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Lagrangian flows driven by BV fields in Wiener spaces

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Abstract We establish the renormalization property for essentially bounded solutions of the continuity equation associated to bounded variation (BV) fields in Wiener spaces, with values in the associated Cameron-Martin space; thus obtaining, by standard arguments, new uniqueness and stability results for correspondent Lagrangian flows.

Keywords Wiener spaces · Continuity equation · Flows · BV functions

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1 Introduction

The theory of flows driven by weakly differentiable fields in Euclidean spaces began with the seminal paper [17], by R.J. DiPerna and P.-L. Lions, where they proved that Sobolev regularity for vector fields in $\mathbb{R}^n$ is sufficient to obtain existence and uniqueness of a generalized notion of flow driven by these fields. Since then it has found many developments and applications; for brevity, we refer to the exposition in [2], mentioning only some notable recent papers: [3,12,18] and [22]. The work of DiPerna and Lions did not cover the case of bounded variation (BV) fields, which arise naturally in many contexts: it was settled by L. Ambrosio in [1] and subsequently BV fields were considered in other settings, e.g. SDEs, in [23], or Fokker-Planck equations, in [24].

On abstract Wiener spaces, a theory of flows was developed somehow independently (see [13–15] for early works and [26] for more recent developments). In [4], tools

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from DiPerna-Lions theory were first applied also in the infinite-dimensional setting, obtaining existence and uniqueness of flows driven by Sobolev fields. A comparison between the two approaches seems difficult in general, due to different assumptions on different norms: one approach might be better than the other, depending on the nature of the driving field (for more details, see the introduction of [4]). Recent developments of similar techniques in infinite dimensional settings deal with Fokker-Planck equations in Wiener spaces [25], continuity equations in Gaussian-Hilbert spaces [16] and also in some non-Gaussian spaces [21].

In Wiener spaces, uniqueness for flows driven by BV fields was left open in [4] and the aim of this article is to settle it. Besides an urge of symmetry with the finite dimensional setting, our interest in dealing with BV fields is mainly due to the lack of localization for the theory involving Sobolev fields obtained in [4]: after giving the statement of the main result, Theorem 2 in Sect. 4, we sketch how BV fields arise naturally by extending smooth fields defined only on open regions of the Wiener space. If extension theorems for Sobolev spaces on infinite dimensional domains were known, one might be able to work without BV fields, but this seems to be a rather delicate subject (see [11] for some recent results, in a negative direction).

Some reasons for the existence of this gap between the finite dimensional theory and the Wiener space theory can be traced in the fact that the theory of BV maps on Wiener spaces (which began with the works by Fukushima and Hino [19,20]) only recently has been studied from a geometric point of view, closer to the finite dimensional setting: see [5,6] and [8].

For a presentation of the general problem of flows and the ideas involved in the Wiener space setting, we refer to the well written introduction of [4] and then to Sect. 5 therein for a rigorous derivation of the links between well-posedness (i.e. existence and uniqueness) of flows driven by a vector field b and that of the associated continuity equation,

$$\partial_t u_t + \operatorname{div}(b_t u_t) = 0, \tag{1}$$

where $\operatorname{div}(= \operatorname{div}_\gamma)$ denotes the distributional divergence with respect to the underlying Gaussian measure.

While existence is settled rather easily, assuming bounds on b and $\operatorname{div} b$, uniqueness is a difficult issue, already in the finite dimensional setting. The DiPerna-Lions argument is based on the notion of renormalized solution, whose definition we recall here: a solution to (1) is said to be renormalized if, for every $\beta \in C^1(\mathbb{R})$, with both $\beta'(z)$ and $\beta(z) - z\beta'(z)$ bounded, it holds, in the distributional sense,

$$\partial_t \beta(u_t) + \operatorname{div}(b_t \beta(u_t)) - (\operatorname{div} b_t)\left[\beta(u_t) - u_t \beta'(u_t)\right] = 0.$$

Roughly speaking, if all the solutions in some class are known to be renormalized, then uniqueness holds in that class (for a precise statement, see e.g. the proof of Theorem 3.1, in [4, Section 3]).

In this article, therefore, we focus on the proof of the renormalization property, the main result being Theorem 2 below: given a BV vector field b with integrable divergence, every essentially bounded solution of (1) is renormalized: from this it is not difficult to recover statements about uniqueness and stability for L^∞-regular Lagrangian flows, as in [4].

Let us comment on the proof technique. In the finite dimensional setting, the approach developed in [1] is based on a refined analysis of error terms arising from smooth approximations, relying on two different estimates: an anisotropic estimate, which is rather good in the regions where the measure-derivative Db is mostly singular with respect to the Lebesgue measure, and an isotropic estimate, which is good instead in the regions where the derivative is mostly absolutely continuous. Then, an optimization procedure on the choice of approximations gives the renormalization property.

In the infinite dimensional setting, a direct implementation of this method fails, because of error terms depending on the dimension of the space. Our contribution may be thus summarized in obtaining a refined anisotropic estimate which is well-behaved at every point and, after a similar optimization procedure, turns out to be sufficient to conclude the renormalization property.

This method works also in the finite dimensional setting and, since the steps might prepare the reader for the Wiener space case, we briefly describe it in Sect. 2. From a technical point of view, we establish first some estimates on smooth functions, obtained via integration by parts, and then we apply Reshetnyak continuity theorem to cover the BV case: to the author's knowledge, this argument is original also in the finite dimensional setting and, although being here a minor deviation from the usual route, it proves to be a good approach to deal with the infinite dimensional case.

Starting from Sect. 3, we deal uniquely with the Wiener space setting: we recall some definitions and facts about Sobolev and BV maps and then, in Sect. 4, we state our main result. In Sect. 5, we establish some technical facts instrumental to its proof, which is finally given in Sect. 6.

2 Renormalized solutions in $\mathbb{R}^d$

Aim of this section is to prove the renormalization property for essentially bounded solutions of the continuity equation associated to a finite dimensional BV field, along the same lines as in Sect. 6 below: there, computations are a bit more involved and thus the hope is to introduce the reader to our approach. In particular, the overall structure of the proof is exactly the same.

For simplicity, we work under global integrability assumptions, but the arguments can be easily adapted to cover the BV_{loc} case. For brevity, we refer to [2, Section 5], for a detailed introduction of the finite dimensional setting. Recall that the class of test functions is given by $C^\infty_c((0,T)\times\mathbb{R}^d)$.

Theorem 1 *Let* $b \in L^1\left((0,T); BV(\mathbb{R}^d;\mathbb{R}^d)\right)$ *with* $\operatorname{div} b \in L^1\left((0,T)\times\mathbb{R}^d\right)$. *Then, any distributional solution* $u = (u_t)_{t\in(0,T)} \in L^\infty\left((0,T)\times\mathbb{R}^d\right)$ *to*

$$\partial_t u_t + \operatorname{div}(b_t u_t) = 0$$

is a renormalized solution.

Proof We sketch the essential steps to prove the result, since they perfectly correspond to those involved in the proof of Theorem 2 below. Then, we slightly expand the arguments in the following subsections.

1. (*Mollification*). We set up a two parameter family of mollified solutions u^ε_ρ, with parameters ρ varying in some set of functions and ε being a positive real number, that solve, in the distributional sense, the equation

$$\partial_t u^\varepsilon_\rho + \operatorname{div}\left(b u^\varepsilon_\rho\right) = r^\varepsilon_\rho. \tag{2}$$

   For simplicity, we omit in what follows the dependence on ρ.
2. (*Approximate renormalization*). We prove that u^ε and all the terms above are sufficiently smooth so that, given any $\beta \in C^1(\mathbb{R})$, with both $\beta'(s)$ and $\beta(s) - s\beta'(s)$ uniformly bounded, standard calculus applies entailing

$$\partial_t \beta\left(u^\varepsilon\right) + \operatorname{div}\left(b\beta\left(u^\varepsilon\right)\right) - (\operatorname{div} b)\left[\beta(u^\varepsilon) - \beta'(u^\varepsilon)u^\varepsilon\right] = \beta'(u^\varepsilon)r^\varepsilon. \tag{3}$$

3. (*Refined anisotropic estimate*). We prove that, for some function $\Lambda_\rho(t,x)$ it holds, for every test function φ,

$$\limsup_{\varepsilon\to 0} \int_{(0,T)\times\mathbb{R}^d} \left|\varphi\beta'\left(u^\varepsilon\right) r^\varepsilon\right| dx dt \leq \|u\|_\infty \left\|\beta'\right\|_\infty \int_{(0,T)\times\mathbb{R}^d} |\varphi| \Lambda_\rho \, d\left|Db_t\right| dt. \tag{4}$$

   This inequality implies that the distribution

$$\sigma := \partial_t\beta(u) + \operatorname{div}(b\beta(u)) - (\operatorname{div} b)\left[\beta(u) - \beta'(u)u\right] \tag{5}$$

   is a measure, with total variation smaller than $\Lambda_\rho d\left|Db_t\right| dt$.
4. (*Optimization*). The null measure is the infimum among all $\Lambda_\rho |Db|$, as ρ varies: this relies on a (by now stardard) argument due to G. Alberti, and settles the renormalization property.

As already remarked in the introduction, when compared with Ambrosio's original strategy to prove the renormalization property (see [1] or[2]) our approach differs mainly in the refined anisotropic estimate, whose derivation seems to be new, to the author's knowledge: such a minor deviation from the usual route plays a key role in the Wiener setting.

2.1 Mollification

Let ρ be any smooth non-negative function on $\mathbb{R}^d$, compactly supported, with $\int \rho = 1$. Given $\varepsilon > 0$, and $f \in L^1_{loc}\left((0,T)\times\mathbb{R}^d\right)$, we mollify by convolution on the space variables, defining

$$T^\varepsilon_\rho \varphi(t,x) := \int_{\mathbb{R}^d} \varphi(t, x_\varepsilon)\rho(y)\, dy,$$

where we write here and below, $x_\varepsilon := x - \varepsilon y$. We frequently omit the dependence on $t \in (0, T)$ since it plays no role.

The adjoint in $L^2((0,T) \times \mathbb{R}^d)$ of T^ε_ρ is

$$(T^\varepsilon_\rho)^* \varphi\,(t,x) := \int_{\mathbb{R}^d} \varphi\,(t, x + \varepsilon y)\,\rho\,(y)\,dy,$$

Since T^ε_ρ preserves test functions, we naturally extend $(T^\varepsilon_\rho)^*$ on distributions. We set therefore $u^\varepsilon_\rho = (T^\varepsilon_\rho)^* u$, so that in (2) it holds

$$r^\varepsilon_\rho := \operatorname{div}\left(b (T^\varepsilon_\rho)^* u\right) - (T^\varepsilon_\rho)^* \operatorname{div}(bu)\,.$$

2.2 Approximate renormalization

To keep notation simple, we omit to explicit the dependence on ρ, since its plays no role in this section. The function u^ε is smooth with respect to the variable $x \in \mathbb{R}^d$, so that to justify (3) above we show that the distribution r^ε is (induced by) an integrable function: actually both $\operatorname{div}(bu^\varepsilon)$ and $(T^\varepsilon)^* \operatorname{div}(bu)$ are integrable functions.

Equivalent expressions for $\operatorname{div}(bu^\varepsilon)$ *and* $(T^\varepsilon)^* \operatorname{div}(bu)$ *via integration by parts.* Let φ be a test function and compute

$$\int \varphi \operatorname{div}\left(bu^\varepsilon\right) = -\int \langle \nabla\varphi, b\rangle u^\varepsilon = -\int u\, T^\varepsilon \langle \nabla \varphi, b\rangle.$$

Integrating by parts, we obtain

$$T^\varepsilon \langle \nabla\varphi, b\rangle\,(x) = \frac{1}{\varepsilon}\int \varphi\,(x_\varepsilon) \operatorname{div}_y (b\,(x_\varepsilon)\,\rho\,(y))\,dy = \int \varphi\,(x_\varepsilon)\, A^\varepsilon\,(x,y)\,dy.$$

If $A^\varepsilon \in L^1\left(\mathbb{R}^{2d}\right)$, the change of variables $(x,y) \mapsto (x + \varepsilon y, y)$ gives

$$\int \varphi \operatorname{div}\left(bu^\varepsilon\right) = -\int \varphi\,(x) \int u^\varepsilon\,(x + \varepsilon y)\, A^\varepsilon\,(x + \varepsilon y, y)\,dy.$$

thus $\operatorname{div}(bu^\varepsilon) \in L^1$. For $(T^\varepsilon)^* \operatorname{div}(bu)$ we proceed similarly, obtaining

$$\langle \nabla T^\varepsilon \varphi, b\rangle\,(x) = \frac{1}{\varepsilon}\int \varphi\,(x_\varepsilon) \operatorname{div}_y (b\,(x)\,\rho\,(y))\,dy = \int \varphi\,(x_\varepsilon)\, B^\varepsilon\,(x,y)\,dy.$$

If $B^\varepsilon \in L^1\left(\mathbb{R}^{2d}\right)$, we conclude that $(T^\varepsilon)^* \operatorname{div}(bu) \in L^1$. Direct estimates for A^ε and B^ε are trivial, but we prefer to sketch a slightly more abstract argument that is useful in the Wiener setting.

Integrability of A^ε *and* B^ε *via divergence identities.* If M is a linear transformation of $\mathbb{R}^{2d}$ that preserves Lebesgue measure, then, for every vector field $c : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$, the following distributional identity holds:

$$\operatorname{div}(c \circ M)(z) = [\operatorname{div}(Mc)](Mz). \tag{6}$$

In the case $M(x, y) = (x - sy, y)$ and $c(x, y) = (0, v(x, y))$, we obtain

$$\operatorname{div}_y(v(x_s, y)) = -s[\operatorname{div}_x v](x_s, y) + \left[\operatorname{div}_y v\right](x_s, y). \tag{7}$$

Such an identity gives, for $v(x, y) = b(x)$ and $s = \varepsilon$,

$$\operatorname{div}_y(b(x_\varepsilon)) = -\varepsilon[\operatorname{div} b](x_\varepsilon) \in L^1.$$

From this, we deduce the integrability of A^ε (computations for B^ε are similar):

$$\varepsilon A^\varepsilon(x, y) = \rho(y) \operatorname{div}_y(b(x_\varepsilon)) + \langle b(x_\varepsilon), \nabla\rho(y)\rangle \in L^1.$$

2.3 Refined anisotropic estimate

The crucial step in our argument is prove that (4) above holds, with

$$\Lambda_\rho(t, x) = \int \left|\operatorname{div}_y(M_t(x) y\rho(y))\right| dy,$$

where M_t is given by the polar decomposition $Db_t(dx)\, dt = M_t(x) |Db_t|(dx)\, dt$. To this aim, we proceed as follows. First, we fix $\varepsilon > 0$ and assume b to be smooth, in order to obtain an estimate for r^ε in terms of Db. Then, still keeping ε fixed, we extend the validity of this estimate to any BV vector field. Finally, we let $\varepsilon \to 0$ and conclude.

Without any loss of generality, we assume both $\|u\|_\infty \le 1$ and $\|\beta'\|_\infty \le 1$. As above, we omit to write as subscripts both ρ and t.

Fix $\varepsilon > 0$ and let b be smooth. In Sect. 2.2, we obtained that

$$r^\varepsilon(x) = \int u^\varepsilon(x + \varepsilon y)\left[B^\varepsilon(x + \varepsilon y, y) - A^\varepsilon(x + \varepsilon y, y)\right] dy.$$

Therefore, we estimate

$$\int \left|\varphi\beta'(u_\varepsilon) r^\varepsilon\right| \le \int |\varphi|(x) \left|B^\varepsilon(x + \varepsilon y, y) - A^\varepsilon(x + \varepsilon y, y)\right| dxdy. \tag{8}$$

After a change of variables $(x, y) \mapsto (x_\varepsilon, y)$ (recall that $x_\varepsilon = x - \varepsilon y$) and using the expressions for A^ε and B^ε, the right hand side above is equal to

$$\int |\varphi|(x_\varepsilon) \left|\operatorname{div}_y\left(\frac{b(x) - b(x_\varepsilon)}{\varepsilon}\rho(y)\right)\right| dxdy.$$

Since b is assumed to be smooth, we have

$$\frac{b(x)-b(x_\varepsilon)}{\varepsilon}=-\fint_0^\varepsilon \frac{d}{ds}b(x_s)=\fint_0^\varepsilon Db(x_s)\,y,$$

where for brevity we write $\fint_0^\varepsilon f(s) := \frac{1}{\varepsilon}\int_0^\varepsilon f(s)\,ds$. Exchanging divergence and integration with respect to s, it holds

$$\operatorname{div}_y\left(\frac{b(x)-b(x_\varepsilon)}{\varepsilon}\rho(y)\right)=\fint_0^\varepsilon \operatorname{div}_y\left(Db(x_s)\,y\rho(y)\right). \tag{9}$$

Let $c(x,y) := \rho(y)\,Db(x)\,y \left(=\rho(y)\,\partial_y b(x)\right)$. By identity (7),

$$\operatorname{div}_y\left(Db(x_s)\,y\rho(y)\right)=-s\left[\operatorname{div}_x c\right](x_s,y)+\left[\operatorname{div}_y c\right](x_s,y).$$

Since $\operatorname{div}_x c$ involves further derivatives of b, the following identity is crucial:

$$\left[\operatorname{div}_x c\right](x_s,y)=\rho(y)\left[\partial_y \operatorname{div} b\right](x_s)=-\rho(y)\frac{d}{ds}\operatorname{div} b(x_s),$$

because it allows to integrate by parts in (9) to obtain

$$\rho(y)\left[\operatorname{div} b(x_\varepsilon)-\fint_0^\varepsilon \operatorname{div} b(x_s)\right]+\fint_0^\varepsilon \left[\operatorname{div}_y c\right](x_s,y).$$

We estimate therefore (8) with

$$\int\int |\varphi|(x_\varepsilon)\left[\fint_0^\varepsilon \left|\left[\operatorname{div}_y c\right](x_s,y)\right|+\left|\fint_0^\varepsilon \operatorname{div} b(x_s)-\operatorname{div} b(x)\right|\rho(y)\right]dydx. \tag{10}$$

Keeping $\varepsilon > 0$ fixed, we extend the estimate to a general b. We focus on the first term of the sum in (10). Exchanging integrations and changing variables $(x,y)\mapsto(x+sy,y)$, we find an equivalent expression, of the form

$$\int\left[\int |\varphi|_\varepsilon(x,y)\,\Lambda_\rho(x,y)\,dy\right]|Db|(x)\,dx,$$

where we let

$$|\varphi|_\varepsilon(x,y) := \fint_0^\varepsilon |\varphi(x_{\varepsilon-s})|, \quad \Lambda_\rho(x,y) := \left|\operatorname{div}_y\left(M(x)\,y\rho(y)\right)\right|$$

and $M(x)$ is defined by the identity $M(x)\,|Db|(x)=Db(x)$. The crucial observation is that this is an expression of the form

$$\int f\left(x,\frac{Db}{|Db|}(x)\right)|Db|(dx),$$

where $f : \mathbb{R}^d \times \mathbb{S}^{d^2-1}$ is continuous and bounded. By Reshetnyak continuity theorem [7, Theorem 2.39], we extend our estimate to a general BV vector field. More precisely, we approximate b with a sequence of smooth vector fields b_n, such that

$$\lim_{n\to\infty} \|b_n - b\|_1 = 0, \quad \lim_{n\to\infty} \|\mathrm{div} b_n - \mathrm{div} b\|_1 = 0$$

and $|Db_n| \left((0,T)\times\mathbb{R}^d\right) \to |Db_n| \left((0,T)\times\mathbb{R}^d\right)$ (such a sequence exists, e.g. by convolution with a smooth kernel). The second term in (10) is easily seen to be continuous with respect to L^1 convergence of $\mathrm{div} b$.

We obtain that, for a general BV field b, the quantity $\int \left|\varphi\beta'\left(u^\varepsilon\right) r^\varepsilon\right|$, is estimated by the sum of two terms:

$$\int \left[\int |\varphi|_\varepsilon (x,y)\, \Lambda_\rho (x,y)\, dy\right] d\,|Db|\,(x) \tag{11}$$

and

$$\int\int |\varphi|\,(x_\varepsilon) \left| \fint_0^\varepsilon \mathrm{div} b\,(x_s) - \mathrm{div} b\,(x)\right| \rho\,(y)\, dy dx. \tag{12}$$

We let $\varepsilon \to 0$. In (12), we estimate $|\varphi|\,(x)$ with its supremum and then use strong continuity in L^1 of translations, together with the fact that ρ has compact support, to show that it converges to zero. In (11), we exploit the fact that φ a test function, so that $|\varphi|_\varepsilon\,(x,y)$ converges pointwise everywhere to $|\varphi|\,(x)$ and dominated by some constant: by Lebesgue's theorem with respect to $|Db|$, we conclude that (4) holds.

2.4 Optimization

We prove that, for every matrix $M \in \mathbb{R}^{d\times d}$, it holds

$$\inf\left\{\int \left|\mathrm{div}_y\,(My\rho\,(y))\right| dy \;:\; \rho \in C_c^\infty(\mathbb{R}^d),\, \rho\geq 0,\, \int \rho = 1\right\} = 0.$$

This is precisely the mathematical content of [2, Lemma 5.1], due to Alberti. For completeness, we report here its proof, since it is used, almost verbatim, in Section 6 below. We argue that, for any $T>0$, there exists some $\rho \in C_c^\infty\left(\mathbb{R}^d\right)$ such that

$$\int \left|\mathrm{div}_y\,(My\rho\,(y))\right| dy \leq \frac{2}{T}.$$

Consider the vector field $\hat{M} : y \mapsto My$ and let $\bar{u}$ be any smooth, non-negative and compactly supported function with $\int \bar{u} = 1$. The solution (u_t) of the continuity equation with initial datum $\bar{u}$, i.e.,

$$\partial_t u_t + \mathrm{div}(\hat{M} u_t) = 0, \quad u_0 = \bar{u},$$

is smooth, non-negative and compactly supported: the same holds for $\rho := \frac{1}{T}\int_0^T u_t dt$. Finally, to estimate $\int \left|\mathrm{div}_y \left(My\rho\left(y\right)\right)\right| dy$, let φ be smooth and compute $\int \varphi\left(y\right) \mathrm{div}_y \left(My\rho\left(y\right)\right) dy$:

$$\frac{1}{T}\int_0^T \int \varphi\left(y\right) \mathrm{div}_y \left(\hat{M}\left(y\right) u_t\left(y\right)\right) = \frac{1}{T}\left(\int \varphi\left(y\right)\left(u_T - u_0\right) dy\right) \leq \frac{2}{T}\left\|\varphi\right\|_\infty.$$

3 Sobolev and BV spaces on Wiener spaces

In this section, we briefly recall the framework of infinite dimensional analysis on Gaussian spaces. Let (X, γ, H) be a Wiener space, i.e., X is a separable Banach space, γ is a non-degenerate centred Gaussian measure on X and $H \subseteq X$ is the associated Cameron-Martin space, which enjoys a Hilbert space structure with scalar product $\langle\cdot,\cdot\rangle_H$. Let $Q \in \mathcal{L}\left(X^*, X\right)$ be the covariance operator associated with γ; duality between X and X^* is denoted with $\langle\cdot,\cdot\rangle$.

3.1 Smooth cylindrical maps

We fix throughout an orthonormal basis $\left(h_n = Qx_n^*\right)_{n\geq 1} \subseteq H$ induced by elements in X^* and let $\pi_N : X \to X$,for $N \geq 1$, be the map defined on $x \in X$ by

$$\pi_N\left(x\right) := \sum_{n=1}^N \langle x, x_n^*\rangle h_n.$$

With a slight abuse of notation, sometimes we identify the image of π with $\mathbb{R}^N$.

Let K be any separable Hilbert space. A map $b : X \to K$ is said to be cylindrical if, for some $N \geq 1$, there exists $k_i \in K$ and $b_i : \mathbb{R}^N \to \mathbb{R}$, with $i \in \{1, \ldots, N\}$, such that it holds, for γ-a.e. $x \in X$,

$$b\left(x\right) = \sum_{i=1}^N b_i\left(\pi_N\left(x\right)\right) k_i.$$

When $K = H$, we also require that $k_i = h_i$, and use the term *field* to indicate H-valued maps. A cylindrical map is said to be *smooth* if it admits a representation with b_i bounded together with their derivatives, $i \in \{1, \ldots, N\}$.

On smooth cylindrical maps, the Sobolev-Malliavin gradient is defined by

$$\nabla b\left(x\right) := \sum_{i=1}^N k_i \otimes \left(QD_F b_i\left(x\right)\right) \in K \otimes H,$$

where $D_F b\,(x) \in X^*$ is the Fréchet derivative of b_i at x. The Gaussian divergence of a smooth cylindrical field b (thus, H-valued) is defined by

$$\mathrm{div} b\,(x) := \sum_{i=1}^{N} \langle \nabla b_i\,(x)\,, h_i \rangle_H - x_i^*\,(x)\, b_i\,(x)\,.$$

Given a smooth cylindrical map b, with values in $K \otimes H$, in order to define its divergence, we represent it first as $b = \sum_i k_i \otimes b_i$ and then we let $\mathrm{div} b = \sum_i k_i \mathrm{div} b_i$.

It is customary and useful to endow the product spaces $K \otimes K'$ with the Hilbert-Schmidt norm, thus defining a Hilbert space structure.

The following integration by parts formula justifies the definition of divergence: for any smooth cylindrical function u and field b, it holds

$$\int \langle \nabla u, b \rangle_H d\gamma = -\int u\, \mathrm{div} b\, d\gamma.$$

It is useful to generalize this identity to smooth cylindrical K-valued maps u and $K \otimes H$-valued maps b, arguing componentwise:

$$\int \langle \nabla u, b \rangle_{K\otimes H} d\gamma = -\int \langle u, \mathrm{div} b \rangle_K d\gamma. \tag{13}$$

3.2 Sobolev, BV spaces and the distributional divergence

We recall some basic facts about Sobolev-Malliavin spaces, referring to [10, Chapter 5] for a detailed introduction. Given $p \in [1, \infty[$, one considers either the abstract completion of smooth K-valued maps u with respect to the norm

$$\|u\|_{1,p} := \left\| |u|_K + |\nabla u|_{K\otimes H} \right\|_{L^p(X,\gamma)},$$

or the space of maps $u \in L^p\,(X, \gamma; K)$ such that there exists some $\nabla u \in L^p\,(X, \gamma; K \otimes H)$ which satisfies (13), for every smooth cylindrical $K \otimes H$-valued map b. A Meyers-Serrin type result in this setting (see [10, Proposition 5.4.6]) shows that the two definitions are equivalent: the space $W^{1,p}\,(X, \gamma : K)$ is well defined and smooth cylindrical K-valued maps are dense.

To introduce BV maps, we refer to [8] for details: $BV(X, \gamma; K)$ is defined as the space of $u \in L \log^{1/2} L\,(X, \gamma; K)$ such that there exists some $K \otimes H$-valued measure Du on X, with finite total variation, such that, for every smooth $K \otimes H$-valued cylindrical map b, it holds

$$\int \langle b, dDu \rangle_{K\otimes H} = -\int \langle u, \mathrm{div} b \rangle_K\, d\gamma.$$

Theorem 4.1 in [8] provides the following alternative characterization: $u \in BV(X, \gamma; K)$ if and only if there exists some sequence u_n of smooth cylindrical maps such that, as $n \to \infty$, $\int |u_n - u| \, d\gamma \to 0$ and $\int |\nabla u_n|_H \, d\gamma$ is uniformly bounded (the smallest bound among all the sequences being the total variation $|Du|(X)$). Actually, in [8], only the scalar case is considered, but the argument easily extends to general K-valued maps. Finally, we let $\|u\|_{BV} := \int |u| \, d\gamma + |Du|(X)$.

Finally, we introduce a notion of distributional divergence for any $K \otimes H$ valued Borel map b using (13) as a definition. In particular, we may consider sufficient conditions to ensure that (13) holds, for some (necessarily unique) function $\operatorname{div} b \in L^p(X, \gamma)$. For $p \in]1, \infty[$, it can be proved that, if $b \in W^{1,p}(X, \gamma; H)$, then $\operatorname{div} b \in L^p(X, \gamma)$: the case $p = 2$ is elementary, the others following from the boundedness of the Riesz transform on Wiener spaces, see [10, Section 5.8]. To the author's knowledge, the validity of a correspondent statement for $p = 1$ is open.

4 Statement of the main result

Let us first recall the precise meaning of distributional solution to the continuity equation (1), in the Wiener space setting. We introduce a suitable class of *test functions*, consisting of those of the form $\varphi(t, \pi_N(x))$ for some smooth bounded $\varphi(t, y)$, supported in some strip $[\delta, T - \delta] \times \mathbb{R}^N$, for some $\delta \in]0, T[$. Then, given a (possibly time-dependent) field $b = (b_t) \in L^1((0, T) \times X; H)$, we say that $u = (u_t) \in L^\infty((0, T) \times X)$ is a distributional solution to (1) if, for every test function φ, it holds

$$\int_0^T \int (\partial_t \varphi + \langle \nabla \varphi, b_t \rangle_H) \, u_t \, d\gamma = 0.$$

As already mentioned in the introduction, for a general overview of the Sobolev case and the links between continuity equations and regular Lagrangian flows, we refer to [4]. Here, we focus only on the renormalization property for solutions $u \in L^\infty((0, T) \times X)$, our main result being the following

Theorem 2 *Let $p > 1$ and $b \in L^1((0, T); BV \cap L^p(X, \gamma; H))$, with* $\operatorname{div} b \in L^1((0, T) \times X)$*. Then, any distributional solution $u = (u_t) \in L^\infty((0, T) \times X)$ to*

$$\partial_t u_t + \operatorname{div}(b_t u_t) = 0$$

is a renormalized solution.

Let us remark that the assumption $b \in L^p(X, \gamma; H)$ is made only for technical convenience and the proof below suggests that it can be removed, exploiting the natural integrability condition $b \in L \log^{1/2} L(X, \gamma, H)$ valid for general BV maps: we just prefer to avoid the introduction of Orlicz spaces.

As discussed in the introduction, the arguments in [4, Section 4], combined with the result above entail existence (assuming a stronger bound on the div$b$), uniqueness and stability of associated L^∞-regular Lagrangian flows: we do not enter into details since this is rather straightforward.

Instead, we briefly show in which sense the introduction of BV vector fields is expected to be useful, even in contexts where higher regularity would be expected. A problem with the results in [4] is that they are of global nature. Given a Sobolev field $b \in W^{1,2}(X, \gamma; H)$ with $\operatorname{div} b \in L^\infty(X, \gamma)$, a global, i.e. on all X, L^∞-regular Lagrangian flow is well-defined: however, it is not clear at all how to proceed e.g. if b is regular, or even defined, only in an open (even regular) set $\Omega \subseteq X$. The natural strategy is to consider the field $\chi_\Omega b$, whose regularity should be then only BV.

To sketch a concrete example of such a situation, one can consider a gradient vector field $\nabla\eta$, where η solves the following elliptic problem with Neumann boundary conditions:

$$\begin{cases} -\operatorname{div}\nabla\eta + \lambda\eta = f & \text{in } \Omega, \\ \frac{\partial}{\partial n}\eta = 0 & \text{in } \partial\Omega. \end{cases}$$

Such problems arise as Kolmogorov equations associated to stochastic processes reflected at the boundary of Ω, as investigated e.g. in [9]: to our purpose, here, they only provide non-trivial examples to which our theorem applies. Indeed, using the results from [9], it is possible to show that if Ω is sufficiently regular and $f \in L^\infty(\Omega; \gamma)$, then $b = \chi_\Omega \nabla\eta$ enjoys a well-posed regular Lagrangian flow. For brevity, we do not enter into a detailed description of this fact: we only remark that a key step is to recognize that $\operatorname{div} b = \chi_\Omega(\lambda\eta - f)$ thanks to the boundary conditions and that $Db = \nabla\eta \otimes D\chi_\Omega + \chi_\Omega(\nabla^2\eta)\gamma$ is a measure.

5 Some technical results

Before we address the proof of Theorem 2, we give some auxiliary facts, about approximations of fields and the exponential map in Wiener spaces.

5.1 Cylindrical approximations

We establish two propositions, the first one being a slight generalization of the approximation procedure employed in the proof of [4, Proposition 3.5].

Recall that, in Sect. 3, we introduced an orthonormal basis in H of the form $\left(h_n = Qx_n^*\right)_{n\geq 1}$ and related projections operators π_N, for $N \geq 1$: we let in all what follows $\mathcal{F}_N$ be the σ-algebra generated by the map π_N and let $\mathbb{E}_N$ be the conditional expectation operator with respect to $\mathcal{F}_N$.

Moreover, the map $x \mapsto (\pi_N(x), x - \pi_N(x))$ induces decompositions $X = \operatorname{Im}\pi_N \oplus \ker\pi_N$ and $H = \operatorname{Im}\pi_N \oplus \operatorname{Im}\pi_N^\perp$. Recall that we also identify $\operatorname{Im}\pi_N = \mathbb{R}^N$ via $h_i \mapsto e_i$. The map π_N also induces a decomposition $\gamma = \gamma_N \otimes \gamma_N^\perp$, where γ_N is the standard N-dimensional normal law on $\mathbb{R}^N$ and $\gamma_N^\perp$ is a non-degenerate Gaussian measure on $\ker\pi_N$, with Cameron-Martin space given by $\operatorname{Im}\pi_N^\perp$.

Let K be a Hilbert space and let μ be a K-valued measure on X. The push-forward $(\pi_N)_\sharp\,\mu$ is defined by $(\pi_N)_\sharp\,\mu(A) := \mu\left(\pi_N^{-1}A\right)$, for A Borel. Since push-forwards commute with linear operators on K, for any $H \otimes H$-valued measure, it holds

$$[\pi_N \otimes \pi_N]\,\mu\,(A) = [\pi_N \mu\,\pi_N]\,(A) = \pi_N \mu\,(A)\,\pi_N.$$

In the next two propositions, let $b \in BV \cap L^p\,(X, \gamma; H)$, with $\mathrm{div} b \in L^q\,(\gamma)$, for some $p, q \in [1, \infty[$, and, for $N \geq 1$, let

$$b^N := \mathbb{E}_N\,[\pi_N b]$$

be a cylindrical approximation of b.

Proposition 1 *Let b and b^N be as above. Then, b^N is a cylindrical BV vector field, with*

$$Db^N = \left[(\pi_N)_\sharp\,(\pi_N\, Db\,\pi_N)\right] \otimes \gamma_N^\perp \quad \textit{and} \quad \mathrm{div} b^N = \mathbb{E}_N\,[\mathrm{div} b]\,.$$

Moreover, it holds

$$\lim_{N\to\infty} \left\| b^N - b \right\|_p + \left\| \mathrm{div} b^N - \mathrm{div} b \right\|_q = 0.$$

Proof Once the identity involving the divergence is proved, the last statement follows at once by the martingale convergence theorem (or because conditional expectations are contractions and convergence is true for cylindrical fields).

Therefore, we focus on the two identities: being N fixed we write $\pi := \pi_N$ and $\mathbb{E} := \mathbb{E}_N$.

Notice that the field b^N is at least as integrable as b, since projections and conditional expectations reduce norms. In what follows, we use duality with smooth cylindrical functions, self-adjointness of π and $\mathbb{E}$, and the commutation relation

$$\pi\mathbb{E}\,[\nabla\varphi] = \mathbb{E}\,[\pi\nabla\varphi] = \nabla\mathbb{E}\,[\varphi]\,.$$

It holds

$$\begin{aligned}\int \varphi\,\mathrm{div} b^N d\gamma &= -\int \langle\nabla\varphi, b^N\rangle d\gamma = -\int \langle\pi\mathbb{E}\,[\nabla\varphi]\,, b\rangle d\gamma \\ &= -\int \langle\nabla\mathbb{E}\,[\varphi]\,, b\rangle = \int \varphi\,\mathbb{E}\,[\mathrm{div} b]\,,\end{aligned}$$

that gives the identity for the divergence.

Similar computations can be performed on $H \otimes H$ smooth maps with $\pi \otimes \mathrm{Id}$ (where Id denotes the identity map) in place of π. It holds $D\,(\pi b) = (\pi \otimes \mathrm{Id})\, Db$:

$$\int \langle\varphi, dD\pi b\rangle = -\int \langle\pi\,\mathrm{div}\varphi, b\rangle = -\int \langle\mathrm{div}\,(\pi \otimes \mathrm{Id}\,\varphi)\,, b\rangle = \int \langle\varphi, (\pi \otimes \mathrm{Id})\, dDb\rangle,$$

therefore

$$\int \langle \mathrm{div}\varphi, b^N \rangle = \int \langle \mathbb{E}[\mathrm{div}\varphi], \pi b \rangle = \int \langle \mathrm{div}\,(\mathrm{Id} \otimes \pi\, \mathbb{E}[\varphi]), \pi b \rangle$$
$$= \int \langle \mathrm{Id} \otimes \pi\, \mathbb{E}[\varphi], dD\pi\, b \rangle = \int \langle \mathbb{E}[\varphi], (\pi \otimes \pi)\, dDb \rangle.$$

and so we conclude. □

The next result is actually measure-theoretical, its proof being based on disintegration of measures and Jensen's inequality. Notice that it generalizes the inequality $\left|Db^N\right|(X) \leq |Db|(X)$.

Proposition 2 *Let*

$$f : X \times (H \otimes H) \to [0, \infty[$$

be Borel, positively homogeneous and convex in the second variable, keeping fixed the first. For any $N \geq 1$, *it holds*

$$\int f\left(\pi_N, \frac{Db^N}{\left|Db^N\right|}\right) d\left|Db^N\right| \leq \int f\left(\pi_N, \pi_N \frac{Db}{|Db|} \pi_N\right) d\,|Db|\,.$$

Proof We write for brevity $\pi := \pi_N$. Let $\mu = (\pi \otimes \pi) Db$, $\nu = Db^N$ and $\rho = \gamma_N^{\perp}$ so that Proposition 1 gives $\nu = \left(\pi_\sharp \mu\right) \otimes \rho$. The total variation and the polar decomposition of ν factorize respectively as

$$|\nu|\,(dx, dy) = \left|\pi_\sharp \mu\right|(dx) \otimes \rho\,(dy) \quad \text{and} \quad \frac{\nu}{|\nu|}(x, y) = \frac{\pi_\sharp \mu}{\left|\pi_\sharp \mu\right|}(x)\,.$$

Therefore, it holds

$$\int f\left(x, \frac{\nu}{|\nu|}(x, y)\right) d\,|\nu\,(x, y)| = \int f\left(x, \frac{\pi_\sharp \mu}{\left|\pi_\sharp \mu\right|}(x)\right) \left|\pi_\sharp \mu\right|(dx)\,.$$

Since $\left|\pi_\sharp \mu\right| \leq \pi_\sharp\,|\mu|$, it holds $\frac{\pi_\sharp \mu}{\pi_\sharp |\mu|} = \frac{\pi_\sharp \mu}{\left|\pi_\sharp \mu\right|} \frac{\left|\pi_\sharp \mu\right|}{\pi_\sharp |\mu|}$ which, by positive homogeneity of f, gives

$$\int f\left(x, \frac{\pi_\sharp \mu}{\left|\pi_\sharp \mu\right|}(x)\right) \left|\pi_\sharp \mu\right|(dx) = \int f\left(x, \frac{\pi_\sharp \mu}{\pi_\sharp\,|\mu|}(x)\right) \pi_\sharp\,|\mu|\,(dx)\,.$$

We now disintegrate $|\mu|$ with respect to π, and apply Jensen's inequality. More precisely, since X is a separable Banach space, there exists a probability kernel $(N\,(x, dy))_x$ such that, for every bounded Borel map $g\,(z)$ it holds

$$\int g\,(z)\, d\,|\mu|\,(z) = \int \pi_\sharp\,|\mu|\,(dx) \int g\,(x, y)\, N\,(x, dy)\,.$$

Moreover, if $\sigma\,|\mu| = \mu$ is the polar decomposition, using $g(z) = h(\pi(z))\,\sigma(z)$, we obtain

$$\frac{\pi_\sharp \mu}{\pi_\sharp |\mu|}(x) = \int \sigma(x, y)\, N(x, dy).$$

By convexity of f and Jensen's inequality,

$$f\left(x, \frac{\pi_\sharp \mu}{\pi_\sharp |\mu|}(x)\right) \le \int f(x, \sigma(x, y))\, N(x, dy).$$

Integrating with respect to $\pi_\sharp |\mu|$, the right hand side above gives

$$\int f(\pi(z), \sigma(z))\, d|\mu|(z) = \int f\left(\pi(z), \frac{\mu}{|\mu|}(z)\right) d|\mu|(z),$$

The homogeneity of f and the identities

$$\frac{\pi Db\pi}{|\pi Db\pi|}\frac{|\pi Db\pi|}{|Db|} = \frac{\pi Db\pi}{|Db|} = \pi\frac{Db}{|Db|}\pi,$$

allow to conclude. □

5.2 Exponentials maps in Wiener spaces

It is well known that the Cameron-Martin space $H \subseteq X$ is isomorphic to a subspace $\mathcal{H} \subseteq L^2(X, \gamma)$ via

$$h \mapsto \hat{h} := -\mathrm{div} h \in L^2(X, \gamma).$$

Notice that the divergence of a constant field is not zero, reflecting the fact that Gaussian measures are not invariant under translations.

We extend the notation $\hat{h}$ as follows: given $b \in L^1(X, \gamma; H)$, we define $\hat{b} \in L^1\left(X \times X, \gamma^2\right)$ by $\hat{b}(x, y) := -\mathrm{div}_y(b(x))(y)$. This provides an embedding $L^1(X, \gamma; H) \subseteq L^1\left(X \times X, \gamma^2\right)$. Similarly, given a Hilbert-Schmidt operator $M \in H \otimes H$, we let $\hat{M} = -\mathrm{div} M \in L^2(X, \gamma; H)$. On cylindrical operators in the form $M = \sum_{i,j} m_{ij} h_i \otimes h_j$, it holds

$$\hat{M}(x) = \sum_i h_i \sum_j m_{ij} x_j^*(x). \tag{14}$$

In particular, it holds $\|\hat{M}\|_2 = \|M\|$ (where $\|\cdot\|$ denotes the Hilbert-Schmidt norm) and $\nabla \hat{M} = M$.

In this section, we focus on integrability result for the solution of a continuity equation driven by $\hat{M}$,

$$\partial_t u_t + \mathrm{div}(\hat{M} u_t) = 0, \quad u_0 = 1. \tag{15}$$

Since M is regular and integrable, the theory developed in [4] provides existence, uniqueness and stability for solutions up to a time T which depends on the exponential integrability of $\mathrm{div}\hat{M}$. Looking for integral bounds on u_T for fixed $T > 0$, already in the case $M = h_i \otimes h_i$, one finds

$$\mathrm{div}\hat{M} = |h_i|^2 - \hat{h}_i^2,$$

whose negative part is exponentially integrable only up to a factor $\alpha < 1/2$ and so the bound developed in [4] does not seem to help. The following proposition provides L^p-bounds for u_T and $|\nabla u_T|$ for some some $p(T) > 1$. Although the proof makes explicit use of the exponential form of solutions, the key ingredient is a well-known consequence of the so-called concentration of measure, and we claim (but not prove here) that one could prove results of this kind for rather general H-Lipschitz fields.

Proposition 3 *Let $M \in H \otimes H$. For every $T > 0$, there exists $p(T, \|M\|) > 1$ such that (15) admits a (unique) solution $u \in L^\infty\left((0,T); W^{1,p}(\gamma)\right)$.*

Proof It is sufficient to assume that $X = \mathbb{R}^N$, $\gamma = \gamma_N$ is a standard Gaussian and M is a square matrix, provided we obtain bounds that are independent of N, the general case following by stability via cylindrical approximation.

Recall that the Hilbert-Schmidt norm is stronger than the usual operator norm, that $\|AB\| \leq \|A\| \|B\|$ and that that products of Hilbert-Schmidt operators have finite trace, in particular $\mathrm{Tr}\left[A^2\right] \leq \|A\|^2$.

Identity (14) shows that $\hat{M}$ is the linear operator given by matrix multiplication.

We begin by rewriting a linear change of variables in a convenient way (see also [27, Chapter 10]). If C is any square matrix in $\mathbb{R}^N$, the following identity holds true:

$$\frac{d(I+C)_\sharp(\gamma_N)}{d\gamma_N}(x + Cx) = |\det_2(I+C)|^{-1} \exp\left(\mathrm{div}(Cx) + |Cx|^2/2\right). \tag{16}$$

where $\det_2(I+C) = \det(I+C)\exp(-\mathrm{Tr}[C])$ is the Carleman-Fredholm determinant. As a consequence, we have

$$\int |\det_2(I+C)| \exp\left(-\mathrm{div}(Cx) - |Cx|^2/2\right) d\gamma_N(x) = 1. \tag{17}$$

The unique solution of (15) is $u_t = (X(t,\cdot)_\sharp \gamma_N)/\gamma_N$, where $X(t,x)$ is the classical (finite dimensional) exponential flow,

$$X(t,x) := \exp(tM)x = x + E_t x,$$

where we write $E_t := \sum_{k=1}^{\infty}(tM)^k/k!$, because in this form we may apply (16) to obtain

$$u_t(X(t,x)) = |\det{}_2\,(I+E_t)|^{-1}\exp\left(\operatorname{div}(E_t x) + |E_t x|^2/2\right). \tag{18}$$

We compute first the determinant, that gives

$$\det{}_2\,(I+E_t) = \exp\left(\operatorname{Tr}[tM - E_t]\right) = \exp\left(\operatorname{Tr}\left[(tM)^2\sum_{k=0}^{\infty}(tM)^k/(k+2)!\right]\right)$$

and estimate the trace,

$$\left|\operatorname{Tr}\left[(tM)^2\sum_{k=0}^{\infty}(tM)^k/(k+2)!\right]\right| \le t^2\,\|M\|^2\exp\left(t\,\|M\|\right),$$

so that the determinant is bounded below and above.

We focus on the quantity

$$\int u_t^p = \int |\det{}_2\,(I+E_t)|^{-p-1}\exp\left((p-1)\operatorname{div}(E_t x) + (p-1)\,|E_t x|^2/2\right)\gamma_N(dx).$$

As we add and subtract a term $(p-1)^2\,|E_t x|^2$ inside the exponential, apply Cauchy-Schwartz and (17), so that we see that we need to estimate only

$$\int \exp\left(2(p-1)^2\,|E_t x|^2 + (p-1)\,|E_t x|^2\right)\gamma_N(dx),$$

since all the determinant terms that appear are bounded, arguing as above.

Exponential integrability of $(p-1)(2p-1)\,|E_t x|^2$ follows from the fact that $x \mapsto E_t x$ is Lipschitz, with constant bounded by $t\,\|M\|\exp(t\,\|M\|)$. Arguing at fixed $T > 0$, we may consider $p = 1+\varepsilon$, with ε so small that [10, Theorem 4.5.7] applies providing a bound, which does not depend on the dimension of the space.

To obtain bounds on the gradient ∇u_t, we notice that (18) gives

$$u_t(y) = |\det{}_2\,(I+E_t)|^{-1}\exp\left(\operatorname{div}E_t(\exp(-tM)\,y) + |E_t\exp(-tM)\,y|^2/2\right).$$

Differentiating with respect to y, we obtain

$$\nabla u_t(y) = u_t(y)\nabla\left[\operatorname{div}E_t(\exp(-tM)\,y) + |E_t\exp(-tM)\,y|^2/2\right].$$

Since we already have a bound on u_t, it is sufficient to bound the gradient terms, but these are all linear expressions in y, which can be explicitly computed and bounded in every L^p space ($p < \infty$) with some constant depending on p, T and $\|M\|$ only. $\square$

6 Proof of Theorem 2

In this final section we provide a complete proof of Theorem 2. The line of reasoning mirrors that of the proof of Theorem 1: in particular, we follow exactly the main steps stated there, and we split this section to give details.

6.1 Mollification

Let ρ be any cylindrical smooth function on X, with $\rho \geq 0$ and $\int \rho d\gamma = 1$, and introduce a modified Ornstein-Uhlenbeck, acting only on the space variables,

$$T_\rho^\varepsilon \varphi (t, x) := \int \varphi (t, x_\varepsilon) \rho (y) d\gamma (y),$$

where we write, here and in what follows,

$$x_\varepsilon := e^{-\varepsilon} x + \sqrt{1 - e^{-2\varepsilon}} y, \quad y_\varepsilon := -\sqrt{1 - e^{-2\varepsilon}} x + e^{-\varepsilon} y.$$

Its adjoint in $L^2((0, T) \times X)$ reads as

$$\left(T_\rho^\varepsilon\right)^* \varphi_t (x) := \int \varphi \left(t, x^\varepsilon\right) \rho \left(y^\varepsilon\right) d\gamma (y),$$

where we introduce the notation

$$x^\varepsilon := e^{-\varepsilon} x - \sqrt{1 - e^{-2\varepsilon}} y, \quad y^\varepsilon := \sqrt{1 - e^{-2\varepsilon}} x + e^{-\varepsilon} y.$$

Since T_ρ^ε preserves test functions on $(0, T) \times X$, we can define $(T_\rho^\varepsilon)^*$ on distributions, by duality. We let therefore $u_\rho^\varepsilon = (T_\rho^\varepsilon)^* u$, so that (2) holds, with

$$r_\rho^\varepsilon := \operatorname{div} \left(b u_\rho^\varepsilon\right) - (T_\rho^\varepsilon)^* \operatorname{div} (bu).$$

6.2 Approximate renormalization

To keep notation simple, we frequently omit here and below the dependence on t and ρ, since they play no role.

As $u \in L^\infty ((0, T) \times X)$, an integration by parts shows that u^ε belongs to every Sobolev space $W^{1,p}(X, \gamma)$ with respect to the space variables, so that the only thing to prove, in order to justify the usual calculus rules that we perform in this step, is that the distribution r^ε is (induced by) an integrable function. It is also enough to prove that both $\operatorname{div} (bu^\varepsilon)$ and $(T^\varepsilon)^* \operatorname{div} (bu)$ are integrable functions (this is a standard argument, compare with Lemma 3.6 and the computations in Theorem 3.7 in [4]). For brevity, we do not enter into details, but remark that to perform these computations and get the approximate renormalization we make use of the integrability assumption $b \in L^p(X, \gamma; H)$.

Equivalent expressions for $\mathrm{div}\,(bu^{\varepsilon})$ *and* $(T^{\varepsilon})^{*}\,\mathrm{div}\,(bu)$ *via integration by parts.* Here and in what follows, the function $C_{\varepsilon} = e^{\varepsilon}\sqrt{1-e^{-2\varepsilon}}$ frequently appears due to various integration by parts. Notice that $C_{\varepsilon} \sim \sqrt{2\varepsilon}$, as $\varepsilon \to 0$.

Given a test function φ, we obtain

$$T^{\varepsilon}_{\rho}\langle\nabla\varphi, b\rangle\,(x) = -\frac{1}{C_{\varepsilon}}\int\varphi\,(x_{\varepsilon})\,\mathrm{div}_{y}\left(e^{\varepsilon}b\,(x_{\varepsilon})\,\rho\,(y)\right) =: \int\varphi\,(x_{\varepsilon})\,A^{\varepsilon}\,(x,y)\,d\gamma\,(y)\,,$$

and

$$\langle\nabla T^{\varepsilon}_{\rho}\varphi, b\rangle\,(x) = -\frac{1}{C_{\varepsilon}}\int\varphi\,(x_{\varepsilon})\,\mathrm{div}_{y}\,(b\,(x)\,\rho\,(y))\,d\gamma\,(y) =: \int\varphi\,(x_{\varepsilon})\,B^{\varepsilon}(x,y)\,d\gamma(y).$$

We then show that A^{ε} and B^{ε} are integrable and change variables $(x, y) \mapsto (x^{\varepsilon}, y^{\varepsilon})$, to conclude as in the finite dimensional case.

Integrability of A^{ε} *and* B^{ε} *via divergence identities.* By rotational invariance of the Gaussian measures, an analogue of identity (6) holds true in Wiener spaces, for vector fields $c : X \times X \to H \oplus H$ and rotations $M = M_s$ defined on $X \times X$ (and then on $H \oplus H$) by

$$M_s\,(x, y) = (x_s, y_s) = \left(e^{-s}x + \sqrt{1-e^{-2s}}\,y, -\sqrt{1-e^{-2s}}\,x + e^{-s}y\right).$$

Once we take $c\,(x, y) = (0, v\,(x, y))$ we get the following analogue of (7),

$$\mathrm{div}_{y}\,(v\,(x_s, y_s)) = \sqrt{1-e^{-2s}}\,[\mathrm{div}_{x}\,(v)]\,(x_s, y_s) + e^{-s}\left[\mathrm{div}_{y}\,(v)\right](x_s, y_s)\,. \qquad (19)$$

If $v\,(x, y) = b\,(x)$ and $s = \varepsilon$, we obtain

$$\mathrm{div}_{y}\,(b\,(x_{\varepsilon})) = \sqrt{1-e^{-2\varepsilon}}\mathrm{div} b\,(x_{\varepsilon}) - e^{-\varepsilon}\hat{b}\,(x_{\varepsilon}, y_{\varepsilon}) \in L^{1}(X \times X, \gamma \otimes \gamma)$$

because of the integrability assumptions on b and its divergence. This shows that A^{ε} is integrable; computations involving B^{ε} are similar.

6.3 Refined anisotropic estimate

We prove that (4) holds true, in the Wiener setting, with

$$\Lambda_{\rho}\,(t, x) := \int_{X}\left|\mathrm{div}_{y}\left(\hat{M}_{t,x}\,(y)\,\rho\,(y)\right)\right|d\gamma\,(y)\,,$$

where $M_{t,x}\,|Db_t|\,(dx)\,dt = Db_t\,(x)\,dt$ gives the polar decomposition of $Db_t dt$ with respect to its total variation measure $|Db_t|\,dt$, a finite measure on $(0, T) \times X$. Here, $\hat{M}$ denotes the field associated to M, as defined in Sect. 5.

The proof goes similarly as in the finite dimensional case. First, we let $\varepsilon > 0$ and b be cylindrical smooth, and obtain an estimate for r^{ε} in terms of Db. Then, keeping

ε fixed, we extend its validity to BV vector fields, first cylindrical and then general. Finally, we let $\varepsilon \to 0$ and conclude with (4).

For simplicity, we assume both $\|u\|_\infty \leq 1$ and $\|\beta'\|_\infty \leq 1$. As above, we frequently omit to write as subscripts both ρ and t.

Let $\varepsilon > 0$ be fixed and b be cylindrical smooth. We perform some computations that give an estimate involving three terms, two of them being error terms, i.e., negligible as $\varepsilon \to 0$, and the third providing the anisotropic estimate. Since Sobolev and BV spaces are well-behaved with respect to push-forwards by linear maps, we may safely work in some fixed finite-dimensional Gaussian space $(\mathbb{R}^N, \gamma_N)$.

If we write explicitly the expressions obtained in the previous step, we obtain the estimate

$$\int \left|\varphi\beta'(u_\varepsilon) r^\varepsilon\right| \leq \int |\varphi|(x_\varepsilon) \left|\operatorname{div}_y \left(\frac{b(x) - e^\varepsilon b(x_\varepsilon)}{C_\varepsilon} \rho(y)\right)\right| dxdy. \tag{20}$$

We add subtract $b(x_\varepsilon)$ in the difference and we split

$$\int |\varphi|(x_\varepsilon) \left\{\frac{e^\varepsilon - 1}{C_\varepsilon} \left|\operatorname{div}_y (b(x_\varepsilon)\rho(y))\right| + \left|\operatorname{div}_y \left(\frac{b(x_\varepsilon) - b(x)}{C_\varepsilon} \rho(y)\right)\right|\right\} dxdy.$$

The first term in the sum above gives the an error term which is smaller than

$$\sqrt{\varepsilon}\, \|\varphi\|_\infty \left[\|b\|_1 \|\nabla\rho\|_\infty + \|\operatorname{div} b\|_1 \|\rho\|_\infty\right], \tag{21}$$

using (19) and noticing that $C_\varepsilon \leq C\sqrt{\varepsilon}$, for $\varepsilon \in (0, 1]$ and some constant C.

We focus then on the expression

$$\int |\varphi|(x_\varepsilon) \left|\operatorname{div}_y \left(\frac{b(x_\varepsilon) - b(x)}{C_\varepsilon} \rho(y)\right)\right| dxdy. \tag{22}$$

Since b is cylindrical smooth, write

$$b(x_\varepsilon) - b(x) = \int_0^\varepsilon \frac{d}{ds} b(x_s)\, ds = \int_0^\varepsilon Db(x_s)\, y_s \frac{ds}{C_s},$$

because of the identity $\frac{d}{ds} x_s = y_s / C_s$. In all what follows, for brevity, we write

$$⨍_0^\varepsilon f(s) := \frac{1}{C_\varepsilon} \int_0^\varepsilon f(s) \frac{ds}{C_s},$$

where the notation is justified by the fact that, as $\varepsilon \to 0$,

$$\frac{1}{C_\varepsilon} \int_0^\varepsilon \frac{ds}{C_s} \to 1. \tag{23}$$

Exchanging divergence and integration, we obtain

$$\mathrm{div}_y \left(\frac{b\left(x_\varepsilon\right) - b\left(x\right)}{C_\varepsilon} \rho\left(y\right) \right) = \fint_0^\varepsilon \mathrm{div}_y \left(Db\left(x_s\right) y_s \rho\left(y\right) \right)$$

$$= \fint_0^\varepsilon \left[\mathrm{div}_y \left(Db\left(x_s\right) y_s \right) \rho\left(y\right) + \left\langle Db\left(x_s\right) y_s, \nabla \rho\left(y\right) \right\rangle \right]. \tag{24}$$

Let us consider the first term in the sum above: write $c\left(x, y\right) := Db\left(x\right) y = \partial_y b\left(x\right)$ and for $s \in \left(0, \varepsilon\right)$, apply identity (19), to obtain

$$\mathrm{div}_y \left(Db\left(x_s\right) y_s \right) = \sqrt{1 - e^{-2s}} \left[\mathrm{div}_x c \right] \left(x_s, y_s\right) + e^{-s} \left[\mathrm{div}_y c \right] \left(x_s, y_s\right). \tag{25}$$

Since the term $\mathrm{div}_x c$ involves further spatial derivatives of b, the following identity,obtained by inspection in coordinates, is crucial:

$$\mathrm{div}_x c\left(x_s, y_s\right) = C_s \frac{d}{ds} \mathrm{div} b\left(x_s\right) + \hat{b}\left(x_s, y_s\right),$$

where we used the notation $\hat{b}$ introduced in the previous section. This allows to integrate by parts and conclude that

$$\fint_0^\varepsilon \sqrt{1 - e^{-2s}} \mathrm{div}_x c\left(x_s, y_s\right) \rho\left(y\right)$$

$$= \left[e^{-\varepsilon} \mathrm{div} b\left(x_\varepsilon\right) - \fint_0^\varepsilon \mathrm{div} b\left(x_s\right) e^{-s} \right] \rho\left(y\right) + \fint_0^\varepsilon \sqrt{1 - e^{-2s}} \hat{b}\left(x_s, y_s\right) \rho\left(y\right),$$

since $\frac{d}{ds} \sqrt{1 - e^{-2s}} = e^{-s} / C_s$. Thanks to these computations we separate from (22) another error term, smaller than

$$\left\| \varphi \right\|_\infty \left\| \rho \right\|_\infty \left[\int \left| e^{-\varepsilon} \mathrm{div} b\left(x_\varepsilon\right) - \fint_0^\varepsilon \mathrm{div} b\left(x_s\right) e^{-s} \right| dx dy + \frac{\varepsilon}{2C_\varepsilon} \left\| b \right\|_1 \right].$$

The integrand above is a linear expression in $\mathrm{div} b$, which reminds of some averaged Ornstein-Uhlenbeck. By rotational invariance of Gaussian measures and by (23) above, its L^1 norm is bounded by some absolute constant, uniformly in $\varepsilon \in (0, 1]$. By density of smooth cylindrical functions in L^1, it defines therefore some a family of continuous operators, that is $R_\varepsilon \left(\mathrm{div} b\right)\left(x, y\right)$, and we estimate

$$\left\| \varphi \right\|_\infty \left\| \rho \right\|_\infty \left[\left\| R_\varepsilon \left(\mathrm{div} b\right) \right\|_1 + \frac{\varepsilon}{C_\varepsilon} \left\| b \right\|_1 \right]. \tag{26}$$

The following expression contains precisely what remains to be estimated from (22), i.e. the second term in the second line of (24) and the second term in the right hand side of (25),

$$\int |\varphi| (x_\varepsilon) \fint_0^\varepsilon \left| e^{-s} \left[\mathrm{div}_y c \right] (x_s, y_s) \rho(y) + \langle c(x_s, y_s), \nabla \rho(y) \rangle \right| dx dy.$$

Once we exchange integration and perform a change of variables $(x, y) \mapsto (x^s, y^s)$, which maps x_ε to $x_{\varepsilon-s}$, we rewrite this expression in a way that easily allows an extension to the BV case, namely,

$$\int f\left(x, \frac{Db}{|Db|}(x)\right) |Db| (dx), \tag{27}$$

where

$$f(x, M) := \fint_0^\varepsilon \int |\varphi| (x_{\varepsilon-s}) \left| e^{-s} \mathrm{div}_y \left(\hat{M}(y) \right) \rho\left(y^s\right) + \langle \hat{M}(y), \nabla \rho\left(y^s\right) \rangle \right| dy,$$

recalling that $\hat{M}(y) = My$ in the finite dimensional setting.

Keep $\varepsilon > 0$ fixed and extend the estimate to a general b. The expression in (20) is smaller than the sum of three terms, namely (21), (26) and (27). We extend the validity of this fact to cylindrical BV fields, and then to the general case.

Under the assumption that b is cylindrical, everything reduces to a computation in $\mathbb{R}^N$, so that it is possible to find smooth cylindrical fields (b_n) such that, as $n \to \infty$,

$$\|b_n - b\|_1 \to 0, \quad \|\mathrm{div} b_n - \mathrm{div} b\|_1 \to 0, \quad |Db_n|(X) \to |Db|(X)$$

and (Db_n) weakly-* converge to Db (an approximating sequence extracted from the usual mollification via Ornstein-Uhlenbeck semigroup provides such a sequence). The left hand side in (20), together with the first and second error terms (21), (26) pass to the limit with respect to this convergence. The only trouble might be caused by (27), but the usual Reshetnyak continuity theorem applies, [7, Theorem 2.39]).

We now extend the estimate to cover general BV fields. We consider $b^N := \mathbb{E}_N[\pi_N b]$ and let $N \to \infty$. Again, (20), together with the first and second error terms (21), (26), pass to the limit because of Proposition 1. To handle the term (27), we prove first that, for every N large enough so that both φ and ρ are N-cylindrical, it holds

$$\int f\left(x, \frac{Db^N}{\left|Db^N\right|}(x)\right) d\left|Db^N\right|(x) \le \int f\left(x, \frac{Db}{|Db|}(x)\right) d|Db|(x).$$

This follows from Proposition 2, since by direct inspection, the left hand side above coincides with

$$\int f_N\left(\pi_N(x), \frac{Db^N}{\left|Db^N\right|}(x)\right) \left|Db^N\right| (dx),$$

where

$$f_N(x, M) := \fint_0^{\varepsilon} \int |\varphi|(x_{\varepsilon-s}) \left| e^{-s} \mathrm{div}_y (My) \rho \left(y^s\right) + \langle My, \nabla\rho\left(y^s\right)\rangle \right| d\gamma_N(y),$$

which is positively homogeneous and convex in the second variable, thus

$$\int f\left(x, \frac{Db^N}{\left|Db^N\right|}(x)\right)\left|Db^N\right|(dx) \leq \int f_N\left(x, \pi_N \frac{Db}{|Db|}(x)\, \pi_N\right)|Db|(dx).$$

Since γ is centred, we recognize that $\pi_N M \pi_N$ can be obtained as a conditional expectation, namely it holds $\pi_N M \pi_N = \mathbb{E}_N[\pi_N \hat{M}]$ and so, again by Proposition 1, applied this time to $\hat{M}$, we obtain the identity

$$\mathrm{div}_y (\pi_N M \pi_N y) = \mathbb{E}_N\left[\mathrm{div}_y \hat{M}(y)\right].$$

Combining these identities in the expression for f_N and recalling that φ and ρ are N-cylindrical we conclude, since the conditional expectation $\mathbb{E}_N$ is a contraction in $L^1(\gamma(dy))$.

We let $\varepsilon \to 0$. The first error term (21) is infinitesimal, but also the term (26), because $\|R_\varepsilon(\mathrm{div} b)\|_1 \to 0$ when b is smooth and cylindrical, by dominated convergence and (23). By uniform boundedness of R_ε in L^1 and again by the approximation provided by Proposition 1, this holds also for any field b with $\mathrm{div} b \in L^1(X, \gamma)$.

The term (27) converges to

$$\int |\varphi|(x)\left[\int \left|\mathrm{div}_y\left(\hat{M}_x(y)\right)\rho(y) + \langle \hat{M}_x(y), \nabla\rho(y)\rangle\right| d\gamma(y)\right]|Db|(dx),$$

since the integrand converges everywhere, being φ and ρ cylindrical smooth, uniformly bounded by some constant because, for any $p \in]1, \infty[$, it holds

$$f(x, M) \leq c_p \|\varphi\|_\infty \left(\|\rho\|_p + \|\nabla\rho\|_p\right)\|M\|.$$

and $\|M\| \leq 1$, $|Db|$-a.e., as provided by the polar decomposition.

Before we address the final step of the proof, let us spend some words on why (5) is a measure also in the Wiener setting, since the usual Euclidean argument exploits local compactness of the space and Riesz theorem. In general, one can consider distributions on $(0, T) \times X$ as linear functionals L on test functions such that, for some $k \geq 0$ and $p \in (1, \infty)$ it holds

$$|L(\varphi)| \leq C(L) \sum_{i=0}^{k} \left\|\nabla^i \varphi\right\|_p.$$

For example, the continuity equation is precisely required to be satisfied in the sense of distributions on $(0, T) \times X$, or σ in (5) defines a distribution. To show that σ is a measure, we use the following fact: if, for some positive measure μ on $(0, T) \times X$, a distribution L satisfies

$$|L(\varphi)| \leq \int |\varphi| \, d\mu,$$

for every test function φ, then it must coincide with a measure (actually, it is the restriction to test functions of the integration with respect such a measure), which is uniquely determined and absolutely continuous with respect to μ. Indeed, it is sufficient to remark that in such a case $L(\varphi)$ defines a continuous functional on $L^1((0, T) \times X, \mu)$, on a dense subspace (see [8, Section 2.1]). This argument, together with the validity of (4), entails that σ in (5) is a well-defined measure.

6.4 Optimization

We prove that, for any Hilbert-Schmidt operator $M \in H \otimes H$, it holds

$$\inf_{\rho} \int \left|\mathrm{div}(\hat{M}\rho)(y)\right| d\gamma(y) = 0,$$

where $\inf_\rho$ runs along all smooth cylindrical functions ρ with $\rho \geq 0$ and $\int \rho = 1$.

The proof goes as in the Euclidean setting, remarking that, for any $p > 1$,

$$\rho \mapsto \int \left|\mathrm{div}\left(\hat{M}\rho\right)(y)\right| d\gamma(y)$$

is continuous with respect to convergence in $W^{1,p}(X, \gamma)$ and so, by density, $\inf_\rho$ may run along all non-negative $\rho \in \bigcup_{p>1} W^{1,p}$, with $\int \rho = 1$.

Therefore, for fixed $T > 0$, we repeat the same construction as in the finite dimensional case, with $\bar{u} = 1$, using Proposition 3 to ensure that $\rho = \frac{1}{T}\int_0^T u_t dt \in W^{1,p}(X, \gamma)$ for some $p(T) > 1$, which gives that ρ is admissible.

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