

Hypoellipticity for filtering problems of partially observable diffusion processes

N. V. Krylov

Received: 5 November 2013 / Revised: 13 March 2014 / Published online: 30 March 2014
© Springer-Verlag Berlin Heidelberg 2014

Abstract We prove that under Hörmander’s type conditions on the coefficients of the unobservable component of a partially observable diffusion process the filtering density is infinitely differentiable and can be represented as the integral of an infinitely differentiable kernel against the prior initial distribution. These results are derived from more general results obtained for SPDEs. One of the main novelties of the paper is the existence and smoothness of the kernel, another one is that we allow the coefficients of our partially observable process to be just measurable with respect to the time variable and Lipschitz continuous with respect to the observation variable.

Keywords Hypoellipticity · SPDEs · Filtering of partially observable diffusion processes · Filtering kernel

Mathematics Subject Classification (1991) 60G35 · 60H15 · 35R60

1 Introduction

Let (Ω, \mathcal{F}, P) be a complete probability space with an increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete with respect to (\mathcal{F}, P) σ -fields $\mathcal{F}_t \subset \mathcal{F}$. Let $w_t^k, k = 1, 2, \dots, d_1$, be independent one-dimensional Wiener processes with respect to $\{\mathcal{F}_t\}$, where $d_1 \geq 1$ is an integer.

Let $d, d' \geq 1$ be integers. Consider a $d + d'$ -dimensional two-component process $z_t = (x_t, y_t)$ with x_t being d -dimensional and y_t d' -dimensional. We assume that z_t

N.V. Krylov was partially supported by NSF Grant DMS-1160569.

N. V. Krylov (✉)
University of Minnesota, 127 Vincent Hall, Minneapolis, MN 55455, USA
e-mail: krylov@math.umn.edu

is a diffusion process defined as a solution of the system

$$\begin{aligned} dx_t &= b(t, z_t) dt + \theta^k(t, z_t) dw_t^k, \\ dy_t &= B(t, z_t) dt + \Theta^k(t, y_t) dw_t^k \end{aligned} \tag{1.1}$$

with some initial data independent of the process w_t , where and below the summation convention over repeated indices is enforced regardless of whether they stand at the same level or at different ones. The coefficients of (1.1) are assumed to be vector-valued functions of appropriate dimensions defined on $[0, \infty) \times \mathbb{R}^{d+d'}$. Actually $\Theta^k(t, y)$ are assumed to be independent of x , so that they are functions on $[0, \infty) \times \mathbb{R}^{d'}$ rather than $[0, \infty) \times \mathbb{R}^{d+d'}$ but as always we may think of $\Theta^k(t, y)$ as functions of (t, z) as well.

One of the main goals of the paper is to show that under Hörmander’s type conditions satisfied for x lying in a ball B , in some sense uniformly with respect to t and y , there exists a function $p(t, y, x) = p(\omega, t, y, x) \geq 0$, which is infinitely differentiable with respect to $(y, x) \in B^2$ for any $t > 0$ and ω , such that for any $f \in C_0^\infty(B)$ and $t > 0$ with probability one

$$E\{f(x_t) \mid \mathcal{F}_t^y\} = \int_B \int_B f(x)p(t, y, x) P_0(dy) dx,$$

where P_0 is the conditional distribution of x_0 given y_0 and $\mathcal{F}_t^y = \sigma\{y_s, s \leq t\}$. Naturally,

$$\int_B p(t, y, x) P_0(dy) \tag{1.2}$$

turns out to be infinitely differentiable with respect to $x \in B$ and represent the conditional density $\pi_t(x)$ of x_t given \mathcal{F}_t^y .

In the literature two approaches to prove infinite differentiability of $\pi_t(x)$ for degenerate processes under Hörmander’s type conditions are known. The first one is based on filtering equations for π_t , which are stochastic partial differential equations (SPDEs). This approach was initiated by Wentzell [17] and in a more general and *time inhomogeneous* case outlined by Kunita in [12] and [13]. It is worth noting that in [17] the coefficient B is supposed to be independent of x and in [13] the functions b and θ are independent of y . Equations in [12] seem not to cover general filtering equations either, since it is assumed there that Θ is an identity matrix and the coefficients are continuously differentiable with respect to t . In [17], [12], and [13] the SPDE is reduced to an ordinary parabolic equation with random coefficients by using a random change of coordinates. Without this reduction Chaleyat-Maurel and Michel in [3] achieve the goal in the *time homogeneous case* by mimicking some steps which are used in the proof of the deterministic Hörmander theorem. In their case as well as in a recent publication [2] the matrix $(\Theta^1, \dots, \Theta^{d'})$ is assumed to have form $(0, I)$ where I is $d' \times d'$ identity matrix.

However, some of the arguments in [12] and [13] are based on the claim that Hörmander's type theorem holds and can be proved by using the Malliavin calculus for equations whose coefficients are only continuous with respect to t . Such a proof is unknown even now, and for us assuming that the coefficients are continuous in t is no better than assuming that they are just measurable. It also looks like in [3] there is a gap at the point where the authors claim without proof that, roughly speaking, what holds for the unknown function also holds for its fractional derivatives.

Our approach is also based on using filtering equations but since we allow the coefficients of (1.1) to be just measurable with respect to t and be only Lipschitz continuous with respect to y , our type of Hörmander's condition is more restrictive than in [3] where the coefficients are time independent and infinitely differentiable in z . In contrast with [12] and [13] we do not appeal to the Malliavin calculus and instead rely on some analytical facts which we prove for more general SPDEs.

The second approach to proving infinite differentiability of $\pi_t(x)$ almost completely ignores filtering equations and is based on the Malliavin calculus and first appeared in the paper by Bismut and Michel [1]. In their model the coefficients are time independent and infinitely differentiable with respect to z , and the Hörmander type condition imposed is the most general in their situation and is much weaker than ours. Kusuoka and Stroock [15] further relax the Hörmander type condition in [1] (see our Remark 4.3) again in the time independent case but in what concerns filtering they assume that $B(t, x, y)$ is independent of x , so that the problem becomes a problem in the theory of conditional Markov processes rather than a more or less general filtering problem, because the coefficients of the equation for the observation process y_t are supposed to be *independent* of the signal process x_t . This result can also be found in [16]. In the recent publication by Chaleyat-Maurel [2] and references therein one can find a detailed account of the progress concerning Malliavin calculus and filtering equations. In particular, in [2] local versions of Hörmander's type condition from [3] are used to obtain the local regularity of solutions. It seems that these methods are not applicable in our case of coefficients only measurable with respect to t and Lipschitz in y .

It is worth reiterating that in [1–3, 15] the coefficients are supposed to be infinitely differentiable with respect to z . In our case they are only infinitely differentiable with respect to x and Lipschitz continuous with respect to $z = (x, y)$. This alone, even without allowing the coefficients to arbitrarily depend on t , makes applying the methods from [1–3, 15] impossible.

Apart from this novelty concerning t and z dependence, the fact that the conditional density $\pi_t(x)$ is represented as (1.2) with infinitely differentiable kernel seems to be new for degenerate diffusions z_t under Hörmander's type condition. Also our Hörmander's type conditions are only required on a part Ω_0 of Ω and all smoothness assertions then are also made for almost all $\omega \in \Omega_0$. It seems that this localization in ω was never addressed before.

We derive our results about filtering densities in Sect. 4 from the results of Sects. 2, 3, and 5. In these sections we treat SPDEs more general than the filtering equation. In the beginning of each section we list the assumptions which are supposed to hold throughout the section. In the formulations of some lemmas, theorems, ... *additional*

assumptions are imposed and they are supposed to hold only in these lemmas, theorems,....

The reader understands that \mathbb{R}^d is a Euclidean space of column-vectors (written in a common abuse of notation as) $x = (x^1, \dots, x^d)$. Denote

$$D_i = \partial/\partial x^i, \quad D_{ij} = D_i D_j,$$

and for an \mathbb{R}^d -valued function $\sigma_t(x) = \sigma_t(\omega, x)$ on $\Omega \times [0, \infty) \times \mathbb{R}^d$ and functions $u_t(x) = u_t(\omega, x)$ on $\Omega \times [0, \infty) \times \mathbb{R}^d$ set

$$L_{\sigma_t} u_t(x) = [Du_t(x)]\sigma_t(x) = [D_i u_t(x)]\sigma_t^i(x),$$

where Du is the row-vector gradient of u .

Next take an integer $d_2 \geq 1$ and assume that we are given \mathbb{R}^d -valued functions $\sigma_t^k = (\sigma_t^{ik}), k = 0, \dots, d_2 + d_1$, on $\Omega \times [0, \infty) \times \mathbb{R}^d$, which are infinitely differentiable with respect to x for any (ω, t) , and define the operator

$$L_t = (1/2) \sum_{k=1}^{d_2+d_1} L_{\sigma_t^k}^2 + L_{\sigma_t^0}. \tag{1.3}$$

Assume that on $\Omega \times [0, \infty) \times \mathbb{R}^d$ we are also given certain real-valued functions $c_t(x)$ and $v_t^k(x), k = 1, \dots, d_1$, which are infinitely differentiable with respect to x , and that on $\Omega \times [0, \infty) \times \mathbb{R}^d$ we are given real-valued functions f_t and $g_t^k, k = 1, \dots, d_1$. Then under natural additional assumptions which will be specified later the SPDE

$$du_t = (L_t u_t + c_t u_t + f_t) dt + (L_{\sigma_t^k} u_t + v_t^k u_t + g_t^k) dw_t^k \tag{1.4}$$

makes sense.

One of the main results of this paper is Theorem 2.4 saying that if the initial condition is a generalized functions of class H_2^n , then (1.4) has a unique solution with this initial data without any nondegeneracy or Hörmander’s type condition. Before this result was known only if $n \geq 1$ is an integer (see [11]). The result is important because it allows one to take a δ -function as the initial condition.

After the existence of solutions is secured we continue our investigation under Hörmander’s type condition and in Sect. 3 prove, roughly speaking, that, if $(s_1, s_2) \in (0, T)$ and for any $\omega \in \Omega_0$ and $t \in (s_1, s_2)$ the Lie algebra generated by the vector-fields $\sigma_t^{d_1+k}, k = 1, \dots, d_2$, has dimension d everywhere in $B_R = \{x : |x| < R\}$ and f_t and g_t^k are infinitely differentiable in B_R for any $\omega \in \Omega$ and $t \in (s_1, s_2)$, then the generalized function u_t satisfying (1.4) coincides on $(s_1, s_2) \times B_R$, for almost all $\omega \in \Omega_0$, with a function which is infinitely differentiable with respect to x . In Sect. 4 we apply this result to filtering problems. In the same section we apply the results of Sect. 5 to derive the existence of smooth filtering kernels. The results of Sect. 5 bear on kernels (or fundamental solutions) for more general SPDEs.

In the whole article T is a fixed number from $(0, \infty)$.

2 An existence theorem for degenerate SPDEs

Denote by \mathcal{D} the space of generalized functions on \mathbb{R}^d , and as usual introduce $\Lambda = (1 - \Delta)^{1/2}$, $H_2^n = \Lambda^{-n} \mathcal{L}_2$, where \mathcal{L}_2 is the Hilbert space of real-valued square integrable functions on \mathbb{R}^d with usual norm. The scalar product and the norm in H_2^n will be denoted by $(\cdot, \cdot)_n$ and $\|\cdot\|_n$, respectively.

Denote by $H_2^n(T)$ the set of \mathcal{D} -valued functions u_t on $\Omega \times [0, T]$ such that (u_t, ϕ) is predictable and (u_0, ϕ) is \mathcal{F}_0 -measurable for any $\phi \in C_0^\infty(\mathbb{R}^d)$ and

$$\int_0^T \|u_t\|_n^2 dt < \infty \quad (\text{a.s.}).$$

Sometimes it is necessary to indicate which filtration of σ -fields is involved in the definition of predictable functions. In these cases we write $H_2^n(T) = H_2^n(T, \mathcal{F})$.

Introduce \mathcal{H}_2^n as the set of \mathcal{F}_0 -measurable H_2^n -valued function on Ω . For an open ball B by $\mathcal{H}_2^n(B)$ we mean the subset of \mathcal{H}_2^n consisting of generalized functions with (closed) support in B . Define

$$\mathbb{H}_2^n(T) = \left\{ u \in H_2^n(T) : E \int_0^T \|u_t\|_n^2 dt < \infty \right\}.$$

In this section n is a fixed number.

Assumption 2.1

- (i) The functions $\sigma_t^k(x)$, $k = 0, \dots, d_2 + d_1$, c_t , v_t^k , $k = 1, \dots, d_1$, are infinitely differentiable with respect to x and each of their derivatives of any order is bounded on $\Omega \times [0, T] \times \mathbb{R}^d$. These functions are predictable with respect to (ω, t) for any $x \in \mathbb{R}^d$;
- (ii) We have that $f \in H_2^n(T)$, $g^k \in H_2^{n+1}(T)$, $k = 1, \dots, d_1$;
- (iii) We have that $u_0 \in \mathcal{H}_2^n$.

Definition 2.1 By a normal solution of (1.4) of class $H_2^n(T)$ with initial condition u_0 we mean a function u which belongs to $H_2^n(T)$, such that (a.s.) u_t is a continuous H_2^{n-1} -valued function and with probability one

$$u_t = u_0 + \int_0^t (L_s u_t + c_s u_s + f_s) ds + \int_0^t (L_{\sigma_s^k} u_s + v_s^k u_s + g_s^k) dw_s^k \quad (2.1)$$

for all $t \in [0, T]$.

Remark 2.1 The usual and stochastic integrals of Hilbert space valued functions are well defined, so that the right-hand side of (2.1) is a continuous H_2^{n-2} -valued process.

Remark 2.2 We say that a function u of class $H_2^n(T)$ is a *generalized* solution of (1.4) with initial condition u_0 if for any $\phi \in C_0^\infty(\mathbb{R}^d)$

$$(u_t, \phi) = (u_0, \phi) + \int_0^t (L_s u_t + c_s u_s + f_s, \phi) ds + \int_0^t (L_{\sigma_s^k} u_s + v_s^k u_s + g_s^k, \phi) dw_s^k \tag{2.2}$$

for almost all $(\omega, t) \in \Omega \times [0, T]$, where by (\cdot, \cdot) we mean the pairing between test functions and generalized ones.

By the way, recall that if $u \in H_2^n$ and $\phi \in C_0^\infty(\mathbb{R}^d)$, then

$$(u, \phi) = (\Lambda^k u, \Lambda^m \phi)_0,$$

as long as $k \leq n$ and $k + m = 2n$.

It is a well-known result (see, for instance, [10]) that if a function u of class $H_2^n(T)$ is a generalized solution of (1.4) with f, g , and u_0 as in Assumption 2.1, then there exists a normal solution \hat{u} of (1.4) of class $H_2^n(T)$ with initial condition u_0 such that \hat{u}_t and u_t coincide as generalized functions for almost all (ω, t) .

This result implies, in particular, that if a generalized solution of class $H_2^n(T)$ is such that u_t is a continuous H_2^m -valued function for some m , then (a.s.) $\hat{u}_t = u_t$ for all $t \in [0, T]$, so that u_t itself is a continuous H_2^{n-1} -valued function (a.s.) and thus is a normal solution of class $H_2^n(T)$. These observations will be crucial in the proof of Theorem 2.4.

Next we need the following technical lemma which enables us to integrate by parts in H_2^n -spaces.

Lemma 2.3 *Let $n \in \mathbb{R}$ and $v \in C_b^\infty(\mathbb{R}^d)$. Then there exists a constant N such that for any $u \in H_2^{n+1}$, $k = 0, 1, \dots, d_1 + d_2$, $t \in [0, T]$ and $\omega \in \Omega$ we have*

$$|(\Lambda^n L_{\sigma_t^k} u, \Lambda^n (vu))_0| \leq N \|u\|_n^2, \tag{2.3}$$

$$\langle \Lambda^n u, \Lambda^n (L_{\sigma_t^k}^2 u) \rangle + \|\Lambda^n L_{\sigma_t^k} u\|_0^2 \leq N \|u\|_n^2, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between H_2^1 and H_2^{-1} .

Proof By obvious reasons we may assume that $u \in C_0^\infty(\mathbb{R}^d)$ and we drop the indices k and t to simplify notation.

We are going to rely on some well-known properties of pseudo-differential operators. The order of a pseudo-differential operator S is a number $n \in \mathbb{R}$ such that $\Lambda^{-n} S$ and $S \Lambda^{-n}$ are bounded operators in \mathcal{L}_2 . If the orders of two operators S_1 and S_2 are n_1 and n_2 , respectively, then the order of $[S_1, S_2] = S_1 S_2 - S_2 S_1$ is at most $n_1 + n_2 - 1$. One also knows that the first order linear differential operators with coefficients whose every derivative of any order is bounded are pseudodifferential operator of order one.

Next, observe that if a pseudo-differential operator S is self adjoint, then for any $u \in C_0^\infty(\mathbb{R}^d)$

$$\begin{aligned} (L_\sigma u, S(vu))_0 &= (L_\sigma Su, vu)_0 + ([S, L_\sigma]u, vu)_0 \\ &= -(Su, vL_\sigma u)_0 + (aSu, u)_0 + ([S, L_\sigma]u, vu)_0 \\ &= -(S(vu), L_\sigma u)_0 - ([v \cdot, S]u, L_\sigma u)_0 + (aSu, u)_0 + ([S, L_\sigma]u, vu)_0, \end{aligned}$$

where $a = L_\sigma^* v$ is a smooth bounded function. It follows that

$$(L_\sigma u, S(vu))_0 = (1/2) [(aSu, u)_0 + ([S, L_\sigma]u, vu)_0 - ([v \cdot, S]u, L_\sigma u)_0]. \tag{2.5}$$

It is important to note for the future that if the order of S is $2n$, then the order of $[S, L_\sigma]$ is at most $2n$, the order of $[v \cdot, S]$ is at most $2n - 1$, and consequently

$$|(L_\sigma u, S(vu))_0| \leq N \|u\|_n^2. \tag{2.6}$$

This with $S = \Lambda^{2n}$ yields (2.3).

By applying (2.5) with $S = \Lambda^{2n}$ and $v \equiv 1$, we get

$$(\Lambda^n L_\sigma u, \Lambda^n u)_0 = (1/2)(a\Lambda^{2n}u, u)_0 + (1/2)([\Lambda^{2n}, L_\sigma]u, u)_0,$$

which after being polarized yields that

$$\begin{aligned} (\Lambda^n L_\sigma v, \Lambda^n u)_0 + (\Lambda^n L_\sigma u, \Lambda^n v)_0 &= (a\Lambda^{2n}u, v)_0 \\ &+ (a\Lambda^{2n}v, u)_0 + ([\Lambda^{2n}, L_\sigma]u, v)_0 + ([\Lambda^{2n}, L_\sigma]v, u)_0 \end{aligned}$$

if $u, v \in C_0^\infty(\mathbb{R}^d)$. We plug in here $v = L_\sigma u$ and obtain

$$\begin{aligned} (L_\sigma^2 u, u)_n + \|L_\sigma u\|_n^2 &= (a\Lambda^{2n}u, L_\sigma u)_0 \\ &+ (a\Lambda^{2n}L_\sigma u, u)_0 + ([\Lambda^{2n}, L_\sigma]u, L_\sigma u)_0 + ([\Lambda^{2n}, L_\sigma]L_\sigma u, u)_0. \end{aligned}$$

After introducing the self adjoint operators

$$S_1 = a\Lambda^{2n} + (a\Lambda^{2n})^*, \quad S_2 = [\Lambda^{2n}, L_\sigma] + ([\Lambda^{2n}, L_\sigma])^*$$

we rewrite the last equation as

$$(L_\sigma^2 u, u)_n + \|L_\sigma u\|_n^2 = (L_\sigma u, S_1 u)_0 + (L_\sigma u, S_2 u)_0$$

and obtain (2.4) owing to (2.6). The lemma is proved.

Theorem 2.4 *In class $H_2^n(T)$ there exists an (a.s.) unique normal solution u of (1.4) with initial condition u_0 . Furthermore, there exists a constant N independent of u, f, g*

such that

$$E \sup_{t \leq T} \|u_t\|_n^2 \leq NE \|u_0\|_n^2 + NE \int_0^T \left(\|f_t\|_n^2 + \sum_{k=1}^{d_1} \|g_t^k\|_{n+1}^2 \right) dt. \tag{2.7}$$

Proof Step 1. First we want to derive an a priori estimate assuming that we are given a normal solution of (1.4) of class $H^{n+1}(T)$. We apply the operator Λ^n to both sides of (1.4) written in the integral form and observe that after that the stochastic integral will belong to \mathcal{L}_2 , whereas the deterministic integral will belong to H_2^{-1} . This allows us to apply Itô’s formula for Banach space valued processes and shows that

$$d\|u_t\|_n^2 = d\|\Lambda^n u_t\|_0^2 = I_t dt + 2 \left(\Lambda^n u_t, \Lambda^n (L_{\sigma_t^k} u_t + v_t^k u_t + g_t^k) \right)_0 dw_t^k,$$

where

$$\begin{aligned} I_t &= \langle \Lambda^n u_t, 2\Lambda^n (L_t u_t + c_t u_t + f_t) \rangle + \sum_{k=1}^{d_1} \|\Lambda^n (L_{\sigma_t^k} u_t + v_t^k u_t + g_t^k)\|_0^2 \\ &= I_t^1 + I_t^2 + I_t^3 + 2I_t^4 + 2I_t^5 + I_t^6, \\ I_t^1 &= \sum_{k=1}^{d_1} \left[\langle \Lambda^n u_t, \Lambda^n (L_{\sigma_t^k}^2 u_t) \rangle + \|\Lambda^n L_{\sigma_t^k} u_t\|_0^2 \right], \\ I_t^2 &= (\Lambda^n u_t, 2\Lambda^n L_{\sigma_t^0} u_t)_0, \\ I_t^3 &= (\Lambda^n u_t, 2\Lambda^n (c_t u_t + f_t))_0, \\ I_t^4 &= (\Lambda^n L_{\sigma_t^k} u_t, \Lambda^n (v_t^k u_t))_0, \\ I_t^5 &= (\Lambda^n L_{\sigma_t^k} u_t, \Lambda^n g_t^k)_0, \\ I_t^6 &= \sum_{k=1}^{d_1} \|\Lambda^n (v_t^k u_t + g_t^k)\|_0^2. \end{aligned}$$

The term I_t^1 is estimated in (2.4) and I_t^2 in (2.3) (with $\nu \equiv 1$), which also provides an estimate for I_t^4 . Almost obviously

$$|I_t^3| + |I_t^6| \leq N \|u_t\|_n^2 + N \|f_t\|_n^2 + N \sum_{k=1}^{d_1} \|g_t^k\|_n^2,$$

where and below we denote by N various constants independent of u, f, g^k, t , and ω . Finally,

$$\begin{aligned} I_t^5 &= (L_{\sigma_t^k} \Lambda^n u_t, \Lambda^n g_t^k)_0 + ([\Lambda^n, L_{\sigma_t^k}] u_t, \Lambda^n g_t^k)_0 \\ &= (\Lambda^n u_t, (L_{\sigma_t^k})^* \Lambda^n g_t^k)_0 + ([\Lambda^n, L_{\sigma_t^k}] u_t, \Lambda^n g_t^k)_0 \end{aligned}$$

and, since the order of the operator $[\Lambda^n, L_{\sigma_t^k}]$ is at most n , we have

$$|I_t^5| \leq N \|u_t\|_n^2 + N \sum_{k=1}^{d_1} \|g_t^k\|_{n+1}^2.$$

Upon collecting our estimates we conclude that

$$\begin{aligned} d\|u_t\|_n^2 &\leq N \left(\|u_t\|_n^2 + \|f_t\|_n^2 + \sum_{k=1}^{d_1} \|g_t^k\|_{n+1}^2 \right) dt \\ &\quad + 2 \left(\Lambda^n u_t, \Lambda^n (L_{\sigma_t^k} u_t + v_t^k u_t + g_t^k) \right)_0 dw_t^k. \end{aligned} \tag{2.8}$$

Step 2. Uniqueness. Now assume that we are given two natural solutions of (1.4) of class $H_2^n(T)$ with the same initial condition. Then for their difference, say u_t , we have

$$d\|u_t\|_{n-1}^2 \leq N_1 \|u_t\|_{n-1}^2 dt + dm_t,$$

where m_t is a local martingale. Next, comparing the differentials we obtain

$$\|u_t\|_{n-1}^2 e^{-N_1 t} \leq \int_0^t e^{-N_1 s} dm_s.$$

Since the right-hand side is a local martingale starting at zero and the left-hand side is nonnegative, it follows, that the right-hand side is zero as is the left-hand side, which proves uniqueness.

Step 3. Here we prove (2.7) as an a priori estimate under the assumptions of Step 1. We follow an absolutely standard and well-known by now way first introduced by Pardoux. With N from (2.8) we have

$$d(e^{-Nt} \|u_t\|_n^2) \leq N \left(\|f_t\|_n^2 + \sum_{k=1}^{d_1} \|g_t^k\|_{n+1}^2 \right) dt + dm_t,$$

where m_t is a local martingale. Since the left-hand side is nonnegative, for any $t \in [0, T]$,

$$\begin{aligned} e^{-Nt} E\|u_t\|_n^2 &\leq E\|u_0\|_n^2 + NE \int_0^t \left(\|f_s\|_n^2 + \sum_{k=1}^{d_1} \|g_s^k\|_{n+1}^2 \right) ds, \\ \sup_{t \in [0, T]} E\|u_t\|_n^2 &\leq E\|u_0\|_n^2 + NE \int_0^T \left(\|f_s\|_n^2 + \sum_{k=1}^{d_1} \|g_s^k\|_{n+1}^2 \right) ds. \end{aligned} \tag{2.9}$$

Next, notice that since $\|u_t\|_n$ is continuous

$$\tau_m := T \wedge \inf\{t \geq 0 : \|u_t\|_n \geq m\}$$

are stopping times and $\tau_m \uparrow T$ as $m \rightarrow \infty$. By Davis’s inequality (2.8) and (2.9) imply that

$$\begin{aligned} E \sup_{t \leq \tau_m} \|u_t\|_n^2 &\leq NE \|u_0\|_n^2 + NE \int_0^T \left(\|f_s\|_n^2 + \sum_{k=1}^{d_1} \|g_s^k\|_{n+1}^2 \right) ds \\ &\quad + 6E \left(\int_0^{\tau_m} \sum_{k=1}^{d_1} \left| \left(\Lambda^n u_t, \Lambda^n (L_{\sigma_t^k} u_t + v_t^k u_t + g_t^k) \right)_0 \right|^2 dt \right)^{1/2}. \end{aligned} \tag{2.10}$$

By the above what is inside the square by magnitude is dominated by

$$N \|u_t\|_n (\|u_t\|_n + \|g_t^k\|_n).$$

Hence the last term in (2.10) is less than

$$\begin{aligned} &NE \left(\int_0^{\tau_m} \|u_t\|_n^2 \left(\|u_t\|_n^2 + \sum_{k=1}^{d_1} \|g_t^k\|_n^2 \right) dt \right)^{1/2} \\ &\leq NE \left(\sup_{t \leq \tau_m} \|u_t\|_n \int_0^{\tau_m} \left(\|u_t\|_n^2 + \sum_{k=1}^{d_1} \|g_t^k\|_n^2 \right) dt \right)^{1/2} \\ &\leq (1/2)E \sup_{t \leq \tau_m} \|u_t\|_n^2 + NE \int_0^{\tau_m} \left(\|u_t\|_n^2 + \sum_{k=1}^{d_1} \|g_t^k\|_n^2 \right) dt, \end{aligned}$$

which after coming back to (2.10), using again (2.9), and sending $m \rightarrow \infty$, by Fatou’s lemma yields the a priori estimate (2.7).

Step 4. Existence in a particular case. If the norms on the right in (2.7) are finite, $u_0 \in L_2(\Omega, \mathcal{F}_0, H_2^{n+1})$, and our equation is uniformly nondegenerate, then (see, for instance, [6]) there exists a unique normal solution of our problem of class $H_2^{n+2}(T)$. For this solution (2.7) is valid.

If the norms on the right in (2.7) are finite, but $u_0 \in L_2(\Omega, \mathcal{F}_0, H_2^n)$ and there is no nondegeneracy assumption, we approximate u_0 in the $L_2(\Omega, \mathcal{F}_0, H_2^n)$ -norm by a sequence $u_0^m \in L_2(\Omega, \mathcal{F}_0, H_2^{n+1})$, $m = 1, 2, \dots$, and add into the right-hand side of (1.4) the term $(1/m)\Delta u_t dt$ to make the equation uniformly nondegenerate. Denote by u^m the normal solutions of the so modified problems. Then (2.7) will hold with N independent of m because no constant of nondegeneracy was involved in the derivation of (2.7).

According to (2.7) the sequence u^m is bounded in $\mathbb{H}_2^n(T)$. In particular, $(1/m)\Delta u^m \rightarrow 0$ in $\mathbb{H}_2^{n-2}(T)$. Having in mind this fact and applying (2.7) to the difference $u_t^m - u_t^k$ and $n - 2$ in place of n we see that the sequence u^m is Cauchy in the space with norm, whose square is given by

$$E \sup_{t \leq T} \|u_t\|_{n-2}^2,$$

in particular, in $\mathbb{H}_2^{n-2}(T)$. Let u be its limit in $\mathbb{H}_2^{n-2}(T)$ such that

$$E \sup_{t \leq T} \|u_t - u_t^m\|_{n-2}^2 \rightarrow 0 \tag{2.11}$$

as $n \rightarrow \infty$. Then Eq. (1.4) in the integral form holds in $\mathbb{H}_2^{n-4}(T)$ so that u is a generalized solution. Now, since the sequence u^m is bounded in $\mathbb{H}_2^n(T)$ and converges to u in $\mathbb{H}_2^{n-2}(T)$, $u \in \mathbb{H}_2^n(T)$. After that we apply a classical result saying that if $u \in \mathbb{H}_2^n(T)$ satisfies (2.7) in the generalized sense with initial condition $u_0 \in H_2^{n-1}$, then (a.s.) u_t is a continuous H_2^{n-1} -valued function (see Remark 2.2).

To establish (2.7) for thus found normal solution take a sequence $\phi^r \in C_0^\infty(\mathbb{R}^d)$ such that it is dense in the unit ball of H_2^n . Then owing to (2.7) write for any $j = 1, 2, \dots$

$$E \sup_{t \leq T} \max_{r=1, \dots, j} (\Lambda^n u_t^m, \Lambda^n \phi^r)_0^2 \leq NE \|u_0^m\|_n^2 + I, \tag{2.12}$$

where I is the second term on the right in (2.7). Since

$$(\Lambda^n (u_t^m - u_t), \Lambda^n \phi^r)_0^2 \leq \|\Lambda^{n-2}(u_t^m - u_t)\|_0^2 \|\Lambda^{n+2} \phi^r\|_0^2,$$

estimate (2.11) allows us to conclude from (2.12) that

$$E \sup_{t \leq T} \max_{r=1, \dots, j} (\Lambda^n u_t, \Lambda^n \phi^r)_0^2 \leq NE \|u_0\|_n^2 + I.$$

By letting $j \rightarrow \infty$ and using the monotone convergence theorem and the fact that ϕ^r are dense in the unit ball of H_2^n , we get (2.7).

Step 5. Existence in the general case. The first assertion of the theorem in the general case is proved as always by using stopping times like

$$\gamma_m = \inf \left\{ t \geq 0 : \|u_0\|_n^2 + \int_0^t \left(\|f_s\|_n^2 + \sum_{k=1}^{d_1} \|g_s^k\|_{n+1}^2 \right) ds \geq m \right\}.$$

The theorem is proved.

In the remaining part of this section by u_t we mean the normal solution from Theorem 2.4. We remind the reader that the common way of saying that a generalized function in a domain is smooth means that there is a smooth function which, as a

generalized function, coincides with the given generalized one in the domain under consideration.

Theorem 2.4 and Sobolev embedding theorems immediately imply the following.

Corollary 2.5 *Suppose that Assumption 2.1 is satisfied for all n . Then (a.s.) the solution u_t is infinitely differentiable with respect to x and every its derivative is a bounded continuous function of (t, x) .*

Corollary 2.6 *Suppose that Assumption 2.1 is satisfied for all n . Let D be a domain in \mathbb{R}^d with $\partial D \neq \emptyset$ and assume that for $x \in D, t \in [0, T]$, and $\omega \in \Omega$ we have*

$$u_0(x), c_t(x), f_t(x) \leq 0, \quad v_t^k(x) = g_t^k(x) = 0, \quad k = 1, \dots, d.$$

Then (a.s.) for all $t \in [0, T]$ on D we have

$$u_t \leq \max_{s \leq t} \max_{\partial D} (u_s)_+. \tag{2.13}$$

This result follows from Theorem 1.2 of [7] in which one takes $\xi = 0, \bar{u} \equiv 1, \bar{f} \equiv -c, \bar{f}^i \equiv 0, \bar{g} \equiv 0$, and ρ_t equal the right-hand side of (2.13) plus a constant $\varepsilon > 0$. One adds ε to be sure that

$$(u_t - \rho_t \bar{u}_t)_+ = (u_t - \rho_t)_+$$

vanishes near the boundary of D and hence belongs to $\overset{0}{W}_2^1(D)$. Then after applying Theorem 1.2 of [7] one sets $\varepsilon \downarrow 0$.

Theorem 2.7 *Take an $R \in [0, \infty)$ and suppose that $\sigma_t^k(x) = 0$ and $Dv_t^k(x) = 0$ for $k = 1, \dots, d_1, t \in [0, T]$, and $\omega \in \Omega$ as long as $|x| > R$. Also assume that $g^k \equiv 0$ for $k = 1, \dots, d_1$. Then there exists a (random finite) constant N independent of f and u_0 such that (a.s.)*

$$\sup_{t \leq T} \|u_t\|_n^2 \leq N \|u_0\|_n^2 + N \int_0^T \|f_t\|_n^2 dt. \tag{2.14}$$

Proof For smooth \mathbb{R}^d -valued functions $\sigma(x)$ on \mathbb{R}^d (whose points are always considered as column vectors) by $D\sigma$ we mean a matrix with entries $(D\sigma)^{ij} = D_j \sigma^i$ and if we are given two such functions σ and γ , then we set

$$D\sigma \gamma := [D\sigma]\gamma. \tag{2.15}$$

Consider the equation

$$x_t = x - \int_0^t \sigma_s^k(x_s) dw_s^k - \int_0^t b_t(x_s) ds, \tag{2.16}$$

where

$$b_t(x) = \sigma_t^0(x) - (1/2) \sum_{k=1}^{d_1} D\sigma_t^k(x)\sigma_t^k(x).$$

As is well known (see, for instance, [14] for more advanced treatment of the subject or see [8]), there exists a function $X_t(x)$ on $\Omega \times [0, T] \times \mathbb{R}^d$, such that

- (i) it is continuous in (t, x) for any ω along with each derivative of $X_t(x)$ of any order with respect to x ,
- (ii) for each ω and t the mapping $x \rightarrow X_t(x)$ of \mathbb{R}^d to \mathbb{R}^d is one-to-one and onto and its inverse mapping $X_t^{-1}(x)$ has bounded and continuous in (t, x) derivatives of any order with respect to x for any ω .
- (iii) it is \mathcal{F}_t -adapted for any x ,
- (iv) for each x with probability one it satisfies (2.16) for all $t \in [0, T]$.

Observe that $X_t(x) = x$ for $|x| \geq R$, and $X_t^{-1}(x) = x$ for all $t \in [0, T]$ if $|x|$ is large enough (depending on ω).

Next, define the operations “hat” and “check” which transform any function $\phi_t(x)$ into

$$\hat{\phi}_t(x) := \phi_t(X_t(x)), \quad \check{\phi}_t = \phi_t(X_t^{-1}(x)).$$

Also define $\rho_t(x)$ from the equation

$$\rho_t(X_t(y)) \det DX_t(y) = 1$$

and observe that by the change of variables formula

$$\int_{\mathbb{R}^d} F(X_t(y))\phi(y) dy = \int_{\mathbb{R}^d} F(x)\check{\phi}_t(x)\rho_t(x) dx, \tag{2.17}$$

whenever at least one side of the equation makes sense. Finally, define the mapping “bar” which transforms any \mathbb{R}^d -valued function $\sigma_t(x)$ into

$$\bar{\sigma}_t(x) = Y_t(x)\hat{\sigma}_t(x) = Y_t(x)\sigma_t(X_t(x)), \tag{2.18}$$

where

$$Y = (DX)^{-1}.$$

By Corollary 6.5 of [8] (also see Remark 2.2) the function \hat{u}_t is well defined and is a normal solution of class $H_2^n(T)$ of the equation

$$d\hat{u}_t = \left[\sum_{k=1}^{d_2} L_{\bar{\sigma}_t^{d_1+k}}^2 \hat{u}_t + \hat{c}_t \hat{u}_t + \hat{f}_t \right] dt + \hat{u}_t \hat{v}_t^k dw_t^k.$$

By using Kolmogorov’s continuity theorem for random fields one easily shows that there exists a function $I_t(x) = I_t(\omega, x)$ which along with each its derivative of any order with respect to x is continuous with respect to $(t, x) \in [0, T] \times \mathbb{R}^d$ for each ω and such that for each (t, x) with probability one

$$I_t(x) = \int_0^t \hat{v}_s^k(x) dw_s^k.$$

Then define

$$\gamma_t(x) = \exp \left[-I_t(x) - (1/2) \sum_{k=1}^{d_1} \int_0^t |\hat{v}_s^k(x)|^2 ds \right], \quad v_t = \hat{u}_t \gamma_t.$$

Of course, $D\gamma_t(x) = 0$ if $|x| \geq R$, so that $\gamma_t(x)$ along with each its derivative of any order with respect to x is continuous and bounded with respect to $(t, x) \in [0, T] \times \mathbb{R}^d$ for each ω . We want to apply Itô’s formula to write an equation for v_t , that is, for any $\phi \in C_0^\infty(\mathbb{R}^d)$ find the stochastic differential of

$$I_t(\phi) := (v_t, \phi) = (\hat{u}_t, \gamma_t \phi) = \left(\Lambda^{-(n-1)} \hat{u}_t, \Lambda^{n-1}(\gamma_t \phi) \right)_0.$$

Here $\xi_t := \Lambda^{-(n-1)} \hat{u}_t$ and $\eta_t := \Lambda^{n-1}(\gamma_t \phi)$ are continuous \mathcal{L}_2 -valued processes, admitting stochastic differentials such that the classical formula for the squared norm is applicable. Then this formula is also applicable to $\xi_t + \lambda \eta_t$ for any number λ . By comparing the coefficients of λ in $\|\xi_t + \lambda \eta_t\|_0^2$ and in the formula we obtain the stochastic differential of $(\xi_t, \eta_t)_0$ that is of $I_t(\phi)$.

In this way we get that with probability one for all $t \in [0, T]$

$$v_t = v_0 + \int_0^t \left[\sum_{k=1}^{d_2} \gamma_s L_{\sigma_s}^2(\gamma_s^{-1} v_s) + \hat{c}_s v_s + \gamma_s \hat{f}_s \right] ds. \tag{2.19}$$

Fix an ω such that (2.19) holds for all $t \in [0, T]$,

$$\int_0^T \|u_t\|_n^2 dt < \infty,$$

$u_0(\omega, \cdot) \in H_2^n$, and $u_t(\omega, \cdot)$ is an H_2^{n-1} -valued continuous function. Then (2.19) becomes a deterministic equation, to which Theorem 2.4 is applicable because the differential operators in (2.19) can be rewritten in a canonical form as in (1.4) with coefficients satisfying Assumption 2.1. Then applying Theorem 2.4 to each particular ω , the set of which has full probability, we obtain (2.14). The theorem is proved.

Lemma 2.8 *Take*

$$f_t(x) = u_0(x) = (1 + |x|^2)^{-d}, \quad g_t^k(x) = 0$$

and call v_t the normal solution of (1.4) on $[0, T]$ with so prescribed data. By Corollary 2.5 the function v_t is (a.s.) infinitely differentiable with respect to x and every its derivative is a bounded continuous function of (t, x) . We assert that with probability one for every $r \in (0, \infty)$ there exists a (random constant) $\varepsilon > 0$ such that $v_t(x) \geq \varepsilon$ for $t \in [0, T]$ and $x \in \bar{B}_r$.

Proof First observe that $v_t \geq 0$ by the maximum principle. We take an $R > 2r$ and concentrate on Eq. (1.4) only for $x \in B_{2r}$. Then equation in B_{2r} will still hold if we change all σ_t^k and v_t^k outside B_{2r} so that they will vanish outside B_R . For simplicity of notation we assume that $\sigma^k = 0$ and $v^k = 0$ outside B_R for $k = 1, \dots, d_1$ for the original coefficients.

Then making the same transformations as in the proof of Theorem 2.7 we come to the conclusion that for almost any ω

- (i) Equation (2.19) holds on $\{(t, x) : t \in [0, T], x \in X_t^{-1}(\bar{B}_{2r})\}$ with $\hat{v}_t \gamma_t$ in place of v_t .

For each ω this is a deterministic (degenerate) parabolic equation with bounded coefficients. Furthermore, for almost any ω .

- (ii) $\gamma_t \hat{f}_t > 0$ and $\hat{v}_t \gamma_t \geq 0$ on $\Gamma := \{(t, x) : t \in [0, T], x \in X_t^{-1}(\bar{B}_{2r})\}$ and $\gamma_t \hat{f}_t, \hat{v}_t \gamma_t$ are continuous on this set.

Now we want to use the maximum principle to show that $v_t(x)$ cannot take zero value in $[0, T] \times \bar{B}_r$ whenever ω is such that (i) and (ii) hold. Were the coefficients of (2.19) continuous in t , this would be just a trivial matter. In our case we still need a little argument. Assume the contrary: there is a point $(t_0, x_0) \in [0, T] \times \bar{B}_r$ such that $v_{t_0}(x_0) = 0$. Obviously, $t_0 > 0$. Then, for $y_0 = X_{t_0}(x_0)$ we have that $(t, y_0) \in \Gamma$ for $t \leq t_0$ and sufficiently close to t_0 . Furthermore, since $\hat{v}_t(y_0) \geq 0$ for those t , we get

$$\begin{aligned} 0 &= \hat{v}_t(y_0) \gamma_t(y_0) + \int_t^{t_0} \gamma_s(y_0) \left[\sum_{k=1}^{d_2} L_{\hat{\sigma}_s}^{2, d_1+k} \hat{v}_s + \hat{c}_s \hat{v}_s + \hat{f}_s \right] (y_0) ds \\ &\geq \int_t^{t_0} \gamma_s(y_0) \left[\sum_{k=1}^{d_2} L_{\hat{\sigma}_s}^{2, d_1+k} \hat{v}_{t_0} + \hat{c}_s \hat{v}_{t_0} + \hat{f}_s \right] (y_0) ds + I_t, \end{aligned}$$

where I_t is defined as the difference of the above two integrals. Since $\hat{v}_t(x)$ and its derivatives with respect to x are continuous with respect to t , we have

$$\lim_{t \uparrow t_0} \frac{1}{t_0 - t} I_t = 0.$$

By taking into account that $\hat{v}_{t_0}(y_0) = 0$, the first-order derivatives of $\hat{v}_{t_0}(x)$ vanish at y_0 , and the Hessian is nonnegative at y_0 , we conclude that

$$0 \geq \overline{\lim}_{t \uparrow t_0} \frac{1}{t_0 - t} \int_t^{t_0} \gamma_s \hat{f}_s(y_0) ds,$$

which is impossible because the limit is just $\gamma_{t_0} \hat{f}_{t_0}(y_0) > 0$. The lemma is proved.

Theorem 2.9 *Suppose that Assumption 2.1 is satisfied for all n . Also assume that there is an $r \in (0, \infty)$ such that $u_0(x), f_t(x) \leq 0$, and $g_t^k(x) = 0$ for $k = 1, \dots, d_1$ and $x \notin B_r$. Then there exists a (random finite) constant N independent of f, g^k , and u_0 such that (a.s.) for $t \in [0, T]$ and $|x| \geq r$ we have*

$$u_t(x) \leq N \max_{s \leq t} \max_{\partial B_r} (u_s)_+. \tag{2.20}$$

Proof Take v_t from Lemma 2.8 and set

$$\rho_t = \max_{s \leq t} \max_{|x|=r} (u_t(x)/v_t(x))_+.$$

By Lemma 2.8 the process ρ_t is finite, nonnegative, increasing, and continuous with probability one. Furthermore, $(u_t - \rho_t v_t)_+$ vanishes on ∂B_r . This along with the fact that $u_0(x) = f_t(x) \leq 0$ and $g_t^k(x) = 0$ for $k = 1, \dots, d_1$ and $x \notin B_r$ by Theorem 1.2 of [7] implies that $u_t - \rho_t v_t \leq 0$ in $[0, T] \times B_R^c$ (a.s.), which obviously proves the theorem.

3 Hypocoercivity

Recall the notation associated with (2.15) and for two smooth \mathbb{R}^d -valued functions σ and γ on \mathbb{R}^d set, as usual,

$$[\sigma, \gamma] = D\gamma\sigma - D\sigma\gamma.$$

Assumption 3.1 Assumption 2.1 (i) is satisfied, Assumption 2.1 (ii) is satisfied for all n and, for an n , the function u_0 is an \mathcal{F}_0 -measurable H_2^n -valued function on Ω .

Introduce the collections of \mathbb{R}^d -valued functions defined on $\Omega \times [0, T] \times \mathbb{R}^d$ inductively as $\mathbb{L}_0 = \{\sigma_t^{d_1+1}, \dots, \sigma_t^{d_1+d_2}\}$,

$$\mathbb{L}_{n+1} = \mathbb{L}_n \cup \{[\sigma_t^{d_1+k}, M] : k = 1, \dots, d_2, M \in \mathbb{L}_n\}, \quad n \geq 0.$$

For any multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$, $\alpha_i \in \{0, 1, \dots\}$, introduce as usual

$$D^\alpha = D_1^{\alpha_1} \dots D_d^{\alpha_d}, \quad |\alpha| = \alpha_1 + \dots + \alpha_d.$$

Also define BC_b^∞ as the set of real-valued measurable functions a on $\Omega \times [0, T] \times \mathbb{R}^d$ such that for each $t \in [0, T]$ and $\omega \in \Omega$, $a_t(x)$ is infinitely differentiable with respect to x , and for any $\omega \in \Omega$ and multi-index α we have

$$\sup_{t,x \in [0,T] \times \mathbb{R}^d} |D^\alpha a_t(x)| < \infty.$$

We denote by Lie_n the set of (finite) linear combinations of elements of \mathbb{L}_n with coefficients which are of class BC_b^∞ . Observe that the vector-field σ^0 is *not* explicitly included into Lie_n . Finally, fix $S \in [0, T)$, $R_0 \in (0, \infty)$, introduce

$$G = (S, T) \times B_{R_0},$$

and fix an $\Omega_0 \in \mathcal{F}$.

Assumption 3.2 For every $\omega \in \Omega_0$ and $\zeta \in C_0^\infty(B_{R_0})$ there exists an n such that we have $\xi I_{[S,T]} \zeta \in \text{Lie}_n$ for any $\xi \in \mathbb{R}^d$.

The following Theorem 2.4 of [8] will be used a few times. By u here and everywhere below in this section we mean the normal solution which exists due to Theorem 2.4.

Theorem 3.1 Take $s_0 \in (S, T)$, $r \in (0, R_0)$ and take a $\zeta \in C_0^\infty(B_{R_0})$ such that $\zeta = 1$ on a neighborhood of \bar{B}_r . Then

- (i) for almost any $\omega \in \Omega_0$, $u_t(x)$ is infinitely differentiable with respect to x for $(t, x) \in (S, T] \times B_{R_0}$ and each derivative is a continuous function in $(S, T] \times B_{R_0}$.
- (ii) for any multi-index α and l such that

$$2(l - |\alpha| - 2) > d + 1, \tag{3.1}$$

there exists a (random, finite) constant N , independent of u, f , and g^k , such that, for almost any $\omega \in \Omega_0$,

$$\sup_{(t,x) \in [s_0, T] \times B_r} |D^\alpha u_t(x)|^2 \leq N \int_S^T \left[\|f_t \zeta\|_{H_2^l}^2 + \|u_t \zeta\|_{H_2^n}^2 \right] dt, \tag{3.2}$$

provided that $g_t^k \zeta I_{\Omega_0} \equiv 0, k = 1, \dots, d_1$.

If we additionally assume that u_S is infinitely differentiable in B_{R_0} for every $\omega \in \Omega_0$, then assertion (i) holds true with $[S, T] \times B_{R_0}$ in place of $(S, T] \times B_{R_0}$, and assertion (ii) holds true with $s_0 = S$ if we add to the right-hand side of (3.2) a constant (independent of u) times $\|\zeta u_S\|_{H_2^{l+1}}^2$.

Here is a generalization of the corresponding results of paper [4], where there is no stochastic terms in the equation. This is a generalization because no continuity hypothesis in time on the coefficients is imposed. The types of Hörmander’s condition

imposed in [4] and here coincide. It is worth noting, however, that the result of [4] bears on the equation formally adjoint to the one we consider when there is no stochastic terms and no dependence on ω . Such equations have the same form as ours and have the same Lie n .

Recall that $\mathcal{H}_2^n(B_r)$ is introduced before Assumption 2.1.

Theorem 3.2 *Suppose that $S = 0$, take $s_0 \in (0, T)$, $r \in (0, R_0)$, and suppose that $u_0 \in \mathcal{H}_2^n(B_r)$. Then, for almost any $\omega \in \Omega_0$, u is infinitely differentiable with respect to x for $(t, x) \in ([s_0, T] \times \mathbb{R}^d) \cup ([0, T] \times B_{R_0}^c)$ and each derivative is a bounded continuous function in $([s_0, T] \times \mathbb{R}^d) \cup ([0, T] \times B_{R_0}^c)$.*

Proof Take $r < r_1 < r_2 < R$ and observe that $B_{r_2} \setminus B_{r_1}$ can be covered by a finite number of balls lying inside $B_{R_0} \setminus B_r$, where $u_0 = 0$. By applying Theorem 3.1 to each such ball we conclude that, for almost any $\omega \in \Omega_0$, u_t is infinitely differentiable with respect x in $[0, T] \times (B_{r_2} \setminus B_{r_1})$ and each its derivatives is bounded and continuous in $[0, T] \times (B_{r_2} \setminus B_{r_1})$.

Then take a $\zeta \in C_0^\infty(B_{r_2})$ such that $\zeta = 1$ in a neighborhood of \bar{B}_{r_1} and set $\eta = 1 - \zeta$. In this case $u_t \eta$ satisfies an equation similar to (1.4) but with different f and g^k which are obtained by adding to the original ones u_t or its first-order derivatives multiplied by $C_0^\infty(\mathbb{R}^d)$ functions which vanish outside $B_{r_2} \setminus B_{r_1}$. The initial condition for $u_t \eta$ is obviously zero. By Theorem 2.4 and embedding theorems we conclude that, for almost any $\omega \in \Omega$, $u_t \eta$ is infinitely differentiable with respect to x for $(t, x) \in [0, T] \times \mathbb{R}^d$ and each derivative is a bounded continuous function in $[0, T] \times \mathbb{R}^d$.

The function $u_t \zeta$ satisfies an equation with the properties similar to those of the equation for $u_t \eta$ and by Theorem 3.1, for almost any $\omega \in \Omega_0$, it is infinitely differentiable with respect to x for $(t, x) \in [s_0, T] \times B_{R_0}$ and each derivative is a continuous function in $[s_0, T] \times B_{R_0}$. This proves the present theorem since $u_t = u_t \eta + u_t \zeta$ and $\zeta = 0$ outside B_{R_0} .

4 Applications to filtering problems

Here we come back to system (1.1). Let $K, \delta > 0$ be some fixed constants. We denote by θ and Θ the matrix-valued functions having θ^k and Θ^k , respectively, as their k th columns.

Assumption 4.1 The functions b, θ, B , and Θ are Borel measurable and bounded functions of their arguments. Each of them satisfies the Lipschitz condition with respect to z with constant K . These functions are infinitely differentiable with respect to x and each derivative of any order is a bounded function of (t, z) .

Assumption 4.2 For all $\xi \in \mathbb{R}^{d'}$ and (t, y)

$$|\Theta^*(t, y)\xi| \geq \delta|\xi|.$$

Notice that in [3] and [2] this condition is satisfied since there $\Theta = (0, I)$, where I is the unit $d' \times d'$ -matrix. In [1] and [15] this nondegeneracy condition is not imposed and the filtering density is not characterized as a solution of an SPDE.

Remark 4.1 Owing to Assumption 4.2, $d' \leq d_1$, the symmetric matrix $\Theta\Theta^*$ is invertible, and

$$\Psi := (\Theta\Theta^*)^{-\frac{1}{2}} \tag{4.1}$$

is a bounded function of (t, y) .

Assumption 4.3 The random vector $z_0 = (x_0, y_0)$ is independent of the process w_t .

Next, we introduce a few more notation. Let

$$\begin{aligned} \Psi_t &= \Psi(t, y_t), \quad \Theta_t = \Theta(t, y_t), \\ a_t(x) &= \frac{1}{2}\theta\theta^*(t, x, y_t), \quad b_t(x) = b(t, x, y_t), \quad B_t(x) = B(t, x, y_t), \\ \theta_t(x) &= \theta(t, x, y_t), \quad \sigma_t(x) = -\theta_t(x)\Theta_t^*\Psi_t, \quad \beta_t(x) = \Psi_t B_t(x). \end{aligned}$$

The columns $\sigma_t^k(x)$, $k = 1, \dots, d_1$, of the matrix-valued function $\sigma_t(x)$ will play the role of $\sigma_t^k(x)$, $k = 1, \dots, d_1$, in the setting of Sect. 2. For $i = 1, \dots, d_1$ the vector-valued functions $\sigma_t^{d_1+i}(x)$ are defined as the i th columns of

$$\theta_t(x) - \theta_t(x)\Theta_t^*\Psi_t^2\Theta_t$$

(so that $d_2 = d_1$ in the notation of Sect. 2, σ_t^0 will be introduced later). Observe that if $d' = d_1$, then Θ_t takes values in the set of square $d_1 \times d_1$ matrices and owing to Assumption 4.2 is nondegenerate. It follows that $\Theta^*\Psi^2\Theta$ is the identity operator for all (t, y) . In that case $\sigma_t^{d_1+i} \equiv 0$, $i = 1, \dots, d_1$, and there is no hope to get any smoothness of the posterior distribution of x_t unless the initial distribution has a smooth density. Therefore, we impose the following.

Assumption 4.4 We have $d' < d_1$ and, for an $S \in [0, T)$ and $R_0 \in (0, \infty)$, Assumption 3.2 is satisfied.

Remark 4.2 In a very popular so-called triangular scheme in which $\Theta = (0, \hat{\Theta})$, where $\hat{\Theta}$ is a nondegenerate square $d' \times d'$ -matrix valued function, one can easily check that for $i = 1, \dots, d_1 - d'$

$$\sigma_t^{d_1+i}(x) = \theta_t^i(x)$$

and $\sigma_t^{d_1+i}(x) \equiv 0$ for $i = d_1 - d' + 1, \dots, d_1$.

Remark 4.3 If the system (1.1) has a triangular form, say, as in Remark 4.2 with coefficients that are independent of t and are infinitely differentiable with respect to (x, y) a wider sets \mathbb{L}_{n+1} , $n \geq 0$, are allowed in [1] in the analog of our Assumption 3.2.

Introduce $d + d'$ -dimensional vector function $Y^i(x, y)$, $i = 1, \dots, d_1$, as the i th column of the matrix

$$\begin{pmatrix} \theta(x, y) \\ \Theta(y) \end{pmatrix}.$$

Fix a point $z_0 = (x_0, y_0) \in \mathbb{R}^d \times \mathbb{R}^{d'}$ and denote by T_{z_0} the subspace of $\mathbb{R}^d \times \mathbb{R}^{d'}$ generated by $Y^i(z_0)$, $i = 1, \dots, d_1 - d'$, and by the Lie brackets at z_0 of $Y^i(z)$, $i = 1, \dots, d_1$, in which at least one of $Y^i(z)$, $i = 1, \dots, d_1 - d'$ appears. Then the requirement in [1] is that $T_{z_0} = \mathbb{R}^d$ (more precisely $T_{z_0} = \mathbb{R}^d \times \{0\}$).

If, in addition, B is independent of x , then in [15] the requirement in [1] is further relaxed to include $\begin{pmatrix} b \\ B \end{pmatrix}$ into the set of vector-fields participating in forming the Lie brackets.

In our situation adding these vector-fields is impossible because our coefficients are only Lipschitz continuous with respect to y .

Introduce \mathcal{F}_t^y as the completion of $\sigma\{y_s : s \leq t\}$ and denote by P_0 the regular version of the conditional distribution of x_0 given y_0 .

Theorem 4.4 *Take an $s_0 \in (S, T]$. Let n be a negative integer such that $n < -1 - d/2$. Then there exists a function π of class $H_2^{n+1}(T, \mathcal{F}_t^y)$, such that*

- (i) π_t is a continuous H_2^n -valued function on $[0, T]$,
- (ii) for almost any $\omega \in \Omega_0$, $\pi_t(x)$ is infinitely differentiable with respect to x for $(t, x) \in [s_0, T] \times B_{R_0}$ and each derivative is a continuous function in $[s_0, T] \times B_{R_0}$,
- (iii) if $S = 0$ and the closed support of P_0 is a subset of B_{R_0} , then, for almost any $\omega \in \Omega_0$, $\pi_t(x)$ is infinitely differentiable with respect to x for $(t, x) \in ([s_0, T] \times \mathbb{R}^d) \cup ([0, T] \times B_{R_0}^c)$ and each derivative is a bounded continuous function in $([s_0, T] \times \mathbb{R}^d) \cup ([0, T] \times B_{R_0}^c)$,
- (iv) for any $f \in C_0^\infty(\mathbb{R}^d)$ and $t \in [0, T]$ with probability one

$$(\pi_t, f) = E\{f(x_t) \mid \mathcal{F}_t^y\}.$$

Before proving the theorem we prove the following.

Lemma 4.5 *There exists an H_2^{n+1} -valued weakly continuous \mathcal{F}_t^y -adapted process π_t such that assertion (iv) of Theorem 4.4 holds.*

Proof Take an $f \in C_0^\infty(\mathbb{R}^d)$. By the famous Fujisaki–Kallianpur–Kunita theorem the process $E\{f(x_t) \mid \mathcal{F}_t^y\}$ has a continuous modification, which we denote by $P_t(f)$. Then a well-known procedure (see, for instance, Chapter 5, §3.3 [16] or the Appendix in [9]) allows us to further modify, if necessary, $P_t(f)$, so that the new modification for which we use the same notation

- (i) is continuous in t and \mathcal{F}_t^y -adapted,

(ii) for any $\omega \in \Omega, t \geq 0$, and any nonnegative $f \in C_0^\infty(\mathbb{R}^d)$ we have

$$0 \leq P_t(f) \leq \sup f,$$

(iii) for any $\omega \in \Omega, t \geq 0, f, g \in C_0^\infty(\mathbb{R}^d)$, and numbers α, β we have

$$P_t(\alpha f + \beta g) = \alpha P_t(f) + \beta P_t(g).$$

Then by Riesz–Markov theorem, there exists a measure-valued function $P_t(dx)$ with $P_t(\mathbb{R}^d) \leq 1$ such that

$$P_t(f) = \int_{\mathbb{R}^d} f(x) P_t(dx)$$

for any $\omega \in \Omega, t \geq 0$, and $f \in C_0^\infty(\mathbb{R}^d)$.

By recalling that finite measures on \mathbb{R}^d belong to H_2^{n+1} , we identify $P_t(dx)$ with a generalized function $\pi_t \in H_2^{n+1}$. Observe that

$$\|\pi_t\|_{H_2^{n+1}} \leq N P_t(\mathbb{R}^d),$$

where N is the embedding constant. This and the continuity of (π_t, f) for $f \in C_0^\infty(\mathbb{R}^d)$ shows that π_t is weakly continuous as a H_2^{n+1} -valued function. All other assertions of the lemma follow from the above. The lemma is proved.

Remark 4.6 If $f(t, x, y)$ is a Borel bounded function, then for any $t \geq 0$ with probability one

$$E\{f(t, x_t, y_t) \mid \mathcal{F}_t^y\} = \int_{\mathbb{R}^d} f(t, x, y_t) P_t(dx), \tag{4.2}$$

where the right-hand side is a predictable function with respect to $\{\mathcal{F}_t^y\}$.

Indeed, we have seen that

$$E\{f(x_t) \mid \mathcal{F}_t^y\} = \int_{\mathbb{R}^d} f(x) P_t(dx)$$

(a.s.) for any $t \geq 0$ and $f \in C_0^\infty(\mathbb{R}^d)$, where the right-hand side is \mathcal{F}_t^y -predictable. This implies our claim in an absolutely standard way.

Proof of Theorem 4.4 Set

$$L_t(x) = a_t^{ij}(x)D_{ij} + b_t^i(x)D_i, \tag{4.3}$$

$$L_t^*(x)u_t(x) = D_{ij}(a_t^{ij}(x)u_t(x)) - D_i(b_t^i(x)u_t(x)),$$

$$\Lambda_t^k(x)u_t(x) = \beta_t^k(x)u_t(x) + \sigma_t^{ik}(x)D_iu_t(x), \tag{4.4}$$

$$\Lambda_t^{k*}(x)u_t(x) = \beta_t^k(x)u_t(x) - D_i(\sigma_t^{ik}(x)u_t(x))$$

where $t \in [0, \infty)$, $x \in \mathbb{R}^d$, $k = 1, \dots, d_1$, and as above we use the summation convention over all “reasonable” values of repeated indices, so that the summation in (4.3) and (4.4) is done for $i, j = 1, \dots, d$ (whereas in (4.5) for $k = 1, \dots, d_1$).

By the Fujisaki–Kallianpur–Kunita theorem, Lemma 4.5, and Remark 4.6 for any $f \in C_0^\infty(\mathbb{R}^d)$ with probability one for all $t \geq 0$

$$(\pi_t, f) = (\pi_0, f) + \int_0^t (\pi_s, L_s f) ds + \int_0^t (\pi_s, \Lambda_s^k f - \bar{\beta}_s^k f) d\hat{w}_s^k, \tag{4.5}$$

where $(\hat{w}_t^1, \dots, \hat{w}_t^{d_1})$ is a standard Wiener process with respect to the filtration $\{\mathcal{F}_t^y\}$ and

$$\bar{\beta}_t = \int_{\mathbb{R}^d} \beta(t, x, y_t) P_t(dx).$$

By a classical result about the Itô formula for the squared norm of Banach space-valued processes, Eq. (4.5) implies that (a.s.) π_t is a continuous H_2^n -valued process (see Remark 2.2). Next, elementary computations show that

$$a_t = (1/2) \sum_{k=1}^{2d_1} \sigma_t^k \sigma_t^{k*}, \quad L_t^* = (1/2) \sum_{k=1}^{2d_1} L_{\sigma_t^k}^2 + L_{\sigma_t^0} + c_t,$$

where

$$\sigma_t^0 = (1/2)L_{\sigma_t^k} \sigma_t^k + (\text{div} \sigma_t^k) \sigma_t^k - b_t, \quad c_t = D_{ij} a_t^{ij} - \text{div} b_t$$

and for $k = 1, \dots, d_1$

$$\Lambda_t^{k*} - \bar{\beta}_t^k = L_{\sigma_t^k} + v_t^k,$$

where

$$v_t^k = \beta_t^k - \bar{\beta}_t^k - \text{div} \sigma_t^k.$$

After that the remaining assertion (ii) and (iii) of the present theorem follows directly from Theorems 3.1 and 3.2. The theorem is proved.

In the following two theorems we, actually, speak about $\pi_t(y, x)$ which is defined as $\pi_t(x)$ when P_0 is the δ -function concentrated at y . These theorems are just direct consequences of Theorems 5.4 and 5.5 and of what was said in the proof of Theorem 4.4. It is worth drawing the reader’s attention to the fact that no continuity with respect to (t, y, x) is claimed in assertions (i) and (iv) of Theorem 4.7 and no continuity with respect to y is claimed in assertion (iii).

Theorem 4.7 *Assume that $\Omega_0 = \Omega, S = 0$ and take $r \in (0, R_0)$ and $s_0 \in (0, T)$. Then there exists a nonnegative function $p_t(y, x) = p_t(\omega, y, x)$ defined for*

$$(\omega, t, y, x) \in \Omega \times (0, T] \times B_r \times \mathbb{R}^d$$

such that

- (i) *it is infinitely differentiable with respect to y in B_r , each of its y -derivative of any order is a bounded function of $(t, y, x) \in [s_0, T] \times B_r \times \mathbb{R}^d$ for any $\omega \in \Omega$ and multi-index α ,*
- (ii) *for any probability distribution P_0 , which is concentrated in B_r , for almost any $\omega \in \Omega$, we have*

$$\pi_t(x) = \int_{B_r} p_t(y, x) P_0(dy)$$

- for all $(t, x) \in (0, T] \times \mathbb{R}^d$,*
- (iii) *for any $y \in B_r$, for almost any $\omega \in \Omega$, for any $t \in (0, T]$ and multi-index α the function $D_y^\alpha p_t(y, x)$ is infinitely differentiable with respect to x and each of its x -derivative of any order is a bounded function of $(t, x) \in ([s_0, T] \times \mathbb{R}^d) \cup ((0, T] \times B_{R_0}^c)$,*
- (iv) *the function $p_t(y, x)$ is infinitely differentiable with respect to $(x, y) \in B_r \times B_r$ and each its derivative of any order is a bounded function of $(t, y, x) \in [s_0, T] \times B_r^2$ for almost any $\omega \in \Omega$.*

We have somewhat better properties of $p_t(x, y)$ for a special class of filtering problems.

Theorem 4.8 *Assume that $S > 0$, take $r \in (0, R_0), s_0 \in (S, T)$ and assume that there exists an $R \in (0, \infty)$ such that, if $|x| \geq R$, then both $\sigma_t(x) = 0$ and $B(t, x, y_t)$ is independent of x . Then there exists a nonnegative function $p_t(x, y) = p_t(\omega, y, x)$ defined for*

$$(\omega, t, y, x) \in \Omega_0 \times (S, T] \times \mathbb{R}^d \times B_r$$

such that

- (i) it is infinitely differentiable with respect to $(y, x) \in \mathbb{R}^d \times B_r$, each its derivative of any order is a bounded function of $(t, y, x) \in [s_0, T] \times \mathbb{R}^d \times B_r$ and the functions $\|p_t(\cdot, x)\|_l$ are bounded on $[s_0, T] \times B_r$ for any $\omega \in \Omega_0$ and $l \geq 0$,
- (ii) for almost any $\omega \in \Omega_0$ we have

$$\pi_t(x) = \int_{\mathbb{R}^d} p_t(y, x) P_0(dy)$$

for all $(t, x) \in (S, T] \times B_r$.

Remark 4.9 An example when $\sigma_t = 0$ is given by the block-diagonal matrix

$$\begin{pmatrix} \theta_t \\ \Theta_t \end{pmatrix} = \begin{pmatrix} \theta'_t & 0 \\ 0 & \Theta'_t \end{pmatrix},$$

where θ'_t and Θ'_t are $d \times (d_1 - d')$ - and $d' \times d'$ -matrix valued functions, respectively.

5 Fundamental solutions of SPDEs

Here we continue our investigation of solutions of general equations (1.4) under the assumptions stated in Sect. 3. We also assume that

$$g^k \equiv 0, \quad k = 1, \dots, d_1.$$

Here again u is the normal solution from Theorem 2.4 and n is from Assumption 2.1 (iii).

Lemma 5.1 *Take an $R \in [0, \infty)$, $s_0 \in (S, T)$, $r \in (0, R_0)$, and assume that $\sigma_t^k = 0$ and $Dv_t^k = 0$ outside B_R for any $k = 1, \dots, d_1$, t , and ω . Then, for any l and multi-index α , such that*

$$2(l - |\alpha| - 2) > d + 1, \tag{5.1}$$

there exists a (random, finite) constant N independent of u such that, for almost any $\omega \in \Omega_0$,

$$\sup_{(t,x) \in [s_0, T] \times B_r} |D^\alpha u_t(x)|^2 \leq N \|u_0\|_n^2 + N \int_0^T \|f_t\|_l^2 dt. \tag{5.2}$$

Proof Obviously, it suffices to concentrate on $n \leq l$. Take a $\zeta \in C_0^\infty(B_{R_0})$ such that $\zeta = 1$ in a neighborhood of \bar{B}_r . Then by Theorem 3.1 for any l and multi-index α satisfying (5.1), there exists a (random, finite) constant N independent of u such that,

for almost any $\omega \in \Omega_0$,

$$\sup_{(t,x) \in [S,T] \times B_r} |D^\alpha u_t(x)|^2 \leq N \int_S^T \left[\|f_t \zeta\|_l^2 + \|u_t \zeta\|_n^2 \right] dt. \tag{5.3}$$

Owing to Theorem 2.7

$$\|u_t \zeta\|_n^2 \leq N \|u_t\|_n^2 \leq N \|u_0\|_n^2 + N \int_0^t \|f_s\|_n^2 ds$$

and this proves the lemma.

Lemma 5.2 *In the setting of of Lemma 5.1 suppose that $S = 0$ and u_0 is infinitely differentiable in B_{R_0} for every $\omega \in \Omega_0$. Then for any $\zeta \in C_0^\infty(B_{R_0})$ such that $\zeta = 1$ in a neighborhood of \bar{B}_r , any l and multi-index α satisfying (5.1), there exists a (random, finite) constant N independent of u such that, for almost any $\omega \in \Omega_0$,*

$$\sup_{(t,x) \in [0,T] \times B_r} |D^\alpha u_t(x)|^2 \leq N \|\zeta u_0\|_{l+1}^2 + N \int_0^t \|f_t\|_l^2 dt + N \|u_0\|_n^2.$$

This lemma is derived from Theorems 3.1 and 2.7 in the same way as Lemma 5.1.

Theorem 5.3 *Assume that $\Omega_0 = \Omega$, $S = 0$ and take $s_0 \in (0, T)$ and $0 < r < r_1 < r_2 < R_0$. Suppose that $u_0 \in \mathcal{H}_2^n(B_r)$. Finally, let $f_t(x) = 0$ for $|x| \geq r$.*

Then for any l and multi-index α satisfying (5.1), and $n \in \mathbb{R}$, there exists a (random, finite) constant N independent of u such that, for almost any $\omega \in \Omega$,

$$\sup_{(t,x) \in \Gamma} |D^\alpha u_t(x)|^2 \leq NJ, \tag{5.4}$$

where

$$\Gamma = ([0, T] \times B_{r_2}) \setminus ([0, s_0] \times B_{r_1}), \quad J = \int_0^T \|f_t\|_l^2 dt + N \|u_0\|_n^2.$$

Proof We take $r_2 < r_3 < R_0$, $\zeta \in C_0^\infty(B_{r_3})$, and $\eta \in C_0^\infty(B_{r_2})$ such that $\zeta = 1$ in a neighborhood of \bar{B}_{r_2} and $\eta = 1$ in a neighborhood of \bar{B}_{r_1} .

As in the proof of Theorem 3.2 by covering $B_{r_2} \setminus B_{r_1}$ with appropriate balls and applying Theorem 3.1 we see that, for almost any $\omega \in \Omega$,

$$\sup_{(t,x) \in [0,T] \times (B_{r_2} \setminus B_{r_1})} |D^\alpha u_t(x)|^2 \leq N \int_0^T [\|f_t\|_l^2 + \|\zeta u_t\|_n^2] dt,$$

where the (random) constant N is independent of u, f . All such constants will be denoted by N . Furthermore, by the same theorem

$$\sup_{(t,x) \in [s_0, T] \times B_{r_2}} |D^\alpha u_t(x)|^2 \leq N \int_0^T [\|f_t\|_l^2 + \|\zeta u_t\|_n^2] dt.$$

It follows that to prove (5.4), it suffices to show that there exists a (random, finite) constant N independent of u such that, for almost any $\omega \in \Omega$,

$$\int_0^T \|\zeta u_t\|_n^2 dt \leq NJ. \tag{5.5}$$

Observe that, obviously, we may assume that $n < 0, 2|n| > d$.

Set

$$P_t = \|u_t \zeta\|_n^2, \quad Q_t = \|u_t \eta\|_n^2$$

and observe that

$$P_t \leq 2Q_t + 2\|u_t(\zeta - \eta)\|_n^2,$$

where, as in the proof of Theorem 3.2, for almost any $\omega \in \Omega$, $u_t(\zeta - \eta)$ is infinitely differentiable with respect to x for $(t, x) \in [0, T] \times \mathbb{R}^d$ and each derivative is a bounded continuous function in $[0, T] \times \mathbb{R}^d$. In particular, for almost any $\omega \in \Omega$,

$$\|u_t(\zeta - \eta)\|_n \leq N \sup_{[0,t] \times \mathbb{R}^d} |u_s(\zeta - \eta)(x)| \leq N \sup_{[0,t]} \sup_{B_{r_1}^c} |u_s(x)|$$

where the first inequality holds by embedding theorems since $n < 0$ and $2|n| > d$. Next, we take an infinitely differentiable function $\kappa(x) \geq 0$ such that $\kappa(x) = 1$ for $|x| \geq r_1$ and $\kappa(x) = 0$ for $|x| \leq r$ and apply Theorems 3.1 and 2.9 (for this we needed the condition that $\Omega_0 = \Omega$) to the function $u_t \kappa$ to see that

$$\begin{aligned} \sup_{[0,t]} \sup_{B_{r_1}^c} |u_s(x)| &\leq N \sup_{[0,t]} \sup_{|x|=r_1} |u_s(x)|, \\ P_t &\leq 2Q_t + N \sup_{[0,t]} \sup_{|x|=r_1} |u_s(x)|^2. \end{aligned} \tag{5.6}$$

To continue, we need an auxiliary function. Take a $\xi \in C_0^\infty(B_{2R_0})$ such that $\xi = 1$ on B_{R_0} and define v_t as a normal $H_2^n(T)$ -solution of the equation

$$dv_t = (L_t v_t + c_t v_t + f_t) dt + (L_{\xi \sigma_t^k} v_t + \xi v_t^k) dw_t^k$$

with initial condition u_0 . By Theorem 2.7

$$\sup_{t \leq T} \|v_t\|_n^2 \leq NJ. \tag{5.7}$$

By Theorem 3.1, for almost any $\omega \in \Omega$,

$$\sup_{[0, T]} \sup_{|x|=r_1} |v_s(x)|^2 \leq N \int_0^T [\|f_t\|_l^2 + \|v_t\|_n^2] dt,$$

which after being combined with (5.7) shows that

$$\sup_{[0, T]} \sup_{|x|=r_1} |v_s(x)|^2 \leq NJ$$

and

$$\sup_{[0, t]} \sup_{|x|=r_1} |u_s(x)|^2 \leq 2 \sup_{[0, t]} \sup_{|x|=r_1} |u_s(x) - v_s(x)|^2 + NJ$$

for almost any $\omega \in \Omega$.

By applying Theorem 3.1 to $u_t - v_t$ we conclude that for $t \in [0, T]$ and almost any $\omega \in \Omega$

$$\sup_{[0, t]} \sup_{B_{r_1}} |u_s - v_s|^2 \leq N \int_0^t \|\eta(u_s - v_s)\|_n^2 ds. \tag{5.8}$$

Hence,

$$\sup_{[0, t]} \sup_{|x|=r_1} |u_s(x)|^2 \leq N \int_0^t \|\eta(u_s - v_s)\|_n^2 ds + NJ \leq N \int_0^t \|\eta u_s\|_n^2 ds + NJ,$$

which along with (5.6) show that

$$P_t \leq 2Q_t + N \int_0^t \|\eta u_s\|_n^2 ds + NJ \leq 2Q_t + N \int_0^t P_s ds + NJ. \tag{5.9}$$

Then

$$Q_t \leq 2\|(u_t - v_t)\eta\|_n^2 + 2\|\eta v_t\|_n^2 \leq N \sup_{B_{r_2}} |u_t - v_t|^2 + NJ$$

and similarly to (5.8)

$$\sup_{B_{r_2}} |u_t - v_t|^2 \leq N \int_0^t \|\zeta(u_s - v_s)\|_n^2 ds \leq N \int_0^t P_s ds + NJ.$$

By coming back to (5.9) and using Gronwall’s inequality we conclude $P_t \leq NJ$ and this proves (5.5) and the theorem.

In the following theorem we prove the existence of a kernel for our SPDE. It is worth drawing the reader’s attention to the fact that no continuity with respect to (t, y, x) is claimed in (i) and (iv) and no continuity with respect to y is claimed in (iii).

Theorem 5.4 *Assume that $\Omega_0 = \Omega$, $S = 0$, and take $r \in (0, R_0)$ and $s_0 \in (0, T)$. Then there exists a nonnegative function $p_t(y, x) = p_t(\omega, y, x)$ defined for*

$$(\omega, t, y, x) \in \Omega \times (0, T] \times B_r \times \mathbb{R}^d$$

such that

- (i) *it is infinitely differentiable with respect to y in B_r , each of its y -derivative of any order is a bounded function of $(t, y, x) \in [s_0, T] \times B_r \times \mathbb{R}^d$, $\omega \in \Omega$,*
- (ii) *if $u_0 \in \mathcal{H}_2^0(B_r)$ and $f \equiv g^k \equiv 0$, $k = 1, \dots, d_1$, then, for almost any $\omega \in \Omega$, $u_t(x)$ coincides (as a generalized function with respect to x) with*

$$(u_0, p_t(\cdot, x)) = \int_{B_r} u_0(y) p_t(y, x) dy \tag{5.10}$$

for all $(t, x) \in (0, T] \times \mathbb{R}^d$.

- (iii) *for any $y \in B_r$, for almost any $\omega \in \Omega$, for any $t \in (0, T]$ and multi-index α the function $D_y^\alpha p_t(y, x)$ is infinitely differentiable with respect to x and each of its x -derivative of any order is a bounded and continuous function of $(t, x) \in ([s_0, T] \times \mathbb{R}^d) \cup ((0, T] \times B_{R_0}^c)$,*
- (iv) *the function $p_t(y, x)$ is infinitely differentiable with respect to $(x, y) \in B_r \times B_r$ and each its derivative of any order is a bounded function of $(t, y, x) \in [s_0, T] \times B_r^2$ for any $\omega \in \Omega$.*

Proof Let $\mathcal{Z}_+ = \{\zeta_1, \zeta_2, \dots\}$ be a countable subset of $C_0^\infty(B_r)$ consisting of non-negative functions such that \mathcal{Z}_+ is dense in the sup-norm in the subset of $C_0^\infty(B_r)$ consisting of nonnegative functions. Also set $\mathcal{Z} = \mathcal{Z}_+ - \mathcal{Z}_+$.

For $\zeta \in \mathcal{H}_2^{-l}(B_r)$, $l \geq 0$, denote by $u_t[\zeta](x)$ the normal solution of (1.4) with initial condition ζ and with $f \equiv g^k \equiv 0$, $k = 1, \dots, d_1$. By Theorems 5.3 and 2.9 for any $l \geq 0$ and $t \in (0, T]$, there exists a (random) constant $N_t(l)$ such that, for almost any $\omega \in \Omega$, for any $x \in \mathbb{R}^d$ and $\zeta \in \mathcal{H}_2^{-l}(B_r)$ we have

$$|u_t[\zeta](x)| \leq N_t(l) \|\zeta\|_{-l}, \tag{5.11}$$

and if $\zeta \in \mathcal{Z}_+$ then by the maximum principle

$$u_t[\zeta](x) \geq 0. \tag{5.12}$$

Furthermore, owing to uniqueness for any collection of rational numbers p_1, p_2, \dots only finite number of which is different from zero, for almost any $\omega \in \Omega$,

$$\left| \sum_i p_i u_t[\zeta_i](x) \right| = \left| u_t \left[\sum_i p_i \zeta_i \right] (x) \right| \leq N_t(l) \left\| \sum_i p_i \zeta_i \right\|_{-l}. \tag{5.13}$$

Obviously, one can choose $N_t(l)$ so that it is a monotone function of t . Owing to this and the fact that other expressions entering (5.13) and (5.12) are continuous in (t, x) for almost any $\omega \in \Omega$ and the fact that the set of rational numbers is countable, there is a set $\Omega' \subset \Omega$ with $P(\Omega \setminus \Omega') = 0$ such that (5.13) is satisfied for any l , any collection of rational numbers p_1, p_2, \dots , only finite number of which is different from zero, any $\omega \in \Omega'$, and any $(t, x) \in (0, T] \times \mathbb{R}^d$, and (5.12) is satisfied for any $\omega \in \Omega'$ and any $(t, x) \in (0, T] \times \mathbb{R}^d$ and $\zeta \in \mathcal{Z}_+$. By setting $u_t[\zeta](x) = 0$ on $\Omega \setminus \Omega'$, we may assume that $\Omega' = \Omega$.

By a theorem of Hahn (see, for instance, Section II.5 of [5] (where set $X = H_2^{-l}$, $\alpha_i = p_i$, $c_i = u_t[\zeta_i](x)$, $x_i = \zeta_i$), for any $x \in \mathbb{R}^d$, $t \in (0, T]$, $l \geq 0$, and $\omega \in \Omega$ there exists a linear functional $Q_t[\cdot](x)$ on H_2^{-l} such that

$$Q_t[\zeta_i](x) = u_t[\zeta_i](x), \quad \forall i, \quad \|Q_t[\cdot](x)\| \leq N_t(l).$$

The general form of linear functionals Q on H_2^{-l} is well known and for smooth elements $f \in H_2^{-l}$ it is given by

$$Q(f) = \int_{\mathbb{R}^d} f(x)p(x) dx,$$

where $p \in H_2^l$ and $\|p\|_l = \|Q\|$.

Hence, for any $x \in \mathbb{R}^d$, $t \in (0, T]$, $l \geq 0$, and $\omega \in \Omega$ there exists a function $p_t(\cdot, x) \in H_2^l$ such that

$$\|p_t(\cdot, x)\|_l \leq N_t(l), \quad \int_{\mathbb{R}^d} \zeta_i(y)p_t(y, x) dy = u_t[\zeta_i](x) \quad \forall i. \tag{5.14}$$

In principle $p_t(y, x)$ is not unique, it might be changed for $y \notin B_r$. In addition it may depend on l . To choose a better representative, without losing generality we take $l = 2m$, where $m = 1, 2, \dots$, so that

$$\|u\|_l = \|(1 - \Delta)^m u\|_0$$

and then set $p_t(y, x) = 0$ for $y \notin B_r$. Then the new $p_t(\cdot, x)$ for which we keep the same notation will satisfy the second relation in (5.14), hence will be independent of l . Also, owing to (5.14) and the choice of l , for thus modified $p_t(y, x)$ the functions

$$\int_{B_r} |D_y^\alpha p_t(y, x)|^2 dy$$

are bounded on $[s_0, T] \times \mathbb{R}^d$ for any $\omega \in \Omega$ and multi-index α . Since this holds for any multi-index, applying embedding theorems we obtain assertion (i). The second relation in (5.14) along with (5.12) and the choice of $\{\zeta_i\}$ implies that $p_t(y, x) \geq 0$ for any $x \in \mathbb{R}^d, y \in B_r, t \in (0, T]$, and $\omega \in \Omega$.

The relations in (5.14), and the fact that \mathcal{Z} is dense in $\mathcal{H}_2^0(B_r)$ imply assertion (ii). For almost all $\omega \in \Omega$ the expressions in (5.10) are infinitely differentiable with respect to x and each of the derivatives is bounded and continuous in $([s_0, T] \times \mathbb{R}^d) \cup ([0, T] \times B_{R_0}^c)$ by Theorem 3.2 if $u_0 \in \mathcal{H}_2^0(B_r)$. This proves assertion (iii) if we set $u_0 = D^\alpha \delta_y$, where δ_y is the delta-function concentrated at $y \in B_r$.

To prove assertion (iv) observe that, owing to Theorem 5.3, for $\zeta \in \mathcal{Z}$ we can modify the functions $u_t[\zeta](x)$, if necessary, in such a way that for any $\omega \in \Omega$ and $t \in (0, T]$ they will be infinitely differentiable with respect to x in B_r and for any multi-index α and $l > 0$ satisfy

$$\sup_{B_r} |D^\alpha u_t[\zeta](x)| \leq N_t(|\alpha|, l) \|\zeta\|_{-l}, \tag{5.15}$$

where $N_t(|\alpha|, l)$ are independent of ζ . In particular, for $x, x', x'' \in B_r$ and $\zeta, \zeta', \zeta'' \in \mathcal{Z}$

$$\begin{aligned} |u_t[\zeta](x') - u_t[\zeta](x'')| &\leq N_t(1, l) \|\zeta\|_{-l} |x' - x''|, \\ |u_t[\zeta'](x) - u_t[\zeta''](x)| &\leq N_t(0, l) \|\zeta' - \zeta''\|_{-l}. \end{aligned} \tag{5.16}$$

Since \mathcal{Z} is dense in $\mathcal{H}_2^{-l}(B_r)$ and (5.14) holds, estimates (5.16) hold for all $\zeta, \zeta', \zeta'' \in \mathcal{H}_2^{-l}(B_r)$, which after taking an appropriate l and ζ, ζ', ζ'' as δ -functions show that $p_t(x, y)$ is a continuous function of (x, y) on B_r^2 .

Estimate (5.15) also implies that the generalized derivatives of $(\zeta, p_t(\cdot, x))$ of order $|\alpha|$ are bounded on B_r by the right-hand side of (5.15) for any $\zeta \in \mathcal{H}_2^{-l}(B_r)$. Since this holds for any α and $(\zeta, p_t(\cdot, x))$ is continuous with respect to $x \in B_r$ (owing to (5.16)), the generalized derivatives are, actually, usual ones, which are bounded and continuous.

By taking $\zeta = D_y^\beta \delta_y$ in (5.15), we find that the usual (and hence generalized) functions $D_x^\alpha [D_y^\beta p_t(y, x)]$ are bounded for $(x, y) \in B_r^2$. It follows that $p_t(y, x)$ admits a modification with respect to (y, x) which is infinitely differentiable. However, the modification coincides with $p_t(y, x)$ on B_r^2 since $p_t(y, x)$ is continuous there.

The asserted boundedness of the derivatives of $p_t(y, x)$ is easily derived from the above argument. The theorem is proved.

In the following theorem we assert the regularity of $p_t(y, x)$ not only for $y \in B_r$ but for all $y \in \mathbb{R}^d$ albeit for $x \in B_r$, the latter being of course inevitable. However the result is proved under a somewhat restrictive assumption. This assumption arose because of our inability to control $\|u_t\|_n$ through $\|u_0\|_n$ times a (random) constant independent of u_0 .

Theorem 5.5 *Suppose that $S > 0$, take an $R \in [0, \infty)$, $r \in (0, R_0)$, $s_0 \in (S, T)$, and assume that $\sigma_t^k = 0$ and $Dv_t^k = 0$ outside B_R for any $k = 1, \dots, d_1$, $t \in [0, T]$, and $\omega \in \Omega$. Then there exists a nonnegative function $p_t(y, x) = p_t(\omega, y, x)$ defined for*

$$(\omega, t, y, x) \in \Omega_0 \times (S, T] \times \mathbb{R}^d \times B_r$$

such that

- (i) *it is infinitely differentiable with respect to $(y, x) \in \mathbb{R}^d \times B_r$, each its derivative of any order is a bounded function of $(t, y, x) \in [s_0, T] \times \mathbb{R}^d \times B_r$ and the functions $\|p_t(\cdot, x)\|_l$ are bounded on $[s_0, T] \times B_r$ for any $\omega \in \Omega_0$ and $l \geq 0$,*
- (ii) *if $f \equiv g^k \equiv 0$, $k = 1, \dots, d_1$, then (a.s.) on Ω_0*

$$u_t(x) = \int_{\mathbb{R}^d} p_t(y, x)u_0(y) dy$$

for all $(t, x) \in (S, T] \times B_r$.

Proof It suffices to repeat the proof of Theorem 5.4 using the same notation as there, replacing \mathcal{Z}_+ with a countable subset of $C_0^\infty(\mathbb{R}^d)$, and using Lemma 5.1 to conclude that for any and $l \geq 0$ and $t \in (S, T]$ there exists a (random) constant $N_t(l)$ such that, for any $\zeta \in \mathcal{H}_2^{-l}$ and $x \in B_r$, estimate (5.11) holds for almost any $\omega \in \Omega_0$. After that the proof goes almost exactly the same way as that of Theorem 5.4. The theorem is proved.

Acknowledgments The author is sincerely grateful to the referee for a few comments which helped improve the presentation.

References

1. Bismut, J.-M., Michel, D.: Diffusions conditionnelles. I. Hypocoellipticit e partielle (French). J. Funct. Anal. **44**(2), 174–211 (1981)
2. Chaleyat-Maurel, M.: Malliavin calculus application to the study of nonlinear filtering. In: The Oxford Handbook of Nonlinear Filtering, pp. 195–230. Oxford University Press, Oxford (2011)
3. Chaleyat-Maurel, M., Michel, M.: Hypocoellipticity theorems and conditional laws. Z. Wahrsch. Verw. Gebiete **65**(4), 573–597 (1984)
4. Cattiaux, P., Mesnager, L.: Hypocoelliptic non-homogeneous diffusions. Probab. Theory Relat. Fields **123**(4), 453–483 (2002)
5. Dunford, N., Schwartz, J.T.: Linear Operators, Part I, General Theory. Interscience Publishers, New York (1958)

6. Krylov, N.V.: An analytic approach to SPDEs. In: Stochastic Partial Differential Equations: Six Perspectives, Mathematical Surveys and Monographs, vol. 64, pp. 185–242. AMS, Providence (1999)
7. Krylov, N.V.: Maximum principle for SPDEs and its applications. In: Baxendale, P.H., Lototsky, S.V. (eds.) Stochastic Differential Equations: Theory and Applications, A Volume in Honor of Professor Boris L. Rozovskii, Interdisciplinary Mathematical Sciences, vol. 2, pp. 311–338. World Scientific, Singapore (2007)
8. Krylov, N.V.: Hörmander’s theorem for stochastic partial differential equations. <http://arxiv.org/abs/1309.5543>
9. Krylov, N.V., Rozovsky, B.L.: On conditional distributions of diffusion processes. *Izvestiya Akademii Nauk SSSR, seriya matematicheskaya* **42**(2), 356–378 (1978) [in Russian; English transl. in *Math. USSR Izvestija* **12**(2), 336–356 (1978)]
10. Krylov, N.V., Rozovsky, B.L.: Stochastic evolution equations, “Itogy nauki i tekhniki”, vol. 14, VINITI, Moscow, pp. 71–146 (1979) [in Russian; English translation in *J. Soviet Math.* **16**(4), 1233–1277 (1981)]
11. Krylov, N.V., Rozovsky, B.L.: On the characteristics of degenerate second order parabolic Itô equations. *Trudy seminar imeni Petrovskogo* **8**, 153–168 (1982) [in Russian, English transl. in *J. Soviet Math.* **32**(4), 336–348 (1986)]
12. Kunita, H.: Stochastic partial differential equations connected with nonlinear filtering, Nonlinear filtering and stochastic control (Cortona, 1981). *Lecture Notes Math.* **972**, 100–169 (1982)
13. Kunita, H.: Densities of a measure-valued process governed by a stochastic partial differential equation. *Syst. Control Lett.* **1**(2), 100–104 (1981/82)
14. Kunita, H.: Stochastic flows and stochastic differential equations. *Cambridge Studies in Advanced Mathematics*, vol. 24. Cambridge University Press, Cambridge (1990)
15. Kusuoka, S., Stroock, D.: The partial Malliavin calculus and its application to nonlinear filtering. *Stochastics* **12**(2), 83–142 (1984)
16. Rozovskii, B.L.: Stochastic Evolution Systems. Kluwer, Dordrecht (1990)
17. Ventcel, A.D.: On equations of the theory of conditional Markov processes. *Teoriya Veroyatnostei i ee Primeneniya* **10**(2), 390–393 (1965) [in Russian; English transl. *Theory Probability and Appl.* **10**, 357–361 (1965)]