

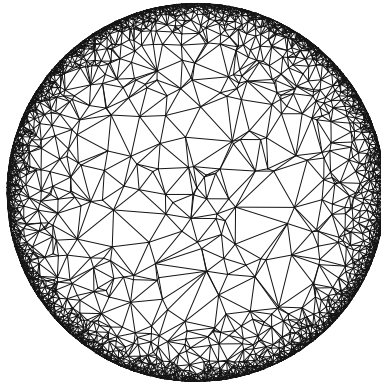
Planar stochastic hyperbolic triangulations

Nicolas Curien¹

Received: 26 January 2014 / Revised: 30 December 2014 / Published online: 30 July 2015
© Springer-Verlag Berlin Heidelberg 2015

Abstract Pursuing the approach of Angel and Ray (Ann Probab, 2015) we introduce and study a family of random infinite triangulations of the full-plane that satisfy a natural spatial Markov property. These new random lattices naturally generalize Angel and Schramm’s uniform infinite planar triangulation (UIPT) and are hyperbolic in flavor. We prove that they exhibit a sharp exponential volume growth, are non-Liouville, and that the simple random walk on them has positive speed almost surely. We conjecture that these infinite triangulations are the local limits of uniform triangulations whose genus is proportional to the size.

Graphical abstract An artistic representation of a random (3-connected) triangulation of the plane with hyperbolic flavor.



✉ Nicolas Curien
nicolas.curien@gmail.com

¹ CNRS and Université Paris 6, Paris, France

Mathematics Subject Classification 05C80 · 05C81 · 60J10

Contents

1	Introduction	510
1.1	Classification of Markovian triangulations of the plane	510
1.2	Peeling process	512
1.3	Properties of the planar stochastic hyperbolic infinite triangulations	512
1.4	A speculation	513
2	Construction of (\mathbf{T}_κ) for $\kappa \in (0, \frac{2}{27}]$	514
2.1	Uniqueness	514
2.2	Interpreting $(C_p^{(\kappa)} : p \geq 2)$ as a harmonic function	516
2.3	Peeling construction	518
2.3.1	General peeling	518
3	Geometric properties	521
3.1	Back to the peeling construction	521
3.2	Volume growth	523
3.3	Anchored expansion	525
4	Simple random walk	527
4.1	Reversibility and ergodicity	528
4.2	Non-intersection by peeling	531
4.3	Proof of Theorem 3 via entropy	535
5	Comments	537
5.1	Local limit of triangulations in high genus	537
5.2	Perspectives	538
	References	539

1 Introduction

Since the introduction of the uniform infinite planar triangulation by Angel and Schramm [9] as the local limit of large uniform triangulations, a large body of work has been devoted to the study of local limits of random maps and especially random triangulations and quadrangulations, see e.g. [3, 5, 15, 17, 18, 29] and the references therein. Recently, Angel and Ray [8] classified all random triangulations of the half-plane that satisfy a natural spatial Markov property and discovered new random lattices of the half-plane exhibiting a “hyperbolic” behavior¹ [35]. Motivated by these works we construct the analogs of these lattices in the full-plane topology and study their properties in fine details.

1.1 Classification of Markovian triangulations of the plane

Recall that a planar map is a proper embedding of a finite connected graph into the sphere seen up to deformations that preserve the orientation. All maps considered here are rooted, i.e. given with a distinguished oriented edge. A triangulation is a planar map whose faces have all degree three. For the sake of simplicity we restrict ourselves to 2-connected triangulations where loops are forbidden (but multiple edges

¹ Here and in the rest of the paper, the “hyperbolic” adjective is loose and should be taken with quotation marks, it does not refer to a precise mathematical notion but to a “phenomenology”

are allowed). We shall also deal with *infinite triangulations of the plane* or equivalently infinite triangulations with one end (a graph G is said to have one end if $G \setminus H$ contains exactly one infinite connected component for any finite subgraph $H \subset G$). They can be realized as proper embeddings \mathcal{X} (seen up to continuous deformations preserving the orientation) of graphs in \mathbb{R}^2 such that every compact subset of \mathbb{R}^2 intersects only finitely many edges of \mathcal{X} and such that the faces have all degree three. For any $p \geq 2$, a triangulation t of the p -gon, also called triangulation with a boundary of perimeter p , is a (rooted) planar map whose faces are all triangles except for one distinguished face of degree p , called the hole, whose boundary is made of a simple cycle (no pinch point). The perimeter $|\partial t|$ of t is the degree of its hole and its size $|t|$ is its number of vertices. The set of all finite triangulations of the p -gon is denoted by \mathcal{T}_p and we set $\mathcal{T}_B = \cup_{p \geq 2} \mathcal{T}_p$ for the set of all finite triangulations with a boundary.

Here comes the key definition of this work: We say that a random infinite triangulation \mathbf{T} of the plane is κ -Markovian for $\kappa > 0$ if there exist non-negative numbers $(C_i^{(\kappa)} : i \geq 2)$ such that for any $t \in \mathcal{T}_p$ we have

$$P(t \subset \mathbf{T}) = C_p^{(\kappa)} \cdot \kappa^{|t|}, \tag{1}$$

where $t \subset \mathbf{T}$ means that \mathbf{T} is obtained from t (with coinciding roots) by filling its hole with a necessary unique infinite triangulation of the p -gon. Our first result which parallels [8] is to show the existence and uniqueness of a one-parameter family of such triangulations:

Theorem 1 *For any $\kappa \in (0, \frac{2}{27}]$ there exists a unique (law of a) random κ -Markovian triangulation \mathbf{T}_κ of the plane. If $\kappa > \frac{2}{27}$ there is none.*

In the special case $\kappa = \frac{2}{27}$, called the critical case, it follows from [9, Theorem 5.1] that the triangulation $\mathbf{T}_{2/27}$ has the law of the uniform infinite planar triangulation (UIPT) introduced by Angel and Schramm [9] as the limit of uniform triangulations of the sphere of growing sizes. The UIPT and its quadrangular analog the UIPQ have received a lot of attention in recent years [4, 12, 22, 27, 31] partially motivated by the connections with the physics theory of 2-dimensional quantum gravity, the Gaussian free field [19] and the Brownian map [20, 30, 34]. Many fundamental problems about the UIPT/Q are still open. We will see below that the qualitative behavior of \mathbf{T}_κ in the regime $\kappa < \frac{2}{27}$, called hyperbolic regime (this terminology will be justified by the following results), is much different from that of the UIPT.

In the work [8], the authors classified all random triangulations of the *half-plane* that satisfy a very natural, but slightly different, spatial Markov property: a random triangulation of the half-plane has the spatial Markov property of [8] if conditionally on a neighborhood of the root with a simple boundary, the remaining lattice has the same law as the original one. Angel and Ray classified these lattices using a single parameter $\alpha \in [0, 1)$ which is equal to the probability that the face adjacent to a given edge on the boundary is a triangle pointing inside the map. Our lattices (\mathbf{T}_κ) for $\kappa \in (0, \frac{2}{27}]$ are the full-plane analogs of the half-planar lattices of [8] for $\alpha \geq 2/3$

and

$$\frac{\alpha^2(1 - \alpha)}{2} = \kappa \quad \text{and} \quad \alpha \in [2/3, 1). \tag{2}$$

Angel and Ray also exhibited subcritical half-planar lattices corresponding to $\alpha \in (0, 2/3)$ which are tree-like [35]. In our full-plane setup, no such subcritical phase exists. Although similar in spirit to Angel and Ray’s spatial Markov property our Markovian assumption (1) is slightly different mainly because of the topology of the plane which forces the presence of a function of the perimeter ($C_p^{(\kappa)} : p \geq 2$) and also because we impose an exponential dependence in the size.

1.2 Peeling process

In the theory of random planar maps, the spatial Markov property is a key feature that has already been thoroughly used, generally under the form of the “peeling process” [4, 6, 12, 19, 21, 33]. The peeling process has been conceived by Watabiki [38] and formalized by Angel [4] in the case of the UIPT. This is an algorithmic procedure that enables to construct the lattice in a Markovian fashion by exploring it face after face (possibly revealing the finite regions enclosed). It turns out that equation (1) implies that \mathbf{T}_κ must admit such a peeling process which yields a path to prove both existence and uniqueness in Theorem 1, as in [8]. Furthermore, we will see in Sect. 2.2 that the function $p \mapsto C_p^{(\kappa)}$ of (1) will be interpreted as a harmonic function of the underlying random walk governing the construction of \mathbf{T}_κ by the peeling process. The peeling process is also a key tool in the proof of the up-coming Theorems 2 and 3.

1.3 Properties of the planar stochastic hyperbolic infinite triangulations

Let us now turn to the properties of these new random lattices in the hyperbolic regime $\kappa \in (0, \frac{2}{27})$. If \mathbb{T} is a finite triangulation or an infinite triangulation of the plane, we let $B_r(\mathbb{T})$ denote the subtriangulation obtained by keeping the faces of \mathbb{T} that contain at least one vertex at graph distance less than or equal to $r - 1$ from the origin of the root edge in \mathbb{T} . Hence, $B_r(\mathbb{T})$ is a triangulation with a finite number of holes. In the infinite case, we also denote by $B_r^\bullet(\mathbb{T})$ the hull of the ball obtained by filling-in all the finite components of $\mathbb{T} \setminus B_r(\mathbb{T})$. Since \mathbb{T} is one-ended, $B_r^\bullet(\mathbb{T})$ belongs to \mathcal{T}_B and its boundary in \mathbb{T} is a simple cycle made of edges whose vertices are at distance exactly r from the origin of the root edge in \mathbb{T} .

Theorem 2 (Sharp exponential volume growth) *For any $\kappa \in (0, \frac{2}{27})$ recall the definition of $\alpha \in (\frac{2}{3}, 1)$ satisfying (2) and let $\delta_\kappa = \sqrt{\alpha(3\alpha - 2)}$. There exists a random variable Π_κ such that $\Pi_\kappa \in (0, \infty)$ almost surely with*

$$\left(\frac{\alpha - \delta_\kappa}{\alpha + \delta_\kappa}\right)^n |\partial B_n^\bullet(\mathbf{T}_\kappa)| \xrightarrow[n \rightarrow \infty]{a.s.} \Pi_\kappa,$$

$$\text{and } \frac{|B_n^\bullet(\mathbf{T}_\kappa)|}{|\partial B_n^\bullet(\mathbf{T}_\kappa)|} \xrightarrow[n \rightarrow \infty]{a.s.} \frac{\alpha(2\alpha - 1)}{\delta_\kappa^2}.$$

In [35], exponential bounds for the volume growth in the half-plane version of \mathbf{T}_κ are obtained but with non-matching exponential factors. It is very likely that the methods used here can be employed to settle [35, Question 6.1]. The results of the last theorem should be compared with the analogous properties for supercritical Galton–Watson trees (whose offspring distribution satisfies the $x \log x$ condition) where the number of individuals at generation n , properly normalized, converges towards a non-degenerate random variable on the event of non-extinction. We also show that in the hyperbolic regime, \mathbf{T}_κ has a positive anchored expansion constant (Proposition 7).

We then turn to the study of the simple random walk on \mathbf{T}_κ : Conditionally on \mathbf{T}_κ we launch a random walker from the target of the root edge and let it choose inductively one of its adjacent oriented edges for the next step. In the critical case $\kappa = \frac{2}{27}$, the simple random walk on the UIPT is known to be recurrent [27]. We show here that the behavior of the simple random walk is drastically different when $\kappa < \frac{2}{27}$. Recall that a graph is non-Liouville if and only if it possesses non-constant bounded harmonic functions.

Theorem 3 (Behavior of random walk) *For $\kappa \in (0, \frac{2}{27})$ there exists $s_\kappa > 0$ (deterministic) such that almost surely*

$$\lim_{n \rightarrow \infty} n^{-1} d_{\text{gr}}(X_0, X_n) = s_\kappa,$$

where $(X_i)_{i \geq 0}$ are the vertices visited by the simple random walk and d_{gr} is the graph metric. Also, \mathbf{T}_κ is almost surely non-Liouville in the hyperbolic regime.

The connoisseurs may remember that a major difficulty towards proving the recurrence of the UIPT [27] was the lack of a uniform bound on the degree. The situation is similar here, as positive speed would directly follow from positive anchored expansion in the bounded degree case by the result of [37]. The unboundedness of the vertex degrees in \mathbf{T}_κ forces us to find a different technique. The proof of Theorem 3 occupies the major part of Sect. 4 and makes extensive use of the fact that the random lattice \mathbf{T}_κ is stationary and reversible with respect to the simple random walk (Proposition 9). In words, re-rooting \mathbf{T}_κ along a simple random walk path does not change its distribution. This stochastic invariance by translation replaces the deterministic invariance of transitive lattices. This is a key feature of the full-plane models compared to the half-plane models of [8]. The proof of Theorem 3 also combines several geometric arguments such as: the exploration process of \mathbf{T}_κ along the simple random walk of [12], the recent results of [13] on intersection properties of planar lattices and the entropy method for stationary random graphs [11].

1.4 A speculation

We end this introduction by stating a conjecture relating our planar stochastic hyperbolic infinite triangulations to local limits of triangulations in high genus. More

precisely, let $\mathcal{T}_{n,g}$ be the set of all (rooted) triangulations of the torus of genus $g \geq 0$ with n vertices (see e.g. [5] for a definition) and denote by $T_{n,g}$ a random uniform element in $\mathcal{T}_{n,g}$. Together with Itai Benjamini, we conjecture that there is a continuous decreasing function $f(\theta) \in (0, \frac{2}{27}]$ with $f(\theta) \rightarrow 0$ as $\theta \rightarrow \infty$ and $f(0) = \frac{2}{27}$ such that we have the following convergence

$$T_{n, \lfloor \theta n \rfloor} \xrightarrow[n \rightarrow \infty]{(d)} \mathbf{T}_{f(\theta)},$$

for the local topology. See Sect. 5 for a more precise statement. Some results in the literature already indicate that large triangulations of high genus should be locally planar such as [24, 28] or the paper [5] in the case of unicellular maps.

In this paper, we restricted ourselves to 2-connected triangulations for sake of simplicity. We believe that our work could be extended to 1-connected triangulations or to other types of maps e.g. quadrangulations. However, in these cases, the classification of Markovian random planar maps may depend on additional parameters exactly as in [8], see Sect. 5. Notice also that a model of “hyperbolic” random quadrangulation has been introduced in [10, Section 6.3] using the Schaeffer construction over a supercritical labeled Galton–Watson tree. We do not know however, whether the random quadrangulations obtained that way are local limits of finite random quadrangulations in high genus or if they are Markovian.

2 Construction of (\mathbf{T}_κ) for $\kappa \in (0, \frac{2}{27}]$

Fix $\kappa > 0$. We will first show that the law (if it exists) of a κ -Markovian random infinite triangulation of the plane is unique and that κ must be less than or equal to $\frac{2}{27}$. In this section we thus *assume* the existence of \mathbf{T}_κ , a κ -Markovian triangulation of the plane.

2.1 Uniqueness

We start with a few pre-requisites on the local topology on triangulations. Following [15], if t and t' are two rooted finite triangulations, the local distance between t and t' is

$$d_{\text{loc}}(t, t') = (1 + \sup\{r \geq 0 : B_r(t) = B_r(t')\})^{-1}.$$

The set of all finite triangulations is not complete for this distance and we shall add infinite triangulations to it. The metric space $(\mathcal{T}_\infty, d_{\text{loc}})$ we obtain is then Polish. Since \mathbf{T}_κ is an infinite random triangulation *with only one end*, it is easy to see that its law over $(\mathcal{T}_\infty, d_{\text{loc}})$ is characterized by the values of $P(t \subset \mathbf{T}_\rho)$ for all triangulations with a boundary $t \in \mathcal{T}_B$. Hence, establishing the uniqueness of the law of \mathbf{T}_κ reduces to showing that the function $(C_i^{(\kappa)} : i \geq 2)$ involved in (1) is uniquely characterized by $\kappa > 0$.

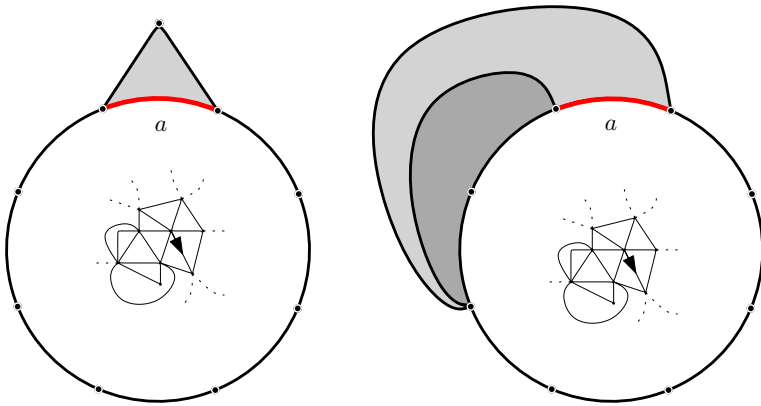


Fig. 1 Peeling the edge a : the triangle revealed is in light gray and the finite enclosed region is in dark gray on the second figure

We begin with a simple remark. Since \mathbf{T}_κ is a 2-connected triangulation, the triangle on the left of the root edge is necessarily a triangle with 3 distinct vertices with the root edge located on one of its side that we see as a triangulation of the 3-gon denoted by t_0 . By (1) we must have

$$1 = P(t_0 \subset \mathbf{T}_\kappa) = \kappa^3 C_3^{(\kappa)}. \tag{3}$$

We will now get another relation linking κ and the $(C_p^{(\kappa)})_{p \geq 1}$. This is done by increasing the map using the so-called peeling mechanism. Let \mathbf{T} be a triangulation of the plane and assume that $t \subset \mathbf{T}$ for some $t \in \mathcal{T}_p$. For any edge a on the boundary of t we consider the triangulation which is obtained by adding to t the triangle adjacent to a in $\mathbf{T} \setminus t$ as well as the finite region this triangle together with t may enclose (recall that \mathbf{T} is one-ended). We call this operation peeling the edge $a \in \partial t$. Two different situations may appear : either the triangle revealed contains a vertex inside $\mathbf{T} \setminus t$ (left on Fig. 1) or this triangle “swallows” k edges on the boundary of t either to the left or to the right of a and encloses a finite triangulation of the $k + 1$ -gon (right on Fig. 1). Note that $k \in \{1, 2, \dots, p - 2\}$ where p is the perimeter of t .

For two triangulations with a boundary t and t' we write $(t, a) \rightarrow t'$ if t' is a possible outcome of the peeling of the edge $a \in \partial t$ in some underlying triangulation \mathbf{T} . It is easy to see that such t' are obtained by either gluing a triangle to a outside t or by gluing a triangle to a with its third vertex identified with a vertex of the boundary of t and filling one of the two holes created with a finite triangulation having the proper perimeter. In the first case the size of the triangulation increases by 1, and in the second case it increases by the number of inner vertices (not located on the boundary) of the enclosed triangulation. This operation is rigid in the sense that two different ways of increasing t yield two distinct maps. For $p \geq 2$ and $n \geq p$, pick a triangulation t of the p -gon with n vertices and fix deterministically an edge a on its boundary. By the previous discussion we have

$$\begin{aligned}
 C_p^{(\kappa)} \cdot \kappa^n &\stackrel{(1)}{=} P(t \in \mathbf{T}_\kappa) = P\left(\bigcup_{(t,a) \rightarrow t'} \{t' \in \mathbf{T}_\kappa\}\right) \\
 &\stackrel{\text{rigidity}}{=} \sum_{(t,a) \rightarrow t'} P(t' \in \mathbf{T}_\kappa) \\
 &\stackrel{(1)}{=} C_{p+1}^{(\kappa)} \kappa^{n+1} + 2 \sum_{i=1}^{p-2} C_{p-i}^{(\kappa)} \cdot \kappa^n \sum_{\tau \in \mathcal{T}_{i+1}} \kappa^{|\tau|-i-1}.
 \end{aligned} \tag{4}$$

Now, for $i \geq 1$ and $\kappa > 0$ introduce the functions

$$Z_{i+1}^{(\kappa)} = \sum_{\tau \in \mathcal{T}_{i+1}} \kappa^{|\tau|-i-1} = \sum_{\tau \in \mathcal{T}_{i+1}} \kappa^{|\tau|-|\partial\tau|}.$$

A closed formula is known for these numbers (see [25] or [9, Proposition 2.4]) and they are finite if and only if $\kappa \in (0, \frac{2}{27}]$. They can be interpreted as the partition function of the following probability measure: The Boltzmann probability distribution of the $i + 1$ -gon with parameter κ is the probability measure that assigns a weight $\kappa^{|\tau|-i-1}/Z_{i+1}^{(\kappa)}$ to each triangulation t of the $i + 1$ -gon. Hence (4) becomes

$$C_p^{(\kappa)} = \kappa \cdot C_{p+1}^{(\kappa)} + 2 \sum_{i=1}^{p-2} C_{p-i}^{(\kappa)} \cdot Z_{i+1}^{(\kappa)} \quad \forall p \in \{2, 3, 4, \dots\}. \tag{5}$$

If $\kappa > \frac{2}{27}$ then $Z_i^{(\kappa)} = \infty$ for any $i \geq 2$, so we must suppose that $\kappa \in (0, \frac{2}{27}]$. Using the last display with $p = 2$ we find that $C_2^{(\kappa)} = \kappa C_3^{(\kappa)}$ which combined with (3) fixes the value of $C_2^{(\kappa)}$. Next, using (5) recursively for $p = 3, 4, \dots$ we see that the values of $C_p^{(\kappa)}$ for $p \geq 4$ are fixed by κ only. This proves uniqueness of the law of \mathbf{T}_κ .

2.2 Interpreting $(C_p^{(\kappa)} : p \geq 2)$ as a harmonic function

Fix $\kappa \in (0, \frac{2}{27}]$ and define the numbers $C_2^{(\kappa)}, C_3^{(\kappa)}, \dots$ using (3) and (5) as in the preceding section. Towards proving the existence of a κ -Markovian triangulation, our first task is to show that

$$C_p^{(\kappa)} > 0 \quad \text{for every } p \geq 2. \tag{6}$$

To do so it will be very useful to interpret them probabilistically. We start by recalling a key calculation that can be found in [8, Section 3.1]. Let $\alpha \in [2/3, 1)$ be given by (2) and let $\beta = \kappa/\alpha$, then we have

$$1 = \alpha + 2 \sum_{i=1}^{\infty} \beta^i Z_{i+1}^{(\alpha\beta)}.$$

This enables us to define a probability distribution $\mathbf{q}^{(\kappa)} = \{\dots, q_{-3}^{(\kappa)}, q_{-2}^{(\kappa)}, q_{-1}^{(\kappa)}, q_1^{(\kappa)}\}$ by setting

$$q_1^{(\kappa)} = \alpha \quad \text{and} \quad q_{-i}^{(\kappa)} = 2\beta^i Z_{i+1}^{(\kappa)} \quad \text{for } i \geq 1.$$

From [8, Equation (3.7)] we even have an exact formula

$$q_{-i}^{(\kappa)} = \frac{2}{4^i} \frac{(2i - 2)!}{(i - 1)!(i + 1)!} \left(\frac{2}{\alpha} - 2\right)^i ((3\alpha - 2)i + 1), \quad \text{for } i \geq 1. \tag{7}$$

Notice that $\mathbf{q}^{(\kappa)}$ has exponential tails as soon as $\alpha > 2/3$ or equivalently $\kappa < 2/27$. Finally we introduce $(\Xi_n^{(\kappa)})_{n \geq 0}$ a random walk started from 2 with independent increments following the distribution $\mathbf{q}^{(\kappa)}$. A computation using (7) done in [35, Lemma 4.2] shows that the drift of this walk is given by

$$\delta_\kappa := \sum_{i \leq 1} i q_i^{(\kappa)} = \sqrt{\alpha(3\alpha - 2)}, \tag{8}$$

which is strictly positive when $\alpha > 2/3$. We can now show that $C_p^{(\kappa)} > 0$ for all $p \geq 2$: In (5) we multiply both sides by β^p and set $\tilde{C}_p^{(\kappa)} = \beta^p C_p^{(\kappa)}$ for $p \geq 2$ and put $\tilde{C}_p^{(\kappa)} = 0$ otherwise, so that (5) becomes

$$\tilde{C}_p^{(\kappa)} = \sum_{i \in \{\dots, -3, -2, -1, 1\}} q_i^{(\kappa)} \cdot \tilde{C}_{p+i}^{(\kappa)} \quad \text{for } p \geq 2. \tag{9}$$

In other words, the function $p \mapsto \tilde{C}_p^{(\kappa)}$ is the (only) function which is harmonic for the random walk $\Xi^{(\kappa)}$ on $\{2, 3, \dots\}$ and null for $p \leq 1$ subject to the condition $\tilde{C}_3^{(\kappa)} = \alpha^{-3}$ given by (3). Note that we have $\tilde{C}_2^{(\kappa)} = \alpha \tilde{C}_3^{(\kappa)} < \tilde{C}_3^{(\kappa)}$ and that the last display can be written as

$$q_1^{(\kappa)} (\tilde{C}_{p+1}^{(\kappa)} - \tilde{C}_p^{(\kappa)}) = \sum_{i=1}^{\infty} q_{-i}^{(\kappa)} (\tilde{C}_p^{(\kappa)} - \tilde{C}_{p-i}^{(\kappa)}).$$

From this, we immediately conclude by induction on $p \geq 2$ that $\tilde{C}_p^{(\kappa)}$ is increasing in p and so $\tilde{C}_p^{(\kappa)}$ is positive for all p 's. It follows that $C_p^{(\kappa)} > 0$ for every $p \geq 2$ as desired.

In the case $\kappa = \frac{2}{27}$ (equivalently $\alpha = \frac{2}{3}$) the function $\tilde{C}_p^{(\kappa)}$ is explicitly known and correspond to the function $9^{-p} C_p$ in [4] and thus grows like \sqrt{p} when $p \rightarrow \infty$. See [21, Proposition 5] for details. In the hyperbolic regime a different behavior appears:

Lemma 4 *When $\kappa < \frac{2}{27}$, the increasing sequence $\tilde{C}_p^{(\kappa)}$ converges to $(\alpha \delta_\kappa)^{-1}$ as $p \rightarrow \infty$.*

Proof By monotonicity $\lim_{p \rightarrow \infty} \tilde{C}_p^{(\kappa)}$ exists in $(0, \infty]$. Let $\Xi^{(\kappa)}$ be the random walk started from 2 with i.i.d. increments distributed as $\mathbf{q}^{(\kappa)}$. Since its drift δ_κ is positive, the stopping time $\tau_2 = \inf\{i \geq 0 : \Xi_i^{(\kappa)} < 2\}$ has a positive chance to be infinite. Since $\tilde{C}_p^{(\kappa)}$ is harmonic on $\{2, 3, 4, \dots\}$, the process $\tilde{C}_{\Xi_n \wedge \tau_2}^{(\kappa)}$ is a martingale and thus by dominated convergence (notice that $\Xi_n^{(\kappa)} \rightarrow \infty$ on the event $\{\tau_2 = \infty\}$) we have

$$\tilde{C}_2^{(\kappa)} = 0 \cdot \mathbb{E}[\mathbf{1}_{\tau_2 \leq n}] + E[\tilde{C}_{\Xi_n^{(\kappa)}}^{(\kappa)} \mathbf{1}_{\tau_2 > n}] \xrightarrow{n \rightarrow \infty} \lim_{p \rightarrow \infty} \tilde{C}_p^{(\kappa)} P(\tau_2 = \infty).$$

To finish the proof and compute $P(\tau_2 = \infty)$ we remark that the random walk $\Xi^{(\kappa)}$ has increments bounded above by 1 so that we can apply the ballot theorem [1, Theorem 2]: if $(\xi_i^{(\kappa)})_{i \geq 0}$ are i.i.d. copies of law $\mathbf{q}^{(\kappa)}$ we have

$$\begin{aligned} P(\tau_2 = \infty) &= P(2 + \xi_1^{(\kappa)} + \xi_2^{(\kappa)} + \dots + \xi_i^{(\kappa)} \geq 2, \forall i \geq 1) \\ &= P(1 + \xi_0^{(\kappa)} + \xi_1^{(\kappa)} + \xi_2^{(\kappa)} + \dots + \xi_i^{(\kappa)} \geq 2, \forall i \geq 0 \mid \xi_0^{(\kappa)} = 1) \\ &= \frac{P(\xi_0^{(\kappa)} + \xi_1^{(\kappa)} + \dots + \xi_i^{(\kappa)} > 0, \forall i \geq 0)}{P(\xi_0^{(\kappa)} = 1)} = \frac{\delta_\kappa}{\alpha}. \end{aligned}$$

□

2.3 Peeling construction

We now construct the desired lattices \mathbf{T}_κ . The method is mimicked from [8] and the idea is to reverse the procedure used in Sect. 2.1 in order to provide an algorithmic device called the *peeling process* [4] that constructs a sequence of growing triangulations with a boundary. For a particular peeling procedure, these triangulations are shown to exhaust the plane and define an infinite triangulation with one end.

2.3.1 General peeling

The peeling process depends on an algorithm \mathcal{A} which associates with every triangulation $t \in \mathcal{T}_B$ one of its boundary edges. From this, we construct a growing sequence of triangulations with a boundary $(\mathbf{T}_n^{(\kappa), \mathcal{A}} : n \geq 0)$ as follows. To start with, $\mathbf{T}_0^{(\kappa), \mathcal{A}}$ is the root triangulation composed by a single oriented edge (seen as a triangulation of the 2-gon). Inductively, assume that $\mathbf{T}_n^{(\kappa), \mathcal{A}}$ is constructed. We write $p = |\partial \mathbf{T}_n^{(\kappa), \mathcal{A}}|$ and denote by $a \in \partial \mathbf{T}_n^{(\kappa), \mathcal{A}}$ the edge chosen by the algorithm. Notice that this choice may depend on an other source of randomness. Independently of $\mathbf{T}_n^{(\kappa), \mathcal{A}}$ and of the possible extra randomness of \mathcal{A} , the next triangulation $\mathbf{T}_{n+1}^{(\kappa), \mathcal{A}}$ is obtained as follows: With probability

$$q_{1,p}^{(\kappa)} := q_1^{(\kappa)} \cdot \frac{\tilde{C}_{p+1}^{(\kappa)}}{\tilde{C}_p^{(\kappa)}},$$

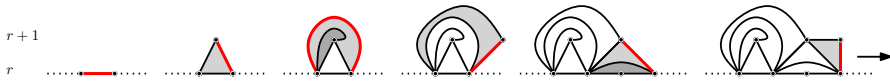


Fig. 2 Illustration of the peeling by layers algorithm. When all the vertices of the boundary are at distance r from the root, we turn around the boundary and create a new layer of vertices at distance $r + 1$ which propagates like a “front”

the triangulation $T_{n+1}^{(\kappa), \mathcal{A}}$ is obtained from $T_n^{(\kappa), \mathcal{A}}$ by gluing a triangle onto the edge a as in Fig. 1 left. Otherwise, for $-p + 2 \leq i \leq -1$ or $1 \leq i \leq p - 2$ with probability

$$\frac{1}{2}q_{-|i|, p}^{(\kappa)} := \frac{1}{2}q_{-|i|}^{(\kappa)} \cdot \frac{\tilde{C}_{p-|i|}^{(\kappa)}}{\tilde{C}_p^{(\kappa)}},$$

we glue a triangle on a and identify its third vertex with the $|i|$ th vertex on the left or on the right of a depending on the sign of i as in Fig. 1 right. Finally, independently of these choices we fill in the hole created with an independent Boltzmann triangulation of the $|i| + 1$ -gon with parameter κ to get $T_{n+1}^{(\kappa), \mathcal{A}}$. In the special case when $p = 1$ we could fill the hole of perimeter 2 with the trivial triangulation made of a single oriented edge which amounts to close the hole on itself. According to (9) these probability transitions sum-up to 1.

Assume now that the algorithm \mathcal{A} is deterministic, i.e. can be seen as a function $\mathcal{A}(t) \in \text{Edges}(\partial t)$. Using the same calculations as in Sect. 2.1 one sees by induction that for every $n \geq 0$ and for every triangulation $t \in \mathcal{T}_p$ that is a possible outcome of the construction at step n (that is $P(T_n^{(\kappa), \mathcal{A}} = t) > 0$) we have

$$P(T_n^{(\kappa), \mathcal{A}} = t) = C_p^{(\kappa)} \cdot \kappa^{|t|}. \tag{10}$$

We remark that the right-hand side of the last display does not depend on the order in which the peeling steps are performed nor on \mathcal{A} as long as $P(T_n^{(\kappa), \mathcal{A}} = t) > 0$. The last display can also be extended to the case when n is replaced by a stopping time τ , that is, a random variable such that $\{\tau = n\}$ is a measurable function of $T_n^{(\kappa), \mathcal{A}}$.

Defining T_κ . The law of the structure of $(T_n^{(\kappa), \mathcal{A}})$ does depend on the algorithm \mathcal{A} and it could happen that the increasing union $\cup_{n \geq 0} T_n^{(\kappa), \mathcal{A}}$ does not create a triangulation of the plane (imagine for example that one edge is never peeled). To prevent this, we now use a particular *deterministic* algorithm called \mathcal{L} for “layers”. Specifically, we first peel the left hand side and then the right hand side of the root edge during the first two steps and then, at step n , peel the right-most edge on $\partial T_n^{(\kappa), \mathcal{L}}$ which belongs to the triangle we just revealed. See Fig. 2 and [21, Section 4.1].

We easily prove by induction that this algorithm associates with every triangulation visited $t \in \mathcal{T}_B$ an edge $\mathcal{L}(t) \in \partial t$ which minimizes $\left\{ d_{\text{gr}}^t(a, e^-) : a \in \partial t \right\}$, where e^- is the origin of the root edge and $d_{\text{gr}}^t(\cdot, \cdot)$ is the graph distance inside t . An argument similar to [8, Proposition 3.6] or [35, Lemma 4.4] shows that using this peeling construction, every vertex on the boundary of the growing triangulations will be swallowed

in the process eventually and so

$$\mathbf{T}_\kappa \stackrel{\text{def.}}{:=} \bigcup_{n \geq 0} \mathbf{T}_n^{(\kappa), \mathcal{L}},$$

defines an infinite triangulation of the plane. We will now check that this random lattice is κ -Markovian. If $\tau_r < \infty$ is the first time when no vertex on $\partial \mathbf{T}_n^{(\kappa), \mathcal{L}}$ is at distance less than $r - 1$ from e^- then an easy geometric argument (see [12, Proposition 6] for a similar result) shows that

$$\mathbf{T}_{\tau_r}^{(\kappa), \mathcal{L}} = B_r^\bullet(\mathbf{T}_\kappa). \tag{11}$$

Hence, by (10) (and the remark following it), for any $t \in \mathcal{T}_p$ which is the hull of a ball of radius r we have

$$P(B_r^\bullet(\mathbf{T}_\kappa) = t) = P(\mathbf{T}_{\tau_r}^{(\kappa), \mathcal{L}} = t) = C_p^{(\kappa)} \cdot \kappa^{|t|}. \tag{12}$$

Although the last display is sufficient to characterize the law of \mathbf{T}_κ , it is not clear how it implies that \mathbf{T}_κ is κ -Markovian. To see this, we will consider another exploration process. Fix a triangulation $\Delta \in \mathcal{T}_B$ and let $t_0 \subset t_1 \subset \dots \subset t_{n_0} = \Delta$ be an increasing sequence of triangulations with a boundary starting from the root edge so that t_{i+1} is obtained from t_i by the peeling of one (necessarily unique) edge $a_i \in \partial t_i$ for $i \leq n_0 - 1$. We consider the following modification of the algorithm \mathcal{L} :

$$\mathcal{L}'(t) = \begin{cases} a_i \text{ if } t = t_i & \text{for } i \in \{0, 1, \dots, n_0 - 1\} \\ \text{an edge } a \text{ minimizing } \{d_{\text{gr}}^t(a, e^-) : a \in \partial t\} & \text{otherwise.} \end{cases}$$

Here also, the peeling process with algorithm \mathcal{L}' will eventually swallow every vertex on the boundary of the growing triangulations and so the increasing union of $\mathbf{T}_n^{(\kappa), \mathcal{L}'}$ defines a random infinite triangulation of the plane denoted by \mathbf{T}'_κ . We first show that \mathbf{T}'_κ has the same distribution as \mathbf{T}_κ . Remark that after step n_0 , the peeling process with \mathcal{L}' reveals an edge minimizing $\{d_{\text{gr}}^t(a, e^-) : a \in \partial t\}$. From this and a few simple geometric considerations we deduce that there exists some $r_0 \geq 1$ (depending on Δ) such that for every $r \geq r_0$ we have

$$\mathbf{T}_{\tau'_r}^{(\kappa), \mathcal{L}'} = B_r^\bullet(\mathbf{T}'_\kappa),$$

where τ'_r is the first time at which no vertex of $\partial \mathbf{T}_n^{(\kappa), \mathcal{L}'}$ is at distance less than or equal to $r - 1$ from the origin. Using (10) again we deduce that for any $t \in \mathcal{T}_p$ which is the hull of a ball of radius r we have $P(B_r^\bullet(\mathbf{T}'_\kappa) = t) = C_p^{(\kappa)} \kappa^{|t|}$. Comparing this with (12) we conclude that $B_r^\bullet(\mathbf{T}'_\kappa)$ and $B_r^\bullet(\mathbf{T}_\kappa)$ have the same law for every $r \geq r_0$. Since \mathbf{T}_κ and \mathbf{T}'_κ are both triangulations of the plane this entails that they have the same law. Coming back to the exploration process with algorithm \mathcal{L}' , a moment's thought

shows that $\Delta \subset \mathbf{T}'_\kappa$ if and only if we have $T_i^{(\kappa), \mathcal{L}'} = t_i$ for every $i \in \{0, 1, \dots, n_0\}$. In particular we have

$$P(\Delta \subset \mathbf{T}'_\kappa) = P(T_{n_0}^{(\kappa), \mathcal{L}'} = \Delta) \stackrel{(10)}{=} C_{|\partial\Delta|}^{(\kappa)} \cdot \kappa^{|\Delta|}.$$

Since $\mathbf{T}_\kappa = \mathbf{T}'_\kappa$ in distribution, the last display still holds with \mathbf{T}'_κ replaced by \mathbf{T}_κ . Because Δ was arbitrary this indeed shows that \mathbf{T}_κ fulfills (1) and completes the construction.

3 Geometric properties

3.1 Back to the peeling construction

We first study in more details the peeling process of \mathbf{T}_κ . This will be used in the proofs of Theorems 2 and 3. Indeed, to study the volume growth of \mathbf{T}_κ we will explore it using the peeling by layers as in [4, 21] and to establish Theorem 3 one shall need to explore \mathbf{T}_κ along a simple random walk path as in [12].

In the last section we *constructed* \mathbf{T}_κ as the increasing union of triangulations given by an abstract peeling process. In the rest of the paper, however, we will think of \mathbf{T}_κ given but unknown and the peeling process as “embedded” in \mathbf{T}_κ and exploring it. In other words, for every algorithm \mathcal{A} , deterministic or using an extra source of randomness, we can couple a realization of \mathbf{T}_κ together with the sequence of growing triangulations $(T_n^{(\kappa), \mathcal{A}})$ such that the latter is a growing subset of \mathbf{T}_κ . Yet another way to express this is that we can explore \mathbf{T}_κ face by face (discovering the enclosed regions when needed) by peeling at each step an edge on the boundary of the current revealed part as long as this choice remains independent of the unexplored part. The proof of the above facts is merely a dynamical reformulation of Sect. 2.1 and is easily adapted from [12, Section 1.2] using the spatial Markov property of \mathbf{T}_κ . Although the law of $(T_n^{(\kappa), \mathcal{A}})$ depends on the algorithm \mathcal{A} , the law of

$$\left(P_n^{(\kappa)}, V_n^{(\kappa)} \right)_{n \geq 0} := \left(|\partial T_n^{(\kappa), \mathcal{A}}|, |T_n^{(\kappa), \mathcal{A}}| \right)_{n \geq 0}$$

does not depend on the algorithm \mathcal{A} . More precisely, from Sect. 2.3 we get that $P^{(\kappa)}$ is a Markov chain with transition probabilities given by

$$P(\Delta P_n^{(\kappa)} = i \mid P_n^{(\kappa)} = p) = q_{i,p}^{(\kappa)} \quad \text{for } i \leq 1, \tag{13}$$

where here and later $\Delta X_n = X_{n+1} - X_n$. Conditionally on $P^{(\kappa)}$ the increments of $V^{(\kappa)}$ are independent and distributed as

$$\Delta V_n^{(\kappa)} \stackrel{(d)}{=} \mathcal{B}_{-\Delta P_n^{(\kappa)}}, \tag{14}$$

where \mathcal{B}_i is the law of the internal volume of a Boltzmann triangulation of the $i + 1$ -gon with $\mathcal{B}_{-1} = 1$ by convention. Also, thanks to Lemma 4 the increments of the chain $P_n^{(\kappa)}$ converge as the perimeter tends to ∞ towards i.i.d. steps of law $q_i^{(\kappa)} = \lim_{p \rightarrow \infty} q_{i,p}^{(\kappa)}$. These numbers have a very natural geometric interpretation: they correspond to the peeling transitions probabilities in the lattice \mathbf{H}_α with α satisfying (2) which is the analogous of \mathbf{T}_κ but in the topology of the half-plane see [8, Section 3.3]. See also [7, Section 2.6]. We recall that $(q_i^{(\kappa)})_{i \leq 1}$ is the step distribution of the random walk $\Xi^{(\kappa)}$ started from 2. Finally, conditionally on $\Xi^{(\kappa)}$ construct $\Omega^{(\kappa)}$ such that $\Delta\Omega_n^{(\kappa)}$ are independent and distributed as $\Delta\Omega_n^{(\kappa)} = \mathcal{B}_{-\Delta\Xi_n^{(\kappa)}}$ for every $n \geq 0$.

Proposition 5 (Perimeter and volume growth during a peeling) *Fix $\kappa \in (0, \frac{2}{27})$ and recall the definitions of α and δ_κ in (2) and (8). We have*

$$\begin{aligned} n^{-1} P_n^{(\kappa)} &\xrightarrow[n \rightarrow \infty]{a.s.} \delta_\kappa, \\ n^{-1} V_n^{(\kappa)} &\xrightarrow[n \rightarrow \infty]{a.s.} \frac{\alpha(2\alpha - 1)}{\delta_\kappa}. \end{aligned}$$

Proof Recall the notation of Sect. 2.2. By (9) and (13) the Markov chain $(P_n^{(\kappa)})_{n \geq 0}$ has the law of Doob’s h -transform of the random walk $\Xi^{(\kappa)}$ (started from 2) of step distribution $q^{(\kappa)}$ by the function $p \mapsto \tilde{C}_p^{(\kappa)}$ which is harmonic on $\{2, 3, \dots\}$ and null for $p \leq 1$. By the results of [16], this process $P^{(\kappa)}$ has the same law as the walk $\Xi^{(\kappa)}$ conditioned on the event $\{\Xi_i^{(\kappa)} \geq 2 : \forall i \geq 0\}$. Since $\Xi^{(\kappa)}$ has a positive drift δ_κ , the last event has a positive probability. Using (14) and the definition of the process $(\Xi^{(\kappa)}, \Omega^{(\kappa)})$ it follows that

$$\left(P_n^{(\kappa)}, V_n^{(\kappa)} \right)_{n \geq 0} \stackrel{(d)}{=} \left(\Xi_n^{(\kappa)}, \Omega_n^{(\kappa)} \right)_{n \geq 0} \text{ conditioned on } \underbrace{\{\Xi_i^{(\kappa)} \geq 2, \forall i \geq 0\}}_{\text{positive proba.}} \tag{15}$$

In particular $P^{(\kappa)}$ and $\Xi^{(\kappa)}$ as well as $V^{(\kappa)}$ and $\Omega^{(\kappa)}$ share the same almost sure properties. Since $\Xi^{(\kappa)}$ and $\Omega^{(\kappa)}$ are random walk with independent increments the statement of the proposition reduces by the law of large number to computing the mean of their increments. The mean of the increment of $\Xi^{(\kappa)}$ is given by (8). For the increment of the random walk $\Omega^{(\kappa)}$ we use [35, Proof of Proposition 3.4]² which gives that for $i \leq 0$

$$E[\mathcal{B}_{-i}] = \frac{i(2i - 1)(1 - \alpha)}{(3\alpha - 2)i + 1}.$$

Plugging this into the definition of $\Omega^{(\kappa)}$ and using the explicit expression of the $q^{(\kappa)}$ given by (7) it follows after a few manipulations using the generating function of

² With the notation in [35, Proposition 3.4], we have $\theta = (1 - \alpha)/2$ where α is given by (2).

Catalan numbers that

$$E[\Delta\Omega_n] = \alpha + \sum_{i \geq 1} i(2i - 1)(1 - \alpha) \frac{2}{4^i} \frac{(2i - 2)!}{(i - 1)!(i + 1)!} \left(\frac{2}{\alpha} - 2\right)^i = \frac{\alpha(2\alpha - 1)}{\delta_\kappa}.$$

□

3.2 Volume growth

The goal of this section is to prove Theorem 2. We suppose that we discover \mathbf{T}_κ using the peeling algorithm with procedure \mathcal{L} which “turns” around the successive boundaries $\partial B_r^\bullet(\mathbf{T}_\kappa)$ for $r \geq 0$ in a cyclic fashion, see Fig. 2. The idea of the proof is similar to that of [21, Theorem 2] but is a little simpler in our case.

Proof of Theorem 2 Recall that the stopping time τ_r is the first time in the exploration process when no vertex on the boundary is at distance less than or equal to $r - 1$ from the origin of \mathbf{T}_κ and that $\mathbf{T}_{\tau_r}^{(\kappa), \mathcal{L}} = B_r^\bullet(\mathbf{T}_\kappa)$ by (11). Recall also that $P_n^{(\kappa)}$ and $V_n^{(\kappa)}$ respectively are the perimeter and the size of the explored triangulation after n steps of peeling. The proof is based on the following estimate:

Lemma 6 (Time to complete a layer) *For any $\varepsilon > 0$ there exists $\eta > 0$ and $c > 0$ such that we have*

$$P \left(\left| \frac{\tau_{r+1} - \tau_r}{P_{\tau_r}^{(\kappa)}} - \frac{2}{\alpha - \delta_\kappa} \right| \geq (P_{\tau_r}^{(\kappa)})^{-1/2+\varepsilon} \left| \mathbf{T}_{\tau_r}^{(\kappa), \mathcal{L}} \right| \right) \leq c \cdot (P_{\tau_r}^{(\kappa)})^{-\eta}.$$

Given the last lemma, the proof of Theorem 2 is easy to complete. Indeed using the fact that $P_{\tau_r} \rightarrow \infty$, choosing $\varepsilon \in (0, \frac{1}{2})$ we can combine the last lemma with the first statement of Proposition 5 to get that

$$\frac{\tau_{r+1}}{\tau_r} \xrightarrow[r \rightarrow \infty]{(P)} \frac{\alpha + \delta_\kappa}{\alpha - \delta_\kappa}.$$

Bootstrapping this into Lemma 6 we get after a few manipulations using the Borel-Cantelli lemma and Proposition 5 that we have $\tau_r^{1/r} \rightarrow \frac{\alpha + \delta_\kappa}{\alpha - \delta_\kappa}$ almost surely and moreover

$$\sum \left| \frac{\tau_{r+1}}{\tau_r} - \frac{\alpha + \delta_\kappa}{\alpha - \delta_\kappa} \right| < \infty, \quad a.s.$$

This implies the almost sure convergence

$$\left(\frac{\alpha - \delta_\kappa}{\alpha + \delta_\kappa} \right)^r \tau_r \xrightarrow[r \rightarrow \infty]{a.s.} \Pi_\kappa$$

where Π_κ is a random variable on $(0, \infty)$. The proof of Theorem 2 is completed using (11) and Proposition 5.

Proof of Lemma 6 The argument is similar to [21, Proposition 10]. Fix $r \geq 0$ and consider the situation at time τ_r . The peeling process will now go cyclically around $\partial B_r^*(\mathbf{T}_\kappa) = \partial T_{\tau_r}^{(\kappa), \mathcal{L}}$ from left to right swallowing the vertices of $\partial B_r^*(\mathbf{T}_\kappa)$ (see Fig. 2) until none is left on the active boundary which happens at time τ_{r+1} . More precisely it can be checked that at a time $\tau_r \leq i \leq \tau_{r+1}$ the boundary of $T_i^{(\kappa), \mathcal{L}}$ is made of two types of vertices : those vertices whose distance to the origin or “height” is r and those having height $r + 1$. In case there are vertices of the two types, they form two connected intervals and the edge that is peeled off at time i is the only edge whose left vertex is height $r + 1$ and whose right one is at height r . We set $x_0 = P_{\tau_r}^{(\kappa)}$, we denote by A_i the number of vertices at height r and by B_i the number of vertices at height $r + 1$ so that we have $A_{\tau_r} = x_0$ and $B_{\tau_r} = 0$. For $\tau_r \leq i \leq \tau_{r+1}$ the process $(A_i, B_i, P_i^{(\kappa)})$ is a Markov chain whose transition kernel Q is given by:

$$\begin{aligned}
 Q((a, b, p), (a, b + 1, p + 1)) &= q_{1,p}^{(\kappa)} \\
 Q((a, b, p), (a - i, b, p - i)) &= \frac{1}{2} \cdot q_{-i,p}^{(\kappa)} \quad \text{for } 1 \leq i \leq a - 1 \\
 Q((a, b, p), \dagger) &= \sum_{i=a}^{p-2} \frac{1}{2} \cdot q_{-i,p}^{(\kappa)} \\
 Q((a, b, p), (a, b - i, p - i)) &= \frac{1}{2} q_{-i,p}^{(\kappa)} \quad \text{for } 1 \leq i \leq b - 1 \\
 Q((a, b, p), (a - (i - b), 0, p - i)) &= \frac{1}{2} q_{-i,p}^{(\kappa)} \quad \text{for } b \leq i \leq p - 2.
 \end{aligned}$$

The third line means that the chain is stopped when A_i reaches 0, which coincides with the completion of the r th layer at time τ_{r+1} and we wish to estimate the time needed to reach this state. Let us ignore for a moment the fifth line in the transition kernel. By (15) we could then approximate $\tau_{r+1} - \tau_r$ by the time θ_{x_0} needed for a random walk started from x_0 and having only negative jumps of size $-i \leq -1$ with probability $\frac{1}{2} q_{-i}^{(\kappa)}$ to reach \mathbb{Z}^- . For this walk, since $\sum_{i \geq 1} i \frac{1}{2} q_{-i}^{(\kappa)} = \frac{\alpha - \delta_\kappa}{2}$ easy estimates show that for every $\varepsilon > 0$

$$P \left(\left| \frac{\theta_{x_0}}{x_0} - \frac{2}{\alpha - \delta_\kappa} \right| \geq x_0^{-1/2+\varepsilon} \right) \leq c \cdot x_0^{-\eta}$$

for some $\eta > 0$ and $c > 0$ and we would thus get the statement of the lemma. Our approximation (forgetting the fifth line in the transition) does not play a big role since the steps corresponding to this transition can only appear in the very beginning of the completion of the layer when a peeling step towards the left swallows all the vertices at height $r + 1$ and reach vertices at height r . See Fig. 3.

However, an easy calculation shows that at step $\tau_r + i$ there are roughly $\frac{\alpha + \delta_\kappa}{2} \cdot i$ edges on $\partial T_{\tau_r+i}^{(\kappa), \mathcal{L}}$ separating the vertices at height r from the left of the current edge to peel. Since $\Delta P_i^{(\kappa)}$ has exponential tails, we deduce that the last phenomenon can

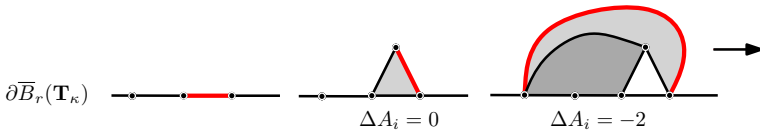


Fig. 3 In the beginning of the peeling of the r th layer, a few peeling steps towards the *left* may contribute to swallowing the vertices of $\partial B_r^*(\mathbf{T}_\kappa)$

only appear in the first few $\ln(\tau_r)$ steps after τ_r . This does not perturb the last display too much and after a few estimates left to the reader we get

$$P\left(\left|\frac{\tau_{r+1} - \tau_r}{P_{\tau_r}^{(\kappa)}} - \frac{2}{\alpha - \delta_\kappa}\right| \geq (P_{\tau_r}^{(\kappa)})^{-1/2+\varepsilon} \mid \mathbf{T}_{\tau_r}^{(\kappa), \mathcal{L}}\right) \leq c \cdot (P_{\tau_r}^{(\kappa)})^{-\eta},$$

for some $\eta > 0$ and $c > 0$ as desired. □

3.3 Anchored expansion

Like in many stochastic examples which are hyperbolic in flavor (e.g. supercritical Galton–Watson trees), the randomness of \mathbf{T}_κ implies that any possible pattern will happen somewhere in the lattice and thus destroys any hope of having a positive Cheeger expansion constant. The latter has to be replaced by a more refined notion: the anchored expansion constant.

If G is a connected graph with an origin vertex ρ and if S is a subset of vertices of G , we denote by $|\partial_E S|$ the number of edges having an endpoint in S and the other outside S . Also, write $|S|_E$ for the sum of the degrees of the vertices of S . The edge anchored expansion constant of G is defined by

$$i_E^*(G) = \liminf_{n \rightarrow \infty} \left\{ \frac{|\partial_E S|}{|S|_E} : S \subset \text{Vertices}(G), S \text{ finite and connected}, \rho \in S, |S|_E \geq n \right\}.$$

It is easy to see that the above definition does not depend on the origin point ρ . See [37] for background on anchored expansion. As in [35, Theorem 2.2] the spatial Markov property enables us to deduce almost effortlessly that our lattices have a.s. a positive anchored expansion constant in the hyperbolic regime:

Proposition 7 (Edge anchored expansion) *For $\kappa < \frac{2}{27}$ we have $i_E^*(\mathbf{T}_\kappa) > 0$ almost surely.*

By ergodicity (see the proof of Proposition 10 below) the variable $i_E^*(\mathbf{T}_\kappa)$ is actually almost surely constant. We do not have a good guess for its correct value. Before proving Proposition 7 we state a lemma. For $p \geq 2$ and $n \geq 0$ we denote by $\mathcal{T}_{n,p}$ the set of all finite rooted triangulations of the p -gon with n vertices where the root edge can be any oriented edge of the map.

Lemma 8 *There exists $m_0 \geq 1$ and $c_1 > 0$ such that for every $m \geq m_0$ and every $n \geq mp$ we have*

$$\#\mathcal{T}_{n+p,p} \leq c_1 \sqrt{\frac{p}{n^3}} 9^p \left(\frac{27}{2}\right)^n.$$

Proof For $p \geq 2$ and $n \geq 0$, if $\mathcal{T}_{n+p,p}^{\rightarrow}$ is the set of all 2-connected triangulations of the p -gon of size $n + p$ such that the hole is on the right-hand side of the root edge then from [25] we read

$$\#\mathcal{T}_{n+p,p}^{\rightarrow} = \frac{(2p - 3)!}{(p - 2)!(p - 2)!} 2^{n+1} \frac{(2p + 3n - 4)!}{n!(2p + 2n - 2)!}.$$

An application of Euler’s formula shows that such a triangulation has exactly $3n + 2p - 3$ edges. Hence we deduce that $\#\mathcal{T}_{n+p,p} \leq 2(3n + 2p - 3)\#\mathcal{T}_{n+p,p}^{\rightarrow}$. Suppose now that $n \geq mp$ for $m \geq 1$. Using the last display and Stirling’s formula we have for a constant $c > 0$ that may vary from line to line

$$\begin{aligned} \#\mathcal{T}_{n+p,p} &\leq 2(3n + 2p - 3)\#\mathcal{T}_{n+p,p}^{\rightarrow} \leq cn \frac{(2p - 3)!}{(p - 2)!(p - 2)!} 2^{n+1} \frac{(2p + 3n - 4)!}{n!(2p + 2n - 2)!} \\ &\leq cn \sqrt{p} 4^p 2^n n^{-1/2} \frac{(2p + 3n - 4)^{2p+3n-4}}{n^n (2p + 2n - 2)^{2p+2n-2}} \\ &\leq c \sqrt{\frac{p}{n^3}} 9^p \left(\frac{27}{2}\right)^n \frac{\left(1 + \frac{2p-4}{3n}\right)^{2p+3n-4}}{\left(1 + \frac{p-1}{n}\right)^{2p+2n-2}} \\ &\leq c \sqrt{\frac{p}{n^3}} 9^p \left(\frac{27}{2}\right)^n \frac{\left(1 + \frac{2p}{3n}\right)^{2p+3n}}{\left(1 + \frac{p}{n}\right)^{2p+2n}}. \end{aligned}$$

We will show that if $n \geq mp$ with m sufficiently large, then the last fraction is the preceding display is smaller than one. To see this, write $n = \alpha p$ so that the fraction can be written as

$$\left(\frac{\left(1 + \frac{2}{3}\alpha\right)^{2\alpha+3}}{\left(1 + \alpha\right)^{2\alpha+2}}\right)^n.$$

An easy series expansion around $\alpha = 0$ shows that $\frac{\left(1 + \frac{2}{3}\alpha\right)^{2\alpha+3}}{\left(1 + \alpha\right)^{2\alpha+2}} = 1 - \frac{\alpha^2}{3} + o(\alpha^2)$. This implies our claim and finishes the proof of the lemma. □

Proof of Proposition 7 First, in the definition of $i_E^*(\mathbf{T}_k)$ we can restrict ourself to those sets S such that $\mathbf{T}_k \setminus S$ has only one (infinite) component because filling-in the finite holes decreases the boundary size and increases the volume. We then consider the triangulation with one hole \bar{S} obtained by adding all the faces adjacent to a vertex of S as well as the finite regions enclosed. One may check that $|\partial \bar{S}| \leq |\partial_E S|$. By Euler’s

relation we also get that $3|\bar{S}| = |\partial\bar{S}| + 3 + \#\text{Edges}(\bar{S})$, hence $|\bar{S}| \geq \#\text{Edges}(\bar{S})/3$. Since we also have $\#\text{Edges}(\bar{S}) \geq \frac{1}{2}|\bar{S}|_E$ we get that $|\bar{S}| \geq \frac{1}{6}|\bar{S}|_E$ and so

$$\frac{|\partial\bar{S}|}{|\bar{S}|} \leq 6 \frac{|\partial_E \bar{S}|}{|\bar{S}|_E}.$$

To prove the proposition, it is thus sufficient to show that the ratio $|\partial A|/|A|$ is bounded away from 0 for all triangulations $A \in \mathcal{T}_B$ such that $A \subset \mathbf{T}_\kappa$. For this we crudely use a first moment method. Fix $m \geq 1$. We have

$$\begin{aligned} P(\exists A \subset \mathbf{T}_\kappa : |A| > (m + 1)|\partial A|) &\leq E\left[\#\{A \subset \mathbf{T}_\kappa : |A| > (m + 1)|\partial A|\}\right] \\ &= \sum_{p \geq 1} \sum_{n > mp} \sum_{A \in \mathcal{T}_{n+p,p}} P(A \subset \mathbf{T}_\kappa) \\ &\stackrel{(1)}{=} \sum_{p \geq 1} \sum_{n > mp} C_p^{(\kappa)} \kappa^{n+p} \#\mathcal{T}_{n+p,p}. \end{aligned}$$

At this point we use Lemma 8 and get for $n \geq mp$ with $m \geq m_0$

$$P(\exists A \subset \mathbf{T}_\kappa : |A| > (m + 1)|\partial A|) \leq c_1 \sum_{p \geq 1} C_p^{(\kappa)} (9\kappa)^p \sqrt{p} \sum_{n > mp} \left(\frac{27}{2} \cdot \kappa\right)^n n^{-3/2}.$$

Since $\kappa < \frac{2}{27}$ the last sum is easily seen to be smaller than $c_2(\frac{27}{2} \cdot \kappa)^{mp}$ for some constant $c_2 > 0$ depending on κ . Also, from Lemma 4 we have $C_p^{(\kappa)} \leq c_3 \cdot \beta^{-p}$ for some constant $c_3 > 0$ still depending on κ . Hence we have

$$P(\exists A \subset \mathbf{T}_\kappa : |A| > (m + 1)|\partial A|) \leq c_1 c_2 c_3 \sum_{p \geq 1} \sqrt{p} \left(\frac{9\kappa}{\beta} \cdot \left(\kappa \frac{27}{2}\right)^m\right)^p.$$

Since $\kappa < \frac{2}{27}$, by choosing m large enough, we can make $\frac{9\kappa}{\beta} \cdot \left(\kappa \frac{27}{2}\right)^m$ as small as we wish and thus the last probability tends to 0 as $m \rightarrow \infty$. This indeed implies that $P(i_E^*(\mathbf{T}_\kappa) = 0) = 0$ and completes the proof of the proposition. \square

4 Simple random walk

In this section, we study the simple random walk on \mathbf{T}_κ . The special case $\kappa = \frac{2}{27}$ has already received a lot of attention: it is known that the UIPT is recurrent [27] and subdiffusive in the quadrangular case [12]. In this section we prove Theorem 3 and thus suppose that $\kappa \in (0, \frac{2}{27})$.

First of all, Proposition 7 combined with the result of [36] shows that

$$\mathbf{T}_\kappa \text{ is almost surely transient for } \kappa < \frac{2}{27}. \tag{16}$$

In the bounded degree case, a positive anchored expansion constant is even sufficient to imply positive speed for the simple random walk as shown by Virag [37]. Unfortunately, the lack of a uniform bound on the degrees in \mathbf{T}_κ prevents us from using this nice result and we shall go through a rather winding but bucolic bypass. The strategy to prove Theorem 3 is the following:

$$\begin{array}{lll}
 \text{study of the peeling along a SRW} & \implies & \text{non (intersection property)} \\
 & \text{Section 4.2} & \\
 & \implies & \text{non Liouville} \\
 & \text{[13]} & \\
 & \implies & \text{positive speed.} \\
 \text{[11] and Section 4.3} & & (17)
 \end{array}$$

Let us introduce a piece of notation. Conditionally on \mathbf{T}_κ consider a simple random walk (at each step, independently of the past, walk through one adjacent oriented edge uniformly at random) started from the *target* of the root edge and denote by $(\mathbf{E}_i)_{i \geq 0}$ the sequence of oriented edges traversed by the walk where by convention \mathbf{E}_0 is the root edge. We also denote by X_0, \dots, X_n the successive vertices visited by the walk, i.e. X_i is the origin of the oriented edge \mathbf{E}_i .

4.1 Reversibility and ergodicity

The notation \overleftarrow{e} stands for the reversed oriented edge \overrightarrow{e} .

Proposition 9 (Reversibility) *For any $\kappa \in (0, \frac{2}{27}]$ and every $i \geq 0$ we have the equalities in distribution*

$$\begin{array}{l}
 \text{(i)} \quad (\mathbf{T}_\kappa; \overrightarrow{E}_0) = (\mathbf{T}_\kappa; \overleftarrow{E}_0) \\
 \text{(ii)} \quad (\mathbf{T}_\kappa; \overrightarrow{E}_0, \dots, \overrightarrow{E}_i) = (\mathbf{T}_\kappa; \overleftarrow{E}_i, \dots, \overleftarrow{E}_0).
 \end{array}$$

Combining the statements of the last proposition we deduce that $(\mathbf{T}_\kappa; \overrightarrow{E}_0) = (\mathbf{T}_\kappa; \overleftarrow{E}_i) = (\mathbf{T}_\kappa; \overrightarrow{E}_i)$ in distribution. This proves that the law of the lattice is unchanged under re-rooting along a simple random walk path. We say in short that \mathbf{T}_κ is a stationary (in our case also reversible) random graph, see [11, Definition 1.3]. We refer to [11, Section 2.1] for more details about the connections between the concepts of stationary (and reversible) random graphs, ergodic theory, unimodularity, mass-transport principle and measured equivalence relations. Note that in the critical case $\kappa = \frac{2}{27}$, the stationarity of the UIPT is an easy consequence of the fact that it is a local limit of uniformly rooted finite graphs (see [9, Theorem 3.2]). Although we conjecture that \mathbf{T}_κ can similarly be obtained as the local limit of uniformly rooted random triangulations in high genus (Conjecture 1) we provide a direct proof of Proposition 9.

Proof Point (i) is easy: the map obtained from \mathbf{T}_κ by reversing the root edge is still κ -Markovian and thus has the same distribution by Theorem 1. Let us now turn to (ii). Let $i, r > 0$. Fix a triangulation with a boundary $t \subset \mathcal{T}_B$ and a path $w = (\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_i)$ such that w could be the result of a i -step random walk inside t (with the convention

that \mathbf{e}_0 is the root edge of t). We denote by x_0, x_1, \dots, x_{i+1} the vertices visited by the path. Furthermore, we assume that t is the hull of the ball of radius r around w in the sense that it is made of all the faces containing a vertex at graph distance (inside t) smaller than or equal to $r - 1$ from the set $\{x_0, x_1, \dots, x_{i+1}\}$ as well as the finite regions enclosed. We write $t = B_r^\bullet(\{x_0, \dots, x_{i+1}\})$. We now ask what is the probability that, inside \mathbf{T}_κ , the first i steps of the walk correspond to w and that the hull of radius r around these is t :

$$\begin{aligned} P(\vec{E}_k = \mathbf{e}_k, \forall k \leq i \text{ and } B_r^\bullet(\{x_0, \dots, x_{i+1}\}) = t) \\ = P(t \subset \mathbf{T}_\kappa) \cdot P(\vec{E}_k = \mathbf{e}_k, \forall k \leq i \mid t \subset \mathbf{T}_\kappa) \\ \stackrel{(1)}{=} C_{|\partial t|}^{(\kappa)} |t| \prod_{k=1}^i \text{deg}(x_k)^{-1}. \end{aligned}$$

We now remark that the last probability is exactly the same if we replace (t, w) with the same triangulation t and the reversed path $\overleftarrow{w} = (\overleftarrow{e}_i, \dots, \overleftarrow{e}_0)$. Since r is arbitrary this proves that $(\mathbf{T}_\kappa; \vec{E}_0, \dots, \vec{E}_i)$ and $(\mathbf{T}_\kappa; \overleftarrow{E}_i, \dots, \overleftarrow{E}_0)$ indeed have the same law.

We consider $(\mathcal{G}^{\leftrightarrow}, d_{\text{loc}}, \mathcal{K})$ the set of all locally finite connected rooted graphs together with an infinite two-sided path $(\dots, \mathbf{e}_1, \mathbf{e}_0, \mathbf{e}_1, \dots)$ made of oriented edges on them, endowed with (an extension of) the local distance and the associated Borel σ -field. There is a natural shift operation θ on this space

$$\theta : (g; (\mathbf{e}_i)_{i \in \mathbb{Z}}) \mapsto (g; (\mathbf{e}_{i-1})_{i \in \mathbb{Z}}).$$

If (g, \mathbf{e}) is a rooted graph we denote by $\mathbb{P}_{(g, \mathbf{e})}$ the law of $(g, (\mathbf{E}_i)_{i \in \mathbb{Z}})$ where $(\vec{E}_i)_{i \geq 1}$ and $(\overleftarrow{E}_{-i})_{i \geq 1}$ are (the oriented edges visited by) two independent simple random walks started respectively from the extremity and the origin of \mathbf{e} . Recall that the underlying probability relative to the lattice \mathbf{T}_κ is denoted by P . We then denote by $P \otimes \mathbb{P}$ the probability measure on $\mathcal{G}^{\leftrightarrow}$:

$$P \otimes \mathbb{P} = \int P(d\mathbf{T}_\kappa) \mathbb{P}_{\mathbf{T}_\kappa}.$$

Under this probability, conditionally on \mathbf{T}_κ the path $(\dots, \overleftarrow{E}_{-1}, \overleftarrow{E}_0, \overrightarrow{E}_1, \dots)$ is a two-sided random walk which is simply obtained by containing two independent simple random walks started from the two extremities of the root edge \overrightarrow{E}_0 .

Proposition 10 (Ergodicity) *For $\kappa \in (0, \frac{2}{27})$, the shift operation θ is ergodic for $P \otimes \mathbb{P}$.*

Proof It follows from the last proposition that the measure $P \otimes \mathbb{P}$ is invariant by θ and θ^{-1} . We will now show that this shift is ergodic. Let $A \in \mathcal{K}$ be such that $A = \theta A$ up to set of zero $P \otimes \mathbb{P}$ -measure. We will first show that

$$\mathbb{P}_{\mathbf{T}_\kappa}(A) \in \{0, 1\} \quad P\text{-almost surely.} \tag{18}$$

To show this, we adapt the proof of [32, Theorem 5.1]. Let $\varepsilon > 0$. Then there exists $r \geq 1$ and an event $A_r \in \mathcal{K}$ which only depends on the steps in $[-r + 1, r - 1]$ of the two-sided walk such that we have

$$P \otimes \mathbb{P}(A \Delta A_r) \leq \varepsilon.$$

Now observe that *conditionally on* \mathbf{T}_κ the two events $\theta^r A_r$ and $\theta^{-r} A_r$ are independent. Using this observation we have

$$\begin{aligned} \int P(d\mathbf{T}_\kappa) \mathbb{P}_{\mathbf{T}_\kappa}(A)^2 &= \int P(d\mathbf{T}_\kappa) \mathbb{P}_{\mathbf{T}_\kappa}(\theta^r A) \mathbb{P}_{\mathbf{T}_\kappa}(\theta^{-r} A) \\ &\geq \int P(d\mathbf{T}_\kappa) \mathbb{P}_{\mathbf{T}_\kappa}(\theta^r A_r) \mathbb{P}_{\mathbf{T}_\kappa}(\theta^{-r} A_r) - 2\varepsilon \\ &= \int P(d\mathbf{T}_\kappa) \mathbb{P}_{\mathbf{T}_\kappa}(\theta^r A_r \cap \theta^{-r} A_r) - 2\varepsilon \\ &\geq \int P(d\mathbf{T}_\kappa) \mathbb{P}_{\mathbf{T}_\kappa}(\theta^r A \cap \theta^{-r} A) - 4\varepsilon = \int P(d\mathbf{T}_\kappa) \mathbb{P}_{\mathbf{T}_\kappa}(A) - 4\varepsilon. \end{aligned}$$

Since ε is arbitrary we get that $\int P(d\mathbf{T}_\kappa) \mathbb{P}_{\mathbf{T}_\kappa}(A)^2 = \int P(d\mathbf{T}_\kappa) \mathbb{P}_{\mathbf{T}_\kappa}(A)$ and so (18) is proved. Now consider the event B defined by $\mathbf{T}_\kappa \in B \iff \mathbb{P}_{\mathbf{T}_\kappa}(A) = 1$. We will show that $P(B) \in \{0, 1\}$ and this will complete the proof. To that end, we adapt [4, Theorem 7.2]. Recall from Sect. 2.3.1 the construction of $\mathbf{T}_\kappa = \cup_n \mathbf{T}_n^{(\kappa), \mathcal{L}}$ using the peeling by layer algorithm. We can encode this construction using an infinite sequence of independent variables $(Z_{m,n,i})_{m \geq 2, n, i \geq 0}$: The variables $Z_{m,\cdot,\cdot}$ are i.i.d. and encode the information (location of the third vertex, the Boltzmann triangulation needed to fill the hole etc...) needed to perform a peeling step when the current boundary is of perimeter m . Then the variable $Z_{m,n,i}$ is used at step t if and only if

- $\partial \mathbf{T}_t^{(\kappa), \mathcal{L}} = m$
- $\max_{s < t} \partial \mathbf{T}_s^{(\kappa), \mathcal{L}} = n$
- if s is the first time such that $\partial \mathbf{T}_t^{(\kappa), \mathcal{L}} = n$ then $t - s = i$.

Using the fact that $\partial \mathbf{T}_t^{(\kappa), \mathcal{L}} \rightarrow \infty$, it is easy to see that if $(Z_{m,n,i})$ and $(\tilde{Z}_{m,n,i})$ are two such sequences which differ by only a finite number of terms then the growing sequences of triangulations $\mathbf{T}_t^{(\kappa), \mathcal{L}}$ and $\tilde{\mathbf{T}}_t^{(\kappa), \mathcal{L}}$ obtained from them differ by a finite change in the sense that there exists $s \geq 0$ such that

$$\mathbf{T}_\kappa \setminus \mathbf{T}_s^{(\kappa), \mathcal{L}} = \tilde{\mathbf{T}}_\kappa \setminus \tilde{\mathbf{T}}_s^{(\kappa), \mathcal{L}}.$$

A moment's though using the transience of \mathbf{T}_κ for $\kappa < 2/27$ shows that the realization of the event B only depends on $\mathbf{T}_\kappa \setminus \mathbf{T}_s^{(\kappa), \mathcal{L}}$ for any $s \geq 1$. Hence the event B is independent of all the $Z_{m,n,i}$ and thus of \mathbf{T}_κ . Thus, $P(B) \in \{0, 1\}$ by Kolmogorov's 0-1 law as desired. □

Let us give an application of the last result and show existence of the speed (but not the positivity of the latter). Combining the stationarity of \mathbf{T}_κ given after Proposition 9

together with Proposition 10, an application of Kingman’s subadditive ergodic theorem (see e.g. [11, Theorem 2.2] or [2, Proposition 4.8]) proves the following convergence

$$n^{-1}d_{gr}(X_0, X_n) \xrightarrow[n \rightarrow \infty]{P \otimes \mathbb{P} \text{ a.s.}} s_\kappa \in [0, 1]. \tag{19}$$

4.2 Non-intersection by peeling

Following the proof-sketch (17) we start by studying the intersection properties of \mathbf{T}_κ . Recall that a graph G is said to have the intersection property if *almost surely* the range of two independent simple random walks intersect infinitely often. It is easy to see that this property does not depend on the starting points of the walks. In this section we show:

Proposition 11 (Non-intersection) *When $\kappa \in (0, \frac{2}{27})$ almost surely \mathbf{T}_κ does not possess the intersection property.*

The key tool to prove Proposition 11 is the peeling process along a simple random path, specifically we explore \mathbf{T}_κ using a peeling algorithm that discovers the triangulation when necessary for the walk to make one more step. This was first used in [12] to establish the subdiffusivity of simple random walk on random quadrangulations. We start with the formal definition of this algorithm denoted \mathcal{W} (for “walk”) and then interpret it in terms of pioneer points. Recall that by convention, the first step of the walk is $\mathbf{E}_0 = (X_0, X_1)$ and so we shall start the process at the target of the root edge.

We define a sequence $\mathbf{e} = \mathbf{T}_0^{(\kappa), \mathcal{W}} \subset \mathbf{T}_1^{(\kappa), \mathcal{W}} \subset \dots \subset \mathbf{T}_n^{(\kappa), \mathcal{W}} \subset \dots \subset \mathbf{T}_\kappa$ of triangulations with boundaries and two random non decreasing functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ such that $f(0) = 0, g(0) = 1$ and

$$X_{g(k)} \in \mathbf{T}_{f(k)}^{(\kappa), \mathcal{W}}, \quad \text{for every } k \geq 0, \tag{20}$$

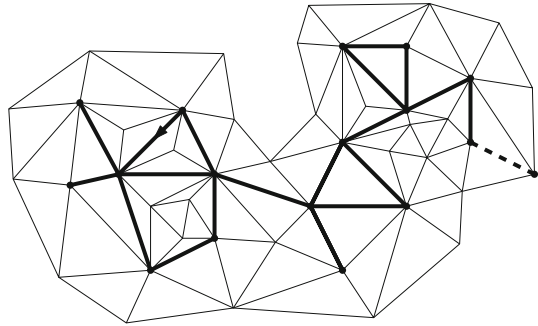
whose evolution is described by induction as follows. We have two cases:

- If the current position $X_{g(k)}$ of the simple random walk belongs to $\partial \mathbf{T}_{f(k)}^{(\kappa), \mathcal{W}}$, then choose an edge a on $\partial \mathbf{T}_{f(k)}^{(\kappa), \mathcal{W}}$ containing $X_{g(k)}$ and set $f(k + 1) := f(k) + 1$ and $g(k + 1) := g(k)$. The triangulation $\mathbf{T}_{f(k+1)}^{(\kappa), \mathcal{W}}$ is the map obtained from $\mathbf{T}_{f(k)}^{(\kappa), \mathcal{W}}$ after peeling the edge a .
- If the current position $X_{g(k)}$ of the simple random walk belongs to $\mathbf{T}_{f(k)}^{(\kappa), \mathcal{W}} \setminus \partial \mathbf{T}_{f(k)}^{(\kappa), \mathcal{W}}$ then we set $f(k + 1) := f(k)$ and $g(k + 1) := g(k) + 1$. In words, we let the walker move for one more step and do not touch the explored triangulation.

Note that we have $f(n) + g(n) = n + 1$ and $f, g \rightarrow \infty$. Although this algorithm has an extra randomness due to the simple random walk, the edges chosen to be revealed in the peeling process are independent of the unknown part, and thus the process $(|\partial \mathbf{T}_n^{(\kappa), \mathcal{W}}|, |\mathbf{T}_n^{(\kappa), \mathcal{W}}|)_{n \geq 0}$ has the same law as $(P_n^{(\kappa)}, V_n^{(\kappa)})_{n \geq 0}$ of Proposition 5. In the following, it will be important to have a geometric interpretation of this algorithm.

Interpretation. For any $k \geq 0$ consider the submap $\text{Hull}(X_1, \dots, X_k) \subset \mathbf{T}_\kappa$ formed by the faces that are adjacent to $\{X_1, X_2, \dots, X_k\}$ as well as the finite holes they enclose.

Fig. 4 The trace of the simple random walk about to reach a pioneer point



By convention $\text{Hull}(\emptyset)$ is the root edge. Then an easy geometric lemma (see [12, Proposition 7]) shows that the peeling times exactly correspond to the times when

$$X_{g(k)} \in \partial\text{Hull}(X_1, \dots, X_{g(k)-1}).$$

These points are called pioneer points. In other words, as soon as the walk reaches a pioneer point, the peeling process starts to discover the neighborhood of the current position (this typically takes a few steps of peeling) enabling the simple random walk to displace again (Fig. 4).

Lemma 12 *We have $P \otimes \mathbb{P}(X_0 \in \partial\text{Hull}(X_1, \dots, X_n) \text{ for all } n \geq 1) > 0$.*

Remark Note that in the peeling process along the simple random walk, the first pioneer point is X_1 and it is indeed possible that X_0 stays on the boundary of the discovered triangulation for ever. The lemma says that this happens with positive probability.

Proof Note that the events $\{X_0 \in \partial\text{Hull}(X_1, \dots, X_n)\}$ are clearly decreasing in n so their $P \otimes \mathbb{P}$ -probabilities tend to some constant $c \in [0, 1]$. We have to show that $c > 0$. By the stationarity and reversibility of the walk on the lattice (Proposition 9) we have

$$\begin{aligned} P \otimes \mathbb{P}(X_0 \in \partial\text{Hull}(X_1, \dots, X_n)) & \stackrel{\text{reversibility}}{=} P \otimes \mathbb{P}(X_n \in \partial\text{Hull}(X_{n-1}, \dots, X_0)) \\ & \stackrel{\text{stationarity}}{=} P \otimes \mathbb{P}(X_{n+1} \in \partial\text{Hull}(X_n, \dots, X_1)) \\ & \stackrel{\text{definition}}{=} P \otimes \mathbb{P}(X_{n+1} \text{ is pioneer}) \end{aligned}$$

and so

$$P \otimes \mathbb{P}(X_n \text{ is pioneer}) \xrightarrow[n \rightarrow \infty]{\text{decreasing}} c. \tag{21}$$

We now combine the transience of the walk with the peeling estimates of Proposition 5. We first claim that there exists a constant $\eta > 0$ such that we have the almost sure

convergence under $P \otimes \mathbb{P}$

$$n^{-1}\#\{X_0, \dots, X_n\} \xrightarrow[n \rightarrow \infty]{a.s.} \eta. \tag{22}$$

Indeed, if $R_n(G, X_0) = \#\{X_0, \dots, X_n\}$ denotes the range of the first n -step of a simple random walk on a graph G started from X_0 then we have $R_{n+m}(G, X_0) \leq R_n(G, X_0) + R_m(G, X_n)$. In our case, the stationarity (Proposition 9) of \mathbf{T}_κ implies that $R_m(\mathbf{T}_\kappa, X_n) = R_m(\mathbf{T}_\kappa, X_0)$ in distribution. Using the ergodicity of the simple random walk (Proposition 10) we deduce from Kingman’s subadditive ergodic theorem that $R_n/n \rightarrow \eta$ almost surely. Finally, since the underlying lattice is transient (16) we must have $\eta > 0$ (see [23] for details).

If we run the peeling algorithm for n steps (either peeling or walk step) then we have

$$\eta g(n) \underset{(22)}{\sim} \#\{X_0, \dots, X_{g(n)}\} \underset{(20)}{\leq} |\mathbf{T}_{f(n)}^{(\kappa), \mathcal{W}}| \underset{\text{Prop. 5}}{\sim} \frac{\alpha(2\alpha - 1)}{\delta_\kappa} f(n) \quad \text{a.s.}$$

We used notation $x_n \sim y_n$ if $x_n/y_n \rightarrow 1$ as $n \rightarrow \infty$. Since $f(n) + g(n) = n + 1$ we have

$$\liminf_{n \rightarrow \infty} \frac{f(n)}{n} \geq \left(1 + \frac{\alpha(2\alpha - 1)}{\eta\delta_\kappa}\right)^{-1} \quad \text{a.s.} \tag{23}$$

Notice that the discovery of a pioneer point automatically triggers at least one peeling step. On the other hand, an estimate similar to [8, Proposition 3.6] or [35, Lemma 4.4] shows that for any $k \geq 0$, when discovering the k th pioneer point, the expected number of peeling steps needed to perform a new random walk step is stochastically dominated by a geometric variable with a fixed parameter (indeed there is a constant $c > 0$, such that if the current point of the SRW is on the boundary of the discovered triangulation, when we peel a triangle adjacent to it, there is a probability at least $\frac{1}{2}q_{-1,p}^{(\kappa)} > c$ that the discovered triangle swallows this point and, after discovering the enclosed region, enables the walk to displace again). It follows that if we put $p(n) = \#\{i \leq g(n) : X_i \text{ is pioneer}\}$ then for some constant $\Lambda \geq 1$ we almost surely have

$$\limsup_{n \rightarrow \infty} \frac{f(n)}{p(n)} \leq \Lambda. \tag{24}$$

We deduce that a.s. the asymptotic proportion of random walk steps which are pioneer satisfies

$$\liminf_{n \rightarrow \infty} \frac{p(n)}{g(n)} \geq \liminf_{n \rightarrow \infty} \frac{p(n)}{n} \underset{(24)}{\geq} \Lambda^{-1} \liminf_{n \rightarrow \infty} \frac{f(n)}{n} \underset{(23)}{\geq} \Lambda^{-1} \left(1 + \frac{\alpha(2\alpha - 1)}{r\delta_\kappa}\right)^{-1}.$$

By Cesàro theorem and (21) we have

$$\begin{aligned}
 c &\stackrel{(21)}{=} \liminf_{n \rightarrow \infty} \frac{1}{n} E \otimes \mathbb{E} \left[\sum_{k=1}^n \mathbf{1}\{X_k \text{ is pioneer}\} \right] \\
 &\geq_{\text{Fatou}} E \otimes \mathbb{E} \left[\liminf_{n \rightarrow \infty} \frac{p(n)}{g(n)} \right] \geq \Lambda^{-1} \left(1 + \frac{\alpha(2\alpha - 1)}{r\delta_\kappa} \right)^{-1},
 \end{aligned}$$

by the last display. This completes the proof of the lemma. □

Let us draw one consequence of the last lemma. With positive probability, when we launch a simple random walk on \mathbf{T}_κ then $\text{Hull}(X_1, \dots, X_n, \dots)$ is a strict subtriangulation of \mathbf{T}_κ with X_0 lying on its boundary. In this case, the remaining lattice $\mathbf{T}_\kappa \setminus \text{Hull}(X_1, \dots, X_n, \dots)$ rooted on the boundary so that the origin of the root edge is X_0 is then a triangulation of the half-plane. A moment's thought shows that the peeling process is still valid in this remaining lattice to the condition of setting $p = \infty$ in the transition probabilities. In fact, this lattice is independent of $\text{Hull}(X_1, \dots, X_n, \dots)$ and has the law of Angel and Ray's infinite triangulation of the half-plane \mathbf{H}_α of parameter α related to κ by (2). The proof of this fact can easily be adapted from [7, Lemma 2.16]. We now derive another consequence of Lemma 12:

Lemma 13 *Conditionally on \mathbf{H}_α consider a simple random walk on \mathbf{H}_α starting from the origin of the root edge and denote by Y_1, Y_2, \dots the vertices visited. Then, with positive probability we have*

$$\forall i \geq 2, Y_i \notin \partial \mathbf{H}_\alpha.$$

The proof of Proposition 11 is easily completed from here: By Lemma 12, with positive probability X_0 stays on the boundary of $\text{Hull}(X_1, \dots, X_n, \dots)$ and the remaining lattice $\mathbf{T}_\kappa \setminus \text{Hull}(X_1, \dots, X_n, \dots)$ has the law of \mathbf{H}_α and is independent of $\text{Hull}(X_1, \dots, X_n, \dots)$. If we now launch another random walk $X_0, X_{-1}, X_{-2}, \dots$ in \mathbf{T}_κ starting from X_0 then by the above lemma, conditionally on the preceding event, this walk has a positive probability to stay inside $\mathbf{T}_\kappa \setminus \text{Hull}(X_1, \dots, X_n, \dots)$ and thus does not intersect the path $\{X_1, X_2, \dots\}$. Consequently, with positive probability \mathbf{T}_κ does not possess the intersection property. By ergodicity (see the proof of Proposition 10) almost surely \mathbf{T}_κ does not possess the intersection property. This completes the proof of Proposition 11.

Proof of Lemma 13 The idea of the proof is to couple the peeling exploration along the SRW $\{Y_1, Y_2, \dots\}$ in \mathbf{H}_α with a peeling exploration along the SRW $\{X_0, X_1, X_2, \dots\}$ in \mathbf{T}_κ on the event in the statement of Lemma 12. Let Y_1 denote the root vertex of \mathbf{H}_α . We can, as for \mathbf{T}_α , discover the lattice \mathbf{H}_α along the simple random walk path Y_1, Y_2, \dots using the peeling of \mathbf{H}_α . In particular the peeling algorithm of \mathbf{H}_α is governed by the transitions $q_i^{(\kappa)}$ for the size of the steps and, exactly as for the \mathbf{T}_κ -case, the holes created are filled in with independent Boltzmann triangulations of the same parameters as for \mathbf{T}_κ (see [35] for details). We denote by $\tilde{\mathbf{H}}_\alpha$ the lattice obtained from \mathbf{H}_α by identifying the two vertices adjacent to Y_1 on the boundary and denote the resulting vertex by

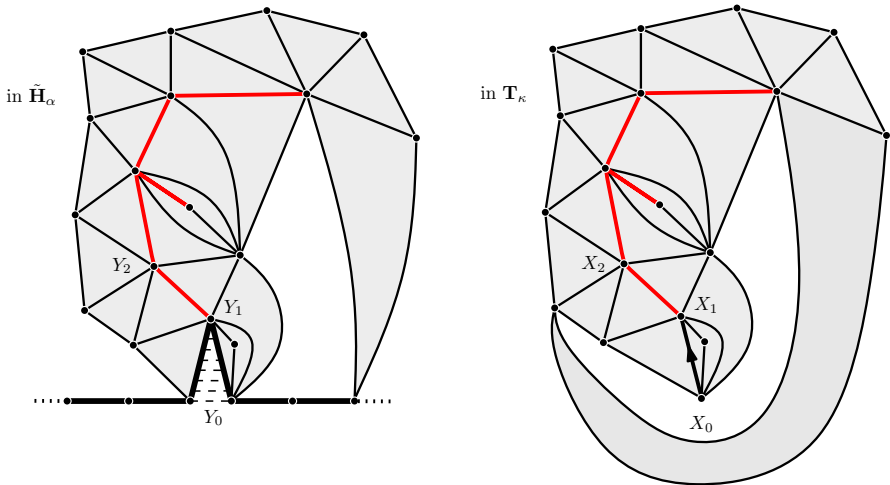


Fig. 5 Coupling between the explorations along SRW in \mathbf{H}_α and in \mathbf{T}_κ until the coupling breaks

Y_0 . We first claim that we can couple \mathbf{T}_κ and $\tilde{\mathbf{H}}_\alpha$ so that the hull of the ball of radius 1 around Y_1 is the same as the hull of the ball of radius 1 around X_1 in \mathbf{T}_κ (recall that X_1 is the origin of the root edge) and that both Y_0 and X_0 are on the boundary of these hulls, see Fig. 5. Using (15) we next argue that we can couple the simple random walk Y on \mathbf{H}_α and its peeling exploration with the simple random walk X on \mathbf{T}_κ and its exploration so that after performing the identification $\mathbf{H}_\alpha \rightarrow \tilde{\mathbf{H}}_\alpha$ these two processes perform the *exact same exploration* i.e. the same steps for the walk and for the peeling (same steps, same Boltzmann triangulations filling in the holes...). This coupling holds as long as X_0 stays on the boundary of $\text{Hull}(X_1, \dots, X_n)$. The reason for this is that whenever a peeling step in \mathbf{T}_κ swallows X_0 , if we were to perform the same peeling step in $\tilde{\mathbf{H}}_\alpha$ this would reach a point on the boundary and would break the coupling, see Fig. 5. By Lemma 12 this coupling can be made for ever with positive probability. On this event it is clear that the simple random walk Y on \mathbf{H}_α does not touch the boundary except at Y_1 as required. \square

4.3 Proof of Theorem 3 via entropy

Combining Proposition 11 and the result of [13] we deduce that \mathbf{T}_κ is non-Liouville a.s. when $\kappa \in (0, \frac{2}{27})$. To finish the proof of Theorem 3 it remains to prove that $s_\kappa > 0$. For this we shall use the notion of entropy. The entropy of the n th position of the simple random walk is the random variable defined by

$$H_n := \sum_{x \in \mathbf{T}_\kappa} \varphi(\mathbb{P}(X_n = x)) \quad \text{where} \quad \varphi(x) = -x \log(x).$$

Since \mathbf{T}_κ is stationary and non-Liouville [11, Theorem 3.2] implies that

$$n^{-1} E[H_n] \xrightarrow{n \rightarrow \infty} h > 0. \tag{25}$$

Actually, the paper [11] deals with simple graphs but the proof goes through *mutatis mutandis*. For technical reasons we turn this convergence in mean into an almost sure statement:

Lemma 14 *We have $\limsup_{n \rightarrow \infty} n^{-1} H_n > 0$ with positive probability.*

Proof We argue by contradiction and suppose $H_n/n \rightarrow 0$ almost surely. The proof of [11, Proposition 3.1] shows that H_n is stochastically bounded by n copies of (dependent) variables $H_{1,i}$ for $i \in \{1, \dots, n\}$ having the same law as H_1 . Hence it follows by Cauchy–Schwarz inequality that

$$E[H_n^2] \leq E \left[\left(\sum_{i=1}^n H_{1,i} \right)^2 \right] = \sum_{1 \leq i, j \leq n} E[H_{1,i} H_{1,j}] \leq n^2 \sqrt{E[H_1^2] E[H_1^2]}.$$

By the standard bound on the entropy we have $H_1 \leq \log(|B_1(\mathbf{T}_\kappa)|) \leq \log(|B_1^\bullet(\mathbf{T}_\kappa)|)$. We leave the reader check that $|B_1^\bullet(\mathbf{T}_\kappa)|$ has an exponential tail, in particular

$$E[H_1^2] \leq E[\log^2(|B_1^\bullet(\mathbf{T}_\kappa)|)] < \infty.$$

Consequently $(H_n/n)_{n \geq 1}$ is bounded in \mathbb{L}^2 hence uniformly integrable. Since we supposed $H_n/n \rightarrow 0$ a.s., by dominated convergence this forces $h = 0$: contradiction with (25)! □

We now adapt the proof of [11, Proposition 3.6] (which was restricted to the bounded degree case) and demonstrate that the last lemma implies positive speed for the simple random walk. For this, fix $\varepsilon > 0$ and introduce the event $A_n^\varepsilon = \{d_{\text{gr}}(X_0, X_n) \leq (s_\kappa + \varepsilon)n\}$. To simplify notation we write B_r for $B_r(\mathbf{T}_\kappa)$. We decompose the entropy H_n as follows

$$\begin{aligned} & \sum_{x \in \mathbf{T}_\kappa} \varphi(\mathbb{P}(X_n = x)) \\ &= \sum_{x \in B_n(s_\kappa + \varepsilon)} \varphi(\mathbb{P}(X_n = x)) + \sum_{x \in B_n \setminus B_n(s_\kappa + \varepsilon)} \varphi(\mathbb{P}(X_n = x)) \\ & \varphi \text{ is concave } \leq \left(\sum_{x \in B_n(s_\kappa + \varepsilon)} \mathbb{P}(X_n = x) \right) \log \left(\frac{|B_n(s_\kappa + \varepsilon)|}{\sum_{x \in B_n(s_\kappa + \varepsilon)} \mathbb{P}(X_n = x)} \right) \\ & \quad + \left(\sum_{x \in B_n \setminus B_n(s_\kappa + \varepsilon)} \mathbb{P}(X_n = x) \right) \log \left(\frac{|B_n \setminus B_n(s_\kappa + \varepsilon)|}{\sum_{x \in B_n \setminus B_n(s_\kappa + \varepsilon)} \mathbb{P}(X_n = x)} \right) \\ &= \varphi(\mathbb{P}(A_n^\varepsilon)) + \mathbb{P}(A_n^\varepsilon) \log(|B_n(s_\kappa + \varepsilon)|) \\ & \quad + \varphi(1 - \mathbb{P}(A_n^\varepsilon)) + (1 - \mathbb{P}(A_n^\varepsilon)) \log(|B_n \setminus B_n(s_\kappa + \varepsilon)|) \end{aligned}$$

We now divide by n and take $\limsup_{n \rightarrow \infty}$. The left-hand side becomes positive with positive probability by the last lemma. On the other hand (the right one), from (19) we deduce that $\mathbb{P}(A_n^\varepsilon) \rightarrow 1$ almost surely under P and so $\varphi(\mathbb{P}(A_n^\varepsilon)) \rightarrow 0$ and $\varphi(1 - \mathbb{P}(A_n^\varepsilon)) \rightarrow 0$ as $n \rightarrow \infty$ almost surely for P . Also Theorem 2 shows that for any $u > 0$ we have

$$\frac{\log(|B_{un}|)}{n} \xrightarrow[n \rightarrow \infty]{P\text{-a.s.}} u \log\left(\frac{\alpha + \delta_\kappa}{\alpha - \delta_\kappa}\right).$$

Finally we get

$$\limsup_{n \rightarrow \infty} \frac{H_n}{n} \leq (s_\kappa + \varepsilon) \log\left(\frac{\alpha + \delta_\kappa}{\alpha - \delta_\kappa}\right).$$

and conclude that $s_\kappa > 0$ with positive probability and thus almost surely by (19). This finishes the proof of Theorem 3.

5 Comments

5.1 Local limit of triangulations in high genus

Recall that $\mathcal{T}_{n,g}$ denotes the set of all (rooted) triangulations of the torus of genus $g \geq 0$ with n vertices and that $T_{n,g}$ is a random uniform element in $\mathcal{T}_{n,g}$. Euler’s formula shows that any triangulation $t \in \mathcal{T}_{n,g}$ has $3(n + 2g - 2)$ edges. Hence, when $g = [\theta n]$, the mean degree of $T_{n,[\theta n]}$ is equal to

$$\frac{6(n + 2[\theta n] + 2)}{n} \xrightarrow[n \rightarrow \infty]{} 6(1 + 2\theta).$$

However, the notion of mean degree is *not* continuous for the local topology and is not even clearly defined for an infinite triangulation. See the phenomenon appearing in the case of unicellular maps [5, Remark 5]. To get a continuous observable for the local topology, we rather look at the mean of the inverse of degree of the root vertex ρ_n in $T_{n,g}$. Indeed, since the root vertex in $T_{n,g}$ is chosen proportionally to its degree we have

$$\begin{aligned} E[\deg(\rho_n)^{-1}] &= \frac{1}{\#\mathcal{T}_{n,g}} \frac{1}{6(n + 2g - 2)} \sum_{t \in \mathcal{T}_{n,g}} \sum_{x \in t} \deg(x) \cdot \frac{1}{\deg(x)} \\ &= \frac{n}{6(n + 2g - 2)} \xrightarrow[n \rightarrow \infty]{} \underbrace{\frac{1}{6(1 + 2\theta)}}_{\text{for } g = [\theta n]}. \end{aligned}$$

Notice that the degree of the root vertex is indeed a continuous function for the local topology. Hence, we can sharpen the conjecture stated at the end of the introduction:

for $\kappa \in (0, 2/27]$, let

$$f(\kappa) = E \left[(\text{degree of the origin in } \mathbf{T}_\kappa)^{-1} \right].$$

It is easy to see from the peeling construction of \mathbf{T}_κ that f is continuous and satisfies $f(0^+) = 0$ and $f(\frac{2}{27}) = 1/6$ (case of the UIPT). We believe that f is in fact strictly decreasing and that

Conjecture 1 (With I. Benjamini) *For any $\theta \geq 0$, let $\kappa \in (0, \frac{2}{27}]$ be such that $f(\kappa) = (6(1 + 2\theta))^{-1}$. Then for any sequence g_n such that $g_n/n \rightarrow \theta$ we have the following convergence in distribution for the local topology*

$$T_{n,g_n} \xrightarrow[n \rightarrow \infty]{(d)} \mathbf{T}_\kappa.$$

This conjecture would follow from precise enumerative formulas on $\#\mathcal{T}_{n,g}$ when n and g are both tending to infinity (the known results focus on asymptotics as $n \rightarrow \infty$ and then $g \rightarrow \infty$, [26]), see the arguments in [5].

5.2 Perspectives

First of all, let us mention that we restricted ourselves to 2-connected triangulations mainly to take advantage of the calculations already performed by Angel and Ray in [8] and by Ray in [35]. This whole work could be extended to other types of maps with simple faces (e.g. quadrangulations with simple faces). In the case of non-simple faces (e.g. 1-connected triangulations where loops are present), one should get the same picture as in the half-planar case [8]: an infinite core-map with simple faces described by a single parameter, and additional parameters to describe the law of the maps inside loops. We do not enter the details and refer to [8, Section 3.5 and 3.6] for details.

Also, it is likely that site percolation on \mathbf{T}_κ can be treated by similar means as in [35] and would yield almost identical results. Contrary to the case of the UIPT which is believed to converge towards the Brownian plane [20] in the scaling limit for the Gromov–Hausdorff topology, it is pretty clear that in the hyperbolic regime \mathbf{T}_κ do not admit any scaling limits in the Gromov–Hausdorff sense.³ However its conformal structure might be of interest (maybe in relation with 2d quantum gravity?). Finally, the geometric relations (underlying the proof of Proposition 11) between the half-planar lattices of [8] and those defined in this work deserve to be explored in more details.

Added in proof: Recently the techniques of Virag [37] to prove positive (liminf) speed for the SRW using anchored expansion have been adapted to the case of stationary random graphs without the bounded degree assumption in [7] and [14]. Using these results and Proposition 9 it is possible to get a quicker proof of the first point of

³ To see this, note that the hull $B_r^\bullet(\mathbf{T}_\kappa)$ contains a number of tentacles reaching distance $2r$ that tends to infinity as $r \rightarrow \infty$. This violates tightness for the Gromov–Hausdorff metric.

Theorem 3. Also [7] builds upon this work to prove that the half-planar versions of \mathbf{T}_κ for $\kappa < \frac{2}{27}$ have positive (liminf) speed.

Acknowledgments I am grateful to Omer Angel, Itai Benjamini, Guillaume Chapuy and Gourab Ray for useful discussions on and around Conjecture 1. Thanks also go to two anonymous referees for a careful reading of this paper.

References

1. Addario-Berry, L., Reed, B.A.: Ballot theorems, old and new. In: Horizons of combinatorics, vol. 17 of Bolyai Soc. Math. Stud., pp. 9–35. Springer, Berlin (2008)
2. Aldous, D., Lyons, R.: Processes on unimodular random networks. *Electron. J. Probab.* **12**(54), 1454–1508 (2007). (electronic)
3. Aldous, D., Steele, J.M.: The objective method: probabilistic combinatorial optimization and local weak convergence. In: Probability on discrete structures, vol. 110 of Encyclopaedia Math. Sci., pp. 1–72. Springer, Berlin (2004)
4. Angel, O.: Growth and percolation on the uniform infinite planar triangulation. *Geom. Funct. Anal.* **13**, 935–974 (2003)
5. Angel, O., Chapuy, G., Curien, N., Ray, G.G.: The local limit of unicellular maps in high genus. *Electron. Commun. Probab.* **18**, 1–8 (2013)
6. Angel, O., Curien, N.: Percolations on infinite random maps, half-plane models. *Ann. Inst. H. Poincaré Probab. Statist.* **51**, 405–431 (2014)
7. Angel, O., Nachmias, A., Ray, G.: Random walks on stochastic hyperbolic half planar triangulations (2014). [arXiv:1408.4196](https://arxiv.org/abs/1408.4196)
8. Angel, O., Ray, G.: Classification of half planar maps. *Ann. Probab.* **43**(3), 1315–1349 (2015)
9. Angel, O., Schramm, O.: Uniform infinite planar triangulation. *Comm. Math. Phys.* **241**, 191–213 (2003)
10. Benjamini, I.: Coarse geometry and randomness. vol. 2100 of Lecture Notes in Mathematics, Springer, Cham (2013) [Saint-Flour Probability Summer School]
11. Benjamini, I., Curien, N.: Ergodic theory on stationary random graphs. *Electron. J. Probab.* **17**, 1–20 (2012)
12. Benjamini, I., Curien, N.: Simple random walk on the uniform infinite planar quadrangulation: subdiffusivity via pioneer points. *Geom. Funct. Anal.* **23**, 501–531 (2013)
13. Benjamini, I., Curien, N., Georgakopoulos, A.: The Liouville and the intersection properties are equivalent for planar graphs. *Electron. Commun. Probab.* **17**, 1–5 (2012)
14. Benjamini, I., Paquette, E., Pfeffer, J.: Anchored expansion, speed, and the hyperbolic poisson voronoi tessellation (2014). [arXiv:1409.4312](https://arxiv.org/abs/1409.4312)
15. Benjamini, I., Schramm, O.: Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* **6**(23), 13 (2001). (electronic)
16. Bertoin, J., Doney, R.A.: On conditioning a random walk to stay nonnegative. *Ann. Probab.* **22**, 2152–2167 (1994)
17. Björnberg, J.E., Stefansson, S.O.: Recurrence of bipartite planar maps. *Electron. J. Probab.* **19**, 1–40 (2014)
18. Chassaing, P., Durhuus, B.: Local limit of labeled trees and expected volume growth in a random quadrangulation. *Ann. Probab.* **34**, 879–917 (2006)
19. Curien, N.: A glimpse of the conformal structure of random planar maps. *Commun. Math. Phys.* **333**, 1417–1463 (2015)
20. Curien, N., Le Gall, J.-F.: The Brownian plane. *J. Theor. Probab.* **27**, 1249–1291 (2014)
21. Curien, N., Le Gall, J.-F.: Scaling limits for the peeling process on random maps (2014). [arXiv:1412.5509](https://arxiv.org/abs/1412.5509)
22. Curien, N., Ménard, L., Miermont, G.: A view from infinity of the uniform infinite planar quadrangulation. *Lat. Am. J. Probab. Math. Stat.* **10**, 45–88 (2013)
23. Derriennic, Y.: Quelques applications du théorème ergodique sous-additif. In: Conference on Random Walks (Kleeback, 1979) (French), vol. 74 of Astérisque, pp. 183–201. Soc. Math. France, Paris 4 (1980)

24. Gamburd, A.: Poisson-dirichlet distribution for random belyi surfaces. *Ann. Probab.* **34**, 1827–1848 (2006)
25. Goulden, I.P., Jackson, D.M.: *Combinatorial enumeration*. A Wiley-Interscience Publication, New York (1983). With a foreword by Gian-Carlo Rota, *Wiley-Interscience Series in Discrete Mathematics*
26. Goulden, I.P., Jackson, D.M.: The KP hierarchy, branched covers, and triangulations. *Adv. Math.* **219**, 932–951 (2008)
27. Gurel-Gurevich, O., Nachmias, A.: Recurrence of planar graph limits. *Ann. Maths* **177**, 761–781 (2013)
28. Guth, L., Parlier, H., Young, R.: Pants decompositions of random surfaces. *Geom. Funct. Anal.* **21**, 1069–1090 (2011)
29. Krikun, M.: Local structure of random quadrangulations. [arXiv:0512304](https://arxiv.org/abs/0512304)
30. Le Gall, J.-F.: Uniqueness and universality of the Brownian map. *Ann. Probab.* **41**, 2880–2960 (2013)
31. Le Gall, J.-F., Ménard, L.: Scaling limits for the uniform infinite quadrangulation. *Ill. J. Math.* **54**, 1163–1203 (2012)
32. Lyons, R., Schramm, O.: Indistinguishability of percolation clusters. *Ann. Probab.* **27**, 1809–1836 (1999)
33. Ménard, L., Nolin, P.: Percolation on uniform infinite planar maps. *Electron. J. Probab.* **19**, 1–27 (2014)
34. Miermont, G.: The Brownian map is the scaling limit of uniform random plane quadrangulations. *Acta Math.* **210**, 319–401 (2013)
35. Ray, G.: Geometry and percolation on half planar triangulations. *Electron. J. Probab.* **19**, 1–28 (2014)
36. Thomassen, C.: Isoperimetric inequalities and transient random walks on graphs. *Ann. Probab.* **20**, 1592–1600 (1992)
37. Virág, B.: Anchored expansion and random walk. *Geom. Funct. Anal.* **10**, 1588–1605 (2000)
38. Watabiki, Y.: Construction of non-critical string field theory by transfer matrix formalism in dynamical triangulation. *Nuclear Phys. B* **441**, 119–163 (1995)