Probab. Theory Relat. Fields (2016) 165:65–116
DOI 10.1007/s00440-015-0627-7

Complex obtuse random walks and their continuous-time limits

S. Attal[1] · J. Deschamps[2] · C. Pellegrini[3]

Received: 26 February 2014 / Revised: 23 March 2015 / Published online: 17 April 2015
© Springer-Verlag Berlin Heidelberg 2015

Abstract We study a particular class of complex-valued random variables and their associated random walks: complex obtuse random variables. They generalize, to the complex case, the real-valued obtuse random variables introduced in Equations de structure pour des martingales vectorielles. (Séminaire de Probabilités, XXVIII, p. 256278. Lecture Notes in Math., vol. 1583. Springer, Berlin (1994)) in order to understand the structure of normal martingales in $\mathbb{R}^N$. The extension to the complex case is motivated by Quantum Statistical Mechanics, in particular for characterizing those quantum baths acting as classical noises. The extension of obtuse random variables to the complex case is far from obvious and makes use of very interesting algebraical structures. We show that complex obtuse random variables are characterized by a 3-tensor which admits certain symmetries; we show that these symmetries are the exact 3-tensor analogue of the normal character for 2-tensors (i.e. matrices), that is, a necessary and sufficient condition for being diagonalizable in some orthonormal basis. We discuss the passage to the continuous-time limit for these random walks and show that they converge in distribution to normal martingales in $\mathbb{C}^N$. We show that the 3-tensor associated to these normal martingales encodes their behavior, in particular the diagonalization directions of the 3-tensor indicate the directions of the space where the

Work supported by ANR project "HAM-MARK" No. ANR-09-BLAN-0098-01.

✉ S. Attal
attal@math.univ-lyon1.fr

1 Université de Lyon, Université de Lyon 1, C.N.R.S. Institut Camille Jordan, 21 av Claude Bernard, 69622 Villeurbanne, Cedex, France

2 Dipartimento di Matematica, Università degli Studi di Genova, Via Dodecaneso, 35, 16146 Genoa, Italy

3 Laboratoire de Statistique et de Probabilité, Institut de Mathématiques de Toulouse, Université Paul Sabatier (Toulouse III), 31062 Toulouse, Cedex 9, France

martingales behaves like a diffusion and those where it behaves like a Poisson process. We finally prove the convergence, in the continuous-time limit, of the corresponding multiplication operators on the canonical Fock space, with an explicit expression in terms of the associated 3-tensor again.

Mathematics Subject Classification 60G46 · 60J10

Contents

1 Introduction and motivations . . . 66
1.1 Generalities . . . 66
1.2 Examples . . . 68
1.3 Structure of the article . . . 70
2 Complex obtuse random variables . . . 71
2.1 Complex obtuse systems . . . 71
2.2 Obtuse random variables . . . 74
2.3 Generic character of obtuse random variables . . . 76
2.4 Associated 3-tensors . . . 78
2.5 Representation of multiplication operators . . . 80
2.6 Back to the example . . . 82
3 Complex doubly-symmetric 3-tensors . . . 83
3.1 The main diagonalization theorem . . . 83
3.2 Back to obtuse random variables . . . 88
3.3 Recovering the real case . . . 90
4 Complex normal martingales . . . 94
4.1 A reminder of normal martingales in $\mathbb{R}^N$ . . . 95
4.2 Normal martingales in $\mathbb{C}^N$ . . . 96
4.3 Complex unitary transforms of real normal martingales . . . 102
5 Continuous-time limit of complex obtuse random walks . . . 106
5.1 Convergence of the tensors . . . 106
5.2 Convergence in distribution . . . 108
5.3 Convergence of the multiplication operators . . . 109
6 Examples . . . 112
References . . . 115

1 Introduction and motivations

1.1 Generalities

Real obtuse random variables are a particular family of random variables which were defined in [4] in order to understand the discrete-time analogue of normal martingales in $\mathbb{R}^N$. They were shown to be deeply connected to the Predictable Representation Property and the Chaotic Representation Property for discrete-time martingales in $\mathbb{R}^N$. They are kind of minimal, centered and normalized random variables in $\mathbb{R}^N$ and they exhibit a very interesting underlying algebraic structure. This algebraic structure is carried by a certain natural 3-tensor associated to the random variable. This 3-tensor has exactly the necessary and sufficient symmetries for being diagonalizable in some orthonormal basis (that is, they obtained the extension to 3-tensors of the condition of being real symmetric for 2-tensors). The corresponding ortho-

normal basis carries the behavior of the associated random walk and in particular of its continuous-time limit. It is shown in [4] that, for the continuous-time limit, in the directions associated to the null eigenvalue, the limit process behaves like a diffusion process, while it behaves like a pure jump process in the directions associated to non-null eigenvalues. In [6] it is concretely shown how the 3-tensor of the discrete-time obtuse random walks converges to the one of normal martingales in $\mathbb{R}^N$.

Since this initial work of [4] was only motivated by Probability Theory and Stochastic Process considerations, there was no real need for an extension of this notion to the complex case. The need for such an extension has appeared naturally through considerations in Quantum Statistical Mechanics. More precisely, the underlying motivation is to characterize the onset of classical noises emerging from quantum baths, in the so-called model of Repeated Quantum Interactions.

Repeated quantum interactions are physical models, introduced and developed in [5], which consist in describing the Hamiltonian dynamics of a quantum system undergoing a sequence of interactions with an environment made of a chain of identical systems. These models were developed for they furnish toy models for quantum dissipative systems, they are at the same time Hamiltonian and Markovian, they spontaneously give rise to quantum stochastic differential equations in the continuous time limit. It has been proved in [7,8] that they constitute a good toy model for a quantum heat bath in some situations and that they can also give an account of the diffusive behavior of an electron in an electric field, when coupled to a heat bath. When adding to each step of the dynamics a measurement of the piece of the environment which has just interacted, one recovers all the discrete-time quantum trajectories for quantum systems [15–17]. Physically this model corresponds exactly to physical experiments such as the ones performed by Haroche et al., on the quantum trajectories of a photon in a cavity [9–11].

The discrete-time dynamics of these repeated interaction systems, as well as their continuous-time limit, give rise to time evolutions driven by quantum noises coming from the environment. These quantum noises emerging from the environment describe all the possible actions inside the environment (excitation, return to ground state, jumps in between two energy levels, ...). It is a remarkable fact that these quantum noises can also be combined together in order to give rise to classical noises. In discrete-time they give rise to any random walk, in continuous-time they give rise to many well-known stochastic processes among which are all Levy processes.

The point is that complex obtuse random variables and their continuous-time limit are the key for understanding what kind of classical noise is appearing at the end from the quantum bath. The 3-tensor helps to read directly from the Hamiltonian which kind of classical noise will be driving the evolution equation. This was our initial motivation for developing the complex theory of obtuse random variables and normal martingales in $\mathbb{C}^N$.

Surprisingly, the extension of obtuse random variables, obtuse random walks and their continuous-time limits, to the complex case is far from obvious. The algebraical properties of the associated 3-tensors give rise to the same kind of behaviors as in the real case, but, as we shall see in this article, many aspects (such as the diagonalization theorem) become now really non-trivial.

1.2 Examples

Let us have here a more detailed discussion on these physical motivations underlying our study. These motivations do not appear anymore in the rest of the article which is devoted entirely to the probabilistic properties of complex obtuse random walks and their continuous-time limit, but we have felt that it could be of interest for the reader to have a clearer picture of the physical motivations which have brought us to consider the complex case extension of obtuse random walks. This part can be skipped by the reader, it has no influence whatsoever on the rest of the article, these physical applications are developed in detail in [2].

Let us illustrate this situation with concrete examples. Consider a quantum system, with state space $\mathcal{H}_{\mathcal{S}}$, in contact with a quantum bath of the form $\mathcal{H}_{\mathcal{E}} = \otimes_{\mathbb{N}} \mathbb{C}^2$, that is, a spin chain. Let us denote by a^i_j, $i, j \in \{0, 1\}$, the usual basis of elementary matrices on $\mathbb{C}^2$ and by $a^i_j(n)$ the corresponding matrix but acting only on the nth copy of $\mathbb{C}^2$. The Hamiltonian for the interaction between $\mathcal{H}_{\mathcal{S}}$ and one copy of $\mathbb{C}^2$ is of the typical form

$$H_{\text{tot}} = H_{\mathcal{S}} \otimes I + I \otimes H_{\mathcal{E}} + V \otimes a^0_1 + V^* \otimes a^1_0.$$

Assume that the interaction between these two parts lasts for a small amount of time h, then the associated unitary evolution operator is $U = e^{-ihH_{\text{tot}}}$ which can be decomposed as $U = \sum_{i,j=0}^{1} U^i_j \otimes a^i_j$ for some operators U^i_j on $\mathcal{H}_{\mathcal{S}}$.

The action of the environment (the spin chain) by acting repeatedly on the system $\mathcal{H}_{\mathcal{S}}$, spin by spin, each time for a time duration h, gives rise to a time evolution driven by a unitary group (V_n) which satisfies (cf [5] for details)

$$V_{n+1} = \sum_{i,j=0}^{1} (U^i_j \otimes I)\, V_n\, (I \otimes a^i_j(n+1)).$$

This describes a rather general discrete time evolution for a quantum system and the operators $a^i_j(n)$ here play the role of discrete time quantum noises, they describe all the possible innovations brought by the environment.

In [5] it is shown that if the total Hamiltonian H_{tot} is renormalized under the form

$$H_{\text{tot}} = H_{\mathcal{S}} \otimes I + I \otimes H_{\mathcal{E}} + \frac{1}{\sqrt{h}} \left(V \otimes a^0_1 + V^* \otimes a^1_0 \right)$$

then the time evolution (V_{nh}) converges, when h tends to 0, to a continuous-time unitary evolution (V_t) satisfying an equation of the form

$$dV_t = K\, V_t\, dt + L\, V_t\, da^*(t) - L^*\, V_t\, da(t),$$

which is a quantum stochastic differential equation driven by quantum noises $da(t)$ and $da^*(t)$, on some appropriate Fock space. In other words, we obtain a perturbation of a Schrödinger equation by some additional quantum noise terms.

The point now is that, in the special case where $V = V^*$, the discrete-time evolution and its continuous-time limit are actually driven by classical noises, for some of the terms in the evolution equation factorize nicely and make appearing classical noises instead of quantum noises (the noises get grouped in order to furnish a family of commuting self-adjoint operators, that is, a classical stochastic process). Indeed, one can show (cf [3] and [6]) that the discrete time evolution can be written under the form of a classical random walk on the unitary group $\mathcal{U}(\mathcal{H}_{\mathcal{S}})$:

$$V_{n+1} = A\, V_n + B\, V_n\, X_{n+1},$$

where (X_n) is a sequence of i.i.d. symmetric Bernoulli random variables. The continuous time limit, with the same renormalization as above, gives rise to a unitary evolution driven by a classical Brownian motion (W_t):

$$dV_t = \left(iH - \frac{1}{2}L^2\right) V_t\, dt + L\, V_t\, dW_t.$$

The equation above is the typical one for the perturbation of a Schrödinger equation by means of a Brownian additional term.

This example is very simple and already well-known (cf [6]); it involves real obtuse random variables and real normal martingales. The point now is that one can consider plenty of much more complicated examples of a choice for the Hamiltonian H_{tot}, which would give rise to classical noises instead of quantum noises. Our motivation was to understand and characterize when such a situation appears and to read on the Hamiltonian which kind of noise is going to drive the dynamics. Let us illustrate this with a more complicated example. Assume now that the environment is made of a chain of 3-level quantum systems, that is, $\mathcal{H}_{\mathcal{E}} = \otimes_{\mathbb{N}} \mathbb{C}^3$. For the elementary interaction between the quantum system $\mathcal{H}_{\mathcal{S}}$ and one copy of $\mathbb{C}^3$ we consider a Hamiltonian of the form

$$H_{\text{tot}} = H \otimes I + A \otimes \begin{pmatrix} 0 & 5 & 0 \\ -1+2i & 0 & 4-2i \\ -2+4i & 0 & 2i \end{pmatrix} + B \otimes \begin{pmatrix} 0 & 0 & 5 \\ -2+4i & 0 & 2+i \\ 1-2i & -i & -1+2i \end{pmatrix},$$

which is self-adjoint under the condition $B = -(1/2)(A + (1+2i)A^*)$.

In this case the quantum dynamics in discrete time happens to be driven by a classical noise too. We shall understand, with the tools developed in this article, that it is of the form

$$V_{n+1} = A\, V_n + B\, V_n\, X_{n+1} + C\, V_n\, Y_{n+1},$$

where the random variables (X_n, Y_n) are i.i.d. in $\mathbb{C}^2$ taking the values

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}, \qquad v_3 = -\frac{1}{5}\begin{pmatrix} 3+4i \\ 1+3i \end{pmatrix}$$

with probabilities $p_1 = 1/3$, $p_2 = 1/4$ and $p_3 = 5/12$ respectively.

Putting a $1/\sqrt{h}$ normalization factor in front of A and B and taking the limit h goes to 0, we shall show in this article, that this gives rise to a continuous time dynamics

$$dV_t = L_0\, V_t\, dt + L_1\, V_t\, dZ_t^1 + L_2\, V_t\, dZ_t^2,$$

where (Z^1, Z^2) is a normal martingale in $\mathbb{C}^2$ given by

$$\begin{cases} Z_t^1 = \frac{2+i}{\sqrt{10}}\, W_t^1 + \frac{i}{\sqrt{2}}\, W_t^2 \\ Z_t^2 = \frac{-1+2i}{\sqrt{10}}\, W_t^1 + \frac{1}{\sqrt{2}}\, W_t^2, \end{cases}$$

where (W^1, W^2) is a 2-dimensional real Brownian motion.

We shall also see in this article how to produce any kind of example in $\mathbb{C}^N$ which mixes Brownian parts and Poisson parts.

The way these random walks and their characteristics are identified, the way the continuous-time limits and their characteristics are identified, are highly non-trivial and make use of all the tools we shall develop along this article: associated doubly-symmetric 3-tensor, diagonalisation of the 3-tensor, probabilistic characteristics of the associated random walk, passage to the limit on the tensor, passage to the limit on the discrete-time martingale, identification of the law of the limit martingale, etc.

1.3 Structure of the article

This article is structured as follows. In Sect. 2 we introduce the notions of *complex obtuse systems*, *complex obtuse random variables* and their associated 3-tensors. We connect complex obtuse systems to real ones. We show a kind of uniqueness result and we show that they generate all finitely supported random variables in $\mathbb{C}^N$.

In Sect. 3 we establish the important symmetries shared by the 3-tensors of obtuse random variables and we show one of our main results: these symmetries are the necessary and sufficient conditions for the 3-tensor to be diagonalizable in some orthonormal basis. We show how to recover the real case, which remarkably does not correspond to the real character of the 3-tensor but to a certain supplementary symmetry.

In Sect. 4 we recall basic results on real normal martingales and deduce the corresponding ones for the complex normal martingales. In particular we establish what is the complex extension of a *structure equation*. We connect the behavior of the complex normal martingale to the diagonalization of its associated 3-tensor.

In Sect. 5 we finally prove our continuous-time convergence theorems. First of all, via the convergence of the tensors, exploiting the results of [19], we prove a convergence in law for the processes. Secondly, in the framework of Fock space approximation by spin chains developed in [1], we prove the convergence of the associated multiplication operators, with explicit formulas in terms of quantum noises.

We finally illustrate our results in Sect. 6 through 2 examples, showing up the different types of behavior.

2 Complex obtuse random variables

2.1 Complex obtuse systems

Definition 2.1 Let $N \in \mathbb{N}^*$ be fixed. An *obtuse system in* $\mathbb{C}^N$ is family of $N+1$ vectors $\{v_1, \ldots, v_{N+1}\}$ in $\mathbb{C}^N$ such that

$$\langle v_i \, , \, v_j \rangle = -1,$$

for all $i \neq j$.

In that case we put

$$\widehat{v}_i = \begin{pmatrix} 1 \\ v_i \end{pmatrix} \in \mathbb{C}^{N+1},$$

so that $\{\widehat{v}_1, \ldots, \widehat{v}_{N+1}\}$ forms an orthogonal basis of $\mathbb{C}^{N+1}$. We put

$$p_i = \frac{1}{\|\widehat{v}_i\|^2} = \frac{1}{1 + \|v_i\|^2},$$

for $i = 1, \ldots N+1$.

Lemma 2.2 *With the definition and notations above we have*

$$\sum_{i=1}^{N+1} p_i = 1 \tag{1}$$

and

$$\sum_{i=1}^{N+1} p_i \, v_i = 0. \tag{2}$$

Proof We have, for all j,

$$\left\langle \sum_{i=1}^{N+1} p_i \, \widehat{v}_i \, , \, \widehat{v}_j \right\rangle = p_j \left\|\widehat{v}_j\right\|^2 = 1 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \, \widehat{v}_j \right\rangle.$$

As the $\widehat{v}_j$'s form a basis, this means that

$$\sum_{i=1}^{N+1} p_i \, \widehat{v}_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This gives the two announced equalities. □

Lemma 2.3 *We also have*

$$\sum_{i=1}^{N+1} p_i \, |v_i\rangle\langle v_i| = I_{\mathbb{C}^N}. \tag{3}$$

Proof As the vectors $(\sqrt{p_i}\,\widehat{v}_i)_{i\in\{1,\dots,N+1\}}$ form an orthonormal basis of $\mathbb{C}^{N+1}$ we have

$$I_{\mathbb{C}^{N+1}} = \sum_{i=1}^{N+1} p_i \, |\widehat{v}_i\rangle\langle\widehat{v}_i|.$$

Now put

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \widetilde{v}_i = \begin{pmatrix} 0 \\ v_i \end{pmatrix},$$

for all $i = 1, \dots, N+1$. We get

$$\begin{aligned} I_{\mathbb{C}^{N+1}} &= \sum_{i=1}^{N+1} p_i \, |u + \widetilde{v}_i\rangle\langle u + \widetilde{v}_i| \\ &= \sum_{i=1}^{N+1} p_i \, |u\rangle\langle u| + \sum_{i=1}^{N+1} p_i \, |u\rangle\langle\widetilde{v}_i| + \sum_{i=1}^{N+1} p_i \, |\widetilde{v}_i\rangle\langle u| + \sum_{i=1}^{N+1} p_i \, |\widetilde{v}_i\rangle\langle\widetilde{v}_i|. \end{aligned}$$

Using (1) and (2), we get

$$I_{\mathbb{C}^{N+1}} = |u\rangle\langle u| + \sum_{i=1}^{N+1} p_i \, |\widetilde{v}_i\rangle\langle\widetilde{v}_i|.$$

In particular we have

$$\sum_{i=1}^{N+1} p_i \, |v_i\rangle\langle v_i| = I_{\mathbb{C}^N},$$

that is, the announced equality. □

Let us just show one example now. On $\mathbb{C}^2$ the 3 vectors

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}, \qquad v_3 = -\frac{1}{5}\begin{pmatrix} 3+4i \\ 1+3i \end{pmatrix}$$

form an obtuse system of $\mathbb{C}^2$. The associated p_i's are then respectively

$$p_1 = \frac{1}{3}, \qquad p_2 = \frac{1}{4}, \qquad p_3 = \frac{5}{12}.$$

We end this subsection with some useful properties of obtuse systems. First, we prove an independence property for obtuse systems.

Proposition 2.4 *Every strict sub-family of an obtuse family is linearly free.*

Proof Let $\{v_1, \ldots, v_{N+1}\}$ be an obtuse family of $\mathbb{C}^N$. Let us show that $\{v_1, \ldots, v_N\}$ is free, which would be enough for our claim. If we had $v_N = \sum_{i=1}^{N-1} \lambda_i \, v_i$ then, taking the scalar product with v_N we would get $\|v_N\|^2 = \sum_{i=1}^{N-1} -\lambda_i$, whereas taking the scalar product with v_{N+1} would give $-1 = \sum_{i=1}^{N-1} -\lambda_i$. This would imply $\|v_N\|^2 = -1$, which is impossible. □

Now we prove a kind of uniqueness result for obtuse systems.

Proposition 2.5 *Let $\{v_1, \ldots, v_{N+1}\}$ be an obtuse system of $\mathbb{C}^N$ having $\{p_1, \ldots, p_{N+1}\}$ as associated probabilities. Then the following assertions are equivalent.*

(i) *The family $\{w_1, \ldots, w_{N+1}\}$ is an obtuse system on $\mathbb{C}^N$ with same respective probabilities $\{p_1, \ldots, p_{N+1}\}$.*
(ii) *There exists a unitary operator U on $\mathbb{C}^N$ such that $w_i = U v_i$, for all $i = 1, \ldots, N+1$.*

Proof One direction is obvious. If $w_i = U v_i$, for all $i = 1, \ldots, N+1$ and for some unitary operator U, then the scalars products $\langle v_i \, , \, v_j \rangle$ and $\langle w_i \, , \, w_j \rangle$ are equal, for each pair (i, j). This shows that $\{w_1, \ldots, w_{N+1}\}$ is obtuse with the same probabilities.

In the converse direction, if $v_1, \ldots, v_{N+1}$ and $w_1, \ldots, w_{N+1}$ are obtuse systems associated to the same probabilities $p_1, \ldots, p_{N+1}$, then

$$\langle v_i \, , \, v_j \rangle = \langle w_i \, , \, w_j \rangle$$

for all i, j. The sub-family $\{v_1, \ldots, v_N\}$ is a basis of $\mathbb{C}^N$, by Proposition 2.4. The map $U : \mathbb{C}^N \to \mathbb{C}^N$ such that $U v_i = w_i$ for all $i = 1, \ldots, N$ is clearly unitary, by the conservation of scalar products. Finally, the vector v_{N+1} is a certain linear combination of the v_i's, $i = 1, \ldots, N$, but w_{N+1} is the same linear combination of the w_i's, $i = 1, \ldots, N$, by the conservation of scalar products. Hence $U v_{N+1}$ is equal to w_{N+1} and the proposition is proved. □

Finaly, we prove that every complex obtuse system is obtained by a unitary transform of $\mathbb{C}^N$ applied to some real obtuse system. This result will be used many times in the article.

Theorem 2.6 *For every obtuse family $\{v_1, \ldots, v_{N+1}\}$ of $\mathbb{C}^N$ there exists a unitary operator U of $\mathbb{C}^N$ such that the vectors $w_i = U v_i$ all have real coordinates. The family $\{w_1, \ldots, w_{N+1}\}$ of $\mathbb{R}^N$ forms a real obtuse system of $\mathbb{R}^N$, with same probabilities as the initial family $\{v_1, \ldots, v_{N+1}\}$.*

Proof Consider the $(N+1) \times (N+1)$ Gram matrix G such that $G_{ij} = \langle v_i \, , \, v_j \rangle$ for all i, j. This matrix is real symmetric and positive. The rank of G is N exactly, as was already noticed in Proposition 2.4. As a consequence G can be written as $G = A^* A$,

where A is a real $(N) \times (N+1)$ matrix. Hence G is the Gram matrix $G_{ij} = \langle w_i \,,\, w_j \rangle$ for some real vectors $w_i \in \mathbb{R}^N$, $i, j = 1, \ldots, N+1$. As the two families have the same Gram matrix, we conclude, as in Proposition 2.5, with the existence of the unitary operator U such as announced. □

2.2 Obtuse random variables

Obtuse families of vectors are actually deeply connected to particular centered and normalized random variables.

Definition 2.7 Consider a random variable X, with values in $\mathbb{C}^N$, which can take only $N+1$ different non-null values $\{v_1, \ldots, v_{N+1}\}$ with strictly positive probability $p_1, \ldots, p_{N+1}$ respectively.

We shall denote by $X^1, \ldots, X^N$ the coordinates of X in $\mathbb{C}^N$. We say that X is *centered* if its expectation is 0, that is, if $\mathbb{E}[X^i] = 0$ for all i. We say that X is *normalized* if its covariance matrix is I, that is, if

$$\mathrm{cov}(X^i, X^j) = \mathbb{E}[\overline{X^i}\, X^j] - \mathbb{E}[\overline{X^i}]\,\mathbb{E}[X^j] = \delta_{ij},$$

for all $i, j = 1, \ldots N$.

Definition 2.8 We consider the canonical version of X, that is, we consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\Omega = \{1, \ldots, N+1\}$, where $\mathcal{F}$ is the full σ-algebra of Ω and where the probability measure $\mathbb{P}$ is given by $\mathbb{P}(\{i\}) = p_i$ and the random variable X is given by $X(i) = v_i$, for all $i \in \Omega$. The coordinates of v_i are denoted by v_i^k, for $k = 1, \ldots, N$, so that $X^k(i) = v_i^k$.

We shall also consider the deterministic random variable X^0 on $(\Omega, \mathcal{F}, \mathbb{P})$ which is constant equal to 1.

With these notations the following result is obvious.

Proposition 2.9 *A random variable X, taking $N+1$ different values in $\mathbb{C}^N$, is centered and normalized if and only if the family $\{X^0, X^1, \ldots, X^N\}$ forms an orthonormal basis of $L^2((\Omega, \mathcal{F}, \mathbb{P});\ \mathbb{C})$.*

We shall now connect these random variables to obtuse systems.

Definition 2.10 In the same way as above we put

$$\widehat{v}_i = \begin{pmatrix} 1 \\ v_i \end{pmatrix} \in \mathbb{C}^{N+1},$$

for all $i = 1, \ldots, N+1$. For $i = 0, \ldots, N$ let $\widetilde{X}^i$ be the random variable defined by

$$\widetilde{X}^i(j) = \sqrt{p_j}\, X^i(j)$$

for all $i = 0, \ldots, N$ and all $j = 1, \ldots, N+1$.

Proposition 2.11 *The following assertions are equivalent.*

(1) *X is centered and normalized.*
(2) *The $(N+1)\times(N+1)$-matrix $\left(\widetilde{X}^i(j)\right)_{i,j}$ is a unitary matrix.*
(3) *The $(N+1)\times(N+1)$-matrix $\left(\sqrt{p_i}\,\widehat{v}_i^j\right)_{i,j}$ is a unitary matrix.*
(4) *The family $\{v_1,\dots,v_{N+1}\}$ is an obtuse system with*

$$p_i = \frac{1}{1+\|v_i\|^2},$$

for all $i = 1,\dots,N+1$.

Proof 1) $\Rightarrow$ 2): Since the random variable X is centered and normalized, each component X^i has a zero mean and the scalar product between two components X^i, X^j is given by the matrix I. Hence, for all i in $\{1,\dots,N\}$, we get

$$\mathbb{E}\left[X^i\right] = 0 \iff \sum_{k=1}^{N+1} p_k\, v_k^i = 0, \tag{4}$$

and for all $i, j = 1,\dots N$,

$$\mathbb{E}[\overline{X^i}\, X^j] = \delta_{ij} \iff \sum_{k=1}^{N+1} p_k\, \overline{v_k^i}\, v_k^j = \delta_{ij}. \tag{5}$$

Now, using Eqs. (4) and (5), we get, for all $i, j = 1,\dots,N$

$$\left\langle \widetilde{X}^0\,,\, \widetilde{X}^0\right\rangle = \sum_{k=1}^{N+1} p_k = 1,$$
$$\left\langle \widetilde{X}^0\,,\, \widetilde{X}^i\right\rangle = \sum_{k=1}^{N+1} \overline{\sqrt{p_k}}\,\sqrt{p_k}\, v_k^i = 0,$$
$$\left\langle \widetilde{X}^i\,,\, \widetilde{X}^j\right\rangle = \sum_{k=1}^{N+1} \overline{\sqrt{p_k}\, v_k^j}\,\sqrt{p_k} v_k^i = \delta_{i,j}.$$

The unitarity follows immediately.
2) $\Rightarrow$ 1): Conversely, if the matrix $\left(\widetilde{X}^i(j)\right)_{i,j}$ is unitary, the scalar products of column vectors give the mean 0 and the covariance I for the random variable X.
2) $\Leftrightarrow$ 3): The matrix $\left(\sqrt{p_j}\,\widehat{v}_i^j\right)_{i,j}$ is the transpose matrix of $\left(\widetilde{X}^i(j)\right)_{i,j}$. Therefore, if one of these two matrices is unitary, its transpose matrix is unitary too.
3) $\Leftrightarrow$ 4): The matrix $\left(\sqrt{p_j}\,\widehat{v}_j^i\right)_{i,j}$ is unitary if and only if

$$\left\langle \sqrt{p_i}\,\widehat{v}_i\,,\, \sqrt{p_j}\,\widehat{v}_j\right\rangle = \delta_{i,j},$$

for all $i, j = 1, \ldots, N+1$. On the other hand, the condition $\left\langle \sqrt{p_i}\,\widehat{v_i}\,,\ \sqrt{p_i}\,\widehat{v_i}\right\rangle = 1$ is equivalent to $p_i\,(1 + \|v_i\|^2) = 1$, whereas the condition $\left\langle \sqrt{p_i}\,\widehat{v_i}\,,\ \sqrt{p_j}\,\widehat{v_j}\right\rangle = 0$ is equivalent to $\sqrt{p_i}\,\sqrt{p_j}\,(1 + \langle v_i\,,\ v_j\rangle) = 0$, that is, $\langle v_i\,,\ v_j\rangle = -1$. This gives the result.

This result motivates the following definition.

Definition 2.12 Random variables in $\mathbb{C}^N$ which take only $N+1$ different values with strictly positive probability, which are centered and normalized, are called *obtuse random variables in* $\mathbb{C}^N$.

Theorem 2.6 extends in the following way to obtuse random variables.

Theorem 2.13 *Let X be an obtuse random variable in $\mathbb{C}^N$. Then there exists an obtuse random variable Y in $\mathbb{C}^N$, taking only real values, but with the same probabilities as those of X, there exists a unitary map U on $\mathbb{C}^N$, such that*

$$X = UY.$$

Proof Theorem 2.6 shows that there exists a unitary operator V of $\mathbb{C}^N$ which transforms the values of X into a real obtuse system. The random variable $Y = VX$ is then a real-valued obtuse random variable in $\mathbb{C}^N$. □

2.3 Generic character of obtuse random variables

We shall here apply the properties of obtuse systems to obtuse random variables, in order to show that these random variables somehow generate all the finitely supported probability distributions on $\mathbb{C}^N$.

First of all, an immediate consequence of Proposition 2.5 is that obtuse random variables on $\mathbb{C}^N$ with a prescribed probability distribution $\{p_1, \ldots, p_{N+1}\}$ are essentially unique.

Proposition 2.14 *Let X be an obtuse random variable of $\mathbb{C}^N$ having $\{p_1, \ldots, p_{N+1}\}$ as associated probabilities. Let Y be a $\mathbb{C}^N$-valued random variable. Then the following assertions are equivalent.*

(i) *The random variable Y is an obtuse random variable on $\mathbb{C}^N$ with same probabilities $\{p_1, \ldots, p_{N+1}\}$.*
(ii) *There exists a unitary operator U on $\mathbb{C}^N$ such that $Y = UX$ in distribution.*

Having proved that uniqueness, we shall now prove that obtuse random variables generate all the random variables (at least with finite support). First of all, a rather simple remark which shows that the choice of taking $N+1$ different values is the minimal one for centered and normalized random variables in $\mathbb{C}^N$.

Proposition 2.15 *Let X be a centered and normalized random variable in $\mathbb{C}^d$, taking n different values. Then we must have*

$$n \geq d + 1.$$

Proof Let X be centered and normalized in $\mathbb{C}^d$, taking the values $v_1, \ldots, v_n$ with probabilities $p_1, \ldots, p_n$. The relation

$$0 = \mathbb{E}[X] = \sum_{i=1}^{n} p_i v_i$$

shows that $\mathrm{Rank}\{v_1, \ldots, v_n\} < n$. On the other hand, the relation

$$I_{\mathbb{C}^d} = \mathbb{E}\left[|X\rangle\langle X|\right] = \sum_{i=1}^{n} p_i |v_i\rangle\langle v_i|$$

shows that $\mathrm{Rank}\{v_1, \ldots, v_n\} \geq d$. This gives the result. □

We can now state the theorem which shows how general, finitely supported, random variables on $\mathbb{C}^d$ are generated by the obtuse ones. We concentrate only on centered and normalized random variables, for they obviously generate all the others, up to an affine transform of $\mathbb{C}^d$.

Theorem 2.16 *Let $n \geq d+1$ and let X be a centered and normalized random variable in $\mathbb{C}^d$, taking n different values $v_1, \ldots, v_n$, with probabilities $p_1, \ldots, p_n$ respectively.*

If Y is any obtuse random variable on $\mathbb{C}^{n-1}$ associated to the probabilities $p_1, \ldots, p_n$, then there exists a partial isometry A from $\mathbb{C}^{n-1}$ to $\mathbb{C}^d$, with $\mathrm{Ran}\, A = \mathbb{C}^d$, *such that*

$$X = AY$$

in distribution.

Proof Assume that the obtuse random variable Y takes the values $w_1, \ldots, w_n$ in $\mathbb{C}^{n-1}$. The family $\{w_1, \ldots, w_{n-1}\}$ is obviously linearly independent, hence there exists a linear map $A : \mathbb{C}^{n-1} \to \mathbb{C}^d$ such that $Aw_i = v_i$ for all $i < n$. Now we have

$$p_n\, v_n = -\sum_{i<n} p_i\, v_i = -\sum_{i<n} p_i\, Aw_i = A\left(-\sum_{i<n} p_i\, w_i\right) = p_n\, Aw_n.$$

Hence the relation $Aw_i = v_i$ holds for all $i \leq n$.

We have proved the relation $X = AY$ in distribution, with A being a linear map from $\mathbb{C}^{n-1}$ to $\mathbb{C}^d$. The fact that X is normalized can be written as $\mathbb{E}[XX^*] = I_d$. But

$$\mathbb{E}[XX^*] = \mathbb{E}[AYY^*A^*] = A\mathbb{E}[YY^*]A^* = A\, I_n\, A^* = AA^*.$$

Hence A must satisfy $AA^* = I_d$, which is exactly saying that A is a partial isometry with range $\mathbb{C}^d$. □

2.4 Associated 3-tensors

Obtuse random variables are naturally associated to some 3-tensors with particular symmetries. This is what we shall prove here.

Definition 2.17 In this article, a *3-tensor on* $\mathbb{C}^N$ is an element of $(\mathbb{C}^N)^* \otimes \mathbb{C}^N \otimes \mathbb{C}^N$, that is, a linear map from $\mathbb{C}^N$ to $\mathbb{C}^N \otimes \mathbb{C}^N$. Coordinate-wise, it is represented by a collection of coefficients $(S_k^{ij})_{i,j,k=1,\dots,N}$. It acts on $\mathbb{C}^N$ as

$$(S(x))^{ij} = \sum_{k=1}^{N} S_k^{ij} x^k.$$

We shall see below that obtuse random variables on $\mathbb{C}^N$ have a naturally associated 3-tensor on $\mathbb{C}^{N+1}$. Note that, because of our notation choice $X^0, X^1, \dots, X^N$, the 3-tensor is indexed by $\{0, 1, \dots, N\}$ instead of $\{1, \dots, N+1\}$.

Proposition 2.18 *Let X be an obtuse random variable in $\mathbb{C}^N$. Then there exists a unique 3-tensor S on $\mathbb{C}^{N+1}$ such that*

$$X^i\, X^j = \sum_{k=0}^{N} S_k^{ij}\, X^k, \tag{6}$$

for all $i, j = 0, \dots, N$. This 3-tensor S is given by

$$S_k^{ij} = \mathbb{E}[X^i\, X^j\, \overline{X^k}], \tag{7}$$

for all $i, j, k = 0, \dots N$.

We also have the relation, for all $i, j = 0, \dots, N$

$$\overline{X^i}\, X^j = \sum_{k=0}^{N} \overline{S_j^{ik}}\, X^k. \tag{8}$$

Proof We know, from Proposition 2.9 that the random variables $\{X^0, X^1, \dots, X^N\}$ form an orthonormal basis of $L^2(\Omega, \mathcal{F}, \mathbb{P})$. These random variables being bounded, the products $X^i\, X^j$ are still elements of $L^2(\Omega, \mathcal{F}, \mathbb{P})$, hence they can be written, in a unique way, as linear combinations of the X^k's. As a consequence, there exists a unique 3-tensor S on $\mathbb{C}^{N+1}$ such that

$$X^i\, X^j = \sum_{k=0}^{N} S_k^{ij}\, X^k$$

for all $i, j = 0, \ldots, N$. In particular we have

$$\mathbb{E}[X^i\, X^j\, \overline{X^k}] = \sum_{l=0}^{N} S_l^{ij}\, \mathbb{E}[X_l\, \overline{X_k}] = S_k^{ij}.$$

This shows Identity (7).

Finally, we have, by the orthonormality of the X^k's

$$\overline{X^i}\, X^j = \sum_{k=0}^{N} \mathbb{E}[\overline{X^i}\, X^j\, \overline{X^k}]\, X^k,$$

that is,

$$\overline{X^i}\, X^j = \sum_{k=0}^{N} \overline{S_j^{ik}}\, X^k,$$

by (7). This gives Identity (8). □

The 3-tensor S has quite some symmetries, let us detail some of them.

Proposition 2.19 *Let S be the 3-tensor associated to an obtuse random variable X on $\mathbb{C}^N$. Then the 3-tensor S satisfies the following relations, for all $i, j, k, l = 0, \ldots N$*

$$S_k^{i0} = \delta_{ik}, \tag{9}$$

$$S_k^{ij} \text{ is symmetric in } (i, j), \tag{10}$$

$$\sum_{m=0}^{N} S_j^{im}\, S_m^{kl} \text{ is symmetric in } (i, k), \tag{11}$$

$$\sum_{m=0}^{N} S_j^{im}\, \overline{S_k^{lm}} \text{ is symmetric in } (i, k). \tag{12}$$

Proof The relation (9) is immediate for

$$S_k^{i0} = \mathbb{E}[X^i\, \overline{X^k}] = \delta_{ik}.$$

Equation (10) comes directly from Formula (7) which shows a clear symmetry in (i, j).

By (8) we have

$$X^i\, \overline{X^j} = \sum_{m=0}^{N} S_j^{im}\, \overline{X^m},$$

whereas

$$X^k\, X^l = \sum_{n=0}^{N} S_n^{kl}\, X^n.$$

Altogether this gives

$$\mathbb{E}\left[X^i\, \overline{X^j}\, X^k\, X^l\right] = \sum_{m=0}^{N} S_j^{im}\, S_m^{kl}.$$

But the left hand side is clearly symmetric in (i, k) and (11) follows.

In order to prove (12), we write, using (8)

$$X^i\, \overline{X^j} = \sum_{m=0}^{N} S_j^{im}\, \overline{X^m}$$

and

$$\overline{X^l}\, X^k = \sum_{n=0}^{N} \overline{S_k^{lm}}\, X^n.$$

Altogether we get

$$\mathbb{E}\left[X^i\, \overline{X^j}\, \overline{X^l}\, X^k\right] = \sum_{m=0}^{N} S_j^{im}\, \overline{S_k^{lm}}.$$

But the left hand side is clearly symmetric in (i, k) and (12) is proved. □

2.5 Representation of multiplication operators

Let X be an obtuse random variable in $\mathbb{C}^N$, with associated 3-tensor S and let $(\Omega, \mathcal{F}, \mathbb{P}_S)$ be the canonical space of X. Note that we have added the dependency on S for the probability measure $\mathbb{P}_S$. The reason is that, when changing the obtuse random variable X on $\mathbb{C}^N$, the canonical space Ω and the canonical σ-field $\mathcal{F}$ do not change, only the canonical measure $\mathbb{P}$ changes.

Definition 2.20 We have seen that the space $L^2(\Omega, \mathcal{F}, \mathbb{P}_S)$ is a $N+1$-dimensional Hilbert space and that the family $\{X^0, X^1, \ldots, X^N\}$ is an orthonormal basis of that space. Hence for every obtuse random variable X, with associated 3-tensor S, we have a natural unitary operator

$$\begin{aligned} U_S : L^2(\Omega, \mathcal{F}, \mathbb{P}_S) &\longrightarrow \mathbb{C}^{N+1} \\ X^i &\longmapsto e_i, \end{aligned}$$

where $\{e_0, \ldots, e_N\}$ is the canonical orthonormal basis of $\mathbb{C}^{N+1}$. The operator U_S is called the *canonical isomorphism* associated to X.

The interesting point with the isomorphisms U_S is that they canonically transport all the obtuse random variables of $\mathbb{C}^N$ onto a common canonical space. But the point is that the probabilistic informations concerning the random variable X are not correctly transferred via this isomorphism: all the informations about the law, the independencies, etc. are lost when identifying X^i to e_i. The only way to recover the probabilistic informations about the X^i's on $\mathbb{C}^{N+1}$ is to consider the *multiplication operator* by X^i, defined as follows. On the space $L^2(\Omega, \mathcal{F}, \mathbb{P}_S)$, for each $i = 0, \ldots, N$, we consider the multiplication operator

$$\begin{aligned} \mathcal{M}_{X^i} : L^2(\Omega, \mathcal{F}, \mathbb{P}_S) &\longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P}_S) \\ Y &\longmapsto X^i\, Y, \end{aligned}$$

These multiplication operators carry all the probabilistic informations on X, even through a unitary transform such as U_S, for we have, by the usual functional calculus for normal operators

$$\begin{aligned} \mathbb{E}[f(X^1, \ldots, X^N)] &= \big\langle X_0\,,\ f(\mathcal{M}_{X^1}, \ldots, \mathcal{M}_{X^N})\, X_0 \big\rangle_{L^2(\Omega, \mathcal{F}, \mathbb{P}_T)} \\ &= \big\langle e_0\,,\ U_S\, f(\mathcal{M}_{X^1}, \ldots, \mathcal{M}_{X^N})\, U_S^*\, e_0 \big\rangle_{\mathbb{C}^{N+1}} \\ &= \big\langle e_0\,,\ f(U_S\, \mathcal{M}_{X^1} U_S^*, \ldots, U_S\, \mathcal{M}_{X^N}\, U_S^*)\, e_0 \big\rangle_{\mathbb{C}^{N+1}}\,. \end{aligned}$$

Definition 2.21 On the space $\mathbb{C}^{N+1}$, with canonical basis $\{e_0, \ldots, e_N\}$ we consider the basic matrices a^i_j, for $i, j = 0, \ldots, N$ defined by

$$a^i_j\, e_k = \delta_{ik}\, e_j.$$

We shall see now that, when carried out on the same canonical space by U_S, the obtuse random variables of $\mathbb{C}^N$ admit a simple and compact matrix representation in terms of their 3-tensor.

Theorem 2.22 *Let X be an obtuse random variable on $\mathbb{C}^N$, with associated 3-tensor S and canonical isomorphism U_S. Then we have, for all $i, j = 0, \ldots, N$*

$$U_S\, \mathcal{M}_{X^i}\, U_S^* = \sum_{j,k=0}^{N} S^{ij}_k\, a^j_k. \tag{13}$$

for all $i = 0, \ldots, N$.

The operator of multiplication by $\overline{X^i}$ is given by

$$U_S\, \mathcal{M}_{\overline{X^i}}\, U_S^* = \sum_{j,k=0}^{N} \overline{S^{ik}_j}\, a^j_k. \tag{14}$$

Proof We have, for any fixed $i \in \{0, \dots, N\}$, for all $j = 0, \dots, N$

$$\begin{aligned} U_S \, \mathcal{M}_{X^i} \, U_S^* \, e_j &= U_S \, \mathcal{M}_{X^i} \, X^j \\ &= U_S \, X^i \, X^j \\ &= U_S \sum_{k=0}^{N} S_k^{ij} \, X^k \\ &= \sum_{k=0}^{N} S_k^{ij} \, e_k . \end{aligned}$$

Hence the operator $U_S \, \mathcal{M}_{X^i} \, U_S^*$ has the same action on the orthonormal basis $\{e_0, \dots, e_N\}$ as the operator

$$\sum_{k=0}^{N} S_k^{ij} \, a_k^j .$$

This proves the representation (13).

The last identity is just an immediate translation of the relation (8). □

2.6 Back to the example

Let us illustrate the previous subsections with our example. To the obtuse system

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}, \qquad v_3 = -\frac{1}{5} \begin{pmatrix} 3+4i \\ 1+3i \end{pmatrix}$$

of $\mathbb{C}^2$ is associated the random variable X on $\mathbb{C}^2$ which takes the values v_1, v_2, v_3 with probability $p_1 = 1/3$, $p_2 = 1/4$ and $p_3 = 5/12$, respectively. The 3-tensor S associated to X is directly computable. We present S as a collection of matrices $S^j = \left(S_k^{i\,j}\right)_{i,k}$, which are then the matrices of multiplication by X^j:

$$S^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad S^1 = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{5}(1-2i) & 0 & -\frac{2}{5}(2+i) \\ -\frac{2}{5}(1-2i) & 0 & \frac{1}{5}(2+i) \end{pmatrix}$$

$$S^2 = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{2}{5}(1-2i) & 0 & \frac{1}{5}(2+i) \\ \frac{1}{5}(1-2i) & -i & -\frac{1}{5}(1-2i) \end{pmatrix}.$$

These matrices are not symmetric (we shall see in Sect. 3.3 what the symmetry of the matrices S^j corresponds to). We recognize the particular form of S^0, for it corresponds to $\mathcal{M}_{X^0} = I$.

3 Complex doubly-symmetric 3-tensors

We are going to leave for a moment the obtuse random variables and concentrate on the symmetries we have obtained above. The relation (9) is really specific to obtuse random variables, we shall leave it for a moment. We concentrate on the relation (10), (11) and (12) which have important consequences for the 3-tensor.

3.1 The main diagonalization theorem

Definition 3.1 A 3-tensor S on $\mathbb{C}^{N+1}$ which satisfies (10), (11) and (12) is called a *complex doubly-symmetric 3-tensor on* $\mathbb{C}^{N+1}$.

The main result concerning complex doubly-symmetric 3-tensors in $\mathbb{C}^{N+1}$ is that they are the exact generalization for 3-tensors of normal matrices for 2-tensors: they are exactly those 3-tensors which can be diagonalized in some orthonormal basis of $\mathbb{C}^{N+1}$.

Definition 3.2 A 3-tensor S on $\mathbb{C}^{N+1}$ is said to be *diagonalizable in some orthonormal basis* $(a_m)_{m=0}^N$ of $\mathbb{C}^{N+1}$ if there exist complex numbers $(\lambda_m)_{m=0}^N$ such that

$$S = \sum_{m=0}^{N} \lambda_m \, a_m^* \otimes a_m \otimes a_m . \tag{15}$$

In other words

$$S(x) = \sum_{m=0}^{N} \lambda_m \, \langle a_m \, , \, x \rangle \, a_m \otimes a_m \tag{16}$$

for all $x \in \mathbb{C}^{N+1}$.

Note that, as opposed to the case of 2-tensors (that is, matrices), the "eigenvalues" λ_m are not completely determined by the representation (16). Indeed, if we put $\widetilde{a}_m = e^{i\theta_m} \, a_m$ for all m, then the $\widetilde{a}_m$'s still form an orthonormal basis of $\mathbb{C}^{N+1}$ and we have

$$S(x) = \sum_{m=1}^{N} \lambda_m \, e^{i\theta_m} \, \langle \widetilde{a}_m \, , \, x \rangle \, \widetilde{a}_m \otimes \widetilde{a}_m .$$

Hence the λ_m's are only determined up to a phase; only their modulus is determined by the representation (16).

Definition 3.3 Actually, there are more natural objects that can be associated to diagonalizable 3-tensors; they are the *orthogonal families in* $\mathbb{C}^N$. Indeed, if S is diagonalizable as above, for all m such that $\lambda_m \neq 0$ put $v_m = \lambda_m \, a_m$. The family $\{v_m; \; m = 1, \ldots, K\}$ is then an orthogonal family in $\mathbb{C}^{N+1}$ and we have

$$S(v_m) = v_m \otimes v_m$$

for all m. In terms of the v_m's, the decomposition (16) of S becomes

$$S(x) = \sum_{m=1}^{K} \frac{1}{\|v_m\|^2} \langle v_m \,,\, x\rangle \; v_m \otimes v_m. \tag{17}$$

This is the form of diagonalization we shall retain for 3-tensors. Be aware that in the above representation the vectors are orthogonal, but not normalized anymore. Also note that they represent the eigenvectors of S associated only to the non-vanishing eigenvalues of S.

We can now state the main theorem.

Theorem 3.4 *A 3-tensor S on $\mathbb{C}^{N+1}$ is diagonalizable in some orthonormal basis if and only if it is doubly-symmetric.*

More precisely, the formulas

$$\mathcal{V} = \left\{ v \in \mathbb{C}^{N+1} \backslash \{0\};\; S(v) = v \otimes v \right\},$$

and

$$S(x) = \sum_{v \in \mathcal{V}} \frac{1}{\|v\|^2} \langle v \,,\, x\rangle \; v \otimes v,$$

establish a bijection between the set of orthogonal systems $\mathcal{V}$ in $\mathbb{C}^{N+1}$ and the set of complex doubly-symmetric 3-tensors S.

Proof First step: let $\mathcal{V} = \{v_m;\; m = 1, \ldots, K\}$ be an orthogonal familly in $\mathbb{C}^{N+1} \backslash \{0\}$. Put

$$S_k^{ij} = \sum_{m=1}^{K} \frac{1}{\|v_m\|^2} \, v_m^i \, v_m^j \, \overline{v_m^k},$$

for all $i, j, k = 0, \ldots, N$. We shall check that S is a complex doubly-symmetric 3-tensor in $\mathbb{C}^{N+1}$. The symmetry of S_k^{ij} in (i, j) is obvious from the definition. This gives (10).

We have

$$\begin{aligned}
\sum_{m=0}^{N} S_j^{im} \, S_m^{kl} &= \sum_{m=0}^{N} \sum_{n,p=1}^{K} \frac{1}{\|v_n\|^2} \frac{1}{\|v_p\|^2} \, v_n^i \, \overline{v_n^j} \, v_n^m \, \overline{v_p^m} \, v_p^k \, v_p^l \\
&= \sum_{n,p=1}^{K} \frac{1}{\|v_n\|^2} \frac{1}{\|v_p\|^2} \, v_n^i \, \overline{v_n^j} \, \langle v_p \,,\, v_n \rangle \, v_p^k \, v_p^l \\
&= \sum_{n=1}^{K} \frac{1}{\|v_n\|^2} \, v_n^i \, \overline{v_n^j} \, v_n^k \, v_n^l
\end{aligned}$$

and the symmetry in (i, k) is obvious. This gives (11).

We have

$$\begin{aligned}\sum_{m=0}^{N} S_j^{im}\,\overline{S_k^{lm}} &= \sum_{m=0}^{N}\sum_{n,p=1}^{K} \frac{1}{\|v_n\|^2}\,\frac{1}{\|v_p\|^2}\, v_n^i\, v_n^m\, \overline{v_n^j}\,\overline{v_p^l}\,\overline{v_p^m}\, v_p^k \\ &= \sum_{n,p=1}^{K} \frac{1}{\|v_n\|^2}\,\frac{1}{\|v_p\|^2}\, v_n^i\,\overline{v_n^j}\,\langle v_n\,,\ v_m\rangle\, v_p^k\,\overline{v_p^l} \\ &= \sum_{n=1}^{K} \frac{1}{\|v_n\|^2}\, v_n^i\,\overline{v_n^j}\, v_n^k\,\overline{v_n^l}\end{aligned}$$

and the symmetry in (i, k) is obvious. This gives (12).

We have proved that the formula

$$S(x) = \sum_{v\in\mathcal{V}} \frac{1}{\|v\|^2}\,\langle v\,,\ x\rangle\ v\otimes v \tag{18}$$

defines a complex doubly-symmetric 3-tensor if $\mathcal{V}$ is any family of (non-vanishing) orthogonal vectors.

Second step: now given a complex doubly-symmetric 3-tensor S of the form (18), we shall prove that the set $\mathcal{V}$ coincides with the set

$$\widehat{\mathcal{V}} = \{v \in \mathbb{C}^N\backslash\{0\};\ S(v) = v\otimes v\}.$$

Clearly, if $y \in \mathcal{V}$ we have by (18)

$$S(y) = y\otimes y.$$

This proves that $\mathcal{V} \subset \widehat{\mathcal{V}}$. Now, let $v \in \widehat{\mathcal{V}}$. On one side we have

$$S(v) = v\otimes v,$$

on the other side we have

$$S(v) = \sum_{z\in\mathcal{V}} \frac{1}{\|z\|^2}\,\langle z\,,\ v\rangle\ z\otimes z.$$

In particular, applying $\langle y| \in \mathcal{V}^*$ to both sides, we get

$$\langle y\,,\ v\rangle\ v = \langle y\,,\ v\rangle\ y$$

and thus either v is orthogonal to y or $v = y$. This proves that v is one of the elements y of $\mathcal{V}$, for it were orthogonal to all the $y \in \mathcal{V}$ we would get $v\otimes v = S(v) = 0$ and v would be the null vector.

We have proved that $\mathcal{V}$ coincides with the set

$$\{v \in \mathbb{C}^N \backslash \{0\};\ S(v) = v \otimes v\}.$$

Third step: now we shall prove that all complex doubly-symmetric 3-tensors S on $\mathbb{C}^{N+1}$ are diagonalizable in some orthonormal basis. The property (10) indicates that the matrices

$$S_k = (S_k^{ij})_{i,j=0,\ldots,N}$$

are symmetric. But, as they are complex-valued matrices, this does not imply any property of diagonalization. Rather we have the following theorem [12].

Theorem 3.5 (Takagi Factorization) *Let M be a complex symmetric matrix, there exist a unitary U matrix and a diagonal matrix D such that*

$$M = U D U^t = U D (\overline{U})^{-1}. \tag{19}$$

Secondly, we shall need to simultaneously "factorize" the S_k's as above. We shall make use of the following criteria (same reference).

Theorem 3.6 (Simultaneous Takagi factorization) *Let $\mathcal{F} = \{A_i\ ;\ i \in \mathcal{J}\}$ be a family of complex symmetric matrices in $\mathcal{M}_n(\mathbb{C})$. Let $\mathcal{G} = \left\{\overline{A_i}\, A_j\ ;\ i, j \in \mathcal{J}\right\}$. Then the following assertions are equivalent.*

(i) *There exists a unitary matrix U such that, for all i in $\mathcal{J}$, the matrix $U A_i U^t$ is diagonal.*
(ii) *The family $\mathcal{G}$ is commuting.*

This is the first part of Step three: proving that in our case the matrices $\overline{S_i}\, S_j$ commute. Using the 3 symmetry properties of S we get

$$\begin{aligned}
\left(\overline{S_i}\, S_j\, \overline{S_k}\, S_l\right)_{m,n} &= \sum_{x,y,z=0}^{N} \overline{S_i^{mx}}\, S_j^{xy}\, \overline{S_k^{yz}}\, S_l^{zn} = \sum_{x,y,z=0}^{N} \overline{S_y^{mx}}\, S_j^{xi}\, \overline{S_k^{yz}}\, S_l^{zn} \\
&= \sum_{x,y,z=0}^{N} \overline{S_y^{zx}}\, S_j^{xi}\, \overline{S_k^{ym}}\, S_l^{zn} = \sum_{x,y,z=0}^{N} \overline{S_n^{zx}}\, S_j^{xi}\, \overline{S_k^{ym}}\, S_l^{zy} \\
&= \sum_{x,y,z=0}^{N} \overline{S_i^{zx}}\, S_j^{xn}\, \overline{S_k^{ym}}\, S_l^{zy} = \sum_{x,y,z=0}^{N} \overline{S_k^{my}}\, S_l^{yz}\, \overline{S_i^{zx}}\, S_j^{xn} \\
&= \left(\overline{S_k}\, S_l\, \overline{S_i}\, S_j\right)_{m,n}.
\end{aligned}$$

This proves that $\overline{S_i}\, S_j\, \overline{S_k}\, S_l = \overline{S_k}\, S_l\, \overline{S_i}\, S_j$. The family $\left\{\overline{S_i} S_j; i, j = 0, \ldots, N\right\}$ is commuting. Thus, by Theorem 3.6, the matrices S_k can be simultaneously Takagi-factorized. There exists then a unitary matrix $U = (u^{ij})_{i,j=0,\ldots,N}$ such that, for all k in $\{0, \ldots, N\}$,

$$S_k = U\, D_k\, \overline{U}^{-1}, \tag{20}$$

where the matrix D_k is a diagonal matrix, $D_k = diag(\lambda_k^0, \ldots, \lambda_k^N)$. Thus, the coefficient S_k^{ij} can be written as

$$S_k^{ij} = \sum_{m=0}^{N} \lambda_k^m \, u^{im} \, u^{jm}.$$

Let us denote by a_m the mth column vector of U, that is, $a_m = (u^{lm})_{l=0,\ldots,N}$. Moreover, we denote by λ^m the vector of λ_k^m, for $k = 0, \ldots, N$. Since the matrix U is unitary, the vectors a_m form an orthonormal basis of $\mathbb{C}^{N+1}$. We have

$$S_k^{ij} = \sum_{m=0}^{N} a_m^i \, \lambda_k^m \, a_m^k.$$

Our aim now is to prove that λ^m is proportional to $\overline{a_m}$. To this end, we shall use the symmetry properties of S. From the simultaneous reduction (20), we get

$$\overline{S_j} \, S_q = \overline{U} \, \overline{D_j} \, D_q \, U^t.$$

Thus, we have

$$(\overline{S_j} \, S_q)_{i,r} = \sum_{m=0}^{N} \overline{S_j^{im}} \, S_q^{mr} = \sum_{m=0}^{N} \overline{a_m^i} \, \overline{\lambda_j^m} \, \lambda_q^m \, a_m^r.$$

In particular we have, for all $p \in \{0, \ldots, N\}$

$$\begin{aligned} &\sum_{i,j,q,r=0}^{N} (\overline{S_j} \, S_q)_{i,r} \, a_p^i \, \lambda_j^p \, \overline{\lambda_q^p} \, \overline{a_p^r} \\ &\quad = \sum_{m=0}^{N} \langle a_m \, , \, a_p \rangle \, \langle \lambda^m \, , \, \lambda^p \rangle \, \langle \lambda^p \, , \, \lambda^m \rangle \, \langle a_p \, , \, a_m \rangle = \left\| \lambda^p \right\|^4. \end{aligned} \tag{21}$$

An immediate consequence of (12) is that

$$\sum_{m=0}^{N} S_j^{im} \, \overline{S_k^{lm}}$$

is also symmetric in (j, l). Applying this, the expression (21) is also equal to

$$\begin{aligned} \sum_{i,j,q,r=0}^{N} \sum_{m=0}^{N} \overline{a_m^q} \, \overline{\lambda_j^m} \, \lambda_i^m \, a_m^r \, a_p^i \, \lambda_j^p \, \overline{\lambda_q^p} \, \overline{a_p^r} &= \sum_{m=0}^{N} \langle \overline{a_p} \, , \, \lambda^m \rangle \, \langle \lambda^m \, , \, \lambda^p \rangle \, \langle a_m \, , \, \overline{\lambda^p} \rangle \, \langle a_p \, , \, a_m \rangle \\ &= \left| \langle \overline{a_p} \, , \, \lambda^p \rangle \right|^2 \, \left\| \lambda^p \right\|^2. \end{aligned}$$

This gives

$$\left|\langle \overline{a_p}\,,\,\lambda^p\rangle\right| = \left\|\lambda^p\right\| = \left\|\overline{a_p}\right\|\,\left\|\lambda^p\right\|.$$

This is a case of equality in Cauchy–Schwarz inequality, hence there exists $\mu_p \in \mathbb{C}$ such that $\lambda^p = \mu_p\,\overline{a_p}$, for all $p = 0, \ldots, N$. This way, the 3-tensor T can be written as

$$S_k^{ij} = \sum_{m=0}^{N} \mu_m\, a_m^i\, a_m^j\, \overline{a_m^k}. \tag{22}$$

In other words

$$S(x) = \sum_{m=0}^{N} \mu_m\, \langle a_m\,,\,x\rangle\; a_m \otimes a_m.$$

We have obtained the orthonormal diagonalization of S. The proof is complete. □

3.2 Back to obtuse random variables

The theorem above is a general diagonalization theorem for 3-tensors. For the moment it does not take into account the relation (9). When we make it enter into the game, we see the obtuse systems appearing.

Theorem 3.7 *Let S be a doubly-symmetric 3-tensor on $\mathbb{C}^{N+1}$ satisfying also the relation*

$$S_k^{i0} = \delta_{ik}$$

for all $i, k = 0, \ldots, N$. Then the orthogonal system $\mathcal{V}$ such that

$$S(x) = \sum_{v\in\mathcal{V}} \frac{1}{\|v\|^2}\,\langle v\,,\,x\rangle\; v \otimes v \tag{23}$$

is made of exactly $N+1$ vectors $v_1, \ldots, v_{N+1}$, all of them satisfying $v_i^0 = 1$. In particular the family of $N+1$ vectors of $\mathbb{C}^N$, obtained by restricting the v_i's to their N last coordinates, forms an obtuse system in $\mathbb{C}^N$.

Proof First assume that $\mathcal{V} = \{v_1, \ldots, v_K\}$. By hypothesis, we have

$$S_k^{ij} = \sum_{m=1}^{K} \frac{1}{\|v_m\|^2}\, v_m^i\, v_m^j\, \overline{v_m^k},$$

for all $i, j, k = 0, \ldots, N$. With hypothesis (9) we have in particular

$$S_k^{i0} = \sum_{m=1}^{K} \frac{1}{\|v_m\|^2} v_m^i \, v_m^0 \, \overline{v_m^k} = \delta_{ik}$$

for all $i, k = 0, \ldots, N$.

Consider the orthonormal family of $\mathbb{C}^{N+1}$ made of the vectors $e_m = v_m / \|v_m\|$. We have obtained above the relation

$$\sum_{m=0}^{K} v_m^0 \, |e_m\rangle\langle e_m| = I$$

as matrices acting on $\mathbb{C}^{N+1}$. The above is thus a spectral decomposition of the identity matrix, this implies that the e_m's are exactly $N+1$ vectors and that all the v_m^0 are equal to 1.

This proves the first part of the theorem. The last part concerning obtuse systems is now obvious and was already noticed when we have introduced obtuse systems.

In particular we have proved the following theorem.

Theorem 3.8 *The set of doubly-symmetric 3-tensors S on $\mathbb{C}^{N+1}$ which satisfy also the relation*

$$S_k^{i0} = \delta_{ik}$$

for all $i, k = 0, \ldots, N$, is in bijection with the set of distributions of obtuse random variables X on $\mathbb{C}^N$. The bijection is described by the following, with the convention $X^0 = \mathbb{1}$:

- *The random variable X is the only (in distribution) random variable satisfying*

$$X^i X^j = \sum_{k=0}^{N} S_k^{ij} X^k,$$

  for all $i, j = 1, \ldots, N$.
- *The 3-tensor S is obtained by*

$$S_k^{ij} = \mathbb{E}[X^i \, X^j \, \overline{X^k}],$$

  for all $i, j, k = 0, \ldots, N$.

In particular the different possible values taken by X in $\mathbb{C}^N$ coincide with the vectors $w_n \in \mathbb{C}^N$, made of the last N coordinates of the eigenvectors v_n associated to S in the representation (23). The associated probabilities are then $p_n = 1/(1 + \|w_n\|^2) = 1/\|v_n\|^2$.

3.3 Recovering the real case

In [4] have been introduced the notions of *real* obtuse random variables and their associated *real* doubly-symmetric 3-tensors. In the same way as we did here, they obtained certain symmetries on the tensor which corresponded exactly to the condition for being diagonalizable in some real orthonormal basis. Note that in [4] the situation for the diagonalization theorem was much easier, for the symmetries associated to the 3-tensor came down to simultaneous diagonalization of commuting symmetric real matrices.

The question we want to answer here is: How do we recover the real case from the complex case? By this we mean: On what condition does a complex doubly-symmetric 3-tensor correspond to a real one, that is, corresponds to real-valued random variables? Surprisingly enough, the answer is not: "When the coefficients S_k^{ij} are all real"! We shall see later on that this condition is necessary, but it is not sufficient as is shown by the following counter-example.

Let us consider the one dimensional random variable X which takes values $i, -i$ with probability $1/2$. As usual denote by X^0 the constant random variable equal to 1 and by X^1 the random variable X. We have the relations

$$\begin{aligned} X^0X^0 &= X^0 \\ X^0X^1 &= X^1 \\ X^1X^0 &= X^1 \\ X^1X^1 &= -X^0 \end{aligned}$$

which give us the following matrices for the associated 3-tensor S:

$$\begin{aligned} S^0 &= (S_k^{i0})_{i,k} = \mathcal{M}_{X^0} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ S^1 &= (S_k^{i1})_{i,k} = \mathcal{M}_{X^1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

They are real-valued matrices, but they are associated to a complex (non real) random variable.

In fact, the major difference between a complex (non real) doubly-symmetric 3-tensor and a real doubly-symmetric 3-tensor lies in the commutation property of indices i and k in the coefficients S_k^{ij}. Let us make this more precise.

Definition 3.9 A doubly symmetric 3-tensor S on $\mathbb{C}^N$ is said to be *totally real* if the associated orthogonal system $\mathcal{V}$ is made of real vectors.

Note that if the 3-tensor S also satisfies (9) then the condition "S is totally real" means that the associated obtuse random variable X on $\mathbb{C}^N$ is real-valued.

We have the following algebraic characterization of totally real tensors.

Proposition 3.10 *Let S be a complex doubly symmetric tensor on $\mathbb{C}^{N+1}$. The following assertions are equivalent.*

(1) *The 3-tensor S is totally real.*
(2) *For all $i, j, k = 0, \ldots, N$ the coefficients S_k^{ij} are real and we have*

$$S_k^{ij} = S_i^{kj}.$$

If S also satisfies the condition (9) *then Condition* (2) *may be reduced to*
(2') *For all $i, j, k = 0, \ldots, N$ we have*

$$S_k^{ij} = S_i^{kj}.$$

In particular obtuse random variables X in $\mathbb{C}^N$ are real-valued if and only if their associated 3-tensor S satisfy (2').

Proof If the tensor S is totally real we then recover the usual definition and properties of real orthogonal families, real obtuse systems and real obtuse random variables. The condition (1) is then an easy consequence.

Conversely, assume that (1) is satisfied. As we have

$$S_k^{ij} = \sum_{m=1}^{K} \frac{1}{\|v_m\|^2}\, v_m^i\, v_m^j\, \overline{v_m^k},$$

the commutation relation implies that

$$\sum_{m=1}^{K} \frac{1}{\|v_m\|^2}\, v_m^i\, v_m^j\, \overline{v_m^k} = \sum_{m=1}^{K} \frac{1}{\|v_m\|^2}\, v_m^k\, v_m^j\, \overline{v_m^i}.$$

Considering

$$\sum_{j=1}^{N} S_k^{ij}\, \overline{v_n^j}$$

with the two different expressions above, gives

$$v_n^i\, \overline{v_n^k} = v_n^k\, \overline{v_n^i}.$$

In particular $v_n^i\, \overline{v_n^k}$ is real for all $i, k = 0, \ldots, N$, all $n = 1, \ldots, K$.

Now, each v_m in the associated orthogonal family $\mathcal{V}$ satisfies

$$S(v_m) = v_m \otimes v_m.$$

In particular, we have

$$\sum_{i,k=0}^{N} S_k^{ij}\, v_m^k\, \overline{v_m^i} = \|v_m\|^2\, v_m^j.$$

As the S_k^{ij}'s and all the $v_m^k\,\overline{v_m^i}$'s are real numbers, we get that all the v_m^j's are real numbers. We have proved the first equivalence.

If one knows that the tensor S satisfies (9), the proof starts in the same way as above, obtaining the relation

$$v_n^i\,\overline{v_n^k} = v_n^k\,\overline{v_n^i}.$$

At that stage, as we have already noticed in Theorem 3.7, the relation (9) gives $v_m^0 = 1$ for all m. Taking $k = 0$ in the equality above proves that the coordinates v_m^i are all real. The equivalence at the end of the proposition is now immediate. □

In the counter-example above, one can check that $S_1^{01} = 1$ and $S_0^{11} = -1$. The commutation condition is not satisfied and the associated random variable is not real.

We know from Theorem 2.13 that there exists a unitary operator of $\mathbb{C}^N$ that can transform a given complex obtuse random variable into a real one. For the moment we have not been very explicit on the construction of this unitary operator, we shall show that the appropriate transform is nicely carried by the associated 3-tensor.

First of all, we establish a general decomposition result for unitary symmetric matrices.

Proposition 3.11 *A matrix S on $\mathbb{C}^N$ is unitary and symmetric if and only if it can be written as*

$$S = V\,V^t$$

for some unitary matrix V.

The map $S \mapsto V$ can be chosen to be Borel.

Proof One direction is obvious, if S is of the form $V\,V^t$ for some unitary V then S is obviously symmetric and unitary.

Conversely, let us assume that S is unitary and symmetric. As it is symmetric, by Takagi Theorem 3.5, the matrix S can be decomposed as $U\,D\,U^t$ for some unitary U and some diagonal matrix D. But as S is unitary we have

$$I = S^*\,S = \overline{U}\,\overline{D}\,U^*\,U\,D\,U^t,$$

hence

$$|D|^2 = U^t\,\overline{U} = I$$

and the matrix D is unitary too. In particular its entries are modulus 1 complex numbers. Let L be the diagonal matrix whose entries are a square root of the entries of D, they are also of modulus 1, so that $L\,\overline{L} = I$.

Put $V = U\,L$, then

$$V\,V^t = U\,L\,L\,U^t = U\,D\,U^t = S,$$

but also

$$V\,V^* = U\,L\,\overline{L}\,U^* = U\,U^* = I.$$

We have proved the announced decomposition of S.

The assertion on the Borel character of the map $S \mapsto V$ comes from the fact that the map $V \mapsto V\,V^t$ is obviously continuous, hence Borel. It is is a surjection between the space of unitary matrices on $\mathbb{C}^n$ and the space of unitary and symmetric matrices on $\mathbb{C}^n$. Both spaces are compact and metrizable. Hence Uniformization Theorems apply and there exists a Borel section of this mapping, that is, a Borel choice for the inverse mapping (cf [18], Corollary 5.2.6). This proves the last claim. □

We can now show how to obtain a real obtuse random variable associated to a complex one.

Theorem 3.12 *Let X be an obtuse random variable in $\mathbb{C}^N$ with associated 3-tensor S. Then the matrix $S_0 = (S_0^{ij})_{i,j=0,\dots,N}$ is symmetric and unitary, it can be decomposed as $V\,V^t$ for some unitary matrix V of $\mathbb{C}^N$. For any such unitary operator V the random variable $R = V^*\,X$ is a real obtuse random variable of $\mathbb{C}^N$.*

Proof By definition we have $S_0^{ij} = \mathbb{E}[X^i\,X^j]$ and hence it is symmetric in (i,j). Let us check it is a unitary matrix. We have

$$\begin{aligned}
\sum_{m=0}^{N} S_0^{im}\,\overline{S_0^{jm}} &= \sum_{m=0}^{N} \mathbb{E}[X^i\,X^m]\,\mathbb{E}[\overline{X^m}\,\overline{X^j}] \\
&= \sum_{m=0}^{N} \left\langle \overline{X^i}\,,\,X^m \right\rangle_{L^2(\Omega,\mathcal{F},\mathbb{P})} \left\langle X^m\,,\,\overline{X^j} \right\rangle_{L^2(\Omega,\mathcal{F},\mathbb{P})} \\
&= \left\langle \overline{X^i}\,,\,\overline{X^j} \right\rangle_{L^2(\Omega,\mathcal{F},\mathbb{P})} \\
&= \mathbb{E}[X^i\,\overline{X^j}] = \delta_{ij}.
\end{aligned}$$

This proves the unitarity.

By Proposition 3.11, there exists a decomposition $S_0 = V\,V^t$ for some unitary matrix V.

We now check the $Y = V^*X$ is real-valued. Recall that, by definition, we have $S_0 = \mathbb{E}[X\,X^t]$. We then have

$$\mathbb{E}[Y\,Y^t] = \mathbb{E}[V^*X\,X^t\,\overline{V}] = V^*\mathbb{E}[X\,X^t]\overline{V} = V^*S_0V^t = I.$$

But as X is an obtuse random variable, then so is Y and in particular we have $\mathbb{E}[\overline{Y}\,Y^t] = I$, also. This shows that the coordinates Y^i of Y satisfy

$$\mathbb{E}\left[(Y^i)^2\right] = \mathbb{E}\left[\left|Y^i\right|^2\right],$$

which means that Y^i is real-valued. □

Speaking of unitary transforms of obtuse random variables, we shall need later on to recall how is transformed the associated 3-tensor under a unitary map of the random variable. The following lemma is just an obvious application of the change of basis formula for 3-tensors, we do not prove it.

Lemma 3.13 *Let X and Y be two obtuse random variables on $\mathbb{C}^N$ such that there exist a unitary operator U on $\mathbb{C}^N$ satisfying $X = UY$. We extend $\mathbb{C}^N$ to $\mathbb{C}^{N+1}$ by adding some e_0 vector to the orthonormal basis; we extend U to a unitary operator on $\mathbb{C}^{N+1}$ by imposing $Ue_0 = e_0$. Let $(u_{ij})_{i,j=0,\dots,N}$ be the coefficients of U on $\mathbb{C}^{N+1}$.*

If S and T are the 3-tensors of X and Y respectively, we then have

$$S_k^{ij} = \sum_{m,n,p=0}^{N} u_{im}\, u_{jn}\, \overline{u_{kp}}\, T_p^{mn}, \tag{24}$$

for all $i, j, k = 0, \dots, N$.

Conversely, the tensor T can be deduced from the tensor S by

$$T_k^{ij} = \sum_{m,n,p=0}^{N} \overline{u_{mi}}\, \overline{u_{nj}}\, u_{pk}\, S_p^{mn}.$$

Definition 3.14 In the following if two 3-tensors S and T are connected by a formula of the form (24) we shall denote it by

$$S = U \circ T.$$

4 Complex normal martingales

The aim of next section is to give explicit results concerning the continuous-time limit of random walks made of sums of obtuse random variables. The continuous time limits will give rise to particular martingales on $\mathbb{C}^N$. In the real case, the limiting martingales are well-understood, they are the so-called *normal martingales* of $\mathbb{R}^N$ satisfying a *structure equation* (cf [4]). In the real case the stochastic behavior of these martingales is intimately related to a certain doubly symmetric 3-tensor and to its diagonalization. To make it short, the directions corresponding to the null eigenvalues of the limiting 3-tensor are those where the limit process behaves like a Brownian motion; the other directions (non-vanishing eigenvalues) correspond to a Poisson process behavior. This was developed in details in [4] and we shall recall their main results below.

In the next section of this article we wish to obtain two types of time-continuous results:

- a limit in distribution for the processes, for which we would like to rely on the results of [19] where is proved that the convergence of the 3-tensors associated to the discrete time obtuse random walks implies the convergence in law of the processes;

– a limit theorem for the multiplication operators, for which we would like to rely on the approximation procedure developed in [1], where is constructed an approximation of the Fock space by means of spin chains and where is proved the convergence of the basic operators $a^i_j(n)$ to the increments of quantum noises.

When considering the complex case we had two choices: either develop a complex theory of normal martingales and structure equations, extend all the results of [4], of [19] and of [1] to the complex case and prove the limit theorems we wished to obtain; or find a way to connect the complex obtuse random walks to the real ones and rely on the results of the real case, in order to derive the corresponding one for the complex case. We have chosen the second scenario, for we have indeed the same connection between the complex obtuse random variables and the complex ones as we have obtained in the discrete time case. In this section we shall present, complex normal martingales and their structure equations, the connection between the complex and the real case, together with their consequences. Only in next section we shall apply these results in order to derive the continuous-time limit theorems.

4.1 A reminder of normal martingales in $\mathbb{R}^N$

We now recall the main results of [4] concerning the behavior of normal martingales in $\mathbb{R}^N$ and their associated 3-tensor.

Definition 4.1 A martingale $X = (X^1, \ldots, X^N)$ with values in $\mathbb{R}^N$ is a *normal martingale* if $X_0 = 0$ a.s. and if its angle brackets satisfy

$$\langle X^i,\ X^j\rangle_t = \delta_{ij}\, t, \tag{25}$$

for all $t \in \mathbb{R}^+$ and all $i, j = 1, \ldots, N$.

This is equivalent to saying that the process $(X^i_t X^j_t - \delta_{ij}\, t)_{t\in\mathbb{R}^+}$ is a martingale, or else that the process $([X^i,\ X^j]_t - \delta_{ij}\, t)_{t\in\mathbb{R}^+}$ is a martingale (where $[\,\cdot,\ \cdot\,]$ here denotes the square bracket).

Definition 4.2 A normal martingale X in $\mathbb{R}^N$ is said to satisfy a *structure equation* if there exists a family $\{\Phi^{ij}_k\ ;\ i,\ j,\ k = 1, \ldots, N\}$ of predictable processes such that

$$[X^i,\ X^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} \int_0^t \Phi^{ij}_k(s)\, dX^k_s. \tag{26}$$

Note that if X has the predictable representation property (i.e. every local integrable martingale is a stochastic integral with respect to X), then X satisfies a structure equation, for (26) is just the integral representation of the square integrable martingale $([X^i,\ X^j]_t - \delta_{ij}\, t)_{t\in\mathbb{R}^+}$.

The following theorem is proved in [4]. It establishes the fundamental link between the 3-tensors $\Phi(s)$ associated to the martingale X and the behavior of X.

Theorem 4.3 *Let X be a normal martingale in $\mathbb{R}^N$ satisfying the structure equation*

$$[X^i,\ X^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} \int_0^t \Phi_k^{ij}(s)\, dX_s^k,$$

for all i, j. Then for almost all (s, ω) the quantities $\Phi(s, \omega)$ are valued in doubly symmetric 3-tensors in $\mathbb{R}^N$.

If one denotes by $\mathcal{V}_s(\omega)$ the orthogonal family associated to the non-vanishing eigenvalues of $\Phi(s, \omega)$ and by $\Pi_s(\omega)$ the orthogonal projector onto $\mathcal{V}_s(\omega)^\perp$, that is, on the null-egeinvalue subspace of $\Phi(s, \omega)$, then the continuous part of X is

$$X_t^c = \int_0^t \Pi_s(dX_s),$$

the jumps of X only happen at totally inaccessible times and they satisfy

$$\Delta X_t(\omega) \in \mathcal{V}_t(\omega).$$

The case we are concerned with is a simple case where the process Φ is actually constant. In that case, things can be made much more explicit, as is proved in [4] again.

Theorem 4.4 *Let Φ be a doubly-symmetric 3-tensor in $\mathbb{R}^N$, with associated orthogonal family $\mathcal{V}$. Let W be a Brownian motion with values in the space $\mathcal{V}^\perp$. For every $v \in \mathcal{V}$, let N^v be a Poisson process with intensity $\|v\|^{-2}$. We suppose W and all the N^v to be independent processes.*

Then the martingale

$$X_t = W_t + \sum_{v \in \mathcal{V}} \left(N_t^v - \frac{1}{\|v\|^2}\, t \right) v \tag{27}$$

satisfies the structure equation

$$[X^i,\ X^j]_t - \delta_{ij}\, t = \sum_{k=1}^{N} \Phi_k^{ij}\, X_t^k, \tag{28}$$

Conversely, any solution of (28) has the same law as X.

The martingale X solution of (28) possesses the chaotic representation property.

4.2 Normal martingales in $\mathbb{C}^N$

Definition 4.5 We consider a martingale X, defined on its canonical space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in $\mathbb{C}^N$, satisfying the following properties (similar to the corresponding definition in $\mathbb{R}^N$):

– the angle bracket $\langle \overline{X^i},\ X^j \rangle_t$ is equal to $\delta_{ij}\, t$,

– the martingale X has the *Predictable Representation Property*; by this we mean that every centered local martingale on the the canonical space of X can be represented as a stochastic integral with respect to X.

A martingale X in $\mathbb{C}^N$ satisfying these properties is called a *normal martingale in* $\mathbb{C}^N$ *satisfying the PRP*.

A direct consequence of the (complex) Kunita–Watanabe Inequality and of the first condition above, is that the processes $\langle X^i,\ X^j\rangle$ are also absolutely continuous with respect to the Lebesgue measure.

Applying all these conditions, we get that, for all $i, j, k = 1, \dots, N$, there exist predictable processes $(\Lambda_t^{ij})_{t\in\mathbb{R}^+}$, $(S_k^{ij}(t))_{t\in\mathbb{R}^+}$ and $(T_k^{ij}(t))_{t\in\mathbb{R}^+}$, such that

$$[X^i,\ X^j]_t = \int_0^t \Lambda_s^{ij}\, ds + \sum_{k=1}^N \int_0^t S_k^{ij}(s)\, dX_s^k, \tag{29}$$

$$[\overline{X^i},\ X^j]_t = \delta_{ij}\, t + \sum_{k=1}^N \int_0^t T_k^{ij}(s)\, dX_s^k. \tag{30}$$

Note that the family of coefficients $\Lambda_t = (\Lambda_t^{ij})_{i,j=1,\dots,N}$ defines a 2-tensor (a matrix), the family $S(t) = (S_k^{ij}(t))_{i,j,k=1,\dots,N}$ defines a 3-tensor, in the sense defined in the article, but that the family $T(t) = (T_k^{ij}(t))_{i,j,k=1,\dots,N}$ defines a 3-tensor of a slightly different nature, that is, it appears here as a natural element of $\mathcal{H}^* \otimes \mathcal{H} \otimes \mathcal{H}^*$, instead as of $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}^*$ with our definition. The process $(\Lambda_t)_{t\in\mathbb{R}^+}$ is locally time-integrable, the processes $(S(t))_{t\in\mathbb{R}^+}$ and $(T(t))_{t\in\mathbb{R}^+}$ are locally square integrable.

Let us recall here a well-known and useful result on uniqueness of stochastic integral representations.

Lemma 4.6 *Let X be a normal martingale. Let A and B^k, $k = 1, \dots, N$ be predictable processes on $(\Omega, \mathcal{F}, \mathbb{P})$ which are locally integrable with respect to dt and dX, respectively. If we have*

$$\int_0^t A_s\, ds + \sum_{k=1}^N \int_0^t B_s^k\, dX_s^k = 0 \tag{31}$$

for all $t \in \mathbb{R}^+$, then $A_t = B_t^k = 0$ almost surely, for almost all $t \in \mathbb{R}^+$, for all $k = 1, \dots, N$.

We now detail the symmetry properties of S, T and Λ, together with some intertwining relations between S and Λ.

Theorem 4.7 (1) *The processes $(S(t))_{t\in\mathbb{R}^+}$ and $(T(t))_{t\in\mathbb{R}^+}$ are connected by the relation*

$$T_k^{ij}(s) = \overline{S_j^{ik}(s)}, \tag{32}$$

almost surely, for a.a. $s \in \mathbb{R}^+$ and for all $i, j, k = 1, \dots, N$.

(2) *The process* $(S(t))_{t\in\mathbb{R}^+}$ *takes its values in the set of doubly-symmetric 3 tensors of* $\mathbb{C}^N$.
(3) *The process* $(\Lambda_t)_{t\in\mathbb{R}^+}$ *takes its values in the set of complex symmetric matrices.*
(4) *We have the relation*

$$\sum_{m=1}^{N} S_m^{ij}(s)\,\Lambda_s^{mk} = \sum_{m=1}^{N} S_m^{kj}(s)\,\Lambda_s^{mi}, \tag{33}$$

almost surely, for a.a. $s \in \mathbb{R}^+$ *and for all* $i, j, k = 1, \dots, N$.
(5) *We have the relation*

$$\sum_{m=1}^{N} \overline{S_j^{km}(s)}\,\Lambda_s^{im} = S_k^{ij}(s), \tag{34}$$

almost surely, for a.a. $s \in \mathbb{R}^+$ *and for all* $i, j, k = 1, \dots, N$.
(6) *The process* $(\overline{X_t^i})_{t\in\mathbb{R}^+}$ *has the predictable representation*

$$\overline{X_t^i} = \sum_{m=1}^{N} \int_0^t \overline{\Lambda_s^{im}}\, dX_s^m.$$

(7) *The matrices* Λ_t *are unitary for a.a.* $t \in \mathbb{R}^+$.

Proof The proof is a rather simple adaptation of the arguments used in Proposition 2.18 and Proposition 2.19. First of all, the symmetry $[X^i,\ X^j]_t = [X^j,\ X^i]_t$ gives

$$\int_0^t \Lambda_s^{ij}\, ds + \sum_{k=1}^{N} \int_0^t S_k^{ij}(s)\, dX_s^k = \int_0^t \Lambda_s^{ji}\, ds + \sum_{k=1}^{N} \int_0^t S_k^{ji}(s)\, dX_s^k$$

for all t. By the uniqueness Lemma 4.6 this gives the symmetry of the matrices Λ_s and the first symmetry relation (10) for the 3-tensors $S(s)$.

Computing $[[X^i,\ X^j],\ \overline{X^k}]_t$ we get

$$\begin{aligned}[[X^i,\ X^j],\ \overline{X^k}]_t &= \sum_{m=1}^{N} \int_0^t S_m^{ij}(s)\, d[X^m,\ \overline{X^k}]_s \\ &= \int_0^t S_k^{ij}(s)\, ds + \sum_{m,n=1}^{N} \int_0^t S_m^{ij}(s)\, T_n^{km}(s)\, dX_s^n.\end{aligned}$$

But this triple bracket is also equal to

$$[X^i,\,[X^j,\,\overline{X^k}]]_t = \sum_{m=1}^{N}\int_0^t \overline{T_m^{jk}(s)}\,d[X^i,\,\overline{X^m}]_s$$
$$= \int_0^t \overline{T_i^{jk}(s)}\,ds + \sum_{m,n=1}^{N}\int_0^t \overline{T_m^{jk}(s)}\,T_n^{mi}(s)\,dX_s^n.$$

Again, by the uniqueness lemma, and the symmetry (10), we get the relation (32).

Now, in the same way as in the proof of Proposition 2.19, we compute

$$[[X^i,\,\overline{X^j}],\,[X^k,\,X^l]]_t$$

in two ways, using the symmetry in (i,k) of that quadruple bracket:

$$[[X^i,\,\overline{X^j}],\,[X^k,\,X^l]]_t = \sum_{m,n=1}^{N}\int_0^t \overline{T_m^{ij}(s)}\,S_n^{kl}(s)\,d[\overline{X^m},\,X^n]_s,$$
$$[[X^k,\,\overline{X^j}],\,[X^i,\,X^l]]_t = \sum_{m,n=1}^{N}\int_0^t \overline{T_m^{kj}(s)}\,S_n^{il}(s)\,d[\overline{X^m},\,X^n]_s.$$

By uniqueness again, the time integral part gives the relation

$$\sum_{m=1}^{N}\overline{T_m^{ij}(s)}\,S_m^{kl}(s) = \sum_{m=1}^{N}\overline{T_m^{kj}(s)}\,S_m^{il}(s),$$

that is,

$$\sum_{m=1}^{N} S_j^{im}(s)\,S_m^{kl}(s) = \sum_{m=1}^{N} S_j^{km}(s)\,S_m^{il}(s),$$

which is the relation (11).

The relation (12) is obtained exactly in the same way, from the symmetry of

$$[[X^i,\,\overline{X^j}],\,[\overline{X^l},\,X^k]]_t$$

in (i,k). We have proved that the 3-tensors $S(s)$ are doubly-symmetric.

Computing $[X^i,\ [X^j,\ X^k]]_t$ in two different ways we get

$$\begin{aligned}[X^i,\ [X^j,\ X^k]]_t &= \sum_{m=1}^{N} \int_0^t S_m^{jk}(s)\, d[X^i,\ X^m]_s \\ &= \sum_{m=1}^{N} \int_0^t S_m^{jk}(s)\, \Lambda_s^{im}\, ds + \sum_{m,n=1}^{N} \int_0^t S_m^{jk}(s)\, S_n^{im}(s)\, dX_s^n,\end{aligned}$$

on one hand, and

$$\begin{aligned}[[X^i,\ X^j],\ X^k]_t &= \sum_{m=1}^{N} \int_0^t S_m^{ij}(s)\, d[X^m,\ X^k]_s \\ &= \sum_{m=1}^{N} \int_0^t S_m^{ij}(s)\, \Lambda_s^{mk}\, ds + \sum_{m,n=1}^{N} \int_0^t S_m^{ij}(s)\, S_n^{mk}(s)\, dX_s^n,\end{aligned}$$

on the other hand. Identifying the time integrals, we get the relation (33).

Finally, computing $[X^i,\ [X^j,\ \overline{X^k}]]_t$ in two different ways we get

$$\begin{aligned}[X^i,\ [X^j,\ \overline{X^k}]]_t &= \sum_{m=1}^{N} \int_0^t \overline{S_j^{km}(s)}\, d[X^i,\ X^m]_s \\ &= \sum_{m=1}^{N} \int_0^t \overline{S_j^{km}(s)}\, \Lambda_s^{im}\, ds + \sum_{m,n=1}^{N} \int_0^t S_k^{jm}(s)\, S_n^{im}(s)\, dX_s^n,\end{aligned}$$

on one hand, and

$$\begin{aligned}[[X^i,\ X^j],\ \overline{X^k}]_t &= \sum_{m=1}^{N} \int_0^t S_m^{ij}(s)\, d[X^m,\ \overline{X^k}]_s \\ &= \sum_{m=1}^{N} \int_0^t S_m^{ij}(s)\, \delta_{mk}\, ds + \sum_{m,n=1}^{N} \int_0^t S_m^{ij}(s)\, \overline{S_k^{mn}(s)}\, dX_s^n,\end{aligned}$$

on the other hand. Identifying the time integrals, we get the relation (34).

Let us prove the result (6). The process $(\overline{X_t^i})_{t\in\mathbb{R}^+}$ is obviously a square integrable martingale and its expectation is 0. Hence it admits a predictable representation of the form

$$\overline{X_t^i} = \sum_{k=1}^{N} \int_0^t H_s^{ik}\, dX_s^k$$

for some predictable processes H^{ik}. We write

$$\begin{aligned}[\overline{X^i},\ \overline{X^j}]_t &= \overline{[X^i,\ X^j]_t} \\ &= \int_0^t \overline{\Lambda_s^{ij}}\, ds + \sum_{k=1}^N \int_0^t \overline{S_k^{ij}(s)}\, d\overline{X_s^k} \\ &= \int_0^t \overline{\Lambda_s^{ij}}\, ds + \sum_{k,l=1}^N \int_0^t \overline{S_k^{ij}(s)}\, H_s^{kl}\, dX_s^l\end{aligned}$$

and

$$\begin{aligned}[\overline{X^i},\ \overline{X^j}]_t &= \sum_{k=1}^N \int_0^t H_s^{ik}\, d[X^k,\ \overline{X^j}]_s \\ &= \sum_{k=1}^N \int_0^t H_s^{ik}\, \delta_{kj}\, ds + \sum_{k,l=1}^N \int_0^t H_s^{ik}\, \overline{S_k^{jl}(s)}\, dX_s^l.\end{aligned}$$

The uniqueness lemma gives the relation $H_s^{ij} = \overline{\Lambda_s^{ij}}$, almost surely, for a.a. s.

We now prove (7). We have

$$\begin{aligned}[\overline{X^i},\ X^j]_t &= \sum_{k=1}^N \int_0^t H_s^{ik}\, d[X^k,\ X^j]_s \\ &= \sum_{k=1}^N \int_0^t H_s^{ik}\, \Lambda_s^{kj}\, ds + \sum_{k,l=1}^N \int_0^t H_s^{ik}\, S_l^{kj}(s)\, dX_s^l,\end{aligned}$$

but also

$$[\overline{X^i},\ X^j]_t = \delta_{ij}\, t + \sum_{k=1}^N \int_0^t \overline{S_j^{ik}(s)}\, dX_s^k.$$

Again, by the uniqueness lemma we get

$$\delta_{ij} = \sum_{k=1}^N H_s^{ik}\, \Lambda_s^{kj} = \sum_{k=1}^N \overline{\Lambda_s^{ik}}\, \Lambda_s^{kj}.$$

This proves the announced unitarity. □

In the same way as for obtuse random variables, complex normal martingales can be brought back to real ones, with the help of a unitary transform which is hidden in the structure equation.

Theorem 4.8 *Let Y be a normal martingale in $\mathbb{C}^N$ satisfying the structure equations*

$$[Y^i,\ Y^j]_t = \int_0^t \Lambda_s^{ij}\, ds + \sum_{k=1}^N \int_0^t S_k^{ij}(s)\, dX_s^k$$

$$[\overline{Y^i},\ Y^j]_t = \delta_{ij}\, t + \sum_{k=1}^N \int_0^t \overline{S_j^{ik}(s)}\, dX_s^k.$$

Then the matrices Λ_s admit a decomposition of the form

$$\Lambda_s = V_s\, V_s^t$$

for some predictable process $(V_s)_{s\in\mathbb{R}^+}$, valued in the set of unitary matrices of $\mathbb{C}^N$.

The process

$$Z_t = \int_0^t V_s^*\, dY_s,$$

$t \in \mathbb{R}^+$, is then a real normal martingale.

Proof By the Theorem above, almost all the Λ_s are symmetric and unitary matrices. The decomposition of each Λ_s into a product $V_s V_s^t$ is then an immediate consequence of Proposition 3.11. The Borel character of the map $S \mapsto V$ in Proposition 3.11 gives the predictable character of the process $(V_s)_{s\in\mathbb{R}^+}$.

Consider the stochastic integral

$$Z_t = \int_0^t V_s^*\, dY_s,$$

which is obviously well-defined. Computing the angle bracket of Z we get

$$\begin{aligned} <Z, Z^t>_t &= \int_0^t V_s^*\, d<Y, Y^t>_s\, \overline{V_s} = \int_0^t V_s^*\, \Lambda_s\, ds\, \overline{V_s} \\ &= \int_0^t I\, ds = tI = <\overline{Z}, Z^t>_t. \end{aligned}$$

Taking expectation of both sides shows that Z is real-valued. □

4.3 Complex unitary transforms of real normal martingales

From the result above concerning real normal martingales, we shall deduce easily the corresponding behavior of complex normal martingales, as they are obtained by unitary transforms of real normal martingales.

In the following we shall be interested in the following objects. Let Y be a normal martingale on $\mathbb{R}^N$, satisfying a structure equation with constant 3-tensor T. Let U be a

unitary operator on $\mathbb{C}^N$. Injecting canonically $\mathbb{R}^N$ into $\mathbb{C}^N$, we consider the complex martingale $X_t = U Y_t$, $t \in \mathbb{R}^+$. We consider the 3-tensor

$$S = U \circ T.$$

We also put

$$\Lambda = U\, U^t.$$

We choose the following notations. Let $\mathcal{W} = \{w_1, \ldots, w_k\} \subset \mathbb{R}^N$ be the orthogonal system associated to T, that is, the directions of non-vanishing eigenvalues of T. Let $\mathcal{W}^\perp$ be its orthogonal space in $\mathbb{R}^N$, that is the null space of T, and let us choose an orthonormal basis $\{\widehat{w}_1, \ldots, \widehat{w}_{N-k}\}$ of $\mathcal{W}^\perp$. Let $\mathcal{V} = U\mathcal{W} = \{U w_1, \ldots, U w_k\} \subset \mathbb{C}^N$, this set coincides with the orthogonal system associated to S that is, the directions of non-vanishing eigenvalues of S. We consider the set $\widehat{\mathcal{V}} = \{\widehat{v}_1, \ldots, \widehat{v}_{N-k}\}$, where $\widehat{v_i} = U\widehat{w_i}$, for all $i = 1, \ldots, N-k$. We denote by $\mathcal{V}_\mathbb{R}$ and $\widehat{\mathcal{V}}_\mathbb{R}$ the following real subspaces of $\mathbb{C}^N$ seen as a $2N$-dimensional real vector space:

$$\begin{aligned}\mathcal{V}_\mathbb{R} &= \mathbb{R}\, v_1 \oplus \cdots \oplus \mathbb{R}\, v_k\\ \widehat{\mathcal{V}}_\mathbb{R} &= \mathbb{R}\widehat{v}_1 \oplus \cdots \oplus \mathbb{R}\widehat{v}_{N-k}.\end{aligned}$$

In particular note that $\mathcal{V}_\mathbb{R} \oplus \widehat{\mathcal{V}}_\mathbb{R} = U\,\mathbb{R}^N$ is a N-dimensional real subspace of $\mathbb{C}^N$.

Finally we denote by Π_S the orthogonal projector from $\mathbb{C}^N$ onto $\widehat{\mathcal{V}}_\mathbb{R}$ where both spaces are seen as real vector spaces.

Theorem 4.9 *With the notations above, the complex martingale X satisfies the following two "structure equations"*

$$[X^i, X^j]_t = \Lambda_{ij}\, t + \sum_{k=1}^N S_k^{ij}\, X_t^k, \tag{35}$$

$$[\overline{X^i}, X^j]_t = \delta_{ij}\, t + \sum_{k=1}^N \overline{S_j^{ik}}\, X_t^k. \tag{36}$$

The solutions to both Eqs. (35) and (36) are unique in distribution. This distribution is described as follows. The process X is valued in $U\,\mathbb{R}^N = \mathcal{V}_\mathbb{R} \oplus \widehat{\mathcal{V}}_\mathbb{R}$. The continuous part of X is

$$X_t^c = \int_0^t \Pi_S(dX_s),$$

which lives in $\widehat{\mathcal{V}}_\mathbb{R}$. The jumps of X only happen at totally inaccessible times and they satisfy, almost surely

$$\Delta X_t(\omega) \in \mathcal{V}_\mathbb{R}.$$

In other words, let W be a $N-k$ dimensional Brownian motion with values in the real space $\widehat{\mathcal{V}}_{\mathbb{R}}$. For every $v \in \mathcal{V}$, let N^v be a Poisson process with intensity $\|v\|^{-2}$. We suppose W and all the N^v to be independent processes. Then the martingale

$$X_t = W_t + \sum_{v \in \mathcal{V}} \left(N_t^v - \frac{1}{\|v\|^2}\, t \right) v \tag{37}$$

satisfies the structure Eqs. (35) *and* (36).

The process X possesses the chaotic representation property.

Proof The martingale Y satisfies the structure equation

$$[Y^i,\ Y^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} T_k^{ij}\, Y_t^k.$$

The process $X_t = U\, Y_t$, $t \in \mathbb{R}^+$, is a martingale with values in $\mathbb{C}^N$. By a standard change of coordinate computation, we have

$$[X^i,\ X^j]_t = S_0^{ij}\, t + \sum_{k=1}^{N} S_k^{ij}\, X_t^k.$$

This gives (35).

With a similar computation, we get

$$[\overline{X^i},\ X^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} \overline{S_j^{ik}}\, X_t^k.$$

This gives (36).

Let us now prove uniqueness in law for the solutions of (35) and (36). Let Z be another solution of the two equations. Consider the unitary operator U associated to the 3-tensor S, that is, for which the 3-tensor $T = U^* \circ S$ is real and doubly-symmetric. Put $A_t = U_t^* Z_t$ for all $t \in \mathbb{R}^p$. Then we get

$$[A^i,\ A^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} R_k^{ij}\, A_t^k$$

and

$$[\overline{A^i},\ A^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} \overline{R_j^{ik}}\, A_t^k.$$

But as R is a real-valued 3-tensor, symmetric in i, j, k, the last expression gives

$$[\overline{A^i},\ A^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} R_k^{ij}\, A_t^k.$$

Decomposing each A_t^j as $B_t^j + i C_t^j$ (real and imaginary parts), the last two relations ought to

$$\begin{aligned}
&[B^i,\ B^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} R_k^{ij}\, B_t^k,\\
&[C^i,\ C^j]_t = 0,\\
&[B^i,\ C^j]_t = \sum_{k=1}^{N} R_k^{ij}\, C_t^k.
\end{aligned}$$

This clearly implies that the C^j's vanish, the processes A^j are real-valued. They satisfy the same structure equation as the real process Y underlying the definition of X. By Theorem 4.4 the processes A and Y have same law. Thus so do the processes $Z = UA$ and $X = UY$. This proves the uniqueness in law for the processes in $\mathbb{C}^N$ satisfying the two Eqs. (35) and (36).

The projector Π_S onto $\widehat{\mathcal{V}}_{\mathbb{R}}$ is given by $\Pi_S = U\, \Pi_T\, U^*$, as can be checked easily, even though we are here considering the vector spaces as real ones. The continuous part of X and the jumps of X are clearly the ones of Y but mapped by U. Hence, altogether we get

$$X_t^c = UY_t^c = \int_0^t U\, \Pi_T(dY_s) = \int_0^t U\, U^* \Pi_S\, U(U^* dX_s) = \int_0^t \Pi_S(dX_s).$$

The part (37) of the theorem is obvious, again by application of the map U.

Finally, let us prove the chaotic representation property. The chaotic representation property for Y says that every random variable $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, the canonical space of Y, can be decomposed as

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{N} \int_{0\le t_1<\ldots<t_n} f_{i_1,\ldots,i_n}(t_1,\ldots,t_n)\, dY_{t_1}^{i_1} \ldots dY_{t_n}^{i_n},$$

for some deterministic functions $f_{i_1,\ldots,i_n}$'s. But decomposing each Y_t^j as $\sum_{m=1}^N \overline{u_{mj}}\, X_t^m$ shows clearly that F can also be decomposed as

$$F = \mathbb{E}[F] + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{N} \int_{0\le t_1<\ldots<t_n} g_{i_1,\ldots,i_n}(t_1,\ldots,t_n)\, dX_{t_1}^{i_1} \ldots dX_{t_n}^{i_n},$$

where the $g_{i_1,\ldots,i_n}$'s are linear combinations of the $f_{i_1,\ldots,i_n}$'s. This proves the chaotic representation property for X and the theorem is completely proved. □

5 Continuous-time limit of complex obtuse random walks

We are now ready to consider the convergence theorem for complexe obtuse random walks.

5.1 Convergence of the tensors

We are now given a time parameter $h > 0$ which is meant to tend to 0 later on. This time parameter is the time step of the obtuse random walk we want to study, but note that h may also appear in the internal parameters of the walk, that is, in the probabilities p_i and the values v_i of X.

Hence, we are given an obtuse random variable $X(h)$ in $\mathbb{C}^N$, with coordinates $X^i(h)$, $i = 1, \dots, N$ and together with the random variable $X^0 = \mathbb{1}$. The associated 3-tensor of $X(h)$ is given by

$$X^i(h)\, X^j(h) = \sum_{k=0}^{N} S_k^{ij}(h)\, X^k(h).$$

Considering the random walk associated to $X(h)$, that is, consider a sequence $(X_n(h))_{n\in\mathbb{N}^*}$ of i.i.d. random variables in $\mathbb{C}^N$ all having the same distribution as $X(h)$, the random walk is the stochastic process with time step h:

$$Z_{nh}^h = \sum_{i=1}^{n} \sqrt{h}\, X_n(h).$$

This calls for defining

$$\widehat{X}^0 = h\, X^0 \qquad \text{and} \qquad \widehat{X}^j(h) = \sqrt{h}\, X^j(h)$$

for all $j = 1, \dots, N$. Putting $\varepsilon_0 = 1$ and $\varepsilon_i = 1/2$ for all $i = 1, \dots, N$, we then have, for all $i, j = 0, \dots, N$

$$\widehat{X}^i(h)\, \widehat{X}^j(h) = \sum_{k=0}^{N} \widehat{S}_k^{ij}(h)\, \widehat{X}^k(h),$$

where

$$\widehat{S}_k^{ij}(h) = h^{\varepsilon_i+\varepsilon_j-\varepsilon_k}\, S_k^{ij}.$$

Finally, we put

$$M_k^{ij} = \lim_{h\to 0} \widehat{S}_k^{ij}(h),$$

if it exists.

Lemma 5.1 *We then get, for all $i, j, k = 1, \dots N$*

$$\begin{aligned}
M_0^{00} &= 0,\\
M_k^{00} &= 0,\\
M_0^{0k} &= 0,\\
M_0^{k0} &= 0,\\
M_0^{ij} &= \lim_{h\to 0} S_0^{ij}(h) \ \ \textit{(if it exists)},\\
M_k^{i0} &= 0,\\
M_k^{0j} &= 0,\\
M_k^{ij} &= \lim_{h\to 0} h^{1/2}\, S_k^{ij}(h) \ \ \textit{(if it exists)}.
\end{aligned}$$

Proof These are direct applications of the definitions and the symmetries verified by the $S_k^{ij}(h)$'s. For example:

$$\widehat{S}_k^{0j}(h) = h\, S_k^{0j}(h) = h\, S_k^{j0} = h\, \delta_{jk}.$$

This gives immediately that $M_k^{0j} = 0$. And so on for all the other cases. □

Proposition 5.2 *Under the hypothesis that the limits above all exist, the 3-tensor M, restricted to its coordinates $i, j, k = 1, \dots, N$, is a doubly-symmetric 3-tensor of $\mathbb{C}^N$.*

Proof Let us check that $(M_k^{ij})_{i,j,k=1,\dots,N}$ satifies the three conditions for being a doubly-symmetric 3-tensor. Recall that for these indices, we have

$$M_k^{ij} = \lim_{h\to 0} h^{1/2}\, S_k^{ij}(h).$$

The first condition $M_k^{ij} = M_i^{kj}$ is obvious from the same property of $S_k^{ij}(h)$ and passing to the limit.

We wish now to prove that $\sum_{m=1}^N M_j^{im}\, M_m^{kl}$ is symmetric in (i, k). The corresponding property for $S(h)$ gives

$$S_j^{i0}(h)\, S_0^{kl}(h) + \sum_{m=1}^N S_j^{im}(h)\, S_m^{kl}(h) = S_j^{k0}(h)\, S_0^{il}(h) + \sum_{m=1}^N S_j^{km}(h)\, S_m^{il}(h).$$

In particular, multiplying by h, we get

$$h\, \delta_{ij}\, S_0^{kl}(h) + \sum_{m=1}^N \widehat{S}_j^{im}(h)\, \widehat{S}_m^{kl}(h) = h\, \delta_{kj}\, S_0^{il}(h) + \sum_{m=1}^N \widehat{S}_j^{km}(h)\, \widehat{S}_m^{il}(h).$$

By hypothesis $\lim_{h\to 0} S_0^{kl}(h)$ and $\lim_{h\to 0} S_0^{il}(h)$ exist hence, passing to the limit, we get

$$\sum_{m=1}^{N} M_j^{im}\, M_m^{kl} = \sum_{m=1}^{N} M_j^{km}\, M_m^{il},$$

which is the second symmetry asked to M for being doubly-symmetric.

The third symmetry is obtained in a similar way. Indeed, we have

$$S_j^{i0}(h)\,\overline{S_k^{l0}}(h) + \sum_{m=1}^{N} S_j^{im}(h)\,\overline{S_k^{lm}}(h) = S_j^{k0}(h)\,\overline{S_i^{l0}}(h) + \sum_{m=1}^{N} S_j^{km}(h)\,\overline{S_i^{lm}}(h).$$

This gives, multiplying by h again

$$h\,\delta_{ij}\,\delta_{lk} + \sum_{m=1}^{N} \widehat{S}_j^{im}(h)\,\overline{\widehat{S}_k^{lm}}(h) = h\,\delta_{kj}$$
$$d_{li} + \sum_{m=1}^{N} \widehat{S}_j^{km}(h)\,\overline{\widehat{S}_i^{lm}}(h).$$

Now, passing to the limit as h tends to 0, we get

$$\sum_{m=1}^{N} M_j^{im}\,\overline{M_k^{lm}} = \sum_{m=1}^{N} M_j^{km}\,\overline{M_i^{lm}}.$$

This gives the last required symmetry. □

5.2 Convergence in distribution

We can now prove a convergence in distribution theorem.

Theorem 5.3 *Let $X(h)$ be an obtuse random variable on $\mathbb{C}^N$, depending on a parameter $h > 0$, let $S(h)$ be its associated doubly symmetric 3-tensor. Let $(X_n(h))_{n\in\mathbb{N}^*}$ be a sequence of i.i.d. random variables with same law as $X(h)$. Consider the discrete-time random walk*

$$Z_{nh}^h = \sum_{i=1}^{n} \sqrt{h}\, X_n(h).$$

If the limits

$$M_0^{ij} = \lim_{h\to 0} S_0^{ij}(h)$$

and

$$M_k^{ij} = \lim_{h\to 0} \sqrt{h}\, S_k^{ij}(h)$$

exist for all $i, j, k = 1, \ldots, N$*, then the process* Z^h *converges in distribution to the normal martingale* Z *in* $\mathbb{C}^N$ *solution of the structure equations*

$$[Z^i, Z^j]_t = M_0^{ij}\, t + \sum_{k=1}^{N} M_k^{ij}\, Z_t^k, \tag{38}$$

$$[\overline{Z^i}, Z^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} \overline{M_j^{ik}}\, Z_t^k. \tag{39}$$

Proof For each $h > 0$, the random variables $X(h)$ can be written $U(h)\,Y(h)$ for a unitary operator $U(h)$ on $\mathbb{C}^N$ and a real obtuse random variable $Y(h)$ in $\mathbb{R}^N$. In terms of the associated 3-tensors, recall that this means

$$S(h) = U(h) \circ T(h)$$

or else

$$T(h) = U(h)^* \circ S(h).$$

Consider any sequence $(h_n)_{n\in\mathbb{N}}$ which tends to 0. By hypothesis $S(h_n)$ converges to M. Furthermore, as the sequence $(U(h_n))_{n\in\mathbb{N}}$ lives in the compact group $\mathcal{U}(\mathbb{C}^N)$ it admits a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ converging to some unitary V. As a consequence the sequence $(T(h_{n_k}))_{k\in\mathbb{N}}$ converges to a real 3-tensor $N = V \circ M$.

The convergence of the 3-tensors $(T(h_{n_k}))_{k\in\mathbb{N}}$ to N imply the convergence in distribution of the associated real martingales $Y_{h_{n_{k_i}}}$, for a subsequence $(h_{n_{k_i}})_{i\in\mathbb{N}}$, by Taviot's Thesis (cf [19, Proposition 4.2.3, Proposition 4.3.2., Proposition 4.3.3]). The limit is a real normal martingale Y whose associated 3-tensor is N.

Applying the unitary operators $U_{h_{n_{k_i}}}$, which converge to V, we have the convergence in law of the process $Z^{h_{n_{k_i}}}$ to the process $Z = VY$. By Theorem 4.9 the process Z is solution of the complex structure equations associated to the tensor S.

For the moment we have proved that for every sequence $(h_n)_{n\in\mathbb{N}}$ there exists a subsequence $(h_{n_j})_{j\in\mathbb{N}}$ such that $Z^{h_{n_j}}$ converges in law to Z solution of the complex structure equations associated to S. As we have proved that the solutions to these equation are unique in law, the limit in law is unique. Hence the convergence is true not only for subsequences, but more generally for h tending to 0. The convergence in law is proved. □

5.3 Convergence of the multiplication operators

Let us first recall very shortly the main elements of the construction and approximation developed in [1], which will now serve us in order to prove the convergence of the

multiplication operators. This convergence of multiplication operators is not so usual in a probabilistic framework, but it is the one interesting in the framework of applications in Quantum Statistical Mechanics, for it shows the convergence of the quantum dynamics of repeated interactions towards a classical Langevin equation, when the unitary interaction is classical (cf [2]).

In Sect. 2.5 we have seen the canonical isomorphism of the canonical space $L^2(\Omega, \mathcal{F}, \mathbb{P}_S)$ of any obtuse random variable X, with the space $\mathbb{C}^{N+1}$. Recall the basic operators a^i_j that were defined there.

When dealing with i.i.d. sequences $(X_n)_{n\in\mathbb{N}}$ of copies of X, the canonical space is then isomorphic to the countable tensor product

$$T\Phi = \bigotimes_{n\in\mathbb{N}} \mathbb{C}^{N+1}.$$

When dealing with the associated random walk with time step h

$$Z^h_{nh} = \sum_{i=1}^{n} \sqrt{h}\, X_{nh}$$

the canonical space is naturally isomorphic to

$$T\Phi(h) = \bigotimes_{n\in h\mathbb{N}} \mathbb{C}^{N+1}.$$

There, natural ampliations of the basic operators a^i_j are defined: the operators $a^i_j(nh)$ is the acting as a^i_j of the copy nh of $\mathbb{C}^{N+1}$ and as the identity of the other copies.

On the other hand, when given a normal martingale A in $\mathbb{R}^N$ with the chaotic representation property, its canonical space $L^2(\Omega', \mathcal{F}', \mathbb{P}')$ is well-known to be naturally isomorphic to the symmetric Fock space

$$\Phi = \Gamma_s(L^2(\mathbb{R}^+;\ \mathbb{C}^N)),$$

via a unitary isomorphism denoted by U_A. This space is the natural space for the quantum noises $a^i_j(t)$, made of the time operator $a^0_0(t) = tI$, the creation noises $a^0_i(t)$, the annihilation noises $a^i_0(t)$ and the exchange processes $a^i_j(t)$, with $i, j = 1, \dots N$ (cf [14]).

The main constructions and results developed in [1] are the following:

- each of the spaces $T\Phi(h)$ can be naturally seen as concrete subspace of Φ;
- when h tends to 0 the subspace $T\Phi(h)$ fills in the whole space Φ, that is, concretely, the orthogonal projector P_h onto $T\Phi(h)$ converges strongly to the identity I;
- the basic operators $a^i_j(nh)$, now concretely acting on Φ, converge to the quantum noises, that is, more concretely the operator

$$\sum_{n;\ nh\le t} h^{\varepsilon^i_j} a^i_j(nh)$$

converges strongly to $a^i_j(t)$ on a certain domain $\mathcal{D}$ (which we shall not make explicit here, please cf [1]), where

$$e^i_j = \begin{cases} 1 & \text{if } i = j = 0, \\ 1/2 & \text{if } i = 0,\ j \neq 0 \text{ or } i \neq 0,\ j = 0, \\ 0 & \text{if } i,\ j \neq 0. \end{cases}$$

Finally recall the representation of the multiplication operators for real-valued normal martingales in $\mathbb{R}^N$ (cf [6]).

Theorem 5.4 *If A is a normal martingale in $\mathbb{R}^N$ with the chaotic representation property and satisfying the structure equation*

$$[A^i,\ A^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} N^{ij}_k\, A^k_t,$$

then its multiplication operator acting of $\Phi = \Gamma_s(L^2(\mathbb{R}^+,\ \mathbb{C}^N))$ is equal to

$$U_A\, \mathcal{M}_{A^i_t}\, U^*_A = a^0_i(t) + a^0_i(t) + \sum_{j,k=1}^{N} N^{ij}_k\, a^j_k(t)$$

or else

$$U_A\, \mathcal{M}_{A^i_t}\, U^*_A = \sum_{j,k=0}^{N} N^{ij}_k\, a^j_k(t)$$

if one extends the coefficients N^{ij}_k to the 0 index, by putting $N^{ij}_0 = \delta_{ij}$.

Once this is recalled, the rest is now rather easy. We can prove the convergence theorem for the multiplication operators.

Theorem 5.5 *The operators of multiplication $\mathcal{M}_{(Z^h_t)^i}$, acting of Φ, converge strongly on $\mathcal{D}$ to the operators*

$$\mathcal{Z}^i_t = \sum_{j,k=0}^{n} M^{ij}_k\, a^j_k(t). \tag{40}$$

These operators are the operators of multiplication by Z, the complex martingale satisfying

$$[Z^i, Z^j]_t = M^{ij}_0\, t + \sum_{k=1}^{N} M^{ij}_k\, Z^k_t \tag{41}$$

and

$$[\overline{Z^i}, Z^j]_t = \delta_{ij}\, t + \sum_{k=1}^{N} \overline{M^{ik}_l}\, Z^k_t. \tag{42}$$

Proof The convergence toward the operator $\mathcal{Z}_t$ given by (40) is a simple application of the convergence theorems of [1], let us detail the different cases.

If $j, k \neq 0$, we know that $\sqrt{h}\, S_k^{ij}$ converges to M_k^{ij} and by [1] we have that $\sum_{m=1}^{[t/h]} a_k^j(m)$ converges to $a_k^j(t)$.

If $j = 0$ and $k \neq 0$, we know that S_0^{ij} converges to M_0^{ij} and that $\sum_{m=1}^{[t/h]} \sqrt{h}\, a_0^j(m)$ converges to $a_0^j(t)$.

If $k = 0$ and $j \neq 0$, we know that S_k^{i0} converges to M_k^{i0} (actually their are all equal to δ_{ik}) and that $\sum_{m=1}^{[t/h]} \sqrt{h}\, a_k^0(m)$ converges to $a_k^0(t)$.

The fact that $\mathcal{Z}_t$ is indeed the multiplication operator by the announced normal martingale comes as follows. The martingale Z is the image UA, under a unitary operator U of some real normal martingale A. The 3-tensor M is the image $U \circ N$, under the unitary operator U, of some real tensor N. The real normal martingale A associated to the real 3-tensor N has its multiplication operator equal to

$$U_A\, \mathcal{M}_{A_t^i}\, U_A^* = \sum_{j,k=0}^{N} N_k^{ij}\, a_k^j(t)$$

by Theorem 5.4. As Z_t is equal to UA_t, its canonical space is the same as the one of A, only the canonical isomorphism is modified by a change of basis. The rest of the proof follows now easily. □

6 Examples

We shall detail 2 examples in dimension 2, showing up typical different behaviors.

The first one is the one we have followed along this article, let us recall it. We are given an obtuse random variable X in $\mathbb{C}^2$ taking the values

$$v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1+i \end{pmatrix}, \qquad v_3 = -\frac{1}{5}\begin{pmatrix} 3+4i \\ 1+3i \end{pmatrix},$$

with probabilities $p_1 = 1/3$, $p_2 = 1/4$ and $p_3 = 5/12$ respectively. Then the 3-tensor S associated to X is given by

$$S^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad S^1 = \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{5}(1-2i) & 0 & -\frac{2}{5}(2+i) \\ -\frac{2}{5}(1-2i) & 0 & \frac{1}{5}(2+i) \end{pmatrix},$$

$$S^2 = \begin{pmatrix} 0 & 0 & 1 \\ -\frac{2}{5}(1-2i) & 0 & \frac{1}{5}(2+i) \\ \frac{1}{5}(1-2i) & -i & -\frac{1}{5}(1-2i) \end{pmatrix}.$$

Now, considering the random walk

$$Z^h_{nh} = \sum_{i=1}^{n} \sqrt{h}\, X_n,$$

where $(X_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. random variables with same law as X, the continuous-time limit of Z^h is the normal martingale in $\mathbb{C}^2$ with associated tensor given by the limits of Lemma 5.1. Here we obtain, for all $i, j, k = 1, 2$

$$M^{ij}_k = 0$$

and

$$M_0 = \begin{pmatrix} -\frac{1}{5}(1-2i) & -\frac{2}{5}(1-2i) \\ -\frac{2}{5}(1-2i) & \frac{1}{5}(1-2i) \end{pmatrix}.$$

The limit process $(Z_t)_{t\in\mathbb{R}^+}$ is a normal martingale in $\mathbb{C}^2$, solution of the structure equations

$$[Z^i,\ Z^j]_t = M^{ij}_0\, t,$$
$$[\overline{Z^i},\ Z^j]_t = \delta_{ij}\, t.$$

It is then rather easy to find a unitary matrix V such that $V\,V^t = M_0$, we find

$$V = \begin{pmatrix} \frac{2+i}{\sqrt{10}} & \frac{i}{\sqrt{2}} \\ \frac{-1+2i}{\sqrt{10}} & \frac{1}{\sqrt{2}} \end{pmatrix},$$

for example. Following our results on complex normal martingales, this means that the process Z has the following distribution: given a 2-dimensional real Brownian motion $W = (W^1,\ W^2)$ then

$$\begin{cases} Z^1_t &= \frac{2+i}{\sqrt{10}}\, W^1_t + \frac{i}{\sqrt{2}}\, W^2_t, \\ Z^2_t &= \frac{-1+2i}{\sqrt{10}}\, W^1_t + \frac{1}{\sqrt{2}}\, W^2_t. \end{cases}$$

For the second example, we consider a fixed parameter $h > 0$. We consider the obtuse random variable $X(h)$ in $\mathbb{C}^2$ whose values are

$$v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}, \qquad v_2 = \frac{1}{\sqrt{2h}}\begin{pmatrix} 1 - i\sqrt{h} \\ i - \sqrt{h} \end{pmatrix}, \qquad v_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} -2\sqrt{h} - i \\ -1 - 2i\sqrt{h} \end{pmatrix},$$

with probabilities $p_1 = 1/2$, $p_2 = h/(1+2h)$ and $p_3 = 1/(2+4h)$ respectively. Then the 3-tensor S associated to X is given by the following, where we have only detailed the leading orders in h

$$S^0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad S^1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2\sqrt{2h}} + O(1) & \frac{-i}{2\sqrt{2h}} + O(1) \\ i & \frac{i}{2\sqrt{2h}} + O(1) & \frac{1}{2\sqrt{2h}} + O(1) \end{pmatrix},$$

$$S^2 = \begin{pmatrix} 0 & 1 & 0 \\ i & \frac{i}{2\sqrt{2h}} + O(1) & \frac{1}{2\sqrt{2h}} + O(1) \\ 0 & \frac{-1}{2\sqrt{2h}} + O(1) & \frac{i}{2\sqrt{2h}} + O(1) \end{pmatrix}.$$

The renormalized 3-tensor converges to the 3-tensor

$$M^1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix},$$
$$M^2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix}$$

and the matrix

$$M^0 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

In order to diagonalize the 3-tensor, we solve

$$(M_1^{ij} x + M_2^{ij} y)_{i,j=1,2} = \begin{pmatrix} x \\ y \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}.$$

There is a unique solution

$$v = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

This means that the continuous-time limit process Z is a compensated Poisson process in the direction v.

This is all for the information which is given by the 3-tensor. If we want to know the direction where the process is Brownian, we need to look at the decomposition of M^0 as $V\,V^t$ for a unitary V. We easily find

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ 1 & i \end{pmatrix}.$$

This unitary operator is the one from which has been rotated a real Brownian motion in order to land in the orthogonal space of $\begin{pmatrix} 1 \\ i \end{pmatrix}$. That is we seek for a direction w in

$\mathbb{R}^2$ such that $V\,w$ is proportional to $\begin{pmatrix} i \\ 1 \end{pmatrix}$. We find $w = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the process Z is a Brownian motion in the direction $\begin{pmatrix} i \\ 1 \end{pmatrix}$.

The process Z is finally described as follows, let N and W be a standard Poisson process and a Brownian motion, respectively, independant of each other. Then

$$\begin{cases} Z_t^1 &= \frac{1}{\sqrt{2}}\,(N_t - t) + i\,W_t \\ Z_t^2 &= \frac{i}{\sqrt{2}}\,(N_t - t) + \;W_t. \end{cases}$$

By choosing an example with two directions whose probabilities are of order h and one direction's probability is of order $1 - 2h$, we shall end up with a 3-tensor M that can be completely diagonalized and a process which is made of two compensated Poisson processes on two orthogonal directions of $\mathbb{C}^2$ (cf the example at the end of [6] for an example in $\mathbb{R}^2$).

Acknowledgments The authors are grateful to the referee for his very careful reading of several versions of this article. His comments helped us to clarify and to shorten many parts of this article

References

1. Attal, S.: Approximating the Fock space with the toy Fock space, Séminaire de Probabilités XXXVI, Springer L.N.M. 1801, p. 477–497 (2003)
2. Attal, S., Deschamps, J., Pellegrini, C.: Classical noises emerging from quantum environments. (preprint)
3. Attal, S., Dhahri, A.: Repeated quantum interactions and unitary random walks. J. Theor. Probab. **23**, 345–361 (2010)
4. Attal, S., Emery, M.: Equations de structure pour des martingales vectorielles. Séminaire de Probabilités, XXVIII, p. 256278. Lecture Notes in Math., vol. 1583. Springer, Berlin (1994)
5. Attal, S., Pautrat, Y.: From repeated to continuous quantum interactions. Ann. Henri Poincaré A J. Theor. Math. Phys. **7**, 59–104 (2006)
6. Attal, S., Pautrat, Y.: From (n+1)-level atom chains to n-dimensional noises. Ann. Inst. H. Poincaré Probab. Stat. **41**(3), 391407 (2005)
7. Bruneau, L., Pillet, C.-A.: Thermal relaxation of a QED cavity. J. Stat. Phys. **134**(5–6), 10711095 (2009)
8. Bruneau, L., De Bièvre, S., Pillet, C.-A.: Scattering induced current in a tight-binding band. J. Math. Phys. **52**(2), 19 (2011). 022109
9. Bauer, B., Benoist, T., Bernard, D.: Iterated stochastic measurements. J. Phys. A Math. Theor. **45**, 494020 (2012)
10. Haroche, S., Gleyzes, S., Kuhr, S., Guerlin, C., Bernu, J., Delglise, S., Busk-Hoff, U., Brune, M., Raimond, J.-M.: Quantum jumps of light recording the birth and death of a photon in a cavity. Nature **446**, 297 (2007)
11. Haroche, S., Sayrin, C., Dotsenko, I., Zhou, X.X., Peaudecerf, B., Rybarczyk, T., Gleyzes, S., Rouchon, P., Mirrahimi, M., Amini, H., Brune, M., Raimond, J.-M.: Real-time quantum feedback prepares and stabilizes photon number states. Nature **477**, 73 (2011)
12. Hong, Y.P., Horn, R.A.: On simultaneous reduction of families of matrices to triangular or diagonal form by unitary congruence. Linear Multilinear Algebra **17**, 271–288 (1985)
13. Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formula and stochastic evolutions. Commun. Math. Phys. **93**, 301–323 (1984)
14. Meyer, P.-A.: Quantum Probability for Probabilists. Lecture Notes in Mathematics, vol. 1538. Springer, Berlin (1993)

15. Pellegrini, C.: Existence, uniqueness and approximation of a stochastic Schrödinger equation: the diffusive case. Ann. Probab. **36**(6), 2332–2353 (2008)
16. Pellegrini, C.: Existence, uniqueness and approximation of the jump-type stochastic Schrödinger equation for two-level systems. Stoch. Process Appl. **120**(9), 1722–1747 (2010)
17. Pellegrini, C.: Markov chain approximations of jump-diffusion stochastic master equations. Ann. l'institut Henri Poincaré Probab. Stat. **46**, 924–948 (2010)
18. Srivastava, S.M.: A Course on Borel Sets, Graduate Texts in Mathematics 180. Springer, New York (1998)
19. Taviot, G.: Martingales et équations de structure : étude géométrique, Thèse de Doctorat de l'Université Louis Pasteur, 29 Mars 1999