty)$),

$$\begin{aligned} J_3(u, \theta) &= \frac{u^\rho}{u^3} \int_u^\infty e^{-\sigma} e^{i\theta\sigma/u} \sigma^{2-\rho} \ell(\sigma/u) d\sigma \\ &= \frac{1}{i\theta} u^{\rho-2} \int_u^\infty \frac{d}{d\sigma} (e^{i\theta\sigma/u}) e^{-\sigma} \sigma^{2-\rho} \ell(\sigma/u) d\sigma \\ &= \frac{1}{i\theta} u^{\rho-2} \int_u^\infty e^{i\theta\sigma/u} e^{-\sigma} \left(\sigma^{2-\rho} \ell(\sigma/u) - C_1 \sigma^{2-\rho} \frac{1}{\sigma} - (2-\rho) \frac{\ell(\sigma/u)}{\sigma^{\rho-1}} \right) d\sigma \\ &\quad + O(|\theta|^{-1}). \end{aligned}$$

⁴ By the argument used in the proof of [23, Proposition A.4] (in estimating I_1 there), we obtain the better estimate $|(u - i\theta) \int_1^\infty e^{-(u-i\theta)x} \{x\} x^{-(\rho-1)} dx| = O(|u - i\theta|^{\rho-1})$. This improved estimate is not needed during this proof.

Writing $\ell(\sigma/u) = \log \sigma - \log u + C$ and proceeding as in the case $|\theta| \leq u$, $\int_u^\infty e^{-\sigma} (\sigma^{2-\rho} \ell(\sigma/u) - C_1 \sigma^{2-\rho} \frac{1}{\sigma} - (2 - \rho) \frac{\log(\sigma/u)}{\sigma^{\rho-1}}) d\sigma \ll |\log u|$. Therefore, $|(u - i\theta)J_3| \ll |\theta J_3| \ll |\log u| u^{\rho-2}$ and the conclusion follows. \square

Proposition 8.5 *Let $M : [0, \infty] \rightarrow \mathbb{R}$ be such that $M(x) = M \in \mathbb{R}$ for $x \in [0, 1)$ and for all $x \geq 1$, $M(x) = x^{-1}$. Let $I(u, \theta) = \int_0^\infty e^{-(u-i\theta)x} M(x) dx$. Then, for all $u \in (0, 1)$, $\theta \in (-\pi, \pi)$,*

- (a) $|(u - i\theta)I(u, \theta)| = O(|u - i\theta| \log(1/|u - i\theta|))$.
- (b) $\frac{d}{d\theta}((u - i\theta)I(u, \theta)) = O(\log(1/|u - i\theta|))$.
- (c) $|\frac{d^2}{d\theta^2}((u - i\theta)I(u, \theta))| = O(|u - i\theta|^{-1} \log(1/|u - i\theta|))$.

Proof Item (a) is contained in the proof of lemma [20, Lemma A.4].

(b) Compute that

$$\frac{d}{d\theta}((u - i\theta)I(u, \theta)) = -iI(u, \theta) + i(u - i\theta) \int_0^\infty e^{-(u-i\theta)x} dx.$$

By item (a), $|I(u, \theta)| = O(\log(1/|u - i\theta|))$. The result follows since $\int_0^\infty e^{-(u-i\theta)x} dx = (u - i\theta)^{-1}$.

(c) Compute that

$$\frac{d^2}{d\theta^2}((u - i\theta)I(u, \theta)) = (u - i\theta) \int_0^\infty e^{-(u-i\theta)x} x dx - 2i \frac{d}{d\theta} I(u, \theta).$$

Integrating by parts gives

$$(u - i\theta) \int_0^\infty e^{-(u-i\theta)x} x dx = \int_0^\infty e^{-(u-i\theta)x} dx = (u - i\theta)^{-1}.$$

From (a) and (b) above, we know that $|\frac{d}{d\theta}(I(u, \theta))| = O(|u - i\theta|^{-1} \log(1/|u - i\theta|))$. Putting the above together, $|\frac{d^2}{d\theta^2}((u - i\theta)I(u, \theta))| \ll |u - i\theta|^{-1}$, ending the proof. \square

Proposition 8.6 *For $\rho \in (0, 1)$, set $\Delta(x) = \lfloor x \rfloor^{-\rho} - x^{-\rho}$, $x \geq 0$ (with the convention $0^{-\rho} = 0$). For $u \in (0, 1)$, $\theta \in (-\pi, \pi)$ define*

$$W(u, \theta) := \int_0^\infty e^{-(u-i\theta)x} \Delta(x) dx.$$

Put $c_\Delta = \int_0^\infty \Delta(x) dx$. The following hold for all $u > 0$, $\theta \in (-\pi, \pi)$.

- (a) $(u - i\theta)W(u, \theta) = c_\Delta(u - i\theta) + O(|u - i\theta|^{\rho+1})$,
- (b) $|\frac{d}{d\theta}((u - i\theta)W(u, \theta) - c_\Delta(u - i\theta))| = O(|u - i\theta|^\rho)$ and
- (c) $|\frac{d^2}{d\theta^2}((u - i\theta)W(u, \theta))| \ll u^{\rho-1}$.

Proof First we note that with the convention $0^{-\rho} = 0$, $\Delta(x) = x^{-\rho}$ for $x \in [0, 1]$. Items (a), (b) are covered by [23, Proposition A.4]. In what follows, we prove (c). Compute that

$$\begin{aligned} \frac{d^2}{d\theta^2}((u - i\theta)W(u, \theta)) &= -2i \frac{d}{d\theta} W(u, \theta) - (u - i\theta) \int_1^\infty e^{-(u-i\theta)x} x^2 \Delta(x) dx \\ &\quad - (u - i\theta) \int_0^1 e^{-(u-i\theta)x} x^2 \Delta(x) dx. \end{aligned}$$

By items (a) and (b), the first term is $O(|u - i\theta|^{\rho-1})$. Clearly, the third term is $O(|u - i\theta|)$. Next, we estimate $W_2(u, \theta) := \int_1^\infty e^{-(u-i\theta)x} x^2 \Delta(x) dx$. Put $\{x\} = x - \lfloor x \rfloor$ and note that

$$\lfloor x \rfloor^{-\rho} - x^{-\rho} = x^{-\rho} \left(1 - \frac{\{x\}}{x}\right)^{-\rho} - x^{-\rho} = \rho \frac{\{x\}}{x^{\rho+1}} + g(x),$$

where $g(x) = O(x^{-(\rho+2)})$. Hence

$$\begin{aligned} (u - i\theta)W_2(u, \theta) &= \rho(u - i\theta) \int_1^\infty e^{-(u-i\theta)x} \{x\}x^{1-\rho} dx \\ &\quad + (u - i\theta) \int_1^\infty e^{-(u-i\theta)x} x^2 g(x) dx. \end{aligned}$$

For the second term we note that

$$|(u - i\theta) \int_1^\infty e^{-(u-i\theta)x} x^2 g(x) dx| \ll |(u - i\theta)| \int_0^\infty e^{-ux} x^{-\rho} dx \ll u^{\rho-1} |u - i\theta|.$$

It remains to estimate $(u - i\theta)I(u, \theta) = (u - i\theta) \int_1^\infty e^{-(u-i\theta)x} \{x\}x^{1-\rho} dx$. By Lemma 8.3(b) (with $\gamma = \rho$), $|(u - i\theta)I(u, \theta)| \ll u^{\rho-1}$, ending the proof. \square

Appendix 2: Tail sequence for (1.2)

The following proposition is an improved version of [20, Proposition C1] and [23, Proposition B.1]. Recall that h denotes the density of the measure μ .

Proposition 8.7 *Suppose that $f : [0, 1] \rightarrow [0, 1]$ is given as in (1.2) with $\beta = 1/\alpha \in (0, 1)$. Set $k = \min\{j \geq 2 : \beta > \frac{1}{j}\}$ and set $N = \min\{\ell \geq 2 : \beta > \frac{2}{k+\ell}\}$ and $N' = \min\{\ell \geq 2 : \beta > \frac{3}{k+\ell}\}$.*

Let Z be a compact subset of $(0, 1]$. Then there exists $Y \subset (0, 1]$ compact with $Z \subset Y$, such that the first return function $\varphi : Y \rightarrow \mathbb{Z}_+$ satisfies

$$\mu(\varphi > n) = \sum_{j=1}^{k+N'} c_j n^{-j\beta} + \sum_{j=1}^{k+N} (\tilde{c}_j^1 \log n + \tilde{c}_j^2) n^{-(j\beta+1)}$$

$$+ \sum_{j=1}^k \left(\hat{c}_j^1 \frac{(\log n)^2}{n^{j\beta+2}} + \hat{c}_j^2 \frac{\log n}{n^{j\beta+2}} + \hat{c}_j^3 \frac{1}{n^{j\beta+2}} \right) + O((\log n)^2/n^{\beta+3}),$$

where $c_j, \tilde{c}_j^1, \tilde{c}_j^2, \hat{c}_j^1, \hat{c}_j^2, \hat{c}_j^3$ are real constants that depend only on f .

Proof First take $Y = [\frac{1}{2}, 1]$. Let $x_n \in (0, \frac{1}{2}]$ be the sequence with $x_1 = \frac{1}{2}$ and $x_n = f(x_{n+1})$ so $x_n \rightarrow 0$. It is well known (see for instance [18]) that $x_n \sim \frac{1}{2}\beta^\beta n^{-\beta}$. In fact, as shown in [23, Proposition B.1],⁵ $x_n = \frac{1}{2}\beta^\beta n^{-\beta} + C_1(\log n)n^{-(\beta+1)} + C_2n^{-(\beta+1)} + O((\log n)^2n^{-(\beta+2)})$, where C_1, C_2 are real constants that depends only on β . To prove item i), we need an improved higher order expansion of x_n . We claim that

$$x_n = \frac{1}{2}\beta^\beta \frac{1}{n^\beta} + C_1 \frac{\log n}{n^{\beta+1}} + C_2 \frac{1}{n^{\beta+1}} + C_3 \frac{(\log n)^2}{n^{\beta+2}} + C_4 \frac{\log n}{n^{\beta+2}} + C_5 \frac{1}{n^{\beta+2}} + O\left(\frac{(\log n)^2}{n^{\beta+3}}\right),$$

where C_1, C_2, C_3, C_4, C_5 are real constants that depend only on $\beta = 1/\alpha$ and whose values will change from line to line.

Recall $x_n = \frac{1}{2}\beta^\beta n^{-\beta} (1 + C_1(\log n)n^{-1} + C_2n^{-1} + O((\log n)^2n^{-2}))$. Put $g(x) = 2^\alpha x^\alpha$. So, $g(x_n) = 2^\alpha x_n^\alpha = \beta n^{-1} + C_1(\log n)n^{-2} + C_2n^{-2} + O((\log n)^2n^{-3})$. Next, put $d(x) = 1/g(x)$. Since $x_n = f(x_{n+1}) = x_{n+1}(1 + g(x_{n+1}))$, we compute that

$$\begin{aligned} d(x_n) &= \frac{1}{2^\alpha x_n^\alpha} = d(x_{n+1})(1 + g(x_{n+1}))^{-\alpha} \\ &= d(x_{n+1})(1 - \alpha g(x_{n+1}) + C_1g(x_{n+1})^2 + C_2g(x_{n+1})^3 + O(g(x_{n+1})^4)) \\ &= d(x_{n+1}) - \alpha + C_1g(x_{n+1}) + C_2g(x_{n+1})^2 + O(n^{-3}). \end{aligned}$$

It follows that for $n \geq 1$

$$\begin{aligned} d(x_{n+1}) - d(x_n) &= \alpha + C_1g(x_{n+1}) + C_2g(x_{n+1})^2 + O(n^{-3}) \\ &= \alpha + C_1 \frac{1}{n} + C_2 \frac{\log n}{n^2} + C_3 \frac{1}{n^2} + O\left(\frac{(\log n)^2}{n^3}\right). \end{aligned}$$

Recalling that $d(x_1) = 1$ and summing from 1 to $n - 1$,

$$d(x_n) = 1 + (n - 1)\alpha + \sum_{j=1}^{n-1} \left(C_1 \frac{1}{j} + C_2 \frac{\log j}{j^2} + C_3 \frac{1}{j^2} + b(j) \right),$$

⁵ The claim in the third line of the proof of [23, Proposition B.1] is missing a logarithmic factor, but the rest of the proof uses the correct formula, and the statement of the result is correct.

where $b(j) = O(\frac{(\log j)^2}{j^3})$. For $r = 0, 1$, one can easily check that $\sum_{j=n}^\infty (\log j)^r / j^2 = \int_n^\infty (\log x)^r / x^2 + O((\log n)^r / n^2)$. Hence, $\sum_{j=n}^\infty \frac{(\log j)^r}{j^2} = \frac{(\log n)^r}{n} + \frac{1}{n} + O(\frac{(\log n)^r}{n^2})$. Similarly, $\sum_{j=n}^\infty \frac{1}{j^2} = \frac{1}{n} + O(\frac{1}{n^2})$ and $\sum_{j=n}^\infty b(j) = O((\log n)^2 / n^2)$.

As shown in [7], $\sum_{j=1}^{n-1} \frac{1}{j} = \gamma + \log n + \frac{1}{2n} + O(n^{-2})$, where γ is the Euler constant. Putting these together,

$$\begin{aligned} d(x_n) &= n\alpha + C_1 \log n + C_2 + C_3 \frac{\log n}{n} + C_4 \frac{1}{n} + O\left(\frac{(\log n)^2}{n^2}\right) \\ &= n\alpha \left(1 + C_1 \frac{\log n}{n} + C_2 \frac{1}{n} + C_3 \frac{\log n}{n^2} + C_4 \frac{1}{n^2} + O\left(\frac{(\log n)^2}{n^3}\right)\right). \end{aligned}$$

Since $\beta = 1/\alpha$ and $x_n = \frac{1}{2}d(x_n)^{-\beta}$, we have

$$x_n = \frac{1}{2}\beta^\beta n^{-\beta} \left(1 + C_1 \frac{\log n}{n} + C_2 \frac{1}{n} + C_3 \frac{\log n}{n^2} + C_4 \frac{1}{n^2} + O\left(\frac{(\log n)^2}{n^3}\right)\right)^{-\beta}$$

and the claim follows.

It is known that the density $h \in C^\infty$ (this follows, for instance, from the argument of [25, Lemma 2]). Hence for $x \in [\frac{1}{2}, 1]$ and k and $N' \geq N$ defined as in the statement of the proposition, we can write

$$h(x) = h(\frac{1}{2}) + h'(\frac{1}{2})(x - \frac{1}{2}) + \dots + \frac{h^{(k+N'-1)}(\frac{1}{2})}{(k+N')!} (x - \frac{1}{2})^{k+N'-1} + O\left((x - \frac{1}{2})^{k+N'}\right).$$

Set $y_n = \frac{1}{2}(x_n + 1)$ (so $f(y_n) = x_n$). Then $\varphi = n$ on $(y_n, y_{n-1}]$, hence $\{\varphi > n\} = [\frac{1}{2}, y_n]$. It follows that

$$\begin{aligned} \mu(\varphi > n) &= \int_{1/2}^{y_n} h(x) dx = \sum_{j=0}^{k+N'-1} \frac{1}{(j+1)!} \frac{h^{(j)}(\frac{1}{2})}{2^{j+1}} x_n^{j+1} + O(x_n^{k+N'+1}) \\ &= \sum_{j=1}^{k+N'} \frac{c_j}{n^{j\beta}} + \sum_{j=1}^{k+N} \left(\tilde{c}_j^1 \frac{\log n}{n^{j\beta+1}} + \tilde{c}_j^2 \frac{1}{n^{j\beta+1}}\right) \\ &\quad + \sum_{j=1}^k \left(\hat{c}_j^1 \frac{(\log n)^2}{n^{j\beta+2}} + \hat{c}_j^2 \frac{\log n}{n^{j\beta+2}} + \hat{c}_j^3 \frac{1}{n^{j\beta+2}}\right) + O\left(\frac{(\log n)^2}{n^{\beta+3}}\right), \end{aligned}$$

where $c_j, \tilde{c}_j^1, \tilde{c}_j^2, \hat{c}_j^1, \hat{c}_j^2, \hat{c}_j^3$ are real constants that depend only on β and $h^{(j)}(1/2)$.⁶ This ends the proof for the choice $Y = [\frac{1}{2}, 1]$. The conclusion follows since the same estimates are obtained by inducing on $Y = [x_q, 1]$ for any fixed $q \geq 1$. \square

⁶ The precise form of the constants can be obtained, but since it is irrelevant to the proof of Theorem 1.4, we skip this straightforward, but tedious, calculation.

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