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Continuity and estimates of the Liouville heat kernel with applications to spectral dimensions

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Abstract The *Liouville Brownian motion (LBM)*, recently introduced by Garban, Rhodes and Vargas and in a weaker form also by Berestycki, is a diffusion process evolving in a planar random geometry induced by the *Liouville measure* M_γ, formally written as $M_\gamma(dz) = e^{\gamma X(z)-\gamma^2\mathbb{E}[X(z)^2]/2}\,dz$, $\gamma \in (0,2)$, for a (massive) Gaussian free field X. It is an M_γ-symmetric diffusion defined as the time change of the two-dimensional Brownian motion by the positive continuous additive functional with Revuz measure M_γ. In this paper we provide a detailed analysis of the heat kernel $p_t(x,y)$ of the LBM. Specifically, we prove its joint continuity, a locally uniform sub-Gaussian upper bound of the form $p_t(x,y) \le C_1 t^{-1}\log(t^{-1})\exp\bigl(-C_2((|x-y|^\beta \wedge 1)/t)^{\frac{1}{\beta-1}}\bigr)$ for $t \in \bigl(0,\frac{1}{2}\bigr]$ for each $\beta > \frac{1}{2}(\gamma+2)^2$, and an on-diagonal lower bound of the form $p_t(x,x) \ge C_3 t^{-1}\bigl(\log(t^{-1})\bigr)^{-\eta}$ for $t \in (0, t_\eta(x)]$, with $t_\eta(x) \in \bigl(0,\frac{1}{2}\bigr]$ *heavily dependent on* x, for each $\eta > 18$ for M_γ*-almost every* x. As applications, we deduce that the pointwise spectral dimension equals 2 M_γ-a.e. and that the global spectral dimension is also 2.

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1 Introduction

One of the main mathematical issues in the theory of two-dimensional Liouville quantum gravity is to construct a random geometry on a two-dimensional manifold (say $\mathbb{R}^2$ equipped with the Euclidian metric dx^2) which can be formally described by a Riemannian metric tensor of the form

$$e^{\gamma X(x)}\, dx^2, \tag{1.1}$$

where X is a massive Gaussian free field on $\mathbb{R}^2$ defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and $\gamma \in (0, 2)$ is a parameter. The study of Liouville quantum gravity is mainly motivated by the so-called KPZ-formula (for Knizhnik, Polyakov and Zamolodchikov), which relates some geometric quantities in a number of models in statistical physics to their formulation in a setup governed by this random geometry. In this context, by the KPZ relation the parameter γ can be expressed in terms of a certain physical constant called the central charge of the underlying model. We refer to [12] and to the survey article [15] for more details on this topic.

However, to give rigorous sense to the expression (1.1) is a highly non-trivial problem. Namely, as the correlation function of the Gaussian free field X exhibits short scale logarithmically divergent behaviour, the field X is not a function but only a random distribution. In other words, the underlying geometry is too rough to make sense in the classical Riemannian framework, so some regularisation is required. While it is not clear how to execute a regularisation procedure on the level of the metric, the method performs well enough to construct the associated volume form. More precisely, using the theory of Gaussian multiplicative chaos established by Kahane in [20] (see also [25]), by a certain cutoff procedure one can define the associated volume measure M_γ for $\gamma \in (0, 2)$, called the *Liouville measure*. It can be interpreted as being given by

$$M_\gamma(A) = \int_A e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]} dz,$$

but this expression for M_γ is only very formal, for M_γ is known to be singular with respect to the Lebesgue measure by a result [20, (141)] by Kahane (see also [25, Theorems 4.1 and 4.2]). Recently, in [17] Garban, Rhodes and Vargas have constructed the natural diffusion process $\mathcal{B} = (\mathcal{B}_t)_{t\geq 0}$ associated with (1.1), which they call the *Liouville Brownian motion (LBM)*. Similar results have been simultaneously obtained in a weaker form also by Berestycki [4]. On a formal level, $\mathcal{B}$ is the solution of the SDE

$$d\mathcal{B}_t = e^{-\frac{\gamma}{2}X(\mathcal{B}_t)+\frac{\gamma^2}{4}\mathbb{E}[X(\mathcal{B}_t)^2]}d\bar{B}_t,$$

where $\bar{B} = (\bar{B}_t)_{t\geq 0}$ is a standard Brownian motion on $\mathbb{R}^2$ independent of X. In view of the Dambis–Dubins–Schwarz theorem this SDE representation suggests defining the LBM $\mathcal{B}$ as a time change of another planar Brownian motion $B = (B_t)_{t\geq 0}$. This has been rigorously carried out in [17], and then by general theory the LBM turns out to be symmetric with respect to the Liouville measure M_γ. In the companion paper [18] Garban, Rhodes and Vargas also identified the Dirichlet form associated with $\mathcal{B}$ and they showed that the transition semigroup is absolutely continuous with respect to M_γ, meaning that the Liouville heat kernel $p_t(x, y)$ exists. Moreover, they observed that the intrinsic metric $d_{\mathcal{B}}$ generated by that Dirichlet form is identically zero, which indicates that

$$\lim_{t\downarrow 0} t \log p_t(x, y) = -\frac{d_{\mathcal{B}}(x, y)^2}{2} = 0, \qquad x, y \in \mathbb{R}^2,$$

and therefore some non-Gaussian heat kernel behaviour is expected. This degeneracy of the intrinsic metric is known to occur typically for diffusions on fractals, whose heat kernels indeed satisfy the so-called sub-Gaussian estimates; see e.g. the survey articles [3,23] and references therein.

In this paper we continue the analysis of the Liouville heat kernel, which has been initiated simultaneously and independently in [24]. As our first main results we obtain the continuity of the heat kernel and a rough upper bound on it.

Theorem 1.1 *Let $\gamma \in (0, 2)$. Then $\mathbb{P}$-a.s. the following hold: A (unique) jointly continuous version $p = p_t(x, y) : (0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ of the Liouville heat kernel exists and is $(0, \infty)$-valued, and in particular the Liouville Brownian motion $\mathcal{B}$ is irreducible. Moreover, the associated transition semigroup $(P_t)_{t>0}$ defined by*

$$P_t f(x) := E_x[f(\mathcal{B}_t)] = \int_{\mathbb{R}^2} p_t(x, y) f(y)\, M_\gamma(dy), \qquad x \in \mathbb{R}^2,$$

is strong Feller, i.e. $P_t f$ is continuous for any bounded Borel measurable $f : \mathbb{R}^2 \to \mathbb{R}$.

Theorem 1.2 *Let $\gamma \in (0, 2)$. Then $\mathbb{P}$-a.s., for any $\beta > \frac{1}{2}(\gamma + 2)^2$ and any bounded $U \subset \mathbb{R}^2$ there exist random constants $C_i = C_i(X, \gamma, U, \beta) > 0$, $i = 1, 2$, such that*

$$p_t(x, y) = p_t(y, x) \leq C_1 t^{-1} \log(t^{-1}) \exp\left(-C_2\left(\frac{|x - y|^\beta \wedge 1}{t}\right)^{\frac{1}{\beta-1}}\right) \tag{1.2}$$

for all $t \in \left(0, \frac{1}{2}\right]$, $x \in \mathbb{R}^2$ and $y \in U$, where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$.

Since $\beta > \frac{1}{2}(\gamma + 2)^2 > 2$, the off-diagonal part $\exp\left(-C_2((|x - y|^\beta \wedge 1)/t)^{\frac{1}{\beta-1}}\right)$ of the bound (1.2) indicates that the process diffuses slower than the two-dimensional Brownian motion, which is why such a bound is called *sub*-Gaussian. We do not expect that the lower bound $\frac{1}{2}(\gamma+2)^2$ for the exponent β is best possible. Unfortunately, Theorem 1.2 alone does not even exclude the possibility that β could be taken arbitrarily close to 2, which in the case of the two-dimensional torus has been in fact disproved in a recent result [24, Theorem 5.1] by Maillard, Rhodes, Vargas and Zeitouni showing that β satisfying (1.2) for small t must be at least $2+\gamma^2/4$. In this sense the Liouville heat kernel does behave anomalously, which is natural to expect from the degeneracy of the intrinsic metric associated with the LBM.

From the conformal invariance of the planar Brownian motion B it is natural to expect that the LBM $\mathcal{B}$ as a time change of B admits two-dimensional behaviour, as was observed by physicists in [1] and in a weak form proved in [26] (see Remark 1.5 below). The on-diagonal part $t^{-1}\log(t^{-1})$ in (1.2) shows a sharp upper bound in this spirit except for a logarithmic correction, and we will also prove the following on-diagonal lower bound valid for M_γ-*a.e.* $x \in \mathbb{R}^2$, which matches (1.2) besides another logarithmic correction.

Theorem 1.3 *Let* $\gamma \in (0,2)$. *Then* $\mathbb{P}$-*a.s., for* M_γ-*a.e.* $x \in \mathbb{R}^2$, *for any* $\eta > 18$ *there exist random constants* $C_3 = C_3(X, \gamma, |x|, \eta) > 0$ *and* $t_0(x) = t_0(X, \gamma, \eta, x) \in \left(0, \frac{1}{2}\right]$ *such that*

$$p_t(x,x) \geq C_3 t^{-1}\left(\log(t^{-1})\right)^{-\eta}, \qquad \forall t \in (0, t_0(x)]. \tag{1.3}$$

Combining the on-diagonal estimates in Theorems 1.2 and 1.3, we can immediately identify the pointwise spectral dimension as 2.

Corollary 1.4 *Let* $\gamma \in (0,2)$. *Then* $\mathbb{P}$-*a.s., for* M_γ-*a.e.* $x \in \mathbb{R}^2$,

$$\lim_{t \downarrow 0} \frac{2 \log p_t(x,x)}{-\log t} = 2.$$

Essentially from Theorems 1.2 and 1.3 we shall further deduce that the global spectral dimension, that is the growth order of the Dirichlet eigenvalues of the generator on bounded open sets, is also 2; see Sect. 6.2 for details.

Remark 1.5 In [26, Theorem 3.6] the following result on the spectral dimension has been proved: $\mathbb{P}$-a.s., for any $\alpha > 0$ and for all $x \in \mathbb{R}^2$,

$$\lim_{y \to x} \int_0^\infty e^{-\lambda t} t^\alpha p_t(x,y)\, dt < \infty, \qquad \forall \lambda > 0, \tag{1.4}$$

and

$$\lim_{y \to x} \int_0^\infty e^{-\lambda t} p_t(x,y)\, dt = \infty, \qquad \forall \lambda > 0. \tag{1.5}$$

In [26] the left hand sides were interpreted as the integrals in t of the on-diagonal heat kernel $p_t(x,x)$, which was needed due to the lack of the knowledge of the continuity of $p_t(x,y)$. By Theorem 1.1 this interpretation can be made rigorous now, and moreover, (1.4) follows immediately from Theorem 1.2. On the other hand, (1.5) is actually an easy consequence of the Dirichlet form theory. Indeed, by [14, Exercises 2.2.2 and 4.2.2] $\int_0^\infty e^{-\lambda t} p_t(x,x)\,dt$ is equal to the reciprocal of the λ-order capacity of the singleton $\{x\}$ with respect to the LBM, and this capacity is zero by [14, Lemma 6.2.4 (i)] and the fact that the same holds for the planar Brownian motion.

The proofs of our main results above are mainly based on the moment estimates for the Liouville measure M_γ by [20,27] and those for the exit times of the LBM $\mathcal{B}$ from balls by [17], together with the general fact from time change theory that the Green operator of the LBM has exactly the same integral kernel as that of the planar Brownian motion (see (2.6) below). To turn those moment estimates into $\mathbb{P}$-almost sure statements, we need some Borel–Cantelli arguments that cannot provide us with uniform control on various random constants over unbounded sets. For this reason we can expect the estimate (1.2) to hold only *locally* uniformly, so that in Theorem 1.2 we cannot drop the dependence of the constants C_1, C_2 on U or the cutoff of $|x-y|$ at 1 in the exponential. Also to remove the logarithmic corrections in (1.2) and (1.3) and the restriction to M_γ-a.e. points in Theorem 1.3 and Corollary 1.4 one would need to have good uniform control on the ratios of the M_γ-measures of concentric balls with different radii. However, we cannot hope for such control in view of [5, Remark A.2], where it is claimed that

$$\limsup_{r\downarrow 0} \sup_{x\in B(0,1)} \frac{M_\gamma(B(x,2r))}{M_\gamma(B(x,r))^{1-\eta}} = \infty$$

for any $\eta \in \left(-\infty, \frac{\gamma^2}{4+\gamma^2}\right)$ $\mathbb{P}$-a.s., with $B(x,R) := \{y \in \mathbb{R}^2 : |x-y| < R\}$ for $x \in \mathbb{R}^2$ and $R > 0$.

The LBM can also be constructed on other domains like the torus, the sphere or planar domains $D \subset \mathbb{R}^2$ equipped with a log-correlated Gaussian field like the (massive or massless) Gaussian free field (cf. [17, Section 2.9]). In fact, Theorem 1.1 has been simultaneously and independently obtained in [24] for the LBM on the torus, where thanks to the boundedness of the space one can utilise the eigenfunction expansion of the heat kernel to prove its continuity and the strong Feller property of the semigroup. On the other hand, in our case of $\mathbb{R}^2$ the Liouville heat kernel $p_t(x,y)$ does not admit such an eigenfunction expansion and the proof of its continuity and the strong Feller property requires some additional arguments. Therefore, although the proofs of our results should directly transfer to the other domains mentioned above, we have decided to work on the plane $\mathbb{R}^2$ in this paper for the sake of simplicity and in order to stress that our methods also apply to the case of unbounded domains.

In [24] Maillard, Rhodes, Vargas and Zeitouni have also obtained upper and lower estimates of the Liouville heat kernel on the torus. Their heat kernel upper bound in [24, Theorem 4.2] involves an on-diagonal part of the form $Ct^{-(1+\delta)}$ for any $\delta > 0$ and an off-diagonal part of the form $\exp\left(-C(|x-y|^\beta/t)^{\frac{1}{\beta-1}}\right)$ for any $\beta > \beta_0(\gamma)$, where $\beta_0(\gamma)$ is a constant larger than our lower bound $\frac{1}{2}(\gamma+2)^2$ on the exponent β and satisfies $\lim_{\gamma\uparrow 2} \beta_0(\gamma) = \infty$. Thus Theorem 1.2 gives a better estimate, and we prove it by self-contained, purely analytic arguments while the proof in [24] relies on

(1.4), whose proof in [26] is technically involved. Concerning lower bounds, an on-diagonal lower bound as in Theorem 1.3 is not treated in [24]. On the other hand, their off-diagonal lower bound [24, Theorem 5.1], which implies the bound $\beta \geq 2 + \gamma^2/4$ for any such exponent β as in (1.2) (in the case of the torus) as mentioned above after Theorem 1.2, is not covered by our results.

The rest of the paper is organised as follows. In Sect. 2 we recall the construction of the LBM in [17] and introduce the precise setup. In Sect. 3 we prove preliminary estimates on the volume decay of the Liouville measure and on the exit times from balls needed in the proofs. In Sect. 4 we show that the resolvent operators of the LBM killed upon exiting an open set have the strong Feller property, which is needed in Sect. 5 to prove Theorems 1.1 and 1.2. In Sect. 5.1 we show the continuity of the Dirichlet heat kernel associated with the killed LBM on a bounded open set by using its eigenfunction expansion, and in Sect. 5.2 we then deduce the continuity of the heat kernel and the strong Feller property on unbounded open sets, as well as Theorem 1.2, using a recent result in [19]. Finally, in Sect. 6 we show the on-diagonal lower bound in Theorem 1.3 and thereby identify the pointwise and global spectral dimensions as 2.

Throughout the paper, we write C for random positive constants depending on the realisation of the field X, which may change on each appearance, whereas the numbered random positive constants C_i will be kept the same. Analogously, non-random positive constants will be denoted by c or c_i, respectively. The symbols $\subset$ and $\supset$ for set inclusion *allow* the case of the equality. We denote by $|\cdot|$ the Euclidean norm on $\mathbb{R}^2$ and by $B(x, R) := \{y \in \mathbb{R}^2 : |x - y| < R\}$, $x \in \mathbb{R}^2$, $R > 0$, open Euclidean balls in $\mathbb{R}^2$ and for abbreviation we set $B(R) := B(0, R)$. Lastly, for non-empty $U \subset \mathbb{R}^2$ and $f : U \to \mathbb{R}$ we write $\|f\|_\infty := \|f\|_{\infty,U} := \sup_{x\in U} |f(x)|$.

2 Liouville Brownian motion

2.1 Massive Gaussian free field and Liouville measure

Consider a massive Gaussian free field X on the whole plane $\mathbb{R}^2$, i.e. a Gaussian Hilbert space associated with the Sobolev space $\mathcal{H}^1_m$ defined as the closure of $C_c^\infty(\mathbb{R}^2)$ with respect to the inner product

$$\langle f, g\rangle_m := m^2 \int_{\mathbb{R}^2} f(x)\, g(x)\, dx + \int_{\mathbb{R}^2} \nabla f(x) \cdot \nabla g(x)\, dx,$$

where $m > 0$ is a parameter called the mass. More precisely, $(\langle X, f\rangle_m)_{f\in\mathcal{H}^1_m}$ is a family of Gaussian random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with mean 0 and covariance

$$\mathbb{E}[\langle X, f\rangle_m \langle X, g\rangle_m] = 2\pi \langle f, g\rangle_m.$$

In other words, the covariance function of X is given by the massive Green function $g^{(m)}$ associated with the operator $m^2 - \Delta$, which can be written as

$$g^{(m)}(x,y) = \int_0^\infty \frac{1}{2u} e^{-\frac{m^2}{2}u - \frac{|x-y|^2}{2u}}\,du = \int_1^\infty \frac{k^{(m)}(u(x-y))}{u}\,du \tag{2.1}$$

with

$$k^{(m)}(z) := \frac{1}{2}\int_0^\infty e^{-\frac{m^2}{2v}|z|^2 - \frac{v}{2}}\,dv.$$

Following [17] we now introduce an n-regularised version of X. To that aim let $(a_n)_{n\geq 0} \subset \mathbb{R}$ be an unbounded strictly increasing sequence with $a_0 = 1$ and let $(Y_n)_{n\geq 1}$ be a family of independent *continuous* Gaussian fields on $\mathbb{R}^2$ defined also on $(\Omega, \mathcal{A}, \mathbb{P})$ with mean 0 and covariance

$$\mathbb{E}[Y_n(x)Y_n(y)] = \int_{a_{n-1}}^{a_n} \frac{k^{(m)}(u(x-y))}{u}\,du =: g_n^{(m)}(x,y); \tag{2.2}$$

here, such Y_n can be constructed by applying a version [22, Problem 2.2.9] of the Kolmogorov–Čentsov continuity theorem to a Gaussian field on $\mathbb{R}^2$ with mean 0 and covariance $g_n^{(m)}$, which in turn exists by the Kolmogorov extension theorem (see e.g. [11, Theorems 12.1.2 and 12.1.3]) since $\big(g_n^{(m)}(x,y)\big)_{x,y\in\Xi}$ is a non-negative definite real symmetric matrix for any finite $\Xi \subset \mathbb{R}^2$. Then for each $n \geq 1$, the n-regularised field X_n is defined as

$$X_n(x) := \sum_{k=1}^{n} Y_k(x), \qquad x \in \mathbb{R}^2,$$

and the associated random Radon measure $M_n = M_{\gamma,n}$ on $\mathbb{R}^2$ is given by

$$M_n(dx) := \exp\Big(\gamma X_n(x) - \tfrac{\gamma^2}{2}\mathbb{E}\big[X_n(x)^2\big]\Big)\,dx \tag{2.3}$$

with a parameter $\gamma \geq 0$. By the classical theory of Gaussian multiplicative chaos established in Kahane's seminal work [20] (see also [25]) we have the following: $\mathbb{P}$-a.s. the family $(M_n)_{n\geq 1}$ converges vaguely on $\mathbb{R}^2$ to a random Radon measure $M = M_\gamma$ called the *Liouville measure*, whose law is uniquely determined by γ and the covariance function $g^{(m)}$ of X, and M has full support $\mathbb{P}$-a.s. for $\gamma \in [0,2)$ and is identically zero $\mathbb{P}$-a.s. for $\gamma \geq 2$. Throughout the rest of this paper, we assume that $\gamma \in (0,2)$ is fixed and we will drop it from our notation, although the quantities defined through the Liouville measure $M = M_\gamma$ will certainly depend on γ.

2.2 Definition of Liouville Brownian motion

The Liouville Brownian motion has been constructed by Garban, Rhodes and Vargas in [17] as the canonical diffusion process under the geometry induced by the measure M. More precisely, they have constructed a positive continuous additive functional $F = (F_t)_{t\geq 0}$ of the planar Brownian motion B naturally associated with the measure

M and they have defined the LBM as $\mathcal{B}_t = B_{F_t^{-1}}$. In this subsection we briefly recall the construction.

Let $\Omega' := C([0,\infty), \mathbb{R}^2)$, let $B = (B_t)_{t\geq 0}$ be the coordinate process on Ω' and set $\mathcal{G}^0_\infty := \sigma(B_s;\ s < \infty)$ and $\mathcal{G}^0_t := \sigma(B_s;\ s \leq t)$, $t \geq 0$. Let $\{P_x\}_{x\in\mathbb{R}^2}$ be the family of probability measures on $(\Omega', \mathcal{G}^0_\infty)$ such that for each $x \in \mathbb{R}^2$, $B = (B_t)_{t\geq 0}$ under P_x is a two-dimensional Brownian motion starting at x. We denote by $\{\mathcal{G}_t\}_{t\in[0,\infty]}$ the minimum completed admissible filtration for B with respect to $\{P_x\}_{x\in\mathbb{R}^2}$ as defined e.g. in [14, Section A.2]. Moreover, let $\{\theta_t\}_{t\geq 0}$ be the family of shift mappings on Ω', i.e. $B_{t+s} = B_t \circ \theta_s$, $s, t \geq 0$. Finally, we write $q_t(x, y) := (2\pi t)^{-1} \exp(-|x-y|^2/(2t))$, $t > 0$, $x, y \in \mathbb{R}^2$, for the heat kernel associated with B.

Definition 2.1 (i) A $[-\infty,\infty]$-valued stochastic process $A = (A_t)_{t\geq 0}$ on $(\Omega', \mathcal{G}_\infty)$ is called a *positive continuous additive functional (PCAF)* of B *in the strict sense*, if A_t is $\mathcal{G}_t$-measurable for every $t \geq 0$ and if there exists a set $\Lambda \in \mathcal{G}_\infty$, called a *defining set* for A, such that

(a) for all $x \in \mathbb{R}^2$, $P_x[\Lambda] = 1$,
(b) for all $t \geq 0$, $\theta_t(\Lambda) \subset \Lambda$,
(c) for all $\omega \in \Lambda$, $[0,\infty) \ni t \mapsto A_t(\omega)$ is a $[0,\infty)$-valued continuous function with $A_0(\omega) = 0$ and

$$A_{t+s}(\omega) = A_t(\omega) + A_s \circ \theta_t(\omega), \qquad \forall s, t \geq 0.$$

(ii) Two such functionals A^1 and A^2 are called *equivalent* if $P_x\big[A^1_t = A^2_t\big] = 1$ for all $t > 0$, $x \in \mathbb{R}^2$, or equivalently, there exists $\Lambda \in \mathcal{G}_\infty$ which is a defining set for both A^1 and A^2 such that $A^1_t(\omega) = A^2_t(\omega)$ for all $t \geq 0$, $\omega \in \Lambda$. Equivalent PCAFs in the strict sense will always be identified hereafter.

(iii) For any such A, a Borel measure μ_A on $\mathbb{R}^2$ satisfying

$$\int_{\mathbb{R}^2} f(y)\, \mu_A(dy) = \lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^2} E_x\bigg[\int_0^t f(B_s)\, dA_s\bigg]\, dx$$

for any non-negative Borel function $f : \mathbb{R}^2 \to [0,\infty]$ is called the *Revuz measure* of A, which exists uniquely by general theory (see e.g. [6, Theorem A.3.5]).

For every $n \in \mathbb{N}$ let now $F^n_t : \Omega \times \Omega' \to [0,\infty)$ be defined as

$$F^n_t := \int_0^t \exp\Big(\gamma X_n(B_s) - \tfrac{\gamma^2}{2}\mathbb{E}\big[X_n(B_s)^2\big]\Big)\, ds, \qquad t \geq 0, \tag{2.4}$$

which is strictly increasing in t. Note that for every n the functional $F^n = (F^n_t)_{t\geq 0}$ considered as a process defined on $(\Omega', \mathcal{G}^0_\infty)$ is a PCAF of B in the strict sense with defining set Ω' and Revuz measure M_n.

Theorem 2.2 ([17, Theorem 2.7]) $\mathbb{P}$-*a.s. the following hold:*

(i) *There exists a unique PCAF F in the strict sense whose Revuz measure is M.*

(ii) *For all* $x \in \mathbb{R}^2$*,* P_x*-a.s.,* F *is strictly increasing and satisfies* $\lim_{t\to\infty} F_t = \infty$.
(iii) *For all* $x \in \mathbb{R}^2$*,* F^n *converges to* F *in* P_x*-probability in the space* $C([0,\infty),\mathbb{R})$ *equipped with the topology of uniform convergence on compact sets.*

The process $(\mathcal{B}, \{P_x\}_{x\in\mathbb{R}^2})$*,* $\mathbb{P}$*-a.s. defined by* $\mathcal{B}_t := B_{F_t^{-1}}$*,* $t \geq 0$*, is called the (massive)* Liouville Brownian motion (LBM).

Thanks to Theorem 2.2, we can apply the general theory of time changes of Markov processes to have the following properties of the LBM: First, it is a recurrent diffusion on $\mathbb{R}^2$ by [14, Theorems A.2.12 and 6.2.3]. Furthermore by [14, Theorem 6.2.1 (i)] (see also [17, Theorem 2.18]), the LBM is M-symmetric, i.e. its transition semigroup $(P_t)_{t>0}$ given by

$$P_t(x, A) := E_x[\mathcal{B}_t \in A]$$

for $t \in (0,\infty)$, $x \in \mathbb{R}^2$ and a Borel set $A \subset \mathbb{R}^2$, satisfies

$$\int_{\mathbb{R}^2} P_t f \cdot g \, dM = \int_{\mathbb{R}^2} f \cdot P_t g \, dM$$

for all Borel measurable functions $f, g : \mathbb{R}^2 \to [0,\infty]$. Here the Borel measurability of $P_t(\cdot, A)$ can be deduced from [17, Corollary 2.20] (or from Proposition 2.4 below).

Remark 2.3 [17, Corollary 2.20] states that $(P_t)_{t>0}$ is a Feller semigroup, meaning that P_t preserves the space of bounded continuous functions. Note that this is different from the notion of a Feller semigroup as for instance in [6,14], i.e. a strongly continuous Markovian semigroup on the space of continuous functions vanishing at infinity. It is not known whether $(P_t)_{t>0}$ is a Feller semigroup in the latter sense.

It is natural to expect that the LBM can be constructed in such a way that it depends measurably on the randomness of the field X. However, this measurability does not seem obvious from the construction in [17], since there the existence of the PCAF F has been deduced from some general theory on the Revuz correspondence for $\mathbb{P}$-a.e. fixed realisation of M. To overcome this issue, in the following proposition we show for $\mathbb{P}$-a.e. environment the pathwise convergence of F^n towards F in an appropriate $\{P_x\}_{x\in\mathbb{R}^2}$-a.s. sense which also ensures the measurability of F_t and $\mathcal{B}_t$ with respect to the product σ-field $\mathcal{A} \otimes \mathcal{G}^0_\infty$ for all $t \geq 0$. The proof is given in Appendix A.

Proposition 2.4 *There exists a set* $\Lambda \in \mathcal{A} \otimes \mathcal{G}^0_\infty$ *such that the following hold:*
(i) *For* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*,* $P_x[\Lambda^\omega] = 1$ *for any* $x \in \mathbb{R}^2$*, where* $\Lambda^\omega := \{\omega' \in \Omega' : (\omega, \omega') \in \Lambda\}$.
(ii) *For every* $(\omega, \omega') \in \Lambda$ *the following limits exist in* $\mathbb{R}$ *for all* $0 < s \leq t$*:*

$$F_{s,t}(\omega,\omega') := \lim_{n\to\infty} \big(F^n_t(\omega,\omega') - F^n_s(\omega,\omega')\big),$$
$$F_t(\omega,\omega') := \lim_{u\downarrow 0} F_{u,t}(\omega,\omega').$$

Moreover, with $F_0(\omega,\omega') := 0$*,* $[0,\infty) \ni t \mapsto F_t(\omega,\omega') \in [0,\infty)$ *is continuous, strictly increasing and satisfies* $\lim_{t\to\infty} F_t(\omega,\omega') = \infty$.

(iii) *Let* $t \geq 0$ *and set* $F_t := t$ *on* Λ^c*. Then* F_t *is* $\mathcal{A} \otimes \mathcal{G}^0_\infty$*-measurable.*
(iv) *For* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*, the process* $(F_t(\omega, \cdot))_{t \geq 0}$ *is a PCAF of* B *in the strict sense with defining set* Λ^ω*.*

The previous proposition implies easily that F indeed has the Revuz measure M. More strongly, we have the following proposition valid for any starting point $x \in \mathbb{R}^2$ $\mathbb{P}$-a.s., which we prove in Appendix B in a slightly more general setting for later use.

Proposition 2.5 *$\mathbb{P}$-a.s., for all* $x \in \mathbb{R}^2$ *and all Borel measurable functions* $\eta : [0, \infty) \to [0, \infty]$ *and* $f : \mathbb{R}^2 \to [0, \infty]$*,*

$$E_x\left[\int_0^\infty \eta(t) f(B_t)\, dF_t\right] = \int_0^\infty \int_{\mathbb{R}^2} \eta(t) f(y) q_t(x, y)\, M(dy)\, dt,$$

and in particular, for any $t > 0$*,*

$$\int_{\mathbb{R}^2} f(y)\, M(dy) = \frac{1}{t} \int_{\mathbb{R}^2} E_x\left[\int_0^t f(B_s)\, dF_s\right] dx.$$

2.3 The Liouville Dirichlet form

By virtue of Propositions 2.4 and 2.5, we can apply the general theory of Dirichlet forms to obtain an explicit description of the Dirichlet form associated with the LBM, as it has been done in [17,18].

Denote by $H^1(\mathbb{R}^2)$ the standard Sobolev space, that is

$$H^1(\mathbb{R}^2) = \left\{f \in L^2(\mathbb{R}^2, dx) : \nabla f \in L^2(\mathbb{R}^2, dx)\right\},$$

on which we define the form

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^2} \nabla f \cdot \nabla g\, dx. \tag{2.5}$$

Recall that $(\mathcal{E}, H^1(\mathbb{R}^2))$ is the Dirichlet form of the planar Brownian motion B. By $H^1_e(\mathbb{R}^2)$ we denote the extended Dirichlet space, that is the set of dx-equivalence classes of Borel measurable functions f on $\mathbb{R}^2$ such that $\lim_{n\to\infty} f_n = f \in \mathbb{R}$ dx-a.e. for some $(f_n)_{n\geq 1} \subset H^1(\mathbb{R}^2)$ satisfying $\lim_{k,l\to\infty} \mathcal{E}(f_k - f_l, f_k - f_l) = 0$. By [6, Theorem 2.2.13] we have the following identification of $H^1_e(\mathbb{R}^2)$:

$$H^1_e(\mathbb{R}^2) = \left\{f \in L^2_{\mathrm{loc}}(\mathbb{R}^2, dx) : \nabla f \in L^2(\mathbb{R}^2, dx)\right\}.$$

The *capacity* of a set $A \subset \mathbb{R}^2$ is defined by

$$\mathrm{Cap}(A) = \inf_{\substack{B\subset\mathbb{R}^2 \text{ open}\\ A\subset B}} \inf_{\substack{f\in H^1(\mathbb{R}^2)\\ f|_B\geq 1\, dx\text{-a.e.}}} \left\{\mathcal{E}(f, f) + \int_{\mathbb{R}^2} f^2\, dx\right\}.$$

A set $A \subset \mathbb{R}^2$ is called *polar* if $\mathrm{Cap}(A) = 0$. We call a function f *quasi-continuous* if for any $\varepsilon > 0$ there exists an open $U \subset \mathbb{R}^2$ with $\mathrm{Cap}(U) < \varepsilon$ such that $f|_{\mathbb{R}^2 \setminus U}$ is real-valued and continuous. By [14, Theorem 2.1.7] any $f \in H_e^1(\mathbb{R}^2)$ admits a quasi-continuous dx-version $\widetilde{f}$, which is unique up to polar sets by [14, Lemma 2.1.4].

Then, as the Liouville measure M is a Radon measure on $\mathbb{R}^2$ and does not charge polar sets by [17, Theorem 2.2] (or by Propositions 2.4, 2.5 and [6, Theorem 4.1.1 (i)]), the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of the LBM $\mathcal{B}$ is a strongly local regular symmetric Dirichlet form on $L^2(\mathbb{R}^2, M)$ which takes on the following explicit form by [14, Theorem 6.2.1]: The domain is given by

$$\mathcal{F} = \left\{u \in L^2(\mathbb{R}^2, M) : u = \widetilde{f}\ M\text{-a.e. for some } f \in H_e^1(\mathbb{R}^2)\right\},$$

which can be identified with $\left\{f \in H_e^1(\mathbb{R}^2) : \widetilde{f} \in L^2(\mathbb{R}^2, M)\right\}$ by [14, Lemma 6.2.1], and for $f, g \in \mathcal{F}$ the form $\mathcal{E}(f, g)$ is given by (2.5).

2.4 The killed Liouville Brownian motion

Let U be a non-empty open subset of $\mathbb{R}^2$ and let $U \cup \{\partial_U\}$ be its one-point compactification. We denote by $T_U := \inf\{s \geq 0 : B_s \notin U\}$ the exit time of the Brownian motion B from U and by $\tau_U := \inf\{s \geq 0 : \mathcal{B}_s \notin U\}$ that of the LBM $\mathcal{B}$, where $\inf \emptyset := \infty$. Since by definition $\mathcal{B}_t = B_{F_t^{-1}}$, $t \geq 0$, and F is a homeomorphism on $[0, \infty)$, we have $\tau_U = F_{T_U}$. Let now $B^U = (B_t^U)_{t \geq 0}$ and $\mathcal{B}^U = (\mathcal{B}_t^U)_{t \geq 0}$ denote the Brownian motion and the LBM, respectively, killed upon exiting U. That is, they are diffusions on U defined by

$$B_t^U := \begin{cases} B_t & \text{if } t < T_U, \\ \partial_U & \text{if } t \geq T_U, \end{cases} \qquad \mathcal{B}_t^U := \begin{cases} \mathcal{B}_t & \text{if } t < \tau_U, \\ \partial_U & \text{if } t \geq \tau_U. \end{cases}$$

Then for $t, \lambda \in (0, \infty)$, the semigroup operator P_t^U and the resolvent operator R_λ^U associated with the killed LBM $\mathcal{B}^U$ are expressed as, for each Borel function $f : U \to [-\infty, \infty]$ and with the convention $f(\partial_U) := 0$,

$$P_t^U f(x) := E_x\left[f(\mathcal{B}_t^U)\right] \quad \text{and} \quad R_\lambda^U f(x) := E_x\left[\int_0^{\tau_U} e^{-\lambda t} f(\mathcal{B}_t)\, dt\right], \qquad x \in \mathbb{R}^2,$$

provided the integrals exist. If U is bounded, as a time change of B^U the killed LBM $\mathcal{B}^U$ has the same integral kernel for its Green operator G_U as B^U, namely for any non-negative Borel function $f : U \to [0, \infty]$ and $x \in \mathbb{R}^2$,

$$G_U f(x) := E_x\left[\int_0^{\tau_U} f(\mathcal{B}_t)\, dt\right] = E_x\left[\int_0^{T_U} f(B_t)\, dF_t\right] = \int_U g_U(x, y) f(y)\, M(dy) \tag{2.6}$$

(cf. Proposition B.1). Here g_U denotes the Euclidean Green kernel given by

$$g_U(x, y) = \int_0^\infty q_t^U(x, y)\, dt, \qquad x, y \in \mathbb{R}^2, \tag{2.7}$$

for the heat kernel $q_t^U(x, y)$ of B^U: $q^U = q_t^U(x, y) : (0, \infty) \times U \times U \to [0, \infty)$ is the jointly continuous function such that $P_x[B_t^U \in dy] = q_t^U(x, y)\, dy$ for $t > 0$ and $x \in U$, and we set $q_t^U(x, y) := 0$ for $t > 0$ and $(x, y) \in (U \times U)^c$. Finally, we recall (see e.g. [14, Example 1.5.1]) that the Green function $g_{B(x_0,R)}$ over a ball $B(x_0, R)$ is of the form

$$g_{B(x_0,R)}(x, y) = \frac{1}{\pi} \log \frac{1}{|x - y|} + \Psi_{x_0,R}(x, y), \qquad x, y \in B(x_0, R), \tag{2.8}$$

for some continuous function $\Psi_{x_0,R} : B(x_0, R) \times B(x_0, R) \to \mathbb{R}$.

3 Preliminary estimates

3.1 Volume decay estimates

For our analysis of the Liouville heat kernel some good control on the volume of small balls under the Liouville measure is needed. An upper estimate has already been established in [17], and we provide a similar lower bound in the next lemma. The argument is based on some bounds on the negative moments of the measure of small balls. Such bounds have been proved in [27] in the case where the limiting random measure is obtained through approximation of the covariance kernel of the Gaussian free field by convolution. Since it is not clear to the authors whether the cutoff procedure producing the approximating measures M_n is covered by the results in [27], we give a comparison argument in Lemma C.1.

In the rest of this section, we write $\tilde{\xi}(q) := \big(2 + \frac{\gamma^2}{2}\big)q + \frac{\gamma^2}{2}q^2$ for $q > 0$.

Lemma 3.1 *Let $\alpha_1 := \frac{1}{2}(\gamma + 2)^2$ and $\alpha_2 := \frac{1}{2}(2 - \gamma)^2$. Then $\mathbb{P}$-a.s., for any $\varepsilon > 0$ and any $R \geq 1$ there exist $C_i = C_i(X, \gamma, R, \varepsilon) > 0$, $i = 4, 5$, such that*

$$C_4 r^{\alpha_1+\varepsilon} \leq M(B(x, r)) \leq C_5 r^{\alpha_2-\varepsilon}, \qquad \forall x \in B(R),\ r \in (0, 1]. \tag{3.1}$$

Proof By the monotonicity of (3.1) in ε and R it suffices to show (3.1) $\mathbb{P}$-a.s. for each ε and R. The upper bound is proved in [17, Theorem 2.2]. We show the lower bound in the same manner. Let $q := 2/\gamma$, so that $\alpha_1 = (2 + \tilde{\xi}(q))/q$. Let $\varepsilon > 0$ and $R \geq 1$ be fixed, and for $n \geq 1$ we set $r_n := 2^{-n} R$ and

$$\Xi_{R,n} := \Big\{\big(\tfrac{k}{2^n} R, \tfrac{l}{2^n} R\big) : k, l \in \mathbb{Z},\ |k|, |l| \leq 2^n\Big\} \subset [-R, R]^2. \tag{3.2}$$

Then for each $n \geq 1$, by Čebyšev's inequality and Lemma C.1,

$$\mathbb{P}\Big[\min_{x\in\Xi_{R,n}} M(B(x,r_n)) \leq 2^{-n(\alpha_1+\varepsilon)}\Big]$$
$$= \mathbb{P}\Big[\max_{x\in\Xi_{R,n}} M(B(x,r_n))^{-q} \geq 2^{n(\alpha_1+\varepsilon)q}\Big] \leq \sum_{x\in\Xi_{R,n}} \mathbb{P}\big[M(B(x,r_n))^{-q} \geq 2^{n(\alpha_1+\varepsilon)q}\big]$$
$$\leq 2^{-n(\alpha_1+\varepsilon)q} \sum_{x\in\Xi_{R,n}} \mathbb{E}\big[M(B(x,r_n))^{-q}\big] \leq 2^{-n(\alpha_1+\varepsilon)q} 2^{2n+3} c 2^{n\tilde{\xi}(q)} = c2^{-\varepsilon qn}$$

for some $c = c(\gamma, R) > 0$. Thus $\sum_{n=1}^{\infty} \mathbb{P}\big[\min_{x\in\Xi_{R,n}} M(B(x,r_n)) \leq 2^{-n(\alpha_1+\varepsilon)}\big] < \infty$, so that by the Borel–Cantelli lemma $\mathbb{P}$-a.s. for some $C = C(X, \gamma, R, \varepsilon) > 0$ we have that $M(B(x,r_n)) \geq C2^{-n(\alpha_1+\varepsilon)}$ for all $n \geq 1$ and all $x \in \Xi_{R,n}$. Since for every $y \in B(R)$ and $r \in (0, 1]$ we have $B(y,r) \supset B(x,r_n)$ for some $x \in \Xi_{R,n}$ with n satisfying $\frac{1}{4}r \leq r_n < \frac{1}{2}r$, the claim follows. □

3.2 Exit time estimates

In this subsection we provide some lower estimates on the exit times from balls which are needed in the proof of Theorems 1.1 and 1.2. More precisely, we establish estimates on the tail behaviour at zero of these exit times by showing certain $\mathbb{P}$-a.s. local uniform bounds on their negative moments.

Let $\{\vartheta_t\}_{t\geq 0}$ denote the family of shift mappings for the LBM $\mathcal{B}$, which is defined by $\vartheta_t(\omega') := \theta_{F_t^{-1}(\omega')}(\omega')$ for $t \geq 0$ and $\omega' \in \Omega'$ and satisfies $F_{s+t}^{-1} = F_s^{-1} + F_t^{-1} \circ \vartheta_s$ and hence $\mathcal{B}_{t+s} = \mathcal{B}_t \circ \vartheta_s$ for $s, t \geq 0$ on Λ^{ω} by virtue of $F_{t+s} = F_t + F_s \circ \theta_t$, $s, t \geq 0$ (cf. [6, Subsection A.3.2]).

Proposition 3.2 *Let $q > 0$. Then $\mathbb{P}$-a.s., for any $\kappa > 2 + \tilde{\xi}(q)$ and any $R \geq 1$ there exists a random constant $C_6 = C_6(X, \gamma, R, q, \kappa) > 0$ such that*

$$E_x\big[\tau_{B(x,r)}^{-q}\big] \leq C_6 r^{-\kappa}, \qquad \forall x \in B(R),\ r \in (0,1], \tag{3.3}$$

Proof Since (3.3) is weaker for larger κ and smaller R, it suffices to show (3.3) $\mathbb{P}$-a.s. for each κ and R. First we note that, letting $n \to \infty$ in [17, Proposition 2.12] by using [17, Lemma 2.8] (see also Theorem A.1 below) and Fatou's lemma, we get

$$\mathbb{E}E_x\big[\tau_{B(x,r)}^{-q}\big] \leq cr^{-\tilde{\xi}(q)}, \qquad \forall x \in \mathbb{R}^2,\ r \in (0,1], \tag{3.4}$$

for some $c = c(\gamma, q) > 0$. As in the proof of Lemma 3.1 above let $r_n := 2^{-n}R$ and $\Xi_{R,n}$ be defined as in (3.2) for any $n \geq 1$. In the sequel we write E_μ for the expectation operator associated with the law P_μ of a Brownian motion with initial distribution μ. Let $x \in \mathbb{R}^2$ and let $\mu_{x,r_n} := P_x[\mathcal{B}_{\tau_{B(x,r_n)}} \in \cdot]$ be the distribution of the LBM upon exiting $B(x,r_n)$. For any $z \in \partial B(x,r_n)$, since $B(z,r_n) \subset B(x,2r_n)$ and hence $\tau_{B(z,r_n)} \leq \tau_{B(x,2r_n)}$, by using (3.4) we get

$$\mathbb{E}E_z\big[\tau_{B(x,2r_n)}^{-q}\big] \le \mathbb{E}E_z\big[\tau_{B(z,r_n)}^{-q}\big] \le c r_n^{-\tilde{\xi}(q)},$$

provided n is large enough so that $r_n \le 1$. By Fubini's theorem, the $\mu_{x,r_n}(dz)$-integral of this inequality becomes

$$\mathbb{E}E_{\mu_{x,r_n}}\big[\tau_{B(x,2r_n)}^{-q}\big] \le c r_n^{-\tilde{\xi}(q)}.$$

Let now $\kappa > 2+\tilde{\xi}(q)$ and $n_0 \ge 1$ satisfying $r_{n_0} \in \big[\frac{1}{2}, 1\big]$ be fixed. Then for all $n \ge n_0$ we obtain by Čebyšev's inequality,

$$\begin{aligned}\mathbb{P}\Big[\max_{x\in\Xi_{R,n+1}} E_{\mu_{x,r_n}}\big[\tau_{B(x,2r_n)}^{-q}\big] \ge r_n^{-\kappa}\Big] &\le r_n^{\kappa} \sum_{x\in\Xi_{R,n+1}} \mathbb{E}E_{\mu_{x,r_n}}\big[\tau_{B(x,2r_n)}^{-q}\big] \\ &\le \frac{c}{2^{n(\kappa-\tilde{\xi}(q)-2)}}\end{aligned}$$

for some $c = c(\gamma, R, q, \kappa) > 0$. Hence by our choice of κ,

$$\sum_{n\ge n_0} \mathbb{P}\Big[\max_{x\in\Xi_{R,n+1}} E_{\mu_{x,r_n}}\big[\tau_{B(x,2r_n)}^{-q}\big] \ge r_n^{-\kappa}\Big] < \infty$$

and we apply the Borel–Cantelli lemma to obtain that $\mathbb{P}$-a.s. for all $n \ge n_0$ and for all $x \in \Xi_{R,n+1}$,

$$E_{\mu_{x,r_n}}\big[\tau_{B(x,2r_n)}^{-q}\big] \le C r_n^{-\kappa} \tag{3.5}$$

for some random constant $C = C(X, \gamma, R, q, \kappa) > 0$.

Now for any $r \in (0, 1]$ we choose $n \ge n_0$ such that $r_n \le \frac{2}{5}r < 2r_n$. For all $y \in B(R)$, by construction there exists $x \in \Xi_{R,n+1}$ such that $|x - y| \le \frac{1}{2}r_n$. Furthermore, from $B(x, r_n) \subset B(x, 2r_n) \subset B(y, r)$ we have $\tau_{B(x,r_n)} \le \tau_{B(x,2r_n)} = \tau_{B(x,r_n)} + \tau_{B(x,2r_n)} \circ \vartheta_{\tau_{B(x,r_n)}} \le \tau_{B(y,r)}$, and therefore by the strong Markov property [6, Theorem A.1.21] of $\mathcal{B}$,

$$\begin{aligned}E_y\big[\tau_{B(y,r)}^{-q}\big] &\le E_y\big[\tau_{B(x,2r_n)}^{-q}\big] \le E_y\Big[\big(\tau_{B(x,r_n)} + \tau_{B(x,2r_n)} \circ \vartheta_{\tau_{B(x,r_n)}}\big)^{-q}\Big] \\ &\le E_y\Big[\big(\tau_{B(x,2r_n)} \circ \vartheta_{\tau_{B(x,r_n)}}\big)^{-q}\Big] = E_y\Big[E_{\mathcal{B}_{\tau_{B(x,r_n)}}}\big[\tau_{B(x,2r_n)}^{-q}\big]\Big] = E_{\mu^y_{x,r_n}}\big[\tau_{B(x,2r_n)}^{-q}\big],\end{aligned}$$

where $\mu^y_{x,r_n} := P_y[\mathcal{B}_{\tau_{B(x,r_n)}} \in \cdot]$. Since $\mu^y_{x,r_n} = P_y[B_{T_{B(x,r_n)}} \in \cdot]$ by $\mathcal{B}_{\tau_{B(x,r_n)}} = B_{T_{B(x,r_n)}}$, the exact formula for the distribution of a Brownian motion upon exiting balls (see e.g. [22, Exercise 4.2.24]) implies that $\mu^y_{x,r_n} \le c\mu_{x,r_n}$ for some explicit constant $c > 0$ (this can be regarded as an application of the scale-invariant elliptic Harnack inequality). Thus $E_y\big[\tau_{B(y,r)}^{-q}\big] \le E_{\mu^y_{x,r_n}}\big[\tau_{B(x,2r_n)}^{-q}\big] \le cE_{\mu_{x,r_n}}\big[\tau_{B(x,2r_n)}^{-q}\big]$ and the claim follows from (3.5). □

Proposition 3.3 $\mathbb{P}$-*a.s., for any* $\beta > \alpha_1$ *and any* $R \geq 1$ *there exist random constants* $C_i = C_i(X, \gamma, R, \beta) > 0$, $i = 7, 8$, *such that*

$$P_x[\tau_{B(x,r)} \leq t] \leq C_7 \exp\Bigl(-C_8 (r^\beta/t)^{\frac{1}{\beta-1}}\Bigr), \qquad \forall x \in B(R),\ r \in (0, 1],\ t > 0.$$

Proof Let $\beta > \alpha_1$ and $R \geq 1$. By [19, Theorem 7.2] it is enough to show that there exist $\varepsilon \in (0, 1)$ and $\delta > 0$ such that $P_x[\tau_{B(x,r)} \leq \delta r^\beta] \leq \varepsilon$ for all $x \in B(R)$ and $r \in (0, 1]$. Indeed, let $\varepsilon \in (0, 1)$ and set $q := 2/\gamma$, $\kappa := \frac{1}{2}(\alpha_1 + \beta)q$ and $\delta := (\varepsilon/C_6)^{1/q}$, so that $\kappa > \alpha_1 q = 2 + \tilde{\xi}(q)$ by $\beta > \alpha_1 = (2 + \tilde{\xi}(q))/q$. Then for any $x \in B(R)$ and $r \in (0, 1]$, by Čebyšev's inequality and Proposition 3.2 we have

$$P_x[\tau_{B(x,r)} \leq \delta r^\beta] = P_x\bigl[(\tau_{B(x,r)})^{-q} \geq (\delta r^\beta)^{-q}\bigr] \leq C_6 \delta^q\, r^{\beta q - \kappa} \leq \varepsilon,$$

proving the claim. □

4 Strong Feller property of the resolvents

In this section we prove that the resolvent operator of the killed LBM $\mathcal{B}^U$ has the strong Feller property. We will mainly follow the arguments in [18, Theorem 2.4], where the strong Feller property of the original LMB $\mathcal{B}$ is established. The essential ingredients are a coupling lemma and the following lemma.

Lemma 4.1 ([16, Lemma 2.19]) $\mathbb{P}$-*a.s., for all* $R > 0$,

$$\lim_{t \downarrow 0} \sup_{n \geq 1} \sup_{x \in B(R)} E_x[F_t^n] = 0.$$

Remark 4.2 The article [16] is an earlier version of [17], but Lemma 4.1 has been removed from the latter, which is why we still cite [16] in this paper.

Proposition 4.3 $\mathbb{P}$-*a.s., for any non-empty open set* $U \subset \mathbb{R}^2$ *and for any* $\lambda > 0$ *the resolvent operator* R_λ^U *is strong Feller, i.e. it maps Borel measurable bounded functions on* U *to continuous bounded functions on* U.

Proof Throughout this proof, we fix any environment $\omega \in \Omega$ such that all the conclusions of Proposition 2.4 (i), (iv) and Lemma 4.1 hold. Note that by Proposition 2.4, Lemma 4.1 and Fatou's lemma we have

$$\lim_{t \downarrow 0} \sup_{x \in B(R)} E_x[F_t] = 0, \qquad \forall R \geq 1. \tag{4.1}$$

Let U be a non-empty open subset of $\mathbb{R}^2$, let $\lambda > 0$ and let $f : U \to \mathbb{R}$ be Borel measurable and bounded. Recall that $T_U = \inf\{s \geq 0 : B_s \notin U\}$ denotes the exit time of the Brownian motion B from U. Since $\tau_U = F_{T_U}$, $R_\lambda^U f$ can be written as

$$\begin{aligned}R_\lambda^U f(x) &= E_x\bigg[\int_0^{\tau_U} e^{-\lambda t} f(\mathcal{B}_t)\,dt\bigg] = E_x\bigg[\int_0^{T_U} e^{-\lambda F_t} f(B_t)\,dF_t\bigg]\\ &= E_x\bigg[\int_0^{T_U\wedge\varepsilon} e^{-\lambda F_t} f(B_t)\,dF_t\bigg] + E_x\bigg[\int_{T_U\wedge\varepsilon}^{T_U} e^{-\lambda F_t} f(B_t)\,dF_t\bigg]\\ &=: N_\varepsilon(x) + R_\lambda^{U,\varepsilon} f(x) \end{aligned} \tag{4.2}$$

for any $x \in \mathbb{R}^2$ and any $\varepsilon > 0$. It is immediate that

$$|N_\varepsilon(x)| \le \|f\|_\infty\, E_x[F_\varepsilon], \tag{4.3}$$

whereas for $R_\lambda^{U,\varepsilon} f(x)$ the Markov property of B gives

$$\begin{aligned}R_\lambda^{U,\varepsilon} f(x) &= E_x\bigg[\mathbb{1}_{\{T_U>\varepsilon\}}\int_\varepsilon^{T_U} e^{-\lambda F_t} f(B_t)\,dF_t\bigg]\\ &= E_x\bigg[\mathbb{1}_{\{T_U>\varepsilon\}} e^{-\lambda F_\varepsilon} E_{B_\varepsilon}\bigg[\int_0^{T_U} e^{-\lambda F_t} f(B_t)\,dF_t\bigg]\bigg]\\ &= E_x\Big[\mathbb{1}_{\{T_U>\varepsilon\}} e^{-\lambda F_\varepsilon} R_\lambda^U f(B_\varepsilon)\Big]. \end{aligned} \tag{4.4}$$

To estimate $R_\lambda^{U,\varepsilon} f(x) - R_\lambda^{U,\varepsilon} f(y)$ we use the coupling lemma [17, Lemma 2.9], which allows to construct for any $x, y \in \mathbb{R}^2$ a couple (B^x, B^y) of Brownian motions $B^x = (B_t^x)_{t\ge0}$ and $B^y = (B_t^y)_{t\ge0}$ with $(B_0^x, B_0^y) = (x, y)$ such that $B_t^x = B_t^y$ for any $t \in [T_{xy}, \infty)$ for a random time T_{xy} satisfying

$$\lim_{\delta\downarrow0}\ \sup_{x,y\in\mathbb{R}^2,\,|x-y|\le\delta} P_{x,y}[T_{xy}\ge\varepsilon] = 0 \tag{4.5}$$

for any $\varepsilon > 0$, where $P_{x,y}$ denotes the law of (B^x, B^y). Let $E_{x,y}$ denote the expectation under $P_{x,y}$ and set $T_U^x := T_U(B^x)$, $T_U^y := T_U(B^y)$, $F_t^x := F_t(B^x)$ and $F_t^y := F_t(B^y)$, with T_U and F_t for $t \ge 0$ regarded as functions on the path space $\Omega' = C([0,\infty), \mathbb{R}^2)$. Then according to [18, Proof of Theorem 2.4], for any $\varepsilon > 0$,

$$\lim_{\delta\downarrow0}\ \sup_{x,y\in B(R),\,|x-y|\le\delta} E_{x,y}\Big[\big|e^{-\lambda F_\varepsilon^x} - e^{-\lambda F_\varepsilon^y}\big|\Big] = 0, \qquad \forall R\ge1, \tag{4.6}$$

whose proof we repeat here for the sake of completeness. Indeed, for any $\varepsilon' \in (0, \varepsilon]$, since $F_\varepsilon^x - F_{T_{xy}}^x = F_\varepsilon^y - F_{T_{xy}}^y > 0$ $P_{x,y}$-a.s. on $\{T_{xy} < \varepsilon\}$ by Proposition 2.4 (i), (ii),

$$\begin{aligned}E_{x,y}\Big[\big|e^{-\lambda F_\varepsilon^x} - e^{-\lambda F_\varepsilon^y}\big|\Big] &\le 2P_{x,y}[T_{xy}\ge\varepsilon] + E_{x,y}\Big[\mathbb{1}_{\{T_{xy}<\varepsilon\}}\big|e^{-\lambda F_\varepsilon^x} - e^{-\lambda F_\varepsilon^y}\big|\Big]\\ &= 2P_{x,y}[T_{xy}\ge\varepsilon] + E_{x,y}\Big[\mathbb{1}_{\{T_{xy}<\varepsilon\}} e^{-\lambda(F_\varepsilon^x - F_{T_{xy}}^x)}\big|e^{-\lambda F_{T_{xy}}^x} - e^{-\lambda F_{T_{xy}}^y}\big|\Big]\\ &\le 2P_{x,y}[T_{xy}\ge\varepsilon] + E_{x,y}\Big[\big|\lambda F_{T_{xy}}^x - \lambda F_{T_{xy}}^y\big|\wedge 1\Big]\end{aligned}$$

$$\leq 2P_{x,y}[T_{xy} \geq \varepsilon] + P_{x,y}[T_{xy} \geq \varepsilon'] + \lambda E_{x,y}\big[\mathbb{1}_{\{T_{xy}<\varepsilon'\}}(F^x_{\varepsilon'} + F^y_{\varepsilon'})\big]$$
$$\leq 3P_{x,y}[T_{xy} \geq \varepsilon'] + \lambda\big(E_x[F_{\varepsilon'}] + E_y[F_{\varepsilon'}]\big)$$

and taking $\lim_{\varepsilon'\downarrow 0} \limsup_{\delta\downarrow 0} \sup_{x,y\in B(R),\,|x-y|\leq\delta}$ yields (4.6) by (4.5) and (4.1).

Now let $x, y \in \mathbb{R}^2$ and $\varepsilon > 0$. From (4.4) we obtain

$$\begin{aligned}\big|R^{U,\varepsilon}_\lambda f(x) - R^{U,\varepsilon}_\lambda f(y)\big| &= \Big|E_{x,y}\Big[\mathbb{1}_{\{T^x_U>\varepsilon\}} e^{-\lambda F^x_\varepsilon} R^U_\lambda f(B^x_\varepsilon) - \mathbb{1}_{\{T^y_U>\varepsilon\}} e^{-\lambda F^y_\varepsilon} R^U_\lambda f(B^y_\varepsilon)\Big]\Big| \\ &\leq \Big|E_{x,y}\Big[\mathbb{1}_{\{T^x_U>\varepsilon\}} e^{-\lambda F^x_\varepsilon}\big(R^U_\lambda f(B^x_\varepsilon) - R^U_\lambda f(B^y_\varepsilon)\big)\Big]\Big| \\ &\quad + \Big|E_{x,y}\Big[\big(\mathbb{1}_{\{T^x_U>\varepsilon\}} e^{-\lambda F^x_\varepsilon} - \mathbb{1}_{\{T^y_U>\varepsilon\}} e^{-\lambda F^y_\varepsilon}\big) R^U_\lambda f(B^y_\varepsilon)\Big]\Big|.\end{aligned} \tag{4.7}$$

Since on the event $\{T_{xy} < \varepsilon\}$ we have $B^x_\varepsilon = B^y_\varepsilon$ and hence $R^U_\lambda f(B^x_\varepsilon) = R^U_\lambda f(B^y_\varepsilon)$, the first term in (4.7) can be estimated from above by

$$\begin{aligned}E_{x,y}\Big[\big|R^U_\lambda f(B^x_\varepsilon) - R^U_\lambda f(B^y_\varepsilon)\big|\Big] &= E_{x,y}\Big[\mathbb{1}_{\{T_{xy}\geq\varepsilon\}}\big|R^U_\lambda f(B^x_\varepsilon) - R^U_\lambda f(B^y_\varepsilon)\big|\Big] \\ &\leq 2\big\|R^U_\lambda f\big\|_\infty P_{x,y}[T_{xy} \geq \varepsilon] \\ &\leq 2\lambda^{-1}\|f\|_\infty P_{x,y}[T_{xy} \geq \varepsilon],\end{aligned} \tag{4.8}$$

where we used the trivial bounds $0 \leq \mathbb{1}_{\{T^x_U>\varepsilon\}} e^{-\lambda F^x_\varepsilon} \leq 1$ and $\big\|R^U_\lambda f\big\|_\infty \leq \lambda^{-1}\|f\|_\infty$. On the other hand, the second term in (4.7) is less than or equal to

$$\begin{aligned}&\lambda^{-1}\|f\|_\infty E_{x,y}\Big[\big|\mathbb{1}_{\{T^x_U>\varepsilon\}} - \mathbb{1}_{\{T^y_U>\varepsilon\}}\big| e^{-\lambda F^x_\varepsilon} + \mathbb{1}_{\{T^y_U>\varepsilon\}}\big|e^{-\lambda F^x_\varepsilon} - e^{-\lambda F^y_\varepsilon}\big|\Big] \\ &\leq \lambda^{-1}\|f\|_\infty E_{x,y}\Big[\big|\big(1 - \mathbb{1}_{\{T^x_U\leq\varepsilon\}}\big) - \big(1 - \mathbb{1}_{\{T^y_U\leq\varepsilon\}}\big)\big| + \big|e^{-\lambda F^x_\varepsilon} - e^{-\lambda F^y_\varepsilon}\big|\Big] \\ &\leq \lambda^{-1}\|f\|_\infty\Big(P_{x,y}[T^x_U \leq \varepsilon] + P_{x,y}[T^y_U \leq \varepsilon] + E_{x,y}\Big[\big|e^{-\lambda F^x_\varepsilon} - e^{-\lambda F^y_\varepsilon}\big|\Big]\Big).\end{aligned} \tag{4.9}$$

Noting $P_{x,y}[T^x_U \leq \varepsilon] = P_x[T_U \leq \varepsilon]$ and $P_{x,y}[T^y_U \leq \varepsilon] = P_y[T_U \leq \varepsilon]$, from (4.2), (4.3), (4.7), (4.8) and (4.9) we get

$$\begin{aligned}&\big|R^U_\lambda f(x) - R^U_\lambda f(y)\big| \\ &\quad\leq \|f\|_\infty\big(E_x[F_\varepsilon] + E_y[F_\varepsilon]\big) + \lambda^{-1}\|f\|_\infty\big(P_x[T_U \leq \varepsilon] + P_y[T_U \leq \varepsilon]\big) \\ &\quad\quad + \lambda^{-1}\|f\|_\infty\Big(2P_{x,y}[T_{xy} \geq \varepsilon] + E_{x,y}\Big[\big|e^{-\lambda F^x_\varepsilon} - e^{-\lambda F^y_\varepsilon}\big|\Big]\Big).\end{aligned} \tag{4.10}$$

Finally, let $x \in U$ and choose $r_x > 0$ so that $B(x, 2r_x) \subset U$. Then for any $y \in B(x, r_x)$, $T_{B(y,r_x)} \leq T_U$ by $B(y, r_x) \subset B(x, 2r_x) \subset U$ and hence

$$P_y[T_U \leq \varepsilon] \leq P_y[T_{B(y,r_x)} \leq \varepsilon] \leq 2\exp\big(-r_x^2/(4\varepsilon)\big) \tag{4.11}$$

(see e.g. [22, Proposition 2.6.19] for the latter inequality). Now we can easily conclude $\limsup_{y\to x}\big|R^U_\lambda f(x) - R^U_\lambda f(y)\big| = 0$ by taking the supremum in $y \in B(x, r_x)$ of the second line of (4.10), using (4.5) and (4.6) to let $y \to x$ and then using (4.1) and (4.11) to let $\varepsilon \downarrow 0$. Thus $R_\lambda^U f$ is continuous on U. □

5 Continuity and upper bounds of the heat kernels

Throughout Sects. 5 and 6 we fix any environment $\omega \in \Omega$ such that all the conclusions of Proposition 2.4 (i), (iv), Lemma 3.1, Propositions 3.3, 4.3 and B.1 hold.

The purpose of this section is to prove Theorem 5.1 below on the continuity of the heat kernels as well as Theorem 1.2. Recall that $\mathcal{F}$ equipped with the norm $\|f\|_{\mathcal{F}}^2 := \mathcal{E}(f, f) + \|f\|_{L^2(\mathbb{R}^2,M)}^2$ is a Hilbert space. For any open set $U \subset \mathbb{R}^2$, we define $\mathcal{F}_U$ to be the closure in $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$ of the set of all functions in $\mathcal{F}$ whose M-essential supports in $\mathbb{R}^2$ are compact subsets of U. It is well known that $(\mathcal{E}, \mathcal{F}_U)$ is the Dirichlet form associated with the killed Liouville Brownian motion $\mathcal{B}^U$ and that it is regular on $L^2(U, M)$ (see e.g. [14, Theorems 4.4.2 and 4.4.3]). The associated non-positive self-adjoint operator on $L^2(U, M)$ is denoted by $\mathcal{L}_U$, its domain by $\mathcal{D}(\mathcal{L}_U)$, and the associated semigroup and resolvent operators by $(T_t^U)_{t>0}$ and $(G_\lambda^U)_{\lambda>0}$, respectively.

Theorem 5.1 *For any non-empty open set $U \subset \mathbb{R}^2$ the following hold:*

(i) *There exists a (unique) jointly continuous function $p^U = p_t^U(x, y) : (0, \infty) \times U \times U \to [0, \infty)$ such that for all $(t, x) \in (0, \infty) \times U$, $P_x[\mathcal{B}_t^U \in dy] = p_t^U(x, y)\, M(dy)$, which we refer to as the* Dirichlet Liouville heat kernel *on U.*
(ii) *The semigroup operator P_t^U is strong Feller, i.e. it maps Borel measurable bounded functions on U to continuous bounded functions on U.*
(iii) *If U is connected, then $p_t^U(x, y) \in (0, \infty)$ for any $(t, x, y) \in (0, \infty) \times U \times U$, and in particular the Dirichlet form $(\mathcal{E}, \mathcal{F}_U)$ of $\mathcal{B}^U$ is irreducible.*

See [14, Section 1.6, p. 55] for the definition of the irreducibility of a symmetric Dirichlet form and [14, Theorem 4.7.1 (i) and Exercise 4.7.1] for its probabilistic consequences.

From now on we will write $p_t(\cdot, \cdot)$ instead of $p_t^{\mathbb{R}^2}(\cdot, \cdot)$ and call it the (global) *Liouville heat kernel*. Note that Theorem 1.1 follows directly from Theorem 5.1 by choosing $U = \mathbb{R}^2$.

5.1 The heat kernel on bounded open sets

In this subsection we will prove Theorem 5.1 for a fixed non-empty bounded open set $U \subset \mathbb{R}^2$. The case of unbounded open sets will be treated later in Sect. 5.2. We denote by $\|f\|_p$ the $L^p(U, M)$-norm for $p \geq 1$ and by $\langle \cdot, \cdot \rangle$ the $L^2(U, M)$-inner product. Let $R \geq 1$ be such that $U \subset B(R)$.

Proposition 5.2 (Faber–Krahn-type inequality) *The spectrum of $-\mathcal{L}_U$ is discrete, and for its smallest eigenvalue $\lambda_1(U)$ there exists $C_9 = C_9(X, \gamma, R) > 0$ such that*

$$\lambda_1(U) \geq \frac{C_9}{M(U) \log\bigl(2 + \frac{1}{M(U)}\bigr)}. \tag{5.1}$$

Proof First, it is elementary to verify that $\sup_{x\in U}\|g_U(x,\cdot)\|_2 < \infty$ by (2.8) and a calculation similar to (B.6) based on Lemma 3.1, so that $g_U \in L^2(U\times U, M\times M)$, $G_U f(x) = \langle g_U(x,\cdot), f\rangle \in \mathbb{R}$ for $x \in U$ for any $f \in L^2(U,M)$, and G_U defines a bounded linear operator on $L^2(U,M)$ which is Hilbert-Schmidt and hence (see e.g. [10, Theorem 4.2.16]) compact. Then in view of [14, (1.5.3) and Theorem 4.2.3 (ii)] the Dirichlet form $(\mathcal{E},\mathcal{F}_U)$ of $\mathcal{B}^U$ is transient in the sense of [14, (1.5.4)], or equivalently in the sense of [14, (1.5.6)] by [14, Theorem 1.5.1], which implies that $\{u \in \mathcal{F}_U : \mathcal{E}(u,u) = 0\} = \{0\}$, namely $\mathcal{L}_U$ is injective. Now by [14, Theorem 4.2.6, Theorem 1.5.4 (i) and Theorem 1.5.2 (iii)], $G_U f \in \mathcal{D}(\mathcal{L}_U)$ and $-\mathcal{L}_U G_U f = f$ for any $f \in L^2(U,M)$, which together with the injectivity of $\mathcal{L}_U$ yields $(-\mathcal{L}_U)^{-1} = G_U$. In particular, $(-\mathcal{L}_U)^{-1}$ is compact, and therefore the spectrum of $-\mathcal{L}_U$ is discrete by [9, Corollary 4.2.3].

For the proof of (5.1), note that by the spectral decomposition of the compact self-adjoint operator $(-\mathcal{L}_U)^{-1} = G_U$ (see e.g. [9, Theorem 4.2.2]) and $g_U \geq 0$,

$$\lambda_1(U)^{-1} = \sup\big\{\langle G_U f, f\rangle : f \in L^2(U,M),\ f \geq 0,\ \|f\|_2 = 1\big\}. \tag{5.2}$$

Let $f \in L^2(U,M)$ satisfy $f \geq 0$ and $\|f\|_2 = 1$. Setting $\nu := \pi\alpha_2/2 = \frac{\pi}{4}(2-\gamma)^2$ and noting that $g_U \leq g_{B(R+1)}$ by $U \subset B(R) \subset B(R+1)$, we have

$$\begin{aligned}\langle G_U f, f\rangle &\leq \langle G_{B(R+1)} f, f\rangle \\ &\leq \int_U\int_U \exp\big(\nu g_{B(R+1)}(x,y)\big)\, M(dy)\, M(dx) \\ &\quad + \int_U\int_U \frac{f(x)f(y)}{\nu}\log\Big(1+\frac{f(x)f(y)}{\nu}\Big)\, M(dy)\, M(dx),\end{aligned} \tag{5.3}$$

where we used the elementary inequality $ab \leq a\log(1+a) + e^b$, valid for any $a, b \in [0,\infty]$, with $a = \frac{f(x)f(y)}{\nu}$ and $b = \nu g_{B(R+1)}(x,y)$. For the first integral in (5.3), we have

$$\int_U\int_U \exp\big(\nu g_{B(R+1)}(x,y)\big)\, M(dy)\, M(dx) \leq \int_U\int_U \frac{c}{|x-y|^{\nu/\pi}}\, M(dy)\, M(dx)$$

with $c = c(\gamma, R) > 0$ by (2.8) and $U \subset B(R)$, and then using Lemma 3.1 with $\varepsilon = \alpha_2/4 \in (0, \alpha_2 - \nu/\pi)$, we further obtain

$$\begin{aligned}\int_U\int_U \frac{1}{|x-y|^{\nu/\pi}}\, M(dy)\, M(dx) &\leq \int_U\int_{B(x,2R)} \frac{1}{|x-y|^{\nu/\pi}}\, M(dy)\, M(dx) \\ &\leq \sum_{n=0}^{\infty}\int_U\int_{B(x,2^{1-n}R)\setminus B(x,2^{-n}R)} (2^{-n}R)^{-\nu/\pi}\, M(dy)\, M(dx) \leq CM(U)\end{aligned}$$

for some $C = C(X, \gamma, R) > 0$. On the other hand, setting $\bar{M}_U := M(\cdot \cap U)/M(U)$, we can write the second term in (5.3) as

$$M(U)^2 \int_U \int_U \frac{f(x)f(y)}{\nu} \log\Big(1 + \frac{f(x)f(y)}{\nu}\Big)\, \bar{M}_U(dy)\, \bar{M}_U(dx).$$

For the homeomorphisms $H, I : [0, \infty) \to [0, \infty)$ defined by $H(s) := s^2$ and $I(s) := s\log(1+s)$, we easily see that the function $H \circ I^{-1}$ is convex, and we apply Jensen's inequality to get

$$\begin{aligned} H \circ I^{-1}&\Big(\int_U \int_U \frac{f(x)f(y)}{\nu} \log\Big(1 + \frac{f(x)f(y)}{\nu}\Big)\, \bar{M}_U(dy)\, \bar{M}_U(dx)\Big) \\ &\leq \int_U \int_U \Big(\frac{f(x)f(y)}{\nu}\Big)^2 \bar{M}_U(dy)\, \bar{M}_U(dx) = \frac{1}{\nu^2 M(U)^2}, \end{aligned}$$

where we used $\|f\|_2 = 1$. Hence

$$\begin{aligned} &\int_U \int_U \frac{f(x)f(y)}{\nu} \log\Big(1 + \frac{f(x)f(y)}{\nu}\Big)\, \bar{M}_U(dy)\, \bar{M}_U(dx) \\ &\quad \leq I \circ H^{-1}\Big(\frac{1}{\nu^2 M(U)^2}\Big) = \frac{1}{\nu M(U)} \log\Big(1 + \frac{1}{\nu M(U)}\Big). \end{aligned}$$

Finally, we combine the above considerations to conclude that

$$\langle G_U f, f\rangle \leq C M(U) + \frac{1}{\nu} M(U) \log\Big(1 + \frac{1}{\nu M(U)}\Big) \leq C M(U) \log\Big(2 + \frac{1}{M(U)}\Big)$$

for some constants C large enough, which together with (5.2) yields the claim. □

In the next proposition we derive from the above Faber–Krahn inequality a Nash-type inequality and thereby an on-diagonal estimate on $(T_t^U)_{t>0}$ of the same form as stated for $p_t = p_t(x, y)$ in Theorem 1.2. In particular, $(T_t^U)_{t>0}$ turns out to be ultra-contractive, i.e. $T_t^U(L^2(U, M)) \subset L^\infty(U, M)$ and $T_t^U : L^2(U, M) \to L^\infty(U, M)$ is a bounded linear operator for all $t > 0$. Recall that for each $t > 0$, T_t^U is a self-adjoint Markovian operator on $L^2(U, M)$ and hence canonically extends to a bounded linear operator on $L^1(U, M)$ with operator norm at most 1 (see e.g. [14, (1.5.2)]). For a bounded linear operator $A : L^1(U, M) \to L^\infty(U, M)$, its operator norm will be denoted by $\|A\|_{L^1(U)\to L^\infty(U)}$.

Proposition 5.3 *There exists a constant* $C_{10} = C_{10}(X, \gamma, R) > 0$ *such that*

$$\big\|T_t^U\big\|_{L^1(U)\to L^\infty(U)} \leq C_{10} t^{-1} \log(t^{-1}), \qquad \forall t \in \big(0, \tfrac{1}{2}\big]. \tag{5.4}$$

Proof Since $\big\|T_t^U\big\|_{L^1(U)\to L^\infty(U)} \leq \big\|T_t^{B(R)}\big\|_{L^1(B(R))\to L^\infty(B(R))}$ by $U \subset B(R)$, it is enough to show (5.4) for $\big(T_t^{B(R)}\big)_{t>0}$. Recall that for any non-empty open subset V of

$B(R)$, the smallest eigenvalue $\lambda_1(V)$ of $-\mathcal{L}_V$ admits the variational expression

$$\lambda_1(V) = \inf\left\{\frac{\mathcal{E}(f,f)}{\|f\|_2^2} : f \in \mathcal{F}_V, f \neq 0\right\}$$

(see e.g. [9, Theorems 4.5.1 and 4.5.3]), so that we can rewrite the Faber–Krahn inequality of Proposition 5.2 for V as

$$\|f\|_2^2 \leq C_9^{-1}\psi(M(V))\mathcal{E}(f,f), \qquad \forall f \in \mathcal{F}_V, \tag{5.5}$$

where $\psi(s) := s\log(2+s^{-1})$ ($\psi(0) := 0$). Next we will verify that

$$\|f\|_2^2 \leq C_9^{-1}\psi\big(M(\mathrm{supp}[f])\big)\mathcal{E}(f,f), \qquad \forall f \in \mathcal{F}_{B(R)}, \tag{5.6}$$

where $\mathrm{supp}[f] := \mathrm{supp}_{B(R)}[f]$ denotes the M-essential support of f in $B(R)$. First, for $f \in \mathcal{F}_{B(R)}$ with $\mathrm{supp}[f]$ compact, (5.6) follows by choosing a decreasing sequence $(V_n)_{n\geq 1}$ of open subsets of $B(R)$ with $\bigcap_{n\geq 1} V_n = \mathrm{supp}[f]$, applying (5.5) with $V = V_n$ and letting $n \to \infty$. Next, for general $f \in \mathcal{F}_{B(R)}$, as $|f| \in \mathcal{F}_{B(R)}$ and $\mathcal{E}(|f|,|f|) \leq \mathcal{E}(f,f)$ we may assume $f \geq 0$. Let $(f_n)_{n\geq 1} \subset \mathcal{F}_{B(R)}$ be a sequence with $\mathrm{supp}[f_n]$ compact and $\lim_{n\to\infty}\|f_n - f\|_{\mathcal{F}} = 0$, where by $f \geq 0$ and [14, Theorem 1.4.2 (v)] we may assume that $f_n \geq 0$ for all n. Then since $f \wedge f_n \in \mathcal{F}_{B(R)}$, $\mathrm{supp}[f \wedge f_n]$ is a compact subset of $\mathrm{supp}[f]$ and

$$\|f - f \wedge f_n\|_{\mathcal{F}} = \|(f-f_n)^+\|_{\mathcal{F}} \leq \|f - f_n\|_{\mathcal{F}} \xrightarrow{n\to\infty} 0,$$

we conclude (5.6) for all $f \in \mathcal{F}_{B(R)}$ by letting $n \to \infty$ in (5.6) for $f \wedge f_n$.

Now, since $\psi : [0,\infty) \to [0,\infty)$ is strictly increasing, [2, Proposition 10.3] and (5.6) together imply that

$$\|f\|_2^2 \leq 8C_9^{-1}\psi\big(4/\|f\|_2^2\big)\mathcal{E}(f,f) \qquad \text{for all } f \in \mathcal{F}_{B(R)} \text{ with } 0 < \|f\|_1 \leq 1.$$

In particular, for such f we have $\theta\big(\|f\|_2^2\big) \leq \mathcal{E}(f,f)$ with $\theta(s) := \frac{1}{32}C_9 s^2/\log(2+s/4)$, and then by [7, Proposition II.1] we obtain

$$\big\|T_t^{B(R)}\big\|_{L^1(B(R))\to L^\infty(B(R))} \leq m(t), \qquad \forall t > 0, \tag{5.7}$$

for the unique differentiable function $m : (0,\infty) \to (0,\infty)$ satisfying

$$m'(t) = -\theta(m(t)), \qquad \lim_{t\downarrow 0} m(t) = \infty. \tag{5.8}$$

It is immediate that $m = \Phi^{-1}$, where $\Phi : (0,\infty) \to (0,\infty)$ is a decreasing diffeomorphism defined by $\Phi(s) := \int_s^\infty \theta(u)^{-1}\,du$, and furthermore for all $s \in (0,\infty)$,

$$\Phi(s) = \int_s^\infty \frac{32C_9^{-1}}{u^2}\log(2+u/4)\,du \leq \frac{80C_9^{-1}}{s}\log(2+s/4) =: \Psi(s),$$

which means that $\Psi^{-1}(t) \geq \Phi^{-1}(t)$ for all $t \in (0, \infty)$ since $\Psi : (0, \infty) \to (0, \infty)$ is also a decreasing diffeomorphism. Finally, for all $t \in \left(0, \frac{1}{2}\right]$ we easily see that $\Psi\left(t^{-1} \log(t^{-1})\right) \leq Ct$ and hence that

$$m(Ct) = \Phi^{-1}(Ct) \leq \Psi^{-1}(Ct) \leq t^{-1} \log(t^{-1}),$$

and the claim then follows from (5.7). □

Now we prove Theorem 5.1 for bounded open sets U. Given the ultracontractivity of $(T_t^U)_{t>0}$ in Proposition 5.3 and the strong Feller property in Proposition 4.3, a general result in [8] provides the existence of a continuous kernel $p^U = p_t^U(x, y)$ for $(T_t^U)_{t>0}$, but we still have to identify this kernel as the transition density of $\mathcal{B}^U$.

Proof of Theorem 5.1 for bounded U We divide the proof of (i) into several steps.

Step 1: In the first step we show the existence of a jointly continuous integral kernel $p^U = p_t^U(x, y)$ for $(T_t^U)_{t>0}$. Being discrete by Proposition 5.2, the spectrum of $-\mathcal{L}_U$ takes the form of an unbounded non-decreasing sequence $(\lambda_n)_{n\geq 1} \subset [0, \infty)$ of eigenvalues repeated according to multiplicity, and there exists a complete orthonormal system $(\varphi_n)_{n\geq 1} \subset \mathcal{D}(\mathcal{L}_U)$ of $L^2(U, M)$ such that $-\mathcal{L}_U \varphi_n = \lambda_n \varphi_n$ for any $n \geq 1$ (see e.g. [9, Corollary 4.2.3]). Then $\varphi_n = e^{\lambda_n} T_1^U \varphi_n \in L^\infty(U, M)$ by Proposition 5.3, so that we may choose a bounded Borel measurable version of φ_n for each n. Further, since $R_\lambda^U \varphi_n$ is continuous on U for any $\lambda > 0$ by Proposition 4.3 and

$$R_\lambda^U \varphi_n = G_\lambda^U \varphi_n = (\lambda + \lambda_n)^{-1} \varphi_n \qquad M\text{-a.e. on } U \tag{5.9}$$

by [14, Theorem 4.2.3 (ii)], there exists a continuous version of φ_n, which is unique, bounded, and still denoted by φ_n. Then by [8, Theorem 2.1.4], the series

$$p_t^U(x, y) := \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y) \tag{5.10}$$

absolutely converges uniformly on $[\varepsilon, \infty) \times U \times U$ for any $\varepsilon > 0$, from which the joint continuity of $p^U = p_t^U(x, y)$ follows, and (5.10) defines an integral kernel for $(T_t^U)_{t>0}$, namely for each $t > 0$ and $f \in L^2(U, M)$,

$$T_t^U f(x) = \int_U p_t^U(x, y) f(y)\, M(dy) \qquad \text{for } M\text{-a.e. } x \in U. \tag{5.11}$$

Note that the boundedness of φ_n together with the uniform convergence of (5.10) implies the boundedness of $p_t^U(x, y)$ on $[\varepsilon, \infty) \times U \times U$ for each $\varepsilon > 0$, and also that $p_t^U(x, y) \geq 0$ by a monotone class argument based on (5.11) and the fact that $T_t^U f \geq 0$ M-a.e. for any $f \in L^2(U, M)$ with $f \geq 0$.

Step 2: In this step we show that R_λ^U is absolutely continuous with respect to the Liouville measure M for any $\lambda > 0$. Let A be a Borel subset of U with $M(A) = 0$. Then $R_\lambda^U \mathbb{1}_A$ is continuous on U by Proposition 4.3, and we also have $R_\lambda^U \mathbb{1}_A = G_\lambda^U \mathbb{1}_A = 0$ M-a.e. on U by $\mathbb{1}_A = 0$ M-a.e. Since M has full support, it follows that $R_\lambda^U \mathbb{1}_A$ is a continuous function on U which is equal to 0 on a dense subset of U and hence it is identically zero on U, proving the absolute continuity of R_λ^U.

Step 3: Next we will show that for *any* $x \in U$,

$$\int_0^\infty e^{-\lambda t}\left(\int_U p_t^U(x,y) f(y)\, M(dy)\right) dt = \int_0^\infty e^{-\lambda t} E_x\big[f(\mathcal{B}_t^U)\big]\, dt, \tag{5.12}$$

for all $\lambda > 0$ and all bounded Borel functions $f : U \to [0,\infty)$. Recall that $P_t^U f(x) = E_x\big[f(\mathcal{B}_t^U)\big]$ denotes the transition semigroup of $\mathcal{B}^U$. Then for any $\varepsilon > 0$, since $P_\varepsilon^U f = T_\varepsilon^U f$ M-a.e., by the absolute continuity of R_λ^U with respect to M we have

$$\begin{aligned}\int_\varepsilon^\infty e^{-\lambda t} P_t^U f(x)\, dt &= e^{-\lambda\varepsilon} R_\lambda^U (P_\varepsilon^U f)(x) = e^{-\lambda\varepsilon} R_\lambda^U (T_\varepsilon^U f)(x)\\ &= e^{-\lambda\varepsilon} R_\lambda^U \left(\sum_{n=1}^\infty e^{-\lambda_n \varepsilon}\langle \varphi_n, f\rangle \varphi_n\right)(x)\\ &= \sum_{n=1}^\infty e^{-(\lambda+\lambda_n)\varepsilon} \frac{1}{\lambda+\lambda_n}\langle \varphi_n, f\rangle \varphi_n(x),\end{aligned}$$

where we also used (5.9) and the uniform convergence of the series in (5.10). Setting $a_n^\varepsilon := e^{-(\lambda+\lambda_n)\varepsilon}\frac{1}{\lambda+\lambda_n} = \int_\varepsilon^\infty e^{-(\lambda+\lambda_n)t}\, dt$ and applying dominated convergence again on the basis of the uniform convergence of (5.10) on $[\varepsilon,\infty)\times U \times U$, we further get

$$\begin{aligned}\int_\varepsilon^\infty e^{-\lambda t} P_t^U f(x)\, dt &= \sum_{n=1}^\infty a_n^\varepsilon \varphi_n(x)\langle \varphi_n, f\rangle\\ &= \lim_{N\to\infty}\sum_{n=1}^N \int_\varepsilon^\infty e^{-(\lambda+\lambda_n)t}\varphi_n(x)\langle\varphi_n, f\rangle\, dt\\ &= \int_\varepsilon^\infty \left(\sum_{n=1}^\infty e^{-\lambda_n t}\varphi_n(x)\langle \varphi_n, f\rangle\right) e^{-\lambda t}\, dt\\ &= \int_\varepsilon^\infty \left(\int_U p_t^U(x,y) f(y)\, M(dy)\right) e^{-\lambda t}\, dt,\end{aligned}$$

and we obtain (5.12) by using monotone convergence to let $\varepsilon \downarrow 0$.

Step 4: Finally, we now prove that $P_x[\mathcal{B}_t^U \in dy] = p_t^U(x,y)\, M(dy)$ for all $(t,x) \in (0,\infty)\times U$. Let $x \in U$. Applying to (5.12) the uniqueness of Laplace transforms for positive measures on $[0,\infty)$ (see e.g. [13, Section XIII.1, Theorem 1a]), we get for all bounded Borel functions $f : U \to [0,\infty)$,

$$\int_U p_t^U(x,y) f(y)\, M(dy) = E_x\big[f(\mathcal{B}_t^U)\big] \qquad \text{for } dt\text{-a.e. } t \in (0,\infty). \tag{5.13}$$

If in addition f is continuous, then we easily see from dominated convergence using the continuity and boundedness of p^U established in Step 1 that (5.13) holds for all $t > 0$. Finally a monotone class argument gives the claim, proving (i).

For (ii), the claim is immediate from dominated convergence in view of the continuity and boundedness of p_t^U for each $t > 0$ and the fact that $M(U) < \infty$. Finally, (iii) follows by [21, Proposition A.3 (2)]. □

5.2 The heat kernel on unbounded open sets

The proof of Theorem 5.1 for unbounded U is based on the following lemma, which essentially contains Theorem 1.2 already.

Lemma 5.4 *For any $\beta > \alpha_1$ and any $R \geq 1$ there exist $C_i = C_i(X, \gamma, R, \beta) > 0$, $i = 11, 12$, such that for any non-empty bounded open subset U of $\mathbb{R}^2$,*

$$p_t^U(x, y) = p_t^U(y, x) \leq C_{11} t^{-1} \log(t^{-1}) \exp\left(-C_{12}\left(\frac{|x-y|^\beta \wedge 1}{t}\right)^{\frac{1}{\beta-1}}\right)$$

for all $t \in \left(0, \frac{1}{2}\right]$, $x \in \mathbb{R}^2$ and $y \in B(R)$, where we extend $p^U = p_t^U(x, y)$ to a function on $(0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$ by setting $p_t^U(x, y) := 0$ for $t > 0$ and $(x, y) \in (U \times U)^c$.

Proof Since for every $R \geq 1$ we have $p_t^{B(R)} \leq C_{10} t^{-1} \log(t^{-1})$ for any $t \in \left(0, \frac{1}{2}\right]$ for $C_{10} = C_{10}(X, \gamma, R) > 0$ by Proposition 5.3 and the continuity of $p_t^{B(R)}$, given the exit time estimates in Proposition 3.3, the result follows from [19, Theorem 1.1]. □

Remark 5.5 The constants appearing in the upper bound in Lemma 5.4 do not depend on the set U. Therefore, for any $R \geq 1$ there exists $C_{13} = C_{13}(X, \gamma, R) > 0$, also not depending on U, such that $p_{1/2}^U(x, y) \leq C_{13}$ for all $x \in \mathbb{R}^2$ and $y \in B(R)$. In particular, by the semigroup property we have for all $t \in \left(\frac{1}{2}, \infty\right)$ and such x and y,

$$p_t^U(x, y) = \int_{\mathbb{R}^2} p_{t-1/2}^U(x, z) p_{1/2}^U(z, y)\, M(dz) \leq C_{13} \int_{\mathbb{R}^2} p_{t-1/2}^U(x, z)\, M(dz) \leq C_{13}.$$

Lemma 5.6 *For any increasing sequence $(U_n)_{n\geq 1}$ of open subsets of $\mathbb{R}^2$ satisfying $\bigcup_{n\geq 1} U_n = \mathbb{R}^2$,*

$$\lim_{n\to\infty} P_x[\tau_{U_n} < t] = 0$$

uniformly in (t, x) over each compact subset of $[0, \infty) \times \mathbb{R}^2$.

Proof It suffices to prove the uniform convergence in (t, x) over $[0, T] \times B(R)$ for any $T, R \in (0, \infty)$. By monotonicity we may assume $t = T$. Then for any $x \in B(R)$ and $n \geq 1$ with $B(2R) \subset U_n$, noting that $\tau_{B(2R)} \leq \tau_{U_n} = \tau_{B(2R)} + \tau_{U_n} \circ \vartheta_{\tau_{B(2R)}}$, by the strong Markov property [6, Theorem A.1.21] of $\mathcal{B}$ we obtain

$$\begin{aligned} P_x[\tau_{U_n} < T] &= P_x\big[\tau_{B(2R)} + \tau_{U_n} \circ \vartheta_{\tau_{B(2R)}} < T\big] \leq P_x\big[\tau_{U_n} \circ \vartheta_{\tau_{B(2R)}} < T\big] \\ &= E_x\big[P_{\mathcal{B}_{\tau_{B(2R)}}}[\tau_{U_n} < T]\big] = P_{\mu^x_{0,2R}}[\tau_{U_n} < T], \end{aligned} \tag{5.14}$$

where $\mu^x_{0,2R} := P_x[\mathcal{B}_{\tau_{B(2R)}} \in \cdot] = P_x[B_{T_{B(2R)}} \in \cdot]$ as in the proof of Proposition 3.2 above. Setting $\mu_{0,2R} := \mu^0_{0,2R}$ and arguing precisely as there, from an explicit formula for the exit distribution of a Brownian motion (see e.g. [22, Exercise 4.2.24]) we get $\mu^x_{0,2R} \leq c\mu_{0,2R}$ for some explicit $c > 0$. Thus by (5.14), for any $n \geq 1$ with $B(2R) \subset U_n$ we obtain $\sup_{x\in B(R)} P_x[\tau_{U_n} < T] \leq c P_{\mu_{0,2R}}[\tau_{U_n} < T]$, which converges to 0 as $n \to \infty$ by dominated convergence since the trajectory of $\{\mathcal{B}_t\}_{t\in[0,T]}$ is bounded and hence contained in U_n for n large enough, completing the proof. □

Proof of Theorem 5.1 for unbounded U (i) Let $R \geq 1$ and let $f : \mathbb{R}^2 \to [0, \infty)$ be bounded and Borel measurable with $f|_{B(R)^c} = 0$. Let $k, l \in \mathbb{N}$ satisfy $l > k \geq R+1$, let $t > 0$ and $x \in B(k)$. Noting that $\tau_{B(k)} \leq \tau_{B(l)} = \tau_{B(k)} + \tau_{B(l)} \circ \vartheta_{\tau_{B(k)}}$ by $B(k) \subset B(l)$ and that $E_x\big[\mathbb{1}_{\{\tau_{B(k)}=t\}} f(\mathcal{B}_t)\big] = 0$ by $P_x[\mathcal{B}_{\tau_{B(k)}} \in B(k)] = 0$ and $f|_{B(k)^c} = 0$, we see from the strong Markov property [19, Proposition 3.4] of $\mathcal{B}$ that

$$\begin{aligned} P^{B(l)}_t f(x) &= P^{B(k)}_t f(x) + E_x\big[\mathbb{1}_{\{\tau_{B(k)}<t<\tau_{B(l)}\}} f(\mathcal{B}_t)\big] \\ &= P^{B(k)}_t f(x) + E_x\big[\mathbb{1}_{\{\tau_{B(k)}<t\}} P^{B(l)}_{t-\tau_{B(k)}} f(\mathcal{B}_{\tau_{B(k)}})\big]. \end{aligned} \tag{5.15}$$

Recall that by Theorem 5.1 (i) for $U = B(l)$ proved in Sect. 5.1,

$$P^{B(l)}_{t-\tau_{B(k)}} f(\mathcal{B}_{\tau_{B(k)}}) = \int_{B(R)} p^{B(l)}_{t-\tau_{B(k)}}(\mathcal{B}_{\tau_{B(k)}}, y) f(y)\, M(dy).$$

We have

$$\sup_{n\geq 1} \sup_{[\frac{1}{2},\infty)\times\mathbb{R}^2\times B(R)} p^{B(n)}_{\cdot}(\cdot,\cdot) \leq C_{13} < \infty$$

by Remark 5.5 and

$$\sup_{n\geq 1} \sup_{(0,\frac{1}{2})\times B(R+1)^c\times B(R)} p^{B(n)}_{\cdot}(\cdot,\cdot) \leq C_{14} < \infty$$

for some $C_{14} = C_{14}(X, \gamma, R) > 0$ by $\mathrm{dist}(B(R+1)^c, B(R)) \geq 1$ and Lemma 5.4. We see therefore from (5.15) that

$$0 \leq P^{B(l)}_t f(x) - P^{B(k)}_t f(x) \leq (C_{13} \vee C_{14}) P_x[\tau_{B(k)} < t] \int_{B(R)} f(y)\, M(dy),$$

and since $f \geq 0$ with $f|_{B(R)^c} = 0$ is arbitrary we obtain

$$0 \leq p^{B(l)}_t(x, y) - p^{B(k)}_t(x, y) \leq (C_{13} \vee C_{14}) P_x[\tau_{B(k)} < t] \tag{5.16}$$

for all $t \in (0, \infty)$, $x \in B(k)$ and $y \in B(R)$ by virtue of the continuity of $p_t^{B(l)}(x, \cdot)$ and $p_t^{B(k)}(x, \cdot)$ proved in the last subsection. Thus it follows from (5.16) and Lemma 5.6 that the limit $p_t(x, y) := \lim_{n\to\infty} p_t^{B(n)}(x, y) \in [0, \infty)$ exists and is continuous on $(0, \infty) \times \mathbb{R}^2 \times B(R)$. Since $R \geq 1$ is arbitrary and the relation $P_x[\mathcal{B}_t \in dy] = p_t(x, y)\, M(dy)$ can be obtained from that for $\mathcal{B}^{B(n)}$ and $p^{B(n)}$ by monotone convergence, statement (i) follows for the global heat kernel $p_t(x, y)$, i.e. for the case $U = \mathbb{R}^2$. For a general unbounded open set $U \subset \mathbb{R}^2$, statement (i) can be obtained by similar arguments and the fact that for any $k, l \in \mathbb{N}$ with $k < l$,

$$0 \leq p_t^{U\cap B(l)}(x, y) - p_t^{U\cap B(k)}(x, y) \leq p_t^{B(l)}(x, y) - p_t^{B(k)}(x, y), \qquad t > 0,\ x, y \in \mathbb{R}^2.$$

In order to see the latter inequality, notice that for $(x, y) \in \big((U \cap B(k)) \times (U \cap B(k))\big)^c$ this inequality holds trivially, and for $(x, y) \in (U \cap B(k)) \times (U \cap B(k))$,

$$\begin{aligned} &p_t^{B(l)}(x, y) - p_t^{B(k)}(x, y) - p_t^{U\cap B(l)}(x, y) + p_t^{U\cap B(k)}(x, y) \\ &= \lim_{r\downarrow 0} \frac{P_x\big[\mathcal{B}_t \in B(y, r),\ \tau_U \vee \tau_{B(k)} \leq t < \tau_{B(l)}\big]}{M(B(y, r))} \geq 0 \end{aligned}$$

by the continuity of the Dirichlet heat kernels on bounded open sets.

(ii) Let $x \in U$ and $t, \varepsilon > 0$. Since

$$\int_{\mathbb{R}^2} p_t(x, y)\, M(dy) = P_x[\mathcal{B}_t \in \mathbb{R}^2] = 1$$

by $P_x[\lim_{s\to\infty} F_s = \infty] = 1$, we can choose $n \in \mathbb{N}$ such that $x \in B(n)$ and

$$\int_{B(n)} p_t(x, y)\, M(dy) > 1 - \varepsilon.$$

Then by the continuity of p_t there exists $r > 0$ such that $B(x, r) \subset U$ and

$$\int_{B(n)} p_t(z, y)\, M(dy) > 1 - \varepsilon, \qquad \forall z \in B(x, r),$$

and hence

$$\int_{U\setminus B(n)} p_t^U(z, y)\, M(dy) \leq \int_{B(n)^c} p_t(z, y)\, M(dy) < \varepsilon, \qquad \forall z \in B(x, r). \quad (5.17)$$

Now, for any bounded Borel function $f : U \to \mathbb{R}$ and $z \in B(x, r)$, writing

$$P_t^U f(z) = \int_{U\setminus B(n)} p_t^U(z, y) f(y)\, M(dy) + \int_{U\cap B(n)} p_t^U(z, y) f(y)\, M(dy)$$

and applying (5.17), we obtain

$$\begin{aligned}
&\left|P_t^U f(x) - P_t^U f(z)\right| \\
&\quad \le 2\|f\|_\infty \varepsilon + \left|\int_{U\cap B(n)} p_t^U(x,y)f(y)\,M(dy) - \int_{U\cap B(n)} p_t^U(z,y)f(y)\,M(dy)\right| \\
&\quad \le (2\|f\|_\infty + 1)\varepsilon
\end{aligned}$$

provided $|x-z|$ is sufficiently small, which proves the continuity of $P_t^U f$ at x. In the last step we used the fact that, since $0 \le p_t^U \le p_t$ on $B(x,r)\times(U\cap B(n))$ where p_t is bounded and p_t^U is continuous, the function $z \mapsto \int_{U\cap B(n)} p_t^U(z,y)f(y)\,M(dy)$ is continuous on $B(x,r)$ by dominated convergence.

(iii) Since U is connected, for any $x, y \in U$ there exists a connected bounded open set $V \subset U$ with $x, y \in V$ and then by the corresponding result for bounded open sets we have $p_t^U(x,y) \ge p_t^V(x,y) > 0$ for any $t > 0$. □

Proof of Theorem 1.2 This is immediate from Lemma 5.4 since, as shown in the above proof, $p_t(x,y) = \lim_{n\to\infty} p_t^{B(n)}(x,y)$ for any $t > 0$ and $x, y \in \mathbb{R}^2$. □

6 On-diagonal lower bounds and spectral dimensions

In this section we prove the on-diagonal lower bound in Theorem 1.3. Indeed, we will show a more general result (Theorem 6.1 below) that also covers the Dirichlet Liouville heat kernels and thereby, in combination with Theorem 1.2, enables us to identify the pointwise and global spectral dimensions as 2. Recall that we have fixed an environment $\omega \in \Omega$ as declared at the beginning of Sect. 5.

Theorem 6.1 *For M-a.e. $x \in \mathbb{R}^2$, for any $\eta > 18$ and any open set $U \subset \mathbb{R}^2$ containing x there exist $C_{15} = C_{15}(X,\gamma,|x|,\eta) > 0$ and $t_0(x,U) = t_0(X,\gamma,\eta,x,U) \in \left(0,\frac{1}{2}\right]$ such that*

$$p_t^U(x,x) \ge C_{15} t^{-1}\left(\log(t^{-1})\right)^{-\eta}, \qquad \forall t \in (0, t_0(x,U)]. \tag{6.1}$$

In particular, Theorem 6.1 immediately implies Theorem 1.3 by choosing $U = \mathbb{R}^2$. Furthermore we can deduce the following result on pointwise spectral dimension.

Corollary 6.2 *For M-a.e. $x \in \mathbb{R}^2$, for any open set $U \subset \mathbb{R}^2$ containing x,*

$$\lim_{t\downarrow 0} \frac{2\log p_t^U(x,x)}{-\log t} = 2. \tag{6.2}$$

Proof This is immediate from the lower bound in Theorem 6.1 and the on-diagonal part of the upper bound in Theorem 1.2 together with $p_t^U(x,x) \le p_t(x,x)$. □

The proof of Theorem 6.1 is given in Sect. 6.1, and then the application to the identification of the global spectral dimension is presented in Sect. 6.2.

6.1 Proof of Theorem 6.1

In order to show Theorem 6.1 we need further moment and tail estimates on the exit times from balls. First, we recall the representation of the expected exit time in terms of the Green kernel.

Lemma 6.3 *For any non-empty open set $U \subset \mathbb{R}^2$ and any $x \in U$,*

$$E_x[\tau_U] = \int_U g_U(x, y)\, M(dy).$$

Proof This follows immediately from Proposition B.1. □

Lemma 6.4 *For any $R \geq 1$ there exist $c_1 = c_1(\gamma) > 0$ and $C_{16} = C_{16}(X, \gamma, R) > 0$ such that*

$$E_x[\tau_{B(x,r)}] \leq M(B(x,r))\big(C_{16} + c_1 \log(r^{-1})\big), \qquad \forall x \in B(R),\ r \in (0, 1].$$

Proof Since $g_{B(x,r)} \leq g_{B(R+2)}$ by $B(x,r) \subset B(R+2)$, we see from Lemma 6.3, (2.8) and $B(x,r) \subset B(R+1)$ that

$$\begin{aligned} E_x[\tau_{B(x,r)}] = \int_{B(x,r)} g_{B(x,r)}(x, y)\, M(dy) &\leq \int_{B(x,r)} g_{B(R+2)}(x, y)\, M(dy) \\ &\leq \int_{B(x,r)} \Big(\frac{1}{\pi} \log \frac{1}{|x-y|} + c\Big)\, M(dy) \end{aligned}$$

with $c = c(R) > 0$. Setting $D_n(x) := B(x, 2^{1-n}r) \setminus B(x, 2^{-n}r)$ for $n \geq 1$ and noting that $M(\{x\}) = 0$ by Lemma 3.1, we further obtain

$$E_x[\tau_{B(x,r)}] \leq cM(B(x,r)) + \frac{1}{\pi} \sum_{n=1}^{\infty} \big(n + \log(r^{-1})\big) M(D_n(x)). \tag{6.3}$$

On the other hand, Lemma 3.1 implies that for $\varepsilon := \alpha_2/2$,

$$\big(n + \log(r^{-1})\big) \frac{M\big(B(x, 2^{1-n}r)\big)}{M(B(x,r))} \leq Cn2^{-n(\alpha_2 - \varepsilon)} r^{\alpha_2 - \alpha_1 - 2\varepsilon} \leq C2^{-n\alpha_2/4}$$

provided $n \geq c \log(r^{-1})$ with $c = c(\gamma) > 0$, which together with (6.3) yields

$$\begin{aligned} E_x[\tau_{B(x,r)}] &\leq cM(B(x,r)) + C \sum\nolimits_{n \geq c \log(r^{-1})} 2^{-n\alpha_2/4} M(B(x,r)) \\ &\quad + cM\Big(\bigcup\nolimits_{1 \leq n < c \log(r^{-1})} D_n(x)\Big) \log(r^{-1}) \\ &\leq M(B(x,r))\big(C + c \log(r^{-1})\big), \end{aligned}$$

completing the proof. □

Lemma 6.5 *There exists a constant* $c_2 > 0$ *such that*

$$E_x[\tau_{B(x,r)}] \geq c_2 M(B(x, r/2)), \qquad \forall x \in \mathbb{R}^2,\ r > 0.$$

Proof By using Lemma 6.3 and the translation and scale invariance of the Green kernel

$$g_{B(x,r)}(x, y) = g_{B(0,r)}(0, y - x) = g_{B(0,\lambda r)}(0, \lambda(y - x))$$

for $x, y \in \mathbb{R}^2$, $r > 0$ and $\lambda > 0$ (see e.g. [14, Example 1.5.1]), we obtain

$$\begin{aligned} E_x[\tau_{B(x,r)}] &= \int_{B(x,r)} g_{B(x,r)}(x, y)\, M(dy) \\ &\geq \int_{B(x,r/2)} g_{B(0,1)}\big(0, r^{-1}(y - x)\big)\, M(dy) \geq c_2 M(B(x, r/2)) \end{aligned}$$

with $c_2 := \inf_{y \in B(0,1/2)} g_{B(0,1)}(0, y) > 0$, which is the claim. □

Proposition 6.6 *For any* $R \geq 1$ *there exists* $C_{17} = C_{17}(X, \gamma, R) > 0$ *such that*

$$P_x[\tau_{B(x,r)} \leq t] \leq 1 - C_{17} \frac{M(B(x, r/2))}{M(B(x, 3r)) \log(r^{-1})}$$

for all $x \in B(R)$, $r \in \big(0, \frac{1}{2}\big]$ *and* $0 < t \leq \frac{1}{2} E_x[\tau_{B(x,r)}]$.

Proof For any $t > 0$, by the obvious relation $\tau_{B(x,r)} \leq t + \mathbb{1}_{\{\tau_{B(x,r)} > t\}}(\tau_{B(x,r)} - t) = t + \mathbb{1}_{\{\tau_{B(x,r)} > t\}}(\tau_{B(x,r)} \circ \vartheta_t)$ and the Markov property [6, Theorem A.1.21] of $\mathcal{B}$,

$$\begin{aligned} E_x[\tau_{B(x,r)}] &\leq t + E_x\big[\mathbb{1}_{\{\tau_{B(x,r)} > t\}}(\tau_{B(x,r)} \circ \vartheta_t)\big] = t + E_x\big[\mathbb{1}_{\{\tau_{B(x,r)} > t\}} E_{\mathcal{B}_t}[\tau_{B(x,r)}]\big] \\ &\leq t + P_x[\tau_{B(x,r)} > t] \sup_{y \in B(x,r)} E_y[\tau_{B(x,r)}], \end{aligned}$$

which implies that for $0 < t \leq \frac{1}{2} E_x[\tau_{B(x,r)}]$,

$$P_x[\tau_{B(x,r)} \leq t] \leq 1 + \frac{t - E_x[\tau_{B(x,r)}]}{\sup_{y \in B(x,r)} E_y[\tau_{B(x,r)}]} \leq 1 - \frac{\frac{1}{2} E_x[\tau_{B(x,r)}]}{\sup_{y \in B(x,r)} E_y[\tau_{B(x,r)}]}. \tag{6.4}$$

Then since $B(x, r) \subset B(y, 2r) \subset B(x, 3r)$ and hence $\tau_{B(x,r)} \leq \tau_{B(y,2r)}$ for any $y \in B(x, r)$, from Lemma 6.4 we obtain

$$\begin{aligned} \sup_{y \in B(x,r)} E_y[\tau_{B(x,r)}] \leq \sup_{y \in B(x,r)} E_y[\tau_{B(y,2r)}] &\leq C \sup_{y \in B(x,r)} M(B(y, 2r)) \log(r^{-1}) \\ &\leq C M(B(x, 3r)) \log(r^{-1}), \end{aligned}$$

and the claim follows by applying this estimate and Lemma 6.5 to (6.4). □

We are now in the position to show an on-diagonal lower bound on the Dirichlet Liouville heat kernels.

Proposition 6.7 *For any $R \geq 1$ there exists $C_{18} = C_{18}(X, \gamma, R) > 0$ such that*

$$p_t^{B(x,r)}(x,x) \geq \frac{C_{18}}{M(B(x,r))} \left(\frac{M(B(x,r/2))}{M(B(x,3r)) \log(r^{-1})} \right)^2$$

for all $x \in B(R)$, $r \in \left(0, \frac{1}{2}\right]$ and $0 < t \leq c_2 M(B(x,r/2))$ (with c_2 as in Lemma 6.5).

Proof Let $0 < t \leq \frac{1}{2} c_2 M(B(x,r/2))$. Since $\frac{1}{2} E_x[\tau_{B(x,r)}] \geq \frac{1}{2} c_2 M(B(x,r/2)) \geq t$ by Lemma 6.5, we see from Proposition 6.6, the Cauchy–Schwarz inequality and the symmetry and semigroup property of the Dirichlet heat kernel $p^{B(x,r)}$ that

$$\begin{aligned}
&\left(C_{17} \frac{M(B(x,r/2))}{M(B(x,3r)) \log(r^{-1})} \right)^2 \leq P_x[\tau_{B(x,r)} > t]^2 \\
&\quad = P_x[\mathcal{B}_t \in B(x,r),\ \tau_{B(x,r)} > t]^2 = \left(\int_{B(x,r)} p_t^{B(x,r)}(x,y)\, M(dy) \right)^2 \\
&\quad \leq M(B(x,r)) \int_{B(x,r)} \left(p_t^{B(x,r)}(x,y) \right)^2 M(dy) = M(B(x,r))\, p_{2t}^{B(x,r)}(x,x),
\end{aligned}$$

which gives the result. □

Corollary 6.8 *Let $c_3 > 0$, $x \in \mathbb{R}^2$, $\eta > 18$ and set $\kappa := \frac{1}{8}(\eta - 2)$. If $r_0 \in \left(0, \frac{1}{2}\right]$ and*

$$M(B(x,2r)) \leq c_3 \left(\log(r^{-1}) \right)^{\kappa} M(B(x,r)), \qquad \forall r \in (0, r_0], \tag{6.5}$$

then for any open set $U \subset \mathbb{R}^2$ containing x there exist $C_{15} = C_{15}(X, \gamma, |x|, \eta, c_3) > 0$ and $t_0(x,U) = t_0(X, \gamma, x, U, r_0) \in \left(0, \frac{1}{2}\right]$ such that (6.1) *holds.*

Proof Let U be an open subset of $\mathbb{R}^2$ with $x \in U$ and let $r_1 = r_1(x, U, r_0) \in (0, r_0/2]$ be such that $B(x, r_1) \subset U$. Also, noting that $\lim_{r \downarrow 0} M(B(x,r)) = M(\{x\}) = 0$ by Lemma 3.1, for $0 < t \leq c_2 M(B(x, r_1/2))$ let $n = n(t) \geq 1$ be such that

$$c_2 M\left(B(x, 2^{-n-1} r_1)\right) < t \leq c_2 M\left(B(x, 2^{-n} r_1)\right)$$

and set $r = r(t) := 2^{1-n} r_1$. Then by Proposition 6.7,

$$\begin{aligned}
t p_t^U(x,x) &\geq t p_t^{B(x,r_1)}(x,x) \geq t p_t^{B(x,r)}(x,x) \\
&\geq C \frac{M(B(x,r/4))}{M(B(x,r))} \left(\frac{M(B(x,r/2))}{M(B(x,3r))} \right)^2 \left(\log(r^{-1}) \right)^{-2}.
\end{aligned} \tag{6.6}$$

On the other hand, we see from (6.5) that

$$\frac{M(B(x,r/4))}{M(B(x,r))} = \frac{M(B(x,r/4))}{M(B(x,r/2))} \frac{M(B(x,r/2))}{M(B(x,r))} \geq c \left(\log(r^{-1}) \right)^{-2\kappa}$$

and

$$\frac{M(B(x, r/2))}{M(B(x, 3r))} \geq \frac{M(B(x, r/2))}{M(B(x, 4r))} \geq c\big(\log(r^{-1})\big)^{-3\kappa}$$

with $c = c(c_3, \eta) > 0$. Now (6.1) follows by combining these estimates with (6.6) and noting that $c \log(t^{-1}) \leq \log(r^{-1}) = \log(r(t)^{-1}) \leq c' \log(t^{-1})$ with $c = c(\gamma) > 0$ and $c' = c'(\gamma) > 0$ provided $t \leq t_0'$ for some $t_0' = t_0'(X, \gamma, |x|) \in \big(0, \frac{1}{2}\big]$ by Lemma 3.1. □

Now Theorem 6.1 follows by Lemma 3.1, Corollary 6.8 and the following result.

Proposition 6.9 *Let μ be a Borel measure on $\mathbb{R}^2$ satisfying $\mu(B(x, r)) \in (0, \infty)$ for all $x \in \mathbb{R}^2$ and $r > 0$. Then for μ-a.e. $x \in \mathbb{R}^2$, for any $\kappa > 2$ there exists $r_0(x) = r_0(\mu, \kappa, x) \in \big(0, \frac{1}{2}\big]$ such that*

$$\mu(B(x, 2r)) \leq 8\big(\log(r^{-1})\big)^{\kappa} \mu(B(x, r)), \qquad \forall r \in (0, r_0(x)]. \tag{6.7}$$

Proof Since (6.7) is weaker for larger κ, it suffices to show (6.7) for μ-a.e. $x \in \mathbb{R}^2$ for each $\kappa \in \big(2, \frac{5}{2}\big]$. Fix an arbitrary $x_0 \in \mathbb{R}^2$. Set $r_k := 2^{-k}$ for $k \in \mathbb{Z}$, $\mu_{x_0} := \mu(\cdot \cap B(x_0, 1))$ and, for $n \in \mathbb{N}$,

$$A_n := \big\{x \in B(x_0, 1) : \mu(B(x, r_{n-1})) \geq n^{\kappa/2} \mu(B(x, r_n))\big\},$$
$$\Xi_n := \big\{x_0 + \big(\tfrac{k}{2^n}, \tfrac{l}{2^n}\big) : k, l \in \mathbb{Z}, |k|, |l| \leq 2^n\big\}.$$

Then since $B(x_0, 1) \subset \bigcup_{x \in \Xi_{n+1}} B(x, r_{n+1})$ and furthermore $B(x, r_{n+1}) \subset B(y, r_n) \subset B(y, r_{n-1}) \subset B(x, r_{n-2})$ for $x \in \Xi_{n+1}$ and $y \in B(x, r_{n+1})$, we have

$$\begin{aligned}
\int_{B(x_0,1)} \frac{\mu(B(y, r_{n-1}))}{\mu(B(y, r_n))} \mu_{x_0}(dy) &\leq \sum_{x \in \Xi_{n+1}} \int_{B(x, r_{n+1})} \frac{\mu(B(y, r_{n-1}))}{\mu(B(y, r_n))} \mu(dy) \\
&\leq \sum_{x \in \Xi_{n+1}} \int_{B(x, r_{n+1})} \frac{\mu(B(x, r_{n-2}))}{\mu(B(x, r_{n+1}))} \mu(dy) \\
&= \int_{\mathbb{R}^2} \sum_{x \in \Xi_{n+1}} \mathbb{1}_{B(x, r_{n-2})}(y) \, \mu(dy) \\
&\leq c\mu(B(x_0, 4))
\end{aligned}$$

for some $c > 0$. By Čebyšev's inequality this implies $\mu_{x_0}(A_n) \leq c\mu(B(x_0, 4))n^{-\kappa/2}$, hence $\sum_{n=1}^{\infty} \mu_{x_0}(A_n) < \infty$, and therefore by the Borel–Cantelli lemma for μ-a.e. $x \in B(x_0, 1)$ there exists $n_0(x) = n_0(\mu, \kappa, x) \in \mathbb{N}$ such that

$$\mu(B(x, r_{n-1})) \leq n^{\kappa/2} \mu(B(x, r_n)), \qquad \forall n \geq n_0(x). \tag{6.8}$$

Now let $x \in B(x_0, 1)$ satisfy (6.8), let $r \in (0, r_{n_0(x)}]$ and let $n \geq n_0(x)$ be such that $r_{n+1} < r \leq r_n$. Then by applying (6.8) twice,

$$\mu(B(x, 2r)) \leq \mu(B(x, r_{n-1})) \leq n^{\kappa/2}(n+1)^{\kappa/2}\mu(B(x, r_{n+1})) \leq 2^{3/2}n^{\kappa}\mu(B(x, r))$$

with $n \leq \frac{1}{\log 2}\log(r^{-1})$. Finally, since x_0 is arbitrary, the claim follows. □

6.2 Global spectral dimension

Let $U \subset \mathbb{R}^2$ be non-empty, open and bounded. As in Sect. 5.1 above, let $(\lambda_n(U))_{n\geq 1}$ be the eigenvalues of $-\mathcal{L}_U$ written in increasing order and repeated according to multiplicity, and define

$$Z_U(t) := \int_U p_t^U(x, x)\, M(dx) = \sum_{n=1}^{\infty} e^{-\lambda_n(U)t}, \qquad t > 0.$$

Then we obtain the following estimates of $Z_U(t)$ from Theorems 1.2 and 6.1 and conclude in particular that the global spectral dimension is 2.

Corollary 6.10 *Let $R \geq 1$ and let $U \subset B(R)$ be a non-empty open subset of $\mathbb{R}^2$. Then for any $\eta > 18$ there exist $C_{19} = C_{19}(X, \gamma, R) > 0$, $C_{20} = C_{20}(X, \gamma, R, \eta) > 0$ and $t_1(U) = t_1(X, \gamma, \eta, U) \in \left(0, \frac{1}{2}\right]$ such that*

$$Z_U(t) \leq C_{19}M(U)t^{-1}\log(t^{-1}), \qquad \forall t \in \left(0, \tfrac{1}{2}\right], \tag{6.9}$$

$$Z_U(t) \geq C_{20}M(U)t^{-1}\left(\log(t^{-1})\right)^{-\eta}, \qquad \forall t \in (0, t_1(U)]. \tag{6.10}$$

In particular,

$$\lim_{t\downarrow 0}\frac{2\log Z_U(t)}{-\log t} = 2. \tag{6.11}$$

Proof (6.11) is a direct consequence of (6.9) and (6.10), and (6.9) is immediate from the inequality $p_t^U(x, x) \leq p_t(x, x)$ and the on-diagonal part of the upper bound in Theorem 1.2. Thus it remains to verify (6.10). We may assume that $R = R(U) := \sup_{x\in U}|x|$. Let $\eta > 18$, let $C_{15} = C_{15}(X, \gamma, R, \eta) > 0$ be as in Theorem 6.1 and define an upper semi-continuous function $t_U : U \to \left[0, \frac{1}{2}\right]$ by

$$t_U(x) := \inf\left\{t \in \left(0, \tfrac{1}{2}\right] : t\left(\log(t^{-1})\right)^{\eta}p_t^U(x, x) < C_{15}\right\} \qquad \left(\inf\emptyset := \tfrac{1}{2}\right). \tag{6.12}$$

Then $t_U(x) > 0$ for M-a.e. $x \in U$ by Theorem 6.1 and therefore there exists $t_1 = t_1(X, \gamma, \eta, U) \in \left(0, \frac{1}{2}\right]$ such that $M\left(t_U^{-1}\left(\left[t_1, \frac{1}{2}\right]\right)\right) \geq \frac{1}{2}M(U)$. Now for each $t \in (0, t_1]$, $p_t^U(x, x) \geq C_{15}t^{-1}\left(\log(t^{-1})\right)^{-\eta}$ for any $x \in t_U^{-1}\left(\left[t_1, \frac{1}{2}\right]\right)$ by $t \leq t_1 \leq t_U(x)$

and (6.12), and hence

$$Z_U(t) \geq \int_{t_U^{-1}([t_1,\frac{1}{2}])} p_t^U(x,x)\,M(dx) \geq C_{15}t^{-1}\bigl(\log(t^{-1})\bigr)^{-\eta} M\bigl(t_U^{-1}\bigl([t_1,\tfrac{1}{2}]\bigr)\bigr)$$
$$\geq \tfrac{1}{2}C_{15}M(U)t^{-1}\bigl(\log(t^{-1})\bigr)^{-\eta},$$

proving (6.10). □

Remark 6.11 It is unknown to the authors whether the eigenvalue counting function $N_U(\lambda) := \#\{n \in \mathbb{N} : \lambda_n(U) \leq \lambda\}$ satisfies the counterparts of (6.9), (6.10) and (6.11).

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Appendix A: Proof of Proposition 2.4

The proof will be based on Lemma 4.1 and the following result proved in [17].

Theorem A.1 *For each $x \in \mathbb{R}^2$, $\mathbb{P} \times P_x$-a.s. the following hold:*

(i) *For all $t \geq 0$, $F_t := \lim_{n\to\infty} F_t^n$ exists in $\mathbb{R}$.*
(ii) *The mapping $[0,\infty) \ni t \mapsto F_t \in [0,\infty)$ is continuous, strictly increasing and satisfies $F_0 = 0$ and $\lim_{t\to\infty} F_t = \infty$.*

Proof See [17, Lemma 2.8 and Proof of Theorem 2.7]. □

We start with a preparatory lemma.

Lemma A.2 *$\mathbb{P}$-a.s., for all $x \in \mathbb{R}^2$,*

$$\lim_{t\downarrow 0}\liminf_{n\to\infty} F_t^n = 0 \qquad P_x\text{-a.s.}$$

Proof Fix any environment $\omega \in \Omega$ such that the conclusion of Lemma 4.1 holds, and let $x \in \mathbb{R}^2$. Then since F_t^n for $n \in \mathbb{N}$ are non-decreasing in t and hence so is $\liminf_{n\to\infty} F_t^n$, we see from Fatou's lemma and Lemma 4.1 that

$$0 \leq E_x\Bigl[\lim_{t\downarrow 0}\liminf_{n\to\infty} F_t^n\Bigr] \leq \lim_{t\downarrow 0} E_x\Bigl[\liminf_{n\to\infty} F_t^n\Bigr] \leq \lim_{t\downarrow 0}\liminf_{n\to\infty} E_x[F_t^n] = 0,$$

which implies that $\lim_{t\downarrow 0}\liminf_{n\to\infty} F_t^n = 0$ P_x-a.s. □

For each $t \geq 0$ we denote by Λ_t the set of all $(\omega,\omega') \in \Omega \times \Omega'$ such that:

(i) For all $u \in [t,\infty)$, $F_{t,u}(\omega,\omega') := \lim_{n\to\infty}\bigl(F_u^n(\omega,\omega') - F_t^n(\omega,\omega')\bigr)$ exists in $\mathbb{R}$.

(ii) The mapping $[t,\infty) \ni u \mapsto F_{t,u}(\omega,\omega') \in [0,\infty)$ is continuous, strictly increasing and satisfies $F_{t,t}(\omega,\omega') = 0$ and $\lim_{u\to\infty} F_{t,u}(\omega,\omega') = \infty$.

We also set $\Lambda_t^\omega := \{\omega' \in \Omega' : (\omega,\omega') \in \Lambda_t\}$ for $\omega \in \Omega$. Note that $\Lambda_t^\omega = \theta_t^{-1}(\Lambda_0^\omega)$ thanks to the fact that for all $n \in \mathbb{N}$ and $\omega' \in \Omega'$,

$$F_{s+t}^n(\omega,\omega') = F_t^n(\omega,\omega') + F_s^n(\omega,\theta_t(\omega')), \qquad \forall s,t \geq 0. \tag{A.1}$$

Furthermore we have $\Lambda_t \in \mathcal{A} \otimes \mathcal{G}_\infty^0$, since F_s^n is $\mathcal{A} \otimes \mathcal{G}_s^0$-measurable for any $n \in \mathbb{N}$ and $s \geq 0$ and Λ_t is easily seen to be equal to

$$\left\{ (\omega,\omega') \in \Omega \times \Omega' : \begin{array}{l} F_{t,s+t}(\omega,\omega') := \lim_{n\to\infty}\big(F_{s+t}^n(\omega,\omega') - F_t^n(\omega,\omega')\big) \text{ exists in } \mathbb{R} \\ \text{for all } s \in \mathbb{Q}\cap[0,\infty),\ \mathbb{Q}\cap[0,N] \ni s \mapsto F_{t,s+t}(\omega,\omega') \in [0,\infty) \text{ is} \\ \text{uniformly continuous and strictly increasing for any } N \in \mathbb{N}, \\ \lim_{\mathbb{Q}\ni s\to\infty} F_{t,s+t}(\omega,\omega') = \infty \end{array} \right\}$$

by virtue of the monotonicity of F_s^n in s. Finally, recall that $\mathbb{P} \times P_x[\Lambda_0] = 1$ for all $x \in \mathbb{R}^2$ by Theorem A.1.

Lemma A.3 *For $\mathbb{P}$-a.e. $\omega \in \Omega$, $P_x[\Lambda_t^\omega] = 1$ for all $t > 0$ and $x \in \mathbb{R}^2$.*

Proof Let $\mu(dy) := (2\pi)^{-1} e^{-|y|^2/2}\, dy$. By Fubini's theorem, we have $\mathbb{E}P_x[\Lambda_0^\omega] = \mathbb{P} \times P_x[\Lambda_0] = 1$ for all $x \in \mathbb{R}^2$ and then its $\mu(dx)$-integral results in $\mathbb{E}P_\mu[\Lambda_0^\omega] = 1$ with $P_\mu[\cdot] := \int_{\mathbb{R}^2} P_x[\cdot]\,\mu(dx)$. Thus for $\mathbb{P}$-a.e. $\omega \in \Omega$, $P_\mu[(\Lambda_0^\omega)^c] = 0$, namely

$$P_y[(\Lambda_0^\omega)^c] = 0 \qquad \text{for } dy\text{-a.e. } y \in \mathbb{R}^2. \tag{A.2}$$

Now for any such $\omega \in \Omega$ and for all $t > 0$ and $x \in \mathbb{R}^2$, we have

$$\begin{aligned} P_x[\Lambda_t^\omega] &= P_x\big[\theta_t^{-1}(\Lambda_0^\omega)\big] = E_x\big[\mathbb{1}_{\Lambda_0^\omega} \circ \theta_t\big] = E_x\big[P_{B_t}[\Lambda_0^\omega]\big] \\ &= \int_{\mathbb{R}^2} P_y[\Lambda_0^\omega] q_t(x,y)\,dy = 1 \end{aligned}$$

by the Markov property of B and (A.2), completing the proof. □

Proof of Proposition 2.4 Set $\mathbb{Q}_+ := \mathbb{Q} \cap (0,\infty)$ and

$$\Lambda := \Big\{(\omega,\omega') \in \Omega \times \Omega' : \lim_{t\downarrow 0} \liminf_{n\to\infty} F_t^n(\omega,\omega') = 0\Big\} \cap \bigcap_{q\in\mathbb{Q}_+} \Lambda_q.$$

Then clearly $\Lambda \in \mathcal{A} \otimes \mathcal{G}_\infty^0$, and (i) follows immediately from Lemmas A.2 and A.3.

Let $(\omega,\omega') \in \Lambda$. Then for each $q \in \mathbb{Q}_+$, $(\omega,\omega') \in \Lambda_q$, so that for all $t \in [q,\infty)$ the limit $F_{q,t}(\omega,\omega')$ exists in $\mathbb{R}$, $[q,\infty) \ni t \mapsto F_{q,t}(\omega,\omega') \in [0,\infty)$ is continuous and strictly increasing and $\lim_{t\to\infty} F_{q,t}(\omega,\omega') = \infty$. Thus for all $0 < s \leq t$ the limit

$$F_{s,t}(\omega,\omega') := \lim_{n\to\infty}\big(F_t^n(\omega,\omega') - F_s^n(\omega,\omega')\big) = F_{q,t}(\omega,\omega') - F_{q,s}(\omega,\omega') \tag{A.3}$$

exists in $\mathbb{R}$, where $q \in \mathbb{Q} \cap (0, s]$, and $[s, \infty) \ni t \mapsto F_{s,t}(\omega, \omega') \in [0, \infty)$ is a strictly increasing continuous function satisfying $\lim_{t\to\infty} F_{s,t}(\omega, \omega') = \infty$. Moreover, for any $t > 0$ and $0 < u \leq s \leq t$,

$$0 \leq F_{u,t}(\omega, \omega') - F_{s,t}(\omega, \omega') = \lim_{n\to\infty}\bigl(F_s^n(\omega, \omega') - F_u^n(\omega, \omega')\bigr) \leq \liminf_{n\to\infty} F_s^n(\omega, \omega'),$$

which tends to 0 as $s \downarrow 0$ and thereby verifies Cauchy's convergence criterion for $(F_{s,t}(\omega, \omega'))_{s\in(0,t]}$ as $s \downarrow 0$. Hence the finite limit $F_t(\omega, \omega') := \lim_{s\downarrow 0} F_{s,t}(\omega, \omega')$ exists, and then recalling (A.3), we easily obtain

$$0 \leq F_t(\omega, \omega') = \lim_{s\downarrow 0}\lim_{n\to\infty}\bigl(F_t^n(\omega, \omega') - F_s^n(\omega, \omega')\bigr) \leq \liminf_{n\to\infty} F_t^n(\omega, \omega') \xrightarrow{t\downarrow 0} 0 \tag{A.4}$$

and, for all $0 < s \leq t$,

$$F_t(\omega, \omega') - F_s(\omega, \omega') = \lim_{u\downarrow 0}\bigl(F_{u,t}(\omega, \omega') - F_{u,s}(\omega, \omega')\bigr) = F_{s,t}(\omega, \omega'). \tag{A.5}$$

Now by (A.4), (A.5) and the properties of the function $t \mapsto F_{s,t}(\omega, \omega')$ mentioned above after (A.3), the mapping $[0, \infty) \ni t \mapsto F_t(\omega, \omega') \in [0, \infty)$ with $F_0(\omega, \omega') := 0$ is continuous, strictly increasing and satisfies $\lim_{t\to\infty} F_t(\omega, \omega') = \infty$, proving (ii).

Statement (iii) is clear, so it remains to show (iv). Let $\omega \in \Omega$ satisfy the property in statement (i). First, $F_0(\omega, \cdot) = 0$ is $\mathcal{G}_0$-measurable, and for any $t > 0$, by (i) we have $\Lambda^\omega \in \mathcal{G}_0 \subset \mathcal{G}_t$, which together with the $\mathcal{G}_t^0$-measurability of $F_s^n(\omega, \cdot)$ for $n \in \mathbb{N}$ and $s \in [0, t]$ implies the $\mathcal{G}_t$-measurability of $F_t(\omega, \cdot)$. Next let $\omega' \in \Lambda^\omega$. (A.1) with $t = 0$ results in $F_s^n(\omega, \theta_0(\omega')) = F_s^n(\omega, \omega')$, $s \geq 0$, and then by $(\omega, \omega') \in \Lambda$ we easily see $\theta_0(\omega') \in \Lambda^\omega$ and $F_s(\omega, \omega') = F_0(\omega, \omega') + F_s(\omega, \theta_0(\omega'))$, $s \geq 0$. For $t > 0$, by (A.1), $(\omega, \omega') \in \Lambda$, (A.3) and (A.5) we have

$$\liminf_{n\to\infty} F_s^n(\omega, \theta_t(\omega')) = \lim_{n\to\infty}\bigl(F_{s+t}^n(\omega, \omega') - F_t^n(\omega, \omega')\bigr) = F_{t,s+t}(\omega, \omega') \xrightarrow{s\downarrow 0} 0$$

and, for any $s \geq 0$ and $u \in [s, \infty)$,

$$\begin{aligned} F_u^n(\omega, \theta_t(\omega')) - F_s^n(\omega, \theta_t(\omega')) &= F_{u+t}^n(\omega, \omega') - F_{s+t}^n(\omega, \omega') \\ &\xrightarrow{n\to\infty} F_{s+t,u+t}(\omega, \omega') = F_{u+t}(\omega, \omega') - F_{s+t}(\omega, \omega'), \end{aligned} \tag{A.6}$$

where the limit is a strictly increasing continuous function of $u \in [s, \infty)$ tending to ∞ as $u \to \infty$, proving in particular $(\omega, \theta_t(\omega')) \in \Lambda$, i.e. $\theta_t(\omega') \in \Lambda^\omega$. Finally, for $t, u > 0$ and $s \in (0, u]$, (A.6) shows $F_{s,u}(\omega, \theta_t(\omega')) = F_{u+t}(\omega, \omega') - F_{s+t}(\omega, \omega')$, and letting $s \downarrow 0$ yields $F_{u+t}(\omega, \omega') = F_t(\omega, \omega') + F_u(\omega, \theta_t(\omega'))$. Therefore $(F_t(\omega, \cdot))_{t\geq 0}$ is a PCAF of B in the strict sense with defining set Λ^ω. □

Appendix B: The Revuz correspondence between *M* and *F*

The purpose of this section is to give a proof of the following proposition, which generalises Proposition 2.5 to the LBM $\mathcal{B}^U$ killed upon exiting an open set $U \subset \mathbb{R}^2$.

Proposition B.1 *$\mathbb{P}$-a.s., for any non-empty open set $U \subset \mathbb{R}^2$, for all $x \in \mathbb{R}^2$ and all Borel measurable functions $\eta : [0,\infty) \to [0,\infty]$ and $f : U \to [0,\infty]$,*

$$E_x\left[\int_0^{T_U} \eta(t) f(B_t)\, dF_t\right] = \int_0^\infty \int_U \eta(t) f(y) q_t^U(x,y)\, M(dy)\, dt, \tag{B.1}$$

where $q_t^U(x,y)$ denotes the jointly continuous transition density of B^U as in (2.7).

We need to prepare a few preliminary facts. First, by [17, Theorem 2.2], $\mathbb{P}$-a.s., for any $\varepsilon > 0$ and any $R \geq 1$ there exists $C_{21} = C_{21}(X, \gamma, R, \varepsilon) > 0$ such that

$$M_n(B(x,r)) \leq C_{21} r^{\alpha_2 - \varepsilon}, \qquad \forall x \in B(R),\ r \in (0,1],\ n \in \mathbb{N}. \tag{B.2}$$

In the rest of this section, we fix any environment $\omega \in \Omega$ such that $(M_n)_{n\geq 1}$ converges to M vaguely on $\mathbb{R}^2$, the conclusions of Proposition 2.4 (i), (iv) hold and (B.2) is valid for all $\varepsilon > 0$ and $R \geq 1$. Then by Proposition 2.4 (i), (ii), for all $x \in \mathbb{R}^2$,

$$(dF_s^n)_{n\geq 1} \text{ converges to } dF_s \text{ weakly on } [t,u] \text{ for any } 0 < t \leq u, \quad P_x\text{-a.s.} \tag{B.3}$$

Lemma B.2 *For any non-empty open set $U \subset \mathbb{R}^2$, any $x \in \mathbb{R}^2$, any $t > 0$ and any bounded Borel measurable function $f : U \to [0,\infty)$ with $f^{-1}((0,\infty))$ bounded, $\left\{\int_0^{T_U\wedge t} f(B_s)\, dF_s^n\right\}_{n\geq 1}$ is uniformly P_x-integrable.*

Proof It suffices to prove that

$$\sup_{n\geq 1} E_x\left[\left(\int_0^{T_U\wedge t} f(B_s)\, dF_s^n\right)^2\right] < \infty. \tag{B.4}$$

For any Borel measurable $h : U \to [0,\infty]$, the Markov property of B yields

$$E_x\left[\left(\int_0^{T_U\wedge t} h(B_s)\, ds\right)^2\right] \leq 2\int_U \int_U h(y)h(z) \int_0^t \int_s^t q_s(x,y) q_{u-s}(y,z)\, du\, ds\, dz\, dy.$$

Then since

$$\begin{aligned}\int_0^t \int_s^t q_s(x,y) q_{u-s}(y,z)\, du\, ds &\leq \int_0^t q_s(x,y)\, ds \int_0^t q_u(y,z)\, du \\ &= \frac{1}{4\pi^2} \int_0^{t/|y-x|^2} s^{-1} e^{-\frac{1}{2s}}\, ds \int_0^{t/|z-y|^2} u^{-1} e^{-\frac{1}{2u}}\, du \\ &\leq \frac{1}{4\pi^2}\left(1+\log^+ \frac{t}{|y-x|^2}\right)\left(1+\log^+ \frac{t}{|z-y|^2}\right),\end{aligned}$$

where $\log^+ = \log(\cdot \vee 1)$, setting $h(y) := f(y)\exp\bigl(\gamma X_n(y) - \frac{\gamma^2}{2}\mathbb{E}\bigl[X_n(y)^2\bigr]\bigr)$, recalling (2.4) and (2.3) and choosing $R \geq 1$ such that $\{x\} \cup f^{-1}((0,\infty)) \subset B(R)$, we obtain

$$\begin{aligned} E_x\biggl[\biggl(\int_0^{T_U\wedge t} f(B_s)\,dF_s^n\biggr)^2\biggr] &= E_x\biggl[\biggl(\int_0^{T_U\wedge t} h(B_s)\,ds\biggr)^2\biggr] \\ &\leq \frac{1}{2\pi^2}\int_U\int_U h(y)h(z)\Bigl(1+\log^+\frac{t}{|y-x|^2}\Bigr)\Bigl(1+\log^+\frac{t}{|z-y|^2}\Bigr)\,dz\,dy \\ &\leq \frac{\|f\|_\infty^2}{2\pi^2}\int_{B(R)}\int_{B(R)}\Bigl(1+\log^+\frac{t}{|y-x|^2}\Bigr)\Bigl(1+\log^+\frac{t}{|z-y|^2}\Bigr)\,M_n(dz)\,M_n(dy). \end{aligned} \tag{B.5}$$

Using (B.2) with $\varepsilon = \alpha_2/2$, for all $y \in B(R)$ and $n \geq 1$ we further get

$$\begin{aligned} &\int_{B(R)}\Bigl(1+\log^+\frac{t}{|z-y|^2}\Bigr)\,M_n(dz) \\ &\quad\leq M_n(B(R)) + \sum_{k=0}^{\infty}\int_{B(y,2^{1-k}R)\setminus B(y,2^{-k}R)}\log^+\frac{t}{(2^{-k}R)^2}\,M_n(dz) \\ &\quad\leq C + C\sum_{k=0}^{\infty}\Bigl(2k+\log^+\frac{t}{R^2}\Bigr)(2^{1-k}R)^{\alpha_2/2} =: C'(X,\gamma,R,t) < \infty \end{aligned} \tag{B.6}$$

for some constant $C = C(X,\gamma,R) > 0$. (B.6) is in fact valid with $y = x$ by $x \in B(R)$, and then (B.4) is immediate from (B.5) and (B.6), completing the proof. □

Now we prove Proposition B.1 on the basis of (B.3), Lemma B.2 and the vague convergence on $\mathbb{R}^2$ of M_n to M.

Proof of Proposition B.1 By a monotone class argument it suffices to consider continuous functions η and f with compact supports in $(0,\infty)$ and U, respectively. First note that by (2.4), Fubini's theorem and (2.3) we have for every $n \in \mathbb{N}$,

$$E_x\biggl[\int_0^{T_U}\eta(t)f(B_t)\,dF_t^n\biggr] = \int_0^\infty\int_U \eta(t)f(y)q_t^U(x,y)\,M_n(dy)\,dt, \tag{B.7}$$

and we need to show that letting $n \to \infty$ on both sides of (B.7) results in (B.1). The left-hand side of (B.7) indeed converges to that of (B.1) by (B.3) and the uniform P_x-integrability of $\bigl\{\int_0^{T_U}\eta(t)f(B_t)\,dF_t^n\bigr\}_{n\geq 1}$ implied by Lemma B.2. On the other hand, the convergence of the right-hand side of (B.7) to that of (B.1) follows from the vague convergence on $\mathbb{R}^2$ of M_n to M together with the fact that the function $U \ni y \mapsto \int_0^\infty \eta(t)f(y)q_t^U(x,y)\,dt$ is continuous with compact support in U by virtue of dominated convergence using the continuity of $q_t^U(x,\cdot)$ on U and $0 \leq q_t^U(x,y) \leq q_t(x,y)$. Thus the proof of Proposition B.1 is complete. □

Appendix C: Negative moments of the Liouville measure

Lemma C.1 *Let $q > 0$ and set $\tilde{\xi}(q) := \big(2 + \frac{\gamma^2}{2}\big)q + \frac{\gamma^2}{2}q^2$. Then there exists $c_4 = c_4(\gamma, q) > 0$ such that for any $x \in \mathbb{R}^2$ and any $r \in (0, 1]$,*

$$\mathbb{E}\big[M(B(x,r))^{-q}\big] \vee \sup_{n\geq 1} \mathbb{E}\big[M_n(B(x,r))^{-q}\big] \leq c_4 r^{-\tilde{\xi}(q)}. \tag{C.1}$$

Proof Since the left-hand side of (C.1) is independent of $x \in \mathbb{R}^2$ by the translation invariance of the laws of M and M_n, $n \geq 1$, it suffices to show (C.1) for $x = 0$.

The proof is based on a comparison with the moment estimates established in [27], where the random Radon measure $M^0 = M^0_\gamma$ on $\mathbb{R}^2$ associated with the covariance function $\gamma^2 g^{(m)}$ has been constructed as follows. Note that $g^{(m)}$ can be written as $g^{(m)}(x, y) = h^{(m)}(x - y)$ with $h^{(m)} := g^{(m)}(\cdot, 0)$, which is easily seen from (2.1) to be of the form $h^{(m)}(x) = \log^+(|x|^{-1}) + \Psi^{(m)}(x)$ for some bounded continuous function $\Psi^{(m)} : \mathbb{R}^2 \to \mathbb{R}$. Define $\psi : \mathbb{R}^2 \to [0, \infty)$ by $\psi(x) := u * u(x) = \int_{\mathbb{R}^2} u(y)u(x - y)\, dy$ with $u(x) := \frac{3}{\pi}(1 - |x|)^+$, so that ψ is Lipschitz continuous, $\psi|_{B(0,2)^c} = 0$, $\int_{\mathbb{R}^2} \psi(x)\, dx = 1$ and it is positive definite, i.e. such that $(\psi(x - y))_{x,y\in\Xi}$ is a non-negative definite real symmetric matrix for any finite $\Xi \subset \mathbb{R}^2$. Now for each $\varepsilon > 0$, let X^0_ε be a continuous Gaussian field on $\mathbb{R}^2$ with mean 0 and covariance

$$\mathbb{E}\big[X^0_\varepsilon(x)X^0_\varepsilon(y)\big] = \psi_\varepsilon * h^{(m)}(x - y)$$

for $\psi_\varepsilon := \varepsilon^{-2}\psi(\varepsilon^{-1}(\cdot))$, where such X^0_ε can be constructed in exactly the same way as that described after (2.2) since $\psi_\varepsilon * h^{(m)}$ is easily shown to be positive definite and Lipschitz continuous. Then [27, Theorem 2.1] (see also [25, Theorem 3.2]) states that, as $\varepsilon \downarrow 0$, the associated random Radon measure $M^0_\varepsilon = M^0_{\gamma,\varepsilon}$ on $\mathbb{R}^2$ defined by

$$M^0_\varepsilon(dx) := \exp\Big(\gamma X^0_\varepsilon(x) - \frac{\gamma^2}{2}\mathbb{E}\big[X^0_\varepsilon(x)^2\big]\Big)\, dx$$

converges to some $M^0 = M^0_\gamma$ in law in the space $\mathcal{M}(\mathbb{R}^2)$ of Radon measures on $\mathbb{R}^2$ equipped with the topology of vague convergence, and M^0 satisfies the moment estimates as in (C.1) by [27, Proposition 3.7].

Returning to (2.2), for each $n \geq 1$ define $h^{(m)}_n := \sum_{k=1}^n g^{(m)}_k(\cdot, 0)$, which is the covariance kernel of $X_n = \sum_{k=1}^n Y_k$, and let $R \geq 1$ and $n \in \mathbb{N}$. Then $h^{(m)}_{n+1} - h^{(m)}_n$ is $(0, \infty)$-valued and continuous, $\lim_{\varepsilon\downarrow 0} \psi_\varepsilon * h^{(m)}_{n+1} = h^{(m)}_{n+1}$ uniformly on $\mathbb{R}^2$ by the uniform continuity of $h^{(m)}_{n+1}$ on $\mathbb{R}^2$, and $h^{(m)}_{n+1}(x) < h^{(m)}(x)$ for any $x \in \mathbb{R}^2$, so that there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$,

$$h^{(m)}_n(x) \leq \psi_\varepsilon * h^{(m)}_{n+1}(x) \leq \psi_\varepsilon * h^{(m)}(x), \qquad \forall x \in B(0, 2R). \tag{C.2}$$

Let $f : \mathbb{R}^2 \to [0, \infty)$ be continuous and satisfy $f|_{B(0,R)^c} = 0$ and let $\eta : [0, \infty) \to \mathbb{R}$ be bounded, continuous and convex. Also let $\varepsilon \in (0, \varepsilon_0]$. Then by (C.2), we can apply

Kahane's convexity inequality (see [25, Theorem 2.1] or [20]) to get

$$\mathbb{E}\Big[\eta\Big(\sum_{x\in k^{-1}\mathbb{Z}^2}\frac{f(x)}{k^2}e^{\gamma X_n(x)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\Big)\Big] \le \mathbb{E}\Big[\eta\Big(\sum_{x\in k^{-1}\mathbb{Z}^2}\frac{f(x)}{k^2}e^{\gamma X^0_\varepsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X^0_\varepsilon(x)^2]}\Big)\Big]$$

for all $k \in \mathbb{N}$, and by using dominated convergence to let $k \to \infty$, we obtain

$$\mathbb{E}\Big[\eta\Big(\int_{\mathbb{R}^2} f(x)e^{\gamma X_n(x)-\frac{\gamma^2}{2}\mathbb{E}[X_n(x)^2]}\,dx\Big)\Big] \le \mathbb{E}\Big[\eta\Big(\int_{\mathbb{R}^2} f(x)e^{\gamma X^0_\varepsilon(x)-\frac{\gamma^2}{2}\mathbb{E}[X^0_\varepsilon(x)^2]}\,dx\Big)\Big],$$

which means that $\mathbb{E}\big[\eta\big(\Phi_f(M_n)\big)\big] \le \mathbb{E}\big[\eta\big(\Phi_f(M^0_\varepsilon)\big)\big]$ for the continuous function $\Phi_f : \mathcal{M}(\mathbb{R}^2) \to [0,\infty)$ given by $\Phi_f(\mu) := \int_{\mathbb{R}^2} f\,d\mu$. Now since M^0_ε converges in law to M^0 as $\varepsilon \downarrow 0$ and $\eta\circ\Phi_f : \mathcal{M}(\mathbb{R}^2) \to \mathbb{R}$ is bounded and continuous, letting $\varepsilon \downarrow 0$ yields

$$\mathbb{E}\big[\eta\big(\Phi_f(M_n)\big)\big] \le \lim_{\varepsilon\downarrow 0}\mathbb{E}\big[\eta\big(\Phi_f(M^0_\varepsilon)\big)\big] = \mathbb{E}\big[\eta\big(\Phi_f(M^0)\big)\big], \qquad \forall n \in \mathbb{N}, \tag{C.3}$$

whose limit as $n \to \infty$ results in

$$\mathbb{E}\big[\eta\big(\Phi_f(M)\big)\big] \le \mathbb{E}\big[\eta\big(\Phi_f(M^0)\big)\big] \tag{C.4}$$

by dominated convergence together with the fact that $\lim_{n\to\infty} M_n = M$ in $\mathcal{M}(\mathbb{R}^2)$ $\mathbb{P}$-a.s. Finally, letting $\eta(t) = \frac{1}{\Gamma(q)}\lambda^{q-1}e^{-\lambda t}$ with $\lambda > 0$ and taking the $d\lambda$-integrals on $(0,\infty)$ in (C.3) and (C.4), by $\frac{1}{\Gamma(q)}\int_0^\infty \lambda^{q-1}e^{-\lambda t}\,d\lambda = t^{-q}$ we conclude that

$$\mathbb{E}\big[\Phi_f(M)^{-q}\big] \vee \sup_{n\ge 1}\mathbb{E}\big[\Phi_f(M_n)^{-q}\big] \le \mathbb{E}\big[\Phi_f(M^0)^{-q}\big], \tag{C.5}$$

and (C.1) for $x = 0$ follows from (C.5) with $f(y) = (2 - 2|y|/r)^+ \wedge 1$ and the corresponding bound for $\mathbb{E}\big[M^0(B(0,r/2))^{-q}\big]$ implied by [27, Proposition 3.7]. □

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