

Backward SLE and the symmetry of the welding

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Abstract The backward chordal Schramm–Loewner evolution naturally defines a conformal welding homeomorphism of the real line. We show that this homeomorphism is invariant under the automorphism $x \mapsto -1/x$, and conclude that the associated solution to the welding problem (which is a natural renormalized limit of the finite time Loewner traces) is reversible. The proofs rely on an analysis of the action of analytic circle diffeomorphisms on the space of hulls, and on the coupling techniques of the second author.

Mathematics Subject Classification 30C · 60D

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1 Introduction

1.1 Introduction and results

The Schramm–Loewner evolution SLE_κ , first introduced in [16], is a stochastic process of random conformal maps that has received a lot of attention over the last decade. We refer to the introductory text [7] for basic facts and definitions. In this paper we are largely concerned with chordal SLE_κ , which can be viewed as a family of random curves γ that join 0 and ∞ in the closure of the upper half plane $\overline{\mathbb{H}}$. A fundamental property of chordal SLE is *reversibility*: The law of γ is invariant under the automorphism $z \mapsto -1/z$ of \mathbb{H} , modulo time parametrization. This has first been proved by the second author in [20] for $\kappa \leq 4$, and recently by Miller and Sheffield for $4 < \kappa \leq 8$ in [11]. It is known to be false for $\kappa > 8$ [15, 21].

In the early years of SLE, Oded Schramm, Wendelin Werner and the first author made an attempt to prove reversibility along the following lines: The “backward” flow

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \sqrt{\kappa} B_t}, \quad f_0(z) = z, \quad 0 \leq t \leq T,$$

generates curves $\beta_T = \beta[0, T]$ whose law is that of the chordal SLE trace $\gamma[0, T]$ (up to translation by $\sqrt{\kappa} B_T$). When $\kappa \leq 4$, these curves are simple, and each point of β (with the exception of the endpoints) corresponds to two points on the real line under the conformal map f_t . The *conformal welding homeomorphism* ϕ of β_T is the auto-homeomorphism of the interval $f_T^{-1}(\beta_T)$ that interchanges these two points. In other words, it is the rule that describes which points on the real line are to be identified (laminated) in order to form the curve β_T . It is known [15] that, for $\kappa < 4$, the welding almost surely uniquely determines the curve. The welding homeomorphism can be obtained by restricting the backward flow to the real line: Two points $x \neq y$ on the real line are to be welded if and only if their swallowing times coincide, $\phi(x) = y$ if and only if $\tau_x = \tau_y$, see Sect. 3.5. An idea to prove reversibility was to prove the

invariance of ϕ under $x \mapsto -1/x$, and to relate this to reversibility of a suitable limit of the curves β_T . But the attempts to prove invariance of ϕ failed, and this program of proving reversibility was never completed successfully.

Instead, other methods of proving reversibility became available. In this paper, we turn the above strategy around: We use the coupling techniques of the second author, introduced in [20] for his proof of reversibility of (forward) SLE traces, to prove the invariance of the welding. The main result of this paper is the following:

Theorem 1.1 *Let $\kappa \in (0, 4]$, and ϕ be a backward chordal SLE_κ welding. Let $h(z) = -1/z$. Then $h \circ \phi \circ h$ has the same distribution as ϕ .*

As a consequence, in the range $\kappa \in (0, 4)$ where the SLE trace is conformally removable, we obtain the reversibility of suitably normalized limits of the β_T (see Sect. 6 for details):

Theorem 1.2 *Let $\kappa \in (0, 4)$, and β be a normalized global backward chordal SLE_κ trace. Let $h(z) = -1/z$. Then $h(\beta \setminus \{0\})$ has the same distribution as $\beta \setminus \{0\}$ as random sets.*

In the important paper [18], Sheffield obtains a representation of the SLE welding in terms of a quantum gravity boundary length measure, and also relates it to a simple Jordan arc, which differs from our β only through normalization. However, Theorems 1.1 and 1.2 do not follow easily from his work. A similar random welding homeomorphism is constructed in [3], where the main point is the very difficult existence of a curve solving the welding problem. Our approach to the welding is different: In order to prove Theorem 1.1, in Sect. 2 we develop a framework to study the effect of analytic perturbations of weldings on the corresponding hulls. We show in Sect. 4 that a Möbius image of a backward chordal SLE_κ process is a backward radial $SLE(\kappa, -\kappa - 6)$ process, and the welding is preserved under this conformal transformation. In Sect. 5 we apply the coupling technique to show that backward radial $SLE(\kappa, -\kappa - 6)$ started from an ordered pair of points (a, b) commutes with backward radial $SLE(\kappa, -\kappa - 6)$ started from (b, a) , and use this in Sect. 6 to prove Theorem 1.1.

In a subsequent paper [25] of the second author, Theorem 1.1 is used to study the ergodic properties of a forward SLE_κ trace near the tip at a fixed capacity time.

1.2 Notation

Let $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, $\mathbb{D}^* = \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, and $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$. Let $I_{\mathbb{R}}(z) = \bar{z}$ and $I_{\mathbb{T}}(z) = 1/\bar{z}$ be the reflections about \mathbb{R} and \mathbb{T} , respectively. Let e^i denote the map $z \mapsto e^{iz}$. Let $\cot_2(z) = \cot(z/2)$ and $\sin_2(z) = \sin(z/2)$. For a real interval J , let $C(J)$ denote the space of real valued continuous functions on J . An increasing or decreasing function in this paper is assumed to be strictly monotonic. We use $B(t)$ to denote a standard real Brownian motion. By $f : D \xrightarrow{\text{Conf}} E$ we mean that f maps D conformally onto E . By $f_n \xrightarrow{\text{l.u.}} f$ in U we mean that f_n converges to f uniformly on every compact subset of U . We will frequently use the notation $D_n \xrightarrow{\text{Cara}} D$ as in Definition 7.1.

The outline of this paper is the following. In Sect. 2, we derive some fundamental results in Complex Analysis, which are interesting on their own. In Sect. 3, we review the properties of forward Loewner processes, and derive some properties of backward Loewner processes. In Sect. 4, we discuss how are backward Loewner processes transformed by conformal maps. In Sect. 5, we present and prove certain commutation relations between backward SLE($\kappa; \rho$) processes. In the last section, we prove the reversibility of backward chordal SLE $_{\kappa}$ processes for $\kappa \in (0, 4]$ and propose questions in other cases. In the appendix, we discuss some results on the topology of domains and hulls.

2 Extension of conformal maps

2.1 Interior hulls in \mathbb{C}

An interior hull (in \mathbb{C}) is a nonempty compact connected set $K \subset \mathbb{C}$ such that $\mathbb{C} \setminus K$ is also connected. For every interior hull K in \mathbb{C} , there are a unique $r \geq 0$ and a unique $\phi_K : \widehat{\mathbb{C}} \setminus K \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus r\overline{\mathbb{D}}$ such that $\phi_K(\infty) = \infty$ and $\phi'_K(\infty) := \lim_{z \rightarrow \infty} z/\phi_K(z) = 1$. We call $\text{rad}(K) := r$ the radius of K and $\text{cap}(K) := \ln(r)$ the capacity of K . The radius is 0 iff K contains only one point. In general, we have $\text{rad}(K) \leq \text{diam}(K) \leq 4 \text{rad}(K)$. We call K nondegenerate if it contains more than one point. For such K , there is a unique $\varphi_K : \widehat{\mathbb{C}} \setminus K \xrightarrow{\text{Conf}} \mathbb{D}^*$ such that $\varphi_K(\infty) = \infty$ and $\varphi'_K(\infty) > 0$. In fact, $\varphi_K = \phi_K/\text{rad}(K)$. Let $\psi_K = \varphi_K^{-1}$ for such K .

For any Jordan curve J in \mathbb{C} , let D_J denote the Jordan domain bounded by J , and let $D_J^* = \widehat{\mathbb{C}} \setminus (D_J \cup J)$. Suppose $f_J : \mathbb{D} \xrightarrow{\text{Conf}} D_J$ and $f_J^* = \psi_{D_J} : \mathbb{D}^* \xrightarrow{\text{Conf}} D_J^*$. Then both f_J and f_J^* extend continuously to a homeomorphism from \mathbb{T} onto J . Let $h = (f_J^*)^{-1} \circ f_J$. Then h is an orientation-preserving automorphism of \mathbb{T} . We call such h a conformal welding. Not every homeomorphism of \mathbb{T} is a conformal welding, but it is well-known (and an easy consequence of the uniformization theorem) that every analytic automorphism is a conformal welding, and that the associated Jordan curve is analytic (c.f. [2, Chapter II, Section 1, 3D]). Also see [10] for the quasiconformal theory of conformal welding, and [4] for deep generalizations and further references.

Lemma 2.1 *Let β be an analytic Jordan curve. Let $\Omega \subset \mathbb{C}$ be a neighborhood of \mathbb{T} . Suppose W is a conformal map defined in Ω , maps \mathbb{T} onto \mathbb{T} , and preserves the orientation of \mathbb{T} . Let $\Omega^\beta = \beta \cup D_\beta \cup \psi_{D_\beta}(\Omega \cap \mathbb{D}^*)$. Then there is a conformal map V defined in Ω^β such that $V \circ \psi_{D_\beta} = \psi_{D_{V(\beta)}} \circ W$ in $\Omega \cap \mathbb{D}^*$ (Fig. 1).*

Proof Fix a conformal map $f_\beta : \mathbb{D} \xrightarrow{\text{Conf}} D_\beta$ and let $h_\beta = \varphi_{D_\beta} \circ f_\beta$ be the associated conformal welding homeomorphism. Define $h = W \circ h_\beta$. Since β is analytic, h is analytic and there is an analytic Jordan curve γ and a conformal map $f_\gamma : \mathbb{D} \xrightarrow{\text{Conf}} D_\gamma$ such that $h = h_\gamma = \varphi_{D_\gamma} \circ f_\gamma$. Define $V = f_\gamma \circ f_\beta^{-1}$ on D_β . Since β and γ are analytic curves, V extends conformally to a neighborhood of β with $V(\beta) = \gamma$. On β , this

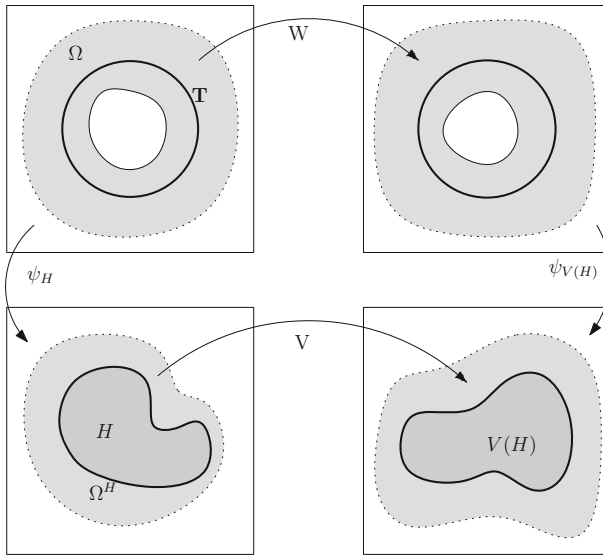


Fig. 1 The situation of Theorem 2.2 and Lemma 2.1. Given H and W , V can be constructed to be analytic in H . In Lemma 2.1, the boundary of H is assumed to be an analytic Jordan curve, while in Theorem 2.2, no regularity assumption is made

extension (still denoted V) satisfies $V = (\psi_{D_\gamma} \circ h_\gamma) \circ (h_\beta^{-1} \circ \psi_{D_\beta}^{-1}) = \psi_{D_\gamma} \circ W \circ \psi_{D_\beta}^{-1}$. Therefore V extends conformally to all of Ω^β and satisfies the desired property. \square

Theorem 2.2 *Let H be a nondegenerate interior hull. Let $\Omega \subset \mathbb{C}$ be a neighborhood of \mathbb{T} . Suppose W is a conformal map defined in Ω , maps \mathbb{T} onto \mathbb{T} , and preserves the orientation of \mathbb{T} . Let $\Omega^H = H \cup \psi_H(\Omega \cap \mathbb{D}^*)$. Then there is a conformal map V defined in Ω^H such that $V \circ \psi_H = \psi_{V(H)} \circ W$ in $\Omega \cap \mathbb{D}^*$. If another conformal map \tilde{V} satisfies the properties of V , then $\tilde{V} = aV + b$ for some $a > 0$ and $b \in \mathbb{C}$ (Fig. 1).*

Proof First, define a sequence of analytic Jordan curves (β_n) by

$$\beta_n = \psi_H(\{e^{1/n+i\theta} : 0 \leq \theta \leq 2\pi\}), \quad n \in \mathbb{N}.$$

Then $\beta_n \cup D_{\beta_n} \rightarrow H$ in $d_{\mathcal{H}}$ (see Appendix B). From Lemma 2.1, for each $n \in \mathbb{N}$, there is a conformal map V_n defined in $\Omega^{\beta_n} := \beta_n \cup D_{\beta_n} \cup \psi_{\beta_n}(\Omega \cap \mathbb{D}^*)$ such that $V_n \circ \psi_{\beta_n} = \psi_{V_n(\beta_n)} \circ W$ in $\Omega \cap \mathbb{D}^*$. Note that for any $a_n > 0$ and $b_n \in \mathbb{C}$, $a_n V_n + b_n$ satisfies the same property as V_n . Thus, we may assume that $0 \in V_n(\beta_n) \subset \mathbb{D}$ and $V_n(\beta_n) \cap \mathbb{T} \neq \emptyset$. Let $\gamma_n = V_n(\beta_n)$, $n \in \mathbb{N}$. Then each γ_n is an interior hull contained in the interior hull \mathbb{D} , and $\text{diam}(\gamma_n) \geq 1$. So $\text{rad}(\gamma_n) \geq 1/4$. From Corollary 8.2, (γ_n) contains a subsequence which converges to some interior hull K contained in \mathbb{D} with radius at least $1/4$. So K is nondegenerate. By passing to a subsequence, we may assume that $\gamma_n \rightarrow K$. From $\beta_n \rightarrow H$ and $\gamma_n \rightarrow K$ we get $\psi_{\beta_n} \xrightarrow{\text{l.u.}} \psi_H$ in $\Omega \cap \mathbb{D}^*$ and $\psi_{\gamma_n} \xrightarrow{\text{l.u.}} \psi_K$ in $W(\Omega \cap \mathbb{D}^*)$. Thus, $\psi_{\beta_n}(\Omega \cap \mathbb{D}^*) \xrightarrow{\text{Cara}} \psi_H(\Omega \cap \mathbb{D}^*)$ by Lemma 7.2.

Since $V_n \circ \psi_{\beta_n} = \psi_{\gamma_n} \circ W$ in $\Omega \cap \mathbb{D}^*$, we find that $V_n = \psi_{\gamma_n} \circ W \circ \psi_{\beta_n}^{-1}$ in $\psi_{\beta_n}(\Omega \cap \mathbb{D}^*)$. Let $V = \psi_K \circ W \circ \psi_H^{-1}$ in $\psi_H(\Omega \cap \mathbb{D}^*)$. Then $V_n \xrightarrow{1.u.} V$ in $\psi_H(\Omega \cap \mathbb{D}^*)$. We may find $r > 1$ such that for any $s \in (1, r]$, $s\mathbb{T} \subset \Omega \cap \mathbb{D}^*$. Then $\psi_H(r\mathbb{T})$ is a Jordan curve in $\psi_H(\Omega \cap \mathbb{D}^*)$ surrounding H , and the Jordan domain bounded by $\psi_H(r\mathbb{T})$ is contained in $\Omega^H = H \cup \psi_H(\Omega \cap \mathbb{D}^*)$. Since $\psi_H(r\mathbb{T})$ is a compact subset of $\psi_H(\Omega \cap \mathbb{D}^*)$, we have $V_n \rightarrow V$ uniformly on $\psi_H(r\mathbb{T})$. It is easy to see that $\Omega^{\beta_n} \xrightarrow{\text{Cara}} \Omega^H$. For n big enough, $\psi_H(r\mathbb{T})$ together with its interior is contained in Ω^{β_n} . From the maximum principle, V_n converges uniformly in the interior of $\psi_H(r\mathbb{T})$ to a conformal map which extends V . We still use V to denote the extended conformal map. Then V is a conformal map defined in Ω^H , and $V_n \xrightarrow{1.u.} V$ in Ω^H . Letting $n \rightarrow \infty$ in the equality $V_n \circ \psi_{\beta_n} = \psi_{\gamma_n} \circ W$ in $\Omega \cap \mathbb{D}^*$, we conclude that $V \circ \psi_H = \psi_{V(H)} \circ W$ in $\Omega \cap \mathbb{D}^*$. So the existence part is proved.

If $\tilde{V} = aV + b$ for some $a > 0$ and $b \in \mathbb{C}$, then $\psi_{\tilde{V}(H)} = a\psi_{V(H)} + b$, which implies $\tilde{V} \circ \psi_H = \psi_{\tilde{V}(H)} \circ W$. Finally, suppose \tilde{V} satisfies the properties of V . Then $\tilde{V} \circ V^{-1}$ is a conformal map in $V(\Omega^H)$. Since $V \circ \psi_H = \psi_{V(H)} \circ W$ and $\tilde{V} \circ \psi_H = \psi_{\tilde{V}(H)} \circ W$ in $\Omega \cap \mathbb{D}^*$, we find that $\tilde{V} \circ V^{-1} = \psi_{\tilde{V}(H)} \circ \psi_{V(H)}^{-1}$ in $\psi_{V(H)}(W(\Omega \cap \mathbb{D}^*)) = V(\Omega^H) \setminus V(H)$. Note that $\psi_{\tilde{V}(H)} \circ \psi_{V(H)}^{-1}$ is a conformal map defined in $\widehat{\mathbb{C}} \setminus V(H)$. Since $V(\Omega^H) \cup (\widehat{\mathbb{C}} \setminus V(H)) = \widehat{\mathbb{C}}$, we may define an analytic function h in \mathbb{C} such that $h = \tilde{V} \circ V^{-1}$ in $V(\Omega^H)$ and $h = \psi_{\tilde{V}(H)} \circ \psi_{V(H)}^{-1}$ in $\mathbb{C} \setminus V(H)$. From the properties of $\psi_{\tilde{V}(H)}$ and $\psi_{V(H)}$, we have $h(\infty) = \infty$ and $h'(\infty) > 0$. Thus, $h(z) = az + b$ for some $a > 0$ and $b \in \mathbb{C}$, which implies that $\tilde{V} = aV + b$. \square

Now we obtain a new proof of the following well-known result about conformal welding.

Corollary 2.3 *Let W be conformal in a neighborhood of \mathbb{T} , maps \mathbb{T} onto \mathbb{T} , and preserves the orientation of \mathbb{T} . If h is a conformal welding, then $W \circ h$ and $h \circ W$ are also conformal weldings.*

Proof Apply Theorem 2.2 to $H = \overline{D_J}$, where J is the Jordan curve for the conformal welding h . We find a conformal map V defined in $\Omega^H = D_J \cup f_J^*(\Omega \cap \mathbb{D}^*)$ such that $V \circ f_J^* = \psi_{V(H)} \circ W$ in $\Omega \cap \mathbb{D}^*$. Let $J' = V(J)$. Then J' is also a Jordan curve, $V(H) = \overline{D_{J'}}$, and $\psi_{V(H)} = f_{J'}^*$. Let $f_{J'} = V \circ f_J$. Then $f_{J'} : \mathbb{D} \xrightarrow{\text{Conf}} D_{J'}$. Thus,

$$W \circ h = W \circ (f_J^*)^{-1} \circ f_J = \psi_{H(H)}^{-1} \circ V \circ f_J = (f_{J'}^*)^{-1} \circ f_{J'},$$

which implies that $W \circ h$ is a conformal welding. As for $h \circ W$, note that $(h \circ W)^{-1} = W^{-1} \circ h^{-1}$ and that h is a conformal welding if and only if h^{-1} is a conformal welding. \square

2.2 Hulls in the upper half plane

Let $\mathbb{H} = \{z \in \mathbb{C} : \text{Im } z > 0\}$. A subset K of \mathbb{H} is called an \mathbb{H} -hull if it is bounded, relatively closed in \mathbb{H} , and $\mathbb{H} \setminus K$ is simply connected. For every \mathbb{H} -hull K , there are a

unique $c \geq 0$ and a unique $g_K : \mathbb{H} \setminus K \xrightarrow{\text{Conf}} \mathbb{H}$ such that $g_K(z) = z + \frac{c}{z} + O(\frac{1}{z^2})$ as $z \rightarrow \infty$. The number c is called the \mathbb{H} -capacity of K , and is denoted by $\text{hcap}(K)$. Let $f_K = g_K^{-1}$. The empty set is an \mathbb{H} -hull with $\text{hcap}(\emptyset) = 0$ and $g_\emptyset = f_\emptyset = \text{id}_{\mathbb{H}}$.

Definition 2.4 Let K_1 and K_2 be \mathbb{H} -hulls.

1. If $K_1 \subset K_2$, define $K_2/K_1 = g_{K_1}(K_2 \setminus K_1)$. We call K_2/K_1 a quotient hull of K_2 , and write $K_2/K_1 < K_2$.
2. The product $K_1 \cdot K_2$ is defined to be $K_1 \cup f_{K_1}(K_2)$.

The following facts are easy to check.

1. K_2/K_1 and $K_1 \cdot K_2$ in the definition are also \mathbb{H} -hulls.
2. For any two \mathbb{H} -hulls K_1 and K_2 , $K_1 \subset K_1 \cdot K_2$ and $K_2 = (K_1 \cdot K_2)/K_1 < K_1 \cdot K_2$. If $K_1 \subset K_2$, then $K_1 \cdot (K_2/K_1) = K_2$.
3. The space of \mathbb{H} -hulls with the product “ \cdot ” is a semigroup with identity element \emptyset , and “ $<$ ” is a transitive relation of this space.
4. $f_{K_1 \cdot K_2} = f_{K_1} \circ f_{K_2}$ in \mathbb{H} ; $g_{K_1 \cdot K_2} = g_{K_2} \circ g_{K_1}$ in $\mathbb{H} \setminus (K_1 \cdot K_2)$.
5. $\text{hcap}(K_1 \cdot K_2) = \text{hcap}(K_1) + \text{hcap}(K_2)$. If $K_1 \subset K_2$ or $K_1 < K_2$, then $\text{hcap}(K_1) \leq \text{hcap}(K_2)$.

From $f_{K_1 \cdot K_2} = f_{K_1} \circ f_{K_2}$ in \mathbb{H} we can conclude that $f_{K_1} = f_{K_1 \cdot K_2} \circ g_{K_2}$ in $\mathbb{H} \setminus K_2$. So f_{K_1} is an analytic extension of $f_{K_1 \cdot K_2} \circ g_{K_2}$, which means that K_1 is uniquely determined by $K_1 \cdot K_2$ and K_2 . So the following definition makes sense.

Definition 2.5 Let K_1 and K_2 be \mathbb{H} -hulls such that $K_1 < K_2$. We use $K_2 : K_1$ to denote the unique \mathbb{H} -hull $K \subset K_2$ such that $K_2/K = K_1$.

For an \mathbb{H} -hull K , the base of K is the set $B_K = \overline{K} \cap \mathbb{R}$. Let the double of K be defined by $K^{\text{db}} = K \cup I_{\mathbb{R}}(K) \cup B_K$, where $I_{\mathbb{R}}(z) := \bar{z}$. Then g_K extends to a conformal map (still denoted by g_K) in $\widehat{\mathbb{C}} \setminus K^{\text{db}}$, which satisfies $g_K(\infty) = \infty$, $g'_K(\infty) = 1$, and $g_K \circ I_{\mathbb{R}} = I_{\mathbb{R}} \circ g_K$. Moreover, $g_K(\widehat{\mathbb{C}} \setminus K^{\text{db}}) = \widehat{\mathbb{C}} \setminus S_K$ for some compact $S_K \subset \mathbb{R}$, which is called the support of K . So f_K extends to a conformal map from $\widehat{\mathbb{C}} \setminus S_K$ onto $\widehat{\mathbb{C}} \setminus K^{\text{db}}$.

Lemma 2.6 f_K can not be extended analytically beyond $\widehat{\mathbb{C}} \setminus S_K$.

Proof Suppose f_K can be extended analytically near $x_0 \in \mathbb{R}$, then the image of f_K contains a neighborhood of $f_K(x_0) \in \mathbb{R}$. So $f_K(\mathbb{H}) = \mathbb{H} \setminus K$ contains a neighborhood of $f_K(x_0)$ in \mathbb{H} . This then implies that $f_K(x_0) \in \mathbb{R} \setminus B_K$. Thus, there is $y_0 \in \mathbb{R} \setminus S_K$ such that $f_K(y_0) = f_K(x_0)$. Since f_K is conformal in \mathbb{H} , we must have $x_0 = y_0 \in \mathbb{R} \setminus S_K$. □

Lemma 2.7 If $K_1 = K_2/K_0 < K_2$, then $S_{K_1} \subset S_{K_2}$, $f_{K_2} = f_{K_0} \circ f_{K_1}$ in $\widehat{\mathbb{C}} \setminus S_{K_2}$, and $g_{K_2} = g_{K_1} \circ g_{K_0}$ in $\widehat{\mathbb{C}} \setminus K_2^{\text{db}}$.

Proof Since $K_2 = K_0 \cdot K_1$, we have $f_{K_2} = f_{K_0} \circ f_{K_1}$ in \mathbb{H} , which implies that $g_{K_0} \circ f_{K_2} = f_{K_1}$ in \mathbb{H} . Since f_{K_2} maps $\widehat{\mathbb{C}} \setminus S_{K_2}$ conformally onto $\widehat{\mathbb{C}} \setminus K_2^{\text{db}} \subset \widehat{\mathbb{C}} \setminus K_0^{\text{db}}$, and g_{K_0} is analytic in $\widehat{\mathbb{C}} \setminus K_2^{\text{db}}$, we see that $g_{K_0} \circ f_{K_2}$ is analytic in $\widehat{\mathbb{C}} \setminus S_{K_2}$. Since $g_{K_0} \circ f_{K_2} = f_{K_1}$ in \mathbb{H} , from Lemma 2.6 we have $S_{K_1} \subset S_{K_2}$, and $g_{K_0} \circ f_{K_2} = f_{K_1}$ in $\widehat{\mathbb{C}} \setminus S_{K_2}$. Composing f_{K_0} to the left of both sides, we get $f_{K_2} = f_{K_0} \circ f_{K_1}$ in $\widehat{\mathbb{C}} \setminus S_{K_2}$. Taking inverse, we obtain the equality for g_K 's. □

Definition 2.8 $S \subset \widehat{\mathbb{C}}$ is called \mathbb{R} -symmetric if $I_{\mathbb{R}}(S) = S$. An \mathbb{R} -symmetric map W is a function defined in an \mathbb{R} -symmetric domain Ω , which commutes with $I_{\mathbb{R}}$, and maps $\Omega \cap \mathbb{H}$ into \mathbb{H} .

Remarks

1. For any \mathbb{H} -hull K , g_K and f_K are \mathbb{R} -symmetric conformal maps.
2. Let W be an R -symmetric conformal map defined in Ω . If an \mathbb{H} -hull K satisfies $K^{db} \subset \Omega$ and $\infty \notin W(K^{db})$, then $W(K)$ is also an \mathbb{H} -hull and $W(K)^{db} = W(K^{db})$.

Definition 2.9 Let Ω be an \mathbb{R} -symmetric domain and K be an \mathbb{H} -hull. If $K^{db} \subset \Omega$, we write Ω_K or $(\Omega)_K$ for $S_K \cup g_K(\Omega \setminus K^{db})$, and call it the collapse of Ω via K . If $S_K \subset \Omega$, we write Ω^K or (Ω^K) for $K^{db} \cup f_K(\Omega \setminus S_K)$, and call it the lift of Ω via K .

Remarks

1. In the definition, Ω_K is an \mathbb{R} -symmetric domain containing S_K ; Ω^K is an \mathbb{R} -symmetric domain containing K^{db} .
2. $(\Omega_K)^K = \Omega$ and $(\Omega^K)_K = \Omega$ if the lefthand sides are well defined.
3. $\Omega_{K_1 \cdot K_2} = (\Omega_{K_1})_{K_2}$ and $\Omega^{K_1 \cdot K_2} = (\Omega^{K_2})^{K_1}$ if either sides are well defined.

Definition 2.10 Let W be an \mathbb{R} -symmetric conformal map with domain Ω . Let K be an \mathbb{H} -hull such that $K^{db} \subset \Omega$ and $\infty \notin W(K^{db})$. We write W_K or $(W)_K$ for the conformal extension of $g_{W(K)} \circ W \circ f_K$ to Ω_K , and call it the collapse of W via K .

Remarks

1. Since $g_{W(K)} \circ W \circ f_K : \Omega_K \setminus S_K \xrightarrow{\text{Conf}} W(\Omega) \setminus S_{W(K)}$, the existence of W_K follows from the Schwarz reflection principle. W_K is an \mathbb{R} -symmetric conformal map, and $W_K(S_K) = S_{W(K)}$.
2. The g_K and f_K defined at the beginning of this section should not be understood as the collapse of g and f via K .
3. $W_{K_1 \cdot K_2} = (W_{K_1})_{K_2}$ if either side is well defined.
4. $V_{W(K)} \circ W_K = (V \circ W)_K$ if either side is well defined. In particular, $(W^{-1})_{W(K)} = (W_K)^{-1}$.

Let B_K^{cv} and S_K^{cv} be the convex hulls of B_K and S_K , respectively. Let $K^{db,cv} = K^{db} \cup B_K^{cv}$. Then $g_K : \widehat{\mathbb{C}} \setminus K^{db,cv} \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus S_K^{cv}$. If $K \neq \emptyset$, then S_K^{cv} is a bounded closed interval, $K^{db,cv}$ is a nondegenerate interior hull, and $\psi_{K^{db,cv}} = f_K \circ \psi_{S_K^{cv}}$. If $S_K^{cv} \subset \Omega$, then $\Omega^K = K^{db,cv} \cup f_K(\Omega \setminus S_K^{cv})$. The lemma below is a part of Lemma 5.3 in [19], where S_K^{cv} was denoted by $[c_K, d_K]$.

Lemma 2.11 *If $K_1 \subset K_2$, then $S_{K_1}^{cv} \subset S_{K_2}^{cv}$.*

Theorem 2.12 *Let W be an \mathbb{R} -symmetric conformal map with domain Ω . Let K be an \mathbb{H} -hull such that $S_K \subset \Omega$ and $\infty \notin W(S_K)$. Then there is a unique \mathbb{R} -symmetric conformal map V defined in Ω^K such that $V_K = W$ (Fig. 2).*

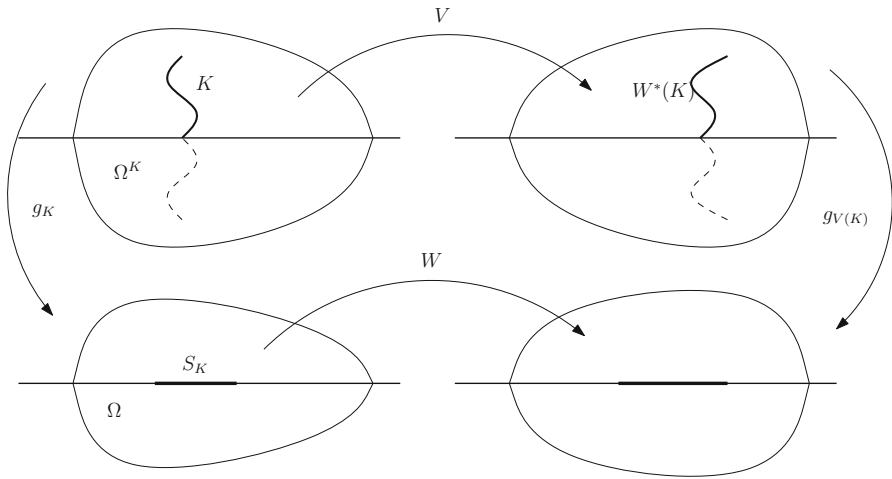


Fig. 2 The situation of Theorem 2.12. Given K and W , there is a unique V , also denoted W^K , which is analytic across K and its reflection $I_{\mathbb{R}}(K)$, see Definition 2.13

Proof We first consider the existence. If $K = \emptyset$, since $f_{\emptyset} = \text{id}$ and $\Omega^{\emptyset} = \Omega$, $V = W$ is what we need. Now suppose $K \neq \emptyset$ and $S_K^{\text{cv}} \subset \Omega$. Note that S_K^{cv} is a bounded closed interval, and so is $W(S_K^{\text{cv}})$. Let $\Omega_{\mathbb{T}} = \psi_{S_K^{\text{cv}}}^{-1}(\Omega \setminus S_K^{\text{cv}})$. Define a conformal map $W_{\mathbb{T}}$ in $\Omega_{\mathbb{T}}$ by $W_{\mathbb{T}} = \psi_{W(S_K^{\text{cv}})}^{-1} \circ W \circ \psi_{S_K^{\text{cv}}}$. Then $W_{\mathbb{T}}(z) \rightarrow \mathbb{T}$ as $\Omega_{\mathbb{T}} \ni z \rightarrow \mathbb{T}$. Thus, $W_{\mathbb{T}}$ extends conformally across \mathbb{T} , maps \mathbb{T} onto \mathbb{T} , and preserves the orientation of \mathbb{T} . Apply Theorem 2.2 to $W_{\mathbb{T}}$ and $K^{\text{db,cv}}$. We find a conformal map \widehat{V} defined in

$$K^{\text{db,cv}} \cup \psi_{K^{\text{db,cv}}}(\Omega_{\mathbb{T}}) = K^{\text{db,cv}} \cup f_K(\Omega \setminus S_K^{\text{cv}}) = \Omega^K$$

such that $\widehat{V} \circ \psi_{K^{\text{db,cv}}} = \psi_{\widehat{V}(K^{\text{db,cv}})} \circ W_{\mathbb{T}}$ in $\Omega_{\mathbb{T}}$. Let $\widetilde{V} = I_{\mathbb{R}} \circ \widehat{V} \circ I_{\mathbb{R}}$. Then $\widetilde{V}(K^{\text{db,cv}}) = I_{\mathbb{R}} \circ \widehat{V}(K^{\text{db,cv}})$. So $\psi_{\widetilde{V}(K^{\text{db,cv}})} = I_{\mathbb{R}} \circ \psi_{\widehat{V}(K^{\text{db,cv}})} \circ I_{\mathbb{R}}$. Since $I_{\mathbb{R}}$ commutes with $\psi_{K^{\text{db,cv}}}$ and $W_{\mathbb{T}}$, we see that \widetilde{V} also satisfies the properties of \widehat{V} . So $\widetilde{V} = a\widehat{V} + b$ for some $a > 0$ and $b \in \mathbb{C}$. Thus, $I_{\mathbb{R}} \circ \widehat{V} \circ I_{\mathbb{R}} = a\widehat{V} + b$. Considering the values of \widehat{V} on $\Omega^K \cap \mathbb{R}$, we find that $a = 1$ and $\text{Re } b = 0$. Note that $\widehat{V} - \frac{b}{2}$ satisfies the property of \widehat{V} , and commutes with $I_{\mathbb{R}}$. By replacing \widehat{V} with $\widehat{V} - \frac{b}{2}$, we may assume that \widehat{V} is an \mathbb{R} -symmetric conformal map.

Since $\widehat{V} \circ \psi_{K^{\text{db,cv}}} = \psi_{\widehat{V}(K^{\text{db,cv}})} \circ W_{\mathbb{T}}$ in $\Omega_{\mathbb{T}}$, from $\psi_{K^{\text{db,cv}}} = f_K \circ \psi_{S_K^{\text{cv}}}$, $\psi_{\widehat{V}(K^{\text{db,cv}})} = f_{\widehat{V}(K)} \circ \psi_{S_{\widehat{V}(K)}^{\text{cv}}}$, and the definitions of $W_{\mathbb{T}}$ and $\Omega_{\mathbb{T}}$, we have

$$\widehat{V} \circ f_K = f_{\widehat{V}(K)} \circ \psi_{S_{\widehat{V}(K)}^{\text{cv}}} \circ \psi_{W(S_K^{\text{cv}})}^{-1} \circ W \tag{2.1}$$

on $\Omega \setminus S_K^{\text{cv}}$. Let $h = \psi_{S_{\widehat{V}(K)}^{\text{cv}}} \circ \psi_{W(S_K^{\text{cv}})}^{-1}$. Since $S_{\widehat{V}(K)}^{\text{cv}}$ and $W(S_K^{\text{cv}})$ are both bounded closed intervals, we have $h(z) = az + b$ for some $a > 0$ and $b \in \mathbb{R}$. Let $V = h^{-1} \circ \widehat{V}$. Then V is also an \mathbb{R} -symmetric conformal map defined on Ω^K , and $f_{V(K)} = h^{-1} \circ f_{\widehat{V}(K)} \circ h$. From (2.1) we have

$$f_{V(K)} \circ W = h^{-1} \circ f_{\widehat{V}(K)} \circ h \circ W = h^{-1} \circ \widehat{V} \circ f_K = V \circ f_K.$$

This finishes the existence part in the case that $K \neq \emptyset$ and $S_K^{cv} \subset \Omega$.

Now we still assume that $K \neq \emptyset$ but do not assume that $S_K^{cv} \subset \Omega$. Let $\Omega_0 = \Omega$ and $W_0 = W$. We will construct \mathbb{H} -hulls K_1, \dots, K_n and \mathbb{R} -symmetric domains $\Omega_1, \dots, \Omega_n$ such that $K_n \cdot K_{n-1} \cdots K_1 = K$, $\Omega_j = \Omega_{j-1}^{K_j}$, and $S_{K_j}^{cv} \subset \Omega_{j-1}$, $1 \leq j \leq n$. When they are constructed, using the above result, we can obtain \mathbb{R} -symmetric conformal maps W_j defined on Ω_j , $1 \leq j \leq n$, such that $(W_j)_{K_j} = W_{j-1}$, $1 \leq j \leq n$. Let $V = W_n$. Then V is defined in $\Omega_n = \Omega_0^{K_n \cdots K_1} = \Omega^K$, and $V_K = (W_n)_{K_n \cdots K_1} = W_0 = W$. So V is what we need.

It remains to construct K_j and Ω_j with the desired properties. Since $\Omega \cap \mathbb{R}$ is a disjoint union of open intervals, and S_K is a compact subset of $\Omega \cap \mathbb{R}$, we may find finitely many components of $\Omega \cap \mathbb{R}$ which cover S_K . There exist mutually disjoint \mathbb{R} -symmetric Jordan curves J_1, \dots, J_n in Ω such that their interiors D_{J_1}, \dots, D_{J_n} are mutually disjoint and contained in Ω , and $S_K \subset \bigcup_{k=1}^n D_{J_k}$. Then $J_j^K := f_K(J_j)$, $1 \leq j \leq n$ are \mathbb{R} -symmetric Jordan curves, which together with their interiors are mutually disjoint, and $K^{db} \subset \bigcup_{k=1}^n D_{f_K(J_k)}$. Let $H_j = K \cap \bigcup_{k=j}^n D_{J_k^K}$, $1 \leq j \leq n$. Then each H_j is an \mathbb{H} -hull, and $K = H_1 \supset H_2 \supset \dots \supset H_n$. Let $K_j = H_j/H_{j+1}$, $1 \leq j \leq n - 1$, and $K_n = H_n$. Then we have $K_n \cdots K_1 = H_1 = K$.

Construct Ω_j , $1 \leq j \leq n$, such that $\Omega_j = \Omega_{j-1}^{K_j}$, $1 \leq j \leq n$. Then

$$\Omega_{j-1} = (\Omega_0)^{K_{j-1} \cdots K_1} = (\Omega^{K_n \cdots K_1})_{K_n \cdots K_j} = (\Omega^K)_{H_j}, \quad 1 \leq j \leq n.$$

It suffices to show that $S_{K_j}^{cv} \subset \Omega_{j-1}$. We have

$$K_j = H_j/H_{j+1} = g_{H_{j+1}}(H_j \setminus H_{j+1}) = g_{H_{j+1}}(K \cap D_{J_j^K}).$$

Thus, $K_j^{db} \subset D_{g_{H_{j+1}}(J_j^K)}$, which implies that $S_{K_j} \subset D_{g_{H_{j+1}}(J_j^K)}$. Since $\mathbb{R} \cap D_{g_{H_j}(J_j^K)}$ is an interval, we have $S_{K_j}^{cv} \subset D_{g_{H_j}(J_j^K)}$. Since $\overline{D_{J_j^K}} \subset \Omega^K$, and J_j^K has positive distance from H_j , we have $D_{g_{H_j}(J_j^K)} \subset (\Omega^K)_{H_j} = \Omega_{j-1}$. So K_j and Ω_j satisfy the properties we need. This finishes the proof of the existence part.

Now we prove the uniqueness. Suppose \tilde{V} is another \mathbb{R} -symmetric conformal map defined on Ω^K such that $\tilde{V}_K = W$. Then

$$g_{V(K)} \circ V = W \circ g_K = g_{\tilde{V}(K)} \circ \tilde{V}$$

on $\Omega \setminus K^{db}$. Thus, $\tilde{V} \circ V^{-1} = f_{\tilde{V}(K)} \circ g_{V(K)}$ on $V(\Omega \setminus K^{db}) = V(\Omega) \setminus V(K^{db})$. We know that $\tilde{V} \circ V^{-1}$ is a conformal map defined on $V(\Omega)$, while $f_{\tilde{V}(K)} \circ g_{V(K)}$ is a conformal map defined on $\widehat{\mathbb{C}} \setminus V(K)^{db} = \widehat{\mathbb{C}} \setminus V(K^{db})$. Since $V(\Omega)$ and $\widehat{\mathbb{C}} \setminus V(K^{db})$ cover $\widehat{\mathbb{C}}$, we may define an analytic function h on \mathbb{C} such that $h = \tilde{V} \circ V^{-1}$ on $V(\Omega)$ and $h = f_{\tilde{V}(K)} \circ g_{V(K)}$ on $\widehat{\mathbb{C}} \setminus V(K^{db})$. From the properties of $f_{\tilde{V}(K)}$ and $g_{V(K)}$, we see that $h(z) - z \rightarrow 0$ as $z \rightarrow \infty$. So $h = \text{id}$, which implies that $\tilde{V} = V$. So the uniqueness is proved. \square

Definition 2.13 We use W^K to denote the unique V in Theorem 2.12, and call it the lift of W via K . Let $W^{\mathcal{H}}$ be the map defined by $W^{\mathcal{H}}(K) = W^K(K)$.

Remarks

1. $(W_K)^K = W$ and $(W^K)_K = W$.
2. The range of W^K is $W^K(\Omega^K) = (W(\Omega))^{W^{\mathcal{H}}(K)}$.
3. $W^{K_1 \cdot K_2} = (W^{K_2})^{K_1}$, $V^{W^K(K)} \circ W^K = (V \circ W)^K$, and $(W^K)^{-1} = (W^{-1})^{W(K)}$.
4. The domain (resp. range) of $W^{\mathcal{H}}$ is the set of \mathbb{H} -hulls whose supports are contained in the domain (resp. range) of W ; and $S_{W^{\mathcal{H}}(K)} = W(S_K)$.
5. $V^{\mathcal{H}} \circ W^{\mathcal{H}} = (V \circ W)^{\mathcal{H}}$; $(W^{\mathcal{H}})^{-1} = (W^{-1})^{\mathcal{H}}$.

Lemma 2.14 Suppose $K_1 \prec K_2$, S_{K_2} lies in the domain of an \mathbb{R} -symmetric conformal map W , and $\infty \notin W(S_{K_2})$. Then $W^{\mathcal{H}}(K_1) \prec W^{\mathcal{H}}(K_2)$, and

$$W^{\mathcal{H}}(K_2) : W^{\mathcal{H}}(K_1) = W^{K_2}(K_2 : K_1). \tag{2.2}$$

Proof From Lemma 2.7, $S_{K_1} \subset S_{K_2}$. So W^{K_1} and W^{K_2} exist. Let $K_0 = K_2 : K_1 \subset K_2$. Then $W^{K_2}(K_0) \subset W^{K_2}(K_2)$ and

$$\begin{aligned} W^{K_2}(K_2)/W^{K_2}(K_0) &= g_{W^{K_2}(K_0)} \circ W^{K_2}(K_2 \setminus K_0) \\ &= g_{W^{K_2}(K_0)} \circ W^{K_2} \circ f_{K_0}(K_2/K_0) = (W^{K_2})_{K_0}(K_2/K_0) \\ &= (W^{K_0 \cdot K_1})_{K_0}(K_1) = W^{K_1}(K_1). \end{aligned}$$

Thus, $W^{K_1}(K_1) \prec W^{K_2}(K_2)$ and $W^{K_2}(K_2) : W^{K_1}(K_1) = W^{K_2}(K_0)$. □

Definition 2.15 Let \mathcal{P}^* denote the set of pair of \mathbb{H} -hulls (H_1, H_2) such that $H_1^{\text{db}} \cap H_2^{\text{db}} = \emptyset$. Let \mathcal{P}_* denote the set of pair of \mathbb{H} -hulls (K_1, K_2) such that $S_{K_1} \cap S_{K_2} = \emptyset$. Define $g_{\mathcal{P}}$ on \mathcal{P}^* by $g_{\mathcal{P}}(H_1, H_2) = (g_{H_2}(H_1), g_{H_1}(H_2))$. Define $f^{\mathcal{P}}$ on \mathcal{P}_* by $f^{\mathcal{P}}(K_1, K_2) = (f_{K_2}^{\mathcal{H}}(K_1), f_{K_1}^{\mathcal{H}}(K_2))$ (Fig. 3).

Remarks

1. $g_{\mathcal{P}}$ is well defined on \mathcal{P}^* because for $j = 1, 2$, K_{3-j}^{db} is contained in the domain of $g_{K_j} : \widehat{\mathbb{C}} \setminus K_j^{\text{db}}$. The value of $g_{\mathcal{P}}$ is a pair of \mathbb{H} -hulls.
2. $f^{\mathcal{P}}$ is well defined on \mathcal{P}_* because for $j = 1, 2$, $S_{K_{3-j}}$ is contained in the domain of $f_{K_j} : \widehat{\mathbb{C}} \setminus S_{K_j}$. The value of $f^{\mathcal{P}}$ is a pair of \mathbb{H} -hulls.

Theorem 2.16 $g_{\mathcal{P}}$ and $f^{\mathcal{P}}$ are bijections between \mathcal{P}^* and \mathcal{P}_* , and are inverse of each other. Moreover, if $(H_1, H_2) = f^{\mathcal{P}}(K_1, K_2)$, then

- (i) $H_1 \cdot K_2 = H_2 \cdot K_1 = H_1 \cup H_2$;
- (ii) $f_{K_2}(S_{K_1}) = S_{H_1}$ and $f_{K_1}(S_{K_2}) = S_{H_2}$;
- (iii) $S_{H_1 \cup H_2} = S_{K_1} \cup S_{K_2}$.

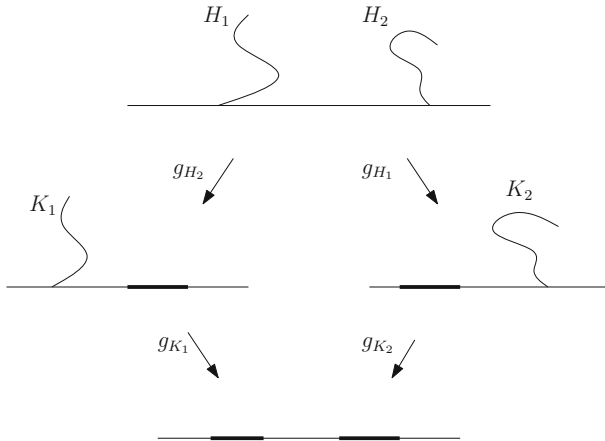


Fig. 3 The pair (H_1, H_2) uniquely determines the pair (K_1, K_2) , and vice versa, see Definition 2.15 and Theorem 2.16

Proof Let $(H_1, H_2) \in \mathcal{P}^*$ and $(K_1, K_2) = g_{\mathcal{P}}(H_1, H_2)$. Then $(\widehat{\mathbb{C}} \setminus H_1^{\text{db}})_{H_2} = \widehat{\mathbb{C}} \setminus K_1^{\text{db}}$, $S_{H_1} \subset \widehat{\mathbb{C}} \setminus K_2^{\text{db}}$, and $(\widehat{\mathbb{C}} \setminus S_{H_1})_{K_2} = \widehat{\mathbb{C}} \setminus g_{K_2}(S_{H_1})$. Since $g_{H_1} : \widehat{\mathbb{C}} \setminus H_1^{\text{db}} \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus S_{H_1}$ and $g_{H_1}(H_2) = K_2$, we get $(g_{H_1})_{H_2} : \widehat{\mathbb{C}} \setminus K_1^{\text{db}} \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus g_{K_2}(S_{H_1})$. From the normalization of $g_{H_1}, g_{H_2}, g_{K_2}$ at ∞ , we conclude that

$$(g_{H_1})_{H_2} = g_{K_1}, \quad g_{K_2}(S_{H_1}) = S_{K_1}. \tag{2.3}$$

From $S_{H_1} \subset \widehat{\mathbb{C}} \setminus K_2^{\text{db}}$ and $g_{K_2}(S_{H_1}) = S_{K_1}$, we see that $S_{K_1} \cap S_{K_2} = \emptyset$, i.e., $(K_1, K_2) \in \mathcal{P}_*$. Since $f_{H_1} = g_{H_1}^{-1}$, $f_{K_1} = g_{K_1}^{-1}$, and $g_{H_1}(H_2) = K_2$, from (2.3) we get $(f_{H_1})_{K_2} = f_{K_1}$, which implies that $(f_{K_1})^{K_2} = f_{H_1}$. Thus, $f_{K_1}^{\mathcal{H}}(K_2) = f_{H_1}(K_2) = H_2$. Similarly, $f_{K_2}^{\mathcal{H}}(K_1) = H_1$. Thus, $f^{\mathcal{P}}(K_1, K_2) = (H_1, H_2)$. So $f^{\mathcal{P}} \circ g_{\mathcal{P}} = \text{id}_{\mathcal{P}^*}$.

Let $(K_1, K_2) \in \mathcal{P}_*$ and $H_1 = f_{K_2}^{\mathcal{H}}(K_1)$. Then $S_{H_1} = f_{K_2}(S_{K_1})$ is disjoint from K_2^{db} . Thus, we may define another \mathbb{H} -hull $H_2 := f_{H_1}(K_2)$. Then $H_2^{\text{db}} \subset \widehat{\mathbb{C}} \setminus H_1^{\text{db}}$. So $(H_1, H_2) \in \mathcal{P}^*$. We have $(\widehat{\mathbb{C}} \setminus S_{K_2})^{K_1} = \widehat{\mathbb{C}} \setminus f_{K_1}(S_{K_2})$ and $(\widehat{\mathbb{C}} \setminus K_2^{\text{db}})^{H_1} = \widehat{\mathbb{C}} \setminus H_2^{\text{db}}$. Since $f_{K_2} : \widehat{\mathbb{C}} \setminus S_{K_2} \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus K_2^{\text{db}}$ and $f_{K_2}^{\mathcal{H}}(K_1) = H_1$, we see that $(f_{K_2})^{K_1} : \widehat{\mathbb{C}} \setminus f_{K_1}(S_{K_2}) \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus H_2^{\text{db}}$. From the normalization of $f_{K_1}, f_{H_1}, f_{K_2}$ at ∞ , we conclude that

$$(f_{K_2})^{K_1} = f_{H_2}, \quad f_{K_1}(S_{K_2}) = S_{H_2}. \tag{2.4}$$

Since $H_1 = f_{K_2}^{\mathcal{H}}(K_1)$, we get $f_{K_2} = g_{H_1} \circ f_{H_2} \circ f_{K_1}$ on $(\widehat{\mathbb{C}} \setminus S_{K_2}) \setminus S_{K_1}$, which implies that $f_{H_1} \circ f_{K_2} = f_{H_2} \circ f_{K_1}$ on $\widehat{\mathbb{C}} \setminus (S_{K_1} \cup S_{K_2})$. So

$$H_2 \cdot K_1 = H_1 \cdot K_2 = H_1 \cup f_{H_1}(K_2) = H_1 \cup H_2. \tag{2.5}$$

Thus, $K_1 = g_{H_2}(H_1)$ and $K_2 = g_{H_1}(H_2)$, i.e., $(K_1, K_2) = g_{\mathcal{P}}(H_1, H_2)$. This shows that the range of $g_{\mathcal{P}}$ is \mathcal{P}_* , which combining with $f^{\mathcal{P}} \circ g_{\mathcal{P}} = \text{id}_{\mathcal{P}^*}$ shows that $f^{\mathcal{P}} = (g_{\mathcal{P}})^{-1}$ and $g_{\mathcal{P}} = (f^{\mathcal{P}})^{-1}$.

In the previous paragraph, since $(K_1, K_2) = g_{\mathcal{P}}(H_1, H_2)$, $f^{\mathcal{P}}(K_1, K_2) = (H_1, H_2)$. Thus, (i) follows from (2.5); the second parts of (ii) follow from (2.4), and the first part follows from symmetry. Finally, since $g_{K_2} \circ g_{H_1} = g_{H_1 \cdot K_2} = g_{H_1 \cup H_2}$, from $g_{H_1} : \widehat{\mathbb{C}} \setminus (H_1^{\text{db}} \cup H_2^{\text{db}}) \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus (S_{H_1} \cup K_2^{\text{db}})$, $g_{K_2} : \widehat{\mathbb{C}} \setminus (S_{H_1} \cup K_2^{\text{db}}) \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus (g_{K_2}(S_{H_1}) \cup S_{K_2})$, and (2.3), we get (iii). \square

Definition 2.17 For $(K_1, K_2) \in \mathcal{P}_*$, we define the quotient union of K_1 and K_2 to be $K_1 \vee K_2 = H_1 \cup H_2$, where $(H_1, H_2) = f^{\mathcal{P}}(K_1, K_2)$.

Remark From Theorem 2.16, $K_1, K_2 \prec K_1 \vee K_2$ and $S_{K_1 \vee K_2} = S_{K_1} \cup S_{K_2}$.

The space of \mathbb{H} -hulls has a natural metric $d_{\mathcal{H}}$ described in Appendix C. Let \mathcal{H}_S denote the set of \mathbb{H} -hulls whose supports are contained in S . From Lemma 9.2, if F is compact, $(\mathcal{H}_F, d_{\mathcal{H}})$ is compact, and $H_n \rightarrow H$ in \mathcal{H}_F implies that $f_{H_n} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{C} \setminus F$.

Theorem 2.18 (i) *Let $F \subset \mathbb{R}$ be compact. Let W be an \mathbb{R} -symmetric conformal map whose domain contains F . Then $W^{\mathcal{H}} : \mathcal{H}_F \rightarrow \mathcal{H}_{W(F)}$ is continuous.*
 (ii) *Let E and F be two nonempty compact subsets of \mathbb{R} with $E \cap F = \emptyset$. Then $f^{\mathcal{P}}$ and $(K_1, K_2) \mapsto K_1 \vee K_2$ are continuous on $\mathcal{H}_E \times \mathcal{H}_F$.*

Proof (i) First, $W^{\mathcal{H}}$ is well defined on \mathcal{H}_F , and the range of $W^{\mathcal{H}}$ is $\mathcal{H}_{W(F)}$. Suppose (H_n) is a sequence in \mathcal{H}_F and $H_n \rightarrow H_0 \in \mathcal{H}_F$. To prove the continuity of $W^{\mathcal{H}}$, we need to show that $W^{\mathcal{H}}(H_n) \rightarrow W^{\mathcal{H}}(H_0)$. Suppose this is not true. Since $\mathcal{H}_{W(F)}$ is compact, by passing to a subsequence, we may assume that $W^{\mathcal{H}}(H_n) \rightarrow K_0 \neq W^{\mathcal{H}}(H_0)$. For each n_k , $W^{H_{n_k}} = f_{W^{\mathcal{H}}(H_{n_k})} \circ W \circ g_{H_{n_k}}$ on $f_{H_{n_k}}(\Omega \setminus F)$. We have $g_{H_{n_k}} \xrightarrow{\text{l.u.}} g_{H_0}$ in $f_{H_0}(\Omega \setminus F)$ and $f_{W^{\mathcal{H}}(H_{n_k})} \xrightarrow{\text{l.u.}} f_{K_0}$ in $W(\Omega) \setminus W(F)$. Thus, $W^{H_{n_k}} \xrightarrow{\text{l.u.}} f_{K_0} \circ W \circ g_{H_0} =: V$ in $f_{H_0}(\Omega \setminus F)$. The domain of $W^{H_{n_k}}$ is $\Omega^{H_{n_k}} = H_{n_k}^{\text{db}} \cup f_{H_{n_k}}(\Omega \setminus S_{H_{n_k}})$, which converges to $\Omega^{H_0} = H_0^{\text{db}} \cup f_{H_0}(\Omega \setminus S_{H_0}) \supset f_{H_0}(\Omega \setminus F)$. It is clear that $\Omega^{H_0} \setminus f_{H_0}(\Omega \setminus F)$ is compact. Since $W^{H_{n_k}} \xrightarrow{\text{l.u.}} V$ in $f_{H_0}(\Omega \setminus F)$, from the maximum principle, $W^{H_{n_k}}$ converges locally uniformly in Ω^{H_0} . We still let V denote the limit function. Since $H_{n_k} \rightarrow H_0$ and $W^{H_{n_k}}(H_{n_k}) \rightarrow K_0$, we have $V(H_0) = K_0$. Since $f_{K_0} \circ W \circ g_{H_0} = V$ in $f_{H_0}(\Omega \setminus F)$, we see that $f_{V(H_0)} \circ W \circ g_{H_0} = V$ in $f_{H_0}(\Omega \setminus S_{H_0})$. Thus, $V = W^{H_0}$. So $K_0 = W^{H_0}(H_0) = W^{\mathcal{H}}(H_0)$. This is the contradiction we need.

(ii) To show $f^{\mathcal{P}}$ is continuous, it suffices to show that, if (K_1^n, K_2^n) is a sequence in $\mathcal{H}_E \times \mathcal{H}_F$ which converges to $(K_1^0, K_2^0) \in \mathcal{H}_E \times \mathcal{H}_F$, then it has a subsequence $(K_1^{(n_k)}, K_2^{(n_k)})$ such that $f^{\mathcal{P}}(K_1^{(n_k)}, K_2^{(n_k)}) \rightarrow f^{\mathcal{P}}(K_1^0, K_2^0)$. Let $(H_1^n, H_2^n) = f^{\mathcal{P}}(K_1^n, K_2^n)$, $n \in \mathbb{N}$. From Theorem 2.16 (iii), $S_{H_1^n \cup H_2^n} = S_{K_1^n} \cup S_{K_2^n} \subset E \cup F$. From Lemma 9.2, $(H_1^n \cup H_2^n)$ has a convergent subsequence with limit in $\mathcal{H}_{E \cup F}$. From Lemma 2.11, $S_{H_1^n} \subset S_{H_1^n \cup H_2^n}^{\text{cv}} \subset A$, where A is the convex hull of $E \cup F$. From Lemma 9.2, (H_1^n) has a convergent subsequence. For the same reason, (H_2^n) also has a convergent subsequence. By passing to subsequences, we may assume that $H_1^n \cup H_2^n \rightarrow M^0 \in \mathcal{H}_{E \cup F}$ and $H_j^n \rightarrow H_j^0$, $j = 1, 2$.

From Theorem 2.16 (i) and the continuity of the dot product, we get $H_1^0 \cdot K_2^0 = H_2^0 \cdot K_1^0 = M^0$. This implies that $M^0 = H_1^0 \cup f_{H_1^0}(K_2^0)$. The measures $(\mu_{H_1^n})$ (see Appendix C) converges to $\mu_{H_1^0}$ weakly. Each $\mu_{H_1^n}$ is supported by $S_{H_1^n}$. From Theorem 2.16 (ii), $S_{H_1^n} = f_{K_2^n}(S_{K_1^n}) \subset f_{K_2^n}(E)$. Since E is a compact subset of $\mathbb{C} \setminus F$, we have $f_{K_2^n} \rightarrow f_{K_2^0}$ uniformly on E . Thus, $f_{K_2^n}(E) \rightarrow f_{K_2^0}(E)$ in the Hausdorff metric. So $\mu_{H_1^0}$ is supported by $f_{K_2^0}(E)$, which implies that $S_{H_1^0} \subset f_{K_2^0}(E)$. Hence $f_{H_1^0}(K_2^0)$ is another \mathbb{H} -hull, which is bounded away from H_1^0 . From $K_2^n \rightarrow K_2^0$ we have $\mathbb{H} \setminus K_2^n \xrightarrow{\text{Cara}} \mathbb{H} \setminus K_2^0$. From (9.1) we get $f_{H_1^n} \xrightarrow{\text{u.}} f_{H_1^0}$ in $\mathbb{C} \setminus S_{H_1^0}$. Thus, $\mathbb{H} \setminus f_{H_1^n}(K_2^n) \xrightarrow{\text{Cara}} \mathbb{H} \setminus f_{H_1^0}(K_2^0)$. Since $H_2^n = f_{H_1^n}(K_2^n)$, we have $\mathbb{H} \setminus H_2^n \xrightarrow{\text{Cara}} \mathbb{H} \setminus f_{H_1^0}(K_2^0)$. On the other hand, $\mathbb{H} \setminus H_2^n \xrightarrow{\text{Cara}} \mathbb{H} \setminus H_2^0$. Since $\mathbb{H} \setminus H_2^0$ and $\mathbb{H} \setminus f_{H_1^0}(K_2^0)$ both contain a neighborhood of ∞ in \mathbb{H} , they must be the same domain. Thus, $H_2^0 = f_{H_1^0}(K_2^0)$ is bounded away from H_1^0 , i.e., $(H_1^0, H_2^0) \in \mathcal{P}^*$. For the same reason, $H_1^0 = f_{H_2^0}(K_1^0)$. Thus, $(H_1^n, H_2^n) \rightarrow (H_1^0, H_2^0) = f^{\mathcal{P}}(K_1^0, K_2^0)$. This shows that $f^{\mathcal{P}}$ is continuous. Finally, since $K_1 \vee K_2 = H_1 \cdot K_2$ if $(H_1, H_2) = f^{\mathcal{P}}(K_1, K_2)$, we see that $(K_1, K_2) \mapsto K_1 \vee K_2$ is also continuous. \square

Corollary 2.19 (i) *Let W be an \mathbb{R} -symmetric conformal map with domain Ω . Then $W^{\mathcal{H}}$ is measurable on $\mathcal{H}_{\Omega \cap \mathbb{R}}$.*
 (ii) *$f^{\mathcal{P}}$ and $(K_1, K_2) \mapsto K_1 \vee K_2$ are measurable on \mathcal{P}_* .*

Proof (i) We may find an increasing sequence of compact subsets (F_n) of $\Omega \cap \mathbb{R}$ such that $\mathcal{H}_{\Omega \cap \mathbb{R}} = \bigcup_{n=1}^{\infty} \mathcal{H}_{F_n}$. From Theorem 2.18 (i), $W^{\mathcal{H}}$ is continuous on each \mathcal{H}_{F_n} . Thus, $W^{\mathcal{H}}$ is measurable on $\mathcal{H}_{\Omega \cap \mathbb{R}}$.

(ii) We may find a sequence of pairs of disjoint bounded closed intervals of \mathbb{R} : (E_n, F_n) , $n \in \mathbb{N}$, such that $\mathcal{P}_* = \bigcup_{n=1}^{\infty} \mathcal{H}_{E_n} \times \mathcal{H}_{F_n}$. From Theorem 2.18 (ii), $f^{\mathcal{P}}$ and $(K_1, K_2) \mapsto K_1 \vee K_2$ are continuous on each $\mathcal{H}_{E_n} \times \mathcal{H}_{F_n}$, and so they are measurable on \mathcal{P}_* . \square

2.3 Hulls in the unit disc

A subset K of $\mathbb{D} = \{|z| < 1\}$ is called a \mathbb{D} -hull if $\mathbb{D} \setminus K$ is a simply connected domain containing 0. For every \mathbb{D} -hull K , there is a unique $g_K : \mathbb{D} \setminus K \xrightarrow{\text{Conf}} \mathbb{D}$ such that $g_K(0) = 0$ and $g'_K(0) > 0$. Then $\ln g'_K(0) \geq 0$ is called the \mathbb{D} -capacity of K , and is denoted by $\text{dcap}(K)$. Let $f_K = g_K^{-1}$.

We may define $K_1 \cdot K_2$, K_2/K_1 (when $K_1 \subset K_2$), and $K_1 \prec K_2$ on the space of \mathbb{D} -hulls as in Definition 2.4. Then the remarks after Definition 2.4 still hold if \mathbb{H} is replaced by \mathbb{D} and hcap is replaced by dcap . Then we may define $K_2 : K_1$ (when $K_1 \prec K_2$) as in Definition 2.5. For a \mathbb{D} -hull K , the base B_K of K is $\overline{K} \cap \mathbb{T}$, and the double of K is $K^{\text{db}} = K \cup I_{\mathbb{T}}(K) \cup B_K$, where $I_{\mathbb{T}}(z) := 1/\bar{z}$. Then g_K extends to a conformal map (still denoted by g_K) on $\widehat{\mathbb{C}} \setminus K^{\text{db}}$, which commutes with $I_{\mathbb{T}}$. Moreover, $g_K(\widehat{\mathbb{C}} \setminus K^{\text{db}}) = \widehat{\mathbb{C}} \setminus S_K$ for some compact $S_K \subset \mathbb{T}$, which is called the support of K .

So f_K extends to a conformal map from $\widehat{\mathbb{C}} \setminus S_K$ onto $\widehat{\mathbb{C}} \setminus K^{\text{db}}$, which commutes with $I_{\mathbb{T}}$. Then Lemmas 2.6 and 2.7 still hold here.

We may define \mathbb{T} -symmetric sets and \mathbb{T} -symmetric conformal maps using Definition 2.8 with \mathbb{R} and \mathbb{H} replaced by \mathbb{T} and \mathbb{D} , respectively. For a \mathbb{T} -symmetric domain Ω and a \mathbb{D} -hull K , we may define domains Ω_K (when $K^{\text{db}} \subset \Omega$) and Ω^K (when $S_K \subset \Omega$) using Definition 2.9. If W is a \mathbb{T} -symmetric conformal map with domain Ω , and if Ω_K is defined, we may then define W_K using Definition 2.10, which is a \mathbb{T} -symmetric conformal map on Ω_K . The remarks after Definitions 2.8, 2.9, and 2.10 hold here with minor modifications. We claim that Theorem 2.12 holds here with modifications. We need several lemmas.

The theorem below relates the \mathbb{H} -hulls with \mathbb{D} -hulls. To distinguish the two set of symbols, we use $f_K^{\mathbb{H}}, g_K^{\mathbb{H}}, B_K^{\mathbb{R}}, S_K^{\mathbb{R}}$, and $K^{\mathbb{R db}}$ for \mathbb{H} -hulls, and $f_K^{\mathbb{D}}, g_K^{\mathbb{D}}, B_K^{\mathbb{T}}, S_K^{\mathbb{T}}$, and $K^{\mathbb{T db}}$ for \mathbb{D} -hulls.

- Theorem 2.20** (i) *Let W be a Möbius transformation that maps \mathbb{D} onto \mathbb{H} , and K be a \mathbb{D} -hull such that $W^{-1}(\infty) \notin S_K^{\mathbb{T}}$. Then there is a unique Möbius transformation W^K that maps \mathbb{D} onto \mathbb{H} such that $W^K(K)$ is an \mathbb{H} -hull, $g_{W^K(K)}^{\mathbb{H}} \circ W^K \circ f_K^{\mathbb{D}} = W$ in $\widehat{\mathbb{C}} \setminus S_K^{\mathbb{T}}$, and $S_{W^K(K)}^{\mathbb{R}} = W(S_K^{\mathbb{T}})$.*
- (ii) *Let W be a Möbius transformation that maps \mathbb{H} onto \mathbb{D} , and K be an \mathbb{H} -hull. Then there is a unique Möbius transformation W^K that maps \mathbb{H} onto \mathbb{D} such that $W^K(K)$ is a \mathbb{D} -hull, $g_{W^K(K)}^{\mathbb{D}} \circ W^K \circ f_K^{\mathbb{H}} = W$ in $\widehat{\mathbb{C}} \setminus S_K^{\mathbb{R}}$, and $S_{W^K(K)}^{\mathbb{T}} = W(S_K^{\mathbb{R}})$.*

Proof (i) Let $z_0 = W^{-1}(\infty) \in \mathbb{T} \setminus S_K^{\mathbb{T}}$. Then $w_0 := f_K^{\mathbb{D}}(z_0) \in \mathbb{T} \setminus B_K^{\mathbb{T}}$ is well defined. Let $W_0^K(z) = i \frac{w_0 + z}{w_0 - z}$. Then W_0^K is a Möbius transformation that maps \mathbb{D} onto \mathbb{H} and takes w_0 to ∞ . Let $L_0 = W_0^K(K)$. Since w_0 is bounded away from K , we see that L_0 is an \mathbb{H} -hull. We have $W_0^K : \widehat{\mathbb{C}} \setminus K^{\mathbb{T db}} \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus L_0^{\mathbb{R db}}$. Define $G = g_{L_0}^{\mathbb{H}} \circ W_0^K \circ f_K^{\mathbb{D}} \circ W^{-1}$ on $\widehat{\mathbb{C}} \setminus W(S_K^{\mathbb{T}})$. Then $G : \widehat{\mathbb{C}} \setminus W(S_K^{\mathbb{T}}) \xrightarrow{\text{Conf}} \widehat{\mathbb{C}} \setminus S_{L_0}^{\mathbb{R}}$, fixes ∞ , and maps \mathbb{H} onto \mathbb{H} . So $G(z) = az + b$ for some $a > 0$ and $b \in \mathbb{R}$. Let $W^K = G^{-1} \circ W_0^K$. Then W^K is also a Möbius transformation that maps \mathbb{D} onto \mathbb{H} , and $W^K(K)$ is also an \mathbb{H} -hull with $S_{W^K(K)}^{\mathbb{R}} = G^{-1}(S_{L_0}^{\mathbb{R}}) = W(S_K^{\mathbb{T}})$ and $g_{W^K(K)}^{\mathbb{H}} = G^{-1} \circ g_{L_0}^{\mathbb{H}} \circ G$. Thus,

$$\begin{aligned} g_{W^K(K)}^{\mathbb{H}} \circ W^K \circ f_K^{\mathbb{D}} \circ W^{-1} &= G^{-1} \circ g_{L_0}^{\mathbb{H}} \circ G \circ G^{-1} \circ W_0^K \circ f_K^{\mathbb{D}} \circ W^{-1} \\ &= G^{-1} \circ g_{L_0}^{\mathbb{H}} \circ W_0^K \circ f_K^{\mathbb{D}} \circ W^{-1} = G^{-1} \circ G \\ &= \text{id}_{\widehat{\mathbb{C}} \setminus W(S_K)}. \end{aligned}$$

This implies that $g_L^{\mathbb{H}} \circ W^K \circ f_K^{\mathbb{D}} = W$ in $\widehat{\mathbb{C}} \setminus S_K$. So we proved the existence. On the other hand, if W^K satisfies the desired property, then from $W^K = f_L^{\mathbb{H}} \circ W \circ g_K^{\mathbb{D}}$ we get $W^K(w_0) = \infty$. So $W^K = G_0 \circ W_0^K$, where $G_0(z) = az + b$ for some $a > 0$ and $b \in \mathbb{R}$. The above argument shows that $G_0 = G^{-1}$. So we get the uniqueness.

(ii) We may use the proof of (i) with slight modifications: replace ∞ by 0, swap \mathbb{H} and \mathbb{D} , swap \mathbb{R} and \mathbb{T} , and define $W_0^K(z) = \frac{z - w_0}{z + w_0}$. □

We also use $W^{\mathcal{H}}(K)$ to denote the hull $W^K(K)$ in the above lemma. The following lemma is similar to Lemma 2.14.

Lemma 2.21 *Let K_1 and K_2 be two \mathbb{H} -(resp. \mathbb{D} -)hulls such that $K_1 \prec K_2$. Let W be a Möbius transformation that maps \mathbb{H} onto \mathbb{D} (resp. maps \mathbb{D} onto \mathbb{H}) such that $\infty \notin W(S_{K_2})$. Then $W^{\mathcal{H}}(K_1) \prec W^{\mathcal{H}}(K_2)$ and (2.2) still holds.*

The following lemma is used to treat the case $S_K = \mathbb{T}$ in Theorem 2.23.

Lemma 2.22 *Let W be a \mathbb{T} -symmetric conformal map with domain $\Omega \supset \mathbb{T}$. Let (K_n) be a sequence of \mathbb{D} -hulls which converges to K . Suppose that for each n , there is a \mathbb{T} -symmetric conformal map $V^{(n)}$ defined on Ω^{K_n} such that $V_{K_n}^{(n)} = W$. Then there is a \mathbb{T} -symmetric conformal map V defined on Ω^K such that $V_K = W$. Moreover, $V(K)$ is a subsequential limit of $(V^{(n)}(K_n))$.*

Proof Since $K_n \rightarrow K$, $\Omega^{K_n} \xrightarrow{\text{Cara}} \Omega^K$. Since $V^{(n)}$ maps $\Omega^{K_n} \cap \mathbb{D}$ into \mathbb{D} , the family $(V^{(n)}|_{\Omega^{K_n} \cap \mathbb{D}})$ is uniformly bounded. Thus, $(V^{(n)})$ contains a subsequence, which convergence locally uniformly in $\Omega^K \cap \mathbb{D}$. To save the symbols, we assume that $(V^{(n)})$ itself converges locally uniformly in $\Omega^K \cap \mathbb{D}$. Since each $V^{(n)}$ is \mathbb{T} -symmetric, the sequence also converges locally uniformly in $\Omega^K \cap \mathbb{D}^*$. From the maximum principle, $(V^{(n)})$ converges locally uniformly in Ω^K . Let V be the limit function. Since each $V^{(n)}$ maps \mathbb{T} onto \mathbb{T} , and $V^{(n)} \rightarrow V$ uniformly on \mathbb{T} , V can not be constant. From Lemma 7.2, V is a conformal map. It is \mathbb{T} -symmetric because each $V^{(n)}$ is \mathbb{T} -symmetric. Since $K_n \rightarrow K$, we have $V^{(n)}(K_n) \rightarrow V(K)$. From $V_{K_n}^{(n)} = W$ we have $g_{V^{(n)}(K_n)} \circ V^{(n)} \circ f_{K_n} = W$ in $\Omega \setminus \mathbb{T}$. Letting $n \rightarrow \infty$ we get $g_{V(K)} \circ V \circ f_K = W$ in $\Omega \setminus \mathbb{T}$. By continuation, this equality also holds on $\Omega \setminus S_K$. Thus, $V_K = W$. □

Theorem 2.23 *Let W be a \mathbb{T} -symmetric conformal map with domain Ω . Let K be a \mathbb{D} -hull such that $S_K \subset \Omega$. Then there is a unique \mathbb{T} -symmetric conformal map V defined on Ω^K such that $V_K = W$.*

Proof We first consider the existence. Case 1. $S_K^{\mathbb{T}} \neq \mathbb{T}$. We will apply Theorems 2.12 and 2.20 for this case. Pick $z_0 \in \mathbb{T} \setminus S_K^{\mathbb{T}}$ and let $h(z) = i \frac{z_0 + z}{z_0 - z}$. From Theorem 2.20 (i), there is a Möbius transformation h^K that maps \mathbb{D} onto \mathbb{H} such that $L := h^K(K)$ is an \mathbb{H} -hull, and $g_L^{\mathbb{H}} \circ h^K \circ f_K^{\mathbb{D}} = h$ in $\widehat{\mathbb{C}} \setminus S_K^{\mathbb{T}}$. Since W is a homeomorphism on S_K , $W(S_K) \neq \mathbb{T}$. So there is $z_W \in \mathbb{T} \setminus W(S_K)$. Let $h_W(z) = z_W \cdot \frac{z-i}{z+i}$. Then h_W is a Möbius transformation that maps \mathbb{H} onto \mathbb{D} and takes ∞ to z_W . Let $\tilde{W} = h_W^{-1} \circ W \circ h^{-1}$. Then \tilde{W} is an \mathbb{R} -symmetric conformal map with domain $h(\Omega)$, and $\tilde{W}(S_L^{\mathbb{R}}) = h_W^{-1} \circ W(S_K^{\mathbb{T}}) \neq \infty$. From Theorem 2.12, there is an \mathbb{R} -symmetric conformal map \tilde{V} with domain $L^{\mathbb{R} \text{ db}} \cup f_L^{\mathbb{R}}(h(\Omega) \setminus S_L^{\mathbb{R}})$ such that $L^* := \tilde{V}(L)$ is an \mathbb{H} -hull, and $\tilde{V} = f_{L^*}^{\mathbb{H}} \circ \tilde{W} \circ g_L^{\mathbb{H}}$ in $\widehat{\mathbb{C}} \setminus L^{\mathbb{R} \text{ db}}$. From Theorem 2.20 (ii), there is a Möbius transformation $h_W^{L^*}$ that maps \mathbb{H} onto \mathbb{D} such that $K^* := h_W^{L^*}(L^*)$ is a \mathbb{D} -hull, and $g_{K^*}^{\mathbb{D}} \circ h_W^{L^*} \circ f_{L^*}^{\mathbb{H}} = h_W$ in $\widehat{\mathbb{C}} \setminus S_{L^*}^{\mathbb{R}}$. Finally, let $V = h_W^{L^*} \circ \tilde{V} \circ h^K$. Then

$$V(K) = h_W^{L^*} \circ \tilde{V}(L) = h_W^{L^*}(L^*) = K^*,$$

and

$$\begin{aligned}
 g_{K^*} \circ V \circ f_K &= g_{K^*} \circ h_W^{L^*} \circ \tilde{V} \circ h^K \circ f_K \\
 &= g_{K^*} \circ h_W^{L^*} \circ f_{L^*}^{\mathbb{H}} \circ \tilde{W} \circ g_L^{\mathbb{H}} \circ h^K \circ f_K \\
 &= h_W \circ \tilde{W} \circ h = W
 \end{aligned}$$

in $\widehat{\mathbb{C}} \setminus K^{\mathbb{T} \text{ db}}$. This finishes the existence part for Case 1.

Case 2. $S_K = \mathbb{T}$. First, we may approximate K using \mathbb{D} -hulls bounded by \mathbb{T} and a Jordan curve in \mathbb{D} . For example, let $J_n = f_K(\{|z| = 1 - 1/(2n)\})$, and let $K_n = \mathbb{D} \setminus D_{J_n}$. Then each K_n is a \mathbb{D} -hull, and $K_n \rightarrow K$. Second, if K' has the form of $\mathbb{D} \setminus D_J$ for some Jordan curve J , then we may define a curve β , which starts from $\beta(0) = z_0 \in \mathbb{T}$, then follows a simple curve in $\mathbb{D} \cap D_J^*$ to a point on J , and then follows J in the clockwise direction, and ends when it finishes one round. Suppose the domain of β is $[0, 1]$. Then β is simple on $[0, 1 - \varepsilon]$ for any $\varepsilon > 0$. Let $K_n = \beta((0, 1 - 1/n])$, $n \in \mathbb{N}$. Then each K_n is a \mathbb{D} -hull with $S_{K_n} \neq \mathbb{T}$, and $K_n \rightarrow K'$. Thus, K can be approximated by a sequence of \mathbb{D} -hulls (K_n) such that $S_{K_n} \neq \mathbb{T}$ for each K_n . Then the existence of V follows from Case 1 and Lemma 2.22.

Now we prove the uniqueness. Suppose \tilde{V} is another \mathbb{T} -symmetric conformal map defined on Ω^K such that $\tilde{V}_K = W$. We may use the argument in the proof of Theorem 2.12 to construct an analytic function h on \mathbb{C} such that $h = \tilde{V} \circ V^{-1}$ on $V(\Omega)$ and $h = f_{\tilde{V}(K)} \circ g_{V(K)}$ on $\mathbb{C} \setminus V(K^{\text{db}})$. Then h is \mathbb{T} -symmetric. From the properties of $f_{\tilde{V}(K)}$ and $g_{V(K)}$, we see that $h(0) = 0$ and $h'(0) > 0$. So $h = \text{id}$, which implies that $\tilde{V} = V$. □

We may then define W^K and $W^{\mathcal{H}}$ using Definition 2.13 with Theorem 2.23 in place of Theorem 2.12 and \mathbb{D} in place of \mathbb{H} . The remarks after Definition 2.13 hold here with minor modifications, and so does Lemma 2.14. Then we define \mathcal{P}^* , \mathcal{P}_* , $g_{\mathcal{P}}$, and $f^{\mathcal{P}}$ using Definition 2.15 with \mathbb{H} replaced by \mathbb{D} . Then Theorem 2.16 still holds here, and we may define the quotient union $K_1 \vee K_2$ for $(K_1, K_2) \in \mathcal{P}_*$.

The space of \mathbb{D} -hulls has a natural metric $d_{\mathcal{H}}$ described in Appendix D. Let \mathcal{H}_S denote the set of \mathbb{D} -hulls whose supports are contained in S . We claim that Theorem 2.18 still holds here if every \mathbb{R} is replaced by \mathbb{T} . For part (i), if $F \neq \mathbb{T}$, then the proof of Theorem 2.18 (i) still goes through with Lemma 10.2 in place of Lemma 9.2; if $F = \mathbb{T}$, then the continuity of $W^{\mathcal{H}}$ follows from Lemma 2.22. For part (ii), the proof of Theorem 2.18 still goes through with some modifications. The relatively compactness of $(H_n \cup J_n)$ follows from Lemma 10.2 instead of Lemma 9.2 because $S_{H_n \cup J_n} \subset E \cup F \subsetneq \mathbb{T}$. To show the relatively compactness of (H_n) and (J_n) , instead of applying Lemma 2.11, we now apply Lemma 10.1, and use the relatively compactness of $(H_n \cup J_n)$ and the inequalities $\text{dcap}(H_n), \text{dcap}(J_n) \leq \text{dcap}(H_n \cup J_n)$. In addition, (10.2) will be used in place of (9.1). This finishes the proof of Theorem 2.18 in the radial case. Then Corollary 2.19 in the radial case immediately follows.

The proof of Theorem 2.18 (i) may also be used to show that the map $K \mapsto W^K(K)$ in Theorem 2.20 (i) (resp. (ii)) is continuous if restricted to $\mathcal{H}_F^{\mathbb{D}}$ (resp. $\mathcal{H}_F^{\mathbb{H}}$), where F is a compact subset of $\mathbb{T} \setminus W^{-1}(\infty)$ (resp. \mathbb{R}). We then can conclude that the maps $K \mapsto W^K(K)$ in Theorem 2.20 (i) and (ii) are both measurable.

3 Loewner equations and Loewner chains

3.1 Forward Loewner equations

We review the definitions and basic facts about (forward) Loewner equations. The reader is referred to [7] for details. Let $\lambda \in C([0, T])$, where $T \in (0, \infty]$. The chordal Loewner equation driven by λ is

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda(t)}, \quad g_0(z) = z.$$

We assume that $g_t(\infty) = \infty$ for $0 \leq t < \infty$. For $z \in \mathbb{C}$, suppose that the maximal interval for $t \mapsto g_t(z)$ is $[0, \tau_z)$. Let $K_t = \{z \in \mathbb{H} : \tau_z \leq t\}$, i.e., the set of $z \in \mathbb{H}$ such that $g_t(z)$ is not defined. Then g_t and K_t , $0 \leq t < T$, are called the chordal Loewner maps and hulls driven by λ . It is known that each K_t is an \mathbb{H} -hull with $\text{hcap}(K_t) = 2t$, and for $0 < t < T$, $g_t = g_{K_t}$ with exactly the same domain: $\widehat{\mathbb{C}} \setminus K_t^{\text{db}}$. At $t = 0$, $K_0 = \emptyset$ and $g_0 = \text{id}_{\widehat{\mathbb{C}} \setminus \{\lambda(0)\}}$.

We say that λ generates a chordal trace β if

$$\beta(t) := \lim_{\mathbb{H} \ni z \rightarrow \lambda(t)} g_t^{-1}(z) \in \overline{\mathbb{H}}$$

exists for $0 \leq t < T$, and β is a continuous curve. We call such β the chordal trace driven by λ . If the chordal trace β exists, then for each t , $\mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \beta((0, t])$, and f_t extends continuously from \mathbb{H} to $\mathbb{H} \cup \mathbb{R}$. The trace β is called simple if it is a simple curve and $\beta(t) \in \mathbb{H}$ for $0 < t < T$, in which case $K_t = \beta((0, t])$ for $0 \leq t < T$.

The radial Loewner equation driven by λ is

$$\partial_t g_t(z) = g_t(z) \frac{e^{i\lambda(t)} + g_t(z)}{e^{i\lambda(t)} - g_t(z)}, \quad 0 \leq t < T; \quad g_0(z) = z.$$

We assume that $g_t(\infty) = \infty$ for $0 \leq t < \infty$. For each $t \in [0, T)$, let K_t be the set of $z \in \mathbb{D} := \{|z| < 1\}$ at which g_t is not defined. Then g_t and K_t , $0 \leq t < T$, are called the radial Loewner maps and hulls driven by λ . It is known that, each K_t is a \mathbb{D} -hull with $\text{dcap}(K_t) = t$, and for $0 < t < T$, $g_t = g_{K_t}$ with exactly the same domain: $\widehat{\mathbb{C}} \setminus K_t^{\text{db}}$. At $t = 0$, $K_0 = \emptyset$ and $g_0 = \text{id}_{\widehat{\mathbb{C}} \setminus \{e^{i\lambda(0)}\}}$.

We say that λ generates a radial trace β if

$$\beta(t) := \lim_{\mathbb{D} \ni z \rightarrow e^{i\lambda(t)}} g_t^{-1}(z) \in \overline{\mathbb{D}}$$

exists for $0 \leq t < T$, and β is a continuous curve. We call such β the radial trace driven by λ . If the radial trace β exists, then for each t , $\mathbb{D} \setminus K_t$ is the component of $\mathbb{D} \setminus \beta((0, t])$ that contains 0. The trace β is called simple if it is a simple curve and $\beta(t) \in \mathbb{D}$ for $0 < t < T$, in which case $K_t = \beta((0, t])$ for $0 \leq t < T$.

Let $\cot_2(z) = \cot(z/2)$. The covering radial Loewner equation driven by λ is

$$\partial_t \tilde{g}_t(z) = \cot_2(\tilde{g}_t(z) - \lambda(t)), \quad 0 \leq t < T, \quad \tilde{g}_0(z) = z.$$

For each $t \in [0, T)$, let \tilde{K}_t be the set of all $z \in \mathbb{H}$ at which \tilde{g}_t is not defined. Then \tilde{g}_t and \tilde{K}_t , $0 \leq t < T$, are called the covering radial Loewner maps and hulls driven by λ . We have $\tilde{g}_t : \mathbb{H} \setminus \tilde{K}_t \xrightarrow{\text{Conf}} \mathbb{H}$. If g_t and K_t , $0 \leq t < T$, are the radial Loewner maps and hulls driven by λ , then $\tilde{K}_t = (e^t)^{-1}(K_t)$ and $e^t \circ \tilde{g}_t = g_t \circ e^t$, where e^t denotes the map $z \mapsto e^{tz}$.

For $\kappa > 0$, chordal (resp. radial) SLE $_{\kappa}$ is defined by solving the chordal (resp. radial) Loewner equation with $\lambda(t) = \sqrt{\kappa}B(t)$. Such driving function a.s. generates a chordal (resp. radial) trace, which is simple if $\kappa \in (0, 4]$.

3.2 Backward Loewner equations

Let $\lambda \in C([0, T))$. The backward chordal Loewner equation driven by λ is

$$\partial_t f_t(z) = \frac{-2}{f_t(z) - \lambda(t)}, \quad f_0(z) = z. \tag{3.1}$$

We assume that $f_t(\infty) = \infty$ for $0 \leq t < T$. Let $L_t = \mathbb{H} \setminus f_t(\mathbb{H})$. We call f_t and L_t , $0 \leq t < T$, the backward chordal Loewner maps and hulls driven by λ .

Define a family of maps f_{t_2, t_1} , $t_1, t_2 \in [0, T)$, such that, for any fixed $t_1 \in [0, T)$ and $z \in \widehat{\mathbb{C}} \setminus \{\lambda(t_1)\}$, the function $t_2 \mapsto f_{t_2, t_1}(z)$ is the maximal solution of the ODE

$$\partial_{t_2} f_{t_2, t_1}(z) = \frac{-2}{f_{t_2, t_1}(z) - \lambda(t_2)}, \quad f_{t_1, t_1}(z) = z.$$

Note that $f_{t, 0} = f_t$ and $f_{t, t} = \text{id}_{\widehat{\mathbb{C}} \setminus \{\lambda(t)\}}$, $0 \leq t < T$. If $t_1 \in (0, T)$, then t_2 could be bigger or smaller than t_1 . Some simple observations give the following lemma.

- Lemma 3.1** (i) For any $t_1, t_2, t_3 \in [0, T)$, $f_{t_3, t_2} \circ f_{t_2, t_1}$ is a restriction of f_{t_3, t_1} . In particular, this implies that $f_{t_1, t_2} = f_{t_2, t_1}^{-1}$.
- (ii) For any fixed $t_0 \in [0, T)$, f_{t_0+t, t_0} , $0 \leq t < T - t_0$, are the backward chordal Loewner maps driven by $\lambda(t_0 + t)$, $0 \leq t < T - t_0$.
- (iii) For any fixed $t_0 \in [0, T)$, f_{t_0-t, t_0} , $0 \leq t \leq t_0$, are the (forward) chordal Loewner maps driven by $\lambda(t_0 - t)$, $0 \leq t \leq t_0$.

Let $L_{t_2, t_1} = \mathbb{H} \setminus f_{t_2, t_1}(\mathbb{H})$ for $0 \leq t_1 \leq t_2 < T$. From (i), (iii), and the properties of forward chordal Loewner maps, we see that, if $0 \leq t_1 < t_2 < T$, then L_{t_2, t_1} is an \mathbb{H} -hull with $\text{hcap}(L_{t_2, t_1}) = 2(t_2 - t_1)$, and $f_{t_2, t_1} = f_{L_{t_2, t_1}}$. If $t_1 = t_2$, this is almost still true except that $f_{t_1, t_1} = \text{id}_{\widehat{\mathbb{C}} \setminus \{\lambda(t_1)\}}$ and $f_{L_{t_1, t_1}} = f_{\emptyset} = \text{id}_{\widehat{\mathbb{C}}}$. Since $L_{t, 0} = L_t$, and $\lambda(t) \in \mathbb{R}$ does not lie in the range of f_t , which is $\widehat{\mathbb{C}} \setminus L_t^{\text{db}}$ for $t > 0$, we get the following lemma.

Lemma 3.2 For $0 \leq t < T$, L_t is an \mathbb{H} -hull with $\text{hcap}(L_t) = 2t$. If $t \in (0, T)$, then $f_t = f_{L_t}$ with the same domain: $\widehat{\mathbb{C}} \setminus S_{L_t}$, and $\lambda(t) \in B_{L_t}$.

If $t_2 \geq t_1 \geq t_0$, from $f_{t_2,t_1} \circ f_{t_1,t_0} = f_{t_2,t_0}$ we get $L_{t_2,t_0} = L_{t_2,t_1} \cdot L_{t_1,t_0}$. From Lemmas 2.7 and 3.1, we obtain the following lemma.

Lemma 3.3 For any $0 \leq t_1 < t_2 < T$, $L_{t_1} \prec L_{t_2}$ and $S_{L_{t_1}} \subset S_{L_{t_2}(t_2)}$. For any fixed $t_0 \in [0, T)$, the family $L_{t_0} : L_{t_0-t} = L_{t_0,t_0-t}$, $0 \leq t \leq t_0$, are the chordal Loewner hulls driven by $\lambda(t_0 - t)$, $0 \leq t \leq t_0$.

Note that $S_{L_0} = S_\emptyset = \emptyset$, and it is easy to see that, for $0 < t_0 < T$, $S_{L_{t_0}}$ is the set of $x \in \mathbb{R}$ such that the solution $f_t(x)$ to (3.1) blows up before or at t_0 , i.e., $S_{L_{t_0}} = \{x \in \mathbb{R} : \tau_x \leq t_0\}$. So every S_{L_t} , $0 < t < T$, is a real interval, and $\bigcap_{0 < t < T} S_{L_t} = \{\lambda(0)\}$.

If for every $t_0 \in [0, T)$, $\lambda(t_0 - t)$, $0 \leq t \leq t_0$, generates a (forward) chordal trace, which we denote by $\beta_{t_0}(t_0 - t)$, $0 \leq t \leq t_0$, then we say that λ generates backward chordal traces β_{t_0} , $0 \leq t_0 < T$. If this happens, then for any $0 \leq t_1 \leq t_2 < T$, $\mathbb{H} \setminus L_{t_2,t_1}$ is the unbounded component of $\mathbb{H} \setminus \beta_{t_2}([t_1, t_2])$, and f_{t_2,t_1} extends continuously from \mathbb{H} to \mathbb{H} such that

$$f_{t_2,t_1}(\lambda(t_1)) = \beta_{t_2}(t_1), \quad 0 \leq t_1 \leq t_2 < T. \tag{3.2}$$

Here we still use f_{t_2,t_1} to denote the continuation if there is no confusion. For $0 \leq t_0 \leq t_1 \leq t_2 < T$, the equality $f_{t_2,t_0} = f_{t_2,t_1} \circ f_{t_1,t_0}$ still holds after continuation, which together with (3.2) implies that

$$f_{t_2,t_1}(\beta_{t_1}(t)) = \beta_{t_2}(t), \quad 0 \leq t \leq t_1 \leq t_2 < T. \tag{3.3}$$

Remark One should keep in mind that each β_t is a continuous function defined on $[0, t]$, $\beta_t(0)$ is the tip of β_t , and $\beta_t(t)$ is the root of β_t , which lies on \mathbb{R} . The parametrization is different from a forward chordal trace β , of which $\beta(0)$ is the root.

The backward radial Loewner equations and the backward covering radial Loewner equation driven by $\lambda \in C([0, T))$ are the following two equations respectively:

$$\begin{aligned} \partial_t f_t(z) &= -f_t(z) \frac{e^{i\lambda(t)} + f_t(z)}{e^{i\lambda(t)} - f_t(z)}, & f_0(z) &= z; \\ \partial_t \tilde{f}_t(z) &= -\cot_2(\tilde{f}_t(z) - \lambda(t)), & \tilde{f}_0(z) &= z. \end{aligned}$$

We have $f_t \circ e^i = e^i \circ \tilde{f}_t$. Let $L_t = \mathbb{D} \setminus f_t(\mathbb{D})$. We call f_t and L_t , $0 \leq t < T$, the backward radial Loewner maps and hulls driven by λ , and call \tilde{f}_t , $0 \leq t < T$, the backward covering radial Loewner maps driven by λ .

By introducing f_{t_2,t_1} in the radial setting, we find that Lemma 3.1 holds if the word ‘‘chordal’’ is replaced by ‘‘radial’’. The following lemma is similar to Lemma 3.2.

Lemma 3.4 For $0 \leq t < T$, L_t is a \mathbb{D} -hull with $\text{dcap}(L_t) = t$. If $t \in (0, T)$, then $f_t = f_{L_t}$ with the same domain: $\widehat{\mathbb{C}} \setminus S_{L_t}$, and $e^{i\lambda(t)} \in B_{L_t}$.

We find that Lemma 3.3 holds here if the word ‘‘chordal’’ is replaced by ‘‘radial’’. So we may define backward radial traces β_t , $0 \leq t < T$, in a similar manner.

The following lemma holds only in the radial case.

Lemma 3.5 *If $T = \infty$, then $\mathbb{T} \setminus \bigcup_{0 < t < \infty} S_{L_t}$ contains at most one point.*

Proof Let $S_\infty = \bigcup_{0 < t < \infty} S_{L_t}$. From Koebe’s 1/4 theorem, as $t \rightarrow \infty$, $\text{dist}(0, L_t) \rightarrow 0$, which implies that the harmonic measure of $\mathbb{T} \setminus B_{L_t}$ in $\mathbb{D} \setminus L_t$ seen from 0 tends to 0. Since $f_t : \mathbb{D} \xrightarrow{\text{Conf}} \mathbb{D} \setminus L_t$, $f_t(0) = 0$, and $f_t(\mathbb{T} \setminus S_{L_t}) = \mathbb{T} \setminus B_{L_t}$, the above harmonic measure at time t equals to $|\mathbb{T} \setminus S_{L_t}| / (2\pi)$. Thus, $|\mathbb{T} \setminus S_\infty| = \lim_{t \rightarrow \infty} |\mathbb{T} \setminus S_{L_t}| = 0$. \square

For $\kappa > 0$, the backward chordal (resp. radial) SLE $_\kappa$ is defined by solving backward chordal (resp. radial) Loewner equation with $\lambda(t) = \sqrt{\kappa} B(t)$, $0 \leq t < \infty$. Since for any fixed $t_0 > 0$, $(\lambda(t_0 - t) - \lambda(t_0), 0 \leq t \leq t_0)$ has the distribution of $(\sqrt{\kappa} B(t), 0 \leq t \leq t_0)$, using the existence of forward chordal (resp. radial) SLE $_\kappa$ traces, we conclude that λ a.s. generates a family of backward chordal (resp. radial) traces.

3.3 Normalized global backward trace

First we consider a backward chordal Loewner process generated by $\lambda(t)$, $0 \leq t < T$. Let $S_t = S_{L_t}$, $0 \leq t < T$, and $S_T = \bigcup_{0 \leq t < T} S_t$. Then (S_t) is an increasing family, and S_T is an interval. The following Lemma is similar in spirit to Proposition 5.1 in [18].

Lemma 3.6 *There exists a family of conformal maps $F_{T,t}$, $0 \leq t < T$, on \mathbb{H} such that $F_{T,t_1} = F_{T,t_2} \circ f_{t_2,t_1}$ in \mathbb{H} if $0 \leq t_1 \leq t_2 < T$. Let $D_t = F_{T,t}(\mathbb{H})$, $0 \leq t < T$, and $D_T = \bigcup_{t < T} D_t$. If $(\widehat{F}_{T,t})$ satisfies the same property as $(F_{T,t})$, then there is a conformal map h_T defined on D_T such that $\widehat{F}_{T,t} = h_T \circ F_{T,t}$, $0 \leq t < T$. If there is $z_0 \in \mathbb{H}$ such that*

$$\lim_{t \rightarrow T} \frac{\text{Im } f_t(z_0)}{|f'_t(z_0)|} = \infty, \tag{3.4}$$

then we may construct $(F_{T,t})$ such that $D_T = \mathbb{C}$, and we have $S_T = \mathbb{R}$.

Proof Fix $z_0 \in \mathbb{H}$. Let $z_t = f_t(z_0)$ and $u_t = f'_t(z_0)$, $0 \leq t < T$. For $t \in [0, T)$, let $M_t(z) = \frac{z - z_t}{u_t}$ and $F_t = M_t \circ f_t$. Then F_t maps z_0 to 0 and has derivative 1 at z_0 . For $0 \leq t_1 \leq t_2 < T$, define $F_{t_2,t_1} = M_{t_2} \circ f_{t_2,t_1}$. Then $F_{t_2,t_1} \circ f_{t_1,t_0} = F_{t_2,t_0}$ if $t_0 \leq t_1 \leq t_2$. Setting $t_0 = 0$ we get $F_{t_2,t_1} \circ f_{t_1} = F_{t_2}$. Thus, F_{t_2,t_1} is a conformal map on \mathbb{H} with $F_{t_2,t_1}(z_{t_1}) = 0$ and $F'_{t_2,t_1}(z_{t_1}) = 1/u_{t_1}$. By Koebe’s distortion theorem, for any $t_1 \in [0, T)$, $\{F_{t_2,t_1} : t_2 \in [t_1, T)\}$ is uniformly bounded on each compact subset of \mathbb{H} . This implies that every sequence in this family contains a subsequence which converges locally uniformly, and the limit function is also conformal on \mathbb{H} , maps z_{t_1} to 0, and has derivative $1/u_{t_1}$ at z_{t_1} .

From a diagonal argument, we can find a sequence (t_n) in $[0, T)$ such that $t_n \rightarrow T$ and for any $q \in \mathbb{Q} \cap [0, T)$, $(F_{t_n,q})$ converges locally uniformly on \mathbb{H} . Let $F_{T,q}$, $q \in \mathbb{Q} \cap [0, T)$, denote the limit functions, which are conformal on \mathbb{H} . Since $F_{t_n,q_2} \circ f_{q_2,q_1} = F_{t_n,q_1}$ for each n , we have $F_{T,q_2} \circ f_{q_2,q_1} = F_{T,q_1}$. For $t \in [0, T)$, choose $q \in \mathbb{Q} \cap [t, T)$ and define the conformal map $F_{T,t} = F_{T,q} \circ f_{q,t}$ on \mathbb{H} . If $q_1 \leq q_2 \in \mathbb{Q} \cap [t, T)$, then $F_{T,q_1} \circ f_{q_1,t} = F_{T,q_2} \circ f_{q_2,q_1} \circ f_{q_1,t} = F_{T,q_2} \circ f_{q_2,t}$. Thus, the definition of $F_{T,t}$ does not depend on the choice of q . If $0 \leq t_1 \leq t_2 < T$, by choosing $q \in \mathbb{Q} \cap [0, T)$ with $q \geq t_1 \vee t_2$, we get $F_{T,t_2} \circ f_{t_2,t_1} = F_{T,q} \circ f_{q,t_2} \circ f_{t_2,t_1} = F_{T,q} \circ f_{q,t_1} = F_{T,t_1}$.

If (3.4) holds, then we start the construction of $(F_{T,t})$ with such z_0 . Since $F_{T,t} : (\mathbb{H}; z_t) \xrightarrow{\text{Conf}} (D_t; 0)$ and $F'_{T,t}(z_t) = 1/u_t$, Koebe's 1/4 theorem implies that $\text{dist}(0, \partial D_t) \geq \frac{1}{4} \text{Im } z_t / |u_t| = \frac{1}{4} \frac{\text{Im } f_t(z_0)}{|f'_t(z_0)|}$, which tends to ∞ as $t \rightarrow T$. So D_T has to be \mathbb{C} .

Suppose $\widehat{F}_{T,t}, 0 \leq t < T$, satisfies the same property as $F_{T,t}, 0 \leq t < T$. Let $h_t = \widehat{F}_{T,t} \circ F_{T,t}^{-1}, 0 \leq t < T$. Then each h_t is a conformal map defined on D_t . If $0 \leq t_1 < t_2 < T$, then

$$h_{t_1} \circ F_{T,t_1} = \widehat{F}_{T,t_1}(z) = \widehat{F}_{T,t_2} \circ f_{t_2,t_1} = h_{t_2} \circ F_{T,t_2} \circ f_{t_2,t_1} = h_{t_2} \circ F_{T,t_1}$$

in \mathbb{H} , which implies that $h_{t_1} = h_{t_2}|_{D_{t_1}}$. So we may define a conformal map h_T on D_T such that $h_t = h_T|_{D_t}$ for $0 \leq t < T$. Such h_T is what we need.

Suppose that (3.4) holds but $S_T \neq \mathbb{R}$. Since S_T is an interval, $\overline{S_T} \neq \mathbb{R}$. Choose $\widehat{z}_0 \in \widehat{\mathbb{C}} \setminus \overline{S_T}$, and start the construction with \widehat{z}_0 in place of z_0 at the beginning of this proof. Let $\widehat{F}_{T,t}, 0 \leq t < T$, denote the family of maps constructed in this way. Then each $\widehat{F}_{T,t}$ is an \mathbb{R} -symmetric conformal map, which implies that $\widehat{D}_T \subset \mathbb{H}$. However, now $D_T = \mathbb{C}$ and $h_T : D_T \xrightarrow{\text{Conf}} \widehat{D}_T$, which is impossible. Thus, $S_T = \mathbb{R}$ when (3.4) holds. □

Let $(F_{T,t}), D_t$, and D_T be as in Lemma 3.6. Let $F_T = F_{T,0}$. Suppose λ generates backward chordal traces $\beta_{t_0}, 0 \leq t_0 < T$, which satisfy

$$\forall t_0 \in [0, T), \quad \exists t_1 \in (t_0, T), \quad \beta_{t_1}([0, t_0]) \subset \mathbb{H}. \tag{3.5}$$

We may define $\beta(t), 0 \leq t < T$, as follows. For every $t \in [0, T)$, pick $t_0 \in (t, T)$ such that $\beta_{t_0}(t) \in \mathbb{H}$, which is possible by (3.5), and define

$$\beta(t) = F_{T,t_0}\beta_{t_0}(t) \in D_{t_0} \subset D_T. \tag{3.6}$$

Since $F_{T,t_1} = F_{T,t_2} \circ f_{t_2,t_1}$ in \mathbb{H} , from (3.3) we see that the definition of β does not depend on the choice of t_0 . Let $t_0 \in [0, T)$. From (3.5), there is $t_1 > t_0$ such that $\beta_{t_1}([0, t]) \in \mathbb{H}$. Since $\beta(t) = F_{T,t_0}(\beta_{t_0}(t)), 0 \leq t \leq t_0$, we see that β is continuous on $[0, t_0]$. Thus, $\beta(t), 0 \leq t < T$, is a continuous curve in D_T .

Fix any $x \in S_T$. Then $x \in S_{t_0}$ for some $t_0 \in (0, T)$. So $f_{t_0}(x)$ lies on the outer boundary of L_{t_0} , which implies that $f_{t_0}(x) \in \beta_{t_0}(t)$ for some $t \in [0, t_0]$. From (3.5), there is $t_1 \in (t_0, T)$ such that $\beta_{t_1}([0, t_0]) \subset \mathbb{H}$. Then $f_{t_1}(x) = f_{t_1,t_0}(\beta_{t_0}(t)) = \beta_{t_1}(t) \in \mathbb{H}$. From the continuity of f_{t_1} on $\mathbb{H} \cup \mathbb{R}$, there is a neighborhood U of x in $\mathbb{H} \cup \mathbb{R}$ such that $f_{t_1}(U) \subset \mathbb{H}$. This shows that $U \cap \mathbb{R} \subset S_{t_1} \subset S_T$. Since $F_T = F_{T,t_1} \circ f_{t_1}$ in \mathbb{H} , we find that F_T has continuation on U . Since $x \in S_T$ is arbitrary, we conclude that S_T is an open interval, and F_T has continuation to $\mathbb{H} \cup S_T$.

Now we assume that λ generates backward chordal traces, and both (3.4) and (3.5) hold. Then $D_T = \mathbb{C}, S_T = \mathbb{R}$, a continuous curve $\beta(t), 0 \leq t < T$, is well defined, and F_T extends continuously to $\mathbb{H} \cup \mathbb{R}$. Moreover, F_T is unique up to a map $z \mapsto az + b$ for some $a, b \in \mathbb{C}, a \neq 0$. With some suitable normalization condition, the family $F_{T,t}$ and the curve β will be determined by λ . We will use the following normalization:

$$F_T(\lambda(0)) = \lambda(0), \quad F_T(\lambda(0) + i) = \lambda(0) + i. \tag{3.7}$$

If (3.7) holds, we call β the normalized global backward chordal trace generated by λ . From (3.7) we see that $\beta(0) = \lambda(0)$, and β does not pass through $\lambda(0) + i$.

For the radial case, Lemma 3.6 still holds with \mathbb{H} replaced by \mathbb{D} , and (3.4) replaced by $T = \infty$. For the construction, we choose $z_0 = 0$ and let $F_{t_2, t_1}(z) = e^{t_2} f_{t_2, t_1}(z)$. If λ generates backward radial traces $\beta_t, 0 \leq t < T$, which satisfy

$$\forall t_1 \in [0, T), \quad \exists t_2 \in (t_1, T), \quad \beta_{t_2}(t_1) \in \mathbb{D}, \tag{3.8}$$

then we may define a continuous curve $\beta(t), 0 \leq t < T$, in D_T using (3.6). If $T = \infty$, then $D_T = \mathbb{C}$, and such β is determined by λ up to a map $z \mapsto az + b$ for some $a, b \in \mathbb{C}, a \neq 0$, which means that we may define a normalized global backward radial trace once a normalization condition is fixed.

3.4 Forward and backward Loewner chains

In this section, we review a condition on a family of hulls that corresponds to continuously driven (forward) Loewner hulls, and discuss the corresponding condition for backward Loewner chains.

Let $D \subset \widehat{\mathbb{C}}$ be a simply connected domain such that $\widehat{\mathbb{C}} \setminus D$ contains more than one point. A relatively closed subset H of D is called a (boundary) hull in D if $D \setminus H$ is simply connected. For example, a hull in \mathbb{H} is an \mathbb{H} -hull iff it is bounded; a hull in \mathbb{D} is a \mathbb{D} -hull iff it does not contain 0. Let $T \in (0, \infty]$. A family of hulls in $D: K_t, 0 \leq t < T$, is called a Loewner chain in D if

1. $K_0 = \emptyset$ and $K_{t_1} \subsetneq K_{t_2}$ whenever $0 \leq t_1 < t_2 < T$;
2. for any fixed $a \in [0, T)$ and a compact set $F \subset D \setminus K_a$, the extremal length (c.f. [1]) of the family of curves in $D \setminus K_t$ that separate F from $K_{t+\varepsilon} \setminus K_t$ tends to 0 as $\varepsilon \rightarrow 0$, uniformly in $t \in [0, a]$.

If $K_t, 0 \leq t < T$, is a Loewner chain in D , and $a \in [0, T)$, then we also call the restriction $K_t, 0 \leq t \leq a$, a Loewner chain in D .

There are two important properties for Loewner chains. If $K_t, 0 \leq t < T$, is a Loewner chain in D , and u is a continuous increasing function defined on $[0, T)$ with $u(0) = 0$, then $K_{u^{-1}(t)}, 0 \leq t < u(T)$, is also a Loewner chain in D , which is called a time-change of (K_t) via u . If W maps D conformally onto E , then $W(K_t), 0 \leq t < T$, is a Loewner chain in E .

An \mathbb{H} - (resp. \mathbb{D} -) Loewner chain is a Loewner chain in \mathbb{H} (resp. \mathbb{D}) such that each hull is an \mathbb{H} - (resp. \mathbb{D} -) hull. An \mathbb{H} - (resp. \mathbb{D} -) Loewner chain (K_t) is said to be normalized if $\text{hcap}(K_t) = 2t$ (resp. $\text{dcap}(K_t) = t$) for each t .

The conditions for the conformal invariance property of \mathbb{H} - (resp. \mathbb{D} -) Loewner chains can be slightly weakened as below.

Proposition 3.7 *If $K_t, 0 \leq t < T$, is an \mathbb{H} - (resp. \mathbb{D} -) Loewner chain, and W is an \mathbb{R} - (resp. \mathbb{T} -) symmetric conformal map, whose domain contains K_t^{db} for each t and*

whose image does not contain ∞ (resp. 0), then $W(K_t), 0 \leq t < T$, is also an \mathbb{H} - (resp. \mathbb{D} -)Loewner chain.

The following proposition combines some results in [8, 12].

Proposition 3.8 *Let $T \in (0, \infty]$. The following are equivalent.*

- (i) $K_t, 0 \leq t < T$, are chordal (resp. radial) Loewner hulls driven by some $\lambda \in C([0, T])$.
- (ii) $K_t, 0 \leq t < T$, is a normalized \mathbb{H} - (resp. \mathbb{D} -)Loewner chain.

If either of the above holds, with $\mathring{\lambda}(t) = \lambda(t)$ (resp. $\mathring{\lambda}(t) = e^{i\lambda(t)}$ in the radial case) we have

$$\{\mathring{\lambda}(t)\} = \bigcap_{\varepsilon > 0} \overline{K_{t+\varepsilon}/K_t}, \quad 0 \leq t < T.$$

In addition, if $K_t, 0 \leq t < T$, is any \mathbb{H} - (resp. \mathbb{D} -)Loewner chain, then the function $u(t) := \text{hcap}(K_t)/2$ (resp. $u(t) := \text{dcap}(K_t)$), $0 \leq t < T$, is continuous increasing with $u(0) = 0$, which implies that $K_{u^{-1}(t)}, 0 \leq t < u(T)$, is a normalized \mathbb{H} - (resp. \mathbb{D} -)Loewner chain.

Definition 3.9 A family of \mathbb{H} - (resp. \mathbb{D} -)hulls: $L_t, 0 \leq t < T$, is called a backward \mathbb{H} - (resp. \mathbb{D} -)Loewner chain if they satisfy

1. $L_0 = \emptyset$ and $L_{t_1} \subset L_{t_2}$ if $0 \leq t_1 \leq t_2 < T$;
2. $L_{t_0} : L_{t_0-t}, 0 \leq t \leq t_0$, is an \mathbb{H} - (resp. \mathbb{D} -)Loewner chain for any $t_0 \in (0, T)$.

If u is a continuous increasing function defined on $[0, T)$ with $u(0) = 0$, then $L_{u^{-1}(t)}, 0 \leq t < u(T)$, is also a backward \mathbb{H} - (resp. \mathbb{D} -)Loewner chain, and is called a time-change of (L_t) via u . A backward \mathbb{H} - (resp. \mathbb{D} -)Loewner chain (L_t) is said to be normalized if $\text{hcap}(L_t) = 2t$ (resp. $\text{dcap}(L_t) = t$) for any $t \in [0, T)$.

Using Lemma 3.3 and Proposition 3.8, we obtain the following.

Proposition 3.10 *Let $T \in (0, \infty]$. The following are equivalent.*

- (i) $L_t, 0 \leq t < T$, are backward chordal (resp. radial) Loewner hulls driven by some $\lambda \in C([0, T])$.
- (ii) $L_t, 0 \leq t < T$, is a normalized backward \mathbb{H} - (resp. \mathbb{D} -)Loewner chain.

If either of the above holds, with $\mathring{\lambda}(t) = \lambda(t)$ (resp. $\mathring{\lambda}(t) = e^{i\lambda(t)}$ in the radial case) we have

$$\{\mathring{\lambda}(t)\} = \bigcap_{\varepsilon > 0} \overline{L_t : L_{t-\varepsilon}}, \quad 0 < t < T, \tag{3.9}$$

In addition, if $L_t, 0 \leq t < T$, is any backward \mathbb{H} - (resp. \mathbb{D} -)Loewner chain, then the function $u(t) := \text{hcap}(K_t)/2$ (resp. $u(t) := \text{dcap}(K_t)$), $0 \leq t < T$, is continuous increasing with $u(0) = 0$, which implies that $L_{u^{-1}(t)}, 0 \leq t < u(T)$, is a normalized backward \mathbb{H} - (resp. \mathbb{D} -)Loewner chain.

We say that f_t and L_t , $0 \leq t < T$, are backward chordal (resp. radial) Loewner maps and hulls, via a time change u , driven by λ , if u is a continuous increasing function defined on $[0, T)$ with $u(0) = 0$, such that $f_{u^{-1}(t)}$ and $L_{u^{-1}(t)}$, $0 \leq t < u(T)$, are backward chordal (resp. radial) Loewner maps and hulls driven by $\lambda \circ u^{-1}$. From the above proposition, if (L_t) is any \mathbb{H} -(resp. \mathbb{D} -)Loewner chain, then L_t , $0 \leq t < T$, are backward chordal (resp. radial) Loewner hulls, via a time change $u(t) := \text{hcap}(L_t)/2$ (resp. $\text{dcap}(L_t)$), driven by λ , which satisfies (3.9).

3.5 Simple curves and weldings

An \mathbb{H} -simple (resp. \mathbb{D} -simple) curve is a half-open simple curve in \mathbb{H} (resp. $\mathbb{D} \setminus \{0\}$), whose open side approaches a single point on \mathbb{R} (resp. \mathbb{T}). Every \mathbb{H} (resp. \mathbb{D} -)simple curve β is an \mathbb{H} (resp. \mathbb{D} -)hull, whose base B_β is a single point, and whose support S_β is an \mathbb{R} (resp. \mathbb{T} -)interval. Here an \mathbb{T} -interval is an arc on \mathbb{T} . The function f_β extends continuously from \mathbb{H} (resp. \mathbb{D}) to $\overline{\mathbb{H}}$ (resp. $\overline{\mathbb{D}}$), which maps S_β onto $\overline{\beta}$, sends the two ends of S_β to B_β , and sends a unique point, say $z_\beta \in S_\beta$ to the tip of β . The point z_β divides S_β into two \mathbb{R} (resp. \mathbb{T} -)intervals such that the restriction of f_β to either interval is a homeomorphism onto $\overline{\beta}$. Thus, there is a unique involution (an auto homeomorphism whose inverse is itself) ϕ_β of S_β , which fixes only one point: z_β , swaps the two end points of S_β , and satisfies that $y = \phi_\beta(x)$ implies that $f_\beta(x) = f_\beta(y)$. We call ϕ_β the welding induced by β .

Suppose K is an \mathbb{H} - or \mathbb{D} -simple curve. Let W be as in Theorems 2.12, 2.20, or 2.23. Then $W^{\mathcal{H}}(K)$ is also an \mathbb{H} - or \mathbb{D} -simple curve. The equality $W^K \circ f_K = f_{W^{\mathcal{H}}(K)} \circ W$ holds after continuous extension from \mathbb{H} or \mathbb{D} to its closure. So the weldings induced by K and $W^{\mathcal{H}}(K)$ satisfy $\phi_{W^{\mathcal{H}}(K)} = W \circ \phi_K \circ W^{-1}$.

Suppose the hulls (L_t) generated by a backward chordal (resp. radial) Loewner process driven by λ are all \mathbb{H} (resp. \mathbb{D} -)simple curves. Then the process generates backward chordal (resp. radial) traces (β_t) such that every β_t is a simple curve, and $L_t = \beta_t([0, t))$, $0 \leq t < T$.

Let ϕ_t be the welding induced by L_t , which is an involution of $S_t := S_{L_t}$. Recall that (S_t) is an increasing family because $L_{t_1} < L_{t_2}$ for $t_1 < t_2$. If $0 \leq t_1 < t_2 < T$, then from $f_{t_2, t_1} \circ f_{t_1} = f_{t_2}$ we see that $\phi_{t_2}|_{S_{t_1}} = \phi_{t_1}$. Thus, there is a unique involution ϕ of $S_T := \bigcup_t S_t$ such that $\phi|_{S_t} = \phi_t$ for each $t \in [0, T)$. In other words, $y = \phi(x)$ implies that $f_t(x) = f_t(y)$ for some $t \geq 0$, where f_t is the continuous extension of the Loewner map at time t from \mathbb{H} (resp. \mathbb{D}) to $\overline{\mathbb{H}}$ (resp. $\overline{\mathbb{D}}$). We say that ϕ is the welding induced by this process. In the case that $S_T = \mathbb{R}$ (resp. $\mathbb{T} \setminus \{z_0\}$ for some $z_0 \in \mathbb{T}$), we will extend ϕ to an involution of $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ (resp. \mathbb{T}) such that ∞ (resp. z_0) is the other fixed point of ϕ .

Here is another way to view the welding ϕ . For every $t \in (0, T)$, ϕ swaps the two end points of S_t . Let $\dot{\lambda}(0) = \lambda(0)$ (resp. $e^{i\lambda(0)}$). Since $f_t(\dot{\lambda}(0)) = \beta_t(0)$ is the tip of L_t for each t , we see that $\dot{\lambda}(0)$ is the only fixed point of ϕ . On the other hand, it is easy to see that, x and y are end points of S_t if and only if $\tau_x = \tau_y = t$, $0 < t < T$; and every point on $S_T \setminus \{\dot{\lambda}(0)\}$ is an end point of some S_t , $0 < t < T$. Thus, for $x \neq y \in S_T \setminus \{\dot{\lambda}(0)\}$, $y = \phi(x)$ if and only if $\tau_x = \tau_y$, i.e., x and y are swallowed at the same time.

Let $\kappa \in (0, 4]$. Since the backward chordal (resp. radial) SLE_κ traces are \mathbb{H} (resp. \mathbb{D} -)simple curves, so the process induces a random welding, which we call a backward chordal (resp. radial) SLE_κ welding. In the chordal case, For any $x \in \mathbb{R} \setminus \{\lambda(0)\} = \mathbb{R} \setminus \{0\}$, the process $X_t^x := \lambda(t) - f_t(x)$ is a rescaled Bessel process of dimension $1 - \frac{4}{\kappa} < 1$, which implies that a.s. $X_t^x \rightarrow 0$ at some finite time. Thus, $S_\infty = \mathbb{R}$. which implies that a chordal SLE_κ welding is an involution of $\widehat{\mathbb{R}}$ with two fixed points: $\lambda(0) = 0$ and ∞ . In the radial case, since $T = \infty$, Lemma 3.5 says that $S_\infty = \mathbb{T}$ or $\mathbb{T} \setminus \{z_0\}$ for some $z_0 \in \mathbb{T}$. The first case can not happen since ϕ has only one fixed point on S_∞ . Thus, a radial SLE_κ welding is an involution of \mathbb{T} with two fixed points, one of which is $e^{i\lambda(0)} = 1$.

Suppose a backward chordal (resp. radial) Loewner process generates \mathbb{H} (resp. \mathbb{D} -)simple backward traces $\beta_t, 0 \leq t < T$. Then (3.5) (resp. (3.8)) is satisfied because $\beta_{t_2}(t_1)$ lies in \mathbb{H} (resp. \mathbb{D}) if $t_2 > t_1$. It is clear that the curve β defined by (3.6) is simple, and $D_t = D_T \setminus \beta([t, T])$ for $0 \leq t < T$. Let ϕ be the welding induced by the process. If $y = \phi(x)$, there is $t \in [0, T)$ such that $y, x \in S_t$ and $f_t(y) = f_t(x)$. From $F_{T,t} \circ f_t = F_T$, we get $F_T(y) = F_T(x)$. This means that ϕ can be realized by the conformal map F_T .

If a backward chordal (resp. radial) Loewner chain (L_t) is composed of \mathbb{H} (resp. \mathbb{D} -)simple curves, then (L_t) induces a welding ϕ , which is an involution of $\bigcup S_{L_t}$, and agrees with ϕ_{L_t} on S_{L_t} for each t . To see this, one may first normalized the backward Loewner chain so that it is generated by a backward Loewner process.

4 Conformal transformations

In this section, we will study how a backward $SLE(\kappa; \rho)$ process changes under a Möbius transformation, and derive the backward SLE counterpart of the results of [17]. For this purpose, we will first define the conformal transformation of a backward Loewner chain. The rest of the arguments are the same as the ones for forward SLE, up to negating κ . We will use the ideas in [17] and some results in [8].

Proposition 4.1 *Suppose $L_t, 0 \leq t < T$, is a backward \mathbb{H} -(resp. \mathbb{D} -)Loewner chain, W is an \mathbb{R} -(resp. \mathbb{T} -)symmetric conformal map whose domain contains every S_{L_t} , and $\infty \notin W(S_{L_t})$ for $0 \leq t < T$. Then $W^{\mathcal{H}}(L_t), 0 \leq t < T$, is also a backward \mathbb{H} -(resp. \mathbb{D} -)Loewner chain.*

Proof From Theorem 2.12, W^{L_t} and $W^{\mathcal{H}}(L_t)$ are well defined. Since $L_0 = \emptyset$, $W^{\mathcal{H}}(L_0) = \emptyset$. Let $0 \leq t_1 \leq t_2 < T$. Since $L_{t_1} \prec L_{t_2}$, from Lemma 2.14, $W^{\mathcal{H}}(L_{t_1}) \prec W^{\mathcal{H}}(L_{t_2})$. Fix $t_0 \in (0, T)$. Since $L_{t_0} : L_{t_0-t}, 0 \leq t \leq t_0$, is an \mathbb{H} -(resp. \mathbb{D} -)Loewner chain, from Lemma 2.14 and Proposition 3.7 we see that

$$W^{\mathcal{H}}(L_{t_0}) : W^{\mathcal{H}}(L_{t_0-t}) = W^{L_{t_0}}(L_{t_0} : L_{t_0-t}), \quad 0 \leq t \leq t_0,$$

is also an \mathbb{H} -(resp. \mathbb{D} -)Loewner chain. This finishes the proof. □

We call $W^{\mathcal{H}}(L_t), 0 \leq t < T$, the conformal transformation of $L_t, 0 \leq t < T$, under W . Using Lemma 2.21 instead of Lemma 2.14, we can show that a similar proposition holds.

Proposition 4.2 *Suppose $L_t, 0 \leq t < T$, is a backward \mathbb{H} -(resp. \mathbb{D} -)Loewner chain, W is a Möbius transform that maps \mathbb{H} onto \mathbb{D} (resp. maps \mathbb{D} onto \mathbb{H}) such that $\infty \notin W(S_{L_t})$ for $0 \leq t < T$. Then $W^{\mathcal{H}}(L_t), 0 \leq t < T$, is a backward \mathbb{D} -(resp. \mathbb{H} -)Loewner chain.*

Suppose (L_t) is composed of \mathbb{H} - or \mathbb{D} -simple curves. Then $(W^{\mathcal{H}}(L_t))$ is also composed of \mathbb{H} or \mathbb{D} -simple curves. Let ϕ and ϕ_W be the weldings induced by these two chains, which are involutions of $\bigcup S_{L_t}$ and $\bigcup S_{W^{\mathcal{H}}(L_t)}$, respectively. Since for each $t \in (0, T)$, $\phi|_{S_{L_t}} = \phi_{L_t}$, $\phi_W|_{S_{W^{\mathcal{H}}(L_t)}} = \phi_{W^{\mathcal{H}}(L_t)}$, $S_{W^{\mathcal{H}}(L_t)} = W(S_{L_t})$, and $\phi_{W^{\mathcal{H}}(L_t)} = W \circ \phi_{L_t} \circ W^{-1}$, we see that $\bigcup S_{W^{\mathcal{H}}(L_t)} = W(\bigcup S_{L_t})$ and

$$\phi_W = W \circ \phi \circ W^{-1}. \tag{4.1}$$

This means that the conformal transformation preserves the welding.

The following proposition is essentially Lemma 2.8 in [8].

Proposition 4.3 *Let W be an \mathbb{R} -symmetric conformal map, whose domain contains $z_0 \in \mathbb{R}$, such that $W(z_0) \neq \infty$. Then*

$$\lim_{H \rightarrow z_0} \frac{\text{hcap}(W(H))}{\text{hcap}(H)} = |W'(z_0)|^2,$$

where $H \rightarrow z_0$ means that $\text{diam}(H \cup \{z_0\}) \rightarrow 0$ with H being a nonempty \mathbb{H} -hull.

Using the integral formulas for capacities of \mathbb{H} -hulls and \mathbb{D} -hulls, it is not hard to derive the following similar proposition.

Proposition 4.4 (i) *Let W be a conformal map on a \mathbb{T} -symmetric domain Ω , which satisfies $I_{\mathbb{R}} \circ W = W \circ I_{\mathbb{T}}$ and $W(\Omega \cap \mathbb{D}) \subset \mathbb{H}$. Let $z_0 \in \Omega \cap \mathbb{T}$ be such that $W(z_0) \neq \infty$. Then*

$$\lim_{H \rightarrow z_0} \frac{\text{hcap}(W(H))}{\text{dcap}(H)} = 2|W'(z_0)|^2,$$

where $H \rightarrow z_0$ means that $\text{diam}(H \cup \{z_0\}) \rightarrow 0$ with H being a nonempty \mathbb{D} -hull.

(ii) *Proposition 4.3 holds with \mathbb{R} replaced by \mathbb{T} , hcap replaced by dcap, and $H \rightarrow z_0$ understood as in (i).*

4.1 Transformations between backward \mathbb{H} -Loewner chains

Suppose L_t and $f_t, 0 \leq t < T$, are backward chordal Loewner hulls and maps driven by $\lambda \in C([0, T])$. From Proposition 3.10, (L_t) is a backward \mathbb{H} -Loewner chain. Let W be an \mathbb{R} -symmetric conformal map, whose domain Ω contains the support of every L_t . Write W_t for W^{L_t} . The domain of W_t is Ω^{L_t} , which contains L_t^{db} . If $t > 0$, $\lambda(t) \in L_t^{\text{db}}$, so $\lambda(t)$ is contained in the domain of W_t . This is also true for $t = 0$ because $W_0 = W$

and $\{\lambda(0)\} = S_0 \subset S_t = S_{L_t} \subset \Omega$ for any $t \in (0, T)$. Let $L_t^* = W^{\mathcal{H}}(L_t) = W_t(L_t)$, $0 \leq t < T$. From Proposition 4.1, (L_t^*) is a backward \mathbb{H} -Loewner chain, and

$$W_t(L_t : L_{t-\varepsilon}) = L_t^* : L_{t-\varepsilon}^*, \quad 0 \leq t - \varepsilon < t < T. \tag{4.2}$$

From Proposition 3.10, L_t^* , $0 \leq t < T$, are backward chordal Loewner hulls via a time change $u(t) := \text{hcap}(L_t^*)/2$, driven by some λ^* , which satisfies

$$\{\lambda^*(t)\} = \bigcap_{\varepsilon > 0} \overline{L_t^* : L_{t-\varepsilon}^*}, \quad 0 < t < T.$$

From (3.9), (4.2), and continuity, we find that

$$\lambda^*(t) = W_t(\lambda(t)), \quad 0 \leq t < T. \tag{4.3}$$

Since (L_t) and $(L_{u^{-1}(t)}^*)$ are normalized, we know that $\text{hcap}(L_t : L_{t-\varepsilon}) = 2\varepsilon$ and $\text{hcap}(L_t^* : L_{t-\varepsilon}^*) = 2u(t) - 2u(t - \varepsilon)$. From (4.2) and Proposition 4.3, we find that

$$u'(t) = W_t'(\lambda(t))^2, \quad 0 \leq t < T. \tag{4.4}$$

Let $f_t^* = f_{L_t^*}$. From the definition of $W_t = W^{L_t}$, we have the equality

$$W_t \circ f_t = f_t^* \circ W, \tag{4.5}$$

which holds in $\Omega \setminus S_{L_t}$. Differentiating (4.5) w.r.t. t , and using (4.3) and (4.4), we get

$$\begin{aligned} [\partial_t W_t](f_t(z)) + W_t'(f_t(z)) \frac{-2}{f_t(z) - \lambda(t)} &= \frac{-2u'(t)}{f_t^*(W(z)) - \lambda^*(t)} \\ &= \frac{-2W_t'(\lambda(t))^2}{W_t(f_t(z)) - W_t(\lambda(t))}. \end{aligned}$$

Thus, for any $w = f_t(z) \in f_t(\Omega \setminus S_{L_t}) = \Omega^{L_t} \setminus L_t^{\text{db}}$,

$$\partial_t W_t(w) = \frac{-2W_t'(\lambda(t))^2}{W_t(w) - W_t(\lambda(t))} - W_t'(w) \frac{-2}{w - \lambda(t)}. \tag{4.6}$$

By analytic extension, the above equality holds for any $w \in \Omega^{L_t} \setminus \{\lambda(t)\}$. Letting $w \rightarrow \lambda(t)$, we find that

$$\partial_t W_t(\lambda(t)) = 3W_t''(\lambda(t)), \quad 0 \leq t < T. \tag{4.7}$$

Differentiating (4.6) w.r.t. w and letting $w \rightarrow \lambda(t)$, we get

$$\frac{\partial_t W_t'(\lambda(t))}{W_t'(\lambda(t))} = -\frac{1}{2} \left(\frac{W_t''(\lambda(t))}{W_t'(\lambda(t))} \right)^2 + \frac{4}{3} \frac{W_t'''(\lambda(t))}{W_t'(\lambda(t))}. \tag{4.8}$$

4.2 Transformations involving backward \mathbb{D} -Loewner chains

Now suppose $L_t, 0 \leq t < T$, are backward radial Loewner hulls driven by λ . Let f_t and \tilde{f}_t be the corresponding radial Loewner maps and covering maps. Suppose W is a \mathbb{T} -symmetric conformal map, whose domain Ω contains the support of every L_t . Let $W_t = W^{L_t}, L_t^* = W_t(L_t) = W^{\mathcal{H}}(L_t)$, and $u(t) = \text{dcap}(L_t^*), 0 \leq t < T$. Then $L_t^*, 0 \leq t < T$, are backward radial Loewner hulls via a time change $u(t) := \text{dcap}(L_t^*)$, driven by some λ^* , which satisfies

$$\{e^{i\lambda^*(t)}\} = \bigcap_{\varepsilon>0} \overline{L_t^* : L_{t-\varepsilon}^*}, \quad 0 < t < T.$$

Let f_t^* (resp. \tilde{f}_t^*), $0 \leq t < T$, denote the backward radial (resp. covering radial) Loewner hulls via the time change u driven by λ^* . The argument in the last subsection still works with Proposition 4.4 in place of Proposition 4.3. We can conclude that $e^{i\lambda(t)}$ lies in the domain of W_t for $0 \leq t < T$; $W_t(e^{i\lambda(t)}) = e^{i\lambda^*(t)}$; $u'(t) = |W_t'(e^{i\lambda(t)})|^2$; and (4.5) still holds. Suppose \tilde{W} is an \mathbb{R} -symmetric conformal map defined on $\tilde{\Omega} = (e^i)^{-1}(\Omega)$, which satisfies $e^i \circ \tilde{W} = W \circ e^i$. Define \tilde{W}_t to be the analytic extension of $\tilde{f}_t^* \circ \tilde{W} \circ \tilde{f}_t^{-1}$ to $\tilde{\Omega}_t := (e^i)^{-1}(\Omega^{L_t})$. Then we get

$$\tilde{W}_t \circ \tilde{f}_t = \tilde{f}_t^* \circ \tilde{W}; \tag{4.9}$$

Comparing this with (4.5) we find $e^i \circ \tilde{W}_t = W_t \circ e^i$. So $\lambda(t)$ lies in the domain of \tilde{W}_t , and

$$u'(t) = \tilde{W}_t'(\lambda(t))^2, \quad 0 \leq t < T. \tag{4.10}$$

Since $W_t(e^{i\lambda(t)}) = e^{i\lambda^*(t)}$, from the continuity, there is $n \in \mathbb{N}$ such that $\tilde{W}_t(\lambda(t)) = \lambda^*(t) + 2n\pi$ for $0 \leq t < T$. Since λ^* and $\lambda^* + 2n\pi$ generate the same backward radial Loewner objects via the time change u , by replacing λ^* with $\lambda^* + 2n\pi$, we may assume that

$$\tilde{W}_t(\lambda(t)) = \lambda^*(t), \quad 0 \leq t < T. \tag{4.11}$$

Differentiating (4.9) w.r.t. t and letting $w = \tilde{f}_t(z)$, we get

$$\partial_t \tilde{W}_t(w) = -\tilde{W}_t'(\lambda(t))^2 \cot_2(\tilde{W}_t(w) - \tilde{W}_t(\lambda(t))) + \tilde{W}_t'(w) \cot_2(w - \lambda(t)), \tag{4.12}$$

which holds for $w \in (e^i)^{-1}(\Omega^{L_t} \setminus \{e^{i\lambda(t)}\})$. Letting $w \rightarrow \lambda(t)$, we get

$$\partial_t \tilde{W}_t(\lambda(t)) = 3\tilde{W}_t''(\lambda(t)), \quad 0 \leq t < T. \tag{4.13}$$

Differentiating (4.12) w.r.t. w and letting $w \rightarrow \lambda(t)$, we get

$$\frac{\partial_t \tilde{W}_t'(\lambda(t))}{\tilde{W}_t'(\lambda(t))} = -\frac{1}{2} \left(\frac{W_t''(\lambda(t))}{W_t'(\lambda(t))} \right)^2 + \frac{4}{3} \frac{W_t'''(\lambda(t))}{W_t'(\lambda(t))} + \frac{1}{6} \tilde{W}_t'(\lambda(t))^2 - \frac{1}{6}. \tag{4.14}$$

The number $\frac{1}{6}$ comes from the Laurent series of $\cot_2(z): \frac{2}{z} - \frac{z}{6} + O(z^3)$.

Let $(L_t), (f_t)$, and (\tilde{f}_t) be as above. Now suppose W is a Möbius transformation that maps \mathbb{D} onto \mathbb{H} such that $W^{-1}(\infty) \notin S_{L_t}$ for every t . Let W^{L_t} be as in Theorem 2.20. Let $W_t = W^{L_t}$ and $L_t^* = W_t(L_t) = W^{\tilde{L}_t}(L_t), 0 \leq t < T$. Then $L_t^*, 0 \leq t < T$, are backward chordal Loewner hulls via a time change $u(t) := \text{hcap}(L_t^*)/2$, driven by some λ^* . Let $f_t^* = f_{L_t^*}$. Then (4.5) still holds, and we have $u'(t) = |W_t'(e^{i\lambda(t)})|^2$ and $W_t(e^{i\lambda(t)}) = e^{i\lambda^*(t)}$. Let $\tilde{W} = W \circ e^i$ and $\tilde{W}_t = W_t \circ e^i$. We get (4.10), (4.11), and $\tilde{W}_t \circ \tilde{f}_t = f_t^* \circ \tilde{W}$. Differentiating this equality w.r.t. t and letting $w = \tilde{f}_t(z)$ tend to $\lambda(t)$, we find that (4.13) still holds.

4.3 Möbius invariance of backward SLE($\kappa; \rho$) processes

We now define backward chordal and radial SLE($\kappa; \rho$) processes, where $\rho = (\rho_1, \dots, \rho_n) \in \mathbb{R}^n$. Let $x_0, q_1, \dots, q_n \in \mathbb{R}$ such that $q_k \neq x_0$ for all k . Let $\lambda(t), 0 \leq t < T$, be the maximal solution of the equation

$$d\lambda(t) = \sqrt{\kappa}dB(t) + \sum_{k=1}^n \frac{-\rho_k}{\lambda(t) - f_t^\lambda(q_k)} dt; \quad \lambda(0) = x_0. \tag{4.15}$$

Here $f_t^\lambda, 0 \leq t < T$, are the backward chordal Loewner maps driven by λ . Then we call the backward chordal Loewner process driven by λ the chordal SLE($\kappa; \rho$) process started from x_0 with force points (q_1, \dots, q_n) , or simply started from $(x_0; q_1, \dots, q_n)$. We allow some q_k to be ∞ . In that case, $f_t^\lambda(q_k)$ is always ∞ , and the term $\frac{-\rho_k}{\lambda(t) - f_t^\lambda(q_k)}$ vanishes.

Let $x_0, q_1, \dots, q_n \in \mathbb{R}$ be such that $q_k \notin x_0 + 2\pi\mathbb{Z}$ for all k . Let $\lambda(t), 0 \leq t < T$, be the maximal solution of the equation

$$d\lambda(t) = \sqrt{\kappa}dB(t) + \sum_{k=1}^n \frac{-\rho_k}{2} \cot_2(\lambda(t) - \tilde{f}_t^\lambda(q_k)) dt; \quad \lambda(0) = x_0. \tag{4.16}$$

Here $\tilde{f}_t^\lambda, 0 \leq t < T$, are the covering backward radial Loewner maps driven by λ . Then the backward radial Loewner process driven by λ is called the radial SLE($\kappa; \rho$) process started from e^{ix_0} with marked points $(e^{iq_1}, \dots, e^{iq_n})$, or simply started from $(e^{ix_0}; e^{iq_1}, \dots, e^{iq_n})$.

The existence of backward chordal (resp. radial) SLE $_\kappa$ traces and Girsanov’s Theorem imply the existence of a backward chordal (resp. radial) SLE($\kappa; \rho$) traces. The traces are \mathbb{H} (resp. \mathbb{D})-simple curves if $\kappa \in (0, 4]$.

The following lemma is easy to check.

Lemma 4.5 *Let W be a Möbius transformation. Then the following hold.*

- (i) *For any $z \in \mathbb{C} \cap W^{-1}(\mathbb{C})$ and $w \in \hat{\mathbb{C}}$,*

$$\frac{2W'(z)}{W(z) - W(w)} - \frac{2}{z - w} = \frac{W''(z)}{W'(z)}.$$

(ii) Let $\tilde{W} = W \circ e^i$. For any $z \in \mathbb{C} \cap \tilde{W}^{-1}(\mathbb{C})$ and $w \in \mathbb{C}$,

$$\frac{2\tilde{W}'(z)}{\tilde{W}(z) - \tilde{W}(w)} - \cot_2(z - w) = \frac{\tilde{W}''(z)}{\tilde{W}'(z)}.$$

(iii) Suppose an analytic function $\tilde{W} : \Omega \rightarrow \mathbb{C}$ satisfies $e^i \circ \tilde{W} = W \circ e^i$ in Ω . Then for any $z, w \in \Omega$,

$$\tilde{W}'(z) \cot_2(\tilde{W}(z) - \tilde{W}(w)) - \cot_2(z - w) = \frac{\tilde{W}''(z)}{\tilde{W}'(z)}.$$

Theorem 4.6 Let $L_t, 0 \leq t < T$, be the backward chordal SLE($\kappa; \rho$) hulls started from $(x_0; q_1, \dots, q_n)$. Suppose $\sum \rho_k = -\kappa - 6$. Let W be a Möbius transformation from \mathbb{H} onto \mathbb{H} such that $\{\infty, W^{-1}(\infty)\} \subset \{q_1, \dots, q_n\}$. Then, after a time change, $W^{\mathcal{H}}(L_t), 0 \leq t < T$, are the backward chordal SLE($\kappa; \rho$) hulls started from $(W(x_0); W(q_1), \dots, W(q_n))$.

Proof Since $W^{-1}(\infty)$ is a force point, it is not contained in the support of any L_t . So $\infty \notin W(S_{L_t}), 0 \leq t < T$. Let λ be the driving function, and $f_t = f_t^\lambda, 0 \leq t < T$, be the corresponding maps. We may and now adopt the notation in Sect. 4.1. Let (\mathcal{F}_t) be the complete filtration generated by $B(t)$ in (4.15). Then (λ_t) and (L_t) are (\mathcal{F}_t) -adapted. From Corollary 2.19 (i), $(W^{\mathcal{H}}(L_t))$ is also (\mathcal{F}_t) -adapted. Since $W_t = W^{L_t} = f_{W^{\mathcal{H}}(L_t)} \circ W \circ g_{L_t}$ on $\Omega^{L_t} \setminus L_t^{db}$, (W_t) is (\mathcal{F}_t) -adapted. So we may apply Itô's formula (c.f. [14]). From (4.3) and (4.7), we get

$$d\lambda^*(t) = W'_t(\lambda(t))d\lambda(t) + \left(\frac{\kappa}{2} + 3\right) W''_t(\lambda(t))dt, \quad 0 \leq t < T.$$

Applying (4.15) and Lemma 4.5 (i), and using the condition that $\sum \rho_k = -\kappa - 6$, we find that

$$\begin{aligned} d\lambda^*(t) &= W'_t(\lambda(t))\sqrt{\kappa}dB(t) + \sum_{k=1}^n \frac{-\rho_k W'_t(\lambda(t))^2}{W_t(\lambda(t)) - W_t \circ f_t^\lambda(q_k)} dt \\ &= W'_t(\lambda(t))\sqrt{\kappa}dB(t) + \sum_{k=1}^n \frac{-\rho_k W'_t(\lambda(t))^2}{\lambda^*(t) - f_t^* \circ W(q_k)} dt, \quad 0 \leq t < T. \end{aligned}$$

From (4.3) we get $\lambda^*(0) = W_0(\lambda(0)) = W(x_0)$. Since $L_t^* = W^{\mathcal{H}}(L_t)$ and f_t^* are backward chordal Loewner hulls and maps via the time change u driven by λ^* , from (4.4) and the above equation, we conclude that, after a time change, $W^{\mathcal{H}}(L_t), 0 \leq t < T$, are the backward chordal SLE($\kappa; \rho$) hulls started from $(W(x_0); W(q_1), \dots, W(q_n))$ and stopped at some time.

It remains to show that the above process is completed. If not, the process can be extended without swallowing the force points $W(q_1), \dots, W(q_n)$. From the condition, $W(\infty)$ is among these force points. So $(W^{-1})^{\mathcal{H}}$ is well defined at the hulls of the

extended process. From Propositions 4.1 and 3.10, this implies that the backward chordal Loewner hulls L_t , $0 \leq t < T$, can be extended without swallowing any of q_1, \dots, q_n , which is a contradiction. \square

The following theorem can be proved using the above proof with minor modifications: we now use the argument in Sect. 4.2 instead of that in Sect. 4.1, apply Lemma 4.5 (ii) and (iii) instead of (i), and use Proposition 4.2 in addition to Proposition 4.1.

Theorem 4.7 *Suppose $\sum \rho_k = -\kappa - 6$. Let (L_t) be the backward radial SLE($\kappa; \rho$) hulls started from $(e^{ix_0}; e^{iq_1}, \dots, e^{iq_n})$. Let W map \mathbb{D} conformally onto \mathbb{H} (resp. \mathbb{D}) such that $\{W^{-1}(\infty)\} \cap \mathbb{T} \subset \{e^{iq_1}, \dots, e^{iq_n}\}$. Then, after a time change, $(W^{\mathcal{H}}(L_t))$ are the backward chordal (resp. radial) SLE($\kappa; \rho$) hulls started from $(W(e^{ix_0}); W(e^{iq_1}), \dots, W(e^{iq_n}))$.*

Corollary 4.8 *Let (L_t) be the backward radial SLE($\kappa; -\kappa - 6$) hulls started from $(e^{ix_0}; e^{iq_0})$. Let W map \mathbb{D} conformally onto \mathbb{H} such that $W(e^{ix_0}) = 0$ and $W(e^{iq_0}) = \infty$. Then, after a time change, $(W^{\mathcal{H}}(L_t))$ are the backward chordal SLE $_{\kappa}$ hulls started from 0.*

Remarks

1. The above theorems are the backward SLE counterpart of the work in [17] for forward SLE($\kappa; \rho$) processes. The condition in their paper for Möbius invariance is $\sum \rho_k = \kappa - 6$. This is one reason why we may view backward SLE $_{\kappa}$ as SLE $_{-\kappa}$.
2. The definition of backward SLE($\kappa; \rho$) process differ from Sheffield’s definition in [18] by a minus sign in (4.15) and (4.16) before the ρ_k ’s. If Sheffield’s definition were used, the condition for conformal invariance would be $\sum \rho_k = \kappa + 6$ instead of $\sum \rho_k = -\kappa - 6$.
3. We may allow interior force points as in [17]. For the chordal (resp. radial) SLE($\kappa; \rho$) process, if $q_k \in \mathbb{H}$ (resp. $e^{iq_k} \in \mathbb{D}$) is a force point, we use $\text{Re } f_t^{\lambda}(q_k)$ (resp. $\text{Re } \tilde{f}_t^{\lambda}(q_k)$) instead of $f_t^{\lambda}(q_k)$ (resp. $\tilde{f}_t^{\lambda}(q_k)$) in (4.15) (resp. (4.16)). In the radial case, adding 0 to be a force point or change the force for 0 does not affect the process. Theorems 4.6 and 4.7 still hold if some or all force points lie inside \mathbb{H} or \mathbb{D} . For the proofs, we apply Lemma 4.5 with real parts taken on the displayed formulas. One particular example is the following corollary.

Corollary 4.9 *Let L_t , $0 \leq t < \infty$, be a backward radial SLE $_{\kappa}$ process. Let W be a Möbius transformation that maps \mathbb{D} onto \mathbb{H} such that $W(1) \neq \infty$. Let T be the maximum number such that $W^{-1}(\infty) \notin S_{L_t}$, $0 \leq t < T$. Then, after a time change, $W^{\mathcal{H}}(L_t)$, $0 \leq t < T$, are the backward chordal SLE($\kappa; -\kappa - 6$) hulls started from $(W(1); W(0))$.*

4. Using the properties of Bessel process and applying Girsanov’s theorem, one may define backward chordal or radial SLE($\kappa; \rho$) processes with exactly one degenerate force point, if the corresponding force ρ_1 satisfies $\rho_1 < -2$ (which corresponds to a Bessel or Bessel-like process of dimension $d = 1 - \frac{4+2\rho_1}{\kappa} > 1$). Theorems 4.6 and 4.7 still hold when a degenerate force point exists. Unlike forward SLE($\kappa; \rho$) process, it is impossible to define a backward SLE($\kappa; \rho$) process with two different degenerate force points.

- Consider the radial case with one force point. Suppose the force $\rho_1 \leq -\frac{\kappa}{2} - 2$. Let $X_t = \lambda(t) - \tilde{f}_t^\lambda(q_1)$. Then X_t is a Bessel-like process with dimension $d = 1 - \frac{4+2\rho_1}{\kappa} \geq 2$, which implies that X_t never hits $2\pi\mathbb{Z}$. So $T = \infty$ and $e^{iq_1} \notin S_t$ for any t . From Lemma 3.5, $S_\infty = \mathbb{T} \setminus \{e^{iq_1}\}$. If, in addition, $\kappa \in (0, 4]$, then a backward radial SLE($\kappa; \rho_1$) process induces a random welding ϕ , which is an involution of \mathbb{T} with exactly two fixed point, $e^{i\lambda(0)}$ and e^{iq_1} , which are the initial point and the force point of the process.

5 Commutation relations

Definition 5.1 Let $\kappa_1, \kappa_2 > 0, n \in \mathbb{N}$, and $\rho_1, \rho_2 \in \mathbb{R}^n$. Let $z_1, z_2, w_k, 2 \leq k \leq n$, be distinct points on \mathbb{R} (resp. \mathbb{T}). We say that a backward chordal (resp. radial) SLE($\kappa_1; \rho_1$) started from $(z_1; z_2, w_2, \dots, w_n)$ commutes with a backward chordal (resp. radial) SLE($\kappa_2; \rho_2$) started from $(z_2; z_1, w_2, \dots, w_n)$ if there exists a coupling of two processes $(L_1(t); 0 \leq t < T_1)$ and $(L_2(t); 0 \leq t < T_2)$ such that

- For $j = 1, 2, (L_j(t), 0 \leq t < T_j)$ is a complete backward chordal (resp. radial) SLE($\kappa_j; \rho_j$) process started from $(z_j; z_{3-j}, w_2, \dots, w_n)$.
- For $j \neq k \in \{1, 2\}$, if $\bar{t}_k < T_k$ is a stopping time w.r.t. the complete filtration (\mathcal{F}_t^k) generated by $(L_k(t))$, then conditioned on $\mathcal{F}_{\bar{t}_k}^k$, after a time change, $f_k(\bar{t}_k, \cdot)^{\mathcal{H}}(L_j(t_j)), 0 \leq t_j < T_j(\bar{t}_k)$, has the distribution of a partial backward chordal (resp. radial) SLE($\kappa_j; \rho_j$) process started from

$$(f_k(\bar{t}_k, (z_j)); \mathring{\lambda}_k(\bar{t}_k), f_k(\bar{t}_k, w_2), \dots, f_k(\bar{t}_k, w_n)),$$

where $f_k(\bar{t}_k, \cdot) := f_{L_k(\bar{t}_k)}, T_j(\bar{t}_k) := \sup\{t_j < T_j : S_{L_j(t_j)} \cap S_{L_k(\bar{t}_k)} = \emptyset\}$, $\mathring{\lambda}_k(\bar{t}_k) = \lambda_k(\bar{t}_k)$ in the chordal case (resp. $e^{i\lambda_k(\bar{t}_k)}$ in the radial case), and λ_k is the driving function for $(L_k(t))$.

Here a partial backward SLE($\kappa; \rho_j$) process is a complete SLE($\kappa; \rho_j$) process stopped at a positive stopping time. If the commutation holds for any distinct points $z_1, z_2, w_k, 2 \leq k \leq n$ on \mathbb{R} (resp. \mathbb{T}), then we simply say that backward chordal (resp. radial) SLE($\kappa_1; \rho_1$) commutes with backward chordal (resp. radial) SLE($\kappa_2; \rho_2$).

Remark The definition is similar to the definition of the commutation relation between forward SLE($\kappa; \rho$) processes that first appeared in [5], where implicitly $g_k(\bar{t}_k, L_j(t_j)) = g_{L_k(\bar{t}_k)}(L_j(t_j))$ was used instead of the $f_k(\bar{t}_k, \cdot)^{\mathcal{H}}(L_j(t_j))$ here, and $T_j(\bar{t}_k)$ was defined to be the first t_j such that $L_j(t_j)^{\text{db}}$ intersects $L_k(\bar{t}_k)^{\text{db}}$.

Theorem 5.2 For any $\kappa > 0$, backward chordal (resp. radial) SLE($\kappa; -\kappa - 6$) commutes with backward chordal (resp. radial) SLE($\kappa; -\kappa - 6$).

We will prove this theorem in the next two subsections.

5.1 Ensemble

In this subsection we will study how two backward SLE processes interact with each other. The argument relies extensively on ideas and techniques from [9]. We first consider the radial case. Fix $\kappa > 0$ and $z_1 \neq z_2 \in \mathbb{T}$. Write $z_j = e^{i\tilde{z}_j}$, $j = 1, 2$. For $j = 1, 2$, let $L_j(t)$, $0 \leq t < T_j$, be a backward radial SLE($\kappa; -\kappa - 6$) process started from $(z_j; z_{3-j})$; let λ_j be the driving function, and let $f_j(t, \cdot)$ and $\tilde{f}_j(t, \cdot)$, $0 \leq t < T_j$, be the corresponding maps and covering maps. At first, we suppose that the two processes are independent. Then for $j = 1, 2$, λ_j satisfies $\lambda_j(0) = \tilde{z}_j$ and the SDE:

$$d\lambda_j(t) = \sqrt{\kappa}dB_j(t) - \frac{-\kappa - 6}{2} \cot_2(\lambda_j(t) - \tilde{f}_j(t, \tilde{z}_{3-j}))dt, \quad 0 \leq t < T_j, \tag{5.1}$$

where $B_1(t)$ and $B_2(t)$ are independent standard Brownian motions. For $j = 1, 2$, let (\mathcal{F}_t^j) denote the complete filtration generated by $B_j(t)$.

Define $\mathcal{D} = \{(t_1, t_2) \in [0, T_1] \times [0, T_2] : S_{L_1(t_1)} \cap S_{L_2(t_2)} = \emptyset\}$. Then for $(t_1, t_2) \in \mathcal{D}$, we have $(L_1(t_1), L_2(t_2)) \in \mathcal{P}_*$. So we may define

$$(L_{1,t_2}(t_1), L_{2,t_1}(t_2)) = f^{\mathcal{P}}(L_1(t_1), L_2(t_2)).$$

Let $f_{1,t_2}(t_1, \cdot) = f_{L_{1,t_2}(t_1)}$ and $f_{2,t_1}(t_2, \cdot) = f_{L_{2,t_1}(t_2)}$. From a radial version of Theorem 2.16, we see that

$$f_{1,t_2}(t_1, \cdot) \circ f_{2,t_1}(t_2, \cdot) = f_{L_1(t_1) \vee L_2(t_2)} = f_{2,t_1}(t_2, \cdot) \circ f_{1,t_1}(t_1, \cdot). \tag{5.2}$$

Recall that $L_1(t_1) \vee L_2(t_2)$ is the quotient union of $L_1(t_1)$ and $L_2(t_2)$, i.e., the unique hull which is the disjoint union of two hulls such that the corresponding two quotient hulls are $L_1(t_1)$ and $L_2(t_2)$. Fix $j \neq k \in \{1, 2\}$. From a radial version of Corollary 2.19 (ii), the random map $f_{j,t_k}(t_j, \cdot)$ is $\mathcal{F}_{t_j}^j \times \mathcal{F}_{t_k}^k$ -measurable. Let $u_{j,t_k}(t_j) = \text{dcap}(L_{j,t_k}(t_j))$. From Propositions 3.10 and 4.1, for any fixed $t_k \in [0, T_k)$, $f_{j,t_k}(t_j, \cdot)$ are backward radial Loewner maps via the time change u_{j,t_k} . Let $\tilde{f}_{j,t_k}(t_j, \cdot)$ be the corresponding covering maps. So $e^i \circ \tilde{f}_{j,t_k}(t_j, \cdot) = f_{j,t_k}(t_j, \cdot) \circ e^i$. From continuity, we see that $\tilde{f}_{j,t_k}(t_j, \cdot)$ is also $\mathcal{F}_{t_j}^j \times \mathcal{F}_{t_k}^k$ -measurable, and from (5.2) we have

$$\tilde{f}_{1,t_2}(t_1, \cdot) \circ \tilde{f}_{2,t_1}(t_2, \cdot) = \tilde{f}_{2,t_1}(t_2, \cdot) \circ \tilde{f}_{1,t_1}(t_1, \cdot). \tag{5.3}$$

Define m on \mathcal{D} by $m(t_1, t_2) = \text{dcap}(L_1(t_1) \vee L_2(t_2))$. From (5.2) we get

$$m(t_1, t_2) = u_{1,t_2}(t_1) + t_2 = u_{2,t_1}(t_2) + t_1. \tag{5.4}$$

Apply the argument in the first paragraph of Sect. 4.2 with $\lambda = \lambda_j$, $L_{t_j} = L_j(t_j)$, $W = f_k(t_k, \cdot)$, and $\tilde{W} = \tilde{f}_k(t_k, \cdot)$, where $t_k \in [0, T_k)$ is fixed. Then we have correspondence: $L_{t_j}^* = L_{j,t_k}(t_j)$, $u = u_{j,t_k}$, and $f_{t_j}^* = \tilde{f}_{j,t_k}(t_j, \cdot)$. Since \tilde{W}_{t_j} is an analytic

extension of $\tilde{f}_{t_j}^* \circ \tilde{W} \circ \tilde{f}_{t_j}^{-1}$, from (5.3), we find that $\tilde{W}_{t_j} = \tilde{f}_{k,t_j}(t_k, \cdot)$. Thus, $e^{i\lambda_j(t_j)}$ (resp. $\lambda_j(t_j)$) lies in the domain of $f_{k,t_j}(t_k, \cdot)$ (resp. $\tilde{f}_{k,t_j}(t_k, \cdot)$) as long as $(t_1, t_2) \in \mathcal{D}$.

Write $\tilde{F}_{k,t_k}(t_j, \cdot) = \tilde{f}_{k,t_j}(t_k, \cdot)$. We will use ∂_t to denote the partial derivative w.r.t. the first variable inside the parentheses, and use $'$ and the superscript (h) to denote the partial derivatives w.r.t. the second variable inside the parentheses. For $h = 0, 1, 2, 3$, define $A_{j,h}$ on \mathcal{D} by

$$A_{j,h}(t_1, t_2) = \tilde{f}_{k,t_j}^{(h)}(t_k, \lambda_j(t_j)) = \tilde{F}_{k,t_k}^{(h)}(t_j, \lambda_j(t_j)). \tag{5.5}$$

Use Sf to denote the (partial) Schwarzian derivative of f . Define $A_{j,S}$ on \mathcal{D} by

$$A_{j,S}(t_1, t_2) = S\tilde{f}_{k,t_j}(t_k, \lambda_j(t_j)) = S\tilde{F}_{k,t_k}(t_j, \lambda_j(t_j)) \tag{5.6}$$

From Sect. 4.2, we know that $L_{j,t_k}(t_j)$ are backward radial Loewner hulls via the time change u_{j,t_k} driven by λ_{j,t_k} , which can be chosen such that

$$\lambda_{j,t_k}(t_j) = A_{j,0}(t_1, t_2). \tag{5.7}$$

Moreover, from (4.10), (4.13), and (4.14), we have

$$u'_{j,t_k}(t_j) = A_{j,1}^2, \tag{5.8}$$

$$\partial_t \tilde{F}_{k,t_k}(t_j, \lambda_j(t_j)) = 3A_{j,2}, \tag{5.9}$$

$$\frac{\partial_t \tilde{F}'_{k,t_k}(t_j, \lambda_j(t_j))}{\tilde{F}'_{k,t_k}(t_j, \lambda_j(t_j))} = -\frac{1}{2} \left(\frac{A_{j,2}}{A_{j,1}} \right)^2 + \frac{4}{3} \frac{A_{j,3}}{A_{j,1}} + \frac{1}{6} A_{j,1}^2 - \frac{1}{6}, \tag{5.10}$$

where all $A_{j,h}$ are valued at (t_1, t_2) .

From now on, we fix an (\mathcal{F}_t^k) -stopping time t_k with $t_k < T_k$. Then the process of conformal maps $(\tilde{F}_{k,t_k}(t_j, \cdot))$ is $(\mathcal{F}_{t_j}^j \times \mathcal{F}_{t_k}^k)_{t_j \geq 0}$ -adapted. Let $T_j(t_k)$ be the maximal number such that for any $t_j < T_j(t_k)$, we have $(t_1, t_2) \in \mathcal{D}$. Then $T_j(t_k)$ is an $(\mathcal{F}_{t_j}^j \times \mathcal{F}_{t_k}^k)_{t_j \geq 0}$ -stopping time. Recall that $(\lambda_j(t))$ is an $(\mathcal{F}_{t_j}^j)$ -adapted local martingale with $\langle \lambda_j \rangle_t = \kappa t$. From now on, we will apply Itô's formula repeatedly. All SDEs below are $(\mathcal{F}_{t_j}^j \times \mathcal{F}_{t_k}^k)_{t_j \geq 0}$ -adapted, and t_j runs in the interval $[0, T_j(t_k))$.

From (5.7), (5.5), and (5.9), we get

$$d\lambda_{j,t_k}(t_j) = A_{j,1} d\lambda_j(t_j) + \left(\frac{\kappa}{2} + 3 \right) A_{j,2} dt, \quad 0 \leq t < T_j(t_k). \tag{5.11}$$

From (5.5) and (5.10) we get

$$\frac{\partial_{t_j} A_{j,h}}{A_{j,h}} = \frac{A_{j,2}}{A_{j,1}} d\lambda_j + \left[-\frac{1}{2} \left(\frac{A_{j,2}}{A_{j,1}} \right)^2 + \left(\frac{\kappa}{2} + \frac{4}{3} \right) \frac{A_{j,3}}{A_{j,1}} + \frac{1}{6} A_{j,1}^2 - \frac{1}{6} \right] dt_j. \tag{5.12}$$

Let

$$\alpha = \frac{6 - (-\kappa)}{2(-\kappa)}, \quad c = \frac{(8 - 3(-\kappa))(-\kappa - 6)}{2(-\kappa)}.$$

Note that if $-\kappa$ is replaced by κ , then c becomes the central charge for forward SLE_κ , which runs in the interval $(-\infty, 1]$. The c here falls in the interval $[25, \infty)$. Since $A_{j,S} = \frac{A_{j,3}}{A_{j,1}} - \frac{3}{2}(\frac{A_{j,2}}{A_{j,1}})^2$, from (5.12) we get

$$\frac{\partial_{t_j} A_{j,1}^\alpha}{A_{j,1}^\alpha} = \alpha \frac{A_{j,2}}{A_{j,1}} d\lambda_j + \left[-\frac{c}{6} A_{j,S} + \frac{\alpha}{6} A_{j,1}^2 - \frac{\alpha}{6} \right] dt_j. \tag{5.13}$$

Now we study $\partial_{t_j} A_{k,h}$ and $\partial_{t_j} A_{k,S}$. From (5.5) we have $A_{k,h}(t_1, t_2) = \tilde{F}_{j,t_j}^{(h)}(t_k, \lambda_k(t_k))$. Recall that $\tilde{F}_{j,t_j}(t_k, \cdot) = \tilde{f}_{j,t_k}(t_j, \cdot)$, and $\tilde{f}_{j,t_k}(t_j, \cdot)$ are backward covering radial Loewner maps via the time change u_{j,t_k} driven by λ_{j,t_k} . From (5.7) and (5.8), we get

$$\partial_t \tilde{f}_{j,t_k}(t_j, z) = -A_{j,1}^2 \cot_2(\tilde{f}_{j,t_k}(t_j, z) - A_{j,0}). \tag{5.14}$$

Differentiate the above formula w.r.t. z , we get

$$\frac{\partial_t \tilde{f}'_{j,t_k}(t_j, z)}{\tilde{f}'_{j,t_k}(t_j, z)} = -A_{j,1}^2 \cot'_2(\tilde{f}_{j,t_k}(t_j, z) - A_{j,0}). \tag{5.15}$$

Differentiating the above formula w.r.t. z , we get

$$\frac{\partial_t \tilde{f}''_{j,t_k}(t_j, z)}{\tilde{f}''_{j,t_k}(t_j, z)} = -A_{j,1}^2 \cot''_2(\tilde{f}_{j,t_k}(t_j, z) - A_{j,0}) \tilde{f}'_{j,t_k}(t_j, z).$$

Since $Sf = (\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2$, from the above formula, we get

$$\partial_t S \tilde{f}_{j,t_k}(t_j, z) = -A_{j,1}^2 \cot'''_2(\tilde{f}_{j,t_k}(t_j, z) - A_{j,0}) \tilde{f}'_{j,t_k}(t_j, z)^2. \tag{5.16}$$

Letting $z = \lambda_k(t_k)$ in (5.14), (5.15), and (5.16), we get

$$\partial_{t_j} A_{k,0} = -A_{j,1}^2 \cot_2(A_{k,0} - A_{j,0}) dt_j; \tag{5.17}$$

$$\frac{\partial_{t_j} A_{k,1}}{A_{k,1}} = -A_{j,1}^2 \cot'_2(A_{k,0} - A_{j,0}) dt_j; \tag{5.18}$$

$$\partial_{t_j} A_{k,S} = -A_{j,1}^2 A_{k,1}^2 \cot'''_2(A_{k,0} - A_{j,0}) dt_j. \tag{5.19}$$

Define X_j on \mathcal{D} such that $X_j = A_{j,0} - A_{k,0}$. Then $X_1 + X_2 = 0$. Since $e^{i\lambda_j(t_j)}$ lies in the domain of $f_{k,t_j}(t_k, \cdot)$, $e^{iA_{j,0}} = f_{k,t_j}(t_k, e^{i\lambda_j(t_j)})$ lies in the range of $f_{k,t_j}(t_k, \cdot)$, i.e., $\mathbb{C} \setminus L_{k,t_j}(t_k)$. On the other hand, since via a time change, $L_{k,t_j}(t_k)$ are backward radial

Loewner hulls driven by $\lambda_{k,t_j}(t_k) = A_{k,0}$, from Lemma 3.4 we have $e^{iA_{k,0}} \in L_{k,t_j}(t_k)$ when $t_k > 0$. Thus, $e^{iA_{j,0}} \neq e^{iA_{k,0}}$ if $t_k > 0$. Switching j and k , the inequality also holds if $t_j > 0$. If $t_j = t_k = 0$, then $e^{iA_{j,0}} = e^{i\tilde{z}_j} \neq e^{i\tilde{z}_k} = e^{iA_{k,0}}$. Thus, $X_j, X_k \notin 2\pi\mathbb{Z}$. So we may define

$$Y = |\sin_2(X_1)|^{-2\alpha} = |\sin_2(X_2)|^{-2\alpha}.$$

From (5.7), (5.11), and (5.17), we get

$$\partial_{t_j} X_j = A_{j,1} d\lambda_j + \left(\frac{\kappa}{2} + 3\right) A_{j,2} dt - A_{j,1}^2 \cot_2(X_j) dt.$$

From Itô’s formula, we get

$$\begin{aligned} \frac{\partial_{t_j} Y}{Y} &= -\alpha \cot_2(X_j) A_{j,1} d\lambda_j - \alpha \left(\frac{\kappa}{2} + 3\right) A_{j,2} \cot_2(X_j) dt_j \\ &\quad - \frac{\alpha}{2} A_{j,1}^2 \cot_2^2(X_j) dt_j + \frac{\alpha\kappa}{4} A_{j,1}^2 dt_j. \end{aligned} \tag{5.20}$$

Define Q and F on \mathcal{D} such that $Q = \cot_2'''(X_1) = \cot_2'''(X_2)$ and

$$F(t_1, t_2) = \exp\left(\int_0^{t_1} \int_0^{t_2} A_{1,1}(s_1, s_2)^2 A_{2,1}(s_1, s_2)^2 Q(s_1, s_2) ds_1 ds_2\right). \tag{5.21}$$

Since $S\tilde{F}_{k,t_k}(0, \cdot) = \text{id}$, from (5.6) we have $A_{j,s} = 0$ when $t_j = 0$. From (5.19) we get

$$\frac{\partial_{t_j} F}{F} = -A_{j,s} dt_j. \tag{5.22}$$

Define a positive function \widehat{M} on \mathcal{D} by

$$\widehat{M} = A_{1,1}^\alpha A_{2,1}^\alpha Y F^{-\frac{\kappa}{6}} e^{\frac{\kappa}{12} m}. \tag{5.23}$$

From (5.4), (5.8), (5.13), (5.18), (5.20), and (5.22), we have

$$\frac{\partial_{t_j} \widehat{M}}{\widehat{M}} = \alpha \frac{A_{j,2}}{A_{j,1}} d\lambda_j - \alpha \cot_2(X_j) A_{j,1} d\lambda_j - \frac{\alpha}{6} dt_j. \tag{5.24}$$

When $t_k = 0$, we have $A_{j,1} = 1, A_{j,2} = 0, m = t_j$, and $X_j = \lambda_j(t_j) - \tilde{f}_j(t_j, \tilde{z}_k)$, so the RHS of (5.24) becomes

$$\frac{1}{\kappa} \left(\frac{\kappa}{2} + 3\right) \cot_2(\lambda_j(t_j) - \tilde{f}_j(t_j, \tilde{z}_k)) d\lambda_j - \frac{\alpha}{6} dt_j. \tag{5.25}$$

Define another positive function M on \mathcal{D} by

$$M(t_1, t_2) = \frac{\widehat{M}(t_1, t_2) \widehat{M}(0, 0)}{\widehat{M}(t_1, 0) \widehat{M}(0, t_2)}. \tag{5.26}$$

Then $M(\cdot, 0) \equiv M(0, \cdot) \equiv 1$. From (5.1), (5.24), and (5.25), we have

$$\begin{aligned} \frac{\partial_{t_j} M}{M} = & \left[- \left(3 + \frac{\kappa}{2} \right) \frac{A_{j,2}}{A_{j,1}} - \frac{-\kappa - 6}{2} \cot_2(X_j) A_{j,1} \right. \\ & \left. + \frac{-\kappa - 6}{2} \cot_2(\lambda_j(t_j) - \tilde{f}_j(t_j, \tilde{z}_k)) - \tilde{f}_j(t_j, \tilde{z}_k) \right] \frac{dB_j(t_j)}{\sqrt{\kappa}}. \end{aligned} \tag{5.27}$$

So when $t_k \in [0, p]$ is a fixed (\mathcal{F}_t^k) -stopping time, M as a function in t_j is an $(\mathcal{F}_{t_j}^j \times \mathcal{F}_{t_k}^k)_{t_j \geq 0}$ -local martingale.

5.2 Coupling measures

Let JP denote the set of disjoint pairs of closed arcs (J_1, J_2) on \mathbb{T} such that $z_j = e^{i\tilde{z}_j}$ is contained in the interior of J_j , $j = 1, 2$. Let $T_j(J_j)$ denote the first time that $S_{L_j(t)}$ intersects $\overline{\mathbb{T} \setminus J_j}$. Then for every $(J_1, J_2) \in \text{JP}$, if $t_j \leq T_j(J_j)$, then $S_{L_j(t_j)} \subset J_j$, which implies that $L_j(t_j) \in \mathcal{H}_{J_j}$. So $[0, T_1(J_1)] \times [0, T_2(J_2)] \subset \mathcal{D}$.

Proposition 5.3 (Boundedness) *For any $(J_1, J_2) \in \text{JP}$, $|\ln(M)|$ is uniformly bounded on $[0, T_1(J_1)] \times [0, T_2(J_2)]$ by a constant depending only on J_1 and J_2 .*

Proof Fix $(J_1, J_2) \in \text{JP}$. In this proof, all constants depend only on (J_1, J_2) , and we say a function is uniformly bounded if its values on $[0, T_1(J_1)] \times [0, T_2(J_2)]$ are bounded in absolute value by a constant. From (5.23) and (5.26), it suffices to show that $\ln(A_{1,1}), \ln(A_{2,1}), \ln(Y), \ln(F)$, and m are all uniformly bounded.

Note that if $t_j \leq T_j(J_j)$, then $L_j(t_j) \in \mathcal{H}_{J_j}$. From a radial version of Theorem 2.16 (iii), we have

$$\{L_1(t_1) \vee L_2(t_2) : t_j \in [0, T_j(J_j)], j = 1, 2\} \subset \mathcal{H}_{J_1 \cup J_2}. \tag{5.28}$$

Since $J_1 \cup J_2 \subsetneq \mathbb{T}$, from Lemma 10.2, the righthand side is a compact set. So the lefthand side is relatively compact. Since $H \mapsto \text{dcap}(H)$ is continuous, and $m(t_1, t_2) = \text{dcap}(L_1(t_1) \vee L_2(t_2))$, we see that m is uniformly bounded. For $j = 1, 2$, since $t_j \leq m$, $T_j(J_j)$ is also uniformly bounded.

Let S_1 and S_2 be the two components of $\mathbb{T} \setminus (J_1 \cup J_2)$. For $s = 1, 2$, let $E_s \subset S_s$ be a compact arc. From Lemma 10.3, $L_n \rightarrow L$ in $\mathcal{H}_{J_1 \cup J_2}$ implies that $f_{L_n} \xrightarrow{\text{l.u.}} f_L$ in $\mathbb{C} \setminus (J_1 \cup J_2)$, which then implies that $f'_{L_n} \xrightarrow{\text{l.u.}} f'_L$ in $\mathbb{C} \setminus (J_1 \cup J_2)$. From (5.28), the compactness of $\mathcal{H}_{J_1 \cup J_2}$, and that $E_1 \cup E_2$ are compact subsets of $\mathbb{C} \setminus (J_1 \cup J_2)$, we conclude that there is a constant $c_1 > 0$ such that $|f'_{L_1(t_1) \vee L_2(t_2)}(z)| \geq c$ for any $t_j \leq T_j(J_j)$, $j = 1, 2$, and $z \in E_1 \cup E_2$. Thus, for $t_j \in [0, T_j(J_j)]$, $j = 1, 2$, the length of $f_{L_1(t_1) \vee L_2(t_2)}(E_s)$, $s = 1, 2$, is bounded below by a constant $c_2 > 0$. Suppose $t_j \in (0, T_j(t_j)]$, $j = 1, 2$. From Lemma 3.4, $e^{iA_{j,0}} \in B_{L_j, t_{3-j}(t_j)}$, $j = 1, 2$. Note that $f_{L_1(t_1) \vee L_2(t_2)}(E_1 \cup E_2)$ disconnects $B_{L_1, t_2(t_1)}$ from $B_{L_2, t_1(t_2)}$ on \mathbb{T} . Thus, there is a constant $c_3 > 0$ such that $|e^{iA_{1,0}(t_1, t_2)} - e^{iA_{2,0}(t_1, t_2)}| \geq c_3$ for $t_j \in (0, T_j(t_j)]$,

$j = 1, 2$. From continuity, this still holds if $t_j \in [0, T_j(J_j)]$, $j = 1, 2$. Thus, $\ln(Y) = -2\alpha \ln |\sin_2(X_j)|$, $|\cot'_2(X_j)|$, and $|\cot''_2(X_j)|$, $j = 1, 2$, are all uniformly bounded.

We may find a Jordan curve σ , which is disjoint from $J_1 \cup J_2$, such that its interior contains J_1 and its exterior contains J_2 . From compactness, $\sup_{z \in \sigma} \ln |f'_j(t_j, z)|$ and $\sup_{z \in \sigma} \ln |f'_{L_1(t_1) \vee L_2(t_2)}(z)|$ are both uniformly bounded. From (5.2) we see that the value $\sup_{w \in f_j(t_j, \sigma)} \ln |f'_{3-j, t_j}(t_k, w)|$ is also uniformly bounded. Note that the interior of $f_j(t_j, \sigma)$ contains $L_j(t_j)^{\text{db}}$, which contains $e^{i\lambda_j(t_j)}$ if $t_j > 0$. From maximum principle, there is $c_4 \in (0, \infty)$ such that $A_{j,1}(t_1, t_2) = |f'_{3-j, t_j}(t_{3-j}, e^{i\lambda_j(t_j)})| \leq c_4$ if $t_j \in (0, T_j(J_j)]$ and $t_{3-j} \in [0, T_{3-j}(J_{3-j})]$. From continuity, $A_{j,1}$ is uniformly bounded, $j = 1, 2$. From (5.18) and the uniformly boundedness of $|\cot'_2(X_j)|$ we see that $\ln(A_{j,1})$ is uniformly bounded, $j = 1, 2$. From (5.21) and the uniformly boundedness of $|\cot''_2(X_j)|$ we see that $\ln(F)$ is also uniformly bounded, which completes the proof. \square

Let μ_j denote the distribution of (λ_j) , $j = 1, 2$. Let $\mu = \mu_1 \times \mu_2$. Then μ is the joint distribution of (λ_1) and (λ_2) , since λ_1 and λ_2 are independent. Fix $(J_1, J_2) \in \text{JP}$. From the local martingale property of M and Proposition 5.3, we have $\mathbf{E}_\mu[M(T_1(J_1), T_2(J_2))] = M(0, 0) = 1$. Define ν_{J_1, J_2} by $d\nu_{J_1, J_2}/d\mu = M(T_1(J_1), T_2(J_2))$. Then ν_{J_1, J_2} is a probability measure. Let ν_1 and ν_2 be the two marginal measures of ν_{J_1, J_2} . Then $d\nu_1/d\mu_1 = M(T_1(J_1), 0) = 1$ and $d\nu_2/d\mu_2 = M(0, T_2(J_2)) = 1$, so $\nu_j = \mu_j$, $j = 1, 2$. Suppose temporarily that the joint distribution of (λ_1) and (λ_2) is ν_{J_1, J_2} instead of μ . Then the distribution of each (λ_j) is still μ_j .

Fix an (\mathcal{F}_t^2) -stopping time $t_2 \leq T_2(J_2)$. From (5.1), (5.27), and Girsanov theorem (c.f. [14]), under the probability measure ν_{J_1, J_2} , there is an $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)_{t_1 \geq 0}$ -Brownian motion $\tilde{B}_{1, t_2}(t_1)$ such that $\lambda_1(t_1), 0 \leq t_1 \leq T_1(J_1)$, satisfies the $(\mathcal{F}_t^1 \times \mathcal{F}_t^2)_{t_1 \geq 0}$ -adapted SDE:

$$d\lambda_1(t_1) = \sqrt{\kappa} d\tilde{B}_{1, t_2}(t_1) - \left(3 + \frac{\kappa}{2}\right) \frac{A_{1,2}}{A_{1,1}} dt_1 - \frac{-\kappa - 6}{2} \cot_2(X_1) A_{1,1} dt_1,$$

which together with (5.5), (5.7), (5.9), and Itô's formula, implies that

$$d\lambda_{1, t_2}(t_1) = A_{1,1} \sqrt{\kappa} d\tilde{B}_{1, t_2}(t_1) - \frac{-\kappa - 6}{2} \cot_2(X_1) A_{1,1}^2 dt_1.$$

From (5.5) and (5.7) we get $X_1 = A_{1,1} - A_{2,1} = \lambda_{1, t_2}(t_1) - \tilde{f}_{1, t_2}(t_1, \lambda_2(t_2))$. Note that $\lambda_{1, t_2}(0) = \tilde{f}_{2, 0}(t_2, \tilde{z}_1) = f_2(t_2, \tilde{z}_1)$. Since $L_{1, t_2}(t_1)$ and $\tilde{f}_{1, t_2}(t_1, \cdot)$ are backward radial Loewner hulls and covering maps via the time change u_{1, t_2} , from (5.8) and the above equation, we find that, under the measure ν_{J_1, J_2} , conditioned on $\mathcal{F}_{t_2}^1$ for any (\mathcal{F}_t^2) -stopping time $t_2 \leq T_2(J_2)$, via the time change u_{1, t_2} , $L_{1, t_2}(t_1) = f_2(t_2, \cdot)^{\mathcal{H}}(L_1(t_1))$, $0 \leq t_1 \leq T_1(J_1)$, is a partial backward radial SLE($\kappa; \frac{-\kappa - 6}{2}$) process started from $e^i \circ \tilde{f}_2(t_2, \tilde{z}_1) = f_2(t_2, z_1)$ with marked point $e^i(\lambda_2(t_2))$. Similarly, the above statement holds true if the subscripts "1" and "2" are exchanged.

The joint distribution ν_{J_1, J_2} is a local coupling such that the desired properties in the statement of Theorem 5.2 holds true up to the stopping times $T_1(J_1)$ and $T_2(J_2)$.

Then we can apply the maximum coupling technique developed in [20] to construct a global coupling using the local couplings within different pairs (J_1, J_2) . To be more specific, the construction is composed of two steps:

1. Prove that for any finite sequence $(J_1^{(k)}, J_2^{(k)})$, $1 \leq k \leq n$, in JP, there is a global coupling, say $\nu^{(n)}$, of $(L_1(t_1))$ and $(L_2(t_2))$, such that for every $1 \leq k \leq n$, if the two processes in the coupling are stopped at $T_1(J_1^{(k)})$ and $T_2(J_2^{(k)})$, respectively, then we get the joint distribution ν_{J_1, J_2} . Such coupling is obtained by doing operations on the process M . From Theorem 4.5 in [21], we see that there is a bounded positive process $M^{(n)}$ defined on $[0, T_1] \times [0, T_2]$ such that
 - (a) When t_1 is fixed, $M^{(n)}$ is a martingale in t_2 , and vice versa.
 - (b) When t_1 or t_2 equals 0, $M^{(n)}$ is constant 1.
 - (c) For every $1 \leq k \leq n$, $M^{(n)}$ agrees with M on $[0, T_1(J_1^{(k)})] \times [0, T_2(J_2^{(k)})]$.
 Then the $\nu^{(n)}$ is defined by weighting the independent coupling of $(L_1(t_1))$ and $(L_2(t_2))$ by $M^{(n)}(T_1, T_2)$.
2. Choose a dense sequence $(J_1^{(k)}, J_2^{(k)})$, $k \in \mathbb{N}$, in JP. For each $n \in \mathbb{N}$, we get a global coupling $\nu^{(n)}$ using the previous step for the pairs $(J_1^{(k)}, J_2^{(k)})$ up to n . Then we choose a suitable topology such that the space of coupling measures is tight. Then the desired commutation coupling is any subsequential limit of the sequence $(\nu^{(n)})$.

The reader is referred to Section 4.3 in [21] for more details of the technique. This finishes the proof of Theorem 5.2 in the radial case.

Now we briefly describe the proof for the chordal case. The proof in this case is simpler because there are no covering maps. Suppose the two backward chordal SLE($\kappa; -\kappa - 6$) processes start from $(z_j; z_k)$, where $z_1 \neq z_2 \in \mathbb{R}$. Formula (5.1) holds with all tildes removed and the function \cot_2 replaced by $z \mapsto \frac{2}{z}$. The domain \mathcal{D} and the \mathbb{H} -hulls $L_{1,t_2}(t_1)$ and $L_{2,t_1}(t_2)$ are defined in the same way. Then (5.2) still holds. From Corollary 2.19 (ii), $f_{1,t_2}(t_1, \cdot)$ and $f_{2,t_1}(t_2, \cdot)$ are $\mathcal{F}_{t_1}^1 \times \mathcal{F}_{t_2}^2$ -measurable. Define $m(t_1, t_2) = \text{hcap}(L_1(t_1) \vee L_2(t_2))/2$. Then (5.4) holds with $u_{j,t_k}(t_j) := \text{hcap}(L_{j,t_k}(t_j))/2$.

Now we apply the argument in Sect. 4.1 with $W = f_k(t_k, \cdot)$. Then $W_t = f_{k,t_j}(t_k, \cdot)$. Let $F_{k,t_k}(t_j, \cdot) = f_{k,t_j}(t_k, \cdot)$, and define $A_{j,h}$ and $A_{j,S}$ using (5.5) and (5.6) with all tildes removed. Using (4.3), (4.4), (4.7), and (4.8), we see that (5.7) still holds here; (5.8) and (5.9) hold with all tildes removed; and (5.10) holds without the tildes and the terms $+\frac{1}{6}A_{j,1}^2 - \frac{1}{6}$. Then we get the SDEs (5.11) and (5.13) without the terms $+\frac{\alpha}{6}A_{j,1}^2 - \frac{\alpha}{6}$. Formulas (5.17), (5.18), and (5.19) hold with \cot_2 replaced by $z \mapsto \frac{2}{z}$. We still define $X_j = A_{j,1} - A_{k,1}$. Then $X_j \neq 0$ in \mathcal{D} . Define Y on \mathcal{D} by $Y = |X_1|^{-2\alpha} = |X_2|^{-2\alpha}$. Then (5.20) holds with \cot_2 replaced by $z \mapsto \frac{2}{z}$ and the term $+\frac{\alpha\kappa}{4}A_{j,1}^2 dt_j$ removed. Define F using (5.21) with $Q = -\frac{12}{X_1^4} = -\frac{12}{X_2^4}$. Then (5.22) still holds. Define \widehat{M} using (5.23) without the factor $e^{\frac{c}{12}m}$. Then (5.24) holds with \cot_2 replaced by $z \mapsto \frac{2}{z}$ and the term $-\frac{\alpha}{6}dt_j$ removed. Define M using (5.26). Then (5.27) holds with all tildes removed and \cot_2 replaced by $z \mapsto \frac{2}{z}$.

We define JP to be the set of disjoint pairs of closed real intervals (J_1, J_2) such that z_j is contained in the interior of J_j . Then Proposition 5.3 holds with a similar proof,

where Lemma 9.2 is applied here, and we can show that $|X_1|$ is uniformly bounded away from 0. The argument on the local couplings hold with all tildes and e^j removed and \cot_2 replaced by $z \mapsto \frac{2}{z}$. Finally, we may apply the maximum coupling technique to construct a global coupling with the desired properties. This finishes the proof in the chordal case.

5.3 Other results

Besides Theorem 5.2, one may also prove the following two theorems, which are similar to the couplings for forward SLE that appear in [5,21].

Theorem 5.4 *Let $\kappa_1, \kappa_2 > 0$ satisfy $\kappa_1\kappa_2 = 16$, and $c_1, \dots, c_n \in \mathbb{R}$ satisfy $\sum_{k=1}^n c_k = \frac{3}{2}$. Let $\rho_j = (\frac{\kappa_j}{2}, c_1(-\kappa_j - 4), \dots, c_n(-\kappa_j - 4))$, $j = 1, 2$. Then backward chordal (resp. radial) SLE($\kappa_1; \rho_1$) commutes with backward chordal (resp. radial) SLE($\kappa_2; \rho_2$).*

Theorem 5.5 *Let $\kappa > 0$ and $\rho \in \mathbb{R}^n$, whose first coordinate is 2. Then backward chordal (resp. radial) SLE($\kappa; \rho$) commutes with backward chordal (resp. radial) SLE($\kappa; \rho$).*

6 Reversibility of backward chordal SLE

Theorem 6.1 *Let $\kappa \in (0, 4]$ and $z_1 \neq z_2 \in \mathbb{T}$. Suppose a backward radial SLE($\kappa; -\kappa - 6$) process $(L_1(t))$ started from $(z_1; z_2)$ commutes with a backward radial SLE($\kappa; -\kappa - 6$) process $(L_2(t))$ started from $(z_2; z_1)$. Then a.s. they induce the same welding.*

Proof For $j = 1, 2$, let $S_t^j = S_{L_j(t)}$ and $f_t^j = f_{L_j(t)}$. Let $T_j(\cdot)$, $j = 1, 2$, be as in Definition 5.1. Let ϕ_j be the welding induced by $(L_j(t))$. Since $-\kappa - 6 \leq -\kappa/2 - 2$, from the last remark in Sect. 4.3, we see that, for $j = 1, 2$, a.s. $T_j = \infty$, $S_\infty^j = \mathbb{T} \setminus \{z_{3-j}\}$, and ϕ_j is an involution of \mathbb{T} with exactly two fixed points: z_1 and z_2 .

Fix $t_2 > 0$. Since $(L_1(t))$ and $(L_2(t))$ commute, the following is true. Conditioned on $(L_2(t))_{t \leq t_2}$, $(f_{t_2}^2)^{\mathcal{H}}(L_1(t_1))$, $0 \leq t_1 < T_1(t_2)$, is a partial backward radial SLE($\kappa; -\kappa - 6$) process, after a time change, started from $(f_{t_2}^2(z_1); e^{i\lambda_2(t_2)})$, where λ_2 is a driving function for $(L_2(t_2))$. We have

$$S := \bigcup_{0 \leq t_1 < T_1(t_2)} S_{(f_{t_2}^2)^{\mathcal{H}}(L_1(t_1))} = f_{t_2}^2 \left(\bigcup_{0 \leq t_1 < T_1(t_2)} S_{t_1}^1 \right) = f_{t_2}^2(S_{T_1(t_2)-}^1). \tag{6.1}$$

Recall that $f_{t_2}^2$ is a homeomorphism from $\mathbb{T} \setminus S_{t_2}^2$ onto $\mathbb{T} \setminus B_{L_2(t_2)} = \mathbb{T} \setminus \{e^{i\lambda_2(t_2)}\}$. From the definition of $T_1(t_2)$, we see that $S_{T_1(t_2)}^1$ intersects $S_{t_2}^2 \neq \emptyset$ at one or two end points of both arcs. If they intersect at only one point, then $S_{T_1(t_2)-}^1$ is a proper subset of $\mathbb{T} \setminus S_{t_2}^2$, and these two arcs share an end point. From (6.1), this then implies that the arc S is a proper subset of $\mathbb{T} \setminus B_{L_2(t_2)}$, and $B_{L_2(t_2)}$ is an end point of S . Recall that, after a time

change, $(f_{t_2}^2)^{\mathcal{H}}(L_1(t_1)), 0 \leq t_1 < T_1(t_2)$, is a partial backward radial SLE $(\kappa; -\kappa - 6)$ process. Since $S \neq \mathbb{T} \setminus B_{L_2(t_2)}$, the process is not complete. Then we conclude that S is contained in a closed arc on \mathbb{T} that does not contain $B_{L_2(t_2)}$ because the force point is not swallowed by the process at any finite time, which contradicts that $B_{L_2(t_2)}$ is an end point of S . Thus, a.s. $S_{T_1(t_2)}^1$ and $S_{t_2}^2$ share two end points. Since ϕ_j swaps the two end points of any $S_t^j, j = 1, 2$, we see that a.s. $\phi_2 = \phi_1$ on $\partial_{\mathbb{T}} S_{t_2}^2$. Let $t_2 > 0$ vary in the set of rational numbers, we see that a.s. $\phi_2 = \phi_1$ on $\bigcup_{t \in \mathbb{Q}_{>0}} \partial_{\mathbb{T}} S_t^2$, which is a dense subset of \mathbb{T} . The conclusion follows since ϕ_1 and ϕ_2 are continuous. \square

We now state the reversibility of backward chordal SLE $_{\kappa}$ for $\kappa \in (0, 4]$ in terms of its welding. Recall that a backward chordal SLE $_{\kappa}$ welding is an involution of $\widehat{\mathbb{R}}$ with two fixed points: 0 and ∞ .

Theorem 6.2 *Let $\kappa \in (0, 4]$, and ϕ be a backward chordal SLE $_{\kappa}$ welding. Let $h(z) = -1/z$. Then $h \circ \phi \circ h$ has the same distribution as ϕ .*

Proof Let $(L_1(t))$ and $(L_2(t))$ be commuting backward radial SLE $(\kappa; -\kappa - 6)$ processes as in Theorem 5.2, which induce the weldings ψ_1 and ψ_2 , respectively. The above theorem implies that a.s. $\psi_1 = \psi_2$. For $j = 1, 2$, let W_j be a Möbius transformation that maps \mathbb{D} onto \mathbb{H} such that $W_j(z_j) = 0$ and $W_j(z_{3-j}) = \infty$, and $W_2 = h \circ W_1$. From Corollary 4.8, $K_j(t) := W_j^{\mathcal{H}}(L_j(t)), 0 \leq t < \infty$, is a backward chordal SLE $_{\kappa}$, after a time change, which then induces backward chordal SLE $_{\kappa}$ welding $\phi_j, j = 1, 2$. Then ϕ_1 and ϕ_2 have the same law as ϕ . From (4.1), we get $\phi_j = W_j \circ \psi_j \circ W_j^{-1}, j = 1, 2$, which implies that a.s. $\phi_2 = h \circ \phi_1 \circ h$. The conclusion follows since ϕ_1 and ϕ_2 has the same distribution as ϕ . \square

Lemma 6.3 *Let $\kappa > 0$. Let $f_t, 0 \leq t < \infty$, be backward chordal SLE $_{\kappa}$ maps. Then for every $z_0 \in \mathbb{H}$, a.s. (3.4) holds.*

Proof Let $Z_t = f_t(z_0) - \lambda(t), X_t = \text{Re } Z_t$, and $Y_t = \text{Im } Z_t$. Then

$$dX_t = -\sqrt{\kappa}dB(t) - \frac{2X_t}{X_t^2 + Y_t^2} dt, \quad dY_t = \frac{2Y_t}{X_t^2 + Y_t^2} dt$$

Let $R_t = |f'_t(z_0)|$. Then $\frac{dR_t}{R_t} = \frac{2(X_t^2 - Y_t^2)}{(X_t^2 + Y_t^2)^2} dt$. Let $N_t = Y_t/R_t$ and $A_t = X_t/Y_t$. Then

$$\frac{dN_t}{N_t} = \frac{4Y_t^2}{(X_t^2 + Y_t^2)^2} dt, \quad dA_t = -\frac{\sqrt{\kappa}dB(t)}{Y_t} - \frac{4A_t}{X_t^2 + Y_t^2} dt.$$

Let $u(t) = \ln(Y_t)$. Then $u'(t) = \frac{2}{X_t^2 + Y_t^2}$. Let $T = \sup u([0, \infty))$ and define $\widehat{N}_s = N_{u^{-1}(s)}$ and $\widehat{A}_s = A_{u^{-1}(s)}$ for $0 \leq s < T$. Then

$$\frac{d\widehat{N}_s}{\widehat{N}_s} = \frac{2}{\widehat{A}_s^2 + 1} ds, \quad d\widehat{A}_s = -\sqrt{1 + \widehat{A}_s^2} \sqrt{\kappa/2} d\widehat{B}(s) - 2\widehat{A}_s ds,$$

where $\widehat{B}(s)$ is another Brownian motion. We claim that $T = \infty$. Suppose $T < \infty$. Then $\lim_{t \rightarrow \infty} Y(t) = e^T \in \mathbb{R}$. From the SDE for A_s , we see that a.s. $\lim_{s \rightarrow T} A_s \in \mathbb{R}$, which implies that $\lim_{t \rightarrow \infty} A_t \in \mathbb{R}$ and $\lim_{t \rightarrow \infty} X_t \in \mathbb{R}$ as $X_t = Y_t A_t$. Then we have a.s. $s'(t) = \frac{2}{X_t^2 + Y_t^2}$ tends to a finite positive number as $t \rightarrow \infty$, which contradicts that $T = \sup\{s(t), 0 \leq t < \infty\} < \infty$. So the claim is proved. Using Itô's formula, we see that $\widehat{A}_s, 0 \leq s < \infty$, is recurrent. Since $(\ln(\widehat{N}_s))' = \frac{2}{\widehat{A}_s^2 + 1}$, we see that a.s. $\widehat{N}_s \rightarrow \infty$ as $s \rightarrow \infty$. So a.s. $N_t = \frac{\text{Im } f_t(z_0)}{|f_t'(z_0)|} \rightarrow \infty$ as $t \rightarrow \infty$, i.e., (3.4) holds. \square

If $\kappa \in (0, 4]$, then since the backward chordal traces are simple, (3.5) holds. From the above lemma and Sect. 3.3, we see that, for $\kappa \in (0, 4]$, the backward chordal SLE_κ a.s. generates a normalized global backward chordal trace β , which we call a normalized global backward chordal SLE_κ trace. Recall that $\beta(t), 0 \leq t < \infty$, is simple with $\beta(0) = 0$, and $i \notin \beta$; and there is $F_\infty : \mathbb{H} \xrightarrow{\text{Conf}} \mathbb{C} \setminus \beta$, whose continuation maps \mathbb{R} onto β such that (3.7) holds, and for any $x \in \mathbb{R}, F_\infty(x) = F_\infty(\phi(x)) \in \beta$. Now we state the reversibility of the backward chordal SLE_κ for $\kappa \in (0, 4)$ in terms of β .

Theorem 6.4 *Let $\kappa \in (0, 4)$, and β be a normalized global backward chordal SLE_κ trace. Let $h(z) = -1/z$. Then $h(\beta \setminus \{0\})$ has the same distribution as $\beta \setminus \{0\}$ as random sets.*

Proof For $j = 1, 2$, let ϕ_j be a backward chordal SLE_κ welding and β_j be the corresponding normalized global trace. Then β_j is a simple curve with one end point 0, and there exists $F_j : \mathbb{H} \xrightarrow{\text{Conf}} \mathbb{C} \setminus \beta_j$ such that $F_j(i) = i, F_j(0) = 0$, and $F_j(x) = F_j(\phi_j(x))$ for $x \in \mathbb{R}$. From Theorem 6.2 we may assume that $\phi_2 = h \circ \phi_1 \circ h^{-1}$. Now it suffices to show that $h(\beta_2 \setminus \{0\}) = \beta_1 \setminus \{0\}$.

Define $G = h \circ F_2 \circ h \circ F_1^{-1}$. Then G is a conformal map defined on $\mathbb{C} \setminus \beta_1$. It has continuation to $\beta_1 \setminus \{0\}$. In fact, if $z \in \mathbb{C} \setminus \beta_1$ and $z \rightarrow z_0 \in \beta_1 \setminus \{0\}$, then $F_1^{-1}(z) \rightarrow \{x, \phi_1(x)\}$ for some $x \in \mathbb{R} \setminus \{0\}$, which then implies that $h \circ F_1^{-1}(z) \rightarrow \{h(x), h \circ \phi_1(x)\}$; since $\phi_2 \circ h = h \circ \phi_1$, we find that $F_2 \circ h \circ F_1^{-1}(z)$ tends to some point on $\beta_2 \setminus \{0\}$, so $G(z)$ tends to some point on $h(\beta_2 \setminus \{0\})$. It was proved in [15] that a forward SLE_κ trace is the boundary of a Hölder domain. Then the same is true for backward chordal SLE_κ traces and the normalized global trace. From the results in [6], we see that $\beta_1 \setminus \{0\}$ is conformally removable, which means that G extends to a conformal map from $(\mathbb{C} \setminus \beta_1) \cup (\beta_1 \setminus \{0\}) = \mathbb{C} \setminus \{0\}$ onto $\mathbb{C} \setminus \{0\}$, and maps $\beta_1 \setminus \{0\}$ to $h(\beta_2 \setminus \{0\})$. Since $G(i) = i$, either $G = \text{id}$ or $G = h$. Suppose $G = h$. Then $F_1 = F_2 \circ h$. Since $F_1(0) = F_2(0) = 0$, for $j = 1, 2, F_j$ maps a neighborhood of 0 in \mathbb{H} onto a neighborhood of 0 in \mathbb{C} without a simple curve. Since $F_1 = F_2 \circ h$, F_1 also maps a neighborhood of ∞ in \mathbb{H} onto a neighborhood of 0 without a simple curve, which contradicts the univalent property of F_1 . Thus, $G = \text{id}$, and we get $h(\beta_2 \setminus \{0\}) = G(\beta_1 \setminus \{0\}) = \beta_1 \setminus \{0\}$, as desired. \square

Now we propose a couple of questions. First, let's consider backward chordal SLE_κ for $\kappa > 4$. Since the process does not generate simple backward chordal traces, the random welding ϕ can not be defined. However, the lemma below and the discussion in Sect. 3.3 show that we can still define a global backward chordal SLE_κ trace.

Lemma 6.5 *Let $\kappa \in (0, \infty)$. Suppose $\beta_t, 0 \leq t < \infty$, are backward chordal traces driven by $\lambda(t) = \sqrt{\kappa}B(t)$. Then a.s. (3.5) holds.*

Proof If $\kappa \in (0, 4]$, a.s. the traces are simple, so (3.5) holds. Now suppose $\kappa > 4$. Let f_t and L_t be the corresponding maps and hulls. It suffices to show that, for any $t_0 > 0$, a.s. there exists $t_1 > t_0$ such that $\beta_{t_1}([0, t_0]) \subset \mathbb{H}$.

Let g_t and $K_t, 0 \leq t < \infty$, be the forward chordal Loewner maps and hulls driven by $\sqrt{\kappa}B(t)$. From Theorem 6.1 in [22], for any deterministic time $t_1 \in (0, \infty)$, the continuation of $g_{t_1}^{-1}$ a.s. maps the interior of $S_{K_{t_1}}$ into \mathbb{H} . From Lemma 3.1 and the property of Brownian motion, we see that, for any $t_1 \in (0, \infty)$, f_{t_1} has the same distribution as $\lambda(t_1) + g_{t_1}^{-1}(\cdot - \lambda(t_1))$, which implies that the continuation of f_{t_1} a.s. maps the interior of $S_{L_{t_1}}$ into \mathbb{H} .

Since a.s. $\bigcup_{n=1}^\infty S_n = S_\infty = \mathbb{R} \supset \lambda([0, t_0])$, and (S_t) is an increasing family of intervals, we see that a.s. there is $N \in \mathbb{N}$ such that the interior of S_N contains $\lambda([0, t_0])$. Let $t_1 = N$. Then f_{t_1} maps $\lambda([0, t_0])$ into \mathbb{H} , which implies that $\beta_{t_1}(t) = f_{t_1}(\lambda(t)) \in \mathbb{H}$ for $0 \leq t \leq t_0$. □

Question 6.6 *Do we have the reversibility of the global backward chordal SLE_κ trace for $\kappa > 4$?*

Second, let's consider backward radial SLE_κ processes. One can show that (3.8) a.s. holds. Since $T = \infty$, we may define a global backward radial SLE_κ trace.

Question 6.7 *Does a global backward radial SLE_κ trace satisfy some reversibility property of any kind?*

Recall that the forward radial SLE_κ trace does not satisfy the reversibility property in the usual sense. However, it's proved in [24] that, for $\kappa \in (0, 4]$, the whole-plane SLE_κ , as a close relative of radial SLE_κ , satisfies reversibility.

Finally, it is worth mentioning the following simple fact. Recall that, if $\kappa \in (0, 4]$, a backward radial SLE_κ welding is an involution of \mathbb{T} with two fixed points, one of which is 1. The following theorem gives the distribution of the other fixed point ζ , and says that a backward radial SLE_κ process conditioned on ζ is a backward radial $SLE(\kappa; -4)$ process with force point ζ . It is similar to Theorem 3.1 in [22], and we omit its proof.

Theorem 6.8 *Let $\kappa \in (0, 4]$. Let μ denote the distribution of a backward radial SLE_κ process. For $\theta \in (0, 2\pi)$, let ν_θ denote the distribution of a backward radial $SLE(\kappa; -4)$ process started from $(1; e^{i\theta})$. Let $f(\theta) = C \sin_2(\theta)^{4/\kappa}$, where $C > 0$ is such that $\int_0^{2\pi} f(\theta)d\theta = 1$. Then*

$$\mu = \int_0^{2\pi} \nu_\theta f(\theta)d\theta.$$

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Appendices

Appendix A: Carathéodory topology

Definition 7.1 Let $(D_n)_{n=1}^\infty$ and D be domains in \mathbb{C} . We say that (D_n) converges to D , and write $D_n \xrightarrow{\text{Cara}} D$, if for every $z \in D$, $\text{dist}(z, \mathbb{C} \setminus D_n) \rightarrow \text{dist}(z, \mathbb{C} \setminus D)$. This is equivalent to the following:

- (i) every compact subset of D is contained in all but finitely many D_n 's;
- (ii) for every point $z_0 \in \partial D$, there exists $z_n \in \partial D_n$ for each n such that $z_n \rightarrow z_0$.

Remark A sequence of domains may converge to two different domains. For example, let $D_n = \mathbb{C} \setminus ((-\infty, n])$. Then $D_n \xrightarrow{\text{Cara}} \mathbb{H}$, and $D_n \xrightarrow{\text{Cara}} -\mathbb{H}$ as well. But two different limit domains of the same domain sequence must be disjoint from each other, because if they have nonempty intersection, then one contains some boundary point of the other, which implies a contradiction.

Lemma 7.2 Suppose $D_n \xrightarrow{\text{Cara}} D$, $f_n : D_n \xrightarrow{\text{Conf}} E_n$, $n \in \mathbb{N}$, and $f_n \xrightarrow{\text{l.u.}} f$ in D . Then either f is constant on D , or f is a conformal map on D . In the latter case, let $E = f(D)$. Then $E_n \xrightarrow{\text{Cara}} E$ and $f_n^{-1} \xrightarrow{\text{l.u.}} f^{-1}$ in E .

Remark The above lemma resembles the Carathéodory kernel theorem [13, Theorem 1.8], but the domains here don't have to be simply connected. The main ingredients in the proof are Rouché's theorem and Koebe's 1/4 theorem. The lemma also holds in the case that D_n and D are domains of any Riemann surface, if the metric in the underlying space is used in place of the Euclidean metric for Definition 7.1 and locally uniform convergence. In particular, if we use the spherical metric, then Lemma 7.2 holds for domains of $\widehat{\mathbb{C}}$.

Appendix B: Topology on interior hulls

Let \mathcal{H} denote the set of all interior hulls in \mathbb{C} . Recall that for any $H \in \mathcal{H}$, ϕ_H^{-1} is defined on $\{|z| > \text{rad}(H)\}$, and for a nondegenerate interior hull, $\psi_H(z) = \phi_H^{-1}(z) = \phi_H^{-1}(\text{rad}(H)z)$ is defined on $\{|z| > 1\}$. It's shown in Section 2.5 of [23] that there is a metric $d_{\mathcal{H}}$ on \mathcal{H} such that for any $H_n, H \in \mathcal{H}$, the followings are equivalent:

1. $d_{\mathcal{H}}(H_n, H) \rightarrow 0$
2. $\text{rad}(H_n) \rightarrow \text{rad}(H)$ and $\phi_{H_n}^{-1} \xrightarrow{\text{l.u.}} \phi_H^{-1}$ in $\{|z| > \text{rad}(H)\}$.
3. $\mathbb{C} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{C} \setminus H$.

In particular, we see that rad is a continuous function on $(\mathcal{H}, d_{\mathcal{H}})$. Thus, for nondegenerate interior hulls, $d_{\mathcal{H}}(H_n, H) \rightarrow 0$ iff $\psi_{H_n} \xrightarrow{\text{l.u.}} \psi_H$ in $\{|z| > 1\}$. The following lemma is Lemma 2.2 in [23].

Lemma 8.1 For any $F \in \mathcal{H}$, the set $\{H \in \mathcal{H} : H \subset F\}$ is compact.

Corollary 8.2 For any $F \in \mathcal{H}$ and $r > 0$, the set $\{H \in \mathcal{H} : H \subset F, \text{rad}(H) \geq r\}$ is compact.

Appendix C: Topology on \mathbb{H} -hulls

From Section 5.2 in [19], there is a metric $d_{\mathcal{H}}$ on the space of \mathbb{H} -hulls such that $d_{\mathcal{H}}(H_n \rightarrow H_\infty)$ iff $f_{H_n} \xrightarrow{1.u.} f_{H_\infty}$ in \mathbb{H} . From Lemma 7.2, this implies that $\mathbb{H} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{H} \setminus H_\infty$. But $\mathbb{H} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{H} \setminus H_\infty$ does not imply $d_{\mathcal{H}}(H_n \rightarrow H_\infty)$. A counterexample is $H_n = \{z \in \mathbb{H} : |z - 2n| \leq n\}$ and $H_\infty = \emptyset$. Since $H_1 \cdot H_2 = H_3$ iff $f_{H_1} \circ f_{H_2} = f_{H_3}$, the dot product is continuous.

Formula (5.1) in [19] states that for any \mathbb{H} -hull H , there is a positive measure μ_H supported by S_H^{cv} , the convex hull of S_H , such that for any $z \in \mathbb{C} \setminus S_H^{\text{cv}}$,

$$f_H(z) = z + \int \frac{-1}{z-x} d\mu_H(x). \tag{9.1}$$

In particular, if H is bounded by a crosscut, then μ_H is absolutely continuous w.r.t. the Lebesgue measure, and $d\mu_H/dx = \frac{1}{\pi} \text{Im } f_H(x)$, where the value of f_H on S_H^{cv} is the continuation of f_H from \mathbb{H} . If H is approximated by a sequence of \mathbb{H} -hulls (H_n) , then μ_H is the weak limit of (μ_{H_n}) . We may choose each H_n to be bounded by a crosscut, whose height is not bigger than $h + 1/n$, where h is the height of H . Then each μ_{H_n} has a density function, whose L^∞ norm is not bigger than $(h + 1/n)/\pi$. Thus, μ_H also has a density function, whose L^∞ norm is not bigger than h/π . We use ρ_H to denote the density function of μ_H . Since $f_H : \mathbb{C} \setminus S_H^{\text{cv}} \xrightarrow{\text{Conf}} \mathbb{C} \setminus H^{\text{db,cv}}$ and $f'_H(\infty) = 1$, we see that $\text{rad}(H^{\text{db,cv}}) = \text{rad}(S_H^{\text{cv}}) = |S_H^{\text{cv}}|/4$. Thus, $\text{diam}(H^{\text{db,cv}}) \leq 4 \text{rad}(H^{\text{db,cv}}) = |S_H^{\text{cv}}|$. On the other hand, the diameter of $H^{\text{db,cv}}$ is at least twice the height of H . So $\|\rho_H\|_\infty \leq \frac{|S_H^{\text{cv}}|}{2\pi}$.

By approximating any \mathbb{H} -hull H using a sequence of \mathbb{H} -hulls (H_n) , each of which is the union of finitely many mutually disjoint \mathbb{H} -hulls bounded by crosscuts in \mathbb{H} , we see that μ_H is in fact supported by S_H . By continuation, (9.1) holds for any $z \in \mathbb{C} \setminus S_H$. Furthermore, the support of μ_H is exactly S_H because from (9.1) f_H extends analytically to the complement of the support of μ_H , while from Lemma 2.6 f_H can not be extended analytically beyond $\mathbb{C} \setminus S_H$. So we obtain the following lemma.

Lemma 9.1 *For any \mathbb{H} -hull H , μ_H has a density function ρ_H , whose support is S_H , and whose L^∞ norm is no more than $\frac{|S_H^{\text{cv}}|}{2\pi}$. Moreover, (9.1) holds for any $z \in \mathbb{C} \setminus S_H$.*

The following lemma extends Lemma 5.4 in [19], and we now give a proof.

Lemma 9.2 *For any compact $F \subset \mathbb{R}$, $\mathcal{H}_F := \{H : S_H \subset F\}$ is compact, and $H_n \rightarrow H$ in \mathcal{H}_F implies that $f_{H_n} \xrightarrow{1.u.} f_H$ in $\mathbb{C} \setminus F$.*

Proof Suppose (H_n) is a sequence in \mathcal{H}_F . Let $|F^{\text{cv}}|$ denote the length of the convex hull of F . Then for each n , ρ_{H_n} is supported by $S_{H_n} \subset F$, and the L^∞ norm of ρ_{H_n} is no more than $\frac{|S_{H_n}^{\text{cv}}|}{2\pi} \leq \frac{|F^{\text{cv}}|}{2\pi}$. Thus, (ρ_{H_n}) contains a subsequence $(\rho_{H_{n_k}})$, which converges in the weak-* topology to a function ρ supported by F . From (9.1) we see that $f_{H_{n_k}}$ converges uniformly on each compact subset of $\mathbb{C} \setminus F$, and if f is the limit function, then $f(z) - z = \int_F \frac{-1}{z-x} \rho(x) dx$, $z \in \mathbb{C} \setminus F$. So $f(z) - z \rightarrow 0$ as $z \rightarrow \infty$. This means that f can not be constant. From Lemma 7.2, f is a conformal map on

$\mathbb{C} \setminus F$. Since $f(z) - z \rightarrow 0$ as $z \rightarrow \infty$, ∞ is a simple pole of f . Thus, $f(\mathbb{C} \setminus F)$ contains a neighborhood of ∞ . Let $G = \mathbb{C} \setminus f(\mathbb{C} \setminus F)$. Then G is compact. Since every $f_{H_{n_k}}$ is \mathbb{R} -symmetric, so is f . Let $H = G \cap \mathbb{H}$. Then f maps \mathbb{H} conformally onto $\mathbb{H} \setminus H$. This implies that H is an \mathbb{H} -hull and $f = f_H$ on \mathbb{H} because $f(z) - z \rightarrow 0$ as $z \rightarrow \infty$. Since f extends $f_H|_{\mathbb{H}}$, from Lemma 2.6, we see that $S_H \subset F$ and $f = f_H$ in $\mathbb{C} \setminus F$. Since $f_{H_{n_k}} \xrightarrow{1.u.} f_H$ in \mathbb{H} , we get $H_{n_k} \rightarrow H \in \mathcal{H}_F$. This shows that \mathcal{H}_F is compact. The above argument also gives $f_{H_{n_k}} \xrightarrow{1.u.} f_H$ in $\mathbb{C} \setminus F$. If $H_n \rightarrow H$, then any subsequence (H_{n_k}) of (H_n) contains a subsequence $(H_{n_{k_l}})$ such that $f_{H_{n_{k_l}}} \xrightarrow{1.u.} f_H$ in $\mathbb{C} \setminus F$, which implies that $f_{H_n} \xrightarrow{1.u.} f_H$ in $\mathbb{C} \setminus F$. □

Appendix D: Topology on \mathbb{D} -hulls

Define a metric $d_{\mathcal{H}}$ on the space of \mathbb{D} -hulls such that

$$d_{\mathcal{H}}(H_1, H_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{|z| \leq 1-1/n} \{|f_{H_1}(z) - f_{H_2}(z)|\}. \tag{10.1}$$

It is clear that $d_{\mathcal{H}}(H_n, H) \rightarrow 0$ iff $f_{H_n} \xrightarrow{1.u.} f_H$ in \mathbb{D} . From Lemma 7.2, this implies that $\mathbb{D} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{D} \setminus H$. On the other hand, from Lemma 10.1 below, one see that $\mathbb{D} \setminus H_n \xrightarrow{\text{Cara}} \mathbb{D} \setminus H$ also implies that $H_n \rightarrow H$. Since $f_{H_n} \xrightarrow{1.u.} f_H$ in \mathbb{D} implies that $f'_{H_n}(0) \rightarrow f'_H(0)$, we see that dcap is a continuous function. Moreover, the dot product is also continuous.

Lemma 10.1 *For any $M < \infty$, $\{H : \text{dcap}(H) \leq M\}$ is compact.*

Proof Suppose (H_n) is a sequence of \mathbb{D} -hulls with $\text{dcap}(H_n) \leq M$ for each n . Then $f'_{H_n}(0) = e^{-\text{dcap}(H_n)} \geq e^{-M}$. Since (f_{H_n}) is uniformly bounded in \mathbb{D} , it contains a subsequence $(f_{H_{n_k}})$, which converges locally uniformly in \mathbb{D} . Let f be the limit. Then $f'(0) = \lim_{k \rightarrow \infty} f'_{H_{n_k}}(0) \geq e^{-M}$. Thus, f is not constant. From Lemma 7.2, f is conformal in \mathbb{D} . Since $f(0) = \lim_{k \rightarrow \infty} f_{H_{n_k}}(0)$ and $f'(0) > 0$, we see that $f = f_H|_{\mathbb{D}}$ for some \mathbb{D} -hull H . Since $f'(0) \geq e^{-M}$, $\text{dcap}(H) \leq M$. From $f_{H_{n_k}} \xrightarrow{1.u.} f_H$ in \mathbb{D} we get $H_{n_k} \rightarrow H$. □

Remark We may compactify the space of \mathbb{D} -hulls by adding one element H_∞ with the associated function $f_{H_\infty} \equiv 0$ in \mathbb{D} , and defining the metric $d_{\mathcal{H}}$ in the extended space using (10.1).

Lemma 10.2 *For any compact $F \subsetneq \mathbb{T}$, $\mathcal{H}_F := \{H : S_H \subset F\}$ is compact.*

Proof Let $H \in \mathcal{H}_F$. From conformal invariance, the harmonic measure of $\mathbb{T} \setminus H^{\text{db}}$ in $\mathbb{D} \setminus H$ seen from 0 equals to the harmonic measure of $\mathbb{T} \setminus S_H$ in \mathbb{D} seen from 0, which is bounded below by $|\mathbb{T} \setminus F|/|\mathbb{T}| > 0$. This implies that the distance between 0 and H

is bounded below by a positive constant r depending on F , which then implies that $\text{dcap}(H)$ is bounded above by $-\ln(r) < \infty$. From Lemma 10.1, we see that \mathcal{H}_F is relatively compact.

It remains to show that \mathcal{H}_F is bounded. Let (H_n) be a sequence in \mathcal{H}_F , which converges to H . We need to show that $H \in \mathcal{H}_F$. Since $H_n \in \mathcal{H}_F$, each f_{H_n} is analytic in $\widehat{\mathbb{C}} \setminus F$. We have $f_{H_n} \xrightarrow{\text{l.u.}} f_H$ in \mathbb{D} . From \mathbb{T} -symmetry, $f_{H_n} \xrightarrow{\text{l.u.}} f_H$ in \mathbb{D}^* . Let $J = \{|z| = 2\} \subset \mathbb{D}^*$. Then $f_{H_{n_k}} \rightarrow f_H$ uniformly on J . Since f_{H_n} maps $\{|z| < 2\} \setminus F$ into the Jordan domain bounded by $f_{H_n}(J)$, we see that the family (f_{H_n}) is uniformly bounded in $\{|z| < 2\} \setminus F$. So it contains a subsequence $(f_{H_{n_k}})$, which converges locally uniformly in $\{|z| < 2\} \setminus F$. The limit function is analytic in $\{|z| < 2\} \setminus F$ and agrees with f_H on \mathbb{D} , which implies that f_H extends analytically across $\mathbb{T} \setminus F$. So $S_H \subset F$, i.e., $H \in \mathcal{H}_F$. \square

There is an integral formula for \mathbb{D} -hulls which is similar to (9.1). For any \mathbb{D} -hull H , there is a positive measure μ_H with support S_H such that

$$f(z) = z \cdot \exp\left(\int_{\mathbb{T}} -\frac{x+z}{x-z} d\mu_H(x)\right), \quad z \in \mathbb{C} \setminus S_H, \quad (10.2)$$

and $H_n \rightarrow H$ iff $\mu_{H_n} \rightarrow \mu_H$ weakly. Moreover, μ_H is absolutely continuous w.r.t. the Lebesgue measure on \mathbb{T} , and the density function is bounded. From this integral formula, it is easy to get the following lemma.

Lemma 10.3 *For any compact $F \subset \mathbb{T}$, $H_n \rightarrow H$ in \mathcal{H}_F implies that $f_{H_n} \xrightarrow{\text{l.u.}} f_H$ in $\mathbb{C} \setminus F$.*

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