Probab. Theory Relat. Fields (2016) 164:741–770
DOI 10.1007/s00440-015-0618-8

Comparison of quenched and annealed invariance principles for random conductance model

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Received: 11 April 2013 / Revised: 21 January 2015 / Published online: 12 February 2015

Abstract We show that there exists an ergodic conductance environment such that the weak (annealed) invariance principle holds for the corresponding continuous time random walk but the quenched invariance principle does not hold.

Mathematics Subject Classification 60K37 · 60F17 · 82C41

1 Introduction

Let $d \geq 2$ and let E_d be the set of all non oriented edges in the d-dimensional integer lattice, that is, $E_d = \{e = \{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$. Let $\{\mu_e\}_{e \in E_d}$ be a random process with non-negative values, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The process $\{\mu_e\}_{e \in E_d}$ represents random conductances. We write $\mu_{xy} = \mu_{yx} = \mu_{\{x,y\}}$ and set $\mu_{xy} = 0$ if $\{x, y\} \notin E_d$. Set

Research supported in part by NSF Grant DMS-1206276, by NSERC, Canada, and Trinity College, Cambridge, and by MTA Rényi "Lendulet" Groups and Graphs Research Group.

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$$\mu_x = \sum_y \mu_{xy}, \qquad P(x, y) = \frac{\mu_{xy}}{\mu_x},$$

with the convention that $0/0 = 0$ and $P(x, y) = 0$ if $\{x, y\} \notin E_d$. For a fixed $\omega \in \Omega$, let $X = \{X_t, t \geq 0, P^x_\omega, x \in \mathbb{Z}^d\}$ be the continuous time random walk on $\mathbb{Z}^d$, with transition probabilities $P(x, y) = P_\omega(x, y)$, and exponential waiting times with mean $1/\mu_x$. The corresponding expectation will be denoted E^x_ω. For a fixed $\omega \in \Omega$, the generator $\mathcal{L}$ of X is given by

$$\mathcal{L} f(x) = \tfrac{1}{2} \sum_y \mu_{xy}(f(y) - f(x)). \tag{1.1}$$

In [4] this is called the *variable speed random walk* (VSRW) among the conductances μ_e. (We have inserted here a factor of $\frac{1}{2}$—see Remark 1.5(5).) This model, of a reversible (or symmetric) random walk in a random environment, is often called the random conductance model (RCM).

We are interested in functional Central Limit Theorems (CLTs) for the process X. Given any process X, for $\varepsilon > 0$, set $X^{(\varepsilon)}_t = \varepsilon X_{t/\varepsilon^2}$, $t \geq 0$. Let $\mathcal{D}_T = D([0, T], \mathbb{R}^d)$ denote the Skorokhod space, and let $\mathcal{D}_\infty = D([0, \infty), \mathbb{R}^d)$. Write d_S for the Skorokhod metric and $\mathcal{B}(\mathcal{D}_T)$ for the σ-field of Borel sets in the corresponding topology. Let X be the canonical process on $\mathcal{D}_\infty$ or $\mathcal{D}_T$, P_{BM} be Wiener measure on $(\mathcal{D}_\infty, \mathcal{B}(\mathcal{D}_\infty))$ and let E_{BM} be the corresponding expectation. We will write W for a standard Brownian motion. It will be convenient to assume that $\{\mu_e\}_{e \in E_d}$ are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and that X is defined on $(\Omega, \mathcal{F}) \times (\mathcal{D}_\infty, \mathcal{B}(\mathcal{D}_\infty))$ or $(\Omega, \mathcal{F}) \times (\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$. We also define the averaged or annealed measure $\mathbf{P}$ on $(\mathcal{D}_\infty, \mathcal{B}(\mathcal{D}_\infty))$ or $(\mathcal{D}_T, \mathcal{B}(\mathcal{D}_T))$ by

$$\mathbf{P}(G) = \mathbb{E}\, P^0_\omega(G).$$

Definition 1.1 For a bounded function F on $\mathcal{D}_T$ and a constant matrix Σ, let $\Psi^F_\varepsilon = E^0_\omega F(X^{(\varepsilon)})$ and $\Psi^F_\Sigma = E_{\mathrm{BM}} F(\Sigma W)$. In the remaining part of the definition we assume that Σ is not identically zero.

(i) We say that the *Quenched Functional CLT* (QFCLT) holds for X with limit ΣW if for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $\Psi^F_\varepsilon \to \Psi^F_\Sigma$ as $\varepsilon \to 0$, with $\mathbb{P}$-probability 1.

(ii) We say that the *Weak Functional CLT* (WFCLT) holds for X with limit ΣW if for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $\Psi^F_\varepsilon \to \Psi^F_\Sigma$ as $\varepsilon \to 0$, in $\mathbb{P}$-probability.

(iii) We say that the *Averaged (or Annealed) Functional CLT* (AFCLT) holds for X with limit ΣW if for every $T > 0$ and every bounded continuous function F on $\mathcal{D}_T$ we have $\mathbb{E}\, \Psi^F_\varepsilon \to \Psi^F_\Sigma$. This is the same as standard weak convergence with respect to the probability measure $\mathbf{P}$.

Since the functions F in this definition are bounded, it is immediate that QFCLT $\Rightarrow$ WFCLT $\Rightarrow$ AFCLT. One could consider a more general form of the WFCLT and QFCLT in which one allows the matrix Σ to depend on the environment $\mu_\cdot(\omega)$. However, if the environment is stationary and ergodic, then Σ is a shift invariant function of the environment, so must be $\mathbb{P}$—a.s. constant.

In [12] it is proved that if μ_e is a stationary ergodic environment with $\mathbb{E}\,\mu_e < \infty$ then the WFCLT holds (here $\Sigma \equiv 0$ is allowed). It is an open question as to whether the QFLCT holds under these hypotheses. For the QFCLT in the case of percolation see [7,15,18], and for the Random Conductance Model with μ_e i.i.d see [2,4,10,16]. In the i.i.d. case the QFCLT holds (with $\Sigma \not\equiv 0$) for any distribution of μ_e provided $p_+ = \mathbb{P}(\mu_e > 0) > p_c$, where p_c is the critical probability for bond percolation in $\mathbb{Z}^d$.

Definition 1.2 For $1 \le i < j \le d$ let T_{ij} be the isometry of $\mathbb{Z}^d$ defined by interchanging the ith and jth coordinates, and T_i be the isometry defined by $T_i(x_1, \dots, x_i, \dots, x_d) = (x_1, \dots, -x_i, \dots, x_d)$. We say that an environment (μ_e) on $\mathbb{Z}^d$ is *symmetric* if the law of (μ_e) is invariant under $T_i, 1 \le i \le d$ and $\{T_{ij}, 1 \le i < j \le d\}$.

If (μ_e) is stationary, ergodic and symmetric, and the WFCLT holds with limit ΣW then the limiting covariance matrix $\Sigma^T \Sigma$ must also be invariant under symmetries of $\mathbb{Z}^d$, so must be a constant $\sigma \ge 0$ times the identity.

Our first result concerns the relation between the weak and averaged FCLT. In general, of course, for a sequence of random variables ξ_n, convergence of $\mathbb{E}\,\xi_n$ does not imply convergence in probability. However, in the context of the RCM, the AFCLT and WFCLT are equivalent.

Theorem 1.3 *Suppose the AFCLT holds. Then the WFCLT holds.*

A slightly more general result is given in Theorem 2.14 below. Our second result concerns the relation between the weak and quenched FCLT.

Theorem 1.4 *Let $d = 2$ and $p < 1$. There exists a symmetric stationary ergodic environment $\{\mu_e\}_{e \in E_2}$ with $\mathbb{E}(\mu_e^p \vee \mu_e^{-p}) < \infty$ and a sequence $\varepsilon_n \to 0$ such that*

(a) *the WFCLT holds for $X^{(\varepsilon_n)}$ with limit W,*
but
(b) *the QFCLT does not hold for $X^{(\varepsilon_n)}$ with limit ΣW for any Σ.*

Remark 1.5 1. Under the weaker condition that $\mathbb{E}\,\mu_e^p < \infty$ and $\mathbb{E}\,\mu_e^{-q} < \infty$ with $p < 1$, $q < 1/2$ we have the full WFCLT for $X^{(\varepsilon)}$ as $\varepsilon \to 0$, i.e., not just along a sequence ε_n. However, the proof of this is very much harder and longer than that of Theorem 1.4(a)—see [5]. (Since our environment has $\mathbb{E}\,\mu_e = \infty$ we cannot use the results of [12].) We have chosen to use in this paper essentially the same environment as in [5], although for Theorem 1.4 a slightly simpler environment would have been sufficient.

2. Biskup [9] has proved that the QFCLT holds with $\sigma > 0$ if $d = 2$ and (μ_e) are symmetric and ergodic with $\mathbb{E}(\mu_e \vee \mu_e^{-1}) < \infty$.

3. See Remark 6.4 for how our example can be adapted to $\mathbb{Z}^d$ with $d \geq 3$; in that case we have the same moment conditions as in Theorem 1.4.
4. In [1] it is proved that the QFCLT holds (in $\mathbb{Z}^d$, $d \geq 2$) for stationary symmetric ergodic environments (μ_e) under the conditions $\mathbb{E}\,\mu_e^p < \infty$, $\mathbb{E}\,\mu_e^{-q} < \infty$, with $p^{-1} + q^{-1} < 2/d$.
5. If $\mu_e \equiv 1$ then due to the normalisation factor $\frac{1}{2}$ in (1.1), the vertical jumps of X occur at rate 1, and the FCLT holds for X with limit W.

The remainder of the paper after Sect. 2 constitutes the proof of Theorem 1.4. The argument is split into several sections. In the proof, we will discuss the conditions listed in Definition 1.1 for $T = 1$ only, as it is clear that the same argument works for general $T > 0$.

2 Averaged and weak invariance principles

The basic setup will be slightly more general in this section than in the introduction. As in the Introduction, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, fix some $T > 0$ and let $\mathcal{D} = \mathcal{D}_T$ in this section (although we will also use $\mathcal{D}_{2T}$). Recall that X is the coordinate/identity process on $\mathcal{D}$. Let $C(\mathcal{D})$ be the family of all functions $F : \mathcal{D} \to \mathbb{R}$ which are continuous in the Skorokhod topology. In the following definition, P_n^ω will stand for a probability measure (not necessarily arising from an RCM) on $\mathcal{D}$ for $\omega \in \Omega$ and $n \geq 1$. We will also refer to a probability measure P_0 on $\mathcal{D}$. The corresponding expectations will be denoted E_n^ω and E_0. The following definition was first introduced in [14], see also [12].

Definition 2.1 We will say that P_n^ω *converge weakly in measure* to P_0 if for each bounded $F \in C(\mathcal{D})$,

$$E_n^\omega F(X) \to E_0 F(X) \ \text{ in } \ \mathbb{P} \text{ probability}. \tag{2.1}$$

Let $\delta_n \to 0$, let $\Lambda_n = \delta_n \mathbb{Z}^d$, and let λ_n be counting measure on Λ_n normalized so that $\lambda_n \to dx$ weakly, where dx is Lebesgue measure on $\mathbb{R}^d$. Suppose that for each ω and $n \geq 1$ we have Markov processes $X^{(n)} = (X_t, t \geq 0, P_{\omega,n}^x, x \in \Lambda_n)$ with values in Λ_n. The corresponding expectations will be denoted $E_{\omega,n}^x$. Write

$$T_t^{(\omega,n)} f(x) = E_{\omega,n}^x f(X_t)$$

for the semigroup of $X^{(n)}$. Since we are discussing weak convergence, it is natural to put the index n in the probability measures $P_{\omega,n}^x$ rather than the process; however we will sometimes abuse notation and refer to $X^{(n)}$ rather than X under the laws $(P_{\omega,n}^x)$. Recall that W denotes a standard Brownian motion.

For the remainder of this section, we will suppose that the following Assumption holds.

Assumption 2.2 1. For each ω, the semigroup $T_t^{(\omega,n)}$ is self adjoint on $L^2(\Lambda_n, \lambda_n)$.

2. The $\mathbb{P}$ law of the ‘environment’ for $X^{(n)}$ is stationary. More precisely, for $x \in \Lambda_n$ there exist measure preserving maps $T_x : \Omega \to \Omega$ such that for all bounded measurable F on $\mathcal{D}_T$,

$$E^x_{\omega,n} F(X) = E^0_{T_x\omega,n} F(X + x), \tag{2.2}$$

$$\mathbb{E}\, E^0_{T_x\omega,n} F(X) = \mathbb{E}\, E^0_{\omega,n} F(X). \tag{2.3}$$

3. The AFCLT holds, that is for all $T > 0$ and bounded continuous F on $\mathcal{D}_T$,

$$\mathbb{E}\, E^0_{\omega,n} F(X) \to E_{\mathrm{BM}} F(X).$$

Given a function F from $\mathcal{D}_T$ to $\mathbb{R}$ set

$$F_x(w) = F(x + w), \quad x \in \mathbb{R}^d, \ w \in \mathcal{D}_T.$$

Note that combining (2.2) and (2.3) we obtain

$$\mathbb{E}\, E^x_{\omega,n} F(X) = \mathbb{E}\, E^0_{\omega,n} F_x(X), \quad x \in \Lambda_n.$$

Set

$$\mathcal{T}^n_t f(x) = \mathbb{E}\, T^{(\omega,n)}_t f(x).$$

Note that $\mathcal{T}^{(n)}_t$ is not in general a semigroup. Write K_t for the semigroup of Brownian motion on $\mathbb{R}^d$. We also need notation for expectation of general functions F on $\mathcal{D}_T$, so we define

$$\begin{aligned} T^{(\omega,n)} F(x) &= E^x_{\omega,n} F(X), \\ \mathcal{T}^{(n)} F(x) &= \mathbb{E}\, E^x_{\omega,n} F(X), \\ \mathcal{K} F(x) &= E_{BM} F(x + W), \\ U^{(\omega,n)} F(x) &= T^{(\omega,n)} F(x) - \mathcal{K} F(x). \end{aligned}$$

Using this notation, the AFCLT states that for $F \in C(\mathcal{D}_T)$

$$\mathcal{T}^{(n)} F(0) \to \mathcal{K} F(0). \tag{2.4}$$

Definition 2.3 Fix $T > 0$ and recall that $\mathcal{D} = \mathcal{D}_T$. Write d_U for the uniform norm, i.e.,

$$d_U(w, w') = \sup_{0 \le s \le T} |w(s) - w'(s)|.$$

Recall that we defined $d_S(w, w')$ to be the usual Skorokhod metric on $\mathcal{D}$. We have $d_S(w, w') \leq d_U(w, w')$, but the topologies given by the two metrics are distinct. Let $\mathcal{M}(\mathcal{D})$ be the set of measurable F on $\mathcal{D}$. A function $F \in \mathcal{M}(\mathcal{D})$ is uniformly continuous in the uniform norm on $\mathcal{D}$ if there exists $\rho(\varepsilon)$ with $\lim_{\varepsilon\to 0}\rho(\varepsilon) = 0$ such that if $w, w' \in \mathcal{D}_T$ with $d_U(w, w') \leq \varepsilon$ then

$$|F(w) - F(w')| \leq \rho(\varepsilon). \tag{2.5}$$

Write $C_U(\mathcal{D})$ for the set of F in $\mathcal{M}(\mathcal{D})$ which are uniformly continuous in the uniform norm. Note that we do not have $C_U(\mathcal{D}) \subset C(\mathcal{D})$.

Let $C_0^1(\mathbb{R}^d)$ denote the set of continuously differentiable functions with compact support. Let $\mathcal{A}_m$ be the set of F such that

$$F(w) = \prod_{i=1}^{m} f_i(w(t_i)), \tag{2.6}$$

where $0 \leq t_1 \leq \ldots t_m \leq T,\ f_i \in C_0^1(\mathbb{R}^d)$, and let $\mathcal{A} = \bigcup_m \mathcal{A}_m$.

Lemma 2.4 *Let $F \in \mathcal{A}$. Then $F \in C_U(\mathcal{D})$, and $\mathcal{K}F \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$.*

Proof Let $f \in \mathcal{A}_m$. Choose $C \geq 2$ so that $||f_i||_\infty \leq C$ and $|f_i(x) - f_i(y)| \leq C|x - y|$ for all x, y, i. Then

$$|F(w) - F(w')| \leq mC^m d_U(w, w').$$

Since f_i are bounded and continuous, so is $\mathcal{K}F$. Also, $|F| \leq C^{m-1}|f(w(t_1))|$, so

$$\begin{aligned}\left|\int \mathcal{K}F(x)dx\right| &\leq \int \mathcal{K}|F|(x)dx \leq C^{m-1}\langle K_{t_1}|f_1|, 1\rangle \\ &= C^{m-1}\langle |f_1|, 1\rangle = C^{m-1}||f_1||_1 < \infty.\end{aligned}$$

□

Lemma 2.5 *For all $F \in \mathcal{M}(\mathcal{D})$,*

$$\begin{aligned} T^{(\omega,n)}F(x) &\overset{(d)}{=} T^{(\omega,n)}F_x(0), \\ U^{(\omega,n)}F(x) &\overset{(d)}{=} U^{(\omega,n)}F_x(0). \end{aligned} \tag{2.7}$$

Proof By the stationarity of the environment,

$$T^{(\omega,n)}F(x) = E^x_{\omega,n}F(X) = E^0_{T_x\omega,n}F(X + x) =^{(d)} E^0_{\omega,n}F(X + x) = T^{(\omega,n)}F_x(0).$$

The result for $U^{(\omega,n)}$ is then immediate. □

Lemma 2.6 *Let $F \in C_U(\mathcal{D}_T)$. Then $T^{(\omega,n)}F_x(0)$, $U^{(\omega,n)}F_x(0)$, and $\mathcal{T}^{(n)}F(x)$ are uniformly continuous on Λ_n for every $n \in \mathbb{N}$, with a modulus of continuity which is independent of n.*

Proof If $|x-y| \le \varepsilon$ then $d_U(w+x, w+y) \le \varepsilon$, so if $F \in C_U(\mathcal{D}_T)$ and ρ is such that (2.5) holds, then $|F_x(w) - F_y(w)| \le \rho(\varepsilon)$, and hence

$$\begin{aligned}|T_t^{(\omega,n)}F_x(0) - T_t^{(\omega,n)}F_y(0)| &= |E^0_{\omega,n}F(x+X) - E^0_{\omega,n}F(y+X)| \\ &\le E^0_{\omega,n}|F(x+X) - F(y+X)| \le \rho(\varepsilon).\end{aligned}$$

This implies the uniform continuity of $T^{(\omega,n)}F_x(0)$ and $U^{(\omega,n)}F_x(0)$. By (2.7),

$$\mathcal{T}^{(n)}F(x) = \mathbb{E}\,T^{(\omega,n)}F(x) = \mathbb{E}\,T^{(\omega,n)}F_x(0),$$

so the uniform continuity of $\mathcal{T}^{(n)}F(x)$ follows from that of $T^{(\omega,n)}F_x(0)$. □

Lemma 2.7 *Let $F \in \mathcal{A}$. Then*

$$\mathcal{T}^{(n)}F(x) \to \mathcal{K}F(x) \quad \textit{for all } x \in \mathbb{R}^d. \tag{2.8}$$

Proof The AFCLT (in 2.2) implies that $\mathbb{E}\,P^0_{\omega,n}$ converge weakly to P_{BM}. Hence the finite dimensional distributions of $X^{(n)}$ converge to those of W, and this is equivalent to (2.8). □

Let $C_b(\mathbb{R}^d)$ denote the space of bounded continuous functions on $\mathbb{R}^d$.

Lemma 2.8 *Let $F \in \mathcal{A}$, and $h \in C_b(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Then*

$$\int h(x)\mathcal{T}^{(n)}F(x)\lambda_n(dx) \to \int h(x)\mathcal{K}F(x)dx. \tag{2.9}$$

Proof This is immediate from (2.8) and the uniform continuity proved in Lemma 2.6. □

The next Lemma gives the key construction in this section: using the self-adjointness of $T_t^{(\omega,n)}$ we can linearise expectations of products. A similar idea is used in [19] in the context of transition densities.

Let $F \in \mathcal{A}_m$ be given by (2.6). Set $s_j = t_m - t_{m-j}$, and let

$$\widehat{F}(w) = \prod_{j=1}^{m-1} f_{m-j}(w_{s_j}) \prod_{j=1}^{m} f_j(w_{t_m+t_j}).$$

Note that $\widehat{F}$ is defined on functions $w \in \mathcal{D}_{2T}$ (not $\mathcal{D}_T$). Write $\langle f, g\rangle_n$ for the inner product in $L^2(\lambda_n)$ and $\langle f, g\rangle$ for the inner product in $L^2(\mathbb{R}^d)$.

Lemma 2.9 *With F and $\widehat{F}$ as above,*

$$\int (T^{(\omega,n)}F(x))^2\lambda_n(dx) = \int (T^{(\omega,n)}\widehat{F}(x)) f_m(x)\lambda_n(dx), \tag{2.10}$$

$$\int (\mathcal{K}F(x))^2 dx = \int (\mathcal{K}\widehat{F}(x)) f_m(x) dx. \tag{2.11}$$

Proof Using the Markov property of $X^{(n)}$

$$T^{(\omega,n)}F(x) = E^x_{\omega,n}\prod_{j=1}^m f_j(w_{t_j}) = E^x_{\omega,n}\left(\prod_{j=1}^{m-1} f_j(w_{t_j}) T^{(\omega,n)}_{t_m-t_{m-1}} f_m(X_{t_{m-1}})\right).$$

Hence we obtain

$$T^{(\omega,n)}F(x) = T^{(\omega,n)}_{t_1}\left(f_1 T^{(\omega,n)}_{t_2-t_1}\left(f_2 \dots T^{(\omega,n)}_{t_m-t_{m-1}} f_m(x)\dots\right)\right).$$

Using the self-adjointness of $T^{(\omega,n)}_t$ gives

$$\begin{aligned}
&\langle T^{(\omega,n)}F, T^{(\omega,n)}F\rangle_n \\
&\quad = \langle T^{(\omega,n)}_{t_1} f_1 T^{(\omega,n)}_{t_2-t_1} f_2 \dots T^{(\omega,n)}_{t_m-t_{m-1}} f_m, T^{(\omega,n)}_{t_1} f_1 T^{(\omega,n)}_{t_2-t_1} f_2 \dots T^{(\omega,n)}_{t_m-t_{m-1}} f_m\rangle_n \\
&\quad = \langle f_1 T^{(\omega,n)}_{t_1} T^{(\omega,n)}_{t_1} f_1 T^{(\omega,n)}_{t_2-t_1} f_2 \dots T^{(\omega,n)}_{t_m-t_{m-1}} f_m, T^{(\omega,n)}_{t_2-t_1} f_2 \dots T^{(\omega,n)}_{t_m-t_{m-1}} f_m\rangle_n.
\end{aligned}$$

Continuing in this way we obtain

$$\begin{aligned}
&\langle T^{(\omega,n)}F, T^{(\omega,n)}F\rangle_n \\
&\quad = \langle T^{(\omega,n)}_{t_m-t_{m-1}} f_{m-1} T^{(\omega,n)}_{t_{m-1}-t_{m-2}} f_{m-2} \dots f_1 T^{(\omega,n)}_{t_1} T^{(\omega,n)}_{t_1} f_1 \dots T^{(\omega,n)}_{t_m-t_{m-1}} f_m, f_m\rangle_n \\
&\quad = \langle T^{(\omega,n)}\widehat{F}, f_m\rangle_n.
\end{aligned}$$

The proof for $\mathcal{K}$ is exactly the same. □

Lemma 2.10 *Let $F \in \mathcal{A}$. Then*

$$\mathbb{E}\int (T^{(\omega,n)}F(x) - \mathcal{K}F(x))^2\lambda_n(dx) \to 0. \tag{2.12}$$

Proof We have

$$\begin{aligned}
\int (T^{(\omega,n)}F(x) - \mathcal{K}F(x))^2\lambda_n(dx) &= \langle (T^{(\omega,n)}F - \mathcal{K}F), (T^{(\omega,n)}F - \mathcal{K}F)\rangle_n \\
&= \langle T^{(\omega,n)}F, T^{(\omega,n)}F\rangle_n - 2\langle T^{(\omega,n)}F, \mathcal{K}F\rangle_n + \langle \mathcal{K}F, \mathcal{K}F\rangle_n.
\end{aligned}$$

Thus

$$\mathbb{E}\int (T^{(\omega,n)}F(x) - \mathcal{K}F(x))^2\lambda_n(dx)$$
$$= \mathbb{E}\,\langle T^{(\omega,n)}F, T^{(\omega,n)}F\rangle_n - 2\langle \mathcal{T}^{(n)}F, \mathcal{K}F\rangle_n + \langle \mathcal{K}F, \mathcal{K}F\rangle_n. \tag{2.13}$$

Since $\mathcal{K}F$ is continuous we have

$$\langle \mathcal{K}F, \mathcal{K}F\rangle_n \to \langle \mathcal{K}F, \mathcal{K}F\rangle.$$

Taking $h = \mathcal{K}F$ and using Lemma 2.4, Lemma 2.8 gives that

$$\langle \mathcal{T}^{(n)}F, \mathcal{K}F\rangle_n \to \langle \mathcal{K}F, \mathcal{K}F\rangle.$$

Let f_m and $\widehat{F}$ be as in the the previous lemma. Then

$$\mathbb{E}\,\langle T^{(\omega,n)}F, T^{(\omega,n)}F\rangle_n = \mathbb{E}\,\langle T^{(\omega,n)}\widehat{F}, f_m\rangle_n = \langle \mathcal{T}^{(n)}\widehat{F}, f_m\rangle_n.$$

Again by Lemma 2.8 and (2.11),

$$\langle \mathcal{T}^{(n)}\widehat{F}, f_m\rangle_n \to \langle \mathcal{K}\widehat{F}, f_m\rangle = \langle \mathcal{K}F, \mathcal{K}F\rangle.$$

Adding the limits of the three terms in (2.13), we obtain (2.12). □

Lemma 2.11 *Let $F \in \mathcal{A}$. Then*

$$T^{(\omega,n)}F(0) \to \mathcal{K}F(0) \ \ \textit{in} \ \ \mathbb{P}\textit{-probability}. \tag{2.14}$$

Proof The previous lemma gives

$$\mathbb{E}\int (U^{(\omega,n)}F(x))^2\lambda_n(dx) \to 0.$$

Using Lemma 2.5 we have

$$\mathbb{E}\int (U^{(\omega,n)}F_x(0))^2\lambda_n(dx) \to 0, \tag{2.15}$$

and using the uniform continuity of $U^{(\omega,n)}F_x(0)$ gives (2.14). □

Write $\mathbb{D}$ for the set of dyadic rationals.

Proposition 2.12 *Given any subsequence (n_k) there exists a subsequence (n'_k) of (n_k) and a set Ω_0 with $\mathbb{P}(\Omega_0) = 1$, such that for any $\omega \in \Omega_0$ and $q_1 \le q_2 \le \cdots \le q_m$ with $q_i \in \mathbb{D}$, the r.v. $(X_{q_i}, i = 1, \ldots, m)$ under P^0_{ω,n'_k} converge in distribution to $(W_{q_i}, i = 1, \ldots, m)$.*

Proof Let $\mathbb{D}_T = [0, T] \cap \mathbb{D}$. Fix a finite set $q_1 \le \cdots \le q_m$ with $q_i \in \mathbb{D}_T$. Then convergence of $(X_{q_i}, i = 1, \dots, m, P^0_{\omega,n})$ is determined by a countable set of functions $F_i \in \mathcal{A}_m$. So by Lemma 2.11 we can find nested subsequences $(n_k^{(i)})$ of (n_k) such that for each i

$$\lim_{k\to\infty} P^0_{(\omega,n_k^{(i)})} F_j(0) = \mathcal{K} F_j(0) \qquad \mathbb{P}\text{-a.s.}, \quad \text{for } 1 \le j \le i.$$

A diagonalization argument then implies that there exists a subsequence n''_k such that $(X_{q_i}, i = 1, \dots, m, P^0_{\omega,n''_k})$ converge in distribution to $(W_{q_i}, i = 1, \dots, m)$. Since the set of the finite sets $\{q_1, \dots, q_m\}$ is countable, an additional diagonalization argument then implies that there exists a subsequence (n'_k) such that this convergence holds for all such finite sets. □

Lemma 2.13 *If AFCLT holds then "tightness in probability" holds, i.e., for any $\delta > 0$ there exist $\delta_1 > 0$ and n_1 such that for $n \ge n_1$, there is a set A_n of ω with $\mathbb{P}(A_n) \ge 1-\delta$, such that for $\omega \in A_n$,*

$$P^0_{\omega,n}\left(\sup_{0\le s\le t\le T, t-s\le\delta_1} |X_s^{(n)} - X_t^{(n)}| \ge \delta\right) < \delta. \tag{2.16}$$

Proof If AFCLT holds then, by the Skorokhod Lemma, we can construct $X^{(n)}$ and W on a common probability space, in such a way that each $X^{(n)}$ has the distribution $\mathbb{E}\, P^0_{\omega,n}$ and $X^{(n)} \to W$ in the Skorokhod topology, a.s.

Fix any $\delta > 0$. By continuity of Brownian motion there exists $\delta_1 > 0$ such that

$$P_{BM}\left(\sup_{0\le s\le t\le T, t-s\le\delta_1} |W_s - W_t| \ge \delta\right) < \delta. \tag{2.17}$$

If a sequence of processes converges in the Skorokhod topology to a continuous process then it converges also in the uniform sense. Hence, in view of (2.17), there exists n_1 such that for $n \ge n_1$,

$$\mathbb{E}\, P^0_{\omega,n}\left(\sup_{0\le s\le t\le T, t-s\le\delta_1} |X_s^{(n)} - X_t^{(n)}| \ge 2\delta\right) < 2\delta.$$

This implies that for $n \ge n_1$, there is a set A_n of ω with $\mathbb{P}(A_n) \ge 1 - \sqrt{2\delta}$, such that for $\omega \in A_n$,

$$P^0_{\omega,n}\left(\sup_{0\le s\le t\le T, t-s\le\delta_1} |X_s^{(n)} - X_t^{(n)}| \ge 2\delta\right) < \sqrt{2\delta}.$$

It is elementary to convert the form of this estimate to the form given in the lemma. □

Theorem 2.14 *If Assumption 2.2 holds then $P^0_{\omega,n}$ converge weakly in measure to P_{BM}.*

Proof Fix any $T > 0$, an arbitrarily small $\varepsilon > 0$ and any bounded function $F \in C(\mathcal{D}_T)$. Let W denote Brownian motion and suppose that processes Y and W are defined on the same probability space, for which we use the generic notation P and E. It is easy to see that one can find $\delta \in (0, \varepsilon/2)$ so small that if the process Y satisfies

$$P\left(\sup_{0\le t\le T} |Y_t - W_t| \ge 3\delta\right) < 3\delta, \tag{2.18}$$

then

$$|EF(Y) - EF(W)| < \varepsilon. \tag{2.19}$$

Let $\delta_1 > 0$ be so small that (2.16) and (2.17) hold with the present choice of δ. Suppose that $0 = q_1 \le q_2 \le \cdots \le q_m = T$ are dyadic rationals and $q_k - q_{k-1} \le \delta_1$ for all k (note that we can assume that T is a dyadic rational without loss of generality). By Proposition 2.12, we can find a sequence n_k such that the joint distributions of the random variables $(X_{q_i}, i = 1, \ldots, m)$ under P^0_{ω,n_k} converge to the distribution of $(W_{q_i}, i = 1, \ldots, m)$, as $k \to \infty$, $\mathbb{P}$-a.s. By the Skorokhod Lemma, we can construct $(X^{\omega,n_k}_{q_i}, i = 1, \ldots, m)$ and $(W^{\omega,n_k}_{q_i}, i = 1, \ldots, m)$ on the same probability space $(\Omega_\omega, \mathcal{F}_\omega, P_\omega)$ so that

$$(X^{\omega,n_k}_{q_i}, i = 1, \ldots, m) \to (W^{\omega,n_k}_{q_i}, i = 1, \ldots, m), \qquad P_\omega\text{-a.s., } \mathbb{P}\text{-a.s.,} \tag{2.20}$$

$(X^{\omega,n_k}_{q_i}, i = 1, \ldots, m)$ has the same distribution under P_ω as $(X_{q_i}, i = 1, \ldots, m)$ under P^0_{ω,n_k}, and $(W^{\omega,n_k}_{q_i}, i = 1, \ldots, m)$ has the same distribution under P_ω as Brownian motion (sampled at a finite number of times).

Using conditional probabilities and enlarging the probability space, if necessary, we can assume that there exist processes $(X^{\omega,n_k}_t, 0 \le t \le T)$ and $(W^{\omega,n_k}_t, 0 \le t \le T)$ on the same probability space $(\Omega_\omega, \mathcal{F}_\omega, P_\omega)$ such that $(X^{\omega,n_k}_t, 0 \le t \le T)$ has the same distribution under P_ω as $(X_t, 0 \le t \le T)$ under P^0_{ω,n_k}, $(W^{\omega,n_k}_t, 0 \le t \le T)$ is Brownian motion, and all the conditions stated in the previous paragraph hold for these processes sampled at q_i, $i = 1, \ldots, m$; in particular, (2.20) holds.

It follows from (2.20) that there exist an event H with $\mathbb{P}(H) > 1 - \delta$ and k_1 such that for $k \ge k_1$ and each $\omega \in H$,

$$P_\omega(|X^{\omega,n_k}_{q_k} - W^{\omega,n_k}_{q_k}| < \delta, \forall\, k = 1, \ldots, m) \ge 1 - \delta. \tag{2.21}$$

By Lemma 2.13, for $k \ge k_2$, there is a set A_k of ω with $\mathbb{P}(A_k) \ge 1 - \delta$, such that for $\omega \in A_k$,

$$P^0_{\omega,n_k}\left(\sup_{0\le s\le t\le T, t-s\le\delta_1} |X^{(n_k)}_s - X^{(n_k)}_t| \ge \delta\right) < \delta. \tag{2.22}$$

Since $(X^{\omega,n_k}_t, 0 \le t \le T)$ has the same distribution under P_ω as $(X^{(n_k)}_t, 0 \le t \le T)$ under P^0_{ω,n_k}, it follows from (2.22) that for $k \ge k_2$, there is a set A_k of ω with $\mathbb{P}(A_k) \ge 1 - \delta$, such that for $\omega \in A_k$,

$$P_\omega\left(\sup_{0\le s\le t\le T, t-s\le\delta_1}|X_s^{\omega,n_k}-X_t^{\omega,n_k}|\ge\delta\right)<\delta. \tag{2.23}$$

For similar reasons, (2.17) implies that

$$P_\omega\left(\sup_{0\le s\le t\le T, t-s\le\delta_1}|W_s^{\omega,n_k}-W_t^{\omega,n_k}|\ge\delta\right)<\delta. \tag{2.24}$$

We now combine (2.21), (2.23) and (2.24) to conclude that for $k\ge k_1\vee k_2$, there is a set $H\cap A_k$ of ω with $\mathbb{P}(H\cap A_k)\ge 1-2\delta$, such that for $\omega\in H\cap A_k$,

$$P_\omega\left(\sup_{0\le t\le T}|X_t^{\omega,n_k}-W_t^{\omega,n_k}|\ge 3\delta\right)<3\delta.$$

In view of (2.18)–(2.19) this implies that for $k\ge k_1\vee k_2$, there is a set $H\cap A_k$ of ω with $\mathbb{P}(H\cap A_k)\ge 1-2\delta$, such that for $\omega\in H\cap A_k$,

$$|E^0_{\omega,n_k}F(X)-EF(W)|=|E_\omega F(X^{\omega,n_k})-EF(W^{\omega,n_k})|<\varepsilon. \tag{2.25}$$

Set $\xi_n=|E^0_{\omega,n}F(X)-EF(W)|$; since $\delta<\varepsilon/2$, (2.25) implies that

$$\mathbb{P}(\xi_{n_k}>\varepsilon)<\varepsilon\quad\text{for}\ \ k\ge k_1\vee k_2. \tag{2.26}$$

We now extend this result to the whole sequence, and claim that there exists n_1 such that

$$\mathbb{P}(\xi_n>\varepsilon)<\varepsilon\quad\text{for}\ \ n\ge n_1. \tag{2.27}$$

Suppose not: then there exists a subsequence n_k^* with $\mathbb{P}(\xi_{n_k^*}>\varepsilon)\ge\varepsilon$ for all k. However, by Proposition 2.12, we can find a subsequence n_k of n_k^* such that the joint distributions of the random variables $(X_{q_i}, i=1,\dots,m)$ under P^0_{ω,n_k} converge to the distribution of $(W_{q_i}, i=1,\dots,m)$, as $k\to\infty$, $\mathbb{P}$-a.s. Applying the argument above to this subsequence, we have a contradiction to (2.26). Thus (2.27) holds, and this completes the proof of the theorem. □

3 Construction of the environment

The remainder of this paper is concerned with the proof of Theorem 1.4. The main idea of the proof as as follows. We choose a sequence a_n of integers, with $a_n\gg a_{n-1}$, and $a_n/a_{n-1}=m_n\in\mathbb{Z}$. For each n we define an ergodic tiling of $\mathbb{Z}^2$ into (disjoint) squares, each with a_n^2 points. Write $\mathcal{S}_n$ for the collection of these squares; they are defined so that each square in $\mathcal{S}_n$ is the union of m_n^2 squares in $\mathcal{S}_{n-1}$. In each square in $\mathcal{S}_n$ we place 4 obstacles of diameter $O(b_n)$, where $b_n\simeq n^{-1/2}a_n$. The obstacles are chosen so that the resulting environment is symmetric. Let F_n be the event that

0 is within a distance $O(b_n)$ of an obstacle at scale n. The obstacles are such that if F_n holds then the rescaled process $Z_n = (b_n^{-1} X_{b_n^2 t}, 0 \le t \le 1)$ will be far from a Brownian motion. Thus if F_n holds i.o. then the QFCLT will fail. On the other hand, if $\mathbb{P}(F_n) \to 0$ then with high probability Z_n will be close to BM, and (after some work) we do have the WFCLT.

We now begin by giving the construction of the sets $\mathcal{S}_n$ and the associated environment. Let $\Omega = (0, \infty)^{E_2}$, and $\mathcal{F}$ be the Borel σ-algebra defined using the usual product topology. Then every $t \in \mathbb{Z}^2$ defines a translation T_t of the environment by t. Stationarity and ergodicity of the measures defined below will be understood with respect to these transformations.

All constants (often denoted c_1, c_2, etc.) are assumed to be strictly positive and finite. For a set $A \subset \mathbb{Z}^2$ let $E(A)$ be the set of edges in A if regarded as a subgraph of $\mathbb{Z}^2$. Let $E_h(A)$ and $E_v(A)$ respectively be the set of horizontal and vertical edges in $E(A)$. Write $x \sim y$ if $\{x, y\}$ is an edge in $\mathbb{Z}^2$. Define the exterior boundary of A by

$$\partial A = \{y \in \mathbb{Z}^2 - A : y \sim x \quad \text{for some} \quad x \in A\}.$$

Let also

$$\partial_i A = \partial(\mathbb{Z}^2 - A).$$

Finally define balls in the ℓ^∞ norm by $B_\infty(x, r) = \{y : ||x - y||_\infty \le r\}$; of course this is just the square with center x and side $2r$.

Let $\{a_n\}_{n \ge 0}$, $\{\beta_n\}_{n \ge 1}$ and $\{b_n\}_{n \ge 1}$ be strictly increasing sequences of positive integers growing to infinity with n, with

$$1 = a_0 < b_1 < \beta_1 < a_1 \ll b_2 < \beta_2 < a_2 \ll b_3 \dots$$

We will impose a number of conditions on these sequences in the course of the paper. We collect these conditions here so that the reader can check that all conditions can be satisfied simultaneously. There is some redundancy in the conditions, for easy reference. (Some additional conditions on b_n/a_{n-1} are needed for the proof in [5] of the full WFCLT for $(X^{(\varepsilon)})$.)

(i) a_n is even for all n.
(ii) For each $n \ge 1$, a_{n-1} divides b_n, and b_n divides β_n and a_n.
(iii) $b_1 \ge 10^{10}$.
(iv) $a_n/\sqrt{2n} \le b_n \le a_n/\sqrt{n}$ for all n, and $b_n \sim a_n/\sqrt{n}$.
(v) $b_{n+1} \ge 2^n b_n$ for all n.
(vi) $b_n > 40 a_{n-1}$ for all n.
(vii) b_n is large enough so that (5.1) and (6.1) hold.
(viii) $100 b_n < \beta_n \le b_n n^{1/4} < 3\beta_n < a_n/10$ for all n.

These conditions do not define a_n's and b_n's uniquely. It is easy to check that there exist constants that satisfy all the conditions: if a_i, b_i, β_i have been chosen for all $i \in \{1, \dots, n-1\}$, then if b_n is chosen large enough [with care on respecting the divisibility condition in (ii)], it will satisfy all the conditions imposed on it with respect to constants of smaller indices. Then one can choose a_n and β_n so that the remaining conditions are satisfied.

We set

$$m_n = \frac{a_n}{a_{n-1}}, \qquad \ell_n = \frac{a_n}{b_n}. \tag{3.1}$$

We begin our construction by defining a collection of squares in $\mathbb{Z}^2$. Let

$$\begin{aligned} B_n &= [0, a_n]^2, \\ B'_n &= [0, a_n - 1]^2 \cap \mathbb{Z}^2, \\ \mathcal{S}_n(x) &= \{x + a_n y + B'_n : y \in \mathbb{Z}^2\}. \end{aligned}$$

Thus $\mathcal{S}_n(x)$ gives a tiling of $\mathbb{Z}^2$ by disjoint squares of side $a_n - 1$ and period a_n. We say that the tiling $\mathcal{S}_{n-1}(x_{n-1})$ is a refinement of $\mathcal{S}_n(x_n)$ if every square $Q \in \mathcal{S}_n(x_n)$ is a finite union of squares in $\mathcal{S}_{n-1}(x_{n-1})$. It is clear that $\mathcal{S}_{n-1}(x_{n-1})$ is a refinement of $\mathcal{S}_n(x_n)$ if and only if $x_n = x_{n-1} + a_{n-1}y$ for some $y \in \mathbb{Z}^2$.

Take $\mathcal{O}_1$ uniform in B'_1, and for $n \geq 2$ take $\mathcal{O}_n$, conditional on $(\mathcal{O}_1, \dots, \mathcal{O}_{n-1})$, to be uniform in $B'_n \cap (\mathcal{O}_{n-1} + a_{n-1}\mathbb{Z}^2)$. We now define random tilings by letting

$$\mathcal{S}_n = \mathcal{S}_n(\mathcal{O}_n), \quad n \geq 1.$$

Let η_n, K_n be positive constants; we will have $\eta_n \ll 1 \ll K_n$. We define conductances on E_2 as a limit of conductances for $n = 1, 2 \dots$, as follows. For each n, conductances on a tile of $\mathcal{S}_n$ will be the same for each tile. Recall that a_n is even, and let $a'_n = \frac{1}{2}a_n$. Let

$$C_n = \{(x, y) \in B_n \cap \mathbb{Z}^2 : y \geq x, x + y \leq a_n\}.$$

We first define conductances $\nu_e^{0,n}$ for $e \in E(C_n)$. Let

$$\begin{aligned} D_n^{00} &= \{(a'_n - \beta_n, y), a'_n - 10b_n \leq y \leq a'_n + 10b_n\}, \\ D_n^{01} &= \{(x, a'_n + 10b_n), (x, a'_n + 10b_n + 1), (x, a'_n - 10b_n), (x, a'_n - 10b_n - 1), \\ &\qquad a'_n - \beta_n - b_n \leq x \leq a'_n - \beta_n + b_n\}. \end{aligned}$$

Thus the set $D_n^{00} \cup D_n^{01}$ resembles the letter I (see Fig. 1).

For an edge $e \in E(C_n)$ we set

$$\begin{aligned} \nu_e^{n,0} &= \eta_n && \text{if } e \in E_v(D_n^{01}), \\ \nu_e^{n,0} &= K_n && \text{if } e \in E(D_n^{00}), \\ \nu_e^{n,0} &= 1 && \text{otherwise.} \end{aligned}$$

We then extend $\nu^{n,0}$ by symmetry to $E(B_n)$. More precisely, for $z = (x, y) \in B_n$, let $R_1 z = (y, x)$ and $R_2 z = (a_n - y, a_n - x)$, so that R_1 and R_2 are reflections in the

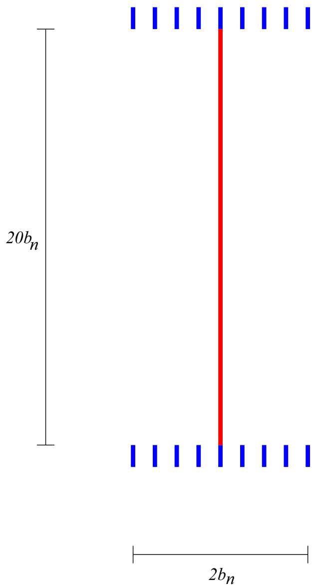

Fig. 1 The set $D_n^{00} \cup D_n^{01}$ resembles the *letter I*. The short vertical (*blue*) edges at the*top* and *bottom* of the *I* have very low conductance. The central (*red*) *line* represents edges with very high conductance. Drawing not to scale (color figure online)

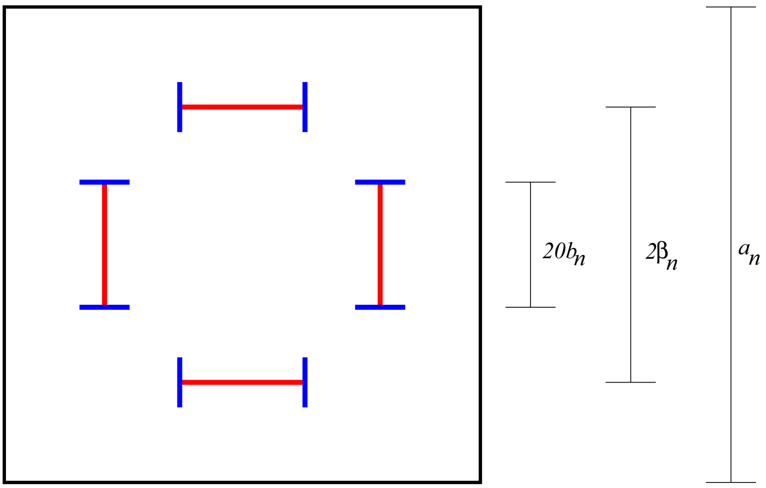

Fig. 2 The obstacle set D_n^0. Each obstacle is a copy, in some cases a rotated one, of the obstacle set given in Fig. 1

lines $y = x$ and $x + y = a_n$. We define R_i on edges by $R_i(\{x, y\}) = \{R_i x, R_i y\}$ for $x, y \in B_n$. We then extend $\nu^{0,n}$ to $E(B_n)$ so that $\nu_e^{0,n} = \nu_{R_1 e}^{0,n} = \nu_{R_2 e}^{0,n}$ for $e \in E(B_n)$. We define the *obstacle* set D_n^0 by setting (see Fig. 2),

$$D_n^0 = \bigcup_{i=0}^{1} (D_n^{0,i} \cup R_1(D_n^{0,i}) \cup R_2(D_n^{0,i}) \cup R_1 R_2(D_n^{0,i})).$$

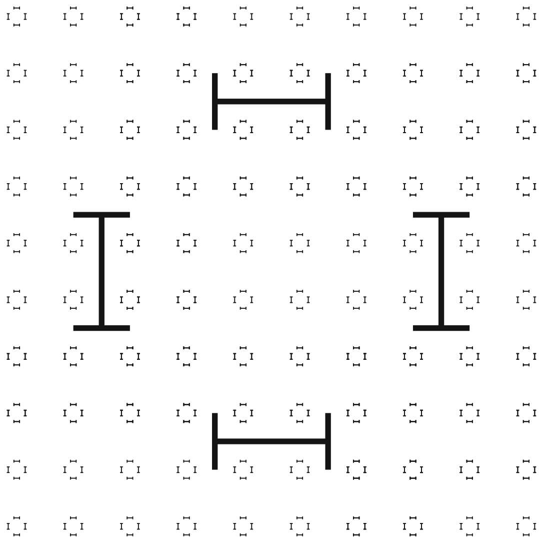

Fig. 3 Two levels of the obstacle set. Drawing not to scale

Note that $\nu_e^{n,0} = 1$ for every edge adjacent to the boundary of B_n, or indeed within a distance $a_n/4$ of this boundary. If $e = (x, y)$, we will write $e - z = (x - z, y - z)$. Next we extend $\nu^{n,0}$ to E_2 by periodicity, i.e., $\nu_e^{n,0} = \nu_{e+a_n x}^{n,0}$ for all $x \in \mathbb{Z}^2$. Finally, we define the conductances ν^n by translation by $\mathcal{O}_n$, so that

$$\nu_e^n = \nu_{e-\mathcal{O}_n}^{n,0}, \quad e \in E_2.$$

We also define the obstacle set at scale n by

$$D_n = \bigcup_{x \in \mathbb{Z}^2} (a_n x + \mathcal{O}_n + D_n^0).$$

We illustrate two levels of construction in Fig. 3.

We define the environment μ_e^n inductively by

$$\begin{aligned} \mu_e^n &= \nu_e^n \quad &&\text{if} \ \ \nu_e^n \neq 1, \\ \mu_e^n &= \mu_e^{n-1} \quad &&\text{if} \ \ \nu_e^n = 1. \end{aligned}$$

Once we have proved the limit exists, we will set

$$\mu_e = \lim_n \mu_e^n. \tag{3.2}$$

Theorem 3.1 (a) *For each n the environments $(\nu_e^n, e \in E_2)$, $(\mu_e^n, e \in E_2)$ are stationary, symmetric and ergodic.*

(b) *The limit (3.2) exists $\mathbb{P}$-a.s.*

(c) *The environment $(\mu_e, e \in E_2)$ is stationary, symmetric in the sense of Definition 1.2, and ergodic with respect to the group of translations of $\mathbb{Z}^2$.*

Proof (a) For $x = (x_1, x_2) \in \mathbb{Z}^2$ define the modulo a value of x as the unique $(y_1, y_2) \in [0, a-1]^2$ such that $x_1 \equiv y_1 \pmod a$ and $x_2 \equiv y_2 \pmod a$. We say that $x, y \in \mathbb{Z}^2$ are equivalent modulo a if their modulo a values are the same, and denote it by $x \equiv y \bmod a$.

Let $\mathcal{K}_n$ be the set of n-tuples $(x_1, \ldots, x_n)$ with $x_i \in (x_{i-1} + a_{i-1}\mathbb{Z}^2) \cap [0, a_i - 1]^2$ (with the convention $a_0 = 1, x_0 = 0$). Denote the uniform measure on $\mathcal{K}_n$ by $\mathbb{P}_n$. Note that $(\mathcal{O}_1, \ldots, \mathcal{O}_n)$ is distributed according to $\mathbb{P}_n$.

Let U_n be a uniformly chosen element of $[0, a_n - 1]^2 \cap \mathbb{Z}^2$. Then since each a_{i-1} divides a_i, the distribution of $(U_n + a_1\mathbb{Z}^2, \ldots, U_n + a_n\mathbb{Z}^2)$ is stationary, symmetric and ergodic with respect to the isometries $(\hat{T}_t, t \in \mathbb{Z}^2)$ defined by

$$\hat{T}_t : (U_n + a_1\mathbb{Z}^2, \ldots, U_n + a_n\mathbb{Z}^2) \to (t + U_n + a_1\mathbb{Z}^2, \ldots, t + U_n + a_n\mathbb{Z}^2).$$

Let β be the bijection between $[0, a_n - 1]^2 \cap \mathbb{Z}^2$ and $\mathcal{K}_n$ defined as $\beta(t) = (x_1, \ldots, x_n)$, where x_i is the mod a_i value of t. The push-forward of the uniform measure for U_n is then the uniform measure on $\mathcal{K}_n$. Furthermore, β commutes with translations in the sense that if $\beta(t) = (x_1, \ldots, x_n)$ and $\tau \in \mathbb{Z}$, then $\beta(t + \tau) = (x_1 + \tau, \ldots, x_n + \tau)$, where addition in the i'th coordinate is understood modulo a_i. Similarly, β commutes with rotations and reflections. Hence symmetry, stationarity and ergodicity of $(O_1 + a_1\mathbb{Z}^2, \ldots, O_n + a_n\mathbb{Z}^2)$ follows from that of $(U_n + a_1\mathbb{Z}^2, \ldots, U_n + a_n\mathbb{Z}^2)$. Symmetry, stationarity and ergodicity of $(\nu_e^n, e \in E_2)$ and $(\mu_e^n, e \in E_2)$ follows from the fact that $(\nu_e^n, e \in E_2)$ and $(\mu_e^n, e \in E_2)$ are deterministic functions of $(O_1 + a_1\mathbb{Z}^2, \ldots, O_n + a_n\mathbb{Z}^2)$, and these functions commute with graph isomorphisms of $\mathbb{Z}^2$.

(b) B_n contains more than $2a_n^2$ edges, of which less than $100b_n$ are such that $\nu_e^{n,0} \neq 1$. So by the stationarity of ν^n,

$$\mathbb{P}(\nu_e^n \neq 1) \leq \frac{50b_n}{a_n^2} \leq \frac{c}{2^n}.$$

The convergence in (3.2) then follows by the Borel–Cantelli lemma.

(c) The definition (3.2) and (a) show that $(\mu_e, e \in E_2)$ is stationary and symmetric, so all that remains to be proved is ergodicity.

Denote by $\mathcal{K}_\infty$ the family of sequences $(x_1, x_2, \ldots)$, satisfying $x_i \in (x_{i-1} + a_{i-1}\mathbb{Z}^2) \cap [0, a_i - 1]^2$ for every i. Let $\mathcal{G}_\infty$ be the σ-field generated by $(\mathcal{O}_1, \mathcal{O}_2, \ldots)$, and (by a slight abuse of notation) for the rest of this proof let $\mathbb{P}$ be the law of $(\mathcal{O}_1, \mathcal{O}_2, \ldots)$. Let $\mathcal{G}_n$ be the sub-σ-field of $\mathcal{G}_\infty$ generated by $(\mathcal{O}_1, \ldots, \mathcal{O}_n)$.

If $(x_1, x_2, \ldots) \in \mathcal{K}_\infty$, $t \in \mathbb{Z}^2$, define the $\mathbb{P}$-preserving transformation $t + (x_1, x_2, \ldots)$ as $(t + x_1, t + x_2, \ldots)$, where in the i'th coordinate is modulo a_i. To show ergodicity of $(\mu_e, e \in E_2)$, it is enough to prove ergodicity of $(\mathcal{O}_1, \mathcal{O}_2, \ldots)$, because $(\mu_e, e \in E_2)$ is a deterministic function of it, and this function commutes with graph isomorphisms of $\mathbb{Z}^2$.

Now let $A \in \mathcal{G}_\infty$ be invariant, and suppose by contradiction that there is some $\varepsilon > 0$ such that $\varepsilon < \mathbb{P}(A) < 1 - \varepsilon$. There exists some n and $B \in \mathcal{G}_n$ with the property that $\mathbb{P}(A \triangle B) < \varepsilon/4$ (where $\triangle$ is the symmetric difference operator). This also implies that $3\varepsilon/4 < \mathbb{P}(B) < 1 - 3\varepsilon/4$. We have for $t \in \mathbb{Z}^2$

$$\begin{aligned}\mathbb{P}(B\triangle(B+t)) &\le \mathbb{P}(A\triangle B) + \mathbb{P}(A\triangle(B+t)) = \mathbb{P}(A\triangle B) + \mathbb{P}((A+t)\triangle(B+t))\\ &= \mathbb{P}(A\triangle B) + \mathbb{P}((A\triangle B)+t) = 2\,\mathbb{P}(A\triangle B) < \varepsilon/2.\end{aligned}$$

We now show that we can choose t so that $\mathbb{P}(B\triangle(B+t)) \ge 2\,\mathbb{P}(B)\,\mathbb{P}(\mathcal{K}_\infty\backslash B) \ge \varepsilon/2$, giving a contradiction.

For an $E \in \mathcal{G}_n$ denote by E_n the subset of $\mathcal{K}_n$ such that $(\mathcal{O}_1, \mathcal{O}_2, \dots) \in E$ if and only if $(\mathcal{O}_1, \dots, \mathcal{O}_n) \in E_n$. Note that $\mathbb{P}(E) = \mathbb{P}_n(E_n)$. So we want to show that for any $B \in \mathcal{G}_n$ there exists a t such that $\mathbb{P}_n(B_n\triangle(B_n+t)) \ge 2\,\mathbb{P}_n(B_n)\,\mathbb{P}_n(\mathcal{K}_n\backslash B_n)$.

Consider the following average:

$$\begin{aligned}\frac{1}{a_n^2}\sum_{t\in[0,a_n-1]^2}\mathbb{P}_n(B_n\triangle(B_n+t)) &= \frac{2}{a_n^2}\sum_{t\in[0,a_n-1]^2}\mathbb{P}_n(B_n\backslash(B_n+t))\\ &= \frac{2}{a_n^4}\sum_{t\in[0,a_n-1]^2}\sum_{x\in\mathcal{K}_n}\mathbb{1}(x\in B_n\backslash(B_n+t)). \quad (3.3)\end{aligned}$$

Use

$$\sum_{x\in\mathcal{K}_n}\mathbb{1}(x\in B_n\backslash(B_n+t)) = \sum_{x\in B_n}\mathbb{1}(x\in B_n\backslash(B_n+t)) = \sum_{x\in B_n}\mathbb{1}(x-t\notin B_n)$$

and change the order of summation to obtain

$$\begin{aligned}\frac{2}{a_n^4}\sum_{t\in[0,a_n-1]^2}\sum_{x\in\mathcal{K}_n}\mathbb{1}(x\in B_n\backslash(B_n+t)) &= \frac{2}{a_n^4}\sum_{x\in B_n}\sum_{t\in[0,a_n-1]^2}\mathbb{1}(x-t\notin B_n)\\ = \frac{2}{a_n^4}\sum_{x\in B_n}(a_n^2-|B_n|) = \frac{2}{a_n^4}|B_n|(a_n^2-|B_n|) &= 2\,\mathbb{P}_n(B_n)\,\mathbb{P}_n(\mathcal{K}_n\backslash B_n). \quad (3.4)\end{aligned}$$

It follows from (3.3)–(3.4) that there exists a $t \in [0, a_n - 1]^2$ such that $\mathbb{P}_n(B_n\triangle(B_n + t)) \ge 2\,\mathbb{P}_n(B_n)\,\mathbb{P}_n(\mathcal{K}_n\backslash B_n)$. □

4 Choice of K_n and η_n

Let

$$\mathcal{L}_n f(x) = \tfrac{1}{2}\sum_y \mu_{xy}^n (f(y) - f(x)), \quad (4.1)$$

and X^n be the associated Markov process.

Proposition 4.1 *For each $n \ge 1$ there exists a constant σ_n, depending only on η_i, K_i, $1 \le i \le n$, such that the QFCLT holds for X^n with limit $\sigma_n W$.*

Proof Since μ_e^n is stationary, symmetric and ergodic, and μ_e^n is uniformly bounded and bounded away from 0, the result follows from [4, Theorem6.1]; see also Remarks 6.2 and 6.5 in that paper. (In fact, while [18, Theorem1.1] is stated for the i.i.d. case, the argument there also works in the ergodic case.) □

Next, we recall (from [6] for example) how σ_n is connected with the electrical conductivity across a square of side a_n. Let $k \in \{a_{n-1}, b_n, a_n\}$, and let

$$\mathcal{Q}_k = \{[0,k]^2 + z, z \in k\mathbb{Z}^2\}.$$

Thus $\mathcal{Q}_k$ gives a tiling of $\mathbb{Z}^2$ by squares of side k which are disjoint except for their boundaries. Given $Q \in \mathcal{Q}_k$ and $m \in \{n-1, n\}$ set

$$\widetilde{\mu}_{xy}^{Q,m} = \begin{cases} \frac{1}{2}\mu_{xy}^m & \text{if } x, y \in \partial_i(Q), \\ \mu_{xy}^m & \text{otherwise.} \end{cases}$$

For $f : Q \to \mathbb{R}$ set

$$\begin{aligned} \tilde{\mathcal{E}}_Q^m(f,f) &= \frac{1}{2}\sum_{x,y\in Q} \widetilde{\mu}_{xy}^{Q,m}(f(y)-f(x))^2, \\ \mathcal{H}_n &= \{f : B_n \to \mathbb{R} \text{ s.t. } f(x,0)=0, f(x,a_n)=1, \quad 0 \le x \le a_n\}, \\ \kappa_n &= \inf\{\tilde{\mathcal{E}}_{B_n}^n(f,f) : f \in \mathcal{H}_n\}. \end{aligned} \tag{4.2}$$

Thus κ_n^{-1} is just the effective resistance across the square B_n when bonds are assigned conductivities $\tilde{\mu}^{B_n,n}$.

Fix $n \ge 1$ and for simplicity consider the environment μ^n in the case when $\mathcal{O}_n = 0$. Then μ^n has period a_n (in both coordinate directions), and μ_{xy}^n for $x, y \in B_n$ is symmetric with respect to all the symmetries on the square B_n. Because of this symmetry, the limiting conductance matrix will be a multiple σ_n of the identity, and it is sufficient to calculate the variance of X^n in one coordinate direction.

We wish to construct an $\mathcal{L}_n$-harmonic function $h_n : \mathbb{Z}^2 \to \mathbb{R}$ so that for all $(x_1, x_2) \in \mathbb{Z}^2$ we have:

$$h_n(x_1, ja_n) = ja_n, \; j \in \mathbb{Z}, \quad h_n(x_1 + a_n, x_2) = h_n(x_1, x_2 + a_n) - a_n = h_n(x_1, x_2). \tag{4.3}$$

It is easy to see by the maximum principle that if such a function exists it is unique. Given such a function h_n, writing $X_t^n = (X_t^{n,1}, X_t^{n,2})$ we have $|h_n(X_t^n) - X_t^{n,2}| \le a_n$ and $h_n(X^n)$ is a martingale. Set

$$g_n(x) = \frac{1}{2}\sum_{y\in\mathbb{Z}^2} \mu_{xy}^n (h_n(x) - h_n(y))^2.$$

The function g_n also has period a_n on $\mathbb{Z}^2$, i.e. $g(x) = g(x')$ if $x - x' \in a_n\mathbb{Z}^2$.

Recall that $B'_n = [0, a_n - 1]^2 \cap \mathbb{Z}^2$, and let $\psi : \mathbb{Z}^2 \to B'_n$ be the natural function which maps $\mathbb{Z}^2$ onto the torus B'_n. So ψ is the identity on B'_n and has period a_n. Let $Y_t = \psi(X^n_t)$; then Y is a Markov process on B'_n with stationary measure $\nu_x = a_n^{-2}$ for each $x \in B'_n$. Then

$$\langle h(X^n)\rangle_t = \int_0^t g_n(X^n_s)ds = \int_0^t g_n(Y_s)ds.$$

So, by the ergodic theorem for Y,

$$\sigma_n^2 = \lim_{t\to\infty} \frac{\langle h(X^n)\rangle_t}{t} = a_n^{-2} \sum_{y\in B'_n} g_n(y) = \tfrac{1}{2} a_n^{-2} \sum_{y\in B'_n} \sum_{x\in\mathbb{Z}^2} \mu^n_{xy}(h_n(x) - h_n(y))^2. \tag{4.4}$$

To construct h_n we use the resistance problem (4.2) in the square $Q = B_n$. Let f_n be the minimising function for (4.2). By the maximum principle f_n is unique, and so using the symmetry of μ^n with respect to reflections in the lines $x_1 = a_n/2$ and $x_2 = a_n/2$ we deduce that for $(x_1, x_2) \in B_n$,

$$f_n(a_n - x_1, x_2) = f(x_1, x_2), \quad f_n(x_1, a_n - x_2) = 1 - f(x_1, x_2).$$

Given this function f_n we construct h_n by setting

$$\begin{aligned} &h_n(x) = a_n f_n(x), \quad x \in B_n, \\ &h_n(x + i a_n e_1 + j a_n e_2) = h(x) + j a_n, \quad x \in B_n, i, \quad j \in \mathbb{Z}. \end{aligned}$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The function h satisfies (4.3) and is clearly $\mathcal{L}_n$-harmonic in the interior of B. Some straightforward calculations show that it is also harmonic at points $x \in \partial_i B_n$, and consequently it is harmonic on $\mathbb{Z}^2$. Since h_n is constant on the lines $\{(i, ja_n), 0 \le i \le a_n\}$ for $j = 0, 1$ we have, using the symmetries of h_n, that

$$\sum_{y\in B'_n} \sum_{x\in\mathbb{Z}^2} \mu_{xy}(h_n(x) - h_n(y))^2 = 2a_n^2 \tilde{\mathcal{E}}^n_{B_n}(f_n, f_n).$$

Thus from (4.4)

$$\sigma_n^2 = \tilde{\mathcal{E}}^n_{B_n}(f_n, f_n) = \inf\{\tilde{\mathcal{E}}^n_{B_n}(f, f) : f \in \mathcal{H}_n\} = \kappa_n. \tag{4.5}$$

We now set

$$\eta_n = b_n^{-(1+1/n)}, \quad n \ge 1. \tag{4.6}$$

Theorem 4.2 *There exist constants* $K_n \in [1, 50b_n]$ *such that* $\sigma_n = 1$ *for all* n.

Proof Let $n \geq 1$; we can assume that K_i, $1 \leq i \leq n-1$ have been chosen so that $\sigma_i = 2$ for $i \leq n-1$.

Since σ_n is non-random, we can simplify our notation and avoid the need for translations by assuming that $\mathcal{O}_k = 0$ for $k = 1, \dots, n$; note that this event has strictly positive probability. For $K \in [0, \infty)$ let $\kappa_n^2(K)$ be the effective conductance across B_n as given by (4.2) if we take $K_n = K$. Since B_n is finite, $\kappa_n^2(K)$ is a continuous non-decreasing function of K. We will show that $\kappa_n^2(1) \leq 1$ and $\kappa_n^2(K) > 1$ for sufficiently large K; by continuity it follows that there exists a K_n such that $\kappa_n^2(K_n) = 1$, and thus $\sigma_n^2(K_n) = 1$.

If $K = 1$ then we have $\mu_e^n \leq \mu_e^{n-1}$, with strict inequality for the edges in D_n. We thus have $\kappa_n^2(1) \leq 1$. To obtain a lower bound on $\kappa_n^2(K)$, we use the dual characterization of effective resistance in terms of flows of minimal energy—see [13], and [3] for use in a similar context to the one here.

Let Q be a square in $\mathcal{Q}_k$, with lower left corner $w = (w_1, w_2)$. Let Q' be the rectangle obtained by removing the top and bottom rows of Q:

$$Q' = \{(x_1, x_2) : w_1 \leq x_1 \leq w_1 + k, w_1 + 1 \leq x_2 \leq w_1 + k - 1\}.$$

A *flow* on Q is an antisymmetric function I on $Q \times Q$ which satisfies $I(x, y) = 0$ if $x \nsim y$, $I(x, y) = -I(y, x)$, and

$$\sum_{y \sim x} I(x, y) = 0 \quad \text{if } x \in Q'.$$

Let $\partial^+ Q = \{(x_1, w_2 + k) : w_1 \leq x_1 \leq w_1 + k\}$ be the top of Q. The *flux* of a flow I is

$$F(I) = \sum_{x \in \partial^+ Q} \sum_{y \sim x} I(x, y).$$

For a flow I and $m \in \{n-1, n\}$ set

$$E_Q^m(I, I) = \frac{1}{2} \sum_{x \in Q} \sum_{y \in Q} (\widetilde{\mu}_{xy}^{Q,m})^{-1} I(x, y)^2.$$

This is the energy of the flow I in the electrical network given by Q with conductances $(\widetilde{\mu}_e^{m,Q})$. If $\mathcal{I}(Q)$ is the set of flows on Q with flux 1, then

$$\kappa_n(K)^{-2} = \inf\{E_{B_n}^n(I, I) : I \in \mathcal{I}(B_n)\}.$$

Let I_{n-1} be the optimal flow for κ_{n-1}^{-2}. The square B_n consists of $m_n^2 = a_n^2/a_{n-1}^2$ copies of B_{n-1}; define a preliminary flow I' by placing a replica of $m_n^{-1} I_{n-1}$ in each of these copies. For each square $Q \in \mathcal{Q}_{a_{n-1}}$ with $Q \subset B_n$ we have $E_Q^{n-1}(I', I') = m_n^{-2}$, and since there are m_n^2 of these squares we have $E_{B_n}^{n-1}(I', I') = 1$.

We now look at the tiling of B_n by squares in $\mathcal{Q}_{b_n}$; recall that $\ell_n = a_n/b_n$ and that ℓ_n is an integer. For each $Q \in \mathcal{Q}_{b_n}$ we have $E_Q^{n-1}(I', I') = \ell_n^{-2}$. Label these squares by (i, j) with $1 \le i, j \le \ell_n$.

We now describe modifications to the flow I' in a square Q of side b_n. For simplicity, take first $Q = [0, b_n]^2$. Set $A_1 = \{x = (x_1, x_2) \in Q : x_1 \ge x_2\}$, and $A_2 = \{x = (x_1, x_2) \in Q : x_2 \ge x_1\}$. Given any edge $e = (x, y)$ in $E(Q)$, either $x, y \in A_1$ or else $x, y \in A_2$. For $x = (x_1, x_2) \in Q$ set $r(x) = (x_2, x_1)$. Define a new flow by

$$I^*(x, y) = \begin{cases} I(x, y) & \text{if } \; x, y \in A_1, \\ I(r(x), r(y)) & \text{if } \; x, y \in A_2. \end{cases} \tag{4.7}$$

The flow I' runs from bottom to top of the square, and the modified flow I^* begins at the bottom, and emerges on the left side of the square. As in [3, Proposition3.2] we have $E_Q(I^*, I^*) \le E_Q(I', I') = \ell_n^{-2}$. Thus 'making a flow turn a corner' costs no more, in terms of energy, than letting it run on straight.

Suppose we now consider the flow I' in a column (i_1, j), $1 \le j \le \ell_n$, and we wish to make the flow avoid an obstacle square (i_1, j_1). Then we can make the flow make a left turn in $(i_1, j_1 - 1)$, and then a right turn in $(i_1 - 1, j_1 - 1)$ so that it resumes its overall vertical direction. This then gives rise to two flows in $(i_1 - 1, j_1 - 1)$: the original flow I' plus the new flow: as in [3] the combined flow in the square $(i_1 - 1, j_1 - 1)$ has energy less than $4\ell_n^{-2}$. If we carry the combined flow vertically through the square $(i_1 - 1, j_1)$, and make the similar modifications above the obstacle, then we obtain overall a new flow J' which matches I' except on the 6 squares $(i, j), i_1 \le i \le i_1, j_1 - 1 \le j \le j_1 + 1$. The energy of the original flow in these 6 squares is $6\ell_n^{-2}$, while the new flow will have energy less than $14\ell_n^{-2}$: we have a 'cost' of at most $4\ell_n^{-2}$ in the 3 squares $(i_1 - 1, j)$, $j_1 - 1 \le j \le j_1 + 1$, zero in (i_1, j_1) and at most ℓ_n^{-2} in the two remaining squares. Thus the overall energy cost of the diversion is at most $8\ell_n^{-2}$ (see Fig. 4).

We now use a similar procedure to construct a modification of I' in B_n with conductances $(\tilde{\mu}_e^{B_n, n})$. We have four obstacles, two oriented vertically and resembling an I, and two horizontal ones. The crossbars on the I, that is the sets D^{01}, contain vertical edges with conductance $\eta_n \ll 1$. We therefore modify I' to avoid these edges, and the squares with side b_n which contain them.

Consider the left vertical I, which has center $(a_n' - \beta_n, a_n')$. Let (i_1, j_1) be the square which contains at the top the bottom left branch of the I, so that this square has top right corner $(a_n' - \beta_n, a_n' - 10b_n)$. The top of this square contains vertical edges with conductance η_n, so we need to build a flow which avoids these. We therefore (as above) make the flow in the column i_1 take a left turn in square $(i_1, j_1 - 1)$, a right turn in $(i_1 - 1, j_1 - 1)$, carry it vertically through $(i_1 - 1, j_1)$, take a right turn in $(i_1 - 1, j_1 + 1)$ and carry it horizontally through $(i_1, j_1 + 1)$ into the edges of high conductance at the right side of $(i_1, j_1 + 1)$. The same pattern is then repeated on the other 3 branches of the left obstacle I, and on the other vertical obstacle.

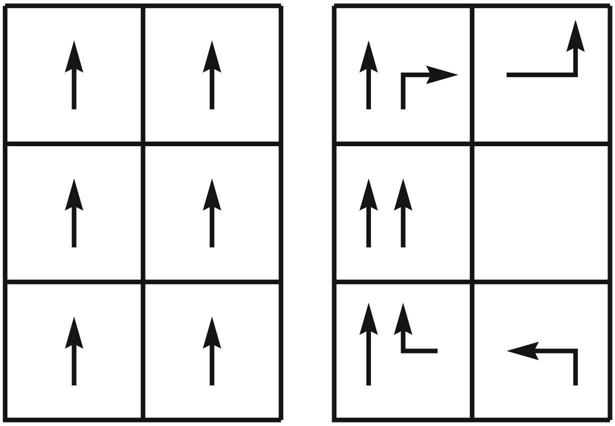

Fig. 4 Diversion of current around an obstacle square

We now bound the energy of the new flow J, and initially will make the calculations just for the change in columns $i_1 - 1$ and i_1 below and to the left of the point $(a_n' - \beta_n, a_n')$. Write $M = 10$ for the half of the overall height of the obstacle. There are $2(M+2)$ squares in this region where I' and J differ; these have labels (i, j) with $i = i_1 - 1, i_1$ and $j_1 - 1 \le j \le j_1 + M$. We begin by calculating the energy if $K = \infty$. In 3 of these squares the new flow J has energy at most $4\ell_n^{-2}$, in $M+1$ of them it has energy at most ℓ_n^{-2}, and in the remaining M it has zero energy. So writing R for this region we have $E_R(I', I') = (2M+4)\ell_n^{-2}$, while

$$E_R(J, J) \le (3 \cdot 4 + M + 1)\ell_n^{-2} = (13 + M)\ell_n^{-2}.$$

So

$$E_R(J, J) - E_R(I', I') \le (9 - M)\ell_n^{-2} = -\ell_n^{-2} < 0. \tag{4.8}$$

This is if $K = \infty$. Now suppose that $K < \infty$. The vertical edge in the obstacle carries a current $2/\ell_n$ and has height Mb_n, so the energy of J on these edges is at most

$$E' = \frac{4\ell_n^{-2} M b_n}{K} \le \frac{4 M b_n}{K n}. \tag{4.9}$$

The last inequality holds because $\ell_n \ge \sqrt{n}$. Finally it is necessary to modify I' near the 4 ends of the two horizontal obstacles. For this, we just modify I' in squares of side a_{n-1}, and arguments similar to the above show that for the new flow J in this region R', which consists of $4 + 2b_n/a_{n-1}$ squares of side a_{n-1}, we have

$$E_{R'}(J, J) - E_{R'}(I', I') \le \frac{9 b_n}{a_{n-1} m_n^2} = \frac{9 a_{n-1}}{b_n} \ell_n^{-2}. \tag{4.10}$$

The new flow J avoids the edges where $\mu_e^n = \eta_n$. Combining these terms we obtain for the whole square B_n, using (4.8)–(4.10),

$$\begin{aligned} E^n_{B_n}(J, J) - E^{n-1}_{B_n}(I', I') &\le -8\ell_n^{-2} + \frac{16Mb_n}{nK} + \frac{40a_{n-1}}{b_n}\ell_n^{-2} \\ &\le -7\ell_n^{-2} + \frac{16Mb_n}{nK} < -\frac{7}{2n} + \frac{160b_n}{nK}. \end{aligned}$$

So if $K' = 50b_n$, we have

$$\kappa_n^{-2}(K') \le E^n_{B_n}(J, J) \le 1 - cn^{-1} < 1.$$

Hence there exists $K_n \in [1, 50b_n)$ such that $\kappa_n^2(K_n) = 1$. □

Lemma 4.3 *Let $p < 1$. Then $\mathbb{E}\,\mu_e^p < \infty$, and $\mathbb{E}\,\mu_e^{-p} < \infty$.*

Proof Since $\mu_e^n = \eta_n = b_n^{-1-1/n}$ on a proportion cb_n/a_n^2 of the edges in B_n, we have

$$\mathbb{E}\,\mu_e^{-p} \le c \sum_n b_n^{p(1+1/n)} \frac{b_n}{a_n^2} \le c \sum_n b_n^{p+p/n-1} < \infty.$$

Here we used the fact that $b_n \ge 2^n$. Similarly,

$$\mathbb{E}\,\mu_e^p \le c \sum_n K_n^p \frac{b_n}{a_n^2} \le c \sum_n \frac{b_n^{1+p}}{a_n^2} < \infty.$$

□

Remark 4.4 Using (4.5) and the methods of [3], one can show that for small enough $\delta\,\kappa_n^2(\delta b_n) < 1$, so that $K_n \asymp b_n$ and consequently $\mathbb{E}\,\mu_e = \infty$. Note that we also have

$$\limsup_{n\to\infty} n\,\mathbb{P}(\mu_e > n) = \limsup_{k\to\infty} b_k\,\mathbb{P}(\mu_e > cb_k) = \lim_{k\to\infty} \frac{b_k^2}{a_k^2} = 0. \qquad (4.11)$$

From now on we take K_n to be such that $\sigma_n = 1$ for all n.

5 Weak invariance principle

Let $X = (X_t, t \in \mathbb{R}_+, P^x_\omega, x \in \mathbb{Z}^d)$ be the process with generator (1.1) associated with the environment (μ_e). Recall (4.1) and the definition of X^n, and define $X^{(n,\varepsilon)}$ by

$$X_t^{(n,\varepsilon)} = \varepsilon X^n_{\varepsilon^{-2}t}, \quad t \ge 0.$$

Let $P_n^\omega(\varepsilon)$ be the law of $X^{(n,\varepsilon)}$ on $\mathcal{D} = \mathcal{D}_1$, and $P^\omega(\varepsilon)$ be the law of $X^{(\varepsilon)}$.

Recall that the Prokhorov distance d_P between probability measures on $\mathcal{D}_1$ is defined as follows (see [8, p. 238]). For $A \subset \mathcal{D}$, let $\mathcal{B}(A,\varepsilon) = \{x \in \mathcal{D} : d_S(x, A) < \varepsilon\}$. For probability measures P and Q on $\mathcal{D}$, $d_P(P, Q)$ is the infimum of $\varepsilon > 0$ such that $P(A) \le Q(\mathcal{B}(A,\varepsilon)) + \varepsilon$ and $Q(A) \le P(\mathcal{B}(A,\varepsilon)) + \varepsilon$ for all Borel sets $A \subset \mathcal{D}$. Recall that convergence in the metric d_P is equivalent to the weak convergence of measures.

To prove the WFCLT it is sufficient to prove:

Theorem 5.1 *There exists a sequence* (b_n) *such that if* $\varepsilon_n = 1/b_n$ *then* $\lim_{n\to\infty} d_P (P^\omega(\varepsilon_n), P_{BM}) = 0$ *in* $\mathbb{P}$*-probability.*

Proof Let $n \ge 1$ and suppose that a_k, b_k have been chosen for $k \le n-1$. By Proposition 4.1 we have for each ω that $d_P(P^\omega_{n-1}(\varepsilon), P_{\mathrm{BM}}) \to 0$. Note that the environment μ^{n-1} takes only finitely many values. So we can choose b_n large enough so that

$$d_P(P^\omega_{n-1}(\varepsilon), P_{\mathrm{BM}}) < n^{-1} \quad \text{for } \ 0 < \varepsilon \le b_n^{-1} \ \text{ and all } \ \omega. \tag{5.1}$$

Now for $\lambda > 1$ set

$$G(\lambda) = \left\{ w \in \mathcal{D}_1 : \sup_{0\le s\le 1} |w(s)| \le \lambda \right\}.$$

We have

$$P_{\mathrm{BM}}(G(\lambda)^c) \le \exp(-c'\lambda^2).$$

We can couple the processes X^{n-1} and X so that the two processes agree up to the first time X^{n-1} hits the obstacle set $\bigcup_{k=n}^\infty D_k$. Let $\xi_n(\omega) = \min\{|x| : x \in \bigcup_{k=n}^\infty D_k(\omega)\}$, and

$$F_n = \{\xi_n > \lambda b_n\}.$$

Let $m \ge n$, and consider the probability that 0 is within a distance λb_n of D_m. Then $\mathcal{O}_m$ has to lie in a set of area $c\lambda b_n b_m$, and so

$$\mathbb{P}\left(\min_{x\in D_m} |x| \le \lambda b_n \right) \le \frac{c b_n b_m}{a_m^2} \le \frac{c b_n}{m b_m}.$$

Thus

$$\mathbb{P}(F_n^c) \le c \sum_{m=n}^\infty \frac{b_n}{m b_m} \le \frac{c}{n}\left(1 + \sum_{m=n+1}^\infty \frac{b_n}{b_m}\right) \le \frac{c'}{n}.$$

Suppose that $\omega \in F_n$ and $n \geq 2$ so that $n^{-1} < \lambda/2$. Then using the coupling above, we have

$$d_P(P^\omega(\varepsilon_n), P^\omega_{n-1}(\varepsilon_n)) \leq P^\omega_0\left(\sup_{0\leq s\leq b_n^2} |X_s^{(n-1)}| > \lambda b_n\right)$$
$$\leq d_P(P^\omega_{n-1}(\varepsilon_n), P_{\text{BM}}) + P_{\text{BM}}(G(\lambda/2)^c).$$

If now $\delta > 0$, choose $\lambda > 1$ such that $P_{\text{BM}}(G(\lambda/2)^c) < \delta/2$, and then $N > 2/\delta$ large enough so that $\mathbb{P}(F_n^c) < \delta$ for $n \geq N$. Then combining the estimates above, if $n \geq N$ and $\omega \in F_n$, $d_P(P^\omega(\varepsilon_n), P_{\text{BM}}) < \delta$, so for $n \geq N$, $\mathbb{P}(d_P(P^\omega(\varepsilon_n), P_{\text{BM}}) > \delta) \leq \mathbb{P}(F_n^c) < \delta$, which proves the convergence in probability. □

6 Quenched invariance principle does not hold

We will prove that the QFCLT does not hold for the processes $X^{(\varepsilon_n)}$, and will argue by contradiction. If the QFCLT holds for X with limit ΣW then since the WFCLT holds for $X^{(\varepsilon_n)}$ with diffusion constant 1 in every direction (by isotropy of the environment), Σ must be the identity matrix.

Let $w_n^0 = (a_n' - 10b_n - 1, a_n' - \beta_n)$ be the centre point on the left edge of the lowest of the four nth level obstacles in the set D_n^0, and let $z_n^0 = w_n - \left(\frac{1}{2}b_n, 0\right)$. Thus z_n^0 is situated a distance $\frac{1}{2}b_n$ to the left of w_n^0—see Fig. 5. Let

$$H_n^0(\lambda) = B_\infty(z_n^0, \lambda b_n), \quad H_n(\lambda) = \bigcup_{x\in a_n\mathbb{Z}^2} (x + \mathcal{O}_n + H_n^0(\lambda)).$$

Lemma 6.1 *For $\lambda > 0$ the event $\{0 \in H_n(\lambda)\}$ occurs for infinitely many n, $\mathbb{P}$-a.s.*

Proof Let $\mathcal{G}_k = \sigma(\mathcal{O}_1, \dots, \mathcal{O}_k)$. Given the values of $\mathcal{O}_1, \dots, \mathcal{O}_{n-1}$, the r.v. $\mathcal{O}_n$ is uniformly distributed over m_n^2 points, with spacing a_{n-1}, and has to lie in a square with side $2\lambda b_n$ in order for the event $\{0 \in H_n(\lambda)\}$ to occur. Thus approximately $(2\lambda b_n/a_{n-1})^2$ of these values of $\mathcal{O}_n$ will cause $\{0 \in H_n(\lambda)\}$ to occur. So

$$\mathbb{P}(0 \in H_n(\lambda) \mid \mathcal{G}_{n-1}) \geq c\frac{(2\lambda b_n/a_{n-1})^2}{(a_n/a_{n-1})^2} = c'\frac{b_n^2}{a_n^2} \geq \frac{c''}{n}.$$

The conclusion then follows from an extension of the second Borel–Cantelli Lemma. □

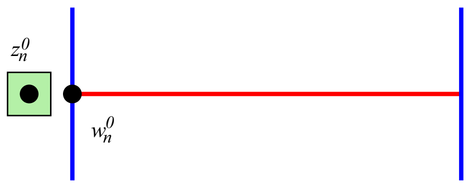

Fig. 5 The *square* represents $H_n^0(\frac{1}{4})$

Lemma 6.2 *With $\mathbb{P}$-probability 1, the event $G_n(\lambda) = \{H_n(\lambda) \cap (\bigcup_{m=n+1}^{\infty} D_m) \neq \emptyset\}$ occurs for only finitely many n.*

Proof Let $m > n$. Then as in the previous lemma, by considering possible positions of $\mathcal{O}_m$, we have

$$\mathbb{P}(H_n(\lambda) \cap D_m \neq \emptyset) \leq c\frac{b_m b_n}{a_m^2} \leq c\frac{b_n}{b_m}.$$

Since $b_m \geq 2^m b_{m-1} > 2^m b_n$,

$$\mathbb{P}\left(H_n(\lambda) \cap \left\{\bigcup_{m=n+1}^{\infty} D_m \neq \emptyset\right\}\right) \leq \sum_{m=n+1}^{\infty} c\frac{b_n}{b_m} \leq c2^{-n},$$

and the conclusion follows by Borel–Cantelli. □

The first two Lemmas have shown, first that 0 is close to a nth level obstacle infinitely often, and next that higher level obstacles do not interfere. Our final task is to show that in this situation, the process X is unlikely to cross the strip of low conductance edges.

Lemma 6.3 *Suppose that $0 \in H_n(1/8)$ and $H_n(4) \cap \left(\bigcup_{m=n+1}^{\infty} D_m\right) = \emptyset$. Write $X_t = (X_t^1, X_t^2)$, and let*

$$F = \{|X_t^2| \leq 3b_n/4, |X_t^1| \leq 2b_n, 0 \leq t \leq b_n^2, X_{b_n^2}^1 > 3b_n/4\}.$$

Then there exists a constant $A_{n-1} = A_{n-1}(\eta_1, K_1, \ldots, \eta_{n-1}, K_{n-1})$ such that

$$P_\omega^0(F) \leq c b_n^{-1/n} A_{n-1} \log A_{n-1}.$$

Proof Let $w_n = (x_n, y_n)$ be the element of $\{w_n^0 + \mathcal{O}_n + a_n x, x \in \mathbb{Z}^2\}$ which is closest to 0. Then, under the hypotheses of the Lemma, we have $3b_n/8 \leq x_n \leq 5b_n/8$, and $|y_n| \leq b_n/8$. Thus the square $B_\infty(0, 2b_n)$ intersects the obstacle set D_n, but does not intersect D_m for any $m > n$. Hence if F holds then we can couple X^n and X so that $X_t^n = X_t$ for $0 \leq t \leq b_n^2$.

Let $\mathbb{H} = \{(x, y) : x \leq x_n\}$, and $J = B \cap \partial_i \mathbb{H}$. If F holds then X^n has to cross the line J, and therefore has to cross an edge of conductance η_n. Let Y be the process with edge conductances μ'_e, where $\mu'_e = \mu_e^{n-1}$ except that $\mu'_e = 0$ if $e = \{(x_n, y), (x_n + 1, y)\}$ for $y \in \mathbb{Z}$. Thus the line $\partial_i \mathbb{H}$ is a reflecting barrier for Y. Let

$$L_t = \int_0^t 1_{(Y_s \in J)} ds$$

be the amount of time spent by Y in J, and

$$G = \{|Y_t^2| \leq 3b_n/4, |Y_t^1| \leq 2b_n, 0 \leq t \leq b_n^2\}.$$

Assuming that G holds, let ξ_1 be a standard $\texttt{exp}(1)$ r.v., set $T = \inf\{s : L_s > \xi_1/\eta_n\}$, and let $X^n_t = Y_t$ on $[0, T)$, and $X^n_T = Y_T + (1, 0)$. Note that one can complete the definition of X^n_t for $t \geq T$ in such a way that the process X^n has the same distribution as the process defined by (4.1). We have

$$P^0_\omega(G \cap \{X^n_s = Y^n_s, 0 \leq s \leq b_n^2\}) = E^0_\omega(1_G \exp(-\eta_n L_{b_n^2})).$$

So

$$P^0_\omega(G \cap \{T \leq b_n^2\}) = E^0_\omega(1_G(1 - \exp(-\eta_n L_{b_n^2})) \leq E^0_\omega(1_G \eta_n L_{b_n^2}) \leq \eta_n E^0_\omega L_{b_n^2}.$$

The process Y has conductances bounded away from 0 and infinity on $\mathbb{H}$, so by [11] Y has a transition probability $p_t(w, z)$ which satisfies

$$p_t(w, z) \leq At^{-1} \exp(-A^{-1}|w - z|^2/t), \quad w, z \in \mathbb{H}, \quad t \geq |w - z|.$$

In addition if $r = |w-z| \geq A$ then $p_t(w, z) \leq p_r(w, z)$. Here $A = A_{n-1}$ is a possibly large constant which depends on $(\eta_i, K_i, 1 \leq i \leq n - 1)$. We can take $A \geq 10$. For $w \in J$ we have $|w| \geq b_n/4$ and so provided $b_n \geq 8A$,

$$\begin{aligned} E^0_\omega \int_0^{b_n^2} 1_{(Y_s = w)} ds &= \int_0^{b_n^2} p_t(0, w) dt \leq b_n p_{b_n}(0, w) + \int_{b_n}^{b_n^2} p_t(0, w) dt \\ &\leq cAe^{-b_n/A} + A \int_0^{b_n^2} t^{-1} \exp(-b_n^2/(16At)) dt \leq cA \log(A). \end{aligned}$$

So since $|J| \leq 2b_n$,

$$P^0_\omega(G \cap \{T \leq b_n^2\}) \leq c\eta_n b_n A \log A \leq cb_n^{-1/n} A \log A.$$

Finally, the construction of X^n from Y gives that $P^0_\omega(F) \leq P^0_\omega(G \cap \{T \leq b_n^2\})$. □

Proof of Theorem 1.4(b) We now choose b_n large enough so that for all $n \geq 2$,

$$b_n^{-1/n} A_{n-1} \log A_{n-1} < n^{-1}. \tag{6.1}$$

Let $W_t = (W^1_t, W^2_t)$ denote two-dimensional Brownian motion with $W_0 = 0$, and let P_{BM} denote its distribution. For a two-dimensional process $Z = (Z^1, Z^2)$, define the event

$$F(Z) = \{|Z^2_s| < 3/4, |Z^1_s| \leq 2, 0 \leq s \leq 1, Z^1_1 > 1\}.$$

The support theorem implies that $p_1 := P_{\text{BM}}(F(W)) > 0$. Write $F_n = F(X^{(\varepsilon_n)})$.

Let $N_1 = N_1(\omega)$ be such that the event $G_n(4)$ defined in Lemma 6.2 does not occur for $n \geq N_1$. Let $\Lambda = \Lambda(\omega)$ be the set of $n > N_1$ such that $0 \in H_n\left(\frac{1}{8}\right)$. Then

$\mathbb{P}(\Lambda \text{ is infinite}) = 1$ by Lemma 6.1. By Lemma 6.3 and the choice of b_n in (6.1) we have $P^0_\omega(F_n) < cn^{-1}$ for $n \in \Lambda$. So

$$P^0_\omega(F_n) \to 0 \quad \text{as} \quad n \to \infty \quad \text{with} \quad n \in \Lambda.$$

Thus whenever $\Lambda(\omega)$ is infinite the sequence of processes $(X^{(\varepsilon_n)}_t, t \in [0,1], P^0_\omega)$, $n \geq 1$, cannot converge to W, and the QFCLT therefore fails. □

Remark 6.4 We can construct similar obstacle sets in $\mathbb{Z}^d$ with $d \geq 3$, and we now outline briefly the main differences from the $d = 2$ case.

We take $b_n = a_n n^{-1/d}$, so that $\sum b_n^d / a_n^d = \infty$, and the analogue of Lemma 6.2 holds. In a cube side a_n we take $2d$ obstacle sets, arranged in symmetric fashion around the centre of the cube. Each obstacle has an associated 'direction' $i \in \{1, \dots, d\}$. An obstacle of direction i consists of a $2b_n^{d-1}$ edges of low conductance η_n, arranged in two $d-1$ dimensional 'plates' a distance Mb_n apart, with each edge in the direction i. The two plates are connected by $d-1$ dimensional plates of high conductance K_n. Thus the total number of edges in the obstacles is cb_n^{d-1}, so taking a_n/a_{n-1} large enough, we have $\sum b_n^{d-1}/a_n^d < \infty$, and the same arguments as in Sect. 3 show that the environment is well defined, stationary and ergodic.

The conductivity across a cube side N in $\mathbb{Z}^d$ is N^{d-2}. Thus if we write $\sigma_n^2(\eta_n, K_n)$ for the limiting diffusion constant of the process X^n, and $R_n = R_n(\eta_n, K_n)$ for the effective resistance across a cube side a_n, then (4.5) is replaced by:

$$\sigma_n^2(\eta_n, K_n) = a_n^{2-d} R_n^{-1}. \tag{6.2}$$

For the QFCLT to fail, we need $\eta_n = o(b_n^{-1})$, as in the two-dimensional case. With this choice we have $R_n(\eta_n, 0)^{-1} < a_n^{d-2}$, and as in Theorem 4.2 we need to show that if K_n is large enough then $R_n(\eta_n, K_n)^{-1} > a_n^{d-2}$.

Recall that $\ell_n = a_n/b_n$. Let I' be as in Theorem 4.2; then I' has flux ℓ_n^{-d+1} across each sub-cube Q' of side b_n. If the sub-cube does not intersect the obstacles at level n, then $E_{Q'}(I', I') = \ell_n^{-d} a_n^{2-d}$. The 'cost' of diverting I' around a low conductance obstacle is therefore of order $c\ell_n^{-d} a_n^{2-d} = cb_n^{-d+2}\ell_n^{-2d+2}$—see [17]. As in Theorem 4.2 we divert the flow onto the regions of high conductance, so as to obtain some cubes in which the new flow has zero energy. To estimate the energy in the high conductance bonds, note that we have $2(d-1)b_n^{d-2}$ sets of parallel paths of edges of high conductance, and each path is of length Mb_n, so the flow in each edge is $F_n = \ell_n^{-d+1}/b_n^{d-2}(2d-2)$. Hence the total energy dissipation in the high conductance edges is

$$K^{-1} M F_n^2 = \frac{c' K^{-1} M b_n^{d-1}}{\ell_n^{2d-2} b_n^{2d-4}} = \frac{c' K^{-1} M}{\ell_n^{2d-2} b_n^{d-3}}.$$

We therefore need

$$\frac{c' K^{-1} M}{\ell_n^{2d-2} b_n^{d-3}} < \frac{c}{b_n^{d-2} \ell_n^{2d-2}},$$

that is we need to choose $K_n > cMb_n$ for some constant c. Since

$$\mathbb{E}\,\mu_e^p \asymp \sum_n \frac{K_n^p b_n^{d-1}}{a_n^d} \asymp M \sum_n \frac{b_n^{d-1+p}}{a_n^d},$$

we find that in $d \geq 3$ our example also has $\mathbb{E}\,\mu_e^{\pm p} < \infty$ if and only if $p < 1$.

Acknowledgments We are grateful to Emmanuel Rio for very helpful advice, and Pierre Mathieu, Jean-Dominique Deuschel and Marek Biskup for some very useful discussions. In addition we want to thank a referee for very careful reading.

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