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Stationary and invariant densities and disintegration kernels

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Abstract Let G be a locally compact topological group, acting measurably on some Borel spaces S and T, and consider some jointly stationary random measures ξ on $S \times T$ and η on S such that $\xi(\cdot \times T) \ll \eta$ a.s. Then there exists a stationary random kernel ζ from S to T such that $\xi = \eta \otimes \zeta$ a.s. This follows from the existence of an invariant kernel φ from $S \times \mathcal{M}_{S\times T} \times \mathcal{M}_S$ to T such that $\mu = \nu \otimes \varphi(\cdot, \mu, \nu)$ whenever $\mu(\cdot \times T) \ll \nu$. Also included are some related results on stationary integration, absolute continuity, and ergodic decomposition.

Keywords Stationary random measures · Differentiation and disintegration · Locally compact topological groups · Skew factorization · Ergodic decomposition

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1 Introduction

Let G be a locally compact, second countable topological group, acting measurably on some Borel spaces S and T, and consider some jointly G-stationary random measures ξ on $S \times T$ and η on S such that $\xi(\cdot \times T) \ll \eta$ a.s. In Theorem 6.1 we prove the existence of a random kernel ζ from S to T, which is G-stationary, jointly with the pair (ξ, η), and such that $\xi = \eta \otimes \zeta$ a.s. Here the *G-stationarity* is defined as invariance in distribution under simultaneous shifts in S and T, whereas the *disintegration* $\xi = \eta \otimes \zeta$ is given, for measurable functions $f \geq 0$ on $S \times T$, by

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$$\xi f \equiv \iint \xi(ds\,dt)\,f(s,t) = \int \eta(ds) \int \zeta_s(dt)\,f(s,t). \tag{1}$$

when T is a singleton, we may regard ξ and η as jointly stationary random measures on S with $\xi \ll \eta$ a.s., in which case the disintegration reduces to $\xi = X \cdot \eta$ a.s. for some measurable random process $X \geq 0$ on S such that (ξ, η, X) is G-stationary. Here $X \cdot \eta$ denotes the random measure on S with η-density X, so that $\xi f = \eta(Xf)$ a.s. for any measurable function $f \geq 0$ on S.

Such problems arise naturally in some areas of stochastic geometry, where they have been solved in special cases by Papangelou [20], pp. 307, 312ff, and in [7], pp. 164f, and [8], pp. 137ff. Here we attempt a more systematic treatment. The mentioned problems are highly non-trivial already for non-random measures ξ and η, where the stationarity reduces to *G-invariance*, defined for measures μ on S by $\mu \circ \theta_r^{-1} = \mu$ for all $r \in G$, and for kernels φ from S to T by the condition

$$\varphi_{rs} = \varphi_s \circ \theta_r^{-1}, \quad r \in G, \;\; s \in S. \tag{2}$$

Here and below, we define the *shifts* θ_r on S and *projections* π_s from G to S by $rs = \theta_r s = \pi_s r$ for any $r \in G$ and $s \in S$. Invariant disintegrations of measures on product spaces, of crucial importance in the context of Palm and moment measures, is the subject of an extensive literature, with contributions by (among others) Ryll-Nardzewski [23], Matthes [15], Mecke [17], Krickeberg [13], Papangelou [19], Rother and Zähle [22], Last [14], Gentner and Last [4], as well as in [11,12].

The random case is more difficult. Here our strategy is to construct a stationary disintegration kernel by substituting ξ and η into a deterministic kernel φ from $S \times \mathcal{M}_{S\times T} \times \mathcal{M}_S$ to T satisfying the disintegration formula

$$\mu = \nu \otimes \varphi(\cdot, \mu, \nu), \quad \mu \in \mathcal{M}_{S\times T}, \;\; \nu \in \mathcal{M}_S, \;\; \mu(\cdot \times T) \ll \nu, \tag{3}$$

along with the more general invariance relation

$$\varphi(s, \mu, \nu) \circ \theta_r^{-1} = \varphi\Big(rs, (\mu, \nu) \circ \theta_r^{-1}\Big), \quad r \in G, \;\; s \in S. \tag{4}$$

Note that (4) reduces to (2) when μ and ν are invariant. The construction of φ is highly non-trivial already for singleton T, when (3) reduces to the differentiation

$$\mu = \varphi(\cdot, \mu, \nu) \cdot \nu, \quad \mu, \nu \in \mathcal{M}_S, \;\; \mu \ll \nu, \tag{5}$$

and the invariance in (4) simplifies to

$$\varphi(s, \mu, \nu) = \varphi\Big(rs, (\mu, \nu) \circ \theta_r^{-1}\Big), \quad r \in G, \;\; s \in S. \tag{6}$$

In Theorem 4.1 we prove the existence of a kernel φ satisfying (3) and (4). The proof is rather intricate and proceeds in several steps. First we consider the density problem when $S = G$ is a Lie group. Here we may start from an arbitrary inner product on the basic tangent space to construct an invariant Riemannian metric on G (cf. [2]). By an extension of Besicovich' covering theorem (cf. [6], Theorem 1.14 and Example

1.15c), the associated geodesic balls form a differentiation basis with respect to any measure $\nu \in \mathcal{M}_G$, which yields the required invariant density.

For general locally compact groups G, there exists an open subgroup G' such that the coset space G/G' is discrete and countable, whereas G' is a projective limit of Lie groups (cf. [18]). In other words, there exists a sequence of compact, normal subgroups $H_n \downarrow \{\iota\}$, such that the quotient groups G'/H_n are isomorphic to Lie groups. The corresponding cosets form a nested sequence of invariant partitions of G', and the associated invariant densities may be regarded as functions φ_n on G', forming densities on the induced σ-fields $\mathcal{G}_n$. By a standard martingale argument (cf. [10], p. 133), the φ_n converge a.e. to a limit φ, which is an invariant density on the Borel σ-field of G'. It remains to extend φ by invariance to a density on all of G.

Next we may extend the density result to any product space $G \times S$ such that G has no action on S. This is accomplished by another martingale argument, based on a nested sequence of dissections of S. Using a skew transformation (cf. [11,12]), we may proceed to product spaces $G \times S$ with arbitrary group actions on S. Finally, we may deduce the density result on the space S alone by attaching a factor (G, λ) to the original measure spaces (S, μ) and (S, ν), where λ denotes a Haar measure on G. The general disintegration case requires an additional perfection argument, similar to the standard construction of regular conditional distributions (cf. [10], p. 107).

In Sect. 5 we consider the seemingly much more elementary integration problem, dealing with random measures of the form $\eta = X \cdot \xi$, where ξ is a random measure and $X \geq 0$ is a measurable process, both defined on a Borel space S. When ξ and X are jointly G-stationary, one would expect η to be G-stationary as well. Quite surprisingly, the conclusion may fail without additional conditions on ξ. In Corollary 5.4 we show that the assertion holds when $\xi \ll \mu$ a.s. for some non-random measure μ, in which case we may choose $\mu = E\xi$.

This suggests that we study the absolute continuity $\xi \ll \eta$ for general random measures ξ and η. In Theorem 5.2 we show that the relation can be replaced by $\xi \ll E(\xi\,|\mathcal{F})$ a.s., as long as η is $\mathcal{F}$-measurable. In particular, when ξ is stationary while η is invariant under the action of a group G, it can be replaced by $\xi \ll E(\xi\,|\mathcal{I})$ a.s., where $\mathcal{I}$ denotes the associated invariant σ-field. Finally, using methods from previous sections, we show in Theorem 6.3 that the conditional expectation $E(\xi\,|\mathcal{I})$ and distribution $\mathcal{L}(\xi\,|\mathcal{I})$ have G-invariant versions.

The theory of Sect. 5 also leads to a significant strengthening of the elementary notion of stationarity, as defined in terms of finite-dimensional distributions. In Corollary 5.3 we show that, if X is a measurable process on S taking values in a space T, then X is G-stationary iff

$$(\mu \circ \theta_r^{-1})(f \circ X) \stackrel{d}{=} \mu(f \circ X), \quad r \in G, \tag{7}$$

for any measure $\mu \in \mathcal{M}_S$ and function $f \in \mathcal{T}_+$. Note that the elementary notion of stationarity corresponds to the case where μ has finite support.

After presenting some technical prerequisites in Sect. 2, we turn in Sect. 3 to some basic results from [11,12] on invariant measures and disintegrations, namely the integral representation of invariant measures on S and the invariant disintegration of a jointly invariant measure on $S \times T$. Using the methods already discussed, we are able to give short and transparent proofs of both results. Our arguments, here and elsewhere, are often simplified by the use of s-finite rather than σ-finite measures. Recall that a measure is said to be *s-finite* if it is a countable sum of bounded measures. Every σ-finite measure is clearly s-finite (but not conversely), and many basic results for σ-finite measures, including Fubini's theorem, extend to the s-finite case. Furthermore, s-finiteness (unlike σ-finiteness) is preserved under measurable mappings and integrations, so that a verification of the desired σ-finiteness can often be postponed until the final result.

We conclude this section with some remarks on terminology and notation, not explained elsewhere. A group G is said to be *measurable* if it is equipped with a σ-field $\mathcal{G}$, such that the group operations $(r, p) \mapsto rp$ and $r \mapsto r^{-1}$ are measurable. For any $r, p \in G$, we write $rp = \theta_r p = \tilde{\theta}_p r$. A measurable group G is said to *act measurably* on a measurable space S, if the action map $(r, s) \mapsto rs$ is jointly measurable from $G \times S$ to S. The shifts θ_r and projections π_s are defined by $rs = \theta_r s = \pi_s r$ for $r \in G$ and $s \in S$.

A *Haar measure* on G is defined as a left-invariant, σ-finite measure λ on G, and we write $\tilde{\lambda} f = \lambda \tilde{f}$ with $\tilde{f}(r) = f(r^{-1})$, so that $\lambda \circ \theta_r^{-1} = \lambda$ and $\tilde{\lambda} \circ \tilde{\theta}_r = \tilde{\lambda}$ for all $r \in G$. Recall that λ exists, uniquely up to a normalization, whenever G is *lcscH* (locally compact, second countable, Hausdorff). Conversely, a measurable group G with a Haar measure is isomorphic to an lcscH group when G is Borel (defined as below), by a theorem of Mackey (cf. [21], first section), but not in general (cf. [5], last section).

For any measurable spaces $S, T, \ldots$, we write $\mathcal{S}, \mathcal{T}, \ldots$ and $\mathcal{S}_+, \mathcal{T}_+, \ldots$ for the associated σ-fields and classes of measurable functions $f \geq 0$, except that the Borel σ-field in R is denoted by $\mathcal{B}$. Absolute continuity, equivalence, and mutual singularity between measures are denoted by $\ll$, $\sim$, and $\perp$, respectively, and we write $\mu \otimes \nu$ for product measure or kernel composition, depending on context. For functions $f, g > 0$, $f \sim g$ means that $f/g \to 1$. By δ_s we mean the Dirac measure or function at s, and we write $f \cdot \mu$ for the measure with μ-density f. Thus, $1_A \mu = 1_A \cdot \mu$ is the restriction of μ to A, where $1_A(s) = 1\{s \in A\}$ is the indicator function of the set A. By $\|\cdot\|$ we mean the total variation or supremum norm of a measure or function. All random elements $\xi, \eta, \ldots$ are defined on an abstract space Ω with probability measure P and associated expectation E, their distributions and conditional distributions are denoted by $\mathcal{L}(\xi)$ and $\mathcal{L}(\xi|\mathcal{F})$, and $\xi \overset{d}{=} \eta$ means that $\mathcal{L}(\xi) = \mathcal{L}(\eta)$. We use $\mathcal{M}_S$ to denote the space of locally finite measures on S, where the local property is explained in Sect. 2. Finally, we write $\mathsf{R}_+ = [0, \infty)$, $\mathsf{Z}_+ = \{0, 1, 2, \ldots\}$, and $\mathsf{N} = \{1, 2, \ldots\}$.

2 Preliminaries

A *Borel space* is defined as a measurable space $(S, \mathcal{S})$ that is Borel isomorphic to a Borel set in R. By a *localized Borel space* we mean a triple $(S, \mathcal{S}, \hat{\mathcal{S}})$, where $(S, \mathcal{S})$ is a Borel space and the class $\hat{\mathcal{S}} \subset \mathcal{S}$ satisfies the conditions

- $\hat{\mathcal{S}}$ is a subring of $\mathcal{S}$,
- $B \in \hat{\mathcal{S}}$ and $C \in \mathcal{S}$ implies $B \cap C \in \hat{\mathcal{S}}$,
- S is a countable union of sets in $\hat{\mathcal{S}}$.

We may think of $\hat{\mathcal{S}}$ as the family of *bounded* Borel sets, and we say that a property holds *locally* on S if it holds on every $B \in \hat{\mathcal{S}}$. To simplify our statements, we assume every Borel space S in this paper to be localized.

Any Polish space S with associated Borel σ-field $\mathcal{S}$ is known to be a Borel space. By definition S can be metrized as a separable and complete metric space, and we may choose $\hat{\mathcal{S}}$ to be the class of all metrically bounded sets in $\mathcal{S}$. An important special case is when S is lcscH, and we may choose $\hat{\mathcal{S}}$ to be the class of relatively compact sets in $\mathcal{S}$.

The class $\mathcal{M}_S$ of locally finite measures on S is endowed with the σ-field $\mathcal{B}_{\mathcal{M}_S}$ induced by all evaluation maps $\pi_B : \mu \mapsto \mu B$ for $B \in \hat{\mathcal{S}}$. Note that the measurable space $(\mathcal{M}_S, \mathcal{B}_{\mathcal{M}_S})$ is again Borel (cf. [9,16]). A *random measure* ξ on S is defined as a random element in $\mathcal{M}_S$. Equivalently, ξ is a locally finite kernel from the basic probability space Ω to S. In other words, ξ is a function on $(\Omega, \mathcal{S})$ such that $\xi(\omega, B)$ is measurable in ω for fixed B and a locally finite measure in B for fixed ω.

By a *dissection system* in S we mean a family $\mathcal{I}$ of subsets $I_{nj} \in \hat{\mathcal{S}},\ n, j \in \mathbb{N}$, such that

- for fixed $n \in \mathbb{N}$, the sets I_{nj} form a partition of S,
- for any $m < n$ in $\mathbb{N}$, the partition (I_{nj}) is a refinement of (I_{mj}),
- for fixed $n \in \mathbb{N}$, any set $B \in \hat{\mathcal{S}}$ is covered by finitely many I_{nj},
- the entire collection $\{I_{nj}\}$ generates the σ-field $\mathcal{S}$.

When $S = \mathbb{R}$, we may choose $\mathcal{I}$ to consist of all dyadic intervals $I_{nj} = 2^{-n}(j-1, j]$, $n \in \mathbb{Z}_+$, $j \in \mathbb{Z}$, and similarly in higher dimensions.

For any dissection system (I_{nj}) in S, let $I_{n,s}$ denote the set I_{nj} containing s. By de Possel's theorem, usually proved by a simple martingale argument (cf. [10], p. 133), the sets $I_{n,s}$ form a differentiation basis for any measure $\nu \in \mathcal{M}_S$. Indeed, for any measures $\mu, \nu \in \mathcal{M}_S$, the associated ratios converge a.e. ν to a density of the absolutely continuous component μ_a of μ with respect to ν. In particular, this yields a measurable version of the Lebesgue decomposition and Radon–Nikodym property (cf. [10], p. 29).

Lemma 2.1 *For any localized Borel space S, there exists a measurable function $h \geq 0$ on $S \times \mathcal{M}_S^2$ such that*

$$\mu = h(\cdot, \mu, \nu) \cdot \nu + \mu_s \equiv \mu_a + \mu_s, \quad \mu, \nu \in \mathcal{M}_S,$$

where $\mu_a \ll \nu$ and $\mu_s \perp \nu$. The measures μ_a and μ_s and relations $\mu \ll \nu$ and $\mu \perp \nu$ all depend measurably on μ and ν, and $h(\cdot, \mu, \nu)$ is unique a.e. ν for any fixed μ and ν.

Proof Fix any dissection system (I_{nj}) in S. For $s \in S$ and $n \in \mathbb{N}$, let $I_n(s)$ be the set I_{nj} containing s, and define

$$h_n(s) = \frac{\mu I_n(s)}{\nu I_n(s)}, \quad s \in S, \ \ n \in \mathbb{N},$$

interpreted as 0 when the denominator vanishes. Then h_n converges a.e. ν to some limit $h \geq 0$ with $\mu_a = h \cdot \nu$, and we may choose the measurable version $h = \limsup_n h_n$. Replacing possibly infinite values by 0, we may ensure that $h < \infty$. The measurability of μ_a and μ_s now follows by Lemma 2.3 below. □

The last result is easily extended to the following measurable disintegration, needed repeatedly below.

Lemma 2.2 *For any Borel spaces S and T, there exists a kernel φ from $S \times \mathcal{M}_{S\times T} \times \mathcal{M}_S$ to T such that*

$$\mu = \nu \otimes \varphi(\cdot, \mu, \nu), \quad \mu \in \mathcal{M}_{S\times T}, \quad \nu \in \mathcal{M}_S \text{ with } \mu(\cdot \times T) \ll \nu.$$

Moreover, $\varphi(\cdot, \mu, \nu)$ is unique a.e. ν for fixed μ and ν.

The stated disintegration may be written more explicitly, for measurable functions $f \geq 0$ on $S \times T$, as

$$\mu f \equiv \iint \mu(ds\, dt)\, f(s,t) = \int \nu(ds) \int \varphi(s, \mu, \nu)(dt)\, f(s,t).$$

Proof By a suitable localization or normalization, we may take μ and ν to be bounded, and by the Borel property of T, we may easily reduce to the case where $T = \mathsf{R}$. For any $B \in \mathcal{T}$ we have $\mu(\cdot \times B) \ll \nu$, and so Lemma 2.1 yields a measurable function $h_B \geq 0$ on $S \times \mathcal{M}_{S\times T} \times \mathcal{M}_S$ with

$$\mu(\cdot \times B) = h_B(\cdot, \mu, \nu) \cdot \nu, \quad B \in \mathcal{S}. \tag{8}$$

Using the measure property on the left and the a.e. uniqueness of the density on the right, we get for any fixed measures μ and ν and sets $B_n \uparrow B$ in $\mathcal{S}$

$$h_{B_n}(\cdot, \mu, \nu) \uparrow h_B(\cdot, \mu, \nu) \text{ a.e. } \nu.$$

Proceeding as in [10], p. 107, we may then construct versions of $h_B(\cdot, \mu, \nu)$ that are measures in B for fixed μ and ν. Since the construction involves changes on at most countably many ν-null sets, the resulting version is again measurable for each $B \in \mathcal{S}$, hence a kernel between the stated spaces. Since (8) is not affected by the changes, the required disintegration formula holds for all measurable product sets $A \times B$ in $S \times T$, and the general formula follows by a monotone-class argument. □

Composition of kernels is defined as in (1). The iterated integration on the right is justified by the measurability of the inner integral, here extended to the s-finite case. A kernel is said to be *s-finite* if it is a countable sum of finite kernels.

Lemma 2.3 *For any s-finite kernel μ from S to T and measurable function $f \geq 0$ on $S \times T$, the integral $\mu_s f(s, \cdot)$ is again measurable.*

Proof We may assume that μ_s is bounded for every s. The measurability of $\mu_s A$, obvious when $A \in \mathcal{S} \times T$, extends by a monotone-class argument to arbitrary $A \in \mathcal{S} \otimes T$, and then by linearity and monotone convergence to any measurable function $f \geq 0$ on $S \times T$. □

The following result shows how densities and disintegration kernels are affected by shifts of the underlying measures, which justifies that we define *invariance* of a kernel ψ from S or $S \times \mathcal{M}_{S\times T} \times \mathcal{M}_S$ to T by the conditions in (2) or (4).

Lemma 2.4 *Let G be a measurable group acting measurably on S and T. Then for any $r \in G$, $\nu \in \mathcal{M}_S$, $h \in \mathcal{S}_+$, and kernels μ from S to T, we have*

(i) $(h \cdot \nu) \circ \theta_r^{-1} = (h \circ \theta_{r^{-1}}) \cdot (\nu \circ \theta_r^{-1})$,
(ii) $(\nu \otimes \mu) \circ \theta_r^{-1} = (\nu \circ \theta_r^{-1}) \otimes (\mu_{r^{-1}(\cdot)} \circ \theta_r^{-1})$.

Proof (i) For any $r \in G$ and $f \in \mathcal{S}_+$,

$$\begin{aligned}\Big((h \cdot \nu) \circ \theta_r^{-1}\Big) f &= (h \cdot \nu)(f \circ \theta_r) = \nu(f \circ \theta_r)h \\ &= (\nu \circ \theta_r^{-1})(h \circ \theta_{r^{-1}}) f \\ &= \Big((h \circ \theta_{r^{-1}}) \cdot (\nu \circ \theta_r^{-1})\Big) f.\end{aligned}$$

(ii) For any $r \in G$ and $f \in (\mathcal{S} \otimes \mathcal{T})_+$,

$$\begin{aligned}\Big((\nu \otimes \mu) \circ \theta_r^{-1}\Big) f &= (\nu \otimes \mu)(f \circ \theta_r) \\ &= \int \nu(ds) \int \mu_s(dt)\, f(rs, rt) \\ &= \int \nu(ds) \int (\mu_s \circ \theta_r^{-1})(dt)\, f(rs, t) \\ &= \int (\nu \circ \theta_r^{-1})(ds) \int (\mu_{r^{-1}s} \circ \theta_r^{-1})(dt)\, f(s, t) \\ &= \Big((\nu \circ \theta_r^{-1}) \otimes (\mu_{r^{-1}(\cdot)} \circ \theta_r^{-1})\Big)\, f.\end{aligned}$$

□

3 Invariant measures and disintegration

For reference we begin with some basic properties of Haar measures, given here without extra effort in a general non-topological setting.

Lemma 3.1 *Let G be a measurable group with Haar measure λ.*

(i) *For any measurable space S and s-finite, G-invariant measure μ on $G \times S$, there exists a unique, s-finite measure ν on S such that $\mu = \lambda \otimes \nu$. The measures μ and ν are simultaneously σ-finite.*
(ii) *There exists a measurable homomorphism $\Delta \colon G \to (0, \infty)$, such that $\tilde{\lambda} = \Delta \cdot \lambda$ and $\lambda \circ \tilde{\theta}_r^{-1} = \Delta_r \lambda$, $r \in G$.*
(iii) *When $\|\lambda\| < \infty$ we have $\Delta \equiv 1$, and λ is also right-invariant with $\tilde{\lambda} = \lambda$.*

The G-invariance in (i) clearly refers to shifts in G only, since there is no group action on S. A similar remark applies to some statements and proofs in subsequent sections. Note that the factorization property in (i) is given for s-finite rather than

σ-finite measures, which simplifies some arguments in [12] and below. In particular, any s-finite, left-invariant measure on G equals $c\lambda$ for some constant $c \in [0, \infty]$. The proof of (i) in [12] is incorrect and may be replaced by the following argument:

Proof of (i): Since λ is σ-finite, we may choose a measurable function $h > 0$ on G with $\lambda h = 1$, and define $\Delta_r = \lambda(h \circ \tilde{\theta}_r) \in (0, \infty]$ for $r \in G$. Using Fubini's theorem (three times), the invariance of μ and λ, and the definitions of Δ and $\tilde{\lambda}$, we get for any $f \geq 0$

$$\begin{aligned}\mu(f\Delta) &= \iint \mu(dr\,ds)\, f(r,s) \int \lambda(dp)\, h(pr)\\ &= \int \lambda(dp) \iint \mu(dr\,ds)\, f(p^{-1}r, s)\, h(r)\\ &= \iint \mu(dr\,ds)\, h(r) \int \lambda(dp)\, f(p^{-1}, s)\\ &= \int \tilde{\lambda}(dp) \iint \mu(dr\,ds)\, h(r)\, f(p,s).\end{aligned}$$

In particular, we may take $\mu = \lambda$ for singleton S to get $\tilde{\lambda} f = \lambda(f\Delta)$, and so $\tilde{\lambda} = \Delta \cdot \lambda$. This gives $\lambda = \tilde{\Delta} \cdot \tilde{\lambda} = \tilde{\Delta}\Delta \cdot \lambda$, and so $\tilde{\Delta}\Delta = 1$ a.e. λ, and finally $\Delta \in (0, \infty)$ a.e. λ. Hence, for general S

$$\mu(f\Delta) = \int \lambda(dp)\, \Delta(p) \iint \mu(dr\,ds)\, h(r)\, f(p,s).$$

Since $\Delta > 0$, we have $\mu(\cdot \times S) \ll \lambda$, and so $\Delta \in (0, \infty)$ a.e. $\mu(\cdot \times S)$, which yields the simplified formula

$$\mu f = \int \lambda(dp) \iint \mu(dr\,ds)\, h(r)\, f(p,s),$$

showing that $\mu = \lambda \otimes \nu$ with $\nu f = \mu(h \otimes f)$. If μ is σ-finite, then $\mu f < \infty$ for some measurable function $f > 0$ on $G \times S$, and Fubini's theorem yields $\nu f(r, \cdot) < \infty$ for $r \in G$ a.e. λ, which shows that even ν is σ-finite. The reverse implication is obvious. □

The following technical result is often useful.

Lemma 3.2 *Let G be a measurable group with Haar measure λ, acting measurably on S and T. Fix a kernel μ from S to T and a measurable, jointly G-invariant function f on $S \times T$. Then the following sets in S and $S \times T$ are G-invariant:*

$$\begin{aligned}A &= \{s \in S;\ \mu_{rs} = \mu_s \circ \theta_r^{-1},\ r \in G\},\\ B &= \{(s,t) \in S \times T;\ f(rs,t) = f(ps,t),\ (r,p) \in G^2 \text{ a.e. } \lambda^2\}.\end{aligned}$$

Proof When $s \in A$, we get for any $r, p \in G$

$$\mu_{rs} \circ \theta_p^{-1} = \mu_s \circ \theta_r^{-1} \circ \theta_p^{-1} = \mu_s \circ \theta_{pr}^{-1} = \mu_{prs},$$

which shows that even $rs \in A$. Conversely, $rs \in A$ implies $s = r^{-1}(rs) \in A$, and so $\theta_r^{-1} A = A$.

Now let $(s, t) \in B$. Then Lemma 3.1(ii) yields $(qs, t) \in B$ for any $q \in G$, and the invariance of λ gives

$$f(q^{-1}rqs, t) = f(q^{-1}pqs, t), \quad (r, p) \in G^2 \text{ a.e. } \lambda^2,$$

which implies $(qs, qt) \in B$ by the joint invariance of f. Conversely, $(qs, qt) \in B$ implies $(s, t) = q^{-1}(qs, qt) \in B$, and so $\theta_q^{-1} B = B$. □

When G is a measurable group with Haar measure λ, acting measurably on S, we say that G acts *properly* on S, if there exists a measurable, *normalizing* function $g > 0$ on S such that $\lambda(g \circ \pi_s) < \infty$ for all $s \in S$. We may then define a kernel φ on S by

$$\varphi_s = \frac{\lambda \circ \pi_s^{-1}}{\lambda(g \circ \pi_s)}, \quad s \in S. \tag{9}$$

Note that φ_s is both G-invariant for every fixed s and a G-invariant function of s. In [11,12] we proved that any s-finite, G-invariant measure ν on S is a unique mixture of the measures φ_s. Here we extend the representation in [11] to the s-finite case:

Proposition 3.3 *Let G be a measurable group with Haar measure λ, acting properly on S, and define the kernel φ by (9), for some normalizing function $g > 0$ on S. Then an s-finite measure ν on S is G-invariant iff*

$$\nu = \int \nu(ds)\, g(s)\, \varphi_s.$$

In that case, there exists a σ-finite, G-invariant measure $\tilde{\nu} \sim \nu$.

The present version may be deduced from some deep results in [12], whose proofs are based on an elaborate approximation and smoothing technique. Here we give a short, elementary proof.

Proof The sufficiency is clear, since the measures φ_s are invariant. Now let ν be s-finite and G-invariant. Using (9), Fubini's theorem, and Lemma 3.1(ii), we get for any $f \in \mathcal{S}_+$

$$\begin{aligned}
\int \nu(ds)\, g(s)\, \varphi_s f &= \int \nu(ds)\, g(s)\, \frac{\lambda(f \circ \pi_s)}{\lambda(g \circ \pi_s)} \\
&= \int \lambda(dr) \int \nu(ds)\, \frac{g(s)\, f(rs)}{\lambda(g \circ \pi_s)} \\
&= \int \lambda(dr) \int \nu(ds)\, \frac{g(r^{-1}s)\, f(s)}{\lambda(g \circ \pi_{r^{-1}s})} \\
&= \int \lambda(dr)\, \Delta_r \int \nu(ds)\, \frac{g(r^{-1}s)\, f(s)}{\lambda(g \circ \pi_s)}
\end{aligned}$$

$$= \int \lambda(dr) \int \nu(ds) \frac{g(rs) f(s)}{\lambda(g \circ \pi_s)}$$
$$= \int \nu(ds) f(s) = \nu f,$$

as required. Since $g \cdot \nu$ is again s-finite, we may choose a bounded measure $\hat{\nu} \sim g \cdot \nu$ and define $\tilde{\nu} = \hat{\nu}\varphi$, which is again G-invariant. It is also σ-finite, since $\tilde{\nu}g = \hat{\nu}\varphi g = \hat{\nu}G < \infty$. □

The last proposition may be combined with the following basic result from [11], where the invariance of kernels is defined as in (2). Extensions will be given in Theorems 4.1 and 6.1 below.

Proposition 3.4 *Let G be a measurable group with Haar measure λ, acting measurably on S and T, where T is Borel. Then for any σ-finite, G-invariant measures μ on $S \times T$ and ν on S with $\mu(\cdot \times T) \ll \nu$, there exists a G-invariant kernel ψ from S to T satisfying $\mu = \nu \otimes \psi$.*

This was proved in [11] by an elaborate smoothing argument. Here we give a more elementary proof based on a simple skew transformation. Similar ideas will be used in the proofs of Lemma 4.8 and Theorem 6.3.

Proof Applying a skew factorization to the G-invariant measures $\lambda \otimes \mu$ on $G \times S \times T$ and $\lambda \otimes \nu$ on $G \times S$ (cf. [11], pp. 293f), we obtain a G-invariant kernel φ from $G \times S$ to T such that $\lambda \otimes \mu = \lambda \otimes \nu \otimes \varphi$. Introducing the kernels $\varphi_r(p, s) = \varphi(rp, s)$, writing $f_r(p, s, t) = f(r^{-1}p, s, t)$ for any measurable functions $f \geq 0$ on $G \times S \times T$, and using the invariance of λ, we get for any $r \in G$

$$\begin{aligned}(\lambda \otimes \nu \otimes \varphi_r) f &= (\lambda \otimes \nu \otimes \varphi) f_r \\ &= (\lambda \otimes \mu) f_r = (\lambda \otimes \mu) f,\end{aligned}$$

which shows that $\lambda \otimes \mu = \lambda \otimes \nu \otimes \varphi_r$. Hence, by a.e. uniqueness

$$\varphi(rp, s) = \varphi(p, s), \quad (p, s) \in G \times S \text{ a.e. } \lambda \otimes \nu, \quad r \in G.$$

Let A be the set of all $s \in S$ satisfying

$$\varphi(rp, s) = \varphi(p, s), \quad (r, p) \in G^2 \text{ a.e. } \lambda^2,$$

and note that $\nu A^c = 0$ by Fubini's theorem. By Lemma 3.1(ii), the defining condition is equivalent to

$$\varphi(r, s) = \varphi(p, s), \quad (r, p) \in G^2 \text{ a.e. } \lambda^2,$$

and so for any $g, h \in \mathcal{G}_+$ with $\lambda g = \lambda h = 1$,

$$(g \cdot \lambda)\varphi(\cdot, s) = (h \cdot \lambda)\varphi(\cdot, s) = \psi(s), \quad s \in A. \tag{10}$$

To make this hold identically, we may redefine $\varphi(r,s) = 0$ when $s \in A^c$, without affecting the disintegration of $\lambda \otimes \mu$. By Lemma 3.2 it even preserves the invariance of φ. Fixing any $g \in \mathcal{G}_+$ with $\lambda g = 1$, we get for all $f \in (\mathcal{S} \otimes \mathcal{T})_+$

$$\begin{aligned}\mu f &= (\lambda \otimes \mu)(g \otimes f)\\ &= (\lambda \otimes \nu \otimes \varphi)(g \otimes f)\\ &= (\nu \otimes (g \cdot \lambda)\varphi) f = (\nu \otimes \psi) f,\end{aligned}$$

which shows that $\mu = \nu \otimes \psi$. Finally, by (10) and the invariance of φ and λ,

$$\begin{aligned}\psi_s \circ \theta_r^{-1} &= \int \lambda(dp)\, g(p)\, \varphi(p,s) \circ \theta_r^{-1}\\ &= \int \lambda(dp)\, g(p)\, \varphi(rp, rs)\\ &= \int \lambda(dp)\, g(r^{-1}p)\, \varphi(p, rs) = \psi_{rs},\end{aligned}$$

which shows that ψ is G-invariant. □

4 Invariant densities and disintegration

Given a measurable group G, acting measurably on some Borel spaces S and T, consider some locally finite measures μ on $S \times T$ and ν on S such that $\mu(\cdot \times T) \ll \nu$. Then Lemma 2.2 yields a disintegration of the form $\mu = \nu \otimes \varphi(\cdot, \mu, \nu)$, where φ is a kernel from $S \times \mathcal{M}_{S\times T} \times \mathcal{M}_S$ to T. When G is lcscH, we show that φ has a G-*invariant* version, in the sense of (4). The result plays a crucial role for our construction of G-stationary densities and disintegration kernels in Sect. 6.

Theorem 4.1 *Let G be an lcscH group, acting measurably on some Borel spaces S and T. Then there exists a G-invariant kernel φ from $S \times \mathcal{M}_{S\times T} \times \mathcal{M}_S$ to T, such that $\mu = \nu \otimes \varphi(\cdot, \mu, \nu)$ for all $\mu \in \mathcal{M}_{S\times T}$ and $\nu \in \mathcal{M}_S$ with $\mu(\cdot \times T) \ll \nu$.*

For invariant measures μ and ν, this essentially reduces to Proposition 3.4. When T is a singleton and G acts measurably on a Borel space S, the invariant disintegration reduces to an *invariant differentiation* $\mu = \varphi(\cdot, \mu, \nu) \cdot \nu$, where for any measurable function $\varphi \geq 0$ on $S \times \mathcal{M}_S^2$, the invariance is defined by (6) for arbitrary $\mu \leq \nu$ in $\mathcal{M}_S$. When G acts transitively on S, it is clearly enough to verify (6) for a fixed element $s \in S$.

Corollary 4.2 *Let G be an lcscH group, acting measurably on a Borel space S. Then there exists a measurable and G-invariant function $\varphi \geq 0$ on $S \times \mathcal{M}_S^2$ such that $\mu = \varphi(\cdot, \mu, \nu) \cdot \nu$ for all $\mu \ll \nu$ in $\mathcal{M}_S$.*

The main theorem will be proved in several steps. First we consider some partial versions of the density result, beginning with the case where $S = G$ is a Lie group.

Lemma 4.3 *An invariant density function exists on any Lie group G.*

Proof Since G is an orientable manifold, any inner product on the basic tangent space generates a left-invariant Riemannian metric ρ on G, given in local coordinates $x = (x^1, \dots, x^d)$ by a smooth family of symmetric, positive definite matrices $\rho_{ij}(x)$ (cf. [2], p. 247). The length of a smooth curve in G is obtained by integration of the length element ds along the curve, where

$$ds^2(x) = \sum_{i,j} \rho_{ij}(x)\, dx^i dx^j, \quad x \in G,$$

(cf. [2], p. 189), and the (geodesic) distance between two points $x, y \in G$ may be defined as the minimum length of all smooth curves connecting x and y. This determines an invariant metric ρ on G (cf. [2], p. 189), which in turn defines the open balls B_r^ε in G of radius $\varepsilon > 0$ centered at $r \in G$.

Besicovich' covering theorem on R^d (cf. [1], p. 364) extends to the compact subsets of any Riemannian manifold (cf. [6], Thm. 1.14 and Ex. 1.15.c), and so the open ρ-balls in G form a differentiation basis for any measure $\nu \in \mathcal{M}_G$ (cf. [1], p. 368). Hence, for measures $\mu \le \nu \in \mathcal{M}_G$, we may define a measurable density function φ on G by

$$\varphi(r, \mu, \nu) = \limsup_{n\to\infty} \frac{\mu B_r^{1/n}}{\nu B_r^{1/n}}, \quad r \in G,$$

where 0/0 is interpreted as 0. To see that φ is invariant, let ι denote the identity element in G, and note that

$$\begin{aligned} \varphi\Big(r, (\mu, \nu) \circ \theta_r^{-1}\Big) &= \limsup_{n\to\infty} \frac{(\mu \circ \theta_r^{-1}) B_r^{1/n}}{(\nu \circ \theta_r^{-1}) B_r^{1/n}} \\ &= \limsup_{n\to\infty} \frac{\mu B_\iota^{1/n}}{\nu B_\iota^{1/n}} = \varphi(\iota, \mu, \nu), \end{aligned}$$

since $r^{-1} B_r^\varepsilon = B_\iota^\varepsilon$ by the invariance of the metric ρ. □

We now extend the last result to the case where G is a *projective limit* of Lie groups, in the sense that every neighborhood of the identity ι contains a compact, invariant subgroup H such that G/H is isomorphic to a Lie group.

Lemma 4.4 *An invariant density function exists on any projective limit G of Lie groups.*

Proof Here G has some compact, invariant subgroups $H_n \downarrow \{\iota\}$ such that the quotient groups G/H_n are isomorphic to Lie groups (cf. [18], p. 177). Since the projection maps $\pi_n : r \mapsto rH_n$ are continuous (cf. [18], p. 27), we have $\mu \circ \pi_n^{-1} \in \mathcal{M}_{G/H_n}$ for any $\mu \in \mathcal{M}_G$. As in Lemma 4.3, we may choose some invariant density functions ψ_n on G/H_n, and introduce the functions

$$\varphi_n(r, \mu, \nu) = \psi_n\Big(rH_n, (\mu, \nu) \circ \pi_n^{-1}\Big), \quad r \in G,\ n \in \mathsf{N},$$

which are ν-densities of μ on the σ-fields $\mathcal{G}_n$ generated by the coset partitions G/H_n of G, so that

$$\mu = \varphi_n(\cdot, \mu, \nu) \cdot \nu \ \text{ on } \mathcal{G}_n, \quad n \in \mathsf{N}.$$

Since $H_n \downarrow \{\iota\}$, the $\mathcal{G}_n$ are non-decreasing and generate the Borel σ-field $\mathcal{G}$ on G. By martingale convergence, we get for fixed μ and ν

$$\varphi_n(r, \mu, \nu) \to \varphi(r, \mu, \nu), \quad r \in G \ \text{ a.e. } \nu,$$

where the limit φ is a density on $\mathcal{G}$. To ensure product measurability, we may choose the version

$$\varphi(r, \mu, \nu) = \limsup_{n\to\infty} \varphi_n(r, \mu, \nu), \quad r \in G, \ \ \mu \le \nu \ \text{ in } \mathcal{M}_G. \tag{11}$$

To show that φ is invariant, we note that an arbitrary shift by $r \in G$ preserves the coset partitions G/H_n of G, since any coset aH_n is mapped into the coset $r(aH_n) = (ra)H_n$. Furthermore, the permutation of G/H_n induced by $r \in G$ agrees with a shift of G/H_n by the group element rH_n, since $(rH_n)(aH_n) = (ra)H_n = r(aH_n)$ by definition of the group operation in G/H_n. We further note that $\pi_n \circ \theta_r = \theta_{rH_n} \circ \pi_n$ on G, since for any $r, s \in G$

$$\pi_n \theta_r s = \pi_n(rs) = rsH_n = (rH_n)(sH_n) = \theta_{rH_n}(sH_n) = \theta_{rH_n}\pi_n s.$$

Using the (G/H_n)-invariance of each ψ_n, we get for any $r \in G$

$$\begin{aligned}\varphi_n\Big(r, (\mu, \nu) \circ \theta_r^{-1}\Big) &= \psi_n\Big(rH_n, (\mu, \nu) \circ \theta_r^{-1} \circ \pi_n^{-1}\Big)\\ &= \psi_n\Big(rH_n, (\mu, \nu) \circ \pi_n^{-1} \circ \theta_{rH_n}^{-1}\Big)\\ &= \psi_n\Big(H_n, (\mu, \nu) \circ \pi_n^{-1}\Big) = \varphi_n(\iota, \mu, \nu),\end{aligned}$$

which shows that the functions φ_n are G-invariant. The G-invariance of φ now follows by (11). □

We proceed to any locally compact group G.

Lemma 4.5 *An invariant density function exists on any lcscH group G.*

Proof Here G contains an open subgroup H that is a projective limit of Lie groups, in the sense of Lemma 4.4 (cf. [18], p. 54, 175). Since the cosets of H are again open and the projection map $\pi : r \mapsto rH$ is both continuous and open (cf. [18], p. 27), the coset space G/H is countable and discrete (cf. [18], p. 28). Choosing an invariant density function ψ on H, as in Lemma 4.4, we define

$$\varphi(r, \mu, \nu) = \psi\Big(\iota, (\mu, \nu)_{rH} \circ \theta_{r^{-1}}^{-1}\Big), \quad r \in G, \ \ \mu \le \nu \text{ in } \mathcal{M}_G, \tag{12}$$

where $\mu_A = \mu(A \cap \cdot)$ denotes the restriction of μ to A.

By (12), the invariance property (6) for $s = \iota$ is equivalent to

$$\psi(\iota, (\mu, \nu)_H) = \psi\Big(\iota, \Big((\mu, \nu) \circ \theta_r^{-1}\Big)_{rH} \circ \theta_{r^{-1}}^{-1}\Big), \quad r \in G.$$

This follows from the relation

$$\mu_H \circ \theta_r^{-1} = (\mu \circ \theta_r^{-1})_{rH}, \quad r \in G, \ \mu \in \mathcal{M}_G, \tag{13}$$

which holds since for any $B \in \mathcal{G}$

$$\begin{aligned}(\mu \circ \theta_r^{-1})_{rH} B &= (\mu \circ \theta_r^{-1})(rH \cap B) = \mu(r^{-1}(rH \cap B)) \\ &= \mu(H \cap r^{-1}B) = \mu_H(r^{-1}B) = (\mu_H \circ \theta_r^{-1})B.\end{aligned}$$

Using (12), (13), and the invariance and density properties of ψ, we get for any measurable function $f \geq 0$ on the coset aH

$$\begin{aligned}\int f(r)\, \varphi(r, \mu, \nu)\, \nu(dr) &= \int f(ar)\, \varphi(r, \mu, \nu)\, (\nu \circ \theta_{a^{-1}}^{-1})(dr) \\ &= \int f(ar)\, \psi\Big(\iota, (\mu, \nu)_{arH} \circ \theta_{(ar)^{-1}}^{-1}\Big)\, (\nu \circ \theta_{a^{-1}}^{-1})(dr) \\ &= \int f(ar)\, \psi\Big(r, (\mu, \nu)_{aH} \circ \theta_{a^{-1}}^{-1}\Big)\, (\nu \circ \theta_{a^{-1}}^{-1})(dr) \\ &= \int f(ar)\, \psi\Big(r, \Big((\mu, \nu) \circ \theta_{a^{-1}}^{-1}\Big)_H\Big)\, (\nu \circ \theta_{a^{-1}}^{-1})(dr) \\ &= \int f(ar)\, (\mu \circ \theta_{a^{-1}}^{-1})(dr) = \int f(r)\, \mu(dr).\end{aligned}$$

Since G/H is countable and $a \in G$ was arbitrary, this proves the density property $\mu = \varphi(\cdot, \mu, \nu) \cdot \nu$. □

Next we extend the density version to any product space $G \times S$.

Lemma 4.6 *A G-invariant density function exists on $G \times S$ when G has no action on S.*

Proof Fix a dissection system (I_{nj}) on S, and put $S' = G \times S$. For any $\mu \leq \nu$ in $\mathcal{M}_{S'}$, the measures $\mu_{nj} = \mu(\cdot \times I_{nj})$ and $\nu_{nj} = \nu(\cdot \times I_{nj})$ belong to $\mathcal{M}_G$ and satisfy $\mu_{nj} \leq \nu_{nj}$. Choosing an invariant density function ψ on G as in Lemma 4.5, we obtain

$$\mu_{nj} = \psi(\cdot, \mu_{nj}, \nu_{nj}) \cdot \nu_{nj} \ \text{ on } \ \mathcal{G}, \quad n, j \in \mathbb{N}. \tag{14}$$

Now define some measurable functions φ_n on $S' \times \mathcal{M}_{S'}^2$ by

$$\varphi_n(r, s, \mu, \nu) = \sum\nolimits_j \psi(r, \mu_{nj}, \nu_{nj})\, 1\{s \in I_{nj}\}, \quad r \in G, \ s \in S, \ n \in \mathbb{N}.$$

Writing $\mathcal{I}_n$ for the σ-field on S generated by the sets I_{nj} for fixed n, we see from (14) that

$$\mu = \varphi_n(\cdot, \mu, \nu) \cdot \nu \text{ on } \mathcal{G} \otimes \mathcal{I}_n, \quad n \in \mathsf{N}.$$

Since the σ-fields $\mathcal{G} \otimes I_n$ are non-decreasing and generate $\mathcal{G} \otimes S$, the φ_n form ν-martingales on bounded sets, and so they converge a.e. ν to a ν-density of μ on $\mathcal{G} \otimes S$. Here we may choose the version

$$\varphi(r, s, \mu, \nu) = \limsup_{n\to\infty} \varphi_n(r, s, \mu, \nu), \quad r \in G, \ s \in S,$$

which inherits the product measurability and G-invariance from the corresponding properties of each φ_n. □

The next step is to go from densities to disintegration kernels.

Lemma 4.7 *When G acts on S but not on T, there exists a G-invariant disintegration kernel from S to T, provided that a G-invariant density function exists on S.*

Proof Define $\theta_r(s, t) = (rs, t)$ for $r \in G$, $s \in S$, and $t \in T$. Suppose that $\mu \in \mathcal{M}_{S\times T}$ and $\nu \in \mathcal{M}_S$ with $\mu(\cdot \times T) \ll \nu$. By hypothesis, there exists for every $B \in \hat{\mathcal{T}}$ a measurable function $\psi_B \geq 0$ on $S \times \mathcal{M}_S^2$ such that

$$\mu(\cdot \times B) = \psi_B(\cdot, \mu, \nu) \cdot \nu, \qquad B \in \hat{\mathcal{T}}, \tag{15}$$
$$\psi_B(s, \mu, \nu) = \psi_B\Big(rs, (\mu, \nu) \circ \theta_r^{-1}\Big), \quad r \in G, \ s \in S. \tag{16}$$

Writing $\mathcal{C}$ for the class of functions $f : \hat{\mathcal{T}} \to \mathsf{R}_+$, we need to construct a measurable mapping $\Phi : \mathcal{C} \to \mathcal{M}_T$, such that the kernel $\varphi = \Phi(\psi)$ from $S \times \mathcal{M}_S^2$ to T satisfies

$$\varphi(s, \mu, \nu)B = \psi_B(s, \mu, \nu), \quad s \in S \text{ a.e. } \nu, \ B \in \hat{\mathcal{T}}, \tag{17}$$

for any $\mu, \nu \in \mathcal{M}_S$. Then by (15)

$$\mu(\cdot \times B) = \varphi(\cdot, \mu, \nu)B \cdot \nu, \quad B \in \hat{\mathcal{T}},$$

which shows that $\mu = \nu \otimes \varphi$. Since $\Phi(f)$ depends only on f, (16) is preserved by Φ, which means that φ is again G-invariant.

To construct Φ, we may first embed T as a Borel set in R_+. Then define $\tilde{f}(B) = f(B \cap T)$ for $f \in \mathcal{C}$ and $B \in \hat{\mathcal{B}}_+$, to extend f to a function $\tilde{f}$ on $\hat{\mathcal{B}}_+$. If $\tilde{m}$ is a corresponding measure on R_+ with the desired properties, its restriction m to T has clearly the required properties on T. Thus, we may henceforth take $T = \mathsf{R}_+$.

For any $f \in \mathcal{C}$, define $F(x) = \inf_{r>x} f[0, r]$ for all $x \geq 0$, where r is restricted to Q_+. Then F is clearly non-decreasing and right-continuous. Hence, on the set where $F(x)$ is finite for all x, there exists a locally finite measure m on R_+ with $m[0, x] = F(x)$ for all $x \geq 0$. Writing A for the class of functions $f \in \mathcal{C}$ with $F(r) = f[0, r]$ for all $r \in \mathsf{Q}_+$, we may put $\Phi(f) = m$ on A and $\Phi(f) = 0$ on A^c, which defines Φ as a measurable function from $\mathcal{C}$ to $\mathcal{M}_T$.

To verify (17), let $B_1, B_2 \in \hat{\mathcal{B}}$ be disjoint with union B. Omitting the arguments of ψ, we may write

$$\begin{aligned}\psi_B \cdot \nu = \mu(\cdot \times B) &= \mu(\cdot \times B_1) + \mu(\cdot \times B_2)\\ &= \psi_{B_1} \cdot \nu + \psi_{B_2} \cdot \nu = (\psi_{B_1} + \psi_{B_2}) \cdot \nu,\end{aligned}$$

which implies $\psi_{B_1} + \psi_{B_2} = \psi_B$ a.e. ν. In particular, $\psi_{[0,r]}$ is a.e. non-decreasing in $r \in \mathbb{Q}_+$. By a similar argument, $B_n \uparrow B$ implies $\psi_{B_n} \uparrow \psi_B$ a.e. ν, which shows that $\psi_{[0,r]}$ is a.e. right-continuous on $\mathbb{Q}_+$. Hence, the kernel $\varphi = \Phi(\psi)$ satisfies $\varphi[0,r] = \psi_{[0,r]}$ a.e. for all $r \in \mathbb{Q}_+$. The extension to (17) now follows by a standard monotone-class argument. □

Combining Lemmas 4.6 and 4.7, we see that a G-invariant disintegration kernel from $G \times S$ to T exists when G has no action on S or T. We turn to the case of a general group action.

Lemma 4.8 *When G acts measurably on S and T, there exists a G-invariant disintegration kernel from $G \times S$ to T.*

Proof Let $\mu(\cdot \times T) \ll \nu$ on $S' = G \times S$. Define $\vartheta(r,s,t) = (r, rs, rt)$ with inverse $\tilde{\vartheta} = \vartheta^{-1}$, and put

$$\begin{aligned}\tilde{\mu} &= \mu \circ \tilde{\vartheta}^{-1}, \qquad \tilde{\nu} = \nu \circ \tilde{\vartheta}^{-1},\\ \mu &= \tilde{\mu} \circ \vartheta^{-1}, \qquad \nu = \tilde{\nu} \circ \vartheta^{-1}.\end{aligned}$$

By Lemma 4.7 there exists a kernel $\psi : S' \times \mathcal{M}_{S' \times T} \times \mathcal{M}_{S'} \to T$ with

$$\begin{aligned}&\tilde{\mu} = \tilde{\nu} \otimes \psi(\cdot, \tilde{\mu}, \tilde{\nu}) \text{ on } G \times S \times T,\\ &\psi(p,s,\mu,\nu) = \psi\Big(rp, s, (\mu,\nu) \circ \theta_r'^{-1}\Big), \quad r, p \in G, \quad s \in S,\end{aligned}$$

where $\theta'_r(p,s,t) = (rp, s, t)$. Define a kernel φ between the same spaces by

$$\varphi(r,s,\mu,\nu) = \psi(r, r^{-1}s, \tilde{\mu}, \tilde{\nu}) \circ \theta_r^{-1}, \quad r \in G, \quad s \in S.$$

Using the definitions of $\tilde{\mu}$, ψ, $\tilde{\nu}$, $\tilde{\vartheta}$, and φ, we get for any $f \in (\mathcal{S}' \otimes T)_+$

$$\begin{aligned}\mu f &= (\tilde{\mu} \circ \vartheta^{-1}) f = \tilde{\mu}(f \circ \vartheta)\\ &= (\tilde{\nu} \otimes \psi(\cdot, \tilde{\mu}, \tilde{\nu}))(f \circ \vartheta)\\ &= \Big((\nu \circ \tilde{\vartheta}^{-1}) \otimes \psi(\cdot, \tilde{\mu}, \tilde{\nu})\Big)(f \circ \vartheta)\\ &= \iint (\nu \circ \tilde{\vartheta}^{-1})(dr\, ds) \int \psi(r,s,\tilde{\mu},\tilde{\nu})(dt)\, f(r, rs, rt)\\ &= \iint \nu(dr\, ds) \int \psi(r, r^{-1}s, \tilde{\mu}, \tilde{\nu})(dt)\, f(r,s,rt)\\ &= \iint \nu(dr\, ds) \int \varphi(r,s,\mu,\nu)(dt)\, f(r,s,t) = (\nu \otimes \varphi) f,\end{aligned}$$

which shows that $\mu = \nu \otimes \varphi$.

Next we get for any $r, p \in G,\ s \in S$, and $t \in T$

$$\begin{aligned}(\tilde{\vartheta} \circ \theta_r)(p, s, t) = \tilde{\vartheta}(rp, rs, rt) &= \Big(rp, (rp)^{-1}rs, (rp)^{-1}rt\Big) \\ &= (rp, p^{-1}s, p^{-1}t) = \theta'_r(p, p^{-1}s, p^{-1}t) \\ &= (\theta'_r \circ \tilde{\vartheta})(p, s, t),\end{aligned}$$

so that $\tilde{\vartheta} \circ \theta_r = \theta'_r \circ \tilde{\vartheta}$. Combining with the definitions of $\varphi,\ \tilde{\mu},\ \tilde{\nu}$ and the G-invariance of ψ, we obtain

$$\begin{aligned}\varphi(p, s, \mu, \nu) \circ \theta_r^{-1} &= \psi(p, p^{-1}s, \tilde{\mu}, \tilde{\nu}) \circ \theta_{rp}^{-1} \\ &= \psi\Big(rp, p^{-1}s, (\tilde{\mu}, \tilde{\nu}) \circ \theta_r'^{-1}\Big) \circ \theta_{rp}^{-1} \\ &= \psi\Big(rp, p^{-1}s, (\mu, \nu) \circ \theta_r^{-1} \circ \tilde{\vartheta}^{-1}\Big) \circ \theta_{rp}^{-1} \\ &= \varphi\Big(rp, rs, (\mu, \nu) \circ \theta_r^{-1}\Big),\end{aligned}$$

which shows that φ is G-invariant in the sense of joint action on S and T. □

It remains to replace the product space $G \times S$ of the previous lemma by a general Borel space S.

Lemma 4.9 *When G acts measurably on S and T, there exists a G-invariant disintegration kernel from S to T.*

Proof Write $S' = G \times S$. By Lemma 4.8 there exists a G-invariant kernel ψ from $S' \times \mathcal{M}_{S' \times T} \times \mathcal{M}_{S'}$ to T, such that for any $\mu \in \mathcal{M}_{S \times T}$ and $\nu \in \mathcal{M}_S$ with $\mu(\cdot \times T) \ll \nu$

$$\lambda \otimes \mu = \lambda \otimes \nu \otimes \psi(\cdot, \mu, \nu) \quad \text{on } S' \times T, \tag{18}$$

where the G-invariance means that

$$\psi(p, s, \mu, \nu) \circ \theta_r^{-1} = \psi\Big(rp, rs, (\mu, \nu) \circ \theta_r^{-1}\Big), \quad r, p \in G, \quad s \in S. \tag{19}$$

Fixing μ and ν and defining

$$\psi_r(p, s) = \psi(rp, s), \qquad f_r(p, s, t) = f(r^{-1}p, s, t),$$

for all $r, p \in G,\ s \in S$, and $t \in T$, we get by the invariance of λ

$$\begin{aligned}(\lambda \otimes \nu \otimes \psi_r) f &= (\lambda \otimes \nu \otimes \psi) f_r \\ &= (\lambda \otimes \mu) f_r = (\lambda \otimes \mu) f,\end{aligned}$$

which shows that (18) remains valid with ψ replaced by any ψ_r. Hence, by uniqueness,

$$\psi(rp, s) = \psi(p, s), \quad (p, s) \in G \times S \ \text{a.e. } \lambda \otimes \nu, \ \ r \in G.$$

Writing A for the set of triples (s, μ, ν) with

$$\psi(rp, s) = \psi(p, s), \quad (r, p) \in G^2 \text{ a.e. } \lambda^2,$$

and putting $A_{\mu,\nu} = \{s \in S;\ (s, \mu, \nu) \in A\}$, we get $\nu A^c_{\mu,\nu} = 0$ by Fubini's theorem. By Lemma 3.1(ii), the defining condition for A is equivalent to

$$\psi(r, s) = \psi(p, s), \quad (r, p) \in G^2 \text{ a.e. } \lambda^2,$$

and so for any $g, g' \in \mathcal{G}_+$ with $\lambda g = \lambda g' = 1$,

$$\lambda(g\,\psi(\cdot, s)) = \lambda(g'\psi(\cdot, s)) \equiv \varphi(s), \quad s \in A_{\mu,\nu}. \tag{20}$$

To make this hold identically, we may redefine $\psi(\cdot, s) = 0$ for $s \notin A_{\mu,\nu}$, without affecting the validity of (18). Condition (19) is not affected either, since A is G-invariant by Lemma 3.2.

Fixing g as above and using (18) and (20), we get for any $f \in (\mathcal{S} \otimes \mathcal{T})_+$

$$\begin{aligned}\mu f &= (\lambda \otimes \mu)(g \otimes f) = (\lambda \otimes \nu \otimes \psi)(g \otimes f)\\ &= (\nu \otimes (g \cdot \lambda)\psi) f = (\nu \otimes \varphi) f,\end{aligned}$$

which shows that $\mu = \nu \otimes \varphi$. By (19), (20), and the invariance of λ, we further obtain

$$\begin{aligned}\varphi(s, \mu, \nu) \circ \theta_r^{-1} &= \int \lambda(dp)\, g(p)\, \psi(p, s, \mu, \nu) \circ \theta_r^{-1}\\ &= \int \lambda(dp)\, g(p)\, \psi\Big(rp, rs, (\mu, \nu) \circ \theta_r^{-1}\Big)\\ &= \int \lambda(dp)\, g(r^{-1}p)\, \psi\Big(p, rs, (\mu, \nu) \circ \theta_r^{-1}\Big)\\ &= \varphi\Big(rs, (\mu, \nu) \circ \theta_r^{-1}\Big),\end{aligned}$$

which shows that φ is again G-invariant. □

This completes the proof of Theorem 4.1. In view of the complexity of the argument, it may be useful to note the following easy proofs when G is compact or countable:

Proof of Theorem 4.1 for compact G: By Lemma 2.2 there exists a kernel $\psi : S \times \mathcal{M}_{S\times T} \times \mathcal{M}_S \to T$ with $\mu = \nu \otimes \psi(\cdot, \mu, \nu)$. Applying this to the measures $\mu \circ \theta_r^{-1}$ and $\nu \circ \theta_r^{-1}$ gives

$$\mu \circ \theta_r^{-1} = (\nu \circ \theta_r^{-1}) \otimes \psi\Big(\cdot, (\mu, \nu) \circ \theta_r^{-1}\Big), \quad r \in G,$$

and so by Lemma 2.4(ii)

$$\mu = \nu \otimes \Big\{\psi\Big(r(\cdot), (\mu, \nu) \circ \theta_r^{-1}\Big) \circ \theta_{r^{-1}}^{-1}\Big\}, \quad r \in G. \tag{21}$$

Assuming $\lambda G = 1$, we may introduce yet another kernel

$$\varphi(s, \mu, \nu) = \int \lambda(dr)\, \psi\Big(rs, (\mu, \nu) \circ \theta_r^{-1}\Big) \circ \theta_{r^{-1}}^{-1}, \quad s \in S,$$

and again $\mu = \nu \otimes \varphi(\cdot, \mu, \nu)$ by (21) and Fubini's theorem. Using the right invariance of λ, we get for any $p \in G$

$$\begin{aligned}\varphi(s, \mu, \nu) \circ \theta_p^{-1} &= \int \lambda(dr)\, \psi\Big(rs, (\mu, \nu) \circ \theta_r^{-1}\Big) \circ \theta_{pr^{-1}}^{-1} \\ &= \int \lambda(dr)\, \psi\Big(rps, (\mu, \nu) \circ \theta_{rp}^{-1}\Big) \circ \theta_{r^{-1}}^{-1} \\ &= \varphi\Big(ps, (\mu, \nu) \circ \theta_p^{-1}\Big),\end{aligned}$$

which shows that φ is G-invariant. □

Proof of Theorem 4.1 for countable G: For ψ as in the compact case, we get by (21) and the a.e. uniqueness of the disintegration

$$\psi(s, \mu, \nu) = \psi\Big(rs, (\mu, \nu) \circ \theta_r^{-1}\Big) \circ \theta_{r^{-1}}^{-1}, \quad s \in S \text{ a.e. } \nu, \ r \in G. \tag{22}$$

Now let A be the set of triples (s, μ, ν) satisfying

$$\psi(s, \mu, \nu) \circ \theta_r^{-1} = \psi\Big(rs, (\mu, \nu) \circ \theta_r^{-1}\Big), \quad r \in G.$$

Since G is countable, (22) yields $\nu\{s \in S;\ (s, \mu, \nu) \notin A\} = 0$ for all μ and ν, which justifies that we choose $\varphi = 1_A \psi$. Since A is G-invariant by Lemma 3.2, we have

$$(s, \mu, \nu) \in A \Leftrightarrow (rs, (\mu, \nu) \circ \theta_r^{-1}) \in A.$$

Thus, φ satisfies (22) identically and is therefore G-invariant. □

5 Random integration and absolute continuity

For any random measure ξ and measurable process $X \geq 0$ on a Borel space S, the integral ξX is again measurable by Lemma 2.3, hence a random variable. Hence, if $X \cdot \xi$ is locally finite, it defines a random measure on S. In general, we may define the distribution of $X \cdot \xi$ as the set of finite-dimensional distributions of the integrals $\xi(fX)$, for arbitrary $f \in \mathcal{S}_+$. Under an additional condition (but not in general), the distribution of $X \cdot \xi$ is uniquely determined by the joint distribution of X and ξ.

Theorem 5.1 *Consider some random measures ξ, η and measurable processes $X, Y \geq 0$ on a Borel space S, such that $(\xi, X) \overset{d}{=} (\eta, Y)$ and $\xi \ll E\xi$ a.s. Then*

$$(\xi, X, X \cdot \xi) \overset{d}{=} (\eta, Y, Y \cdot \eta).$$

Proof Since $E\xi$ is s-finite, we may choose a bounded measure $\mu \sim E\xi$. By monotone convergence, we may further take X and Y to be bounded. Since $\xi \ll \mu$ and hence $X \cdot \xi \ll \mu$ a.s, Lemma 2.1 yields a measurable function $g \geq 0$ on $S \times \mathcal{M}_S$ such that a.s.

$$\xi = g(\cdot, \xi) \cdot \mu, \qquad X \cdot \xi = g(\cdot, X \cdot \xi) \cdot \mu. \tag{23}$$

By the a.e. uniqueness of the density

$$X_s g(s, \xi) = g(s, X \cdot \xi), \quad s \in S \text{ a.e. } \mu, \quad \text{a.s. } P,$$

and so by Fubini's theorem

$$X_s g(s, \xi) = g(s, X \cdot \xi) \text{ a.s. } P, \quad s \in S \text{ a.e. } \mu. \tag{24}$$

Since $(\xi, X) \stackrel{d}{=} (\eta, Y)$ and $\mathcal{M}_S$ is Borel, the transfer theorem (cf. [10], p. 112) yields a random measure ζ on S such that

$$(\xi, X, X \cdot \xi) \stackrel{d}{=} (\eta, Y, \zeta). \tag{25}$$

By (24) and (25) we get

$$Y_s g(s, \eta) = g(s, \zeta) \text{ a.s. } P, \quad s \in S \text{ a.e. } \mu,$$

and so by Fubini's theorem

$$Y_s g(s, \eta) = g(s, \zeta), \quad s \in S \text{ a.e. } \mu, \quad \text{a.s. } P.$$

Since also $\eta = g(\cdot, \eta) \cdot \mu$ by (23) and (25), we obtain

$$Y \cdot \eta = Y g(\cdot, \eta) \cdot \mu = g(\cdot, \zeta) \cdot \mu \quad \text{a.s.} \tag{26}$$

Combining (23), (25), and (26) gives

$$\begin{aligned}(\xi, X, X \cdot \xi) &= (\xi, X, g(\cdot, X \cdot \xi) \cdot \mu) \\ &\stackrel{d}{=} (\eta, Y, g(\cdot, \zeta) \cdot \mu) = (\eta, Y, Y \cdot \eta),\end{aligned}$$

as required. □

The last result is false without the condition $\xi \ll E\xi$ a.s. For a counterexample, let τ be a uniformly distributed random variable on [0, 1], and define $\xi = \delta_\tau$, $X_t \equiv 1\{\tau = t\}$, and $Y_t \equiv 0$. Then X and Y are measurable with $X_t = Y_t = 0$ a.s. for every t, and so $(\xi, X) \stackrel{d}{=} (\xi, Y)$. However, $X \cdot \xi = \xi$ whereas $Y \cdot \xi = 0$, and so the asserted relation fails.

We proceed to show how any a.s. relation $\xi \ll \eta$ can be extended to a broader class of random measures ξ or η.

Theorem 5.2 *Let ξ and η be random measures on a Borel space S such that $\xi \ll \eta$ a.s. Then for any σ-field $\mathcal{F} \supset \sigma(\eta)$, we have*

$$\xi \ll E(\xi \,|\mathcal{F}) \ll \eta \ \textit{a.s.}$$

In particular, the condition $\xi \ll E\xi$ in Theorem 5.1 can be replaced by $\xi \ll \mu$ for any fixed, s-finite measure μ on S. More generally, the a.s. relation $\xi \ll \eta$ implies $\xi \ll E(\xi \mid \eta) \ll \eta$ a.s. Since ξ is locally finite, the conditional intensity $E(\xi \,|\mathcal{F})$ has a measure-valued version that is again a.s. s-finite, though it may fail to be σ-finite. Thus, the asserted relations are measurable by Lemma 2.1.

Proof We may clearly assume that $E\xi$ is σ-finite. By Lemma 2.1 there exists a measurable function $h \geq 0$ on $S \times \mathcal{M}_S$ such that $\xi = h(\cdot, \xi, \eta) \cdot \eta = X \cdot \eta$ a.s., where $X_s = h(s, \xi, \eta)$. Since $\mathcal{M}_S$ is Borel, we may choose an $\mathcal{F}$-measurable random probability measure ν on $\mathcal{M}_S$ such that $\nu = \mathcal{L}(\xi \,|\mathcal{F})$ a.s. Using the disintegration theorem (cf. [10], p. 108) and Fubini's theorem, we get a.s. for any $f \in \mathcal{S}_+$

$$\begin{aligned} E[\,\xi f \,|\mathcal{F}\,] = E[\,\eta(fX)\,|\mathcal{F}\,] &= E\left[\int \eta(ds)\, f(s)\, h(s,\xi,\eta) \,\middle|\, \mathcal{F}\right] \\ &= \int \nu(dm) \int \eta(ds)\, f(s)\, h(s,m,\eta) \\ &= \int \eta(ds)\, f(s) \int \nu(dm)\, h(s,m,\eta) \\ &= \int \eta(ds)\, f(s)\, E[\,h(s,\xi,\eta)\,|\mathcal{F}\,] \\ &= \eta(f\,E[X|\mathcal{F}\,]) = (E[X|\mathcal{F}\,]\cdot\eta) f. \end{aligned}$$

Since f was arbitrary, we obtain $E(\xi \,|\mathcal{F}) = E(X \,|\mathcal{F}) \cdot \eta$ a.s., which implies $E(\xi \,|\mathcal{F}) \ll \eta$ a.s.

The previous calculation shows that the remaining claim, $\xi \ll E(\xi \,|\mathcal{F})$ a.s., is equivalent to

$$h(\cdot,\xi,\eta)\cdot\eta \ll \int \nu(dm)\, h(\cdot,m,\eta)\cdot\eta \quad \text{a.s.}$$

Using the disintegration theorem again, we may write this as

$$E\nu\left\{\mu \in \mathcal{M}_S;\ h(\cdot,\mu,\eta)\cdot\eta \ll \int \nu(dm)\, h(\cdot,m,\eta)\cdot\eta\right\} = 1,$$

which reduces the assertion to $X \cdot \eta \ll EX \cdot \eta$ a.s., for any non-random measure η and measurable process $X \geq 0$ on S.

To prove this, we note that $EX_s = 0$ implies $X_s = 0$ a.s., so that

$$X_s \ll EX_s \ \text{ a.s. } P, \quad s \in S,$$

and hence by Fubini's theorem

$$X_s \ll EX_s, \quad s \in S \text{ a.e. } \eta, \quad \text{a.s. } P.$$

Now fix any $\omega \in \Omega$ outside the exceptional P-null set, and let $B \in \mathcal{S}$ be arbitrary with $\eta(EX; B) = 0$. Splitting B into the four subsets where

$$X_s = EX_s = 0, \quad X_s > EX_s = 0, \quad EX_s > X_s = 0, \quad X_s EX_s > 0,$$

we may easily check that even $\eta(X; B) = 0$. This shows that indeed $X \cdot \eta \ll EX \cdot \eta$ a.s. □

For measurable processes and group actions, Theorem 5.1 yields the following strengthening of the elementary notion of stationarity, defined in terms of finite-dimensional distributions.

Corollary 5.3 *For any Borel spaces S and T and a group G acting measurably on S, let X be a measurable process on S taking values in T. Then X is G-stationary in the sense of finite-dimensional distributions, iff* (7) *holds for any measurable function $f \geq 0$ on T and measure μ on S.*

Proof If X is G-stationary in the sense of finite-dimensional distributions, then so is $f(X)$. Hence, $f \circ X \circ \theta_r \stackrel{d}{=} f \circ X$ for all $r \in G$, and (7) follows by Theorem 5.1 since X is measurable. Conversely, assume (7). Considering measures μ with finite support and using the Cramér–Wold theorem, we see that $f(X)$ is G-stationary in the sense of finite-dimensional distributions. Since T is Borel, the same property holds for X. □

The property in (7) will be referred to as *strong G-stationarity* of X. More generally, a random measure ξ and measurable process X on S are said to be *jointly strongly G-stationary* if

$$\left(\xi \circ \theta_r^{-1}, \mu(f \circ X)\right) \stackrel{d}{=} \left(\xi, (\mu \circ \theta_r^{-1})(f \circ X)\right), \quad r \in G,$$

for any μ and f as above. Note that the shift of ξ by $r \in G$ corresponds to a shift of X by r^{-1}. (This is only a convention, justified by Lemma 2.4.) Under a suitable condition, the joint G-stationarity of ξ and X is preserved by integration:

Corollary 5.4 *Given a group G acting measurably on a Borel space S, consider a random measure ξ and a bounded, measurable process $X \geq 0$ on S, such that (ξ, X) is G-stationary and $\xi \ll E\xi$ a.s. Then $(\xi, X, X \cdot \xi)$ is strongly G-stationary.*

Proof By Lemma 2.4, we have

$$(X \cdot \xi) \circ \theta_r^{-1} = (X \circ \theta_{r^{-1}}) \cdot (\xi \circ \theta_r^{-1}), \quad r \in G.$$

Since $\xi \ll E\xi$ a.s. and $(\xi \circ \theta_r^{-1}, X \circ \theta_{r^{-1}}) \stackrel{d}{=} (\xi, X)$, the assertion follows by Theorem 5.1. □

Again the a.s. condition $\xi \ll E\xi$ cannot be omitted. For a counterexample, let ξ be a unit rate Poisson process on R with first point on R_+ at τ, and define $X_t \equiv 1\{\tau = t\}$. Then $X_t = 0$ a.s. for every $t \in \mathsf{R}$, and so (ξ, X) is stationary, whereas $X \cdot \xi = \delta_\tau$ is not.

6 Stationary densities and disintegration

Using Theorem 4.1, we may construct stationary densities and disintegration kernels for stationary random measures ξ and η. By a *random kernel* $\zeta : S \to T$ we mean a σ-finite kernel from $\Omega \times S$ to T. The triple (ξ, η, ζ) is said to be *strongly G-stationary* if

$$(\xi, \eta, \mu\zeta) \circ \theta_r^{-1} \stackrel{d}{=} \Big(\xi, \eta, (\mu \circ \theta_r^{-1})\zeta\Big), \quad r \in G,\ \mu \in \mathcal{M}_S.$$

By Corollary 5.3, this is equivalent to ordinary G-stationarity. Here the stronger version follows by the same argument.

Theorem 6.1 *Let the lcscH group G act measurably on some Borel spaces S and T, and consider some random measures ξ on $S \times T$ and η on S such that (ξ, η) is G-stationary with $\xi(\cdot \times T) \ll \eta$ a.s. Then there exists a (ξ, η)-measurable random kernel $\zeta : S \to T$, such that (ξ, η, ζ) is strongly G-stationary with $\xi = \eta \otimes \zeta$ a.s.*

For non-random ξ and η, the stationarity becomes invariance, and the result essentially reduces to Proposition 3.4. For singleton T, we obtain conditions for the existence of a stationary density. Our proof simplifies only marginally in this case. In particular, we may then omit Lemma 4.7.

Corollary 6.2 *Let the lcscH group G act measurably on a Borel space S, and consider some random measures $\xi \ll \eta$ on S such that (ξ, η) is G-stationary. Then there exists a (ξ, η)-measurable process $X \geq 0$ on S, such that (ξ, η, X) is strongly G-stationary with $\xi = X \cdot \eta$ a.s.*

Proof of Theorem 6.1 Let φ be such as in Theorem 4.1, and define $\zeta_s = \varphi(s, \xi, \eta)$ for $s \in S$, so that $\xi = \eta \otimes \zeta$. Since φ is G-invariant, we get for any $\mu \in \mathcal{M}_S$ and $r \in G$

$$\begin{aligned}
\mu\zeta \circ \theta_r^{-1} &= \int \mu(ds)\, \varphi(s, \xi, \eta) \circ \theta_r^{-1} \\
&= \int \mu(ds)\, \varphi\Big(rs, (\xi, \eta) \circ \theta_r^{-1}\Big) \\
&= (\mu \circ \theta_r^{-1})\, \varphi\Big(\cdot, (\xi, \eta) \circ \theta_r^{-1}\Big),
\end{aligned}$$

and so the G-stationarity of (ξ, η) yields

$$\begin{aligned}
(\xi, \eta, \mu\zeta) \circ \theta_r^{-1} &= \Big\{(\xi, \eta) \circ \theta_r^{-1}, (\mu \circ \theta_r^{-1})\, \varphi\Big(\cdot, (\xi, \eta) \circ \theta_r^{-1}\Big)\Big\} \\
&\stackrel{d}{=} \Big(\xi, \eta, (\mu \circ \theta_r^{-1})\, \varphi(\cdot, \xi, \eta)\Big) \\
&= \Big(\xi, \eta, (\mu \circ \theta_r^{-1})\zeta\Big),
\end{aligned}$$

which shows that (ξ, η, ζ) is strongly G-stationary. □

If the random measure η of the last result is a.s. G-invariant, then by Theorem 5.2 it can be replaced by the random measure $E(\xi\,|\mathcal{I})$, where $\mathcal{I}$ denotes the associated invariant σ-field. Assuming ξ to be G-stationary, we show that $E(\xi\,|\mathcal{I})$ (and indeed even $\mathcal{L}(\xi\,|\mathcal{I})$) has a G-invariant version. For general results on ergodic decomposition, see [3], pp. 716f, and [10], pp. 195f.

Theorem 6.3 *Let the lcscH group G act measurably on a Borel space S, and consider a G-stationary random measure ξ on S with associated G-invariant σ-field $\mathcal{I}$. Then $E(\xi\,|\mathcal{I})$ and $\mathcal{L}(\xi\,|\mathcal{I})$ have G-invariant, measure-valued versions.*

Proof For any measure-valued version $\Xi = \mathcal{L}(\xi\,|\mathcal{I})$, we note that $E(\xi\,|\mathcal{I})$ has the measure-valued version $\eta = \int m\,\Xi(dm)$. If Ξ is invariant under the shifts θ_r on $\mathcal{M}_S$ given by $\theta_r\mu = \mu\circ\theta_r^{-1}$, then

$$\begin{aligned}\eta\circ\theta_r^{-1} &= \int (m\circ\theta_r^{-1})\,\Xi(dm)\\ &= \int m\,(\Xi\circ\theta_r^{-1})(dm) = \int m\,\Xi(dm) = \eta,\end{aligned}$$

which shows that η is invariant under shifts on S. It is then enough to prove the assertion for $\mathcal{L}(\xi\,|\mathcal{I})$.

Since ξ is stationary and $\mathcal{I}$ is invariant, we have for any measurable function $f \geq 0$ on $\mathcal{M}_S$

$$E[f(\xi\circ\theta_r^{-1})\,|\mathcal{I}] = E[f(\xi)\,|\mathcal{I}] \text{ a.s.,} \quad r\in G. \tag{27}$$

Fixing a countable, measure-determining class $\mathcal{F}$ of functions f and a countable, dense subset $G' \subset G$, we note that (27) holds simultaneously for all $f \in \mathcal{F}$ and $r \in G'$, outside a fixed P-null set. Since the space $\mathcal{M}_S$ is again Borel, we can choose a measure-valued version $\Xi = \mathcal{L}(\xi\,|\mathcal{I})$.

For any non-exceptional realization Q of Ξ, we may choose a random measure η with distribution Q and write the countably many relations (27) in probabilistic form as

$$Ef(\eta\circ\theta_r^{-1}) = Ef(\eta), \quad f\in\mathcal{F}, \ \ r\in G'.$$

Since $\mathcal{F}$ is measure-determining, we conclude that

$$\eta\circ\theta_r^{-1} \stackrel{d}{=} \eta, \quad r\in G'.$$

Using the invariance of Haar measure λ and the measurability of the mapping $\mu \mapsto \lambda\otimes\mu$, we get for every $r \in G'$

$$(\lambda\otimes\eta)\circ\theta_r^{-1} = \lambda\otimes(\eta\circ\theta_r^{-1}) \stackrel{d}{=} \lambda\otimes\eta, \tag{28}$$

where θ_r on the left denotes the joint shift $(p, s) \mapsto (rp, rs)$ on $G \times S$.

Next introduce on $G \times S$ the skew transformation $\vartheta(r, s) = (r, rs)$ with inverse $\tilde{\vartheta}$, and note as in Lemma 4.8 that

$$\tilde{\vartheta} \circ \theta_r = \theta'_r \circ \tilde{\vartheta}, \quad r \in G, \tag{29}$$

where $\theta'_r(p, s) = (rp, s)$ denotes the shift in $G \times S$ acting on G alone. Then (28) yields for any $r \in G'$

$$\begin{aligned}(\lambda \otimes \eta) \circ \tilde{\vartheta}^{-1} \circ \theta_r'^{-1} &= (\lambda \otimes \eta) \circ \theta_r^{-1} \circ \tilde{\vartheta}^{-1} \\ &\overset{d}{=} (\lambda \otimes \eta) \circ \tilde{\vartheta}^{-1},\end{aligned}$$

which means that the random measure $\zeta = (\lambda \otimes \eta) \circ \tilde{\vartheta}^{-1}$ on $G \times S$ is invariant in distribution under shifts by $r \in G'$ in the component G alone.

To extend this to arbitrary $r \in G$, we may assume that $S = \mathsf{R}$, so that even $S' = G \times S$ becomes lcscH. Choosing a metrization of S' such that every bounded set is relatively compact, we see that the ε-neighborhood of a bounded set is again bounded. Then for any measures $\mu \in \mathcal{M}_{S'}$ and continuous functions $f \geq 0$ on S' with compact support, we get by dominated convergence, as $r \to \iota$ in G,

$$(\mu \circ \theta_r'^{-1}) f = \int \mu(dp\, ds)\, f(rp, s) \to \int \mu(dp\, ds)\, f(p, s) = \mu f,$$

which gives $\mu \circ \theta_r'^{-1} \overset{v}{\to} \mu$ by the definition of the vague topology, showing that G acts continuously on $\mathcal{M}_{S'}$ under shifts in G alone. For any $r \in G$, we may choose a sequence $r_n \to r$ in G' and conclude that

$$\zeta \overset{d}{=} \zeta \circ \theta_{r_n}'^{-1} \overset{v}{\to} \zeta \circ \theta_r'^{-1}.$$

Hence, $\zeta \circ \theta_r'^{-1} \overset{d}{=} \zeta$ for all $r \in G$, which means that ζ is stationary under shifts in G alone.

Reversing the skew transformation and using (29) and the invariance of λ, we get for any $r \in G$

$$\lambda \otimes (\eta \circ \theta_r^{-1}) = (\lambda \otimes \eta) \circ \theta_r^{-1} \overset{d}{=} \lambda \otimes \eta.$$

Fixing a $B \in \mathcal{G}$ with $0 < \lambda B < \infty$ and using the measurability of the projection $\mu \mapsto \mu(B \times \cdot)$ on $G \times S$, we obtain

$$\begin{aligned}(\lambda B)\, (\eta \circ \theta_r^{-1}) &= (\lambda \otimes (\eta \circ \theta_r^{-1}))(B \times \cdot) \\ &\overset{d}{=} (\lambda \otimes \eta)(B \times \cdot) = (\lambda B)\, \eta,\end{aligned}$$

which implies $\eta \circ \theta_r^{-1} \overset{d}{=} \eta$. This shows that Q is G-invariant, and the a.s. G-invariance of Ξ follows. □

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