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ERRATUM

Erratum to: Martingale transforms and L^p-norm estimates of Riesz transforms on complete Riemannian manifolds

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In this Erratum, we correct an error in the representation formulas of the Riesz transforms and a gap in the original proof of the L^p-norm estimates of the Riesz transforms obtained in the original paper. We would like to point out that, based on the correct representation formulas of the Riesz transforms stated below, and using a new martingale subordination inequality due to Bañuelos and Osekowski (Theorem 2.2 in [2]), we can correct the gap in the original paper without major change in the line of our original argument and to arrive at the original estimates obtained in the original paper.

As pointed out by [1], various formulas in the original paper of the form $\int_0^\tau e^{a(s-\tau)} M_\tau M_s^{-1} \alpha_s dB_s$ need to be rewritten as $e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} \alpha_s dB_s$ since $e^{-a\tau} M_\tau$ is not $\mathcal{F}_s$-measurable. In view of this, the correct representation formula of the Riesz transforms in Theorem 3.2 in the original paper should be written as follows

$$R_a(L)f(x) = -2 \lim_{y \to +\infty} E_y \left[e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} dQ_a f(X_s, s) dB_s \,\middle|\, X_\tau = x \right]. \tag{1}$$

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Indeed, a careful check of the original proof of Lemma 3.7 in the original paper implies that the correct reformulation of Lemma 3.7 in the original paper should be given by

$$\eta(X_\tau) = e^{a\tau} M_\tau^{*,-1} \eta_a(X_0, B_0) + e^{a\tau} M_\tau^* \int_0^\tau e^{-as} M_s^* \left(\nabla, \frac{\partial}{\partial y}\right) \eta_a(X_s, B_s) \cdot (U_s dW_s, dB_s), \tag{2}$$

and a small modification of the original proof of Theorem 3.8 given in the original paper implies that the correct formulations of Theorem 3.8 in the original paper should be given by

$$\frac{1}{2}\omega(x) = \lim_{y\to\infty} E_y \left[e^{-a\tau} M_\tau \int_0^\tau e^{as} M_s^{-1} \frac{\partial}{\partial y} \omega_a(X_s, B_s) dB_s \middle| X_\tau = x \right]. \tag{3}$$

By the same argument as used in the original paper, one can derive that the correct formulation of the Riesz transform should be given by (1).

The original statement and the original proof of Theorem 1.4 in the original paper remain valid in the case of complete steady or expanding Ricci gradient solitons $Ric + \nabla^2\phi = -a$, where $a \geq 0$ is a constant. In particular, on complete Riemannian manifolds with $Ric + \nabla^2\phi = 0$, it holds $\|R(L)\|_{p,p} \leq 2(p^* - 1)$ for all $p \in (1, \infty)$. This extends the known result from Euclidean spaces to complete steady gradient Ricci solitons.

In general, the original proof of the L^p-estimates in Theorem 1.4 in the original paper contained a gap and need to be corrected. The reason, which was pointed out by Bañuelos and Baudoin [1], is due to the fact that $e^{a\tau} M_\tau$ is not $\mathcal{F}_s$-measurable, so one cannot apply the Burkholder martingale subordinate inequality as was done in the original paper. Based on (1) and the martingale inequality in Theorem 2.6 in Bañuelos and Baudoin [1], we can revise the original argument in the original paper to prove that, on any complete Riemannian manifold with $Ric + \nabla^2\phi \geq -a$, where $a \geq 0$ is a constant, the L^p-norm of the Riesz transform $R_a(L) = \nabla(a - L)^{-1/2}$ satisfies $\|R_a(L)\|_{p,p} \leq C(p^* - 1)^{3/2}$, where C is a universal constant which is independent of a and p. For the detail, see [3].

Recently, Bañuelos and Osekowski [2] proved a new martingale inequality which improves Theorem 2.6 in [1] and derived that if $Ric + \nabla^2\phi \geq 0$, then for all $p \in (1, \infty)$, the original estimate stated in the original paper remains valid, i.e., $\|\nabla(-L)^{-1/2}\|_{p,p} \leq 2(p^* - 1)$. In particular, $\|\nabla(-\Delta)^{-1/2}\|_{p,p} \leq 2(p^* - 1)$ holds on complete Riemannian manifolds with $Ric \geq 0$. We would like to point out, based on the above correct representation formulas of the Riesz transforms, and using the new martingale subordination inequality due to Banuelos and Osekowski (Theorem 2.2 in [2]), we can correct the gap in the original paper the original paper without major change in the line of our original argument and to arrive at the original estimates obtained in the original paper.

To see this, let us first recall Theorem 2.2 in Bañuelos and Osekowski [2]: Let X and Y be $\mathbb{R}^n$-valued martingales with continuous paths such that Y is differentially subordinate to X. Consider the solution of the matrix equation

$$dM_t = V_t M_t dt; \quad M_0 = Id;$$

where $(V_t)_{t\geq 0}$ is an adapted and continuous process taking values in the set of symmetric and non-positive $n \times n$ matrices. For a given $a \geq 0$, consider the process

$$Z_t = e^{-at} M_t \int_0^t e^{as} M_s^{-1} dYs.$$

Then for any $1 < p < \infty$ and $T \geq 0$, we have the sharp bound

$$\|Z_T\|_p \leq (p^* - 1)\|X_T\|_p.$$

Now, let Qf be the Poisson integral of f, $Z_s = (X_s, B_s)$ be the Brownian motion on $M \times \mathbb{R}$. Let $\widehat{X}_t = \int_0^t \nabla Qf(Z_s)dZ_s$ and $\widehat{Y}_t = \int_0^t A\nabla Qf(Z_s)dZ_s$. By the original paper, $\|A\| \leq 1$. So $\widehat{Y}$ is a subordinate of $\widehat{X}$. By Theorem 2.2 in [2] stated as above, when $Ric + \nabla^2\phi \geq 0$, it holds

$$\left\| M_\tau \int_0^\tau M_s^{-1} A\nabla Qf(Z_s)dZ_s \right\|_p \leq (p^* - 1)\|\widehat{X}_\tau\|_p.$$

By the argument used in the original paper in p. 271 and taking $a = 0$ there, Itô's formula implies that

$$\widehat{X}_\tau = f(X_\tau) - Qf(X_0, B_0).$$

Thus

$$\begin{aligned}
\lim_{y\to\infty} \left\| M_\tau \int_0^\tau M_s^{-1} A\nabla Qf(Z_s)dZ_s \right\|_p &\leq (p^* - 1) \lim_{y\to\infty} \|f(X_\tau) - Qf(X_0, B_0)\|_p \\
&\leq (p^* - 1) \lim_{y\to\infty} [\|f(X_\tau)\|_p + \|Qf(X_0, y)\|_p] \\
&= (p^* - 1) \lim_{y\to\infty} \|f(X_\tau)\|_p \\
&= (p^* - 1)\|f\|_p,
\end{aligned}$$

where we have used the facts that $\lim_{y\to\infty} \|Qf(X_0, y)\|_p = 0$ and $\|f(X_\tau)\|_p = \|f\|_p$.

Therefore the original estimates in Theorem 1.4 and Corollary 1.5 obtained in the original paper remain valid. See also Theorem 1.1 in [2]. More precisely, we have

Theorem 0.1 *Let M be a complete Riemannian manifold, $\phi \in C^2(M)$. Suppose that $Ric + \nabla^2\phi \geq 0$. Then for all $1 < p < \infty$ we have*

$$\|\nabla(-L)^{-1/2} f\|_p \leq 2(p^* - 1)\|f\|_p, \quad \forall f \in L^p(M, \mu),$$

where $d\mu(x) = e^{-\phi(x)} dv(x)$, and $L = \Delta - \nabla\phi \cdot \nabla$. In particular, on any complete Riemannian manifold with non-negative Ricci curvature, we have

$$\|\nabla(-\Delta)^{-1/2} f\|_p \leq 2(p^* - 1)\|f\|_p, \quad \forall f \in L^p(M, v).$$

Similarly, using the correct probabilistic representation formulas of the Riesz transforms and Theorem 2.2 in [2], we can re-derive the original estimates of Theorem 1.4 obtained in the original paper for the case $Ric + \nabla^2\phi \geq a > 0$. To save the length of this note, we omit the detail of the proof.

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References

1. Bañuelos, R., Baudoin, F.: Martingale transforms and their projection operators on manifolds. Potential Anal. doi:10.1007/s11118-012-9307-8
2. Bañuelos, R., Osekowski, A.: Sharp martingale inequalities and applications to Riesz transforms on manifolds, Lie groups and Gauss space. arxiv.org:1305.1492
3. Li, X.-D.: On the L^p-estimates for Beurling–Ahlfors and Riesz transforms on Riemannian manifolds. arXiv:1304.1168v2