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Affine Lie algebras and conditioned space-time Brownian motions in affine Weyl chambers

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Abstract We construct a sequence of Markov processes on the set of dominant weights of an affine Lie algebra $\mathfrak{g}$ considering tensor product of irreducible highest weight modules of $\mathfrak{g}$ and specializations of the characters involving the Weyl vector ρ. We show that it converges towards a space-time Brownian motion with a drift, conditioned to remain in a Weyl chamber associated to the root system of $\mathfrak{g}$. This extends in particular the results of Defosseux (arXiv:1401.3115, 2014) to any affine Lie algebras, in the case with a drift.

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1 Introduction

In [2] we have studied a conditioned space-time Brownian motion which appears naturally in the framework of representation theory of the affine Lie algebra $\hat{\mathfrak{sl}}_2$: a space-time Brownian motion $(t, B_t)_{t\geq 0}$ conditioned (in Doob's sense) to remain in a moving boundary domain

$$D = \{(r, z) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 < z < r\},$$

which can be seen as the Weyl chamber associated to the root system of the affine Lie algebra $\hat{\mathfrak{sl}}_2$. The present paper deals with the case of any affine Lie algebras. Let us briefly describe the framework of the paper. First we need an affine Lie algebra $\mathfrak{g}$. As in the finite dimensional case, for a dominant integral weight λ of $\mathfrak{g}$ one defines

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the character of an irreducible highest-weight representation $V(\lambda)$ of $\mathfrak{g}$ with highest weight λ, as a formal series defined for h in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ by

$$\mathrm{ch}_\lambda(h) = \sum_\mu \dim(V(\lambda)_\mu) e^{\langle \mu, h\rangle},$$

where $V(\lambda)_\mu$ is the weight space of $V(\lambda)$ corresponding to the weight μ. This formal series converges for every h in a subset of the Cartan subalgebra which doesn't depend on λ. A particular choice of an element $h \in \mathfrak{h}$ in the region of convergence of the characters is called a specialization. Let us fix a dominant weight ω once for all. For a dominant weight λ, the following decomposition

$$\mathrm{ch}_\omega \mathrm{ch}_\lambda = \sum_{\beta \in P_+} M_\lambda(\beta) \mathrm{ch}_\beta,$$

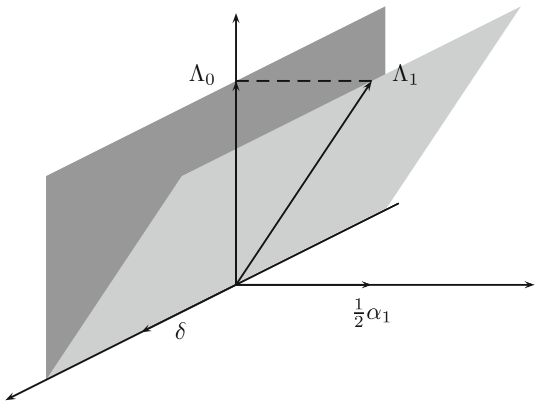
where $M_\lambda(\beta)$ is the multiplicity of the module with highest weight β in the decomposition of $V(\omega) \otimes V(\lambda)$, allows to define a transition probability Q_ω on the set of dominant weights, letting for β and λ two dominant weights of $\mathfrak{g}$,

$$Q_\omega(\lambda, \beta) = \frac{\mathrm{ch}_\beta(h)}{\mathrm{ch}_\lambda(h)\mathrm{ch}_\omega(h)} M_\lambda(\beta), \tag{1}$$

where h is chosen in the region of convergence of the characters. Such a Markov chain has been recently considered by C. Lecouvey, E. Lesigne, and M. Peigné in [4].

It is a natural question to ask if there exists a sequence $(h_n)_n$ of elements of $\mathfrak{h}$ such that the corresponding sequence of Markov chains converges towards a continuous process and what the limit is. One could show that there are basically three cases depending on the scaling factor. Roughly speaking the three cases are the following. When the scaling factor is $n^{-\alpha}$, with $\alpha \in (0, 1)$ (resp. $\alpha > 1$), the limiting process has to do with a Brownian motion conditioned—in Doob's sense—to remain in a Weyl chamber (resp. an alcove) associated to the root system of an underlying finite dimensional Lie algebra. When $\alpha = 1$, the limiting process has to do with a space time Brownian motion conditioned to remain in a Weyl chamber associated to the root system of the affine Lie algebra. Figure 1 below illustrates three distinct asymptotic behaviors in the case when the affine Lie algebra is $\hat{\mathfrak{sl}}_2$. The Weyl chamber $\mathcal{C}$ is the area delimited by gray and light gray half-planes. Essentially, when the scaling factor is $n^{-\alpha}$ with $\alpha \in (0, 1)$, one could show that the Λ_0-component of the limiting process is $+\infty$ and that its projection on $\mathbb{R}\alpha_1$ is a Brownian motion conditioned to remain positive. When the scaling factor is $n^{-\alpha}$ with $\alpha > 1$, one could show that the projection of the limiting process on $\mathbb{R}_+\Lambda_0 + \mathbb{R}\alpha_1$ lives in an interval (dashed interval within Fig. 1) and that its projection on $\mathbb{R}\alpha_1$ is a Brownian motion conditioned to remain in an interval. When the scaling factor is n^{-1}, the projection of the limiting process on $\mathbb{R}_+\Lambda_0 + \mathbb{R}\alpha_1$ is a space-time Brownian motion conditioned to remain in $\mathcal{C}$, the time axis being $\mathbb{R}_+\Lambda_0$ and the space axis being $\mathbb{R}\alpha_1$. This is this last case which is considered in the paper, for any affine Lie algebras. The convergence for the other values of α could be obtained with similar arguments as the ones developed in this

Fig. 1 The affine Weyl Chamber corresponding to $A_1^{(1)}$



paper. Nevertheless the case when $\alpha = 1$ seems the most interesting case in our context as the limiting process in this case, is the only one that is really specific to the affine framework. Thus we prefer to focus on this case. In this way we lose in generality but hope to win in clarity.

The paper is organized as follows. In Sect. 2 we describe the conditioned process occuring in our setting when representations of affine Lie algebra $\hat{\mathfrak{sl}}_2$ are considered. This is a space-time Brownian motion with a positive drift conditioned (in Doob's sense) to remain forever in the time-dependent domain D. We show by purely probabilistic arguments that the theta functions play a crucial role in the construction of the process, which is unlighted by the algebraic point of view developed in the following sections. The vocation of this section is to give an idea of the probabilistic aspects of mathematical objects occuring in the paper. In Sect. 3 we briefly recall the necessary background on representation theory of affine Lie algebras. We introduce in Sect. 4 random walks on the set of integral weights of an affine Lie algebra $\mathfrak{g}$, and Markov chains on the set of its dominant integral weights, considering tensor products of irreducible highest weight representations of $\mathfrak{g}$. We show that the Weyl character formula implies that they satisfy a reflection principle. In Sect. 5 we consider a sequence of random walks obtained for particular specializations involving the Weyl vector ρ of the affine Lie algebra, and prove that its scaling limit is a space-time standard Brownian motion with drift ρ, living on the Cartan subalgebra of $\mathfrak{g}$. We introduce in Sect. 6 a space-time Brownian motion with drift ρ, conditioned to remain in an affine Weyl chamber and prove that it satisfies a reflection principle. We prove in Sect. 7 that this conditioned space-time Brownian motion is the scaling limit of a sequence of Markov processes constructed in Sect. 4 for particular specializations involving ρ.

2 A moving boundary problem

Let $\gamma \in \mathbb{R}$, and $(X_t, t \geq 0) = ((\tau_t, B_t^\gamma), t \geq 0)$ be a space-time Brownian motion. For $(u, x) \in \mathbb{R} \times \mathbb{R}$, $\mathbb{P}_{(u,x)}$ denotes a probability under which $(B_t^\gamma)_{t\geq 0}$ is a standard Brownian motion with drift γ, starting from x, and $\tau_t = u + t$, for all $t \geq 0$. Consider the subset D of $\mathbb{R}^2$ defined by

$$D = \{(r, z) \in \mathbb{R}_+ \times \mathbb{R}_+ : 0 < z < r\},$$

and consider an application h defined on the closure $\bar{D}$ of D by

$$h(u, x) := \mathbb{P}_{(u,x)}(\forall t \geq 0, 0 < B_t^\gamma < \tau_t),$$

$(u, x) \in \bar{D}$. When $\gamma \in (0, 1)$, a classical martingale argument shows that the function h is the unique bounded harmonic positive function for the space-time Brownian motion killed on the boundary ∂D, i.e.

$$\forall (t, x) \in D, \quad \left(\frac{1}{2}\partial_{xx} + \gamma \partial_x + \partial_t\right) h(t, x) = 0,$$

which satisfies the following boundary conditions

$$\forall t \geq 0, \quad h(t, 0) = h(t, t) = 0,$$

and the condition at infinity

$$\lim_{\substack{(t,x) \to +\infty : \\ \frac{x}{t} \to \gamma}} h(t, x) = 1.$$

Such a problem is usually referred to as a moving boundary problem (see for instance [1] for a review of various problems specifically related to time-dependent boundaries). Actually the function h can be determined using a reflection principle involving the group of tranformations W generated by linear transformations s_k, $k \in \mathbb{Z}$, defined on $\mathbb{R}^2$ by

$$s_k(t, x) = (t, 2kt - x), \quad (t, x) \in \mathbb{R}^2.$$

Let us explain how. For $k \in \mathbb{Z}$, define t_k as the transformation on $\mathbb{R}^2$ given by

$$t_k(t, x) = (t, 2kt + x),$$

$(t, x) \in \mathbb{R}^2$. The group W is actually a semi-direct product

$$\{\mathrm{Id}, s_0\} \ltimes \{t_k, k \in \mathbb{Z}\},$$

and $\bar{D}$ is a fundamental domain for the action of W on $\mathbb{R}^2$. The following proposition is immediate.

Proposition 2.1 *For $(u, x), (u + t, y) \in \mathbb{R}^2$,*

$$\mathbb{P}_{s_0(u,x)}(X_t = s_0(u + t, y)) = e^{-2\gamma(y-x)}\mathbb{P}_{(u,x)}(X_t = (u + t, y))$$
$$\mathbb{P}_{t_k(u,x)}(X_t = t_k(u + t, y)) = e^{-2k(y-x)-2k^2t+2k\gamma t}\mathbb{P}_{(u,x)}(X_t = (u + t, y)),$$

for $k \in \mathbb{Z}$, where $\mathbb{P}_{(u,x)}(X_t = (u + t, y))$ stands, by a usual abuse of notation, for the probability semi-group of $(X_t)_{t\geq 0}$.

The probability semi-group of the space-time Brownian motion killed on the boundary of D satisfies a reflection principle. Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$ and $(.,.)$ be the usual inner product on $\mathbb{R}^2$. Let T denote the first exit time of D. The reflection principle is the following.

Proposition 2.2

$$\mathbb{P}_{(u,x)}(X_t = (u+t, y),\ T \geq t) = e^{-\gamma x} \sum \det(r) e^{-2k^2u - 2kx + (\gamma r t_k(u,x), e_1)} \times \mathbb{P}_{rt_k(u,x)}(X_t = (u+t, y)),$$

where the sum runs over $r \in \{Id, s_0\}$, $k \in \mathbb{Z}$.

Proof As $X_T \in \bar{D}$, one obtains using a strong Markov property and Proposition 2.1 that for $(u,x), (u+t, y) \in D$,

$$e^{-\gamma x} \sum_{r \in \{\mathrm{Id}, s_0\}, k \in \mathbb{Z}} \det(r) e^{-2k^2u - 2kx + (\gamma r t_k(u,x), e_1)} \mathbb{P}_{rt_k(u,x)}(X_t = (u+t, y), T \leq t)$$

equals 0. Moreover

$$e^{-\gamma x} \sum_{r \in \{\mathrm{Id}, s_0\}, k \in \mathbb{Z}} \det(r) e^{-2k^2u - 2kx + (\gamma r t_k(u,x), e_1)} \mathbb{P}_{rt_k(u,x)}(X_t = (u+t, y), T > t)$$

equals

$$\mathbb{P}_{(u,x)}(X_t = (u+t, y),\ T \geq t).$$

Proposition follows by summing the two identities. □

One obtains for the function h the following expression.

Proposition 2.3 *For* $(u,x) \in D$,

$$h(u,x) = 2 \sum_{k \in \mathbb{Z}} sh(\gamma(x + 2ku)) e^{-2(kx + k^2u) - \gamma x}.$$

Proof Summing over y such that $(t+u, y) \in D$ in Proposition 2.2 and letting t go to infinity gives the proposition. □

Actually W can be identified with the Weyl group associated to an affine Lie algebra $\hat{\mathfrak{sl}}_2$. Writing $X_t = \tau_t \Lambda_0 + B_t^\gamma \frac{\alpha_1}{2}$, $t \geq 0$, where Λ_0 and α_1 are defined below, the Doob's h-transform of $(X_t)_{t \geq 0}$ is a Markov process conditioned to remain in a Weyl chamber associated to the root system of the affine Lie algebra $\hat{\mathfrak{sl}}_2$. The following sections extend this construction to any affine Lie algebras and relate identities from Propostions 2.2 and 2.3, which are particular cases of Propositions 6.4 and 6.1, to representations theory of affine Lie algebras.

3 Affine Lie algebras and their representations

In order to make the reading more pleasant, we have tried to emphasize only on definitions and properties that we need for our purpose. For more details, we refer the reader to [3], which is our main reference for the whole paper.

3.1 Affine Lie algebras

The following definitions mainly come from chapters 1 and 6 of [3]. Let $A = (a_{i,j})_{0\le i,j\le l}$ be a generalized Cartan matrix of affine type. That is all the proper principal minors of A are positive and $\det A = 0$. Suppose that rows and columns of A are ordered such that $\det \mathring{A} \neq 0$, where $\mathring{A} = (a_{i,j})_{1\le i,j\le l}$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of A with $\Pi = \{\alpha_0, \dots, \alpha_l\} \subset \mathfrak{h}^*$ the set of simple roots, $\Pi^\vee = \{\alpha_0^\vee, \dots, \alpha_l^\vee\} \subset \mathfrak{h}$, the set of simple coroots, which satisfy the following condition

$$\alpha_j(\alpha_i^\vee) = a_{i,j},\ i, j \in \{0, \dots, l\}.$$

Let us consider the affine Lie algebra $\mathfrak{g}$ with generators $e_i, f_i, i = 0, \dots, l$, $\mathfrak{h}$ and the following defining relations:

$$\begin{aligned}
&[e_i, f_i] = \delta_{ij}\alpha_i^\vee, \quad [h, e_i] = \alpha_i(h)e_i, \quad [h, f_i] = -\alpha_i(h)f_i,\\
&[h, h'] = 0,\ \text{for } h, h' \in \mathfrak{h},\\
&(\mathrm{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\mathrm{ad} f_i)^{1-a_{ij}} f_j = 0,
\end{aligned}$$

for all $i, j = 0, \dots, l$. Let Δ (resp. Δ_+) denote the set of roots (resp. positive roots) of $\mathfrak{g}$, Q and $Q^\vee$ the root and the coroot lattices. We denote a_i, $i, \dots, l$ the labels of the Dynkin diagram of A and $a_i^\vee$, $i = 0, \dots, l$ the labels of the Dynkin diagram of ${}^tA$. The numbers

$$h = \sum_{i=0}^{l} a_i \quad \textit{and} \quad h^\vee = \sum_{i=0}^{l} a_i^\vee,$$

are called, respectively, the Coxeter number and the dual Coxeter number. The element

$$K = \sum_{i=0}^{n} a_i^\vee \alpha_i^\vee,$$

is called the canonical central element. The element δ defined by

$$\delta = \sum_{i=0}^{n} a_i \alpha_i,$$

is the smallest positive imaginary root. Fix an element $d \in \mathfrak{h}$ which satisfies the following condition

$$\alpha_i(d) = 0, \text{ for } i = 1, \dots, l, \quad \alpha_0(d) = 1.$$

The elements $\alpha_0^\vee, \dots, \alpha_l^\vee, d$, form a basis of $\mathfrak{h}$. We denote $\mathfrak{h}_\mathbb{R}$ the linear span over $\mathbb{R}$ of $\alpha_0^\vee, \dots, \alpha_l^\vee, d$. We define a nondegenerate symmetric bilinear $\mathbb{C}$-valued form $(.|.)$ on $\mathfrak{h}$ as follows

$$\begin{cases} (\alpha_i^\vee|\alpha_j^\vee) = \frac{a_j}{a_j^\vee} a_{ij} & i, j = 0, \dots, l \\ (\alpha_i^\vee|d) = 0 & i = 1, \dots, l \\ (\alpha_0^\vee|d) = a_0 & (d|d) = 0. \end{cases}$$

We define an element $\Lambda_0 \in \mathfrak{h}^*$ by

$$\Lambda_0(\alpha_i^\vee) = \delta_{0i}, \quad i = 0, \dots, l; \quad \Lambda_0(d) = 0.$$

The linear isomorphism

$$\begin{aligned} \nu : \mathfrak{h} &\to \mathfrak{h}^*, \\ h &\mapsto (h|.) \end{aligned}$$

identifies $\mathfrak{h}$ and $\mathfrak{h}^*$. We still denote $(.|.)$ the induced inner product on $\mathfrak{h}^*$. We record that

$$\begin{aligned} &(\delta|\alpha_i) = 0, \quad i = 0, \dots, l, \quad (\delta|\delta) = 0, \quad (\delta|\Lambda_0) = 1 \\ &(K|\alpha_i) = 0, \quad i = 0, \dots, l, \quad (K|K) = 0, \quad (K|d) = a_0. \end{aligned}$$

The form $(.|.)$ is W-invariant, for W the Weyl group of the affine Lie algebra $\mathfrak{g}$, i.e. the subgroup of $GL(\mathfrak{h}^*)$ generated by fundamental reflections s_α, $\alpha \in \Pi$, defined by

$$s_\alpha(\beta) = \beta - \beta(\alpha^\vee)\alpha, \quad \beta \in \mathfrak{h}^*.$$

We denote $\mathring{\mathfrak{h}}$ (resp. $\mathring{\mathfrak{h}}_\mathbb{R}$) the linear span over $\mathbb{C}$ (resp. $\mathbb{R}$) of $\alpha_1^\vee, \dots, \alpha_l^\vee$. The dual notions $\mathring{\mathfrak{h}}^*$ and $\mathring{\mathfrak{h}}_\mathbb{R}^*$ are defined similarly. Then we have an orthogonal direct sum of subspaces:

$$\mathfrak{h} = \mathring{\mathfrak{h}}_\mathbb{R} \oplus (\mathbb{C}K + \mathbb{C}d); \quad \mathfrak{h}^* = \mathring{\mathfrak{h}}_\mathbb{R}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0).$$

We set $\mathfrak{h}_\mathbb{R} = \mathring{\mathfrak{h}}_\mathbb{R} + \mathbb{R}K + \mathbb{R}d$, and $\mathfrak{h}_\mathbb{R}^* = \mathring{\mathfrak{h}}_\mathbb{R}^* + \mathbb{R}\delta + \mathbb{R}\Lambda_0$.

Notation For $\lambda \in \mathfrak{h}^*$ such that $\lambda = a\Lambda_0 + z + b\delta$, $a, b \in \mathbb{C}$, $z \in \mathring{\mathfrak{h}}_\mathbb{R}^*$, denote $\bar{\lambda}$ the projection of λ on $\mathbb{C}\Lambda_0 + \mathring{\mathfrak{h}}^*$ defined by $\bar{\lambda} = a\Lambda_0 + z$, and by $\bar{\bar{\lambda}}$ its projection on $\mathring{\mathfrak{h}}^*$ defined by $\bar{\bar{\lambda}} = z$.

We denote $\mathring{W}$ the subgroup of $GL(\mathfrak{h}^*)$ generated by fundamental reflections s_{α_i}, $i = 1, \ldots, l$. Let $\mathbb{Z}(\mathring{W}.\theta^\vee)$ denote the lattice in $\mathring{\mathfrak{h}}_\mathbb{R}$ spanned over $\mathbb{Z}$ by the set $\mathring{W}.\theta^\vee$, where

$$\theta^\vee = \sum_{i=1}^{l} a_i^\vee \alpha_i^\vee,$$

and set $M = \nu(\mathbb{Z}(\mathring{W}.\theta^\vee))$. Then W is the semi-direct product $T \ltimes \mathring{W}$ (Proposition 6.5 chapter 6 of [3]) where T is the group of transformations t_α, $\alpha \in M$, defined by

$$t_\alpha(\lambda) = \lambda + \lambda(K)\alpha - \left((\lambda|\alpha) + \frac{1}{2}(\alpha|\alpha)\lambda(K)\right)\delta, \quad \lambda \in \mathfrak{h}^*.$$

3.2 Weights, highest-weight modules, characters

The following definitions and properties mainly come from chapter 9 and 10 of [3]. We denote P (resp. P_+) the set of integral (resp. dominant) weights defined by

$$P = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z},\ i = 0, \ldots, l\},$$
$$(\text{resp. } P_+ = \{\lambda \in P : \langle \lambda, \alpha_i^\vee \rangle \geq 0,\ i = 0, \ldots, l\}),$$

where $\langle ., . \rangle$ is the pairing between $\mathfrak{h}$ and its dual $\mathfrak{h}^*$. The level of an integral weight $\lambda \in P$, is defined as the integer $(\delta|\lambda)$. For $k \in \mathbb{N}$, we denote P^k (resp. P_+^k) the set of integral (resp. dominant) weights of level k defined by

$$P^k = \{\lambda \in P : (\delta|\lambda) = k\}.$$
$$(\text{resp. } P_+^k = \{\lambda \in P_+ : (\delta|\lambda) = k\}.)$$

Recall that a $\mathfrak{g}$-module V is called $\mathfrak{h}$-diagonalizable if it admits a weight space decomposition $V = \oplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ by weight spaces V_λ defined by

$$V_\lambda = \{v \in V : \forall h \in \mathfrak{h},\ h.v = \lambda(h)v\}.$$

The category $\mathcal{O}$ is defined as the set of $\mathfrak{g}$-modules V which are $\mathfrak{h}$-diagonalizable with finite dimensional weight spaces and such that there exists a finite number of elements $\lambda_1, \ldots, \lambda_s \in \mathfrak{h}^*$ such that

$$P(V) \subset \cup_{i=1}^{s} \{\mu \in \mathfrak{h}^* : \lambda_i - \mu \in \mathbb{N}\Delta_+\},$$

where $P(V) = \{\lambda \in \mathfrak{h}^* : V_\lambda \neq \{0\}\}$. One defines the formal character $\mathrm{ch}(V)$ of a module V from $\mathcal{O}$ by

$$\mathrm{ch}(V) = \sum_{\mu \in P(V)} \dim(V_\mu) e^\mu.$$

For $\lambda \in P_+$ we denote $V(\lambda)$ the irreducible module with highest weight λ. It belongs to the category $\mathcal{O}$. The Weyl character's formula (Theorem 10.4, chapter 10 of [3]) states that

$$\mathrm{ch}(V(\lambda)) = \frac{\sum_{w\in W} \det(w) e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)}}, \tag{2}$$

where $\mathrm{mult}(\alpha)$ is the dimension of the root space $\mathfrak{g}_\alpha$ defined by

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : \forall h \in \mathfrak{h},\ [h,x] = \alpha(h)x\},$$

for $\alpha \in \Delta$ and $\rho \in \mathfrak{h}^*$ is chosen such that $\rho(\alpha_i^\vee) = 1$, for all $i \in \{0,\dots,l\}$. In particular

$$\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{\mathrm{mult}(\alpha)} = \sum_{w\in W} \det(w) e^{w(\rho)-\rho}. \tag{3}$$

Letting $e^\mu(h) = e^{\mu(h)}$, $h \in \mathfrak{h}$, the formal character $\mathrm{ch}(V(\lambda))$ can be seen as a function defined on its region of convergence. Actually the series

$$\sum_{\mu\in P} \dim(V(\lambda)_\mu) e^{\langle \mu, h\rangle}$$

converges absolutely for every $h \in \mathfrak{h}$ such that $\mathrm{Re}(\delta(h)) > 0$ (see chapter 11 of [3]). We denote $\mathrm{ch}_\lambda(h)$ its limit. For $\beta \in \mathfrak{h}$ such that $\mathrm{Re}(\beta|\delta) > 0$, let $\mathrm{ch}_\lambda(\beta) = \mathrm{ch}_\lambda(\nu^{-1}(\beta))$.

3.3 Theta functions

Connections between affine Lie algebras and theta functions are developed in chapter 13 of [3]. We recall properties that we need for our purpose. For $\lambda \in P$ such that $(\delta|\lambda) = k$ one defines the classical theta function Θ_λ of degree k by the series

$$\Theta_\lambda = e^{-\frac{(\lambda|\lambda)}{2k}\delta} \sum_{\alpha\in M} e^{t_\alpha(\lambda)}.$$

This series converges absolutely on $\{h \in \mathfrak{h} : \mathrm{Re}(\delta(h)) > 0\}$ to an analytic function. As

$$e^{\frac{(\lambda|\lambda)}{2k}\delta} \sum_{w\in \mathring{W}} \det(w)\Theta_{w(\lambda)} = \sum_{w\in W} \det(w) e^{w(\lambda)},$$

this last series converges absolutely on $\{h \in \mathfrak{h} : \mathrm{Re}(\delta(h)) > 0\}$ to an analytic function too.

4 Markov chains on the sets of integral or dominant weights

Let us choose for this section a dominant weight $\omega \in P_+$ and $h \in \mathfrak{h}_{\mathbb{R}}$ such that $\delta(h) \in \mathbb{R}_+^*$.

Random walks on P We define a probability measure μ_ω on P letting

$$\mu_\omega(\beta) = \frac{\dim(V(\omega)_\beta)}{\mathrm{ch}_\omega(h)} e^{\langle \beta, h\rangle}, \quad \beta \in P. \tag{4}$$

Remark 4.1 If $(X(n), n \geq 0)$ is a random walk on P whose increments are distributed according to μ_ω, keep in mind that the function

$$z \in \mathring{\mathfrak{h}}_{\mathbb{R}} \mapsto \left(\frac{\mathrm{ch}_\omega(iz+h)}{\mathrm{ch}_\omega(h)}\right)^n,$$

is the Fourier transform of the projection of $X(n)$ on $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$.

Markov chains on P_+ Given two irreducible representations $V(\lambda)$ and $V(\omega)$, the tensor product of $\mathfrak{g}$-modules $V(\lambda) \otimes V(\beta)$ decomposes has a direct sum of irreducible modules. The following decomposition

$$V(\lambda) \otimes V(\omega) = \sum_{\beta \in P_+} M_\lambda(\beta) V(\beta),$$

where $M_\lambda(\beta)$ is the multiplicity of the module with highest weight β in the decomposition of $V(\omega) \otimes V(\lambda)$, leads to the definition a transition probability Q_ω on P_+ given by

$$Q_\omega(\lambda, \beta) = \frac{\mathrm{ch}_\beta(h)}{\mathrm{ch}_\lambda(h)\mathrm{ch}_\omega(h)} M_\lambda(\beta), \quad \lambda, \beta \in P_+. \tag{5}$$

For $n \in \mathbb{N}$, $\omega \in P_+$, $\beta \in P$, denote $m_{\omega^{\otimes n}}(\beta)$ the multiplicity of the weight β in $V(\omega)^{\otimes n}$. For $n \in \mathbb{N}$, $\lambda, \beta \in P_+$, denote $M_{\lambda, \omega^{\otimes n}}(\beta)$ the multiplicity defined by

$$V(\lambda) \otimes V(\omega)^{\otimes n} = \sum_{\beta \in P_+} M_{\lambda \otimes \omega^{\otimes n}}(\beta) V(\beta).$$

The Weyl character formula implies the following lemma, which is known as a consequence of the Brauer-Klimyk rule when $\mathfrak{g}$ is a complex semi-simple Lie algebra.

Lemma 4.2 *For* $n \in \mathbb{N}$, $\lambda, \beta \in P_+$ *one has*

$$M_{\lambda \otimes \omega^{\otimes n}}(\beta) = \sum_{w \in W} \det(w) m_{\omega^{\otimes n}}(w(\beta + \rho) - (\lambda + \rho)),$$

Proof See Proposition 2.1 of [5] and remark below. The proof is exactly the same in the framework of Kac-Moody algebras.

Let us consider the random walk $(X(n))_{n\geq 0}$ defined above and its projection $(\bar{X}(n))_{n\geq 0}$ on $(\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}})$. Denote $\bar{P}_\omega$ the transition kernel of this last random walk. The next property is immediate.

Lemma 4.3 *Let β_0, λ_0 be two weights in $(\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}})$. The transition kernel $\bar{P}_\omega$ satisfies for every $n \in \mathbb{N}$,*

$$\bar{P}^n_\omega(\lambda_0, \beta_0) = \sum_{\beta \in P : \bar{\beta} = \beta_0} e^{\langle \beta - \lambda_0, h\rangle} \frac{m_{\omega^{\otimes n}}(\beta - \lambda_0)}{ch^n_\omega(h)}$$

Let us consider a Markov process $(\Lambda(n))_{n\geq 0}$ whose Markov kernel is given by (5). If λ_1 and λ_2 are two dominant weights such that $\lambda_1 = \lambda_2\,(mod\,\delta)$ then the irreducible modules $V(\lambda_1)$ and $V(\lambda_2)$ are isomorphic. Thus if we consider the random process $(\bar{\Lambda}(n), n \geq 0)$, where $\bar{\Lambda}(n)$ is the projection of $\Lambda(n)$ on $(\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}})$, then $(\bar{\Lambda}(n), n \geq 1)$ is a Markov process whose transition kernel is denoted $\bar{Q}_\omega$.

Proposition 4.4 *Let β_0, λ_0 be two dominant weights in $(\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}})$, and n be a positive integer. The transition kernel $\bar{Q}_\omega$ satisfies*

$$\bar{Q}^n_\omega(\lambda_0, \beta_0) = \frac{\mathrm{ch}_{\beta_0}(h)e^{-\langle \beta_0, h\rangle}}{\mathrm{ch}_{\lambda_0}(h)e^{-\langle \lambda_0, h\rangle}} \sum_{w\in W} \det(w) e^{\langle w(\lambda_0+\rho)-(\lambda_0+\rho), h\rangle}\, \bar{P}^n_\omega(\overline{w(\lambda_0+\rho)-\rho}, \beta_0)$$

Proof Using Lemma (4.2), one obtains for any dominant weight $\lambda_0, \beta_0 \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{g}}^*_{\mathbb{R}})$,

$$\begin{aligned}
\bar{Q}^n_\omega(\lambda_0, \beta_0) &= \frac{\mathrm{ch}_{\beta_0}(h)}{\mathrm{ch}_{\lambda_0}(h)\mathrm{ch}^n_\omega(h)} \sum_{\beta \in P_+ : \bar{\beta} = \beta_0} e^{\langle \beta - \bar{\beta}, h\rangle} M_{\lambda, \omega^{\otimes n}}(\beta)\\
&= \frac{\mathrm{ch}_{\beta_0}(h)}{\mathrm{ch}_{\lambda_0}(h)\mathrm{ch}^n_\omega(h)} \sum_{\beta \in P : \bar{\beta} = \beta_0} e^{\langle \beta - \bar{\beta}, h\rangle} \sum_{w\in W} \det(w) m_{\omega^{\otimes n}}(w(\beta+\rho) - (\lambda+\rho)).\\
&= \frac{\mathrm{ch}_{\beta_0}(h)e^{-\langle \beta_0, h\rangle}}{\mathrm{ch}_{\lambda_0}(h)e^{-\langle \lambda_0, h\rangle}} \sum_{w\in W} \det(w) e^{\langle w(\lambda_0+\rho)-(\lambda_0+\rho), h\rangle}\, \bar{P}^n_\omega(\overline{w(\lambda_0+\rho)-\rho}, \beta_0).
\end{aligned}$$

5 Scaling limit of Random walks on *P*

Let us fix $\rho = h^\vee \Lambda_0 + \bar{\bar{\rho}}$, where $\bar{\bar{\rho}}$ is half the sum of positive roots in $\mathring{\mathfrak{h}}^*$. For $n \in \mathbb{N}^*$, we consider a random walk $(X^n(k), k \geq 0)$ starting from 0, whose increments are distributed according to a probability measure μ_ω defined by (4) with $\omega \in P^{h^\vee}_+$ and $h = \frac{1}{n}\nu^{-1}(\rho)$. In particular $X^n(k)$ is an integral weight of level $h^\vee k$ for $k \in \mathbb{N}$. Proposition 5.1 gives the scaling limit of the process $(\bar{\bar{X}}^n(k), k \geq 0)$,

Proposition 5.1 *The sequence of processes* $(\frac{1}{n}\bar{\bar{X}}^n([nt]), t \geq 0)_{n\geq 0}$ *converges towards a standard Brownian motion on* $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$ *with drift* $\bar{\bar{\rho}}$.

Proof The key ingredients for the proof are Theorems 13.8 and 13.9 of [3], which provide a transformation law for normalized characters. The two theorems deal with two different classes of affine Lie algebras. Let us make the proof in the framework of Theorem 13.8. The proof is similar in the framework of Theorem 13.9. For the affine Lie algebras considered in Theorem 13.8 one has that for $n \geq 1$ and $z \in \mathring{\mathfrak{h}}^*$,

$$\begin{aligned}\mathrm{ch}_\omega\left(\frac{1}{n}(\rho+z)\right)\\ &= C_n e^{\frac{1}{2n}||\bar{\bar{\rho}}+z||^2} \sum_{\Lambda\in P_+^{h^\vee} \bmod \mathbb{C}\delta} S_{\omega,\Lambda} e^{-m_\Lambda \frac{4\pi^2 n}{h^\vee}} \mathrm{ch}_\Lambda\left(\frac{4\pi^2 n}{h^\vee}\Lambda_0 + 2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee}\right),\end{aligned}$$

where C_n is a constant independent of z, $m_\Lambda = \frac{||\Lambda+\rho||^2}{4h^\vee} - \frac{||\rho||^2}{2h^\vee}$ and $S_{\omega,\Lambda}$ is a coefficient independent of z and n, for $\Lambda \in P_+^{h^\vee}$. Notice that the sum is well-defined as for $\lambda_1 = \lambda_2 \bmod \mathbb{C}\delta$ one has

$$e^{-m_{\lambda_1}\frac{4\pi^2 n}{h^\vee}} \mathrm{ch}_{\lambda_1}\left(\frac{4\pi^2 n}{h^\vee}\Lambda_0 + 2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee}\right) = e^{-m_{\lambda_2}\frac{4\pi^2 n}{h^\vee}} \mathrm{ch}_{\lambda_2}\left(\frac{4\pi^2 n}{h^\vee}\Lambda_0 + 2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee}\right).$$

Let us prove the convergence. Let $i \in \{1, \dots, l\}$. One has $\langle h^\vee\Lambda_0, \alpha_i^\vee\rangle = 0$, which implies that $V(h^\vee\Lambda_0)_{h^\vee\Lambda_0-\alpha_i} = \{0\}$. Consequentely,

$$\text{if } \beta \in P \text{ and } \dim(V(h^\vee\Lambda)_\beta) \neq 0, \text{ then } \beta = h^\vee\Lambda_0 - \sum_{k=0}^{l} i_k\alpha_k,$$

where i_k is a nonnegative integer, for $k \in \{1, \dots, l\}$, and i_0 is a positive integer, which implies that $(\beta|\Lambda_0) \leq -1$. Moreover, the action of f_k, for $k \in \{0, \dots, l\}$, on an integrable highest weight module being locally nilpotent, the number of weights β such that $\dim(V(h^\vee\Lambda_0)_\beta) \neq 0$ and $(\beta|\Lambda_0) = -1$ is finite. As the characters are defined on the set

$$\{\lambda \in \mathfrak{h}^* : \mathrm{Re}(\lambda|\delta) > 0\},$$

by absolutely convergent series, it implies that

$$\mathrm{ch}_{h^\vee\Lambda_0}\left(\frac{4\pi^2 n}{h^\vee}\Lambda_0 + 2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee}\right)$$

is equal to

$$1 + (1+\epsilon(n))e^{-\frac{4n\pi^2}{h^\vee}} \sum_{\beta:(\beta|\Lambda_0)=-1} \dim V(h^\vee\Lambda_0)_\beta e^{\left(\beta|2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee}\right)},$$

where $\lim_{n\to\infty}\epsilon(n)=0$. Thus

$$\lim_{n\to\infty}\left(\mathrm{ch}_{h^\vee\Lambda_0}\left(\frac{4\pi^2 n}{h^\vee}\Lambda_0+2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee}\right)\right)^{[nt]}=1. \tag{6}$$

Let $\Lambda\in P_+^{h^\vee}$ such that $(\Lambda|\Lambda_0)=0$. As previously, if $\beta\in P$ and $\dim(V(\Lambda)_\beta)\neq 0$ then $(\beta|\Lambda_0)\leq 0$, and the number of weights β such that $\dim(V(\Lambda)_\beta)\neq 0$ and $(\beta|\Lambda_0)=0$ is finite. Thus,

$$\mathrm{ch}_\Lambda\left(\frac{4\pi^2 n}{h^\vee}\Lambda_0+2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee}\right),$$

is bounded independently of n. Besides, one easily verifies that for such a Λ one has $m_\Lambda\geq m_{h^\vee\Lambda_0}$ and that $m_\Lambda=m_{h^\vee\Lambda_0}$ implies $\Lambda=h^\vee\Lambda_0$. Thus

$$\left(1+\sum_{\Lambda\in P_+^{h^\vee}\setminus\{h^\vee\Lambda_0\}\bmod\mathbb{C}\delta}\frac{S_{\omega,\Lambda}}{S_{\omega,h^\vee\Lambda_0}}e^{-(m_\Lambda-m_{h^\vee\Lambda_0})\frac{4\pi^2 n}{h^\vee}}\frac{\mathrm{ch}_\Lambda(\frac{4\pi^2 n}{h^\vee}\Lambda_0+2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee})}{\mathrm{ch}_{h^\vee\Lambda_0}(\frac{4\pi^2 n}{h^\vee}\Lambda_0+2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee})}\right)^{[nt]}$$

converges towards 1 when n goes to infinity. The last convergence and Theorem 13.8 of [3], recalled at the beginning of the proof, imply

$$\lim_{n\to\infty}\left(\frac{\mathrm{ch}_\omega(\frac{1}{n}(\rho+z))}{C_n S_{\omega,h^\vee\Lambda_0}e^{-m_{h^\vee\Lambda_0}\frac{4\pi^2 n}{h^\vee}}\mathrm{ch}_{h^\vee\Lambda_0}(\frac{4\pi^2 n}{h^\vee}\Lambda_0+2i\pi\frac{\bar{\bar{\rho}}+z}{h^\vee})}\right)^{[nt]}=e^{\frac{t}{2}||\bar{\bar{\rho}}+z||^2}.$$

Finally, using convergence (6) one obtains

$$\lim_{n\to\infty}\left(\frac{\mathrm{ch}_\omega(\frac{1}{n}(\rho+z))}{\mathrm{ch}_\omega(\frac{1}{n}\rho)}\right)^{[nt]}=e^{\frac{t}{2}(||\bar{\bar{\rho}}+z||^2-||\bar{\bar{\rho}}||^2)},$$

which achieves the proof by remark (4.1). □

6 A conditioned space-time Brownian motion

Denote $\mathcal{C}$ the fundamental Weyl chamber defined by

$$\mathcal{C}=\{x\in\mathfrak{h}^*:\langle x,\alpha_i^\vee\rangle\geq 0,\ i=0,\ldots,l\}.$$

Let us consider a standard Brownian motion $(B_t)_{t\geq 0}$ on $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$. We consider a random process $(\tau_t\Lambda_0+B_t)_{t\geq 0}$ on $(\mathbb{R}\Lambda_0+\mathring{\mathfrak{h}}_{\mathbb{R}}^*)$. For $x\in(\mathbb{R}\Lambda_0+\mathring{\mathfrak{h}}_{\mathbb{R}}^*)$, denote $\mathbb{P}_x^0$ (resp. $\mathbb{P}_x^\rho$), a probability under which $\tau_t=(x|\delta)+th^\vee$, $\forall t\geq 0$, and $(B_t)_{t\geq 0}$ is a standard Brownian motion (resp. a standard Brownian motion with drift $\bar{\bar{\rho}}$) starting from $\bar{\bar{x}}$.

Under $\mathbb{P}_x^0$ (resp. $\mathbb{P}_x^\rho$), the stochastic process $(\tau_t\Lambda_0 + B_t)_{t\geq 0}$ has a transition probability semi-group $(p_t)_{t\geq 0}$ (resp. $(p_t^\rho)_{t\geq 0}$) defined by

$$p_t(x,y) = \frac{1}{(2\pi t)^{\frac{l}{2}}} e^{-\frac{1}{2t}||y-x||^2} 1_{(y|\delta)=th^\vee+(x|\delta)}, \quad x,y \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}_{\mathbb{R}}^*).$$

$$(\text{resp. } p_t^\rho(x,y) = \frac{1}{(2\pi t)^{\frac{l}{2}}} e^{-\frac{1}{2t}||y-\bar{\bar{\rho}}t-x||^2} 1_{(y|\delta)=th^\vee+(x|\delta)}, \quad x,y \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}_{\mathbb{R}}^*).)$$

Let $X_t = \tau_t\Lambda_0 + B_t$, for $t \geq 0$, and consider the stopping time T defined by

$$T = \inf\{t \geq 0 : X_t \notin \mathcal{C}\}.$$

The following proposition gives the probability for $(X_t)_{t\geq 0}$ to remain forever in $\mathcal{C}$, under $\mathbb{P}_x^\rho$, for $x \in \mathcal{C}$.

Proposition 6.1 *Let $x \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}_{\mathbb{R}}^*) \cap \mathcal{C}$. One has*

$$\mathbb{P}_x^\rho(T = +\infty) = \sum_{w\in W} \det(w) e^{(x,w(\rho)-\rho)}.$$

Proof If we consider the function h defined on $(\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}_{\mathbb{R}}^*)$ by

$$h(\lambda) = \mathbb{P}_\lambda^\rho(T = \infty), \quad \lambda \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}_{\mathbb{R}}^*) \cap \mathcal{C},$$

usual martingal arguments state that h is the unique bounded harmonic function for the killed process $(X_{t\wedge T})_{t\geq 0}$ under $\mathbb{P}_x^\rho$ such that

$$h(\lambda) = 0, \text{ for } \lambda \in \partial\mathcal{C}, \tag{7}$$

and

$$\lim_{t\to\infty} h(X_{t\wedge T}) = 1_{T=\infty}. \tag{8}$$

Let us proves that the function defined by the sum satisfies these properties. First notice that the boundary condition (7) is satisfied. Moreover, as x is in the interior of $\mathcal{C}$, formula (3) implies that

$$\sum_{w\in W} \det(w) e^{(x,w(\rho)-\rho)}$$

is positive and bounded by 1. Choose an orthonormal basis $v_1,\ldots,v_l$ of $\mathring{\mathfrak{h}}_{\mathbb{R}}^*$ and consider for $w \in W$ a function g_w defined on $\mathbb{R}_+^* \times \mathbb{R}^l$ by

$$g_w(t,x_1,\ldots,x_l) = e^{(t\Lambda_0+x,w(\rho)-\rho)},$$

where $x = x_1v_1 + \cdots + x_lv_l$. Letting $\Delta = \sum_{i=1}^{l} \partial_{x_i x_i}$, the function g_w satisfies

$$\left(\frac{1}{2}\Delta + h^\vee \partial_t + \sum_{i=1}^{l} (\rho, v_i)\partial_{x_i}\right) g_w = \frac{1}{2}||w(\rho) - \rho||^2 + (\rho|w(\rho) - \rho) = 0. \quad (9)$$

As the function $g = \sum_w \det(w) g_w$ is analytic on $\mathbb{R}_+^* \times \mathbb{R}^l$, it satisfies (9) too. Ito's Lemma implies that $(g((\tau_{t\wedge T}, B_{t\wedge T}))_{t\geq 0}$ is a local martingale. As the function g is bounded by 1 on $\{(t, x) \in \mathbb{R}_+^* \times \mathbb{R}^l : t\Lambda_0 + x_1v_1 + \cdots + x_lv_l \in \mathcal{C}\}$, $(g((\tau_{t\wedge T}, B_{t\wedge T}))_{t\geq 0}$ is a martingale, i.e. g is harmonic for the killed process under $\mathbb{P}_x^\rho$. It remains to prove that the condition (8) is satisfied. For this, we notice that for any $w \in W$ distinct from the identity, $\rho - w(\rho) = \sum_{i=0}^{l} k_i \alpha_i$, where the k_i are non negative integers not simultaneously equal to zero. As almost surely

$$\lim_{t\to\infty} \frac{X_t}{t} = \rho,$$

one obtains

$$\lim_{t\to\infty} g_w(X_t) = 0$$

for every $w \in W$ distinct from the identity. As the function g is analytic on $\mathbb{R}_+^* \times \mathbb{R}^l$, the expected convergence follows. □

The following lemma is needed to prove a reflection principle for a Brownian motion killed on the boundary of the affine Weyl chamber.

Lemma 6.2 *For $x, y \in \mathfrak{h}_{\mathbb{R}}^*$, $t \in \mathbb{R}_+$, $w \in W$, one has*

$$p_t^0(\overline{wx}, \overline{wy}) = e^{(w(y-x)-(y-x), h^\vee \Lambda_0)} p_t^0(\bar{x}, \bar{y}).$$

Proof Notice that $\overline{wx} = \overline{w\bar{x}}$. For $w \in \mathring{W}$, $\overline{wx} = w\bar{x}$, $p_t^0(w(\bar{x}), w(\bar{y})) = p_t^0(x, y)$ and $(wx - x|\Lambda_0) = (wy - t|\Lambda_0) = 0$, which implies the identity. For $w = t_\alpha$, $\alpha \in M$, one has

$$\begin{aligned}
p_t^0(\overline{wx}, \overline{wy}) &= p_t^0(h^\vee u\alpha + \bar{x}, h^\vee(u + t)\alpha + \bar{y}) \\
&= \frac{1}{(2\pi t)^{\frac{l}{2}}} e^{-\frac{1}{2t}||\bar{y}+th^\vee\alpha-\bar{x}||^2} 1_{(y|\delta)=th^\vee+(x|\delta)} \\
&= p_t^0(\bar{x}, \bar{y}) e^{-\frac{1}{2t}((h^\vee)^2 t^2(\alpha|\alpha)+2h^\vee t(\alpha|y-x))} \\
&= e^{(w(y-x)-(y-x), h^\vee\Lambda_0)} p_t^0(\bar{x}, \bar{y}).
\end{aligned}$$

□

In the following, by a classical abuse of notation,

$$\mathbb{P}_x^\rho(X_t = y, T \geq t), \text{ or } \mathbb{P}_x^0(X_t = y, T \geq t),$$

$x, y \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}})$, $t \geq 0$, stands for the semi-group of the process $(X_t)_{t\geq 0}$, with drift or not, killed on the boundary of $\mathcal{C}$. We first prove a reflection principle for a Brownian motion with no drift.

Lemma 6.3 *For $x, y \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}})$ in the interior of $\mathcal{C}$, such that $(y|\delta) = (x|\delta) + th^\vee$, we have*

$$\mathbb{P}^0_x(X_t = y, T > t) = \sum_{w\in W} \det(w) e^{(wx-x, h^\vee\Lambda_0)} p^0_t(\overline{wx}, y),$$
$$= \sum_{w\in W} \det(w) e^{(y-w(y), h^\vee\Lambda_0)} p^0_t(x, \overline{w(y)}).$$

Proof Lemma 6.2 implies in particular that we need to prove only one of the two identities. Let us prove the second one. Actually Lemma 6.2 implies that for $\alpha \in \Pi$ such that $s_\alpha(X_T) = 0$

$$\mathbb{E}_{X_T}(1_{X_r=\overline{wy}}) = e^{(wy-s_\alpha wy|h^\vee\Lambda_0)} \mathbb{E}_{X_T}(1_{X_r} = \overline{s_\alpha wy}),$$

which implies that

$$\mathbb{E}_x\left(\sum_{w\in W} \det(w) e^{(y-wy, h^\vee\Lambda_0)} 1_{T\leq t,\, X_t=wy}\right) = 0.$$

Then lemma follows from the fact that

$$\mathbb{E}_x\left(\sum_{w\in W} \det(w) e^{(y-w(y), h^\vee\Lambda_0)} 1_{T>t,\, X_t=wy}\right) = \mathbb{E}_x(1_{X_t=y,\, T>t}).$$

□

Proposition 6.4 *For $x, y \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}})$ in the interior of $\mathcal{C}$, such that $(y|\delta) = (x|\delta) + th^\vee$, we have*

$$\mathbb{P}^\rho_x(X_t = y, T > t) = \sum_{w\in W} \det(w) e^{(w(x)-x, \rho)} p^\rho_t(\overline{w(x)}, y)$$
$$= \sum_{w\in W} \det(w) e^{(y-w(y), \rho)} p^\rho_t(x, \overline{w(y)}),$$

Proof The result follows in a standard way from Lemma 6.3 from a Girsanov's theorem. □

7 Scaling limit of the Markov chain on P_+

For $x \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}})$, Proposition 6.1 and identity (3) imply in particular that the probability $P^\rho_x(T = +\infty)$ is positive when x is in the interior of $\mathcal{C}$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of $(X_t)_{t\geq 0}$. Let us fix $x \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}})$ in the interior of $\mathcal{C}$. One considers the following conditioned process.

Definition 7.1 One defines a probability $\mathbb{Q}_x^\rho$ letting

$$\mathbb{Q}_x^\rho(A) = \mathbb{E}_x\left(\frac{\mathbb{P}^\rho_{X_t}(T = +\infty)}{\mathbb{P}^\rho_x(T = +\infty)} 1_{T\geq t,\, A}\right), \quad \text{for } A \in \mathcal{F}_t,\ t \geq 0.$$

Under the probability $\mathbb{Q}_x^\rho$, the process $(X_t)_{t\geq 0}$ is a space-time Brownian motion with drift ρ, conditioned to remain forever in the affine Weyl chamber. Let $(x_n)_{n\geq 0}$ be a sequence of elements of P_+ such that the sequence $(\frac{x_n}{n})_{n\geq 0}$ converges towards x when n goes to infinity. For any $n \in \mathbb{N}^*$, we consider a Markov process $(\Lambda^n(k), k \geq 0)$ starting from x_n, with a transition probability Q_ω defined by (5), with $\omega \in P_+^{h^\vee}$ and $h = \frac{1}{n}\nu^{-1}(\rho)$. Notice that for $n, k \in \mathbb{N}$, $\Lambda^n(k)$ is a dominant weight of level $kh^\vee + (x_n|\delta)$. Then the following convergence holds.

Theorem 7.2 *The sequence of processes* $(\frac{1}{n}\bar{\Lambda}^n([nt]), t \geq 0)$ *converges when n goes to infinity towards the process* $(X_t, t \geq 0)$ *under* $\mathbb{Q}_x^\rho$.

Proof Propositions 4.4 and 5.1 imply that the sequence of processes $(\frac{1}{n}\bar{\Lambda}^n([nt]), t \geq 0)$ converges when n goes to infinity towards a Markov process with transition probability semi-group $(q_t)_{t\geq 0}$ defined by

$$q_t(x, y) = \frac{\psi(y)}{\psi(x)} \sum_{w\in W} \det(w) e^{(w(x)-x|\rho)} p_t^\rho(\overline{w(x)}, y), \quad x, y \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}}),$$

where $\psi(x) = \sum_{w\in W} \det(w) e^{(x|w(\rho)-\rho)}$. Propositions 6.1 and 6.4 imply that

$$q_t(x, y) = \frac{\mathbb{P}^\rho_y(T = +\infty)}{\mathbb{P}^\rho_x(T = +\infty)} \mathbb{P}_x(X_t = y,\ T > t), \quad x, y \in (\mathbb{R}\Lambda_0 + \mathring{\mathfrak{h}}^*_{\mathbb{R}}),$$

which achieves the proof. □

References

1. Crank, J.: Free and Moving Boundary Problems. Clarendon Press, Oxford (1984)
2. Defosseux, M.: The affine Lie algebra $\hat{\mathfrak{sl}}_2$ and a conditioned space-time Brownian motion, arXiv:1401.3115 (2014)
3. Kac, V.G.: Infinite dimensional Lie algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
4. Lecouvey, C., Lesigne, E., Peigné, M.: Conditioned random walks from Kac-Moody root systems, arXiv:1306.3082 [math.CO] (2013)
5. Stembridge, J.R.: Multiplicity-free products and restrictions of Weyl characters. Represent. Theory **7**, 404–439 (2003)