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Elliptic determinantal process of type A

Makoto Katori

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Abstract We introduce an elliptic extension of Dyson's Brownian motion model, which is a temporally inhomogeneous diffusion process of noncolliding particles defined on a circle. Using elliptic determinant evaluations related to the reduced affine root system of type A, we give determinantal martingale representation (DMR) for the process, when it is started at the configuration with equidistant spacing on the circle. DMR proves that the process is determinantal and the spatio-temporal correlation kernel is determined. By taking temporally homogeneous limits of the present elliptic determinantal process, trigonometric and hyperbolic versions of noncolliding diffusion processes are studied.

Keywords Noncolliding diffusion process · Dyson's Brownian motion model · Elliptic determinant evaluations · Determinantal process · Determinantal martingale · Alcove of affine Weyl group

Mathematics Subject Classification 60J65 · 60G44 · 82C22 · 60B20 · 33E05

1 Introduction

Eigenvalue distributions of random-matrix ensembles provide important examples of *determinantal point processes*, in which any correlation function is given by a determinant specified by a single continuous function called the *correlation kernel* [3,15,34,39,40]. Dyson's Brownian motion model with parameter $\beta = 2$ [10,41], which we simply call *the Dyson model* in this paper, and other noncolliding diffusion

M. Katori (✉)
Department of Physics, Faculty of Science and Engineering, Chuo University,
Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan
e-mail: katori@phys.chuo-u.ac.jp

processes [8,18,21,25,29] are dynamical extensions of random-matrix ensembles. There any *spatio-temporal* correlation function is expressed by determinant [12,35] and such processes are said to be *determinantal* [7,26]. The noncolliding diffusion processes have attracted much attention in probability theory also by the fact that they are realized as h-transforms in the sense of Doob of absorbing particle systems in the Weyl chambers [18,25,29]. The relationship between the above mentioned integrability as spatio-temporal models and h-transform constructions as stochastic processes has been clarified by introducing a notion of *determinantal martingales* in [23,24,28]. The purpose of the present paper is to report *elliptic extensions* of these determinantal processes. Since the Dyson model can be regarded as a multivariate extension of the three-dimensional Bessel process, BES(3) [5,27], first we discuss an elliptic extension of BES(3).

Let $i = \sqrt{-1}$, $v, \tau \in \mathbb{C}$ and put

$$z = z(v) = e^{\pi i v}, \quad q = q(\tau) = e^{\pi i \tau}. \tag{1.1}$$

The Jacobi theta function ϑ_1 is defined as

$$\begin{aligned} \vartheta_1(v; \tau) &= i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n-(1/2))^2} z^{2n-1} \\ &= 2 \sum_{n=1}^{\infty} (-1)^{n-1} e^{\pi i \tau (n-(1/2))^2} \sin\{(2n-1)\pi v\}. \end{aligned} \tag{1.2}$$

(Note that the present function $\vartheta_1(v; \tau)$ is represented as $\vartheta_1(\pi v, q)$ in [44].) For $\Im \tau > 0$, $\vartheta_1(v; \tau)$ is holomorphic for $|v| < \infty$ and satisfies the partial differential equation

$$\frac{\partial \vartheta_1(v; \tau)}{\partial \tau} = \frac{1}{4\pi i} \frac{\partial^2 \vartheta_1(v; \tau)}{\partial v^2}. \tag{1.3}$$

With parameters $N \in \mathbb{N} \equiv \{1, 2, \ldots\}$, $\alpha > 0$, and $0 < t_* < \infty$, we introduce the following function of $(t, x) \in [0, t_*) \times \mathbb{R}$,

$$\begin{aligned} A_N^{\alpha}(t_* - t, x) &= \left[\frac{1}{\alpha} \frac{d}{dv} \log \vartheta_1(v; \tau) \right]_{v = x/\alpha, \tau = 2\pi i N(t_* - t)/\alpha^2} \\ &= \frac{1}{\alpha} \frac{\vartheta_1'(x/\alpha; 2\pi i N(t_* - t)/\alpha^2)}{\vartheta_1(x/\alpha; 2\pi i N(t_* - t)/\alpha^2)}, \end{aligned} \tag{1.4}$$

where $\vartheta_1'(v; \tau) = d\vartheta_1(v; \tau)/dv$. As a function of $x \in \mathbb{R}$, it is odd,

$$A_N^{\alpha}(t_* - t, -x) = -A_N^{\alpha}(t_* - t, x), \tag{1.5}$$

and periodic with period α

$$A_N^{\alpha}(t_* - t, x + m\alpha) = A_N^{\alpha}(t_* - t, x), \quad m \in \mathbb{Z}. \tag{1.6}$$

It has only simple poles at $x = m\alpha, m \in \mathbb{Z}$, and simple zeroes at $x = (m + 1/2)\alpha, m \in \mathbb{Z}$.

Let $r > 0$. Suppose that $\check{X}(t), t \in [0, t_*)$ satisfies the stochastic differential equation (SDE)

$$d\check{X}(t) = dB(t) + A_1^{2\pi r}(t_* - t, \check{X}(t))dt \tag{1.7}$$

started at $\check{X}(0) = x \in (0, 2\pi r)$, where B denotes the one-dimensional standard Brownian motion (BM). (From now on, BM means a one-dimensional standard Brownian motion unless specially mentioned). By periodicity (1.6) with period $\alpha = 2\pi r$, $\check{X}$ can be considered to describe a diffusion process of a particle moving around a circle with radius $r > 0$; $\mathrm{S}^1(r) = \{x \in \mathbb{R} : x + 2\pi r = x\}$. Note that this system is temporally inhomogeneous defined only in a time interval $[0, t_*)$. Independently of $t \in [0, t_*)$ and N, however, we have

$$\begin{aligned} A_N^{2\pi r}(t_* - t, x) &\sim \frac{1}{x} \quad \text{as } x \downarrow 0, \\ A_N^{2\pi r}(t_* - t, x) &\sim -\frac{1}{2\pi r - x} \quad \text{as } x \uparrow 2\pi r. \end{aligned} \tag{1.8}$$

It implies that the behavior of $\check{X} \in (0, 2\pi r)$ in the vicinity of 0 (and $2\pi r$) is similar to that of BES(3) near 0. We define a process $X \in [0, 2\pi r)$ by

$$X(t) = \check{X}(t) \mod 2\pi r, \quad t \in [0, t_*). \tag{1.9}$$

It gives a Markov process showing a position on the circumference $[0, 2\pi r)$ of $\mathrm{S}^1(r)$. We write the probability law of $X(t), t \in [0, t_*)$ started at $x = X(0) \in (0, 2\pi r)$ as $\mathbb{P}_x$.

The backward Kolmogorov equation for the SDE (1.7) is given as

$$-\frac{\partial u(s,x)}{\partial s} = \frac{1}{2}\frac{\partial^2 u(s,x)}{\partial x^2} + A_1^{2\pi r}(t_* - s, x)\frac{\partial u(s,x)}{\partial x}, \quad 0 \le s < t_*, \quad x \in (0, 2\pi r). \tag{1.10}$$

Let $q(t - s, y|x)$ be a solution of diffusion equation $-\partial v(s,x)/\partial s = (1/2)\partial^2 v(s,x)/\partial x^2, 0 \le s \le t < \infty, x \in (0, 2\pi r)$ satisfying $\lim_{s \uparrow t} v(s,x) = \delta_y(\{x\}), y \in (0, 2\pi r)$. Then

$$u(s,x) = p(t, y|s, x) \equiv q(t-s, y|x)\frac{\vartheta_1(y/2\pi r; i(t_* - t)/2\pi r^2)}{\vartheta_1(x/2\pi r; i(t_* - s)/2\pi r^2)}, \quad 0 \le s \le t < t_*, \tag{1.11}$$

solves (1.10) under the condition $\lim_{s \uparrow t} u(s,x) = \delta_y(\{x\}), y \in (0, 2\pi r)$, since $\vartheta_1(v;\tau)$ satisfies (1.3). For $0 < y < 2\pi r$, $\vartheta_1(y/2\pi r; i(t_* - t)/2\pi r^2) > 0$ and it has simple zeroes at $y = 0$ and $y = 2\pi r$, $t \in [0, t_*)$. Then, if $q(t - s, y|x)$ is chosen as the transition probability density (tpd) of the absorbing BM in the interval $[0, 2\pi r]$ with absorbing walls at $x = 0$ and $x = 2\pi r$, (1.11) is strictly positive and finite for any $x, y \in (0, 2\pi r), 0 \le s \le t < t_*$, and thus $p(t, y|s, x)$ gives the tpd for the process $X(t), t \in [0, t_*)$. Let $W(t), t \ge 0$ be BM started at $x = W(0) \in (0, 2\pi r)$, where its probability law is denoted by P_x. Consider a filtration $\{\mathcal{F}_W(t) : t \ge 0\}$ generated by $W(t), t \geq 0$, which satisfies the usual conditions, and introduce a stopping time $T_W = \inf\{t > 0 : W(t) \in \{0, 2\pi r\}\}$. Then the above fact implies that, for $t \in [0, t_*)$,

$$\mathbb{P}_x(X(t) \in dy) = \mathrm{P}_x(T_W > t, W(t) \in dy) \frac{\vartheta_1(y/2\pi r; i(t_* - t)/2\pi r^2)}{\vartheta_1(x/2\pi r; it_*/2\pi r^2)}, \quad x, y \in (0, 2\pi r). \tag{1.12}$$

Note that

$$\vartheta_1(v; \tau) \sim 2e^{\pi i \tau/4} \sin(\pi v) \quad \text{as} \quad \Im\tau \to +\infty \, (i.e., \; q = q(\tau) = e^{\pi i \tau} \to 0).$$

Thus, in the limit $t_* \to \infty$, (1.7) becomes a temporally homogeneous SDE,

$$d\check{X}(t) = dB(t) + \frac{1}{2r} \cot\left(\frac{\check{X}(t)}{2r}\right) dt, \quad t \geq 0, \tag{1.13}$$

and (1.12) becomes

$$\mathbb{P}_x(X(t) \in dy) = \mathrm{P}_x(T_W > t, W(t) \in dy) \frac{\sin(y/2r)}{\sin(x/2r)}, \quad t \in [0, \infty), \quad x, y \in (0, 2\pi r). \tag{1.14}$$

If we take the further limit $r \to \infty$ in (1.13), we have $X(t) = \check{X}(t), t \geq 0$ and

$$dX(t) = dB(t) + \frac{dt}{X(t)}, \quad t \geq 0, \tag{1.15}$$

which is the SDE for BES(3) on $\mathbb{R}_+ = \{x > 0 : x \in \mathbb{R}\}$, and (1.14) becomes

$$\mathbb{P}_x(X(t) \in dy) = \mathrm{P}_x(T'_W > t, W(t) \in dy) \frac{y}{x}, \quad t \in [0, \infty), \quad x, y \in \mathbb{R}_+, \tag{1.16}$$

where $T'_W = \inf\{t > 0 : W(t) = 0\}$. The relation (1.16) states that BES(3) is the *Doob h-transform of the absorbing BM* in $[0, \infty)$ with an absorbing wall at the origin, where the harmonic function is given by $h(x) = x, x \geq 0$. We regard (1.13) as a trigonometric extension, and (1.7) as an elliptic extension of (1.15), respectively. The equality (1.16) is generalized to (1.14) and (1.12), respectively. We can also discuss a scaling limit realizing $q \to 1$, in which hyperbolic version of (1.7) is obtained (see Sect. 5).

Let $N \in \{2, 3, \dots\}$ and consider the following bounded region

$$\mathcal{A}_{2\pi r}^{A_{N-1}} = \{\boldsymbol{x} = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 < x_2 < \cdots < x_N < x_1 + 2\pi r\},$$

which is called a *scaled alcove of the affine Weyl group of type* A_{N-1} (with scale $2\pi r$) [19,31]. With an additional condition $x_1 \geq 0$, we also consider the space

$$\mathcal{A}_{[0,2\pi r)^N} = \mathcal{A}_{2\pi r}^{A_{N-1}} \cap \{\boldsymbol{x} \in \mathbb{R}^N : x_1 \geq 0\}$$
$$= \{\boldsymbol{x} \in \mathbb{R}^N : 0 \leq x_1 < x_2 < \cdots < x_N < 2\pi r\}.$$

Note that it is different from the scaled alcove of type C_N defined by $\mathcal{A}_{2\pi r}^{C_N} = \{\boldsymbol{x} \in \mathbb{R}^N : 0 < x_1 < x_2 < \cdots < x_N < 2\pi r\}$ [19,31] which excludes $x_1 = 0$ from $\mathcal{A}_{[0,2\pi r)^N}$.

Now we introduce an N-particle extension of the above process, $\check{\boldsymbol{X}}^A(t) = (\check{X}_1^A(t), \dots, \check{X}_N^A(t))$, $t \in [0, t_*)$. Assume that the initial configuration is chosen in the alcove

$$\check{\boldsymbol{X}}^A(0) = \boldsymbol{u} \in \mathcal{A}_{2\pi r}^{A_{N-1}},$$

and an index $\delta \in \pi r\mathbb{Z}$ is determined so that

$$\overline{u}_\delta \equiv \delta + \sum_{j=1}^N u_j \in (0, 2\pi r).$$

Let

$$\overline{X}_\delta^A(t) = \delta + \sum_{j=1}^N \check{X}_j^A(t), \quad t \in [0, t_*).$$

Then $\check{\boldsymbol{X}}^A(t)$, $t \in [0, t_*)$ is defined as a solution of the following set of SDEs on $\mathbb{R}$,

$$d\check{X}_j^A(t) = dB_j(t) + \sum_{\substack{1 \leq k \leq N, \\ k \neq j}} A_N^{2\pi r}(t_* - t, \check{X}_j^A(t) - \check{X}_k^A(t))dt + A_N^{2\pi r}(t_* - t, \overline{X}_\delta^A(t))dt, \tag{1.17}$$

$1 \leq j \leq N, t \in [0, t_*)$, where $B_j, 1 \leq j \leq N$ are independent BMs on $\mathbb{R}$. By (1.5), $\sum_{1 \leq j,k \leq N, j \neq k} A_N^{2\pi r}(t_* - s, x_j - x_k) = 0$, and the summation of (1.17) over $j = 1, 2, \dots, N$ gives

$$d\overline{X}_\delta^A(t) = \sqrt{N} dB(t) + N A_N^{2\pi r}(t_* - t, \overline{X}_\delta^A(t))dt, \quad t \in [0, t_*), \tag{1.18}$$

where B is BM on $\mathbb{R}$. We then define the process $\boldsymbol{X}^A(t) = (X_1^A(t), \dots, X_N^A(t)) \in [0, 2\pi r)^N$, $t \in [0, t_*)$ by

$$X_j^A(t) = \check{X}_j^A(t) \mod 2\pi r, \quad 1 \leq j \leq N, \quad t \in [0, t_*). \tag{1.19}$$

It represents a Markov process showing the positions of N particles on the circumference $[0, 2\pi r)$ of $\mathrm{S}^1(r)$.

Let $\mathfrak{M}([0, 2\pi r))$ be the space of nonnegative integer-valued Radon measures on the interval $[0, 2\pi r)$, which is a Polish space with the vague topology. Any element

ξ of $\mathfrak{M}([0, 2\pi r))$ can be represented as $\xi(\cdot) = \sum_{j \geq 1} \delta_{x_j}(\cdot)$, in which the sequence of points in $[0, 2\pi r)$, $\boldsymbol{x} = (x_j)_{j \geq 1}$, satisfies $\xi(K) = \sharp\{x_j : x_j \in K\} < \infty$ for any subset $K \subset [0, 2\pi r)$. Now we consider the process $\boldsymbol{X}^A(t)$ as an $\mathfrak{M}([0, 2\pi r))$-valued process and write it as

$$\Xi^A(t, \cdot) = \sum_{j=1}^{N} \delta_{X_j^A(t)}(\cdot), \quad t \in [0, t_*). \tag{1.20}$$

The probability law of $\Xi^A(t, \cdot), t \in [0, t_*)$ starting from a fixed configuration $\xi \in \mathfrak{M}([0, 2\pi r))$ is denoted by $\mathbb{P}_\xi^A$ and the process specified by the initial configuration is expressed by $\left(\Xi^A(t), t \in [0, t_*), \mathbb{P}_\xi^A\right)$. The expectations with respect to $\mathbb{P}_\xi^A$ is denoted by $\mathbb{E}_\xi^A$. We introduce a filtration $\{\mathcal{F}_{\Xi^A}(t) : t \in [0, t_*)\}$ generated by $\Xi^A(t), t \in [0, t_*)$, which satisfies the usual conditions. Let $\mathrm{C}([0, 2\pi r))$ be the set of all continuous real-valued functions on $[0, 2\pi r)$. We set

$$\mathfrak{M}_0([0, 2\pi r)) = \{\xi \in \mathfrak{M}([0, 2\pi r)) : \xi(\{x\}) \leq 1 \text{ for any } x \in [0, 2\pi r)\},$$

which denotes a collection of configurations without any multiple points.

In the present paper, we study the case that the initial state $\eta = \sum_{j=1}^{N} \delta_{v_j}$ is corresponding to the configuration with equidistant spacing on $\mathrm{S}^1(r)$;

$$v_j = \frac{2\pi r}{N}(j-1), \quad 1 \leq j \leq N, \tag{1.21}$$

and we will prove that the process $(\Xi^A(t), t \in [0, t_*), \mathbb{P}_\eta^A)$ is determinantal. In this case, $\sum_{j=1}^{N} v_j = \pi r(N-1)$ and the index $\delta \in \pi r \mathbb{Z}$ is determined as

$$\delta = -\pi r(N-2), \tag{1.22}$$

so that $\overline{v}_\delta = \pi r \in (0, 2\pi r)$. We will present that this determinantal process can be considered as an elliptic extension of the Dyson model. The key lemmas to construct the elliptic determinantal process are obtained from the *elliptic determinant evaluations* related to infinite families of irreducible reduced affine root systems studied in [13–15,30,37,42,43]. According to Macdonald's classification of reduced affine root systems [33], the present process is related to the system of type A_{N-1}. Further study concerning other types is in progress. Connection between the elliptic determinantal processes and probabilistic discrete models with elliptic weights [4,6,38] will be an interesting future problem. We note that the function $A_N^\alpha(t_* - t, z)$ can be regarded as Villat's kernel for an annulus $\mathbb{A}_q = \{z \in \mathbb{C} : q < |z| < 1\}$ with $0 < q = e^{-2\pi^2 N(t_*-t)/\alpha^2} < 1$. It is the reason why it also appears in the study of stochastic Komatu-Loewner evolution in doubly connected domains [2,45].

The paper is organized as follows. In Sect. 2 preliminaries of elliptic functions and their related functions are given. Useful determinantal identities are obtained from the elliptic determinant evaluations related to the affine root system of types A_{N-1} [13–15,30,37,42,43]. There the generalized h-transform and determinantal martingale for

$\left(\Xi^A(t), t \in [0, t_*), \mathbb{P}^A_\eta\right)$ are introduced. The main results are given in Sect. 3. First the determinantal martingale representation (DMR) is given for $\left(\Xi^A(t), t \in [0, t_*), \mathbb{P}^A_\eta\right)$ (Theorem 3.1). As a result of Theorem 1.3 of [23], DMR proves that the process is determinantal and the correlation kernel is determined (Corollary 3.3). Explicit expression for the correlation kernel of $\left(\Xi^A(t), t \in [0, t_*), \mathbb{P}^A_\eta\right)$ is shown and the infinite-particle limit is discussed. In Sect. 4 the temporally homogeneous limit $t_* \to \infty$ is studied both in the level of SDEs and in the level of determinantal process. We study the system of noncolliding Brownian motions on a circle, $\widehat{\Xi}^A(t), t \in [0, \infty)$, obtained from $\Xi^A(t), t \in [0, t_*)$ by this reduction. It is different from the dynamical circular unitary ensemble (CUE) model studied in [20,23,36]. Finally in Sect. 5, the results are expressed by using Gosper's q-sine function [17] as well as the q-gamma function [1]. Then $q \to 1$ limit is discussed, in which temporally homogeneous processes expressed by hyperbolic functions are obtained.

2 Preliminaries

2.1 Elliptic functions and their related functions

The Jacobi theta function ϑ_1 defined by (1.2) has the following infinite-product expressions,

$$\begin{aligned}\vartheta_1(v;\tau) &= -iq^{1/4}q_0 z \prod_{j=1}^{\infty}\left(1-q^{2j}z^2\right)\left(1-q^{2j-2}/z^2\right)\\ &= 2q^{1/4}q_0\sin(\pi v)\prod_{j=1}^{\infty}\left(1-2q^{2j}\cos(2\pi v)+q^{4j}\right)\end{aligned} \tag{2.1}$$

with

$$q_0 = q_0(\tau) \equiv \prod_{n=1}^{\infty}(1-q^{2n}), \quad q = e^{\pi i \tau}.$$

It is easy to see from these expressions that, when $\Re\tau = 0$, $\Im\tau > 0$ (that is, $0 < q < 1$), $\vartheta_1(v;\tau) > 0$ for $v \in (0,1)$ and it has simple zeroes at $v = 0$ and $v = 1$. It is odd with respect to v

$$\vartheta_1(-v;\tau) = -\vartheta_1(v;\tau), \tag{2.2}$$

and has quasi-periodicity

$$\begin{aligned}&\vartheta_1(v+1;\tau) = -\vartheta_1(v;\tau),\\ &\vartheta_1(v+\tau;\tau) = -\frac{1}{z^2 q}\vartheta_1(v;\tau) = -e^{-\pi i(2v+\tau)}\vartheta_1(v;\tau).\end{aligned} \tag{2.3}$$

We define

$$\vartheta_0(v;\tau) \equiv -ie^{\pi i(v+\tau/4)}\vartheta_1\left(v+\frac{\tau}{2};\tau\right) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} z^{2n},$$
$$\vartheta_2(v;\tau) \equiv \vartheta_1\left(v+\frac{1}{2};\tau\right) = \sum_{n\in\mathbb{Z}} q^{(n-(1/2))^2} z^{2n-1}, \tag{2.4}$$
$$\vartheta_3(v;\tau) \equiv e^{\pi i(v+\tau/4)}\vartheta_1\left(v+\frac{1+\tau}{2};\tau\right) = \sum_{n\in\mathbb{Z}} q^{n^2} z^{2n},$$

as usual. (Note that the present functions $\vartheta_\mu(v;\tau)$, $\mu = 1, 2, 3$ are denoted by $\vartheta_\mu(\pi v, q)$, and $\vartheta_0(v;\tau)$ by $\vartheta_4(\pi v, q)$ in [44].) They solve the partial differential equation

$$\frac{\partial \vartheta_\mu(v;\tau)}{\partial \tau} = \frac{1}{4\pi i}\frac{\partial^2 \vartheta_\mu(v;\tau)}{\partial v^2}, \quad \mu = 0, 1, 2, 3. \tag{2.5}$$

We will use the following formulas; $n \in \mathbb{N}$, $q = e^{\pi i \tau}$,

$$\prod_{j=0}^{n-1} \vartheta_1(v+j/n;\tau) = \frac{q_0^n}{(q^n)_0}\vartheta_1(nv;n\tau), \tag{2.6}$$
$$\prod_{j=1}^{n-1} \vartheta_1(j/n;\tau) = \frac{nq_0^n}{(q^n)_0}\frac{\vartheta_1'(0;n\tau)}{\vartheta_1'(0;\tau)}, \tag{2.7}$$

where

$$\vartheta_1'(0;\tau) \equiv \left.\frac{\partial \vartheta_1(v;\tau)}{\partial v}\right|_{v=0} = 2\pi \sum_{j=1}^{\infty}(-1)^{j-1}(2j-1)q^{n-(1/2))^2}, \tag{2.8}$$

and

$$\frac{\vartheta_1(v+w;\tau)\vartheta_1'(0;\tau)}{\pi\vartheta_1(v;\tau)\vartheta_1(w;\tau)} = \cot(\pi v) + \cot(\pi w) + 4\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty} q^{2\ell m}\sin[2\pi(\ell v + mw)]. \tag{2.9}$$

The formula (2.6) is obtained from Eq. (2.3) of [37] (see (2.17) below) and (2.7) is obtained as its $v \to 0$ limit, since $\vartheta_1(v;\tau)$ has a simple root at $v = 0$. The formula (2.9) is found in 'Miscellaneous Examples' of Chapter XXI in [44].

Let ω_1 and ω_3 be fundamental periods and set

$$\tau = \frac{\omega_3}{\omega_1}, \quad \Im\tau > 0,$$
$$\Omega_{m,n} = 2m\omega_1 + 2n\omega_3, \quad m, n \in \mathbb{Z}.$$

The Weierstrass $\wp$ function and zeta function ζ are defined as the following meromorphic functions

$$
\begin{aligned}
\wp(z) &= \wp(z|2\omega_1, 2\omega_3) \\
&= \frac{1}{z^2} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left[\frac{1}{(z-\Omega_{m,n})^2} - \frac{1}{{\Omega_{m,n}}^2} \right], \\
\zeta(z) &= \zeta(z|2\omega_1, 2\omega_3) \\
&= \frac{1}{z} + \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \left[\frac{1}{z-\Omega_{m,n}} + \frac{1}{\Omega_{m,n}} + \frac{z}{{\Omega_{m,n}}^2} \right].
\end{aligned}
$$

Let $\omega_2 = -(\omega_1+\omega_3)$, and put $\zeta(\omega_\nu) = \eta_\nu$, $\nu = 1, 2, 3$. The relation $\eta_1+\eta_2+\eta_3 = 0$ holds. By definition, $\wp(z)$ is an elliptic function with fundamental periods ω_1, ω_2 and ω_3,

$$
\wp(z+2\omega_\nu) = \wp(z), \quad \nu = 1, 2, 3,
$$

and it is even, $\wp(-z) = \wp(z)$. The function ζ is odd, $\zeta(-z) = -\zeta(z)$, and is quasi-periodic in the sense

$$
\zeta(z+2\omega_\nu) = \zeta(z) + 2\eta_\nu, \quad \nu = 1, 2, 3.
$$

By definition,

$$
\wp(z) = -\zeta'(z). \tag{2.10}
$$

Moreover, the relation

$$
\{\zeta(z+u) - \zeta(z) - \zeta(u)\}^2 = \wp(z+u) + \wp(z) + \wp(u) \tag{2.11}
$$

holds (see Section 20.41 in [44]). From this, we obtain the following identity.

Lemma 2.1 *For $a, b, c \in \mathbb{C}$,*

$$
\begin{aligned}
&\zeta(a-b)\zeta(a-c) + \zeta(b-a)\zeta(b-c) + \zeta(c-a)\zeta(c-b) \\
&\quad = \frac{1}{2}\left\{ \zeta(a-b)^2 + \zeta(b-c)^2 + \zeta(a-c)^2 \right\} \\
&\qquad - \frac{1}{2}\left\{ \wp(a-b) + \wp(b-c) + \wp(a-c) \right\}.
\end{aligned}
$$

Proof Put $z = a-b$, $u = b-c$ in (2.11). Then by the fact that ζ is odd, the equality is derived. □

The following relation is established,

$$
\zeta(z|2\omega_1, 2\omega_3) - \frac{\eta_1 z}{\omega_1} = \frac{1}{2\omega_1} \frac{d}{dv} \log \vartheta_1(v;\tau) \Big|_{v=z/2\omega_1}, \quad \tau = \frac{\omega_3}{\omega_1}.
$$

It has the following expansion with respect to $q = q(\tau) = e^{\pi i \tau}$,

$$\zeta(z|2\omega_1, 2\omega_3) - \frac{\eta_1 z}{\omega_1} = \frac{\pi}{2\omega_1} \cot\left(\frac{\pi z}{2\omega_1}\right) + \frac{2\pi}{\omega_1} \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin\left(\frac{n\pi z}{\omega_1}\right). \tag{2.12}$$

The function $A_N^\alpha(t_* - t, x)$ defined by (1.4) is then expressed as

$$\begin{aligned} A_N^\alpha(t_* - t, x) &= \left[\zeta(x|2\omega_1, 2\omega_3) - \frac{\eta_1 x}{\omega_1}\right]_{\omega_1 = \alpha/2, \omega_3 = \pi i N(t_* - t)/\alpha} \\ &= \zeta\left(x \middle| \alpha, \frac{2\pi i N(t_* - t)}{\alpha}\right) - \frac{2\eta_1(t_* - t)x}{\alpha}, \end{aligned} \tag{2.13}$$

for $t \in [0, t_*)$, $x \in \mathbb{R}$, where

$$\begin{aligned} \eta_1(t_* - t) = \eta_1(t_* - t; N, \alpha) &= \frac{\pi^2}{\omega_1}\left(\frac{1}{12} - 2\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}\right)\Bigg|_{\omega_1 = \alpha/2, q = e^{-2\pi^2 N(t_* - t)/\alpha^2}} \\ &= \frac{2\pi^2}{\alpha}\left(\frac{1}{12} - 2\sum_{n=1}^{\infty} \frac{ne^{-4\pi^2 nN(t_* - t)/\alpha^2}}{1 - e^{-4\pi^2 nN(t_* - t)/\alpha^2}}\right). \end{aligned} \tag{2.14}$$

The formula (2.12) gives

$$\begin{aligned} A_N^\alpha(t_* - t, x) = \frac{\pi}{\alpha} \cot\left(\frac{\pi x}{\alpha}\right) + \frac{4\pi}{\alpha} \sum_{n=1}^{\infty} \frac{e^{-4\pi^2 nN(t_* - t)/\alpha^2}}{1 - e^{-4\pi^2 nN(t_* - t)/\alpha^2}} \sin\left(\frac{2\pi n x}{\alpha}\right), \\ t \in [0, t_*), \quad x \in \mathbb{R}. \end{aligned} \tag{2.15}$$

From this expression, we can readily observe (1.5) and (1.6) and other properties of A_N^α. In particular, we can see (1.8).

2.2 Elliptic determinant identities

Let p be a fixed complex number such that $0 < |p| < 1$. We use the standard notations

$$\begin{aligned} (a; p)_\infty &= \prod_{j=0}^{\infty} (1 - ap^j), \\ (a_1, \dots, a_n; p)_\infty &= (a_1; p)_\infty \cdots (a_n; p)_\infty. \end{aligned}$$

Following [37,43], here we use 'multiplicative notation' for theta functions,

$$E(s;p) = (s, p/s; p)_\infty,$$
$$E(s_1, \ldots, s_n; p) = \prod_{j=1}^{n} E(s_j; p).$$

The function $E(s;p)$ is holomorphic for $s \neq 0$. The zero set of $E(s;p)$ is given by $p^{\mathbb{Z}} \equiv \{p^j : j \in \mathbb{Z}\}$, and all zeroes are single. The inversion formula

$$E(1/s;p) = -\frac{1}{s} E(s;p), \tag{2.16}$$

the quasi-periodicity $E(ps;p) = -E(s;p)/s$, and the Laurent expansion

$$E(s;p) = \frac{1}{(p;p)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n p^{\binom{n}{2}} s^n$$

are known. If g_n denotes a primitive nth root of unity, $n \in \mathbb{N}$, the following equality holds [37],

$$E(s^n; p^n) = \prod_{j=0}^{n-1} E(s g_n^j; p). \tag{2.17}$$

With (1.1), the function $E(s;p)$ is related to the Jacobi theta function ϑ_1 by

$$\vartheta_1(v;\tau) = i q^{1/4} q_0 \frac{1}{z} E(z^2; q^2). \tag{2.18}$$

The equality (2.17) is then rewritten as (2.6).

For $N \in \mathbb{N}$, $\boldsymbol{s} = (s_1, \ldots, s_N) \in \mathbb{C}^N$, put

$$W_{A_{N-1}}(\boldsymbol{s};p) = \prod_{1 \leq j < k \leq N} s_k E(s_j/s_k; p) = (-1)^{N(N-1)/2} \prod_{1 \leq j < k \leq N} s_j E(s_k/s_j; p),$$

where the second equality is proved by the inversion formula (2.16). It is the *Macdonald denominator* for the reduced affine root systems of type A_{N-1} [37].

We start with the following two lemmas, which are readily obtained from the results given in [30,37,42,43].

Lemma 2.2 *For* $\boldsymbol{r} = (r_1, \ldots, r_N) \in \mathbb{C}^N$ *and* $\kappa \in \mathbb{C}$ *assume that*

$$\frac{r_j}{r_k} \notin p^{\mathbb{Z}}, \quad 1 \leq j \neq k \leq N, \quad \textit{and} \quad \kappa \prod_{j=1}^{N} r_j \notin p^{\mathbb{Z}}. \tag{2.19}$$

Then for $\mathbf{s} \in \mathbb{C}^N$

$$\det_{1 \le j,k \le N} \left[\frac{E\left(\kappa s_j \prod_{m=1, m \neq k}^N r_m;\, p\right)}{E\left(\kappa \prod_{m=1}^N r_m;\, p\right)} \prod_{1 \le \ell \le N, \ell \neq k} \frac{E(s_j/r_\ell;\, p)}{E(r_k/r_\ell;\, p)} \right]$$
$$= \frac{E\left(\kappa \prod_{j=1}^N s_j;\, p\right) W_{A_{N-1}}(\mathbf{s};\, p)}{E\left(\kappa \prod_{j=1}^N r_j;\, p\right) W_{A_{N-1}}(\mathbf{r};\, p)}. \tag{2.20}$$

Proof of Lemma 2.2 Let $s_1, \dots, s_N, a_1, \dots, a_{N-1}, b_1, \dots, b_N, c_2, \dots, c_N$, and κ be indeterminates, which satisfy

$$\prod_{k=1}^{j-1} a_k \cdot b_j \cdot \prod_{\ell=j+1}^N c_\ell = \kappa, \quad 1 \le j \le N. \tag{2.21}$$

The following equality holds [42] (see Corollary 4.5 and Remark 4.6 in [37], and see also Section 5.11 in [30]),

$$\det_{1 \le j,k \le N} \left[\prod_{\ell=1}^{k-1} E(a_\ell s_j;\, p) \cdot E(b_k s_j;\, p) \cdot \prod_{m=k+1}^N E(c_m s_j;\, p) \right]$$
$$= E\left(\kappa \prod_{j=1}^N s_j;\, p\right) \prod_{k=2}^N E(b_k/c_k;\, p) \prod_{1 \le \ell < m \le N} c_m s_m E(s_\ell/s_m, a_\ell/c_m;\, p). \tag{2.22}$$

We put $a_j = 1/r_j, 1 \le j \le N-1, b_j = \alpha/r_j, 1 \le j \le N, c_j = 1/r_j, 2 \le j \le N$ with $\alpha \in \mathbb{C}$. Then the condition (2.21) gives

$$\prod_{j=1}^N r_j = \frac{\alpha}{\kappa} \tag{2.23}$$

and (2.22) becomes

$$\det_{1 \le j,k \le N} \left[E(\alpha s_j/r_k;\, p) \prod_{1 \le \ell \le N, \ell \neq k} E(s_j/r_\ell;\, p) \right]$$
$$= E\left(\alpha \prod_{j=1}^N s_j/r_j;\, p\right) E(\alpha;\, p)^{N-1} \prod_{1 \le \ell < m \le N} \frac{s_m}{r_m} E\left(\frac{s_\ell}{s_m};\, p\right) E\left(\frac{r_m}{r_\ell};\, p\right). \tag{2.24}$$

Under the assumption (2.19) we divide the both sides of (2.24) by $E(\alpha; p)^N \prod_{1\le k,\ell\le N, k\neq\ell} E(r_k/r_\ell; p)$ and obtain the equality

$$\det_{1\le j,k\le N}\left[\frac{E(\alpha s_j/r_k; p)}{E(\alpha; p)} \prod_{1\le \ell\le N, \ell\neq k} \frac{E(s_j/r_\ell; p)}{E(r_k/r_\ell; p)}\right] = \frac{E\left(\alpha \prod_{j=1}^N s_j/r_j; p\right)}{E(\alpha; p)} \prod_{1\le\ell<m\le N} \frac{s_m}{r_m}\frac{E(s_\ell/s_m; p)}{E(r_\ell/r_m; p)}.$$

If we use (2.23), we obtain (2.20). □

The following equality is given in the first line of Proposition 6.1 in [37].

Lemma 2.3 *For* $\boldsymbol{s} \in \mathbb{C}^N$, $\kappa \in \mathbb{C}$,

$$E\left(\kappa \prod_{j=1}^N s_j; p\right) W_{A_{N-1}}(\boldsymbol{s}; p) = \frac{(p^N; p^N)_\infty^N}{(p; p)_\infty^N} \det_{1\le j,k\le N}\left[s_j^{k-1} E\left((-1)^{N-1} p^{k-1} \kappa s_j^N; p^N\right)\right].$$

From now on, we assume $\Re\tau = 0$, $\Im\tau > 0$, that is, $0 < q = e^{\pi i\tau} < 1$. It is obvious from (1.2) that if $v \in \mathbb{R}$, then $\vartheta_1(v; \tau) \in \mathbb{R}$, $|\vartheta_1(v; \tau)| < \infty$. We set $p = q^2$ and $s_j = e^{ix_j/r}$, $r_j = e^{iu_j/r}$, $\kappa = e^{i\delta/r}$ in Lemma 2.2, $x_j, u_j \in \mathbb{R}$, $1 \le j \le N$, $\delta \in \pi r\mathbb{Z}$. Then, through (2.18), these lemmas are rewritten as follows.

Lemma 2.4 *Assume that* $\boldsymbol{u} = (u_1, \dots, u_N) \in \mathcal{A}_{[0,2\pi r)^N}$ *and* $\overline{u}_\delta = \delta + \sum_{j=1}^N u_j \in (0, 2\pi r)$. *Let* $\overline{x}_\delta = \delta + \sum_{j=1}^N x_j$. *Then*

$$\det_{1\le j,k\le N}\left[\frac{\vartheta_1((\overline{u}_\delta + x_j - u_k)/2\pi r; \tau)}{\vartheta_1(\overline{u}_\delta/2\pi r; \tau)} \prod_{1\le\ell\le N, \ell\neq k} \frac{\vartheta_1((x_j - u_\ell)/2\pi r; \tau)}{\vartheta_1((u_k - u_\ell)/2\pi r; \tau)}\right] = \frac{\vartheta_1(\overline{x}_\delta/2\pi r; \tau)}{\vartheta_1(\overline{u}_\delta/2\pi r; \tau)} \prod_{1\le j<k\le N} \frac{\vartheta_1((x_j - x_k)/2\pi r; \tau)}{\vartheta_1((u_j - u_k)/2\pi r; \tau)}.$$

Similarly Lemma 2.3 gives the following.

Lemma 2.5

$$\vartheta_1\left(\frac{\overline{x}_\delta}{2\pi r}; \tau\right) \prod_{1\le j<k\le N} \vartheta_1\left(\frac{x_j - x_k}{2\pi r}; \tau\right) = C_N^A(\tau) \det_{1\le j,k\le N}\left[e^{i(k-1)x_j/r} \vartheta_1\left(\frac{N-1}{2} + (k-1)\tau + \frac{\delta + Nx_j}{2\pi r}; N\tau\right)\right] \tag{2.25}$$

with

$$C_N^A(\tau) = C_N^A(\tau; r, \delta) = q_0(\tau)^{(N-1)(N-2)/2} i^{(N-1)(3N-2)/2} q(\tau)^{(N-1)(3N-2)/8} e^{i(N-1)\delta/2r}$$
$$= q_0(\tau)^{(N-1)(N-2)/2} \exp\left[(N-1)\left\{\frac{(3N-2)}{8}\tau + \frac{\delta}{2\pi r} + \frac{3N-2}{4}\right\}\pi i\right]. \tag{2.26}$$

Let $\eta(x)$ denotes Dedekind's η-function [11,33],

$$\eta(x) = x^{1/24} \prod_{n=1}^{\infty}(1 - x^n). \tag{2.27}$$

The following equalities were proved as Proposition 5.6.3 in [15] (see also [13,14]).

Lemma 2.6 *Let $\alpha \in \mathbb{C}$. For N odd*

$$\det_{1\le j,k\le N}[\vartheta_3(x_j + \alpha - k/N; \tau)] = N^{N/2} \eta(e^{2N\pi i\tau})^{-(N-1)(N-2)/2}$$
$$\times \vartheta_3\left(\sum_{j=1}^{N}(x_j + \alpha) + \frac{N\tau}{2}; 2N\tau\right) \prod_{1\le j<k\le N} \vartheta_1(x_k - x_j; N\tau), \tag{2.28}$$

while for N even

$$\det_{1\le j,k\le N}[\vartheta_1(x_j + \alpha - k/N; \tau)] = N^{N/2} \eta(e^{2N\pi i\tau})^{-(N-1)(N-2)/2}$$
$$\times \vartheta_0\left(\sum_{j=1}^{N}(x_j + \alpha) + \frac{N\tau}{2}; 2N\tau\right) \prod_{1\le j<k\le N} \vartheta_1(x_k - x_j; N\tau). \tag{2.29}$$

2.3 Generalized h-transform

The backward Kolmogorov equation for (1.17) is given as

$$-\frac{\partial u^A(s, \boldsymbol{x})}{\partial s} = \frac{1}{2}\sum_{j=1}^{N} \frac{\partial^2 u^A(s, \boldsymbol{x})}{\partial x_j^2} + \sum_{\substack{1\le j,k\le N,\\ j\ne k}} A_N^{2\pi r}(t_* - s, x_j - x_k) \frac{\partial u^A(s, \boldsymbol{x})}{\partial x_j}$$
$$+ \sum_{1\le j\le N} A_N^{2\pi r}(t_* - s, \overline{x}_\delta) \frac{\partial u^A(s, \boldsymbol{x})}{\partial x_j}. \tag{2.30}$$

We write the tpd of the process $\Xi^A(t), t \in [0, t_*)$ as $p_N^A(t, \boldsymbol{y}|s, \boldsymbol{x}) = p_N^A(t, \boldsymbol{y}|s, \boldsymbol{x}; r, t_*)$, $0 \le s \le t < t_*$, provided that $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{A}_{[0,2\pi r)^N}$ and $\overline{x}_\delta, \overline{y}_\delta \in (0, 2\pi r)$. Since the configuration of the process $\Xi^A(t), t \in [0, t_*)$ is unlabeled as (1.20), we solve (2.30) to obtain p_N^A under the 'initial condition'

$$\lim_{s\uparrow t} u^A(s,\boldsymbol{x}) = \sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N}\delta_{y_{\sigma(j)}}(\{x_j\}), \tag{2.31}$$

where $\mathcal{S}_N$ denotes a collection of all permutations of N indices. It is a moderated version of the usual one $\lim_{s\uparrow t} u^A(s,\boldsymbol{x}) = \prod_{j=1}^{N}\delta_{y_j}(\{x_j\})$ for processes with labeled configurations.

We consider the Brownian motion $\boldsymbol{V}^r(\cdot)$ started at $\boldsymbol{u}\in\mathcal{A}_{[0,2\pi r)^N}$ with an index $\delta\in\pi r\mathbb{Z}$ chosen as $\overline{u}_\delta\in(0,2\pi r)$, which is killed when it arrives at the boundary of $\mathcal{A}^{A_{N-1}}_{2\pi r}$ and when $\overline{V}^r_\delta(\cdot)\in\{0,2\pi r\}$. Let $q^A_N(t-s,\boldsymbol{y}|\boldsymbol{x}) = q^A_N(t-s,\boldsymbol{y}|\boldsymbol{x};r)$, $\boldsymbol{x},\boldsymbol{y}\in\mathcal{A}_{[0,2\pi r)^N}$, $0<s<t<t_*$ be the tpd of $\boldsymbol{V}^r(\cdot)$, which satisfies

$$\lim_{t\downarrow 0} q^A_N(t,\boldsymbol{y}|\boldsymbol{x}) = \sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^{N}\delta_{y_{\sigma(j)}}(\{x_j\}), \tag{2.32}$$

Lemma 2.7 *The tpd of the process* $\Xi^A(t)$, $t\in[0,t_*)$ *is given by*

$$p^A_N(t,\boldsymbol{y}|s,\boldsymbol{x}) = \frac{h^A_N(t_*-t,\boldsymbol{y})}{h^A_N(t_*-s,\boldsymbol{x})} q^A_N(t-s,\boldsymbol{y}|\boldsymbol{x}),\quad 0\le s\le t<t_*,\quad \boldsymbol{x},\boldsymbol{y}\in\mathcal{A}_{[0,2\pi r)^N},$$

where $\overline{x}_\delta, \overline{y}_\delta\in(0,2\pi r)$ *and*

$$\begin{aligned}
h^A_N(t_*-t,\boldsymbol{x}) &= h^A_N(t_*-t,\boldsymbol{x};r,t_*)\\
&= e^{-N(N-1)(N-2)t_*/48r^2}\eta(e^{-N(t_*-t)/r^2})^{-(N-1)(N-2)/2}\\
&\quad\times\vartheta_1\left(\frac{\overline{x}_\delta}{2\pi r};\frac{iN(t_*-t)}{2\pi r^2}\right)\prod_{1\le j<k\le N}\vartheta_1\left(\frac{x_k-x_j}{2\pi r};\frac{iN(t_*-t)}{2\pi r^2}\right),
\end{aligned} \tag{2.33}$$

$t\in[0,t_*)$, $\boldsymbol{x}\in\mathcal{A}_{[0,2\pi r)^N}$.

Proof Set

$$u^A(s,\boldsymbol{x}) = f^A(s,\boldsymbol{x}) q^A_N(t-s,\boldsymbol{y}|\boldsymbol{x}) \tag{2.34}$$

and put it into (2.30) assuming that f^A is C^1 in t and C^2 in $\boldsymbol{x}$. Then we have

$$\begin{aligned}
&-\frac{\partial f^A(s,\boldsymbol{x})}{\partial s} q^A_N(t-s,\boldsymbol{y}|\boldsymbol{x})\\
&= \frac{1}{2} q^A_N(t-s,\boldsymbol{y}|\boldsymbol{x})\sum_{j=1}^{N}\frac{\partial^2 f^A(s,\boldsymbol{x})}{\partial x_j^2} + \sum_{j=1}^{N}\frac{\partial f^A(s,\boldsymbol{x})}{\partial x_j}\frac{\partial q^A_N(t-s,\boldsymbol{y}|\boldsymbol{x})}{\partial x_j}
\end{aligned}$$

$$+q_N^A(t-s,\boldsymbol{y}|\boldsymbol{x})\sum_{j=1}^N\left\{\sum_{\substack{1\le k\le N,\\ k\neq j}}A_N^{2\pi r}(t_*-s,x_j-x_k)+A_N^{2\pi r}(t_*-s,\overline{x}_\delta)\right\}\frac{\partial f^A(s,\boldsymbol{x})}{\partial x_j}$$
$$+f^A(s,\boldsymbol{x})\sum_{j=1}^N\left\{\sum_{\substack{1\le k\le N,\\ k\neq j}}A_N^{2\pi r}(t_*-s,x_j-x_k)+A_N^{2\pi r}(t_*-s,\overline{x}_\delta)\right\}\frac{\partial q_N^A(t-s,\boldsymbol{y}|\boldsymbol{x})}{\partial x_j}, \tag{2.35}$$

since $q_N^A(t-s,\boldsymbol{y}|\boldsymbol{x})$ satisfies the diffusion equation. We put

$$f^A(s,\boldsymbol{x})=g^A(s)\left\{\vartheta_1\left(\frac{\overline{x}_\delta}{2\pi r};\frac{iN(t_*-s)}{2\pi r^2}\right)\prod_{1\le j<k\le N}\vartheta_1\left(\frac{x_k-x_j}{2\pi r};\frac{iN(t_*-s)}{2\pi r^2}\right)\right\}^{-1}, \tag{2.36}$$

where g^A is a C^1 function of time s to be determined. By definition (1.4), we see

$$\frac{\partial f^A(s,\boldsymbol{x})}{\partial x_j}=-\left\{\sum_{\substack{1\le k\le N,\\ k\neq j}}A_N^{2\pi r}(t_*-s,x_j-x_k)+A_N^{2\pi r}(t_*-s,\overline{x}_\delta)\right\}f^A(s,\boldsymbol{x})$$

and

$$\frac{\partial A_N^{2\pi r}(t_*-s,x_j-x_k)}{\partial x_j}=\frac{1}{(2\pi r)^2}\frac{\vartheta_1''((x_j-x_k)/2\pi r;iN(t_*-s)/2\pi r^2)}{\vartheta_1((x_j-x_k)/2\pi r;iN(t_*-s)/2\pi r^2)}$$
$$-(A_N^{2\pi r}(t_*-s,x_j-x_k))^2. \tag{2.37}$$

Then (2.35) gives the equation

$$-\frac{\partial f^A(s,\boldsymbol{x})}{\partial s}=-\frac{1}{2(2\pi r)^2}\sum_{\substack{1\le j,k\le N,\\ j\neq k}}\frac{\vartheta_1''((x_j-x_k)/2\pi r;iN(t_*-s)/2\pi r^2)}{\vartheta_1((x_j-x_k)/2\pi r;iN(t_*-s)/2\pi r^2)}f^A(s,\boldsymbol{x})$$
$$-\frac{1}{2(2\pi r)^2}N\frac{\vartheta_1''(\overline{x}_\delta/2\pi r;iN(t_*-s)/2\pi r^2)}{\vartheta_1(\overline{x}_\delta/2\pi r;iN(t_*-s)/2\pi r^2)}f^A(s,\boldsymbol{x})$$
$$-\frac{1}{2}\sum_{\substack{1\le j,k,\ell\le N,\\ j\neq k\neq\ell}}A_N^{2\pi r}(t_*-s,x_j-x_k)A_N^{2\pi r}(t_*-s,x_j-x_\ell)f^A(s,\boldsymbol{x}), \tag{2.38}$$

where the sum in the last term denotes the summation over $1\le j,k,\ell\le N$ conditioned that j,k,ℓ are all distinct. By the setting (2.36),

$$
\begin{aligned}
\text{LHS of (2.38)} &= -\frac{1}{g^A(s)} \frac{dg^A(s)}{ds} f^A(s, \boldsymbol{x}) \\
&\quad - \frac{iN}{4\pi r^2} \sum_{\substack{1 \le j,k \le N, \\ j \neq k}} \frac{\dot{\vartheta}_1((x_j - x_k)/2\pi r; iN(t_* - s)/2\pi r^2)}{\vartheta_1((x_j - x_k)/2\pi r; iN(t_* - s)/2\pi r^2)} f^A(s, \boldsymbol{x}) \\
&\quad - \frac{iN}{2\pi r^2} \frac{\dot{\vartheta}_1(\overline{x}_\delta/2\pi r; iN(t_* - s)/2\pi r^2)}{\vartheta_1(\overline{x}_\delta/2\pi r; iN(t_* - s)/2\pi r^2)} f^A(s, \boldsymbol{x}),
\end{aligned}
$$

where $\dot{\vartheta}_1(x; \tau) = d\vartheta_1(x; \tau)/d\tau$ and (2.2) was used. Since (1.3) is satisfied, the above is equal to

$$
\begin{aligned}
&-\frac{1}{g^A(s)} \frac{dg^A(s)}{ds} f^A(s, \boldsymbol{x}) \\
&\quad - \frac{N}{16\pi^2 r^2} \sum_{\substack{1 \le j,k \le N, \\ j \neq k}} \frac{\vartheta_1''((x_j - x_k)/2\pi r; iN(t_* - s)/2\pi r^2)}{\vartheta_1((x_j - x_k)/2\pi r; iN(t_* - s)/2\pi r^2)} f^A(s, \boldsymbol{x}) \\
&\quad - \frac{N}{8\pi^2 r^2} \frac{\vartheta_1''(\overline{x}_\delta/2\pi r; iN(t_* - s)/2\pi r^2)}{\vartheta_1(\overline{x}_\delta/2\pi r; iN(t_* - s)/2\pi r^2)} f^A(s, \boldsymbol{x}).
\end{aligned}
$$

Therefore, (2.38) becomes

$$
\begin{aligned}
\frac{1}{g^A(s)} \frac{dg^A(s)}{ds} &= \frac{1}{2} \sum_{\substack{1 \le j,k,\ell \le N, \\ j \neq k \neq \ell}} A_N^{2\pi r}(t_* - s, x_j - x_k) A_N^{2\pi r}(t_* - s, x_j - x_\ell) \\
&\quad - \frac{N-2}{16\pi^2 r^2} \sum_{\substack{1 \le j,k \le N, \\ j \neq k}} \frac{\vartheta_1''((x_j - x_k)/2\pi r; iN(t_* - s)/2\pi r^2)}{\vartheta_1((x_j - x_k)/2\pi r; iN(t_* - s)/2\pi r^2)}.
\end{aligned}
\tag{2.39}
$$

Now we rewrite RHS of (2.39) by using the functions $\wp$ and ζ through (2.13). First we see

$$
\begin{aligned}
&\frac{1}{2} \sum_{\substack{1 \le j,k,\ell \le N, \\ j \neq k \neq \ell}} A_N^{2\pi r}(t_* - s, x_j - x_k) A_N^{2\pi r}(t_* - s, x_j - x_\ell) \\
&= \frac{1}{2} \sum_{\substack{1 \le j,k,\ell \le N, \\ j \neq k \neq \ell}} \zeta(x_j - x_k)\zeta(x_j - x_\ell) - \frac{\eta_1(t_* - s)}{\pi r} \sum_{\substack{1 \le j,k,\ell \le N, \\ j \neq k \neq \ell}} \zeta(x_j - x_k)(x_j - x_\ell) \\
&\quad + \frac{1}{2} \left(\frac{\eta_1(t_* - s)}{\pi r} \right)^2 \sum_{\substack{1 \le j,k,\ell \le N, \\ j \neq k \neq \ell}} (x_j - x_k)(x_j - x_\ell).
\end{aligned}
$$

For

$$\sum_{\substack{1\le j,k,\ell\le N,\\ j\ne k\ne \ell}} \zeta(x_j-x_k)\zeta(x_j-x_\ell) = 2\sum_{1\le j<k<\ell\le N} \Big\{\zeta(x_j-x_k)\zeta(x_j-x_\ell) + \zeta(x_k-x_\ell)\zeta(x_k-x_j) + \zeta(x_\ell-x_j)\zeta(x_\ell-x_k)\Big\},$$

Lemma 2.1 gives

$$\begin{aligned}\frac{1}{2}\sum_{\substack{1\le j,k,\ell\le N,\\ j\ne k\ne \ell}} \zeta(x_j-x_k)\zeta(x_j-x_\ell) &= \frac{1}{4}\sum_{\substack{1\le j,k,\ell\le N,\\ j\ne k\ne \ell}} \zeta(x_j-x_k)^2 - \frac{1}{4}\sum_{\substack{1\le j,k,\ell\le N,\\ j\ne k\ne \ell}} \wp(x_j-x_k)\\ &= \frac{N-2}{4}\sum_{\substack{1\le j,k\le N,\\ j\ne k}} \zeta(x_j-x_k)^2 - \frac{N-2}{4}\sum_{\substack{1\le j,k\le N,\\ j\ne k}} \wp(x_j-x_k).\end{aligned}$$

On the other hand, differentiation of (2.13) with respect to x gives

$$\frac{\partial A_N^{2\pi r}(t_*-s,x)}{\partial x} = \zeta'(x) - \frac{\eta_1(t_*-s)}{\pi r} = -\wp(x) - \frac{\eta_1(t_*-s)}{\pi r},$$

where (2.10) was used. Combining it with (2.37) gives

$$\begin{aligned}\frac{1}{(2\pi r)^2}\frac{\vartheta_1''(x/2\pi r; iN(t_*-s)/2\pi r^2)}{\vartheta_1(x/2\pi r; iN(t_*-s)/2\pi r^2)} &= A_N^{2\pi r}(t_*-s,x)^2 - \wp(x) - \frac{\eta_1(t_*-s)}{\pi r}\\ &= \zeta(x)^2 - \frac{2\eta_1(t_*-s)}{\pi r}\zeta(x)x + \left(\frac{\eta_1(t_*-s)}{\pi r}\right)^2 x^2 - \wp(x) - \frac{\eta_1(t_*-s)}{\pi r}.\end{aligned}$$

Then

$$\begin{aligned}&-\frac{N-2}{16\pi^2 r^2}\sum_{\substack{1\le j,k\le N,\\ j\ne k}} \frac{\vartheta_1''((x_j-x_k)/2\pi r; iN(t_*-s)/2\pi r^2)}{\vartheta_1((x_j-x_k)/2\pi r; iN(t_*-s)/2\pi r^2)}\\ &= -\frac{N-2}{4}\sum_{\substack{1\le j,k\le N,\\ j\ne k}} \zeta(x_j-x_k)^2 + \frac{N-2}{2}\frac{\eta_1(t_*-s)}{\pi r}\sum_{\substack{1\le j,k\le N,\\ j\ne k}} \zeta(x_j-x_k)(x_j-x_k)\\ &\quad -\frac{N-2}{4}\left(\frac{\eta_1(t_*-s)}{\pi r}\right)^2\sum_{\substack{1\le j,k\le N,\\ j\ne k}} (x_j-x_k)^2 + \frac{N-2}{4}\sum_{\substack{1\le j,k\le N,\\ j\ne k}} \wp(x_j-x_k)\\ &\quad +\frac{\eta_1(t_*-s)}{\pi r}\frac{N(N-1)(N-2)}{4}.\end{aligned}$$

It is easy to prove that

$$\sum_{\substack{1\le j,k,\ell\le N,\\ j\neq k\neq \ell}} \zeta(x_j-x_k)(x_j-x_\ell) - \frac{N-2}{2}\sum_{\substack{1\le j,k\le N,\\ j\neq k}} \zeta(x_j-x_k)(x_j-x_k) = 0,$$
$$\sum_{\substack{1\le j,k,\ell\le N,\\ j\neq k\neq \ell}} (x_j-x_k)(x_j-x_\ell) - \frac{N-2}{2}\sum_{\substack{1\le j,k\le N,\\ j\neq k}} (x_j-x_k)^2 = 0,$$

by using the fact that ζ is odd. Then Eq. (2.39) is reduced to be

$$\frac{d}{ds}\log g^A(s) = \frac{\eta_1(t_*-s)}{4\pi r} N(N-1)(N-2).$$

Since $\eta_1(t_*-t)$ is explicitly given as (2.14) with $\alpha = 2\pi r$, this equation can be solved as

$$g^A(s) = c' e^{N(N-1)(N-2)s/48r^2} \prod_{n=1}^{\infty} \left(\frac{1-e^{-nN(t_*-s)/r^2}}{1-e^{-nNt_*/r^2}} \right)^{(N-1)(N-2)/2}$$

with a constant c'. The solution (2.34) has been determined of the form

$$u^A(s,\boldsymbol{x}) = \prod_{n=1}^{\infty} \left(1-e^{-nNt_*/r^2}\right)^{-(N-1)(N-2)/2} \frac{c'}{h_N^A(t_*-s,\boldsymbol{x})} q_N^A(t-s,\boldsymbol{y}|\boldsymbol{x}).$$

For (2.32), the condition (2.31) is satisfied, if and only if

$$c' = \prod_{n=1}^{\infty} \left(1-e^{-nNt_*/r^2}\right)^{(N-1)(N-2)/2} h_N^A(t_*-t,\boldsymbol{y}).$$

Since $\vartheta_1(x/2\pi r; iN(t_*-t)/2\pi r^2) > 0$ if $x \in (0, 2\pi r)$, $t \in [0, t_*)$, and $q_N^A(t-s,\boldsymbol{y}|\boldsymbol{x})$ is assumed to be the tpd of $\boldsymbol{V}^r(\cdot)$, we can conclude that $0 < p_N^A(t,\boldsymbol{y}|s,\boldsymbol{x}) < \infty$ for any $\boldsymbol{x},\boldsymbol{y} \in \mathcal{A}_{[0,2\pi r)^N}$, $0 \le s \le t < t_*$, where δ is chosen so that $\overline{x}_\delta, \overline{y}_\delta \in (0, 2\pi r)$. Then the proof is completed. □

Let $\check{\boldsymbol{W}}(t) = (\check{W}_1(t), \dots, \check{W}_N), t \ge 0$ be N-dimensional Brownian motion on $(\mathrm{S}^1(r))^N$ started at $\boldsymbol{u} \in \mathcal{A}_{2\pi r}^{A_{N-1}}$. The expectation with respect to this process is denoted by $\check{\mathrm{E}}_{\boldsymbol{u}}$. Consider a stopping time

$$T_{\check{\mathbf{W}}} = \inf\left\{t > 0 : \check{\boldsymbol{W}}(t) \notin \mathcal{A}_{2\pi r}^{A_{N-1}}\right\}.$$

Put $\overline{W}_\delta = \delta + \sum_{j=1}^N \check{W}_j(t)$, where the index $\delta \in \pi r\mathbb{Z}$ is determined so that $\overline{u}_\delta \in (0, 2\pi r)$. Then we also consider the following stopping time

$$T_{\overline{W}_\delta} = \inf\{t > 0 : \overline{W}_\delta \in \{0, 2\pi r\}\}.$$

For the process $\Xi^A(t)$, $t \in [0, t_*)$ is $\mathfrak{M}([0, 2\pi r))$-valued, measurable functions are symmetric functions of N variables X_j^A, $1 \leq j \leq N$ at each time. By the definition (1.19) for $\boldsymbol{X}^A$, they should be periodic with period $2\pi r$. Let $T \in [0, t_*)$. Then any $\mathcal{F}_{\Xi^A}(T)$-measurable function F will be given as follows. With an arbitrary integer $M \in \mathbb{N}$ and arbitrary sequence of times $0 \leq t_1 < \cdots < t_M \leq T$,

$$F(\Xi^A(\cdot)) = \prod_{m=1}^{M} g_m(\boldsymbol{X}^A(t_m)), \tag{2.40}$$

where $g_m(\boldsymbol{x})$, $1 \leq m \leq M$ are symmetric functions and

$$g_m((x_j + 2\pi r n_j)) = g_m(\boldsymbol{x}), \quad n_j \in \mathbb{Z}, \quad 1 \leq j \leq N, \tag{2.41}$$

for $1 \leq m \leq M$.

The indicator function of ω is denoted by $\mathbf{1}(\omega)$; $\mathbf{1}(\omega) = 1$ if ω is satisfied, and $\mathbf{1}(\omega) = 0$ otherwise. Lemma 2.7 implies the following equality.

Proposition 2.8 *Suppose* $\xi = \sum_{j=1}^{N} \delta_{u_j} \in \mathfrak{M}_0([0, 2\pi r))$. *Let* $T \in [0, t_*)$. *For any* $\mathcal{F}_{\Xi^A}(T)$*-measurable observable* F*,*

$$\mathbb{E}_{\xi}^{A}[F(\Xi^A(\cdot))] = \check{\mathrm{E}}_{\boldsymbol{u}}\left[F\left(\sum_{j=1}^{N} \delta_{\check{W}_j(\cdot)}\right) \mathbf{1}(T_{\check{\mathbf{W}}} \wedge T_{\overline{W}_\delta} > T) \frac{h_N^A(t_* - T, \check{\boldsymbol{W}}(T))}{h_N^A(t_*, \boldsymbol{u})}\right].$$

See Remark 2 at the end of Sect. 3.1.

2.4 Markov process $\boldsymbol{W}^r$

We write the tpd of BM on $\mathbb{R}$ as

$$p_{\mathrm{BM}}(t, y|x) = \frac{1}{\sqrt{2\pi t}} e^{-(y-x)^2/2t}, \quad x, y \in \mathbb{R}, \quad t \in [0, \infty).$$

By wrapping it on $\mathrm{S}^1(r)$, we define

$$p_{A_{N-1}}^{r}(t, y|x) = \begin{cases} \sum_{\ell \in \mathbb{Z}} p_{\mathrm{BM}}(t, y + 2\pi r \ell | x), & \text{if } N \text{ is even}, \\ \sum_{\ell \in \mathbb{Z}} (-1)^{\ell} p_{\mathrm{BM}}(t, y + 2\pi r \ell | x), & \text{if } N \text{ is odd}, \end{cases} \tag{2.42}$$

$x, y \in [0, 2\pi r)$, $t \geq 0$. Using the Jacobi theta functions (2.5), it is written as

$$p_{A_{N-1}}^{r}(t, y|x) = \begin{cases} p_{\mathrm{BM}}(t, y|x) \vartheta_3\left(\dfrac{i(y-x)r}{t}; \dfrac{2\pi i r^2}{t}\right), & \text{if } N \text{ is even}, \\ p_{\mathrm{BM}}(t, y|x) \vartheta_0\left(\dfrac{i(y-x)r}{t}; \dfrac{2\pi i r^2}{t}\right), & \text{if } N \text{ is odd}. \end{cases}$$

We find that by Jacobi's imaginary transformations [44],

$$\vartheta_0(v;\tau) = e^{\pi i/4}\tau^{-1/2}e^{-\pi i v^2/\tau}\vartheta_2\left(\frac{v}{\tau}; -\frac{1}{\tau}\right),$$
$$\vartheta_3(v;\tau) = e^{\pi i/4}\tau^{-1/2}e^{-\pi i v^2/\tau}\vartheta_3\left(\frac{v}{\tau}; -\frac{1}{\tau}\right),$$

the above is further rewritten as

$$p^r_{A_{N-1}}(t, y|x) = \begin{cases} \dfrac{1}{2\pi r}\vartheta_3\left(\dfrac{y-x}{2\pi r}; \dfrac{it}{2\pi r^2}\right), & \text{if } N \text{ is even}, \\ \dfrac{1}{2\pi r}\vartheta_2\left(\dfrac{y-x}{2\pi r}; \dfrac{it}{2\pi r^2}\right), & \text{if } N \text{ is odd}. \end{cases} \tag{2.43}$$

Lemma 2.6 given by Forrester [15] implies the following.

Proposition 2.9 *For $N \in \{2, 3, \ldots\}$, $\boldsymbol{v}$ is given by (1.21). Then for $\boldsymbol{y} \in \mathcal{A}_{[0,2\pi r)^N}$, $t > 0$,*

$$q^A_N(t, \boldsymbol{y}|\boldsymbol{v}) = \det_{1\le j,k\le N}\left[p^r_{A_{N-1}}(t, y_j|v_k)\right]. \tag{2.44}$$

Proof For N odd, we put $\alpha = 1/N + \tau/2$ in (2.28), and for N even, $\alpha = 1/N + (1+\tau)/2$ in (2.29). Let $\tau = it/2\pi r^2$. Then we have

$$\det_{1\le j,k\le N}\left[p^r_{A_{N-1}}(t, y_j|v_k)\right] = \left(\frac{\sqrt{N}}{2\pi r}\right)^N \eta(e^{-Nt/r^2})^{-(N-1)(N-2)/2}e^{Nt/8r^2}$$
$$\times\vartheta_1\left(\frac{\overline{y}_{-\pi r(N-2)}}{2\pi r}; \frac{iNt}{\pi r^2}\right)\prod_{1\le j<k\le N}\vartheta_1\left(\frac{y_k - y_j}{2\pi r}; \frac{iNt}{2\pi r^2}\right), \tag{2.45}$$

where quasi-periodicity of $\vartheta_\mu, \mu = 0, 1, 2, 3$ has been used. By expression (2.43) with (2.5), it is obvious that (2.45) satisfies the diffusion equation. This expression (2.45) guarantees the positivity and finiteness of $\det_{1\le j,k\le N}[p^r_{A_{N-1}}(t, y_j|v_k)]$ for $\boldsymbol{y} \in \mathcal{A}_{[0,2\pi r)^N}$ and $\overline{y}_{-\pi r(N-2)} \in (0, 2\pi r)$. Equation (2.45) also shows that it vanishes when $y_j = y_k$ for any $j \ne k$ and when $\overline{y}_{-\pi r(N-2)} \in \{0, 2\pi r\}$. By the expression (2.42) and the argument given by Liechty and Wang [32] (see also [16]), we can prove that (2.45) satisfies the moderated initial configuration

$$\lim_{t\downarrow 0}\det_{1\le j,k\le N}\left[p^r_{A_{N-1}}(t, y_j|v_k)\right] = \sum_{\sigma\in\mathcal{S}_N}\prod_{j=1}^N \delta_{v_{\sigma(j)}}(\{y_j\}).$$

Then the proof is completed. □

Remark 1 The Karlin-McGregor-type [22] determinantal formula (2.44) for $q^A_N(t, \boldsymbol{y}|\boldsymbol{x})$ given in Proposition 2.9 is crucial for DMR which will be proved in Theorem 3.1. In the present paper, we obtained it for the special initial configuration (1.21) due to the explicit evaluation (2.45) given by Lemma 2.6 of Forrester [15]. If we obtain the Karlin-McGregor-type determinantal formula for q_N^A for other initial configuration, we can prove DMR for the process and it will immediately conclude that the process is determinantal by Theorem 1.3 of [23].

Given $N \in \mathbb{N}$, let $W^r(t), t \geq 0$ be a Markov process in $[0, 2\pi r)$ such that its transition density is given by $p^r_{A_{N-1}}(t, y|x), t \geq 0, x, y \in [0, 2\pi r)$, defined by (2.42). Then we introduce an N independent copies of $W^r(t), t \geq 0$, denoted by $W^r_j(t), t \geq 0, 1 \leq j \leq N$ and let $\boldsymbol{W}^r(t) = (W^r_1(t), \dots, W^r_N(t)), t \geq 0$. The probability space of the process is denoted by $(\Omega_{W^r}, \mathcal{F}_{W^r}, \mathrm{P}^r_{\boldsymbol{v}})$, and the expectation is written as $\mathrm{E}^r_{\boldsymbol{v}}$, where the initial configuration is given by $\boldsymbol{v}$ with (1.21). A filtration $\{\mathcal{F}_{W^r}(t) : t \geq 0\}$ is generated by $\boldsymbol{W}^r(t), t \geq 0$, which satisfies the usual conditions.

2.5 Martingales and complex Brownian motions

Let $0 < t_* < \infty$ and $\xi = \sum_{j=1}^N \delta_{u_j} \in \mathfrak{M}_0([0, 2\pi r))$. For $1 \leq k \leq N$, define

$$\begin{aligned}\Phi^A_{\xi,u_k}(z) &= \Phi^A_{\xi,u_k}(z; N, r, t_*)\\ &= \frac{\vartheta_1((\overline{u}_\delta + z - u_k)/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1(\overline{u}_\delta/2\pi r; iNt_*/2\pi r^2)} \prod_{\substack{1\leq \ell \leq N,\\ \ell \neq k}} \frac{\vartheta_1((z-u_\ell)/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1((u_k-u_\ell)/2\pi r; iNt_*/2\pi r^2)}, \quad z \in \mathbb{C},\end{aligned} \tag{2.46}$$

and

$$\begin{aligned}\mathcal{M}^A_{\xi,u_k}(t, x) &= \mathcal{M}^A_{\xi,u_k}(t, x; N, r, t_*)\\ &= \int_{\mathbb{R}} dw \, \frac{e^{-(ix+w)^2/2t}}{\sqrt{2\pi t}} \Phi^A_{\xi,u_k}(iw), \quad (t, x) \in [0, t_*) \times [0, 2\pi r).\end{aligned} \tag{2.47}$$

Since $\Phi^A_{\xi,u_k}(z), 1 \leq k \leq N$ are holomorphic for $|z| < \infty$, (2.47) is written as

$$\begin{aligned}\mathcal{M}^A_{\xi,u_k}(t, x) &= \int_{\mathbb{R}} d\widetilde{w} \, \frac{e^{-\widetilde{w}^2/2t}}{\sqrt{2\pi t}} \Phi^A_{\xi,u_k}(x + i\widetilde{w})\\ &= \widetilde{\mathrm{E}}[\Phi^A_{\xi,u_k}(x + i\widetilde{W}(t))],\end{aligned} \tag{2.48}$$

where $\widetilde{W}$ denotes a BM on $\mathbb{R}$ started at 0, which is independent of $\boldsymbol{W}^r$, and $\widetilde{\mathrm{E}}$ does the expectation for $\widetilde{W}$. Then the following is proved.

Lemma 2.10 *Assume $v_j, 1 \leq j \leq N$ are given by (1.21) and $\eta = \sum_{j=1}^N \delta_{v_j}$. Then*

(i) $\mathcal{M}^A_{\eta,v_k}(t, W^r(t)), 1 \leq k \leq N, t \in [0, t_*)$ *are continuous-time martingales;* $\mathrm{E}^r[\mathcal{M}^A_{\eta,v_k}(t, W^r(t))|\mathcal{F}_{W^r}(s)] = \mathcal{M}^A_{\eta,v_k}(s, W^r(s))$ *a.s. for any two bounded stopping times with* $0 \leq s \leq t < t_*$.

(ii) *For any* $t \in [0, t_*)$, $\mathcal{M}^A_{\eta, v_k}(t, x)$, $1 \le k \le N$, *are linearly independent functions of* $x \in [0, 2\pi r)$,
(iii) $\mathcal{M}^A_{\eta, v_k}(0, v_j) = \delta_{jk}, \quad 1 \le j, k \le N.$

Proof (i) For the quasi-periodicity (2.3) of ϑ_1, the expression (2.48) with (2.46) implies that, for $\ell \in \mathbb{Z}$,

$$\mathcal{M}^A_{\eta, v_k}(t, x + 2\pi r \ell) = (-1)^{\ell N} \mathcal{M}^A_{\eta, v_k}(t, y) = \begin{cases} \mathcal{M}^A_{\eta, v_k}(t, y), & \text{if } N \text{ is even,} \\ (-1)^{\ell} \mathcal{M}^A_{\eta, v_k}(t, y), & \text{if } N \text{ is odd.} \end{cases} \tag{2.49}$$

Then, for $0 \le s \le t < t_*$, $1 \le k \le N$, (2.42) gives

$$\mathrm{E}^r \left[\mathcal{M}^A_{\eta, v_k}(t, W^r(t)) | \mathcal{F}_{W^r}(s) \right] = \int_0^{2\pi r} dw \, \mathcal{M}^A_{\eta, v_k}(t, w) p^r_{A_{N-1}}(t - s, w | W^r(s))$$

$$= \begin{cases} \sum_{\ell \in \mathbb{Z}} \int_{2\pi r \ell}^{2\pi r(\ell+1)} dw \, \mathcal{M}^A_{\eta, v_k}(t, w - 2\pi r \ell) p_{\mathrm{BM}}(t - s, w | W^r(s)), & \text{if } N \text{ is even,} \\ \sum_{\ell \in \mathbb{Z}} \int_{2\pi r \ell}^{2\pi r(\ell+1)} dw \, (-1)^{\ell} \mathcal{M}^A_{\eta, v_k}(t, w - 2\pi r \ell) p_{\mathrm{BM}}(t - s, w | W^r(s)), & \text{if } N \text{ is odd.} \end{cases}$$

By (2.49), it is equal to

$$\int_{\mathbb{R}} dw \, \mathcal{M}^A_{\eta, v_k}(t, w) p_{\mathrm{BM}}(t - s, w | W^r(s)) \quad \text{a.s.}$$

By Definition (1.2) of ϑ_1, we will obtain the following expansions; for $1 \le k \le N$,

$$\frac{\vartheta_1((\overline{u}_\delta + z - u_k)/2\pi r; i N t_*/2\pi r^2)}{\vartheta_1(\overline{u}_\delta/2\pi r; i N t_*/2\pi r^2)} = \sum_{n_0 \in \mathbb{Z}} b^0_{n_0} e^{i(2n_0 - 1)z/2r},$$
$$\frac{\vartheta_1((z - u_\ell)/2\pi r; i N t_*/2\pi r^2)}{\vartheta_1((u_k - u_\ell)/2\pi r; i N t_*/2\pi r^2)} = \sum_{n_\ell \in \mathbb{Z}} b^\ell_{n_\ell} e^{i(2n_\ell - 1)z/2r}, \quad 1 \le \ell \le N, \quad \ell \neq k,$$

where the coefficients $b^\ell_{n_\ell}$, $0 \le \ell \le N$, $\ell \neq k$ are functions of $\{u_\ell\}_{\ell=1}^N$, t_*, N and r. Then, if we introduce an N-component index $\boldsymbol{n} = (n_0, n_1, \dots, n_{k-1}, n_{k+1}, \dots, n_N)$ for each $1 \le k \le N$, and put $B^k_{\boldsymbol{n}} = \prod_{0 \le \ell \le N, \ell \neq k} b^\ell_{n_\ell}$, (2.48) with $\xi = \eta$, $u_k = v_k$, $x = w$ is expanded as

$$\mathcal{M}^A_{\eta,v_k}(t,w) = \sum_{\boldsymbol{n}\in\mathbb{Z}^N} B^k_{\boldsymbol{n}} \exp\left(i \sum_{0\le\ell\le N,\ell\neq k} (2n_\ell - 1)\frac{w}{2r}\right)$$
$$\times \widetilde{\mathrm{E}}\left[\exp\left(-\sum_{0\le\ell\le N,\ell\neq k}(2n_\ell - 1)\frac{\widetilde{W}(t)}{2r}\right)\right]$$
$$= \sum_{\boldsymbol{n}\in\mathbb{Z}^N} B^k_{\boldsymbol{n}} G\left(\frac{i}{2r}\sum_{0\le\ell\le N,\ell\neq k}(2n_\ell - 1); t, w\right), \qquad (2.50)$$

where

$$G(\alpha; t, w) = e^{\alpha w - \alpha^2 t/2}, \quad \alpha \in \mathbb{C}.$$

For any $\alpha \in \mathbb{C}$, it is easy to confirm that

$$\int_{\mathbb{R}} dw\, G(\alpha; t, w) p_{\mathrm{BM}}(t-s, w|x) = G(\alpha; s, x), \quad 0 \le s \le t, \quad x \in \mathbb{R}.$$

Then (2.50) gives

$$\int_{\mathbb{R}} dw\, \mathcal{M}^A_{\eta,v_k}(t,w) p_{\mathrm{BM}}(t-s, w|x) = \mathcal{M}^A_{\eta,v_k}(s,x), \quad 0 \le s \le t, \quad x \in \mathbb{R},$$

and hence (i) is concluded. As a matter of course, $\eta \in \mathfrak{M}_0([0, 2\pi r))$, and then the zeroes of $\Phi^A_{\eta,v_j}(z)$ are distinct from those of $\Phi^A_{\eta,v_k}(z)$, if $j \neq k$. Then (ii) is proved. By (2.48), $\mathcal{M}^A_{\eta,v_k}(0,x) = \lim_{t\downarrow 0}\widetilde{\mathrm{E}}\left[\Phi^A_{\eta,v_k}(x + i\widetilde{W}(t))\right] = \Phi^A_{\eta,v_k}(x)$, $1 \le k \le N$. Since $\Phi^A_{\eta,v_k}(v_j) = \delta_{jk}$, $1 \le j, k \le N$ by Definition (2.46), (iii) is also satisfied. □

Let $\widetilde{W}_j(\cdot)$, $1 \le j \le N$ be independent N copies of $\widetilde{W}(\cdot)$. For $\widetilde{\boldsymbol{W}}(t) = (\widetilde{W}_1(t), \dots, \widetilde{W}_N(t))$, $t \ge 0$, the probability space is denoted by $(\widetilde{\Omega}_W, \widetilde{\mathcal{F}}_W, \widetilde{\mathrm{P}})$ with expectation $\widetilde{\mathrm{E}}$. We put

$$Z^r_j(t) = W^r_j(t) + i\widetilde{W}_j(t), \quad 1 \le j \le N, \quad t \ge 0,$$

which are independent complex Brownian motions on $\mathbb{C}(r) \equiv [0, 2\pi r) \times i\mathbb{R}$. The probability space for $\boldsymbol{Z}^r(t) = (Z^r_1(t), \dots, Z^r_N(t))$, $t \ge 0$ is given by the direct product of the two spaces, $(\Omega_{W^r}, \mathcal{F}_{W^r}, \mathrm{P}^r_{\boldsymbol{u}})$ for $\boldsymbol{W}^r(\cdot)$ and $(\widetilde{\Omega}_W, \widetilde{\mathcal{F}}_W, \widetilde{\mathrm{P}})$ for $\widetilde{\boldsymbol{W}}(\cdot)$, which is denoted by $(\Omega^r, \mathcal{F}^r, \mathbf{P}^r_{\boldsymbol{u}})$ with expectation $\mathbf{E}^r_{\boldsymbol{u}}$.

Proposition 2.11 *Let $\overline{Z}^r_\delta(t) = \delta + \sum_{j=1}^N Z^r_j(t)$ and $\overline{W}^r_\delta = \delta + \sum_{j=1}^N W^r(t)$, $t \ge 0$. Then the following equality holds,*

$$\widetilde{\mathrm{E}}\left[\vartheta_1\left(\frac{\overline{Z}^r_\delta(t)}{2\pi r}; \frac{iNt_*}{2\pi r^2}\right)\prod_{1\le j<k\le N}\vartheta_1\left(\frac{Z^r_j(t)-Z^r_k(t)}{2\pi r}; \frac{iNt_*}{2\pi r^2}\right)\right]$$
$$=\left[e^{Nt/24r^2}\prod_{n=1}^{\infty}\left(\frac{1-e^{-nN(t_*-t)/r^2}}{1-e^{-nNt_*/r^2}}\right)\right]^{-(N-1)(N-2)/2}$$
$$\times\vartheta_1\left(\frac{\overline{W}^r_\delta(t)}{2\pi r}; \frac{iN(t_*-t)}{2\pi r^2}\right)\prod_{1\le j<k\le N}\vartheta_1\left(\frac{W^r_j(t)-W^r_k(t)}{2\pi r}; \frac{iN(t_*-t)}{2\pi r^2}\right).$$

Proof By (2.25) in Lemma 2.5

$$\widetilde{\mathrm{E}}\left[\vartheta_1\left(\frac{\overline{Z}^r_\delta(t)}{2\pi r}; \tau\right)\prod_{1\le j<k\le N}\vartheta_1\left(\frac{Z^r_j(t)-Z^r_k(t)}{2\pi r}; \tau\right)\right]$$
$$= C^A_N(\tau)\det_{1\le j,k\le N}\left[\widetilde{\mathrm{E}}\left[e^{i(k-1)Z^r_j(t)/r}\vartheta_1\left(\frac{N-1}{2}+(k-1)\tau+\frac{\delta+NZ^r_j(t)}{2\pi r}; N\tau\right)\right]\right], \tag{2.51}$$

where the multilinearity of determinant and independence of $Z^r_j(t)$'s have been used. Using the Laurent expansion (1.2), we have

$$\widetilde{\mathrm{E}}\left[e^{i(k-1)Z^r_j(t)/r}\vartheta_1\left(\frac{N-1}{2}+(k-1)\tau+\frac{\delta+NZ^r_j(t)}{2\pi r}; N\tau\right)\right]$$
$$= e^{i(k-1)W^r_j(t)/r} i\sum_{n\in\mathbb{Z}}(-1)^n e^{[(n-(1/2))^2N\tau+(2n-1)\{(N-1)/2+(k-1)\tau+(\delta+NW^r_j(t))/2\pi r\}]\pi i}$$
$$\times\widetilde{\mathrm{E}}\Big[e^{-\{2(k-1)+(2n-1)N\}\widetilde{W}_j(t)/2r}\Big]. \tag{2.52}$$

Here

$$\widetilde{\mathrm{E}}\Big[e^{-\{2(k-1)+(2n-1)N\}\widetilde{W}_j(t)/2r}\Big] = \int_{\mathbb{R}} d\widetilde{w}\,\frac{e^{-\widetilde{w}^2/2t}}{\sqrt{2\pi t}}e^{-\{2(k-1)+(2n-1)N\}\widetilde{w}/2r}$$
$$= e^{t[2(k-1)+(2n-1)N]^2/8r^2}$$
$$= e^{(k-1)^2t/2r^2}\exp\left[\left(n-\frac{1}{2}\right)^2 N\left(-\frac{iNt}{2\pi r^2}\right)\pi i\right.$$
$$\left.+(2n-1)(k-1)\left(-\frac{iNt}{2\pi r^2}\right)\pi i\right].$$

Then (2.52) is equal to

$$
\begin{aligned}
& e^{(k-1)^2 t/2r^2} e^{i(k-1)W_j^r(t)/r} i \sum_{n\in\mathbb{Z}} (-1)^n \exp\left[\left(n-\frac{1}{2}\right)^2 N\left(\tau - \frac{iNt}{2\pi r^2}\right)\pi i \right. \\
& \quad \left. +(2n-1)\left\{\frac{N-1}{2} + (k-1)\left(\tau - \frac{iNt}{2\pi r^2}\right) + \frac{\delta + NW_j^r(t)}{2\pi r}\right\}\pi i\right] \\
& = e^{(k-1)^2 t/2r^2} e^{i(k-1)W_j^r(t)/r} \\
& \quad \times \vartheta_1\left(\frac{N-1}{2} + (k-1)\left(\tau - \frac{iNt}{2\pi r^2}\right) + \frac{\delta + NW_j^r(t)}{2\pi r}; N\left(\tau - \frac{iNt}{2\pi r^2}\right)\right).
\end{aligned}
$$

Put this into (2.51), we have

$$
\begin{aligned}
& \widetilde{\mathrm{E}}\left[\vartheta_1\left(\frac{\overline{Z}_\delta^r(t)}{2\pi r}; \tau\right) \prod_{1\le j<k\le N} \vartheta_1\left(\frac{Z_j^r(t) - Z_k^r(t)}{2\pi r}; \tau\right)\right] = C_N^A(\tau) e^{t\sum_{k=1}^N (k-1)^2/2r^2} \\
& \quad \times \det_{1\le j,k\le N}\left[e^{i(k-1)W_j^r(t)/r} \vartheta_1\left(\frac{N-1}{2} + (k-1)\left(\tau - \frac{iNt}{2\pi r^2}\right)\right.\right. \\
& \qquad\qquad \left.\left. + \frac{\delta + NW_j^r(t)}{2\pi r}; N\left(\tau - \frac{iNt}{2\pi r^2}\right)\right)\right] \\
& = \frac{C_N^A(\tau)}{C_N^A(\tau - iNt/2\pi r^2)} e^{(N-1)N(2N-1)t/12r^2} C_N^A\left(\tau - \frac{iNt}{2\pi r^2}\right) \\
& \quad \times \det_{1\le j,k\le N}\left[e^{i(k-1)W_j^r(t)/r} \vartheta_1\left(\frac{N-1}{2} + (k-1)\left(\tau - \frac{iNt}{2\pi r^2}\right)\right.\right. \\
& \qquad\qquad \left.\left. + \frac{\delta + NW_j^r(t)}{2\pi r}; N\left(\tau - \frac{iNt}{2\pi r^2}\right)\right)\right] \\
& = \frac{C_N^A(\tau)}{C_N^A(\tau - iNt/2\pi r^2)} e^{(N-1)N(2N-1)t/12r^2} \\
& \quad \times \vartheta_1\left(\frac{\overline{W}_\delta^r(t)}{2\pi r}; \tau - \frac{iNt}{2\pi r^2}\right) \prod_{1\le j<k\le N} \vartheta_1\left(\frac{W_j^r(t) - W_k^r(t)}{2\pi r}; \tau - \frac{iNt}{2\pi r^2}\right),
\end{aligned}
$$

where (2.25) of Lemma 2.5 was used again. By (2.26),

$$
\begin{aligned}
\frac{C_N^A(\tau)}{C_N^A(\tau - iNt/2\pi r^2)} &= e^{-(N-1)N(3N-2)t/16r^2} \left(\frac{q_0(\tau)}{q_0(\tau - iNt/2\pi r^2)}\right)^{(N-1)(N-2)/2} \\
&= e^{-(N-1)N(3N-2)t/16r^2} \prod_{n=1}^{\infty} \left(\frac{1 - e^{2n\pi i\tau}}{1 - e^{2n\pi i\tau + nNt/r^2}}\right)^{(N-1)(N-2)/2}.
\end{aligned}
$$

If we set $\tau = iNt_*/2\pi r^2$, the equality is obtained. □

For $\boldsymbol{W}^r(t)$, $t \geq 0$, define

$$\mathcal{D}^A_\xi(t, \boldsymbol{W}^r(t)) = \det_{1 \leq j,k \leq N} [\mathcal{M}^A_{\xi,u_k}(t, W^r_j(t))], \quad t \in [0, t_*), \tag{2.53}$$

which we call the *determinantal martingale* [23]. By Lemma 2.10, it is a continuous-time martingale. Then the following equality is established.

Lemma 2.12 *Assume that* $\xi = \sum_{j=1}^N u_j \in \mathfrak{M}_0([0, 2\pi r))$ *and* $\overline{u}_\delta \in (0, 2\pi r)$. *Then*

$$\mathcal{D}^A_\xi(t, \boldsymbol{W}^r(t)) = \frac{h^A_N(t_* - t, \boldsymbol{W}^r(t))}{h^A_N(t_*, \boldsymbol{u})}, \quad t \in [0, t_*).$$

Proof By multilinearity of determinant and independence of $\widetilde{W}_j(\cdot)$, $1 \leq j \leq N$, (2.53) with (2.46) and (2.48) gives

$$\mathcal{D}^A_\xi(t, \boldsymbol{W}^r(t)) = \widetilde{\mathrm{E}}\left[\det_{1 \leq j,k \leq N}\left[\frac{\vartheta_1((\overline{u}_\delta + Z^r_j(t) - u_k)/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1(\overline{u}_\delta/2\pi r; iNt_*/2\pi r^2)} \times \prod_{\substack{1 \leq \ell \leq N,\\ \ell \neq k}} \frac{\vartheta_1((Z^r_j(t) - u_\ell)/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1((u_k - u_\ell)/2\pi r; iNt_*/2\pi r^2)}\right]\right].$$

By Lemma 2.4, it is equal to

$$\widetilde{\mathrm{E}}\left[\frac{\vartheta_1(\overline{Z}^r_\delta(t)/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1(\overline{u}_\delta/2\pi r; iNt_*/2\pi r^2)} \prod_{1 \leq j < k \leq N} \frac{\vartheta_1((Z^r_j(t) - Z^r_k(t))/2\pi r; iNt_*/2\pi r^2)}{\vartheta_1((u_j - u_k)/2\pi r; iNt_*/2\pi r^2)}\right].$$

Then we apply Proposition 2.11. By definition (2.33) of h^A_N, the equality is obtained. $\square$

3 Main results

3.1 Determinantal martingale representation

By Lemmas 2.12 we obtain the following representation. We call it the *determinantal martingale representation* (DMR) for the process $(\Xi^A(t), t \in [0, t_*), \mathbb{P}^A_\eta)$.

Theorem 3.1 *Suppose that* $N \in \mathbb{N}$, $\eta = \sum_{j=1}^N \delta_{v_j}$ *with* (1.21) *and* (1.22). *Let* $T \in [0, t_*)$. *For any* $\mathcal{F}_{\Xi^A}(T)$*-measurable observable* F,

$$\mathbb{E}_{\eta}^{A}\left[F\left(\Xi^{A}(\cdot)\right)\right] = \mathrm{E}_{\boldsymbol{v}}^{r}\left[F\left(\sum_{j=1}^{N}\delta_{W_j^r(\cdot)}\right)\mathcal{D}_{\eta}^{A}(T, \boldsymbol{W}^r(T))\right]$$
$$= \mathbf{E}_{\boldsymbol{v}}^{r}\left[F\left(\sum_{j=1}^{N}\delta_{\Re Z_j^r(\cdot)}\right)\det_{1\le j,k\le N}[\Phi_{\eta,v_k}^{A}(Z_j^r(T))]\right]. \quad (3.1)$$

Note that the second representation of (3.1) is an elliptic extension of the *complex Brownian motion representation* reported in [28] for the Dyson model (*i.e.* the noncolliding BM).

Proof It is sufficient to consider the case that F is given as (2.40). Moreover, by Markov property, it is enough to prove the case $M = 1$; $0 \le t_1 \le T < \infty$. Here we prove the equalities

$$\mathbb{E}_{\eta}^{A}\left[g_1(\boldsymbol{X}^A(t_1))\right] = \mathrm{E}_{\boldsymbol{v}}^{r}\left[g_1(\boldsymbol{W}^r(t_1))\mathcal{D}_{\eta}^{A}(t_1, \boldsymbol{W}^r(t_1))\right]$$
$$= \mathbf{E}_{\boldsymbol{v}}^{r}\left[g_1(\boldsymbol{W}^r(t_1))\det_{1\le j,k\le N}[\Phi_{\eta,v_k}^{A}(Z_j^r(t_1))]\right], \quad (3.2)$$

where g_1 is a symmetric function having periodicity (2.41). By Proposition 2.8,

$$\mathbb{E}_{\eta}^{A}\left[g_1(\boldsymbol{X}^A(t_1))\right] = \check{\mathrm{E}}_{\boldsymbol{v}}\left[g_1(\check{\mathbf{W}}(t_1))\mathbf{1}(T_{\check{\mathbf{W}}} \wedge T_{\overline{W}_\delta} > t_1)\frac{h_N^A(t_* - t_1, \check{\mathbf{W}}(t_1))}{h_N^A(t_*, \boldsymbol{v})}\right]. \quad (3.3)$$

The definition of h_N^A given by (2.33) and the initial condition η give

$$\text{(RHS)} = \check{\mathrm{E}}_{\boldsymbol{v}}\left[g_1(\check{\mathbf{W}}(t_1))\mathbf{1}(T_{\check{\mathbf{W}}} \wedge T_{\overline{W}_\delta} > t_1)\frac{|h_N^A(t_* - t_1, \check{\mathbf{W}}(t_1))|}{h_N^A(t_*, \boldsymbol{v})}\right].$$

By the determinantal formula (2.44) of q_N^A given in Proposition 2.9, the above is written as

$$\mathrm{E}_{\boldsymbol{v}}^{r}\left[\sum_{\sigma\in\mathcal{S}_N}\mathrm{sgn}(\sigma)g_1(\boldsymbol{W}^r(t_1))\mathbf{1}(\sigma(\boldsymbol{W}^r(t_1)) \in \mathcal{A}_{[0,2\pi r)^N})\frac{|h_N^A(t_* - t_1, \boldsymbol{W}^r(t_1))|}{h_N^A(t_*, \boldsymbol{v})}\right], \quad (3.4)$$

where $\boldsymbol{W}^r(t_1) = (W_1^r(t_1), \dots, W_N^r(t_1))$ and the transition density of each W_j^r is given by (2.42), $1 \le j \le N$. Here we used the notation $\sigma(\boldsymbol{x}) = (x_{\sigma(1)}, \dots, x_{\sigma(N)})$ for $\sigma \in \mathcal{S}_N$. Since

$$\mathrm{sgn}(\sigma)\mathbf{1}(\sigma(\boldsymbol{W}^r(t_1)) \in \mathcal{A}_{[0,2\pi r)^N})|h_N^A(t_* - t_1, \boldsymbol{W}^r(t_1))|$$
$$= \mathbf{1}(\sigma(\boldsymbol{W}^r(t_1)) \in \mathcal{A}_{[0,2\pi r)^N})h_N^A(t_* - t_1, \boldsymbol{W}^r(t_1)), \quad \sigma \in \mathcal{S}_N,$$

(3.4) is equal to

$$
\begin{aligned}
&\mathrm{E}_{\boldsymbol{v}}^{r}\left[\sum_{\sigma\in\mathcal{S}_N}\mathbf{1}(\sigma(\boldsymbol{W}^r(t_1))\in\mathcal{A}_{[0,2\pi r)^N})g_1(\boldsymbol{W}^r(t_1))\frac{h_N^A(t_*-t_1,\boldsymbol{W}^r(t_1))}{h_N^A(t_*,\boldsymbol{u})}\right]\\
&\quad=\mathrm{E}_{\boldsymbol{v}}^{r}\left[g_1(\boldsymbol{W}^r(t_1))\frac{h_N^A(t_*-t_1,\boldsymbol{W}^r(t_1))}{h_N^A(t_*,\boldsymbol{u})}\right].
\end{aligned}
$$

Then by Lemma 2.12, we obtain the first line of (3.2). By definitions of $\mathbf{E}_{\boldsymbol{v}}^r$ and $\mathcal{D}_\eta^A$ given by (2.53) with (2.48), the second line of (3.2) is also obtained. □

Remark 2 The function $h_N^A(t_*-t,\boldsymbol{x})$, $t\in[0,t_*)$ is not a harmonic function of $\boldsymbol{x}$, but Lemma 2.10 proves that $\mathcal{D}_\eta^A(t,\boldsymbol{W}^r(t))$ given by (2.53) is a continuous-time martingale, where $\boldsymbol{W}^r(t)$ is a Markov process defined by using Brownian motion in Sect. 2.4. Then Itô's formula implies

$$
\left(\frac{\partial}{\partial t}+\frac{1}{2}\Delta\right)\mathcal{D}_\eta^A(t,\boldsymbol{x})=0,
$$

where $\Delta=\sum_{j=1}^N\partial^2/\partial x_j^2$. In this sense, DMR is a time-dependent extension of h-transform [23].

3.2 Determinantal process

For any integer $M\in\mathbb{N}$, a sequence of times $\boldsymbol{t}=(t_1,\dots,t_M)$ with $0\le t_1<\cdots<t_M<t_*$, and a sequence of functions $\boldsymbol{f}=(f_{t_1},\dots,f_{t_M})\in\mathrm{C}([0,2\pi r))^M$, the *moment generating function* of multitime distribution of $\left(\Xi^A(t),t\in[0,t_*),\mathbb{P}_\xi^A\right)$ is defined by

$$
\Psi_{\xi,\boldsymbol{t}}^A[\boldsymbol{f}]=\mathbb{E}_\xi^A\left[\exp\left\{\sum_{m=1}^M\int_0^{2\pi r}f_{t_m}(x)\Xi(t_m,dx)\right\}\right]. \tag{3.5}
$$

It is expanded with respect to 'test functions' $\chi_{t_m}(\cdot)=e^{f_{t_m}(\cdot)}-1$, $1\le m\le M$ as

$$
\begin{aligned}
\Psi_{\xi,\boldsymbol{t}}^A[\boldsymbol{f}]=&\sum_{\substack{0\le N_m\le N,\\1\le m\le M}}\int_{\prod_{m=1}^M\mathcal{A}_{[0,2\pi r)^{N_m}}}\prod_{m=1}^M\left\{d\boldsymbol{x}_{N_m}^{(m)}\prod_{j=1}^{N_m}\chi_{t_m}\left(x_j^{(m)}\right)\right\}\\
&\times\rho_\xi\left(t_1,\boldsymbol{x}_{N_1}^{(1)};\dots;t_M,\boldsymbol{x}_{N_M}^{(M)}\right),
\end{aligned}
$$

and it defines the *spatio-temporal correlation functions* $\rho_\xi(\cdot)$ for the process $(\Xi^A(t),t\in[0,t_*),\mathbb{P}_\xi^A)$.

Given an integral kernel $\mathbb{K}(s,x;t,y)$, $(s,x),(t,y)\in[0,t_*)\times[0,2\pi r)$, the *Fredholm determinant* is defined as

$$\underset{\substack{(s,t)\in\{t_1,\dots,t_M\}^2,\\ (x,y)\in[0,2\pi r)^2}}{\mathrm{Det}} \Big[\delta_{st}\delta_x(y) + \mathbb{K}(s,x;t,y)\chi_t(y)\Big]$$
$$= \sum_{\substack{0\le N_m\le N,\\ 1\le m\le M}} \sum_{\substack{\boldsymbol{x}_{N_m}^{(m)}\in\mathcal{A}_{[0,2\pi r)^{N_m}},\\ 1\le m\le M}} \prod_{m=1}^{M}\prod_{j=1}^{N_m} \chi_{t_m}\Big(x_j^{(m)}\Big)$$
$$\times \det_{\substack{1\le j\le N_m, 1\le k\le N_n,\\ 1\le m,n\le M}} \Big[\mathbb{K}(t_m, x_j^{(m)}; t_n, x_k^{(n)})\Big]. \tag{3.6}$$

We put the following definition [7,26].

Definition 3.2 For a given initial configuration ξ, if any moment generating function (3.5) is expressed by a Fredholm determinant, we say the process $\big(\Xi^A(t), t \in [0, t_*), \mathbb{P}_\xi^A\big)$ is determinantal. In this case, all spatio-temporal correlation functions are given by determinants as

$$\rho_\xi\Big(t_1, \boldsymbol{x}_{N_1}^{(1)}; \dots; t_M, \boldsymbol{x}_{N_M}^{(M)}\Big) = \det_{\substack{1\le j\le N_m, 1\le k\le N_n,\\ 1\le m,n\le M}} \Big[\mathbb{K}_\xi(t_m, x_j^{(m)}; t_n, x_k^{(n)})\Big], \tag{3.7}$$

$0 \le t_1 < \dots < t_M < t_*$, $1 \le m \le M$, $1 \le N_m \le N$, $\boldsymbol{x}_{N_m}^{(m)} \in [0, 2\pi r)^{N_m}$, $1 \le m \le M \in \mathbb{N}$. Here the integral kernel $\mathbb{K}_\xi : ([0, t_*) \times [0, 2\pi r))^2 \mapsto \mathbb{R}$ is called the (spatio-temporal) correlation kernel.

By Theorem 1.3 in [23], DMR given by Theorem 3.1 leads to the following result.

Corollary 3.3 *For $\eta = \sum_{j=1}^N \delta_{v_j}$ with (1.21) and (1.22), the process $(\Xi^A(t), t \in [0, t_*), \mathbb{P}_\eta^A)$ is determinantal with the correlation kernel*

$$\mathbb{K}_\eta^A(s, x; t, y) = \mathbb{K}_\eta^A(s, x; t, y; N, r, t_*)$$
$$= \int_0^{2\pi r} \eta(du)\, p_{A_{N-1}}^r(s, x|u)\mathcal{M}_{\eta,u}^A(t, y) - \mathbf{1}(s>t)\, p_{A_{N-1}}^r(s-t, x|y), \tag{3.8}$$

$(s, x), (t, y) \in [0, t_*) \times [0, 2\pi r)$.

3.3 Explicit expression of $\mathbb{K}_\eta^A$ and infinite-particle limit

For $\eta = \sum_{j=1}^N \delta_{v_j}$ with (1.21) and (1.22), the entire functions (2.46) become

$$\begin{aligned}
\Phi^A_{\eta,v_k}(z;N,r,t_*) &= \frac{\vartheta_1(z/2\pi r-(k-1)/N+1/2;\tau)}{\vartheta_1(1/2;\tau)} \\
&\quad\times\prod_{\substack{1\le\ell\le N,\\ \ell\ne k}}\frac{\vartheta_1(z/2\pi r-(\ell-1)/N;\tau)}{\vartheta_1((k-\ell)/N;\tau)} \\
&= \frac{\vartheta_1(z/2\pi r-(k-1)/N+1/2;\tau)}{\vartheta_1(1/2;\tau)} \\
&\quad\times\prod_{n=1}^{N-1}\frac{\vartheta_1(z/2\pi r-(k-1)/N+n/N;\tau)}{\vartheta_1(n/N;\tau)},
\end{aligned}$$

$1 \le k \le N$, with $\tau = \tau(t_*) = iNt_*/2\pi r^2$, where we have used (2.3). Using the formulas (2.6) and (2.7), it is written as

$$\begin{aligned}
\Phi^A_{\eta,v_k}(z;N,r,t_*) &= \frac{\pi}{N\vartheta_1''(0;N\tau)}\vartheta_1(N\{z/2\pi r-(k-1)/N\};N\tau) \\
&\quad\times\frac{\vartheta_1(z/2\pi r-(k-1)/N+1/2;\tau)\vartheta_1'(0;\tau)}{\pi\vartheta_1(z/2\pi r-(k-1)/N;\tau)\vartheta_1(1/2;\tau)},
\end{aligned} \tag{3.9}$$

$1 \le k \le N$. If we apply the formula (2.9) and the Laurent expansion (1.2) of ϑ_1, we have

$$\begin{aligned}
\Phi^A_{\eta,v_k}(z;N,r,t_*) &= \frac{2\pi e^{N\pi i\tau/4}}{N\vartheta_1'(0;N\tau)} \\
&\times\Bigg[\cos(z/2r-(k-1)\pi/N)\sum_{n=1}^{\infty}(-1)^{n-1}e^{N\pi i\tau n(n-1)} \\
&\quad\times\frac{\sin\Big[(2n-1)N\{z/2r-(k-1)\pi/N\}\Big]}{\sin(z/2r-(k-1)\pi/N)} \\
&\quad-4\sum_{n=1}^{\infty}(-1)^{n-1}e^{N\pi i\tau n(n-1)}\sum_{\ell=1}^{\infty}\frac{e^{2\pi i\tau\ell}}{1+e^{2\pi i\tau\ell}} \\
&\quad\times\sin\Big[(2n-1)N\{z/2r-(k-1)\pi/N\}\Big]\sin\Big[2\ell\{z/2r-(k-1)\pi/N\}\Big]\Bigg],
\end{aligned}$$

since $\cot(\pi/2)=0$ and $\sin(\theta+m\pi)=(-1)^m\sin\theta$, $m\in\mathbb{Z}$. For $M\in\mathbb{N}$, let

$$\sigma_M(m) = \begin{cases} m, & \text{if } M \text{ is odd},\\ m-1/2, & \text{if } M \text{ is even}.\end{cases} \tag{3.10}$$

It is easy to confirm the equality

$$\frac{\sin(Mx)}{\sin x} = \sum_{\substack{m\in\mathbb{Z},\\ |\sigma_M(m)|\le (M-1)/2}} e^{2i\sigma_M(m)x}.$$

Then we see

$$\begin{aligned}
\Phi^A_{\eta,v_k}(z;N,r,t_*) &= \frac{2\pi e^{N\pi i\tau/4}}{N\vartheta_1'(0;N\tau)} \\
&\times \Bigg[\sum_{n=1}^{\infty}(-1)^{n-1}e^{N\pi i\tau n(n-1)}\Bigg\{\cos\Big[(2n-1)N\{z/2r-(k-1)\pi/N\}\Big] \\
&+ \sum_{\substack{m\in\mathbb{Z},\\ |\sigma_{N-1}(m)|\le\{(2n-1)N-2\}/2}} e^{2i\sigma_{N-1}(m)\{z/2r-(k-1)\pi/N\}}\Bigg\} \\
&+2\sum_{n\in\mathbb{Z}}(-1)^{n-1}e^{N\pi i\tau n(n-1)}\sum_{\ell=1}^{\infty}\frac{e^{2\pi i\tau\ell}}{1+e^{2\pi i\ell}} \\
&\times\cos\Big[\{(2n-1)N+2\ell\}\{z/2r-(k-1)\pi/N\}\Big]\Bigg].
\end{aligned}$$

Here we have used the fact that, for $M\in\mathbb{N}$, $2\sigma_M(m)+1=2\sigma_{M-1}(m+1)$ if M is odd, and $2\sigma_M(m)+1=2\sigma_{M-1}(m)$ if M is even, and that $\sigma_{(2n-1)N-1}(m)=\sigma_{N-1}(m)$ for $n,N\in\mathbb{N}$.

The functions

$$\begin{aligned}
\mathcal{M}^A_{\eta,v_k}(t,x;N,r,t_*) &= \widetilde{\mathrm{E}}[\Phi^A_{\eta,v_k}(x+i\widetilde{W}(t);N,r,t_*)] \\
&= \int_{\mathbb{R}} d\widetilde{w}\,\frac{e^{-\widetilde{w}^2/2t}}{\sqrt{2\pi t}}\Phi^A_{\eta,v_k}(x+i\widetilde{w};N,r,t_*),\quad 1\le k\le N,
\end{aligned}$$

which give continuous-time martingales if we put $x=W^r_j(\cdot)$, $1\le j\le N$ (Lemma 2.10 (i)), are calculated by performing Gaussian integrals for each term. By setting $\tau=iNt_*/2\pi r^2$, the result is expressed as

$$\begin{aligned}
\mathcal{M}^A_{\eta,v_k}(t,x;N,r,t_*) &= \frac{2\pi}{N\vartheta_1'(0;iN^2t_*/2\pi r^2)} \\
&\times\Bigg[\sum_{n=1}^{\infty}(-1)^{n-1}e^{-(n-(1/2))^2N^2(t_*-t)/2r^2}\cos\Big[(2n-1)N\{x/2r-(k-1)\pi/N\}\Big] \\
&+\sum_{n=1}^{\infty}(-1)^{n-1}e^{-(n-(1/2))^2N^2t_*/2r^2}
\end{aligned}$$

$$
\begin{aligned}
&\times \sum_{\substack{m\in\mathbb{Z},\\ |\sigma_{N-1}(m)|\le\{(2n-1)N-2\}/2}} e^{\sigma_{N-1}(m)^2 t/2r^2+2i\sigma_{N-1}(m)\{x/2r-(k-1)\pi/N\}} \\
&+2\sum_{n\in\mathbb{Z}}(-1)^{n-1}e^{-(n-(1/2))^2N^2(t_*-t)/2r^2}\sum_{\ell=1}^{\infty}\frac{e^{-\ell N t_*/r^2}}{1+e^{-\ell N t_*/r^2}}e^{\ell\{\ell+(2n-1)N\}t/2r^2} \\
&\times\cos\Big[\{(2n-1)N+2\ell\}\{x/2r-(k-1)\pi/N\}\Big]\Bigg]. \qquad (3.11)
\end{aligned}
$$

Then by Corollary 3.3, the correlation kernel is determined as

$$
\mathbb{K}^A_\eta(s,x;t,y;N,r,t_*)=\mathcal{G}^A_\eta(s,x;t,y;N,r,t_*)-\mathbf{1}(s>t)p^r_{A_{N-1}}(s-t,x|y),
$$

$(s,x),(t,y)\in[0,\infty)\times[0,2\pi r)$, where

$$
\mathcal{G}^A_\eta(s,x;t,y;N,r,t_*)=\sum_{k=1}^{N}p^r_{A_{N-1}}(s,x|v_k)\mathcal{M}^A_{\eta,v_k}(t,y;N,r,t_*)
$$

with (2.43). We note that, by using (3.10), (2.43) is written as

$$
p^r_{A_{N-1}}(t,y|x)=\frac{1}{2\pi r}\sum_{\ell\in\mathbb{Z}}e^{-\sigma_{N-1}(\ell)^2t/2r^2+i\sigma_{N-1}(\ell)(y-x)/r},\quad t\ge 0,\quad x,y\in[0,2\pi r). \qquad (3.12)
$$

We find that, by using the identity $\sum_{k=1}^N e^{-2i(k-1)\alpha\pi/N}=N\sum_{k\in\mathbb{Z}}\mathbf{1}(\alpha=kN)$, the above is expressed as follows,

$$
\begin{aligned}
&\mathcal{G}^A_\eta(s,x;t,y;N,r,t_*)=\frac{1}{\vartheta_1'(0;iN^2t_*/2\pi r^2)r} \\
&\times\Bigg[\frac{1}{2}\sum_{k\in\mathbb{Z}}e^{-k^2N^2s/2r^2+ikNx/r}\vartheta_2\left(\frac{N(y-x)}{2\pi r}-\frac{ikN^2s}{2\pi r^2};\frac{iN^2\{t_*-(t-s)\}}{2\pi r^2}\right) \\
&+\sum_{n=1}^{\infty}(-1)^{n-1}e^{-(n-(1/2))^2N^2t_*/2r^2}\sum_{\substack{m\in\mathbb{Z},\\ |\sigma_{N-1}(m)|\le\{(2n-1)N-2\}/2}} e^{\sigma_{N-1}(m)^2(t-s)/2r^2+i\sigma_{N-1}(m)(y-x)/r} \\
&\times\vartheta_3\left(\frac{Nx}{2\pi r}-\frac{iN\sigma_{N-1}(m)s}{2\pi r^2};\frac{iN^2s}{2\pi r^2}\right) \\
&+2\sum_{k\in\mathbb{Z}}e^{-k^2N^2s/2r^2+ikNx/r}\sum_{\ell=1}^{\infty}\sum_{n\in\mathbb{Z}}(-1)^{n-1}\frac{e^{-\ell N t_*/r^2}}{1+e^{-\ell N t_*/r^2}}e^{-(n-(1/2))^2N^2t_*/2r^2} \\
&\times e^{\{(n-(1/2))N+\ell\}^2(t-s)/2r^2}\cos\left[\{(2n-1)N+2\ell\}\pi\left(\frac{y-x}{2\pi r}-\frac{ikNs}{2\pi r^2}\right)\right]\Bigg], \qquad (3.13)
\end{aligned}
$$

where ϑ_2 and ϑ_3 are defined by (2.5).

We consider the infinite-particle limit with fixed particle-density;

$$N \to \infty, \quad r \to \infty \quad \text{with} \quad \rho = \frac{N}{2\pi r} = \text{const.} \tag{3.14}$$

We can see that the first term in (3.13) vanishes in this limit because of the overall factor $1/r$, and obtain

$$\begin{aligned}
\mathbf{G}_\eta^A(s, x; t, y; \rho, t_*) &\equiv \lim_{\substack{N\to\infty, r\to\infty,\\ \rho=\text{const.}}} \mathcal{G}_\eta^A(s, x; t, y; N, r, t_*) \\
&= \frac{2\pi}{\vartheta_1'(0; 2\pi i\rho^2 t_*)} \Bigg[\sum_{n=1}^{\infty} (-1)^{n-1} e^{-2\pi^2(n-(1/2))^2\rho^2 t_*} \\
&\quad \times \int_{|v|\le(2n-1)\rho} dv\, e^{\pi^2 v^2(t-s)/2 + i\pi v(y-x)} \vartheta_3\Big(\rho x - i\pi v\rho s; 2\pi i\rho^2 s\Big) \\
&\quad + \sum_{k\in\mathbb{Z}} e^{-2\pi^2 k^2\rho^2 s + 2\pi k\rho x} \sum_{n\in\mathbb{Z}} (-1)^{n-1} \int_0^\infty dv\, \frac{e^{-2\pi^2 v\rho t_*}}{1+e^{-2\pi^2 v\rho t_*}} \\
&\quad \times e^{\pi^2\{(2n-1)\rho+v\}^2(t-s)/2} \cos\Big[\{(2n-1)\rho+v\}\pi(y-x-2\pi ik\rho s)\Big]\Bigg].
\end{aligned} \tag{3.15}$$

The correlation kernel in this limit (3.14) is given by

$$\mathbf{K}_\eta^A(s, x; t, y; \rho, t_*) = \mathbf{G}_\eta^A(s, x; t, y; \rho, t_*) - \mathbf{1}(s > t) p_{\text{BM}}(s-t, x|y), \tag{3.16}$$

$(s, x), (t, y) \in [0, t_*) \times \mathbb{R}$. The convergence of correlation kernel (3.15) in the limit (3.14) implies well-definedness of elliptic determinantal process with an infinite number of particles. Further study will be reported elsewhere (see Sect. 4.3 below).

4 Reduced processes in temporally homogeneous limit

4.1 Reduction in SDEs

Since we have

$$\lim_{t_*\to\infty} A_N^{2\pi r}(t_* - t, x) = \frac{1}{2r} \cot\left(\frac{x}{2r}\right) \tag{4.1}$$

by (2.15), the limit $t_* \to \infty$ of (1.17) gives a temporally homogeneous system of the SDEs,

$$d\check{X}_j^A(t) = dB_j(t) + \frac{1}{2r} \sum_{\substack{1\le k\le N,\\ k\ne j}} \cot\left(\frac{\check{X}_j^A(t) - \check{X}_k^A(t)}{2r}\right) dt + \frac{1}{2r} \cot\left(\frac{\overline{X}_\delta^A(t)}{2r}\right) dt, \quad t \ge 0, \tag{4.2}$$

$1 \le j \le N$, and (1.18) becomes

$$d\overline{X}_{\delta}^{A}(t) = \sqrt{N} dB(t) + \frac{N}{2r} \cot\left(\frac{\overline{X}_{\delta}^{A}(t)}{2r}\right) dt, \quad t \ge 0. \tag{4.3}$$

The system of SDEs without the third term in RHS of (4.2) has been studied as a dynamical extension of the *circular unitary ensemble (CUE)* of random matrix theory [15,20,34]. It is interesting to see that a one-parameter extension of that system is discussed as a driving system for a multiple Schramm-Loewner evolution by Cardy [9]. Moreover, for

$$\lim_{r\to\infty} \lim_{t_*\to\infty} A_N^{2\pi r}(t_* - t, x) = \lim_{r\to\infty} \frac{1}{2r} \cot\left(\frac{x}{2r}\right) = \frac{1}{x},$$

and $\delta = \pi r n$ with a fixed $n \in \mathbb{Z}$ determined by the initial configuration, the $r \to \infty$ limit of the system (4.2) is given by

$$dX_j^A(t) = dB_j(t) + \sum_{\substack{1\le k\le N, \\ k\neq j}} \frac{1}{X_j^A(t) - X_k^A(t)} dt, \quad 1 \le j \le N, \quad t \ge 0, \tag{4.4}$$

where $X_j^A = \check{X}_j^A$, $1 \le j \le N$ in $r \to \infty$. It is the system of SDEs of the Dyson model (*i.e.*, the noncolliding Brownian motion on $\mathbb{R}$).

4.2 Reduction in correlation kernel of determinantal process

Next we study the reduction as determinantal processes for $(\Xi^A(t), t \in [0, t_*), \mathbb{P}_\eta^A)$. For the configuration $\eta = \sum_{j=1}^N \delta_{v_j}$ with (1.21) and (1.22), an explicit expression of correlation kernel $\mathbb{K}_\eta^A$ was given in Sect. 3.3. In (3.13) we can take the temporally homogeneous limit $t_* \to \infty$ as follows. We see from (2.8) and (2.5) that $\vartheta_1'(0; \tau) \sim 2\pi e^{\pi i\tau/4}$, $\vartheta_2(v; \tau) \sim 2e^{\pi i\tau/4}\cos(\pi v)$, and $\vartheta_3(v; \tau) \to 1$ as $\Im\tau \to +\infty$. Then, we have

$$\begin{aligned}
\widehat{\mathcal{G}}_\eta^A(s, x; t, y; N, r) &\equiv \lim_{t_*\to\infty} \mathcal{G}_\eta^A(s, x; t, y; N, r, t_*) \\
&= \sum_{k\in\mathbb{Z}} e^{-k^2N^2s/2r^2 + ikNx/r} e^{N^2(t-s)/8r^2} \mathcal{K}^A\left((y-x) - \frac{ikNs}{r}; N, r\right) \\
&\quad + \frac{1}{2\pi r} \sum_{\substack{m\in\mathbb{Z}, \\ |\sigma_{N-1}(m)|\le(N-2)/2}} e^{\sigma_{N-1}(m)^2(t-s)/2r^2 + i\sigma_{N-1}(m)(y-x)/r} \\
&\quad \times \vartheta_3\left(\frac{Nx}{2\pi r} - \frac{iN\sigma_{N-1}(m)s}{2\pi r^2}; \frac{iN^2s}{2\pi r}\right),
\end{aligned} \tag{4.5}$$

where

$$\mathcal{K}^A(x; N, r) = \frac{1}{2\pi r} \cos\left(\frac{Nx}{2r}\right).$$

The correlation kernel is given by

$$\widehat{\mathbb{K}}^A_\eta(s, x; t, y; N, r) = \widehat{\mathcal{G}}^A_\eta(s, x; t, y; N, r) - \mathbf{1}(s > t) p^r_{A_{N-1}}(s - t, x|y), \tag{4.6}$$

$(s, x), (t, y) \in [0, \infty) \times [0, 2\pi r)$. The determinantal process defined by the correlation kernel (4.6) is denoted by $(\widehat{\Xi}^A(t), t \in [0, \infty), \widehat{\mathbb{P}}^A_\eta)$. From the explicit expression (4.5), we can show that $\left(\widehat{\Xi}^A(t), t \in [0, \infty)\right), \widehat{\mathbb{P}}^A_\eta)$ exhibits a typical nonequilibrium phenomenon; *relaxation to equilibrium*.

Let $(\widehat{\Xi}^A(t), t \in [0, \infty), \widehat{\mathbb{P}}^A_{\text{eq}})$ be the equilibrium determinantal process, whose correlation kernel is homogeneous both in space $[0, 2\pi r)$ and time $[0, \infty)$ and given by

$$\begin{aligned}
&\widehat{\mathbb{K}}^A_{\text{eq}}(t - s, y - x; N, r) = \widehat{\mathcal{G}}^A_{\text{eq}}(t - s, y - x; N, r) - \mathbf{1}(s > t) p^r_{A_{N-1}}(s - t, x|y) \\
&= \begin{cases}
\dfrac{1}{2\pi r} \displaystyle\sum_{\substack{\ell \in \mathbb{Z}, \\ |\sigma_{N-1}(\ell)| \leq (N-2)/2}} e^{\sigma_{N-1}(\ell)^2 (t-s)/2r^2 + i\sigma_{N-1}(\ell)(y-x)/r} + e^{N^2(t-s)/8r^2} \mathcal{K}^A(y - x; N, r), \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{if } s < t, \\
\dfrac{1}{2\pi r} \dfrac{\sin[(N-1)(y-x)/2r]}{\sin[(y-x)/2r]} + \mathcal{K}^A(y - x; N, r), \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{if } s = t, \\
-\dfrac{1}{2\pi r} \displaystyle\sum_{\substack{\ell \in \mathbb{Z}, \\ |\sigma_{N-1}(\ell)| > (N-2)/2}} e^{\sigma_{N-1}(\ell)^2 (t-s)/2r^2 + i\sigma_{N-1}(\ell)(y-x)/r} + e^{N^2(t-s)/8r^2} \mathcal{K}^A(y - x; N, r), \\
\qquad\qquad\qquad\qquad\qquad\qquad \text{if } s > t,
\end{cases}
\end{aligned} \tag{4.7}$$

$(s, x), (t, y) \in [0, \infty) \times [0, 2\pi r)$. Note that the spatial dependence of (4.7) is on a distance of two points $|y - x|$.

Proposition 4.1 *The process* $(\widehat{\Xi}^A(t + T), t \in [0, \infty), \widehat{\mathbb{P}}^A_\eta)$ *converges to* $(\widehat{\Xi}^A(t), t \in [0, \infty), \widehat{\mathbb{P}}^A_{\text{eq}})$ *as* $T \to \infty$ *weakly in the sense of finite-dimensional distributions.*

Note that if we ignore the terms containing $\mathcal{K}^A$, the kernel (4.7) is equal to the correlation kernel of the equilibrium determinantal process of the noncolliding BM on a circle $S^1(r)$ with $N - 1$ particles, $\mathbb{K}^{\text{CUE}}_{\text{eq}}(t - s, y - x; N - 1, r)$ [23,36]. It is a reversible process with respect to the eigenvalue-angle distribution of unitary random matrices with size $N - 1$ in the CUE (if we set $r = 1$). With contribution from the term $\mathcal{K}^A$, we have the particle density

$$
\begin{aligned}
\rho = \rho(N, r) &= \lim_{|y-x|\to 0} \widehat{\mathbb{K}}^{A}_{\mathrm{eq}}(0, y-x; N, r) \\
&= \lim_{|y-x|\to 0} \mathbb{K}^{\mathrm{CUE}}_{\mathrm{eq}}(0, y-x; N-1, r) + \lim_{|y-x|\to 0} \mathcal{K}^{A}(y-x; N, r) \\
&= \frac{N-1}{2\pi r} + \frac{1}{2\pi r} = \frac{N}{2\pi r},
\end{aligned}
$$

which is uniform on $[0, 2\pi r)$ as it should be.

Proof of Proposition 4.1 Since $e^{-k^2 N^2 T/2r^2} \to 0, k \neq 0$ and $\vartheta_3(v; iN^2 T/2\pi r^2) \to 1$ as $T \to \infty$, (4.5) gives

$$
\begin{aligned}
&\lim_{T\to\infty} \widehat{\mathcal{G}}^{A}_{\eta}(s+T, x; t+T, y; N, r) \\
&= e^{N^2(t-s)/8r^2} \mathcal{K}^{A}(y-x; N, r) \\
&\quad + \frac{1}{2\pi r} \sum_{\substack{\ell\in\mathbb{Z}, \\ |\sigma_{N-1}(\ell)|\le (N-2)/2}} e^{\sigma_{N-1}(\ell)^2 (t-s)/2r^2 + i\sigma_{N-1}(\ell)(y-x)/r},
\end{aligned}
$$

which is $\widehat{\mathcal{G}}^{A}_{\mathrm{eq}}(t-s, y-x; N, r)$. If we set $t = s$, the second term becomes

$$
\frac{1}{2\pi r} \sum_{\substack{\ell\in\mathbb{Z}, \\ |\sigma_{N-1}(\ell)|\le (N-2)/2}} e^{i\sigma_{N-1}(\ell)(y-x)/r} = \frac{1}{2\pi r} \frac{\sin[(N-1)(y-x)/2r]}{\sin[(y-x)/2r]}.
$$

Combining the above results with the expression (3.12) of $p^{r}_{A_{N-1}}$, we obtain (4.7). The convergence of correlation kernel $\widehat{\mathbb{K}}^{A}_{\eta} \to \widehat{\mathbb{K}}^{A}_{\mathrm{eq}}$ in the long-term limit guarantees the convergence of moment generating function of multitime distribution $\Psi^{A}_{\eta,\boldsymbol{t}}[\boldsymbol{f}]$ given by the Fredholm determinant (3.6) of $\widehat{\mathbb{K}}^{A}_{\eta}$. It implies the convergence $\widehat{\mathbb{P}}^{A}_{\eta} \to \widehat{\mathbb{P}}^{A}_{\mathrm{eq}}$ in the long-term limit in the sense of finite-dimensional distributions. Then the proof is completed. □

4.3 Reduction in infinite-particle system

The temporally homogeneous limit $t_* \to \infty$ of (3.16) with (3.15) gives

$$
\begin{aligned}
\widehat{\mathbf{K}}^{A}_{\eta}(s, x; t, y; \rho) &\equiv \lim_{t_*\to\infty} \mathbf{K}^{A}_{\eta}(s, x; t, y; \rho, t_*) \\
&= \int_{|v|\le\rho} dv\, e^{\pi^2 v^2 (t-s)/2 + i\pi v(y-x)} \vartheta_3\Big(\rho x - i\pi v\rho s; 2\pi i \rho^2 s\Big) \\
&\quad - \mathbf{1}(s > t) p_{\mathrm{BM}}(s-t, x|y),
\end{aligned} \tag{4.8}
$$

$(s, x), (t, y) \in [0, \infty) \times \mathbb{R}$. If we set $\rho = 1$, this correlation kernel is exactly the same as Eq. (1.5) in [26]. There we gave a set $\mathfrak{X}$ of infinite-particle configurations, started at which the Dyson model is well defined as a determinantal process. The function (4.8) was derived as the correlation kernel of the Dyson model started at $\xi^{\mathbb{Z}} \equiv \sum_{j\in\mathbb{Z}} \delta_j \in \mathfrak{X}$. As shown in [26], for $(s, x), (t, y) \in [0, \infty) \times \mathbb{R}$,

$$\begin{aligned}
&\lim_{T\to\infty} \widehat{\mathbf{K}}_{\eta}^{A}(s+T, x; t+T, y; \rho) \\
&= \mathbf{K}_{\sin}(t-s, y-x; \rho) \equiv \begin{cases} \displaystyle\int_0^{\rho} dv\, e^{\pi^2 v^2 (t-s)/2} \cos[\pi v(y-x)], & \text{if } s < t, \\[2ex] \displaystyle\frac{\sin[\pi\rho(y-x)]}{\pi(y-x)}, & \text{if } s = t, \\[2ex] \displaystyle -\int_{\rho}^{\infty} dv\, e^{\pi^2 v^2 (t-s)/2} \cos[\pi v(y-x)], & \text{if } s > t. \end{cases}
\end{aligned}$$

The limit kernel is the *extended sine kernel* with density ρ [15]. The convergence implies the infinite-dimensional relaxation phenomenon of the Dyson model from $\xi^{\mathbb{Z}}$ to equilibrium [26]. This observation is consistent with Proposition 4.1, since $\lim_{N\to\infty, r\to\infty, \rho=\text{const.}} \mathcal{K}^A(x; N, r) = 0$.

5 Expressions by Gosper's q-sine function and hyperbolic limit

Gosper defined his q-sine function as [17]

$$\sin_q(\pi z) = q^{(z-1/2)^2} \frac{(q^{2z}; q^2)_\infty (q^{2-2z}; q^2)_\infty}{(q; q^2)_\infty^2}, \quad 0 < q < 1.$$

By the product form (2.1) of $\vartheta_1(v; \tau)$, we find the equality

$$\vartheta_1(v; \tau) = -i(q; q^2)_\infty^2 (q^2; q^2)_\infty e^{-2\pi i v} \sin_q(-\pi v/\tau)$$

with (1.1). Then the function $A_N^{\alpha}(t_* - t, x)$ defined by (1.4) is expressed as

$$A_N^{\alpha}(t_* - t, x) = \frac{i\alpha}{2N(t_*-t)} \frac{d}{dz} \log(\sin_q z)\Big|_{z = i\alpha x/2N(t_*-t)} - \frac{2\pi i}{\alpha}, \quad t \in [0, t_*), \quad x \in \mathbb{R}, \tag{5.1}$$

with

$$q = e^{-2\pi^2 N(t_*-t)/\alpha^2}. \tag{5.2}$$

The entire functions (2.46), with which the determinantal martingales and correlation functions are expressed, are rewritten as

$$\begin{aligned}
\Phi_{\xi, u_k}^{A}(z; N, r, t_*) &= e^{-iN(z-u_k)/r} \frac{\sin_q(\pi i(\overline{u}_\delta + z - u_k) r/N t_*)}{\sin_q(\pi i \overline{u}_\delta r/N t_*)} \\
&\quad \times \prod_{\substack{1\le \ell \le N, \\ \ell \ne k}} \frac{\sin_q(\pi i(z - u_\ell) r/N t_*)}{\sin_q(\pi i(u_k - u_\ell) r/N t_*)},
\end{aligned} \tag{5.3}$$

$1 \leq k \leq N, z \in \mathbb{C}$, with $q = e^{-N(t_*-t)/2r^2}$. We remark that, the q-extension of the gamma function is defined as [1]

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q;q)_\infty}{(q^z;q)_\infty}, \quad 0 < q < 1,$$

and the q-analogue of Euler's reflection formula

$$\sin_q(\pi z) = q^{1/4} \Gamma_{q^2}(1/2)^2 \frac{(q^2)^{\binom{z}{2}}}{\Gamma_{q^2}(z)\Gamma_{q^2}(1-z)} \tag{5.4}$$

holds. Then (5.1) and (5.3) are also expressed using the q-gamma function Γ_{q^2}.

In the previous section, we studied the temporally homogeneous limit $t_* \to \infty$ with fixed $\alpha = 2\pi r$. By (5.2), it corresponded to the limit $q \to 0$. In contrast to it, here we consider the following scaling limit,

$$t_* \to \infty, \quad \alpha \to \infty \quad \text{with} \quad \frac{t_*}{\alpha} = a = \text{const.}$$

By (5.2), it gives the limit $q \to 1$. Since $\lim_{q \uparrow 1} \sin_q(\pi z) = \sin(\pi z)$, and $\sin(\pi i z) = i \sinh(\pi z)$, we obtain the limits

$$A_N(x;a) \equiv \lim_{\substack{t_* \to \infty, \alpha \to \infty, \\ t_*/\alpha = a}} A_N^\alpha(t_* - t, x) = \frac{1}{2Na} \coth\left(\frac{x}{2Na}\right), \tag{5.5}$$

$$\begin{aligned} \Phi^A_{\xi,u_k}(z;N,a) &\equiv \lim_{\substack{t_* \to \infty, r \to \infty, \\ t_*/r = 2\pi a}} \Phi^A_{\xi,u_k}(z;N,r,t_*) \\ &= \frac{\sinh[(\overline{u}_\delta + z - u_k)/2Na]}{\sinh(\overline{u}_\delta/2Na)} \prod_{\substack{1 \leq \ell \leq N, \\ \ell \neq k}} \frac{\sinh[(z-u_\ell)/2Na]}{\sinh[(u_k - u_\ell)/2Na]}, \\ &\qquad 1 \leq k \leq N, \ z \in \mathbb{C}. \end{aligned} \tag{5.6}$$

In this scaling limit the SDE (1.7) discussed in Sect. 1 becomes a temporally homogeneous process on $\mathbb{R}_+$

$$dX(t) = dB(t) + \frac{1}{2a} \coth\left(\frac{X(t)}{2a}\right), \quad t \in [0, \infty).$$

Similarly (5.5) and (5.6) enable us to discuss the determinantal process of a hyperbolic version of the Dyson model such that

$$dX_j^A(t) = dB_j(t) + \frac{1}{2Na} \sum_{\substack{1 \leq k \leq N, \\ k \neq j}} \coth\left(\frac{X_j^A(t) - X_k^A(t)}{2Na}\right) dt + \frac{1}{2Na} \coth\left(\frac{\overline{X}_\delta^A(t)}{2Na}\right) dt, \tag{5.7}$$

$1 \leq j \leq N, t \geq 0$, summation of which over $j = 1, 2, \ldots, N$ gives

$$d\overline{X}_{\delta}^{A}(t) = \sqrt{N} dB(t) + \frac{1}{2a} \coth\left(\frac{\overline{X}_{\delta}^{A}(t)}{2Na}\right) dt. \quad t \geq 0, \tag{5.8}$$

Note that $\lim_{q \uparrow 1} \Gamma_q(z) = \Gamma(z)$ and in this limit (5.4) with $z \to iz$ gives $\sinh(\pi z) = -\pi i / \{\Gamma(iz)\Gamma(1-iz)\}$, and thus the above limit is expressed also by using the gamma functions. If we take the further limit $a \to \infty$ with $\delta/a =$ constant > 0 in (5.7), we obtain the Dyson model (4.4). In the argument in the present paper, analyticity of functions $\Phi_{\xi, u_k}^{A}(z)$, $1 \leq k \leq N$ plays an essential role to determine correlation kernels of determinantal processes. Since (5.3) is an entire function of z, the calculations and results for the trigonometric version of the Dyson model given in Sect. 4.2 will be readily mapped to those for the hyperbolic versions with (5.7) and (5.8) by analytic continuation of z and suitable change of parameters.

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