Probab. Theory Relat. Fields (2014) 158:895–924
DOI 10.1007/s00440-013-0499-7

Uniqueness and blow-up for a stochastic viscous dyadic model

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Received: 3 November 2011 / Revised: 28 January 2013 / Published online: 5 May 2013

Abstract We consider the dyadic model with viscosity and additive Gaussian noise as a simplified version of the stochastic Navier–Stokes equations, with the purpose of studying uniqueness and emergence of singularities. We prove pathwise uniqueness and absence of blow-up in the intermediate intensity of the non-linearity, morally corresponding to the 3D case, and blow-up for stronger intensity. Moreover, blow-up happens with probability one for regular initial data.

Keywords Dyadic model · Infinite dimensional system of stochastic equations · Pathwise uniqueness · Blow-up

Mathematics Subject Classification (1991) Primary 76D03; Secondary 35Q35 · 60H15 · 35R60

1 Introduction

Motivations Uniqueness is a problem with many facets for PDEs and different problems may require different approaches. When turning to stochastic PDEs, the problem acquires new levels of complexity, as uniqueness for stochastic processes can be understood in several ways. We refer to [25] for a recent review.

A prototypical example of PDE for which uniqueness is open are the Navier–Stokes equations, where the issue of uniqueness is mixed with the issue of regularity and emergence of singularities [22]. The stochastic version shares the same problems. In recent years, by means of a clever way to solve the Kolmogorov equation, Da Prato and

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Debussche [13,19] have shown existence of Markov families of solutions. Moreover, such Markov families admit a unique invariant measure, with exponential convergence rate [38]. In [26,28] similar results have been obtained with a completely different method, based on the Krylov selection method [35]. Related results can be found in [1,14,24,27,40–44]. Both methods apply equally well in more general situations [8].

The purpose of this paper is to analyse uniqueness and emergence of blow-up in a much simpler infinite dimensional stochastic equation. We look for a model that retains some characteristics of the original problem and is amenable to the analysis of [13,28]. The main point is the choice of the non-linearity.

The Navier–Stokes non-linearity on the torus with 2π-periodic boundary conditions reads in Fourier series as

$$(u\cdot\nabla)u = \mathrm{i}\sum_k \sum_{n+m=k} (u_n\cdot m)u_m\,\mathrm{e}^{\mathrm{i}k\cdot x}\,.$$

Here the kth mode interacts with almost every other mode. The most reasonable simplification is to reduce the interaction to a finite number of modes, while keeping the orthogonality property in the energy estimate. The simplest possible is the nearest neighbour interaction and this gives the dyadic model.

The dyadic model The dyadic model has been introduced in [29,33] as a model of the interaction of the energy of an inviscid fluid among different packets of wave-modes (shells). It has been lately studied in [4,6,12,34,46] and in the inviscid and stochastically forced case in [3,5,9].

The viscous version has been studied in [10,11,30]. Blow-up of *positive* solutions with non-linearity of strong intensity is proved in [10]. In [7] the authors prove well-posedness and convergence to the inviscid limit, again for positive solutions, with non-linearity of intensity of "Navier–Stokes" type.

In this paper we study the dyadic model with additive noise,

$$dX_n = \left(-\nu\lambda_n^2 X_n + \lambda_{n-1}^\beta X_{n-1}^2 - \lambda_n^\beta X_n X_{n+1}\right)dt + \sigma_n\,dW_n,\quad n\geq 1, \tag{1.1}$$

where $\lambda_n = 2^n$ and $X_0\equiv 0$. The noise coefficients satisfy suitable assumptions and the parameter β measures the relative intensity of the non-linearity with respect to the linear term. Throughout the paper we consider the viscous problem, namely $\nu > 0$. The inviscid limit will be addressed in a future work.

The non-linear term cancels out as in Navier–Stokes providing an a-priori bound in $\ell^2(\mathbf{R})$ independent of β. If $\lambda_n^2 X_n \gtrsim \lambda_n^\beta X_n^2$, the linear term dominates the non-linear term. This is the heuristic reason why local strong solutions exist when the initial condition decays at least as $\lambda_n^{-(\beta-2)}$. If $\beta\leq 2$ this is always true due to the ℓ^2-bound, and the non-random problem has a unique global solution [10]. Likewise, uniqueness holds with noise when $\beta\leq 2$.

By a scaling argument (see for instance [10]), one can "morally" identify the dyadic model with the Navier–Stokes equations when $\beta\approx\frac{5}{2}$. In [7] well-posedness is proved in a range which includes the value $\frac{5}{2}$, but only for *positive* solutions. Positivity is preserved by the unforced dynamics. It is clear that, as is, positivity is broken by the random perturbation.

Main results This paper contains a thorough analysis of the case $\beta > 2$, which can be roughly summarised in the table below.

	$\beta \le 2$	$2 < \beta \le 3$	$\beta > 3$
Blow-up	NO	NO[a]	YES
Uniqueness	YES	YES	?

[a] Absence of blow-up is proved up to $\beta_c < 3$

We prove pathwise uniqueness in the range $\beta \in (2, 3]$ by adapting an idea for positive solutions of [7]. The solution is decomposed in a quasi-positive component and a residual term. Quasi-positivity means that there is a lower bound that decays as a (negative) power of λ_n. This bound is preserved by the system as long as the random perturbation is *not too strong*. Under the same conditions the residual term is small.

Quasi-positivity and the invariant area argument of [7] together imply smoothness of the solution. Here by smoothness we mean that $(\lambda_n^\gamma X_n)_{n\ge1}$ is bounded for every γ. This result holds for $\beta \in (2, \beta_c)$, where $\beta_c \in (2, 3]$ is the value identified in [7].

When $\beta > 3$ we use an idea of [10] for positive solutions. We are able to identify a set of initial conditions that lead to blow-up with positive probability.

Emergence of blow-up has been already proved in several stochastic models. See for instance [16,17] for the Schrödinger equation [21,36,37] for the nonlinear heat equation (the result of [23] is basically one dimensional and no ideas for infinite dimensional systems are involved). All such results ensure that blow-up occurs only with positive probability.

We first state some general conditions that ensure that blow-up occurs with probability one. Roughly speaking, one needs first to identify a set of initial states that lead to blow-up with positive probability. In general, this is not sufficient (see Example 5.6). The crucial idea is to prove that such sets are recurrent for the evolution, *conditional* to nonappearance of blow-up. We believe that these general results may be of independent interest.

Our main result on blow-up for the dyadic model ensures that if at least one component is forced by noise, then blow-up occurs with full probability. The result holds as long as the initial state satisfies $\lambda_n^\alpha X_n(0) \approx O(1)$ for some $\alpha > \beta - 2$. This is optimal since it is the same condition that ensures the existence of a local smooth solution. In different words, "smoothness" is transient.

The main ingredient to prove recurrence for the sets leading to blow-up is a stronger form of quasi-positivity. This ensures that the negative parts of the solution become smaller in a finite time, depending only on the size of the initial condition in H and on the size of the random perturbation. We remark that recurrence is not at all obvious, since for $\beta > 3$ the dissipation of the system is not strong enough to provide existence of a stationary solution.

It remains open to understand uniqueness for $\beta > 3$, since blow-up rules out the use of smooth solutions, making pathwise uniqueness a harder problem. Uniqueness in law may still be achievable.

2 Preliminary results and definitions

The following assumption on the intensity of the noise will be in strength for the whole paper.

Assumption 2.1 There is $\alpha_0 > \max\{\frac{1}{2}(\beta - 3), \beta - 3\}$ such that

$$\sup_{n\geq 1}\big(\lambda_n^{\alpha_0}\sigma_n\big) < \infty. \tag{2.1}$$

2.1 Notations

Set $\lambda = 2$ and $\lambda_n = \lambda^n$. For $\alpha \in \mathbf{R}$ let V_α be the (Hilbert) space

$$V_\alpha = \left\{(x_n)_{n\geq 1} : \sum_{n=1}^{\infty}(\lambda_n^\alpha x_n)^2 < \infty\right\},$$

with scalar product $\langle x, y\rangle_\alpha = \sum_{n=1}^\infty \lambda_n^{2\alpha} x_n y_n$ and norm $\|\cdot\|_\alpha = \langle\cdot,\cdot\rangle_\alpha^{1/2}$. Set in particular $H = V_0$ and $V = V_1$.

2.2 Definitions of solution

We turn to the definition of solution. We consider first strong solutions, which are unique, regular but defined on a (possibly) random interval. Then we will consider weak solutions, which are global in time.

2.2.1 Strong solutions

We first discuss local strong solution.

Definition 2.2 (*Strong solution*) Let $\mathcal{W}$ be an Hilbert sub-space of H. Given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and a cylindrical Wiener process $(W_t, \mathscr{F}_t)_{t\geq 0}$ on H, a *strong solution in* $\mathcal{W}$ with initial condition $x \in \mathcal{W}$ is a pair $(X(\cdot; x), \tau_x^{\mathcal{W}})$ such that

- $\tau_x^{\mathcal{W}}$ is a stopping time with $\mathbb{P}[\tau_x^{\mathcal{W}} > 0] = 1$,
- $X(\cdot; x)$ is a process defined on $[0, \tau_x^{\mathcal{W}})$ with $\mathbb{P}[X(0, x) = x] = 1$,
- $X(\cdot; x)$ is continuous with values in $\mathcal{W}$ for $t < \tau_x^{\mathcal{W}}$,
- $\|X(t; x)\|_{\mathcal{W}} \to \infty$ as $t \uparrow \tau_x^{\mathcal{W}}$, $\mathbb{P}$−a. s.,
- $X(\cdot; x)$ is solution of (1.1) on $[0, \tau_x^{\mathcal{W}})$.

The strong solution turns out to be a Markov process (and even a strong Markov process, but we do not need this fact here) in the following sense (see [31] for further details). Set $\mathcal{W}' = \mathcal{W} \cup \{\triangle\}$, where the terminal state $\triangle$ is an isolated point. Define the set $\overline{W}(\mathcal{W}')$ of all paths $\omega : [0, \infty) \to \mathcal{W}'$ such that there exists a time $\zeta(\omega) \in [0, \infty]$ with ω continuous with values in $\mathcal{W}$ on $[0, \zeta(\omega))$ and $\omega(t) = \triangle$ for $t \geq \zeta(\omega)$. The strong solution defined above can be extended as a process in $[0, \infty)$ with values in $\mathcal{W}'$ in a canonical way, achieving value $\triangle$ for $t \geq \tau_x^{\mathcal{W}}$. We say that the strong solution is Markov when the process on the extended state space $\mathcal{W}'$ is a Markov process.

Theorem 2.3 *Let $\beta > 2$ and assume (2.1). Let $\alpha \in (\beta - 2, \alpha_0 + 1)$, then for every $x \in V_\alpha$ there exists a strong solution $(X(\cdot; x), \tau_x^\alpha)$ with initial condition x. Moreover, the solution is unique in the sense that if $(X(\cdot; x), \tau_x)$ and $(X'(\cdot; x), \tau_x')$ are two solutions, then $\mathbb{P}[\tau_x = \tau_x'] = 1$ and $X(\cdot; x) = X'(\cdot; x)$ for $t < \tau_x$. Finally, the process $(X(\cdot; x))_{x \in V_\alpha}$ is Markov, in the sense given above.*

Proof Existence and uniqueness are essentially based on the same ideas of [42, Theorem 5.1], but with simpler estimates. We give a quick sketch of the proof to introduce some of the definitions we will use later. Let $\chi \in C^\infty([0, \infty))$ be non increasing and such that $\chi(u) = 1$ for $u \leq 1$ and $\chi(u) = 0$ for $u \geq 2$. Consider the problem

$$dX_n^R = -\nu\lambda_n^2 X_n^R \, dt + \chi_R(\|X^R\|_\alpha)\big(\lambda_{n-1}^\beta (X_{n-1}^R)^2 - \lambda_n^\beta X_n^R X_{n+1}^R\big) \, dt + \sigma_n \, dW_n. \tag{2.2}$$

The above equation has a (pathwise) unique global solution for every $x \in V_\alpha$, which is continuous in time with values in V_α. Given $x \in V_\alpha$, define $\tau_x^{\alpha,R}$ as the first time t when $\|X^R(t)\|_\alpha = R$. Then $\tau_x^\alpha = \sup_{R>0} \tau_x^{\alpha,R}$ and the strong solution $X(t; x)$ coincides with $X^R(t; x)$ for $t \leq \tau_x^{\alpha,R}$. By uniqueness the definition makes sense. Markovianity follows by the Markovianity of each X^R. □

By pathwise uniqueness, if $x \in V_\alpha$, then $\tau_x^\alpha \leq \tau_x^{\alpha'}$ for every $\alpha' \in (\beta - 2, \alpha)$. We will be able to deduce that $\tau_x^\alpha = \tau_x^{\alpha'}$ as a consequence of Proposition 4.3.

2.2.2 Weak martingale solutions

The fact that the *blow-up time* τ_x^α associated to a strong solution may (or may not) be infinite is the main topic of discussion of the paper. To consider global solutions we introduce weak solutions.

Given a sequence of independent one-dimensional standard Brownian motions $(W_n)_{n\geq1}$, let $Z = (Z_n)_{n\geq1}$ be the solution of

$$dZ_n + \nu\lambda_n^2 Z_n \, dt = \sigma_n \, dW_n, \quad n \geq 1, \tag{2.3}$$

with $Z_n(0) = 0$ for all $n \geq 1$. Define the functional $\mathcal{G}_t$ as

$$\mathcal{G}_t(y, z) = \|y(t)\|_H^2 + 2\int_0^t \left(\nu\|y(s)\|_V^2 - \sum_{n=1}^\infty \lambda_n^\beta \big(y_n + z_n\big)(y_{n+1}z_n - y_n z_{n+1})\right) ds.$$

If $Y \in L^\infty_{\text{loc}}(0,\infty;H) \cap L^2_{\text{loc}}(0,\infty;V)$ and Assumption 2.1 holds, then by the lemma below $\mathcal{G}_t(Y,Z)$ is finite and jointly measurable in the variables (t,y,z) (see [8,41] for a related problem). The following regularity result for Z is standard [15].

Lemma 2.4 *Assume* (2.1) *with* $\alpha_0 \in \mathbf{R}$. *Given* $\alpha < \alpha_0 + 1$, *then almost surely* $Z \in C([0,T];V_\alpha)$ *for every* $T > 0$. *Moreover, for every* $\epsilon \in (0,1]$, *with* $\epsilon < \alpha_0 + 1 - \alpha$, *there are* $c_{2.4-1.\epsilon} > 0$ *and* $c_{2.4-2.\epsilon} > 0$, *such that for every* $T > 0$,

$$\mathbb{E}\left[\exp\left(\frac{c_{2.4-2.\epsilon}}{T^\epsilon}\sup_{[0,T]}\|Z(t)\|^2_\alpha\right)\right] \le c_{2.4-1.\epsilon}.$$

Definition 2.5 (*Energy martingale solution*) A *weak martingale* solution starting at $x \in H$ is a couple (X,W) on a filtered probability space $(\Omega,\mathscr{F}(\mathscr{F}_t)_{t\ge0},\mathbb{P})$ such that $W = (W_n)_{n\ge1}$ is a sequence of independent standard Brownian motions and $X = (X_n)_{n\ge1}$ is component-wise a solution of (1.1) with $X(0) = x$.

A weak solution is an *energy* solution if $Y = X - Z \in L^\infty_{\text{loc}}(0,\infty;H) \cap L^2_{\text{loc}}([0,\infty);V)]$ with probability one and there is a set $T_{\mathbb{P}} \subset (0,\infty)$ of null Lebesgue measure such that for every $s \notin T_{\mathbb{P}}$ and every $t > s$, the following *energy inequality* holds,

$$\mathbb{P}[\mathcal{G}_t(Y,Z) \le \mathcal{G}_s(Y,Z)] = 1.$$

Remark 2.6 Let $\Omega_\beta = C([0,\infty);V_{-\beta})$ and define on Ω_β the canonical process ξ as $\xi_t(\omega) = \omega(t)$ for all $t > 0$ and $\omega \in \Omega_\beta$. It is a standard interpretation [24] that a weak solution can be seen as a probability on the path space Ω_β. Namely, if $\mathbb{P}_x$ is the law of a weak solution starting at $x \in H$, then ξ is a weak solution on $(\Omega_\beta,\mathbb{P}_x)$. This interpretation will be used in the rest of the paper.

Remark 2.7 The process $Y = X - Z$ satisfies the equations

$$\dot{Y}_n + \nu\lambda_n^2 Y_n = \lambda_{n-1}^\beta (Y_{n-1}+Z_{n-1})^2 - \lambda_n^\beta (Y_n+Z_n)(Y_{n+1}+Z_{n+1}), \tag{2.4}$$

$\mathbb{P}$–almost surely, for every $n \ge 1$ and $t > 0$.

Given $\alpha > \beta - 2$ and $R > 0$, define the following random times on Ω_β,

$$\tau^\alpha_\infty = \inf\{t \ge 0 : \|\omega(t)\|_\alpha = \infty\}, \qquad \tau^{\alpha,R}_\infty = \inf\{t \ge 0 : \|\omega(t)\|_\alpha > R\}, \tag{2.5}$$

and each random time is ∞ if the corresponding set is empty. The *energy inequality* required in Definition 2.5 ensures that all weak solutions with the same initial condition coincide with the strong solution up to the *blow-up* time τ^α_∞.

Theorem 2.8 *Let* $\beta > 2$ *and assume* (2.1). *Then for every* $x \in H$ *there exists at least one energy martingale solution* $\mathbb{P}_x$. *Moreover,*

- *if $\alpha \in (\beta-2, 1+\alpha_0)$, $x \in V_\alpha$ and $\mathbb{P}_x$ is an energy martingale solution with initial condition x, then $\tau_x^\alpha = \tau_\infty^\alpha$ under $\mathbb{P}_x$ and for every $t > 0$,*

$$\xi_s = X(s;x), \quad s \le t, \qquad \mathbb{P}_x - a.s. \quad on\ \{\tau_x^\alpha > t\},$$

  where $(X(\cdot;x), \tau_x^\alpha)$ is the strong solution with initial condition x defined on Ω_β.
- *There exists at least one family $(\mathbb{P}_x)_{x\in H}$ of energy martingale solutions satisfying the almost sure Markov property. Namely for every $x \in H$ and every bounded measurable $\phi : H \to \mathbf{R}$,*

$$\mathbb{E}^{\mathbb{P}_x}\big[\phi(\xi_t)|\mathscr{B}_s\big] = \mathbb{E}^{\mathbb{P}_{\omega(s)}}[\phi(\xi_{t-s})], \quad \mathbb{P}_x - a.s.,$$

  for almost every $s \ge 0$ (including 0) and for all $t \ge s$.

Proof The proof of the first fact can be done as in [2]. The proofs of the other two facts are entirely similar to those of Theorem 2.1 of [41] and Theorem 3.6 of [42] and we refer to these references for further details. □

A natural way to prove existence of weak solution (see [2]) is to use finite dimensional approximations. Consider for each $N \ge 1$ the solution $(X_n^{(N)})_{1\le n\le N}$ to the following finite dimensional system,

$$\begin{cases} \dot{X}_1^{(N)} = -\nu\lambda_1^2 X_1^{(N)} - \lambda_1^\beta X_1^{(N)} X_2^{(N)} + \sigma_1\, dW_1, \\ \dots, \\ \dot{X}_n^{(N)} = -\nu\lambda_n^2 X_n^{(N)} + \lambda_{n-1}^\beta (X_{n-1}^{(N)})^2 - \lambda_n^\beta X_n^{(N)} X_{n+1}^{(N)} + \sigma_n\, dW_n, \\ \dots, \\ \dot{X}_N^{(N)} = -\nu\lambda_N^2 X_N^{(N)} + \lambda_{N-1}^\beta (X_{N-1}^{(N)})^2 + \sigma_N\, dW_N. \end{cases} \tag{2.6}$$

Given $x \in H$, let $\mathbb{P}_x^{(N)}$ be the probability distribution on Ω_β of the solution of the above system with initial condition $x^{(N)} = (x_1, x_2, \dots, x_N)$.

Definition 2.9 (*Galerkin martingale solution*) Given $x \in H$, a *Galerkin* martingale solution is any limit point in Ω_β of the sequence $(\mathbb{P}_x^{(N)})_{N\ge 1}$.

It is easy to verify (it is the proof of existence in Theorem 2.8, see [2] for details in a similar problem) that *Galerkin* martingale solutions are *energy* solutions.

Remark 2.10 All results of this section hold for any polynomial non-linearity with finite modes interaction. On the other hand the rest of the paper is strongly based on the structure of the non-linearity. At least for nearest-neighbour interaction, we are dealing with the difficult case. Indeed every nearest-neighbour interaction can be written [34] as $a_1 B_n^1(X) + a_2 B_n^2(X)$, where B_n^1 is the non-linearity of the dyadic model and $B_n^2(x) = \lambda_{n+1}^\beta x_{n+1}^2 - \lambda_n^\beta x_{n-1} x_n$. In [34] the authors prove that the inviscid problem with non-linearity B_n^2 is well-posed.

3 Control of the negative components

Given $\beta > 2$, $\alpha \in \mathbf{R}$ and $c_0 > 0$, consider the solution Z of (2.3) and define the following process,

$$N_{\alpha,c_0}(t) = min\big\{m \geq 1 : |Z_n(s)| \leq c_0 \nu \lambda_{n-1}^{-\alpha} \text{ for } s \in [0,t] \text{ and } n \geq m\big\}, \quad (3.1)$$

with $N_{\alpha,c_0}(t) = \infty$ if the set is empty.

Lemma 3.1 (Moments of N_{α,c_0}) *Given $\beta > 2$, assume (2.1) and let $\alpha < \alpha_0 + 1$. Then for every $\gamma \in (0, \alpha_0 + 1 - \alpha)$ and $\epsilon \in (0, 1]$, with $\epsilon < \alpha_0 + 1 - \alpha - \gamma$, there are two numbers $c_{3.1-1} > 0$ and $c_{3.1-2} > 0$, depending only on ϵ, γ and α_0, such that*

$$\mathbb{P}[N_{\alpha,c_0}(t) > n] \leq c_{3.1-1} \exp\left(-c_{3.1-2} \frac{c_0 \nu}{t^\epsilon} \lambda_n^\gamma\right),$$

for every $t > 0$ and $n \geq 1$. In particular, $\mathbb{P}[N_{\alpha,c_0}(t) = n] > 0$ for every $n \geq 1$ and

$$\mathbb{E}\big[\exp\big(\lambda^\gamma_{N_{\alpha,c_0}(t)}\big)\big] < \infty.$$

Proof For $n \geq 1$,

$$\{N_{\alpha,c_0}(t) \leq n\} = \left\{\sup_{k \geq n} \sup_{[0,t]} \lambda_{k-1}^\alpha |Z_k(s)| \leq c_0 \nu\right\}.$$

Hence if $\gamma < \alpha_0 + 1 - \alpha$ and $k \geq n$,

$$\sup_{[0,t]} \lambda_{k-1}^\alpha |Z_k(s)| \leq \lambda_{n-1}^{-\gamma} \sup_{[0,t]} \|Z(s)\|_{\alpha+\gamma}.$$

Therefore by Chebychev's inequality and Lemma 2.4,

$$\mathbb{P}[N_{\alpha,c_0}(t) > n] \leq \mathbb{P}\left[\sup_{[0,t]} \|Z(s)\|_{\alpha+\gamma} > c_0 \nu \lambda_{n-1}^\gamma\right] \leq c_{2.4-1.\epsilon} \exp\left(-c_{2.4-2.\epsilon} \frac{c_0 \nu}{t^\epsilon} \lambda_{n-1}^\gamma\right),$$

for every $\epsilon \in (0, 1]$ with $\epsilon < \alpha_0 + 1 - \alpha - \gamma$. The double-exponential moment follows from this estimate.

We finally prove that $\mathbb{P}[N_{\alpha,c_0}(t) = n] > 0$. We prove it for $n = 1$ and all other cases follow similarly. By independence,

$$\mathbb{P}[N_{\alpha,c_0}(t) = 1] = \exp\left(-\sum_{n=1}^\infty -\log \mathbb{P}\big[\sup_{[0,t]} \lambda_{k-1}^\alpha |Z_k(s)| \leq c_0 \nu\big]\right),$$

and it is sufficient to show that the series above is convergent. By (2.1),

$$\mathbb{P}\left[\sup_{[0,t]} \lambda_{k-1}^\alpha |Z_k(s)| \leq c_0 \nu\right] \geq \mathbb{P}\left[\sup_{[0,t]} |\zeta(\lambda_n^2 s)| \leq 2^\alpha c_0 \nu \lambda_n^{\alpha_0+1-\alpha}\right],$$

where ζ is the solution of the one dimensional SDE $d\zeta + \nu\zeta\, dt = dW$, with $\zeta(0) = 0$. The conclusion follows by standard tail estimates on the one dimensional Ornstein–Uhlenbeck process (see for instance [20]), since $\alpha < 1 + \alpha_0$. □

The lemma below is the crucial result of the paper. To formulate its statement, we introduce suitable finite dimensional approximations. Consider for each integer $N \geq 1$ the finite dimensional approximations of (2.4),

$$\begin{cases} \dot{Y}_1^{(N)} = -\nu\lambda_1^2 Y_1^{(N)} - \lambda_1^\beta X_1^{(N)} X_2^{(N)}, \\ \dots, \\ \dot{Y}_n^{(N)} = -\nu\lambda_n^2 Y_n^{(N)} + \lambda_{n-1}^\beta (X_{n-1}^{(N)})^2 - \lambda_n^\beta X_n^{(N)} X_{n+1}^{(N)}, \\ \dots, \\ \dot{Y}_N^{(N)} = -\nu\lambda_N^2 Y_N^{(N)} + \lambda_{N-1}^\beta (X_{N-1}^{(N)})^2. \end{cases} \tag{3.2}$$

In the above system we have set $X_n^{(N)} = Y_n^{(N)} + Z_n$ for $n = 1, \dots, N$. It is easy to verify that the above SDE admits a unique global solution.

Lemma 3.2 (Main lemma) *Let $\beta > 2$, $N \geq 1$ and $T > 0$, and assume (2.1). Let $\alpha \in [\beta - 2, 1 + \alpha_0)$ and consider $c_0 > 0$, $a_0 > 0$ and $n_0 \geq 1$ such that*

$$c_0 \leq a_0 \quad \text{and} \quad c_0 < \sqrt{a_0}\big(\lambda_{n_0}^{\frac{1}{2}(\alpha+2-\beta)} - \sqrt{a_0}\big). \tag{3.3}$$

Assume that $\lambda_{n-1}^\alpha X_n^{(N)}(0) \geq -a_0\nu$ for all $n = n_0, \dots, N$. If $N > N_{\alpha,c_0}(T)$, then $Y_n^{(N)}(t) \geq -a_0\nu\lambda_{n-1}^{-\alpha}$ for all $t \in [0, T]$ and all $n \geq n_0 \vee N_{\alpha,c_0}(T)$.

Proof For simplicity we drop the superscript $^{(N)}$. We can first assume that $\lambda_{n-1}^\alpha Y_n(0) > -\nu a_0$ for $n \geq n_0, \dots, N$ (the case of equality follows by continuity). Then the same is true in a neighbourhood of $t = 0$. Let $t_0 > 0$ be the first time when at least for one n, $\lambda_{n-1}^\alpha Y_n(t_0) = -\nu a_0$. Let $n \geq n_0 \vee N_{\alpha,c_0}(T)$ be one of such indices. Then

$$\begin{aligned} \dot{Y}_n(t_0) &\geq -\nu\lambda_n^2 Y_n(t_0) - \lambda_n^\beta (Y_n(t_0) + Z_n(t_0))(Y_{n+1}(t_0) + Z_{n+1}(t_0)) \\ &\geq a_0\nu^2\lambda^2\lambda_{n-1}^{2-\alpha} + \lambda_n^\beta\big(a_0\nu\lambda_{n-1}^{-\alpha} - Z_n(t_0)\big)(Y_{n+1}(t_0) + Z_{n+1}(t_0)) \\ &\geq a_0\nu^2\lambda^2\lambda_{n-1}^{2-\alpha} - \lambda_n^\beta\big(a_0\nu\lambda_{n-1}^{-\alpha} - Z_n(t_0)\big)(Y_{n+1}(t_0) + Z_{n+1}(t_0))_-, \end{aligned}$$

since $\nu a_0\lambda_{n-1}^{-\alpha} - Z_n(t_0) \geq 0$ for $n \geq N_{\alpha,c_0}(T)$. Here $x_- = \max(-x, 0)$. We also know that $Y_{n+1}(t_0) \geq -a_0\nu\lambda_n^{-\alpha}$, hence $Y_{n+1}(t_0) + Z_{n+1}(t_0) \geq -\nu(a_0 + c_0)\lambda_n^{-\alpha}$ and $(Y_{n+1}(t_0) + Z_{n+1}(t_0))_- \leq \nu(a_0 + c_0)\lambda_n^{-\alpha}$. We also have $a_0\nu\lambda_{n-1}^{-\alpha} - Z_n(t_0) \leq \nu(a_0 + c_0)\lambda_{n-1}^{-\alpha}$, so in conclusion

$$\dot{Y}_n(t_0) \geq \nu^2\lambda^2\lambda_{n-1}^{2-\alpha}\big(a_0 - \lambda_n^{\beta-2-\alpha}(a_0 + c_0)^2\big) > 0.$$

□

The next theorem shows that the process can diverge *only* in the positive area.

Theorem 3.3 *Given $\beta > 2$, assume* (2.1). *Let $\alpha \in (\beta - 2, \alpha_0 + 1)$ and $x \in V_\alpha$, and let $(X(\cdot; x), \tau_x^\alpha)$ be the strong solution in V_α with initial condition x. Then*

$$\mathbb{E}\left[\sup_{n\geq 1}\sup_{t\in[0,T\wedge\tau_x^\alpha]}\left(\lambda_{n-1}^\alpha\big(X_n(t)\big)_-\right)^p\right] < \infty,$$

for every $T > 0$ and $p \geq 1$. In particular,

$$\inf_{n\geq 1}\inf_{t\in[0,\tau_x^\alpha\wedge T]}\lambda_{n-1}^\alpha X_n > -\infty, \qquad \mathbb{P}\text{–a. s.}$$

Proof Fix $x \in V_\alpha$ and $T > 0$. Set $a_0 = \frac{1}{4}$ and $c_0 = \frac{1}{6}$, so that condition (3.3) holds for any n_0. Choose $n_0 \geq 1$ as the smallest integer such that $\lambda_{n-1}^\alpha x_n \geq -\frac{1}{4}\nu$ for all $n \geq n_0$. With the choice $c_0 = \frac{1}{6}$, define the event $\mathcal{Z}_{\alpha,T} = \{N_{\alpha,1/6}(T) < \infty\}$. By Lemma 3.1 $\mathcal{Z}_{\alpha,T}$ has probability one. Lemma 3.2 implies that on $\{\tau_x^\alpha > T\}$,

$$Y_n(t) \geq -\tfrac{1}{4}\nu\lambda_{n-1}^{-\alpha}, \quad \text{for } n \geq n_0 \vee N_{\alpha,\frac{1}{6}}(T).$$

Indeed, we can set $x^{(N)} = (x_1, \dots, x_N)$ and notice that on the event $\{\tau_x^\alpha > T\}$, problem (2.4) has a unique solution. Hence for every N the solution of (3.2) with initial condition $x^{(N)}$ converges to the solution of (2.4) with initial condition x. Here the convergence is component-wise uniform in time on $[0, T]$.

Let $N_1 = n_0 \vee N_{\alpha,1/6}(T)$. It is clear that N_1 has the same finite moments of $N_{\alpha,1/6}(T)$. Moreover on $\{\tau_x^\alpha > T\}$,

$$\lambda_{n-1}^\alpha X_n(t) \geq \begin{cases} -\lambda_{N_1-1}^\alpha \sup_{t\in[0,T]}\|X(t)\|_H, & n < N_1,\\ -\frac{5}{12}\nu, & n \geq N_1,\end{cases}$$

for every $n \geq 1$. Therefore

$$\sup_{n\geq 1}\sup_{t\in[0,T]}\lambda_{n-1}^\alpha\big(X_n(t)\big)_- \leq \nu + \lambda_{N_1-1}^\alpha\sup_{t\in[0,T]}\|X(t)\|_H.$$

From Lemma 3.1 and the fact that $\mathbb{E}[\sup_{[0,T]}\|X(t)\|_H^p]$ is finite for every $p \geq 1$, the estimate in the statement of the theorem readily follows. □

Remark 3.4 Given an initial condition $x \in V_\alpha$, if we set

$$\tau_{x,\pm}^\alpha = \sup\{t : \sup_{n\geq 1}\lambda_n^\alpha(X_n)_\pm < \infty\},$$

then $\tau_x^\alpha = \min(\tau_{x,+}^\alpha, \tau_{x,-}^\alpha)$, and the previous theorem implies that $\tau_x^\alpha = \tau_{x,+}^\alpha$.

Corollary 3.5 *Let $\beta > 2$, $\alpha \in (\beta - 2, \alpha_0 + 1)$ and $x \in V_\alpha$, and assume* (2.1). *If either problem* (2.4)*, with initial state x, admits a unique solution for almost every possible value assumed by Z, or we are dealing with a Galerkin solution, then*

$$\mathbb{E}\left[\sup_{n\geq 1}\sup_{t\in[0,T]}\left(\lambda_{n-1}^{\alpha}\big(X_n(t)\big)_{-}\right)^p\right]<\infty,$$

for every $T>0$ and $p\geq 1$. In particular,

$$\inf_{n\geq 1}\inf_{t\in[0,T]}\lambda_{n-1}^{\alpha}X_n>-\infty,\quad \mathbb{P}\text{–}a.\ s.$$

Proof We simply notice that in the proof of the theorem above we have used the piece of information $\{\tau_x^\alpha>T\}$ only to ensure that (2.4) admits a unique solution.

On the other hand, if we are dealing with a Galerkin solution, then up to a subsequence we still have component-wise uniform convergence in time. □

4 Uniqueness and regularity for $2<\beta\leq\frac{5}{2}$

In this section we prove two extensions of results given in the non-random case. The first concerns path-wise uniqueness, the second is about absence of blow-up. Both extensions are based on the control of negative components shown in Sect. 3.

Theorem 4.1 (Pathwise uniqueness) *Let $\beta\in(2,3]$ and assume that (2.1) holds. Let $X(0)\in V_{\beta-2}$, then there exists a (pathwise) unique solution of (1.1) with initial condition $X(0)$, in the class of Galerkin martingale solutions.*

We do not know if uniqueness holds in some larger class (energy or weak martingale solutions), neither we know if a Galerkin solution develops blow-up. By slightly restricting the range of values of β, we have an improvement.

Theorem 4.2 (Smoothness) *There exists $\beta_c\in(\frac{5}{2},3]$ such that the following statement holds. Assume (2.1) and let $\beta\in(2,\beta_c)$ and $\alpha\in(\beta-2,1+\alpha_0)$. Then $\tau_x^\alpha=\infty$ for all $x\in V_\alpha$ and path-wise uniqueness holds in the class of energy martingale solutions.*

4.1 The proof of Theorem 4.1

The proof is based on [7, Proposition 3.2], which builds up on an idea in [4]. Both results hold for positive solutions and no noise.

Proof of Theorem 4.1 Fix $T>0$. It is sufficient to show uniqueness on $[0,T]$. We will use Lemma 3.2 with $c_0=\frac{1}{6}$ and $a_0=\frac{1}{4}$. With these values (3.3) holds for any n_0. Moreover, the bounds of Lemma 3.2 hold for Galerkin solutions, since they are the component-wise limit of finite dimensional approximations.

Let n_0 be the smallest integer such that $\inf_{n\geq n_0}\lambda_n^{\beta-2}X_n(0)\geq-\frac{1}{4}\nu$ and set $N_0=1+n_0\vee N_{\beta-2,1/6}(T)$. Let X^1, X^2 be two solutions with the same initial condition $X(0)$. By Lemma 3.2, $X_n^i(t)\geq Z_n(t)-\frac{1}{4}\nu\lambda_{n-1}^{2-\beta}$ for $n\geq N_0$, $t\in[0,T]$, and $i=1,2$. Set $A_n=X_n^1-X_n^2$, $B_n=X_n^1+X_n^2$, $D_n=\frac{1}{2}\nu\lambda_{n-1}^{2-\beta}-2Z_n$, and

$$\psi_\ell(t) = \sum_{n=1}^{N_0-1} \frac{A_n^2}{\lambda_n}, \quad \psi_{h,N}(t) = \sum_{N_0}^{N} \frac{A_n^2}{\lambda_n}, \quad \psi_N(t) = \psi_\ell(t) + \psi_{h,N}(t).$$

Notice that $B_n + D_n \geq 0$ if $t \in [0, T]$ and $n \geq N_0$. A simple computation yields

$$\begin{aligned}\frac{d}{dt}\psi_{h,N} + 2\nu \sum_{n=N_0}^{N} \lambda_n A_n^2 &= -\sum_{n=N_0}^{N} \lambda_n^{\beta-1} B_{n+1} A_n^2 - \lambda_N^{\beta-1} B_N A_N A_{N+1} \\ &\quad + \lambda_{N_0-1}^{\beta-1} B_{N_0-1} A_{N_0-1} A_{N_0} = \boxed{1} + \boxed{2}_N + \boxed{3}_{N_0},\end{aligned}$$

for $N > N_0$. For the first term we notice that $D_{n+1} \leq \frac{5}{6}\nu\lambda_n^{2-\beta}$, hence $\boxed{1} \leq \sum_{n=N_0}^{N} \lambda_n^{\beta-1} D_{n+1} A_n^2 \leq \nu \sum_{n=N_0}^{N} \lambda_n A_n^2$. For the second term,

$$\sum_{N=1}^{\infty} \int_0^T \boxed{2}_N \, dt \leq \sup_{[0,T]} \|X^1 + X^2\|_H \int_0^T \|A\|^2_{\frac{1}{2}(\beta-1)} \, ds.$$

The quantity on the right-hand side is a. s. finite since $Z \in C([0, T]; V_{(\beta-1)/2})$ by Lemma 2.4, $V \in L^2([0, T]; V)$ and $\beta \leq 3$. This implies that a. s. $\int_0^T \boxed{2}_N \, dt \to 0$ as $N \to \infty$. Likewise,

$$\frac{d}{dt}\psi_\ell \leq -\sum_{n=1}^{N_0-1} \lambda_n^{\beta-1} B_{n+1} A_n^2 - \boxed{3}_{N_0} \leq \lambda_{N_0-1}^{\beta} \left(\sup_{[0,T]} \|X^1 + X^2\|_H \right) \psi_\ell - \boxed{3}_{N_0},$$

and in conclusion $\frac{d}{dt}\psi_N \leq \lambda_{N_0-1}^{\beta} \big(\sup_{[0,T]} \|X^1 + X^2\|_H\big) \psi_\ell + \boxed{2}_N$. Set $\psi(t) = \|A(t)\|_{-1/2}$, then $\psi_N \uparrow \psi$. Integrate in time the inequality for ψ_N and take the limit as $N \uparrow \infty$ to get

$$\psi(t) \leq \lambda_{N_0-1}^{\beta} \left(\sup_{[0,T]} \|X^1 + X^2\|_H \right) \int_0^t \psi(s) \, ds.$$

By Gronwall's lemma $\psi(t) = 0$ a. s. for all $t \in [0, T]$. □

4.2 The proof of Theorem 4.2

We give a minimal requirement for smoothness of solutions of (1.1). This is analogous to the criterion developed in [7] without noise. Given $T > 0$ define the subspace K_T of Ω_β as

$$K_T = \left\{ \omega \in \Omega_\beta : \lim_n \left(\max_{t \in [0,T]} \lambda_n^{\beta-2} |\omega_n(t)| \right) = 0 \right\}.$$

Proposition 4.3 *Assume* (2.1)*, and let* $\beta > 2$, $\alpha \in (\beta - 2, 1 + \alpha_0)$. *Let* $x \in V_\alpha$ *and* $\mathbb{P}_x$ *be an energy martingale solution starting at* x. *If* τ_∞^α *is the random time defined in* (2.5)*, then* $\{\tau_\infty^\alpha > T\} = K_T$ *under* $\mathbb{P}_x$*, for every* $T > 0$.

Proof Fix $\alpha \in (\beta - 2, \alpha_0 + 1)$, $x \in V_\alpha$ and a solution $\mathbb{P}_x$ starting at x, and let τ_∞^α, $\tau_\infty^{\alpha,R}$ be the random times defined in (2.5). Assume $\tau_\infty^\alpha(\omega) > T$, then $\tau_\infty^{\alpha,R_0}(\omega) > T$ for some $R_0 > \|x\|_\alpha$. In particular $\|\xi_t(\omega)\|_\alpha \leq R_0$ for $t \in [0, T]$. Hence,

$$\lambda_n^{\beta-2} \max_{[0,T]} |\xi_{n,t}(\omega)| \leq \lambda_n^{\beta-2-\alpha} \sup_{[0,T]} \|\xi_t(\omega)\|_\alpha \leq R_0 \lambda_n^{\beta-2-\alpha},$$

and $\omega \in K_T$. Vice versa, let $\omega \in K_T$ and choose $(M_n)_{n\geq 1}$ such that $M_n \downarrow 0$ and $\lambda_n^{\beta-2} \max_{[0,T]} |\xi_n(t)| \leq M_n$. Set $u_n = \lambda_n^\alpha \xi_n$ and $m_n = \max_{[0,T]} |u_n(t)|$, then

$$|u_n(t)| \leq |u_n(0)| + \left(\sup_{[0,T]} \lambda_n^\alpha |Z_n(t)|\right) + \nu^{-1}\lambda^{\alpha-2} M_{n-1} m_{n-1} + \nu^{-1}\lambda^{2-\beta} M_{n-1} m_n,$$

and

$$(1 - \nu^{-1}\lambda^{2-\beta} M_{n-1}) m_n \leq |u_n(0)| + \left(\sup\nolimits_{[0,T]} \lambda_n^\alpha |Z_n(t)|\right) + \nu^{-1}\lambda^{\alpha-2} M_{n-1} m_{n-1}.$$

Set $A_n = 2\lambda_n^\alpha |x_n| + 2\left(\sup_{[0,T]} \lambda_n^\alpha |Z_n(t)|\right)$. By Lemma 2.4 applied with an $\alpha' > \alpha$, we know that $\sum_n A_n^2 < \infty$ with probability one. For n large enough (depending only on λ, ν and β), the above inequality reads $m_n \leq A_n + \frac{1}{2} m_{n-1}$. By solving the recursion we get $\sum_n m_n^2 < \infty$, and in particular $\tau_\infty^\alpha(\omega) > T$. □

The basic idea of the proof of Theorem 4.2 is that given a smooth initial state x, there is a solution $\mathbb{P}_x$ that satisfies $\mathbb{P}_x[K_T] = 1$. Hence is the unique solution.

Proof of Theorem 4.2 Fix $\alpha \in (\beta - 2, \alpha_0 + 1)$, $x \in V_\alpha$, $T > 0$ and an energy martingale solution $\mathbb{P}_x$ starting at x, and let τ_∞^α be the random time defined in (2.5). There is no loss of generality in assuming that $\mathbb{P}_x$ is a Galerkin solution. Indeed, by Theorem 2.8, τ_∞^α is equal a. s. to the lifespan τ_x^α of the strong solution with the same initial state.

Since $\mathbb{P}_x$ is a Galerkin solution, there are $x^{(N_k)}$ and the solution $\mathbb{P}^{(N_k)}$ with initial state $x^{(N_k)}$ of (2.6) with dimension N_k, such that $x^{(N_k)} \to x$ in H and $\mathbb{P}^{(N_k)} \rightharpoonup \mathbb{P}_x$ in Ω_β. By definition we also have that $x_n^{(N_k)} = x_n$ for $n \leq N_k$.

By a standard argument (Skorokhod's theorem) there are a common probability space $(\bar{\Omega}, \bar{\mathscr{F}}, \bar{\mathbb{P}})$ and random variables $X^{(N_k)}$, X on $\bar{\Omega}$ with laws $\mathbb{P}^{(N_k)}$, $\mathbb{P}_x$ respectively, such that $X_n^{(N_k)} \to X_n$, $\bar{\mathbb{P}}$—a. s., uniformly on $[0, T]$ for all $n \geq 1$.

Let $\epsilon > 0$ be such that $\alpha > \beta - 2 + 2\epsilon$ and $6 - 2\beta - 3\epsilon > 0$. We will use Lemma 3.2 with $a_0 < \frac{1}{2}$ (to be chosen later in the proof) and $c_0 = \frac{1}{3} a_0$. Let $\bar{n}$ be the smallest integer such that $\lambda_{n-1}^\alpha |x_n| \leq a_0 \nu$ for all $n \geq \bar{n}$, and set $N_0 = \bar{n} \vee N_{\beta-2+2\epsilon,c_0}(T)$.

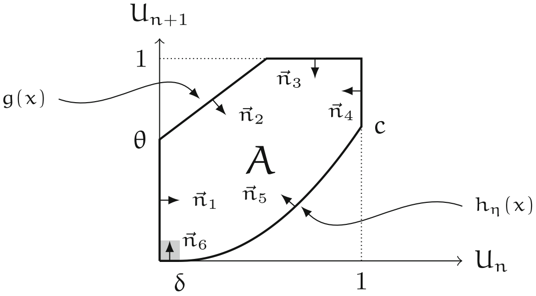
For each integer $n_0 \geq 1$ and real $M > 0$ define the event

$$A_M(n_0) = \left\{ \sup_{[0,T]} \left(|X_{n_0-1}^{(N_k)}| + |X_{n_0}^{(N_k)}|\right) \leq M \quad \text{for all } k \text{ such that } N_k \geq n_0 \right\}.$$

Clearly $\bar{\mathbb{P}}\big[\bigcup_M A_M(n_0)\big] = 1$ since $X_n^{(N_k)} \to X_n$ uniformly for $n \geq 1$, hence

$$\bar{\mathbb{P}}\Bigg[\bigcup_{n_0\geq 1,\ M>0} \big(\{N_0 = n_0\} \cap A_M(n_0)\big)\Bigg] = 1.$$

Fix $n_0 \geq 1$ and $M > 0$, then everything boils down to prove that K_T happens on $\{N_0 = n_0\} \cap A_M(n_0)$ for $(X_n^{(N_k)})_{n_0 \leq n \leq N_k}$ uniformly in k. We work pathwise for $\omega \in \{N_0 = n_0\} \cap A_M(n_0)$ and we adapt the method in [7]. We will prove that the area in Figure 4.2 is invariant for a suitable rescaling of Y. The area



A is defined by $c = \lambda^{-(6-2\beta-3\epsilon)}$, $\delta = \frac{1}{10}$, $\theta = \frac{3}{5}$ and $m = \frac{3}{4}$, and by $g(x) = \min(mx + \theta, 1)$ and h_η specified later in (4.6). In [7] we used the value $\eta = \delta$.

First, we change and rescale the solution. Let $\epsilon_n = \nu\lambda_{n-1}^{-2\epsilon}$ and define

$$\delta_0^{-1} = \max\big\{\delta^{-1}, \lambda_{n_0}^{\beta-2+\epsilon} M + 2a_0\lambda_{n_0}^{\epsilon}\epsilon_{n_0-1}, \sup\nolimits_{n\geq n_0} \lambda_n^{\epsilon}(\lambda_n^{\beta-2}x_n + 2a_0\epsilon_n)\big\},$$
$$U_n(t) = \lambda_n^{\epsilon}\delta_0^2\big(\lambda_n^{\beta-2}Y_n^{(N_k)}(\delta_0 t) + a_0\epsilon_n\big), \quad V_n(t) = \lambda_n^{\epsilon}\big(a_0\epsilon_n - \lambda_n^{\beta-2}Z_n(\delta_0 t)\big).$$

It follows by Lemma 3.2 that

$$U_n \geq 0, \quad \text{and} \quad \tfrac{2}{3}a_0\lambda_n^{\epsilon}\epsilon_n \leq V_n \leq \tfrac{5}{3}a_0\lambda_n^{\epsilon}\epsilon_n, \tag{4.1}$$

for all $n_0 \leq n \leq N_k$. By the choice of δ_0 it follows that $U_n(0) \leq \delta_0 \leq \delta$ for all $n \geq n_0$, $\max_{[0,T]} U_{n_0-1} \leq \delta$, and $\max_{[0,T]} U_{n_0} \leq \delta$.

Consider for $n \geq n_0$ the coupled systems in (U_n, U_{n+1}),

$$\frac{d}{dt}\begin{pmatrix} U_n \\ U_{n+1}\end{pmatrix} = \lambda_n^{2-\epsilon}\left(\delta_0^3\mathfrak{B}_n^0 + \delta_0\mathfrak{B}_n^1 + \frac{1}{\delta_0}\lambda^{\beta-4+2\epsilon}\mathfrak{B}_n^2\right), \tag{4.2}$$

where

$$\mathfrak{B}_n^i = \begin{pmatrix} P_n^i \\ \lambda^{2-\epsilon}P_{n+1}^i\end{pmatrix}, \qquad i = 0, 1, 2,$$
$$P_n^0 = a_0\nu\lambda_n^{2\epsilon}\epsilon_n + \lambda^{\beta-4+2\epsilon}V_{n-1}^2 - \lambda^{2-\beta-\epsilon}V_nV_{n+1},$$

$$P_n^1 = -\nu\lambda_n^\epsilon U_n - 2\lambda^{\beta-4+2\epsilon}V_{n-1}U_{n-1} + \lambda^{2-\beta-\epsilon}(V_{n+1}U_n + V_nU_{n+1}),$$
$$P_n^2 = U_{n-1}^2 - \lambda^{6-2\beta-3\epsilon}U_nU_{n+1}.$$

The goal is to prove that $(U_n(t))_{n_0\le n\le N_k}$ is uniformly bounded in n and t. Indeed, we will see that $0 \le U_n(t) \le 1$ for all n, t. In turns this implies that $-\lambda_n^\epsilon \epsilon_n \le \lambda_n^{\beta-2+\epsilon} Y_n^{(N_k)}(t) \le \delta_0^{-2}$, for all n, t. Since $Y_n^{(N_k)} \to Y_n$ uniformly on $[0, T]$ for each n, the same holds for the limit Y. Due to Lemma 2.4, $X \in K_T$.

By the choice of δ_0 each pair $(U_n(t), U_{n+1}(t))$ is in the interior of A at $t = 0$. If we show that each pair stays in A for all $t > 0$, then $U_n \le 1$. To this end it suffices to show that each vector field on the right hand side of (4.2) points inwards on the boundary of A. By Lemma 3.2 it immediately follows that the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_6$ point inwards. Moreover, since A is convex, it is sufficient to verify that each of the products of $\mathbf{n}_i$, $i = 2, \dots, 5$, with the vector fields $\mathfrak{B}_n^0$, $\mathfrak{B}_n^1$ and $\mathfrak{B}_n^2$ is positive.

The vector field $\mathfrak{B}^1$. We will use (4.1), that $U \le 1$ in A and that ϵ_n is non-increasing. If a_0 is chosen small enough (depending only on m, β and ϵ, but not on M, n_0 or δ_0), then the lower bounds we will obtain are positive numbers.

On the border with normal $\mathbf{n}_2 = (m, -1)$, $\lambda^2 U_{n+1} - mU_n \ge \lambda^2\theta$, hence

$$\mathfrak{B}_n^1 \cdot \mathbf{n}_2 = mP_n^1 - \lambda^{2-\epsilon}P_{n+1}^1 \ge \lambda_n^\epsilon\big(\nu\lambda^2\theta - a_0c(m, \beta, \epsilon)\epsilon_{n-1}\big). \tag{4.3}$$

On the border with normal $\mathbf{n}_3 = (0, -1)$ we have $U_{n+1} = 1$, hence

$$\mathfrak{B}_n^1 \cdot \mathbf{n}_3 = -\lambda^{2-\epsilon}P_{n+1}^1 \ge \lambda^{2-\epsilon}\lambda_{n+1}^\epsilon(\nu - 4a_0\lambda^{2-\beta}\epsilon_{n+1}). \tag{4.4}$$

Similarly, on the border with normal $\mathbf{n}_4 = (-1, 0)$ we have $U_n = 1$, hence

$$\mathfrak{B}_n^1 \cdot \mathbf{n}_4 = -P_n^1 \ge \lambda_n^\epsilon(\nu - 4a_0\lambda^{2-\beta}\epsilon_n). \tag{4.5}$$

Before computing the scalar product with $\mathbf{n}_5$, let us give the definition of h_η. For $\eta \in (0, 1)$ define $\varphi_\eta(x) = \big((x-\eta)/(1-\eta)\big)^{\lambda^2}$, $x \in [\eta, 1]$, and, for $\eta \le \delta$,

$$h_\eta(x) = \frac{c}{1-\varphi_\eta(\delta)}\big(\varphi_\eta(x) - \varphi_\eta(\delta)\big), \quad x \in [\delta, 1]. \tag{4.6}$$

Each h_η is positive, increasing, convex, $h_\eta(\delta) = 0$, $h_\eta(1) = c$ and $h_\eta \to h$ in $C^1([\delta, 1])$ as $\eta \uparrow \delta$. Moreover, there is $c_{\delta,\eta} > 0$ such that $xh_\eta' - \lambda^2 h_\eta \ge c_{\delta,\eta}$. With this inequality in hand, we proceed with the estimate of $\mathfrak{B}_n^1 \cdot \mathbf{n}_5$. On the border with normal $\mathbf{n}_5 = (-h_\eta'(U_n), 1)$ we have $U_n \in [\delta, 1]$ and $U_{n+1} = h_\eta(U_n)$. Since $h_\eta' \le c\lambda^2/(1-2\delta)$, it follows that

$$\mathfrak{B}_n^1 \cdot \mathbf{n}_5 = \lambda^{2-\epsilon}P_{n+1}^1 - h_\eta'(U_n)P_n^1 \ge \lambda_n^\epsilon\big(\nu c_{\delta,\eta} - a_0c(\beta, \epsilon, \delta)\epsilon_n\big). \tag{4.7}$$

The vector field $\mathfrak{B}^0$. Using (4.1) we have that $|P_n^0| \le \lambda_n^{2\epsilon}(a_0\nu\epsilon_n + ca_0^2\epsilon_{n-1}^2)$. This quantity can be made a small fraction of λ_n^ϵ if a_0 is small enough. Therefore, due to formulae (4.3), (4.4), (4.5), (4.7), each product $(\mathfrak{B}_n^0 + \mathfrak{B}_n^1) \cdot \mathbf{n}_i$, $i = 2, \dots, 5$ is positive.

The vector field $\mathfrak{B}^2$. We have chosen the same parameters as in [7], hence the products $\mathfrak{B}_n^2 \cdot \mathbf{n}_3$ and $\mathfrak{B}_n^2 \cdot \mathbf{n}_4$ are positive. A simple computation shows that $\mathfrak{B}_n^2 \cdot \mathbf{n}_2$ and $\mathfrak{B}_n^2 \cdot \mathbf{n}_5$ are continuous functions of h_η, h'_η, and have positive minima for $\eta = \delta$. Then the same is true for η small enough, since $h_\eta \to h$ in $C^1([\delta, 1])$.

The proof we have given (due to the choice of the numbers m, θ, δ) works for $\beta \leq \frac{5}{2}$. Hence we can consider β_c slightly larger than $\frac{5}{2}$. A larger value of β_c may be considered (see [7, Remark 2.2]). □

5 The blow-up time

We analyse in more detail the blow-up time introduced in Definition 2.2. We give some general results that hold beyond the dyadic model. Such results are the key to prove in the next section that blow-up happens with probability one. Example 5.6 shows that the a. s. emergence of blow-up is a property dependent in general on the structure of the drift. Hence it strongly motivates our analysis.

Let $(X(\cdot; x), \tau_x)_{x\in\mathcal{W}}$ be the local strong solution of a stochastic equation on a suitable separable Hilbert space $\mathcal{W}$. Having our case in mind, we assume that

- $\mathbb{P}[\tau_x > 0] = 1$ for all $x \in \mathcal{W}$,
- $X(\cdot; x)$ is continuous for $t < \tau_x$ with values in $\mathcal{W}$,
- $X(\cdot; x)$ is the maximal local solution, namely either $\tau_x = \infty$ or $\|X(t; x)\|_{\mathcal{W}} \to \infty$ as $t \uparrow \infty$, $\mathbb{P}-$a. s.,
- $(X(\cdot; x), \tau_x)_{x\in\mathcal{W}}$ is Markov (in the sense given in Theorem 2.3),
- all martingale solutions coincide with the strong solution up to τ_x.

The last statement plainly implies that the occurrence of blow-up is an intrinsic property of the unique local strong solution. Define

$$\flat(t, x) = \mathbb{P}[\tau_x > t], \quad \text{and} \quad \flat(x) = \inf_{t\geq 0} \flat(t, x) = \mathbb{P}[\tau_x = \infty],$$

for $x \in \mathcal{W}$ and $t \geq 0$. Clearly $\flat(0, x) = 1$ and $\flat(\cdot, x)$ is non-increasing. Next lemma shows a 0–1 law for the supremum of $\flat$ over space and time.

Lemma 5.1 *Consider the family of processes (X, τ) on $\mathcal{W}$ as above. If there is $x_0 \in \mathcal{W}$ such that $\mathbb{P}[\tau_{x_0} = \infty] > 0$, then*

$$\sup_{x\in\mathcal{W}} \mathbb{P}[\tau_x = \infty] = 1.$$

Proof By the Markov property,

$$\flat(t + s, x) = \mathbb{P}[\tau_x > t + s] = \mathbb{E}[\mathbb{1}_{\{\tau_x > t\}} \flat(s, X(t; x))],$$

and in the limit as $s \uparrow \infty$, by monotone convergence,

$$\flat(x) = \mathbb{E}[\mathbb{1}_{\{\tau_x > t\}} \flat(X(t; x))]. \tag{5.1}$$

Set $c = \sup \flat(x)$, then by the above formula,

$$\flat(x_0) = \mathbb{E}[\mathbb{1}_{\{\tau_{x_0} > t\}} \flat(X(t; x_0))] \leq c\mathbb{E}[\mathbb{1}_{\{\tau_{x_0} > t\}}] = c\flat(t, x_0).$$

As $t \uparrow \infty$, we get $\flat(x_0) \leq c\flat(x_0)$, that is $c \geq 1$, hence $c = 1$. □

Remark 5.2 Something more can be said by knowing additionally that there is x_0 with $\flat(x_0) = 1$. Indeed, $\mathbb{1}_{\{\tau_{x_0} > t\}} = 1$ a. s., and, using again formula (5.1),

$$\mathbb{E}[\flat(X(t; x_0))] = \mathbb{E}[\mathbb{1}_{\{\tau_{x_0} > t\}} \flat(X(t; x_0))] = \flat(x_0) = 1.$$

Hence $\flat(X(t; x_0)) = 1$, a. s. for every $t > 0$. This is very close to proving that $\flat \equiv 1$. In fact [28, Theorem 6.8] proves, although with a completely different approach, that $\flat(x_0) = 1$ implies that $\flat \equiv 1$ on $\mathcal{W}$. This holds under the assumptions of strong Feller regularity and conditional irreducibility, namely that $\mathbb{P}[X(t; x) \in A, \tau_x > t] > 0$ for every $x \in \mathcal{W}$, $t > 0$ and every open set $A \subset \mathcal{W}$.

Proposition 5.3 *Consider the family (X, τ) of processes as above. Assume that, given $x \in \mathcal{W}$, there exist a closed set $B_\infty \subset \mathcal{W}$ with non-empty interior and three numbers $p_0 \in (0, 1)$, $T_0 > 0$ and $T_1 > 0$ such that*

- $\mathbb{P}[\sigma_{B_\infty}^{x,T_1} = \infty, \tau_x = \infty] = 0,$
- $\mathbb{P}[\tau_y \leq T_0] \geq p_0$ *for every* $y \in B_\infty$,

where the (discrete) hitting time $\sigma_{B_\infty}^{x,T_1}$ of B_∞, starting from x, is defined as

$$\sigma_{B_\infty}^{x,T_1} = \min\{k \geq 0 : X(kT_1; x) \in B_\infty\},$$

and $\sigma_{B_\infty}^{x,T_1} = \infty$ if the set is empty. Then

$$\mathbb{P}[\tau_x < \infty] \geq \frac{p_0}{1 + p_0}.$$

Remark 5.4 The first condition in the above proposition can be interpreted as recurrence in a conditional sense: knowing that the solution does not explode, it will visit B_∞ in a finite time with probability 1.

Proof The first assumption says that $\mathbb{P}[\sigma_{B_\infty}^{x,T_1} > n, \tau_x > nT_1] \downarrow 0$ as $n \to \infty$. If $\mathbb{P}[\tau_x = \infty] = 0$, there is nothing to prove. If on the other hand $\mathbb{P}[\tau_x = \infty] > 0$, then $\mathbb{P}[\tau_x > nT_1] > 0$ for all $n \geq 1$ and, since $\mathbb{P}[\tau_x > nT_1] \downarrow \mathbb{P}[\tau_x = \infty]$ as $n \to \infty$,

$$\mathbb{P}[\sigma_{B_\infty}^{x,T_1} \leq n | \tau_x > nT_1] = 1 - \frac{\mathbb{P}[\sigma_{B_\infty}^{x,T_1} > n, \tau_x > nT_1]}{\mathbb{P}[\tau_x > nT_1]} \longrightarrow 1, \quad n \to \infty.$$

For $n \geq 1$,

$$\mathbb{P}[\tau_x > nT_1 + T_0] \leq \mathbb{P}[\tau_x > nT_1 + T_0, \ \sigma_{B_\infty}^{x,T_1} \leq n] + \mathbb{P}[\sigma_{B_\infty}^{x,T_1} > n].$$

The strong solution is Markov, hence

$$\mathbb{P}[\tau_x > nT_1 + T_0,\ \sigma_{B_\infty}^{x,T_1} \le n] = \sum_{k=0}^{n} \mathbb{P}[\tau_x > nT_1 + T_0,\ \sigma_{B_\infty}^{x,T_1} = k] \le (1 - p_0)\mathbb{P}[\sigma_{B_\infty}^{x,T_1} \le n].$$

In conclusion

$$\mathbb{P}[\tau_x > nT_1 + T_0] \le (1 - p_0)\mathbb{P}[\sigma_{B_\infty}^{x,T_1} \le n] + \mathbb{P}[\sigma_{B_\infty}^{x,T_1} > n] = 1 - p_0\mathbb{P}[\sigma_{B_\infty}^{x,T_1} \le n] \le 1 - p_0\mathbb{P}[\sigma_{B_\infty}^{x,T_1} \le n | \tau_x > nT_1]\mathbb{P}[\tau_x > nT_1],$$

and, as $n \to \infty$, $\mathbb{P}[\tau_x = \infty] \le 1 - p_0\mathbb{P}[\tau_x = \infty]$, that is $\mathbb{P}_x[\tau_x = \infty] \le \frac{1}{1+p_0}$. □

Corollary 5.5 *Assume that there are* $p_0 \in (0, 1)$, $T_0 > 0$ *and* $B_\infty \subset \mathcal{W}$ *such that the assumptions of the previous proposition hold for every* $x \in \mathcal{W}$ *(the time* T_1 *may depend on* x*). Then for every* $x \in \mathcal{W}$, $\mathbb{P}[\tau_x < \infty] = 1$.

Proof The previous proposition yields that $\sup_{x\in\mathcal{W}} \mathbb{P}[\tau_x = \infty] \le \frac{1}{1+p_0}$. By the dichotomy of Lemma 5.1, $\mathbb{P}[\tau_x < \infty] = 1$ for every $x \in \mathcal{W}$. □

Example 5.6 The following simple one dimensional example shows that the a. s. occurrence of blow-up depends on the structure of the drift. Our proofs below are elementary and mimic the proofs of the next section. Consider the SDEs,

$$dX = f_i(X)\,dt + dW, \quad i = 1, 2,$$

with initial condition $X(0) = x \in \mathbf{R}$, where

$$f_1(x) = \begin{cases} x^2, & x \ge 0, \\ x, & x < 0, \end{cases} \qquad f_2(x) = \begin{cases} x^2, & x \ge 0, \\ -x, & x < 0. \end{cases}$$

The Feller test [32, Proposition 5.22]) yields $0 < \flat_1(x) < 1$ for the blow-up function corresponding to the drift f_1, and $\flat_2(x) \equiv 1$ for the one of the drift f_2.

In view of the results proved above and the analysis of the next section (see Theorem 6.1), we notice that

- if $B_\infty = \{x \ge 1\}$, then for both drifts there are $p_0 > 0$ and $T_0 > 0$ such that $\mathbb{P}[\tau_x \le T_0] \ge p_0$ for all $x \in B_\infty$, that is the second assumption of Proposition 5.3 holds,
- the first assumption of Proposition 5.3 holds for f_2 but *not* for f_1,
- in both cases $\mathbb{E}\big[\sup_{[0,T]}(X_n)_-^p\big] < \infty$ for all $T > 0$ and $p \ge 1$.

Indeed, given an initial condition $x \in [1, \infty)$, we have that

$$\mathbb{P}\left[\left\{\sup_{t\in[0,2]} |W_t| \le \tfrac{1}{4}\right\} \cap \{\tau_x > 2\}\right] = 0.$$

Set $Y_t = X_t - W_t$, so that $Y_0 = x$ and $dY = dX - dW = (Y + W)^2$, in particular $Y_t \geq 1$. On the event $\{\sup_{t\in[0,2]} |W_t| \leq \frac{1}{4}\}$,

$$\dot{Y} \geq Y^2 - 2|W|Y \geq Y\left(Y - \frac{1}{2}\right) \geq \frac{1}{2}Y^2,$$

hence by comparison Y_t (and hence X_t) explodes before time $\frac{2}{x} \leq 2$.

6 Blow-up for $\beta > 3$

In the first part of the section we prove that there are sets in the state space which lead to blow-up with positive probability. The idea is to use Lemma 3.2 to adapt the estimates of [10], which work only for positive solutions.

In the second part of the section we show that such sets are recurrent, when the blow-up time is conditioned to be infinite. The general result of the previous section immediately implies that blow-up occurs with full probability.

6.1 Blow-up with positive probability

Given $\alpha > \beta - 2$, $p \in (0, \beta - 3)$, $a_0 > 0$ and $M_0 > 0$, define the set

$$B_\infty(\alpha, p, a_0, M_0) = \left\{x \in V_\alpha : \|x\|_p \geq M_0 \text{ and } \inf_{n\geq 1}\left(\lambda_{n-1}^{\beta-2} x_n\right) \geq -\nu a_0\right\}. \quad (6.1)$$

We will show that for suitable values of a_0, M_0, each solution of (1.1) with initial condition in the above set blows up in finite time with positive probability.

Theorem 6.1 *Let $\beta > 3$ and assume (2.1). Given $\alpha \in (\beta - 2, \alpha_0 + 1)$, $p \in (0, \beta - 3)$, and $a_0 \in (0, \frac{1}{4}]$, there exist $p_0 > 0$, $T_0 > 0$ and $M_0 > 0$ such that for each $x \in B_\infty(\alpha, p, a_0, M_0)$ and for every energy martingale weak solution $\mathbb{P}_x$ starting at x,*

$$\mathbb{P}_x[\tau_\infty^\alpha \leq T_0] \geq p_0.$$

Proof Choose $c_0 > 0$ with $c_0 \leq a_0 \leq \sqrt{a_0}(1 - \sqrt{a_0})$, and consider the random integer $N_{\alpha,c_0}(T_0)$ defined in (3.1). The value T_0 will be specified later. Set

$$p_0 = \mathbb{P}_x[N_{\alpha,c_0}(T_0) = 1].$$

We recall that $p_0 > 0$ by Lemma 3.1, and that its value depends only on the distribution of the solution of (2.3). The theorem will be proved if we show that

$$\mathbb{P}_x[\tau_\infty^\alpha > T_0,\ N_{\alpha,c_0}(T_0) = 1] = 0. \quad (6.2)$$

Indeed $\mathbb{P}_x[\tau_\infty^\alpha \leq T_0] = 1 - \mathbb{P}_x[\tau_\infty^\alpha \leq T_0,\ N_{\alpha,c_0}(T_0) > 1] \geq 1 - \mathbb{P}_x[N_{\alpha,c_0}(T_0) > 1] = p_0$. We proceed with the proof of (6.2) and we work pathwise on the event

$$\Omega(\alpha, T_0) = \{\tau_\infty^\alpha > T_0\} \cap \{N_{\alpha,c_0}(T_0) = 1\}.$$

Let Z be the solution of (2.3) and $Y = X - Z$. Equation (2.4) has a unique solution on $[0, T_0]$ on $\{\tau_\infty^\alpha > T_0\}$. On $\{N_{\alpha,c_0}(T_0) = 1\}$ we have $\lambda_{n-1}^{\beta-2}|Z_n(t)| \leq c_0$ for every $t \in [0, T_0]$ and every $n \geq 1$. Set

$$\eta_n = X_n - Z_n + a_0\nu\lambda_{n-1}^{2-\beta}.$$

By this position $\eta = (\eta_n)_{n\geq 1}$ satisfies the system

$$\begin{cases} \dot\eta_n = -\nu\lambda_n^2\eta_n + a_0\nu^2\lambda^2\lambda_{n-1}^{4-\beta} + \lambda_{n-1}^\beta X_{n-1}^2 - \lambda_n^\beta X_n X_{n+1}, \\ \eta_n(0) = X_n(0) + a_0\nu\lambda_{n-1}^{2-\beta}, \end{cases} \qquad n \geq 1.$$

Moreover, by Lemma 3.2 (with a_0, c_0 as fixed above), it follows that $\eta_n(t;\omega) \geq 0$ for all $t \in [0, T_0]$, $n \geq 1$ and $\omega \in \Omega(\alpha, T_0)$.

Fix a number $b > 0$, which will be specified later, then

$$\begin{aligned} \frac{d}{dt}\big(\eta_n^2 + b\eta_n\eta_{n+1}\big) &= -2\nu\lambda_n^2\eta_n^2 - b\nu(1+\lambda^2)\lambda_n^2\eta_n\eta_{n+1} \\ &\quad + a_0\lambda^2\nu^2(2 + b\lambda^{4-\beta})\lambda_{n-1}^{4-\beta}\eta_n + a_0 b\lambda^2\nu^2\lambda_{n-1}^{4-\beta}\eta_{n+1} \\ &\quad + 2\lambda_{n-1}^\beta X_{n-1}^2\eta_n + b\lambda_{n-1}^\beta X_{n-1}^2\eta_{n+1} + b\lambda_n^\beta X_n^2\eta_n \\ &\quad - 2\lambda_n^\beta X_n X_{n+1}\eta_n - b\lambda_n^\beta X_n X_{n+1}\eta_{n+1} - b\lambda_{n+1}^\beta\eta_n X_{n+1}X_{n+2}. \end{aligned}$$

Since $(a_0+c_0)^2 \leq a_0$ and $0 \leq (a_0\nu\lambda_{n-1}^{2-\beta} - Z_n) \leq \nu(a_0+c_0)\lambda_{n-1}^{2-\beta}$, Young's inequality and some straightforward computations yield

$$\frac{d}{dt}\big(\eta_n^2 + b\eta_n\eta_{n+1}\big) + 2\nu\lambda_n^2\eta_n^2 + b\nu(1+\lambda^2)\lambda_n^2\eta_n\eta_{n+1} \geq A_n + B_n + C_n,$$

where

$$\begin{aligned} A_n &= b\lambda_n^\beta\eta_n^3 + \tfrac{\lambda^{2p}}{1+\lambda^{2p}}\lambda_{n-1}^\beta\eta_{n-1}^2\eta_n - 2\lambda_n^\beta\eta_n^2\eta_{n+1} - b\lambda_n^\beta\eta_n\eta_{n+1}^2 - b\lambda_{n+1}^\beta\eta_n\eta_{n+1}\eta_{n+2}, \\ B_n &= -2b\lambda^{\beta-2}\nu(a_0+c_0)\lambda_n^2\eta_n^2, \quad \text{and} \quad C_n = -4\nu^2(a_0+c_0)^2\tfrac{\lambda^{2\beta+2p-4}}{\lambda^{2p}-1}\lambda_{n-1}^{4-\beta}\eta_n. \end{aligned}$$

The term A_n is roughly the same as in the deterministic case, hence by proceeding in the same way as in [10] we have

$$\sum_{n=1}^\infty \lambda_n^{2p}A_n \geq k_1\sum_{n=1}^\infty \lambda_n^{\beta+2p}\eta_n^3 + k_1'\sum_{n=1}^\infty \lambda_n^{\beta+2p}\eta_n^2\eta_{n+1} = k_1\sum_{n=1}^\infty \lambda_n^{\beta+2p}\eta_n^3,$$

where we have chosen b so that $k_1' = \lambda^{2p} - 1 - 4b(2 + 2\lambda^\beta + \lambda^{-2p}) = 0$. The other two terms are simpler, indeed

$$\sum_{n=1}^{\infty} \lambda_n^{2p} B_n = -2b\lambda^{\beta-2}\nu(a_0+c_0)\sum_{n=1}^{\infty}\lambda_n^{2+2p}\eta_n^2 = -k_2\|\eta\|_{1+p}^2,$$

and, by the Cauchy–Schwarz inequality and the fact that $p < \beta - 3$,

$$\sum_{n=1}^{\infty} \lambda_n^{2p} C_n \geq -4\nu^2(a_0+c_0)^2 \frac{\lambda^{3\beta+2p-8}}{\lambda^{2p}-1}\left(\lambda^{2(3-\beta+p)}-1\right)^{-\frac{1}{2}}\|\eta\|_{1+p} = -k_3\|\eta\|_{1+p}.$$

On the other hand,

$$2\nu\|\eta\|_{1+p}^2 + b\nu(1+\lambda^2)\sum_{n=1}^{\infty}\lambda_n^{2+2p}\eta_n\eta_{n+1} \leq k_4\|\eta\|_{1+p}^2,$$

$$\|\eta\|_{1+p}^2 = \sum_{n=1}^{\infty}\left(\lambda_n^{\frac{1}{3}(\beta+2p)}\eta_n\right)^2\lambda_n^{-\frac{2}{3}(\beta-3-p)} \leq \left(\frac{1}{k_5}\sum_{n=1}^{\infty}\lambda_n^{\beta+2p}\eta_n^3\right)^{\frac{2}{3}}.$$

If we set $H(t) = \sum_{n=1}^{\infty}\lambda_n^{2p}\left(\eta_n^2 + b\eta_n\eta_{n+1}\right)$ and $\psi(t) = \|\eta\|_{1+p}^2$, the estimates obtained so far together yield

$$\dot{H} + k_4\psi \geq k_1k_5\psi^{\frac{3}{2}} - k_2\psi - k_3\sqrt{\psi}.$$

Finally, $H \leq (1+b\lambda^{-p})\psi = k_6\psi$, and it is easy to show by a simple argument (for instance the one in [10]) that if

$$H(0) > M_0^2 := \frac{k_6}{k_1k_5}\left(k_4+k_2+\sqrt{(k_4+k_2)^2+2k_1k_3k_5}\right) \quad \text{and} \quad T_0 > \frac{4k_6^{\frac{3}{2}}}{k_1k_5\sqrt{H(0)}},$$

then H becomes infinite before time T_0. □

6.2 Ineluctable occurrence of the blow-up

So far we know that if the initial condition is *not too negative* and the noise is *not too strong*, then the deterministic dynamics dominates and the process diverges. In this section we show that the sets that lead to blow-up are recurrent in a conditional sense (as in Remark 5.4).

Theorem 6.2 *Let $\beta > 3$ and assume (2.1). Assume moreover that the set $\{n \geq 1 : \sigma_n \neq 0\}$ is non-empty. Given $\alpha \in (\beta-2, 1+\alpha_0)$, for every $x \in V_\alpha$ and every energy martingale solution $\mathbb{P}_x$ with initial condition x,*

$$\mathbb{P}_x[\tau_\infty^\alpha < \infty] = 1.$$

Our strategy to prove the theorem is based on Corollary 5.5. We will show that the sets (6.1) where blow-up occurs satisfy the assumptions of the corollary. Lemma 6.4 shows that the negative part of the solution becomes small. Lemma 6.5 shows that the size of the solution becomes large. Finally, Lemma 6.6 shows that, without blow-up, the sets (6.1) are visited with probability one.

Lemma 6.3 *Let $\beta > 3$ and assume* (2.1). *There exists $c_{6.3} > 0$ such that for $\alpha \in (\beta - 2, 1 + \alpha_0)$, for every $x \in V_\alpha$, every energy martingale solution $\mathbb{P}_x$ starting at x, every $T > 0$ and every $c_0 > 0$ with $4c_0(1 + \lambda^{\beta-3}) \le 1$,*

$$\sup_{[0,T]} \|X(t)\|_H \le \|x\|_H + c_{6.3}\nu,$$

$\mathbb{P}_x$−a. s. on the event $\{\tau_\infty^\alpha > T\} \cap \{N_{\beta-2,c_0}(T) = 1\}$.

Proof Problem (2.4) has a unique solution on $\{\tau_\infty^\alpha > T\}$, hence we work directly on Y. We know that $\lambda_n^{\beta-2}|Z_n(t)| \le c_0\nu$ for $t \in [0, T]$ and $n \ge 1$, hence

$$\begin{aligned}\frac{d}{dt}\|Y\|_H^2 + 2\nu\|Y\|_1^2 &\le 2\sum_{n=1}^\infty \lambda_n^\beta (Y_n Y_{n+1} Z_n + Y_{n+1} Z_n^2 - Y_n^2 Z_{n+1} - Y_n Z_n Z_{n+1})\\ &\le 4c_0\nu(1 + \lambda^{\beta-3})\|Y\|_1^2 + \tfrac{\lambda^{2\beta-4}}{2(\lambda^{\beta-3}-1)} c_0^3\nu^3.\end{aligned}$$

The assumption on c_0 and the inequality $\|Y\|_1 \ge \lambda\|Y\|_H$ yield

$$\frac{d}{dt}\|Y\|_H^2 + \nu\lambda^2\|Y\|_H^2 \le k_0 c_0^3\nu^3,$$

where the value of k_0 depends only on β. The bound for Y follows by integrating the differential inequality. The lemma then follows using that $X = Y + Z$ and that $N_{\beta-2,c_0}(T) = 1$. □

The next lemma is a slight improvement of Lemma 3.2. We prove that there is a drift towards the positive cone and solutions tend to be *not too negative* if the effect of noise is small, regardless of the sign of the initial condition.

Lemma 6.4 (Contraction of the negative components) *Let $\beta > 3$ and assume* (2.1). *For every $M > 0$, $a_0 \in (0, \frac{1}{4}]$ and $c_0 < a_0$, with $4c_0(1 + \lambda^{\beta-3}) \le 1$, there exists $T_M > 0$ such that for every $x \in V_\alpha$, with $\alpha \in (\beta - 2, 1 + \alpha_0)$ and $\|x\|_H \le M$, and every energy martingale solution $\mathbb{P}_x$,*

$$\inf_{n\ge1}\big(\lambda_{n-1}^{\beta-2} X_n(T_M)\big) \ge -(a_0 + c_0)\nu,$$

$\mathbb{P}_x$−a. s. on the event $\{\tau_\infty^\alpha > T_M\} \cap \{N_{\beta-2,c_0}(T_M) = 1\}$.

Proof Let n_0 be the first integer such that $\inf_{n\ge n_0} \lambda_n^{\beta-2} x_n \ge -a_0\nu$. If $n_0 = 1$ there is nothing to prove, so we consider the case $n_0 > 1$. Lemma 3.2 implies that $\lambda_n^{\beta-2} Y_n(t) \ge -a_0\nu$ holds for every $t \in [0, T_M]$ and every $n \ge n_0$.

The idea to prove the lemma is to show that $(Y_{n_0-1})_-$ becomes closer to 0 within a time T_{n_0-1}. At time T_{n_0-1} we can apply again Lemma 3.2. The same contraction idea yields that the negative part of the component $n_0 - 2$ becomes small as well within a time T_{n_0-2}, and so on. The sequence of times depends only on the size of the initial state in H and turns out to be summable. Therefore it suffices to prove the following statement: *given* $n > 1$, *if we know that for* $t_0 > 0$,

$$\sup_{k\geq 1}\sup_{[t_0,T]} \lambda_{k-1}^{\beta-2}|Z_k| \leq c_0\nu \quad\text{and}\quad \sup_{[t_0,T]} \lambda_n^{\beta-2}(Y_{n+1})_- \leq a_0\nu, \tag{6.3}$$

then at time $t_0 + T_n$ *we have that* $Y_n(t_0 + T_n) \geq -a_0\nu\lambda_{n-1}^{2-\beta}$. Here we have set

$$T_n(\|x\|_H, c_0, a_0) = \tfrac{2}{\nu}(\beta-2)\log\big(\lambda(n-1)\lambda_n^{-2}\big) + \tfrac{2}{\nu}\lambda_n^{-2}\log\big(1 \vee \tfrac{\|x\|_H + c_{6.3}\nu}{(a_0-c_0)\nu}\big).$$

We first notice that $\sum_n T_n < \infty$, hence we can choose T_M as the sum $T_M = \sum_n T_n(M, c_0, a_0)$. We turn to the proof of the above claim. Set

$$\eta_n = Y_n + c_0\nu\lambda_{n-1}^{2-\beta}.$$

Then $X_n = \eta_n - (c_0\nu\lambda_{n-1}^{2-\beta} - Z_n)$ and

$$\begin{aligned}\dot\eta_n &= -\nu\lambda_n^2\eta_n + c_0\nu^2\lambda^2\lambda_{n-1}^{4-\beta} + \lambda_{n-1}^{\beta}X_{n-1}^2 - \lambda_n^{\beta}X_nX_{n+1}\\ &\geq -(\nu\lambda_n^2 + \lambda_n^{\beta}X_{n+1})\eta_n + c_0\nu^2\lambda^2\lambda_{n-1}^{4-\beta} - \lambda_n^{\beta}(c_0\nu\lambda_{n-1}^{2-\beta} - Z_n)(X_{n+1})_-.\end{aligned}$$

By (6.3), $(X_{n+1})_- \leq (a_0 + c_0)\nu\lambda_n^{2-\beta}$ and $(c_0\nu\lambda_{n-1}^{2-\beta} - Z_n) \leq 2c_0\nu\lambda_{n-1}^{2-\beta}$, hence $\dot\eta_n \geq -(\nu\lambda_n^2 + \lambda_n^{\beta}X_{n+1})\eta_n$. Since $a_0 + c_0 \leq \frac{1}{2}$, it follows that

$$\nu\lambda_n^2 + \lambda_n^{\beta}X_{n+1} \geq \nu\lambda_n^2 - \nu(a_0+c_0)\lambda_n^2 = \nu\lambda_n^2\big(1-(a_0+c_0)\big) \geq \frac{1}{2}\nu\lambda_n^2.$$

Therefore for $t \geq t_0$,

$$\eta_n(t) \geq \eta_n(t_0)\exp\left(-\int_{t_0}^{t}(\nu\lambda_n^2 + \lambda_n^{\beta}X_{n+1})\,ds\right) \geq -(\eta_n(t_0))_-\,\mathrm{e}^{-\frac{1}{2}\nu\lambda_n^2(t-t_0)}.$$

Finally, by Lemma 6.3, $(\eta_n(t_0))_- \leq (Y_n(t_0))_- \leq \|Y(t_0)\|_H \leq \|x\|_H + c_{6.3}\nu$. It is elementary now to check that at time $t_0 + T_n$,

$$Y_n(t_0 + T_n) = \eta_n(t_0 + T_n) - c_0\nu\lambda_{n-1}^{2-\beta} \geq -a_0\nu\lambda_n^{2-\beta}.$$

□

The last ingredient to show that the hitting time of sets (6.1) is finite is the fact that the solution can be large enough, while being *not too negative*. At this stage the noise is crucial, although one randomly perturbed component is enough for our purposes. The underlying ideas of the following lemma come from control theory. We do not need sophisticated results [39,45] though, because a quick and strong impulse turns out to be sufficient.

Lemma 6.5 (Expansion in H) *Under the assumptions of Theorem 6.2, let m be equal to $\min\{n \geq 1 : \sigma_n \neq 0\}$. Let $M_1 > 0$, $M_2 > 0$, and $a_0, a_0', c_0 > 0$ be such that $c_0 < a_0 < a_0' < \frac{1}{4}$ and $c_0 + a_0 < a_0'$. For every $X(0) \in V_\alpha$, with $\|X(0)\|_H \leq M_1$ and $\inf_{n\geq 1} \lambda_{n-1}^{\beta-2} X_n(0) \geq -a_0\nu$, there exists $T = T(M_1, M_2, c_0, a_0, a_0', m) > 0$ such that*

- $\lambda_{n-1}^{\beta-2} X_n(t) \geq -(a_0' + c_0)\nu$ *for every $n \geq 1$ and $t \in [0, T]$,*
- $\|X(T)\|_H \geq M_2$,

on the event

$$\{\tau_\infty^\alpha > T_M\} \cap \left\{ \sup_{[0,T]} \lambda_{n-1}^{\beta-2} |Z_n(t)| \leq c_0\nu \text{ for } n \neq m \right\} \cap \left\{ \sup_{[0,T]} \lambda_{m-1}^{\beta-2} |Z_m(t) - \psi(t)| \leq c_0\nu \right\}.$$

Here $\psi : [0, T] \to \mathbf{R}$ is a non-decreasing continuous function such that $\psi(0) = 0$ and $\psi(T)$ large enough depending on the above given data (its value is given in the proof).

Proof We work on the event given in the statement of the theorem.

Step 1: estimate in H. Set $\bar{\psi} = \sup_{[0,T]} \|Z_m\|_H \leq \psi(T) + c_0\nu$, then as in Lemma 6.3,

$$\begin{aligned} \frac{d}{dt}\|Y\|_H^2 + 2\nu\|Y\|_1^2 &\leq 2\sum_{n=1}^{\infty} \lambda_n^\beta \big(|Z_n Y_n Y_{n+1}| + Z_n^2 |Y_{n+1}| + |Z_{n+1}| Y_n^2 + |Z_n Z_{n+1} Y_n|\big) \\ &\quad + 2\lambda_{m-1}^\beta \big(|Z_m| Y_{m-1}^2 + |Z_{m-1} Z_m Y_{m-1}|\big) \\ &\quad + 2\lambda_m^\beta \big(||Z_m Y_m Y_{m+1} + Z_m^2 |Y_{m+1}| + |Z_m Z_{m+1} Y_m|\big) \\ &\leq \nu\|Y\|_1^2 + k_0\nu^3 + 16\lambda_m^\beta (1+\nu)(1+\bar{\psi}^2)(1+\|Y\|_H^2). \end{aligned}$$

If $k_1 = k_0\nu^3$, $k_2 = 16\lambda_m^\beta(1+\nu)$ and $M_3(T, \psi(T))^2 = (M_1^2 + k_1/k_2)\exp\big(k_2 T(1 + \bar{\psi}^2)\big)$, it follows from Gronwall's lemma that $\sup_{[0,T]} \|Y(t)\|_H^2 \leq M_3(T, \psi(T))^2$. Since on the given event we have that $\|Z(t)\|_H \leq \lambda^{2-\beta}(\lambda^{2-\beta} - 1)^{-1} + \bar{\psi}$ for every $t \in [0, T]$, we finally have that

$$\sup_{[0,T]} \|X(t)\|_H \leq c_0\nu + \tfrac{\lambda^{2-\beta}}{\lambda^{2-\beta}-1} + \psi(T) + M_3(T, \psi(T)) =: M_4(T, \psi(T)).$$

Step 2: large size at time T. Using the previous estimate we have

$$\begin{aligned} X_m(t) &= \mathrm{e}^{-\nu\lambda_m^2 t} X_m(0) + Z_m(t) + \int_0^t \mathrm{e}^{-\nu\lambda_m^2(t-s)}\big(\lambda_{m-1}^\beta X_{m-1}^2 - \lambda_m^\beta X_m X_{m+1}\big)\, ds \\ &\geq -a_0\nu\lambda_{m-1}^{2-\beta} + \big(\psi(t) - c_0\nu\lambda_{m-1}^{2-\beta}\big) - \lambda_m^\beta t \sup_{[0,T]} \|X\|_H^2. \end{aligned}$$

For $t = T$ we have $X_m(T) \geq \psi(T) - \nu - \lambda_m^\beta M_4(T, \psi(T))T$. If we choose $\psi(T) = M_2 + 2\nu$, then $M_4(T, M_2 + 2\nu)T \to 0$ as $T \downarrow 0$. Therefore we can choose T small enough so that $\lambda_m^\beta M_4(T, \psi(T))T \leq \nu$, hence $X_m(T) \geq M_2$ and $\|X(T)\|_H \geq M_2$.

Step 3: Bound from below for $n = m$. The choice of $\psi(T)$ and the computations in the above step yield

$$\lambda_{m-1}^{\beta-2} X_m(t) \geq -(a_0 + c_0)\nu - \lambda^{2-\beta} M_4(T, M_2 + 2\nu)\lambda_m^{2\beta-2} T,$$

since ψ is non-negative. By assumption we have that $a_0 + c_0 < a_0'$, hence, possibly fixing a smaller value of T than the one chosen in the previous step, we can ensure that $X_m \geq -a_0'\nu\lambda_{m-1}^{2-\beta}$ on $[0, T]$.

Step 4: Bound from below for $n \neq m$. If $n > m$, the proof proceeds as in Lemma 3.2, since X_m appears in the system of equations for $(Y_n)_{n>m}$ only through the *positive* term $\lambda_m^\beta X_m^2$ in the equation for the $(m+1)$th component.

If $n < m$, the proof follows by finite induction. For $n = m$ the lower bound is true by the previous step. Let now $n \geq m$ and assume that $\lambda_n^{2-\beta} Y_{n+1} \geq -a_0'\nu$ on $[0, T]$. We prove that $\lambda_{n-1}^{2-\beta} Y_n \geq -a_0'\nu$ as in Lemma 6.4. Set $\eta_n = Y_n + a_0'\nu\lambda_{n-1}^{2-\beta}$. Since $\lambda_{n-1}^{\beta-2}|Z_n| \leq c_0\nu$ and $(X_{n+1})_- \leq (a_0' + c_0)\nu\lambda_n^{2-\beta}$ on $[0, T]$,

$$\begin{aligned} \dot\eta_n &\geq -(\nu\lambda_n^2 + \lambda_n^\beta X_{n+1})\eta_n + a_0'\nu^2\lambda^{\beta-2}\lambda_n^{4-\beta} - \lambda_n^\beta(a_0'\nu\lambda_{n-1}^{2-\beta} - Z_n)(X_{n+1})_- \\ &\geq -(\nu\lambda_n^2 + \lambda_n^\beta X_{n+1})\eta_n + \nu\lambda^{\beta-2}\big(a_0' - (a_0' + c_0)^2\big)\lambda_n^{4-\beta} \\ &\geq -(\nu\lambda_n^2 + \lambda_n^\beta X_{n+1})\eta_n, \end{aligned}$$

The fact that $\eta_n(0) \geq 0$ implies that $\eta_n(t) \geq 0$. □

We systematize the random perturbation that, by Lemma 6.4 and Lemma 6.5, moves the solution from a ball in H to sets (6.1). Let $c_0 > 0$, $t_0 > 0$, $T_c > 0$, $T_e > 0$ and $\psi : [0, T_e] \to \mathbf{R}$ be a non-negative non-increasing function, and define

$$\mathcal{N}(t_0; c_0, T_c, T_e, \psi) = \mathcal{N}_c(c_0, t_0, T_c) \cap \mathcal{N}_e(c_0, t_0 + T_c, T_e, \psi),$$

where

$$\mathcal{N}_c(c_0, t_1, t_2) = \big\{\lambda_{n-1}^{\beta-2}|Z_n^c(t)| \leq c_0\nu \text{ for all } n \geq 1 \text{ and } t \in [t_1, t_1 + t_2]\big\},$$

$$\mathcal{N}_e(c_0, t_1, t_2, \psi) = \left\{ \sup_{[t_1, t_1+t_2]} \lambda_{m-1}^{\beta-2} |Z_m^e(t) - \psi_{t_1}(t)| \le c_0 \nu, \ t \in [t_1, t_1 + t_2] \right\}$$
$$\cap \big\{ \lambda_{n-1}^{\beta-2} |Z_n^e(t)| \le c_0 \nu \text{ for all } n \ne m \text{ and } t \in [t_1, t_1 + t_2] \big\}.$$

Here $\psi_s : [s, s + T_e] \to \mathbf{R}$ is defined for $s \ge 0$ as $\psi_s(t) = \psi(t - s)$, for $t \in [s, s + T_e]$, m is the smallest integer of the set $\{n : \sigma_n \ne 0\}$, and for every $n \ge 1$,

$$Z_n^c(t) = \sigma_n \int_{t_0}^{t} e^{-\nu \lambda_n^2 (t - t_0)} \, dW_n, \quad t \in [t_0, t_0 + T_c],$$

$$Z_n^e(t) = \sigma_n \int_{t_0+T_c}^{t} e^{-\nu \lambda_n^2 (t - t_0 - T_c)} \, dW_n, \quad t \in [t_0 + T_c, t_0 + T_c + T_e].$$

Under a martingale solution $\mathbb{P}_x$ starting at x, the two events $\mathcal{N}_c(c_0, t_0, T_c)$ and $\mathcal{N}_e(c_0, t_0 + T_c, T_e, \psi)$ are independent, have positive probability (by Lemma 3.1), and the values of their probability is independent of t_0. Moreover, if t_0, T_c, T_e and t_0' are given such that $t_0 + T_c + T_e \le t_0'$, then the events $\mathcal{N}(t_0; c_0, T_c, T_e, \psi)$ and $\mathcal{N}(t_0'; c_0, T_c, T_e, \psi)$ are independent.

Lemma 6.6 *Assume* (2.1) *and let* $\beta > 3$ *and* $\alpha \in (\beta - 2, \alpha_0 + 1)$. *There exists* $c_{6.6} > 0$ *such that if* $M > 0$, $T_c > 0$, $T_e > 0$, $c_0 > 0$, *and* $\psi : [0, T_e] \to \mathbf{R}$ *is a non-negative non-decreasing function, with*

$$\frac{c_{6.6}}{M^2} + e^{-\nu \lambda^2 T} < 1, \quad (T = T_c + T_e),$$

then for every $x \in V_\alpha$ *and every energy martingale solution* $\mathbb{P}_x$ *starting at* x,

$$\mathbb{P}_x \left[\{\tau_\infty^\alpha = \infty\} \cap \bigcap_{k \ge 1} \left(\{\|X(kT)\|_H \le M\} \cap \mathcal{N}(kT; c_0, T_c, T_e, \psi) \right)^c \right] = 0.$$

Proof We first obtain a quantitative estimate on the return time in balls of H of the Markov process $X^R(\cdot; x)$, solution of problem (2.2), starting at $x \in V_\alpha$. The same estimate will hold for the strong solution and the lemma will follow.

Step 1. Standard computations with Itô's formula and Gronwall's lemma yield

$$\mathbb{E}[\|X^R(t; x)\|_H^2] \le \|x\|_H^2 \, e^{-2\nu \lambda^2 t} + c_{6.6}, \tag{6.4}$$

where $c_{6.6} = (2\nu\lambda^2)^{-1} \sum_{n=1}^{\infty} \sigma_n^2$. The series converges due to (2.1) and $\alpha_0 > \beta - 3$.

Step 2. We use the previous estimate to show that

$$\mathbb{P}\big[\|X^R(kT; x)\|_H \ge M \text{ for } k = 1, \dots, n \big] \le \big(e^{-\nu \lambda^2 T} + \tfrac{c_{6.6}}{M^2} \big)^{n-1}. \tag{6.5}$$

We proceed as in [18, Lemma III.2.4]. Define, for k integer, $C_k = \{\|X^R(kT;x)\|_H \geq M\}$ and $B_k = \bigcap_{j=0}^k C_j$. Set $\alpha_k = \mathbb{E}[\mathbb{1}_{B_k}\|X^R(kT;x)\|_H^2]$ and $p_k = \mathbb{P}[B_k]$. By the Markov property, Chebychev's inequality and (6.4),

$$\mathbb{P}[C_{k+1}|\mathscr{F}_{kT}] \leq \tfrac{1}{M^2}\,\mathrm{e}^{-\nu\lambda^2 T}\,\|X^R(kT;x)\|_H^2 + \tfrac{c_{6.6}}{M^2},$$

hence

$$p_{k+1} = \mathbb{E}\big[\mathbb{1}_{B_k}\mathbb{P}[C_{k+1}|\mathscr{F}_{kT}]\big] \leq \tfrac{1}{M^2}\,\mathrm{e}^{-\nu\lambda^2 T}\,\alpha_k + \tfrac{c_{6.6}}{M^2}p_k.$$

On the other hand, by integrating (6.4) on B_k, we get

$$\alpha_{k+1} \leq \mathbb{E}[\mathbb{1}_{B_k}\|X^R((k+1)T;x)\|_H^2] \leq \mathrm{e}^{-\nu\lambda^2 T}\,\alpha_k + c_{6.6}p_k.$$

Let $(\bar{\alpha}_k)_{k\in\mathbf{N}}$ and $(\bar{p}_k)_{k\in\mathbf{N}}$ be the solutions to the recurrence system

$$\begin{cases}\bar{\alpha}_{k+1} = \mathrm{e}^{-\nu\lambda^2 T}\,\bar{\alpha}_k + c_{6.6}\bar{p}_k,\\ \bar{p}_{k+1} = \tfrac{1}{M^2}\,\mathrm{e}^{-\nu\lambda^2 T}\,\bar{\alpha}_k + \tfrac{c_{6.6}}{M^2}\bar{p}_k,\end{cases}\qquad k \geq 1,$$

with $\bar{p}_1 = p_1$ and $\bar{\alpha}_1 = \alpha_1$. Then $\bar{\alpha}_k = M^2\bar{p}_k$ for $k \geq 2$ and $\alpha_k \leq \bar{\alpha}_k$, $p_k \leq \bar{p}_k$ for all $k \geq 1$. The inequality (6.5) easily follows.

Step 3. We recall that $\tau_x^\alpha = \sup_{R>0}\tau_x^{\alpha,R}$, hence by (6.5),

$$\begin{aligned}&\mathbb{P}[\|X(kT;x)\|_H \geq M, k \leq n, \tau_x^\alpha = \infty]\\ &\leq \lim_{R\uparrow\infty}\mathbb{P}[\|X(kT;x)\|_H \geq M, k \leq n, \tau_x^{\alpha,R} > nT]\\ &= \lim_{R\uparrow\infty}\mathbb{P}[\|X^R(kT;x)\|_H \geq M, k \leq n, \tau_x^{\alpha,R} > nT]\\ &\leq \left(\mathrm{e}^{-\nu\lambda^2 T} + \frac{c_{6.6}}{M^2}\right)^{n-1}.\end{aligned}$$

Define the hitting time $K_1 = \min\{k \geq 0 : \|X(kT;x)\|_H \leq M\}$ of the ball $B_M(0)$ in H ($K_1 = \infty$ if the set is empty). Clearly $K_1 < \infty$ on $\{\tau_x^\alpha = \infty\}$. Likewise, define the return times $K_j = \min\{k > K_{j-1} : \|X(kT;x)\|_H \leq M\}$, $j \geq 2$ ($K_j = \infty$ if the set is empty). By the previous step, $K_j < \infty$ on $\{\tau_x^\alpha = \infty\}$ for each $j \geq 1$.

Step 4. Consider for $k \geq 1$ the events $\mathcal{N}_k = \mathcal{N}(kT; c_0, T_c, T_e, \psi)$. We know that $\mathbb{P}[\mathcal{N}_k]$ is constant in k, so we set $p = \mathbb{P}[\mathcal{N}_k]$. Moreover, by the choice of T, it turns out that $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_k, \ldots$ are independent. Set $\mathcal{N}_\infty = \emptyset$ and define the time

$$L_0 = \min\{j \geq 1 : \mathbb{1}_{\mathcal{N}_{K_j}} = 1\},$$

with $L_0 = \infty$ if the set is empty. Notice that if L_0 is finite, then $\|X(K_{L_0}T;x)\|_H \leq M$ and the random perturbation leads the system to a set (6.1) within time $K_{L_0}T + T_c + T_e$. Hence the lemma is proved if we show that

$$\mathbb{P}[L_0 = \infty, \tau_x^\alpha = \infty] = 0. \tag{6.6}$$

Step 5. Given an integer $\ell \geq 1$, we have that

$$\begin{aligned}\mathbb{P}[L_0 > \ell, \tau_x^\alpha = \infty] &= \mathbb{P}[\mathcal{N}_{K_1}^c \cap \cdots \cap \mathcal{N}_{K_\ell}^c \cap \{\tau_x^\alpha = \infty\}] \\ &= \sum_{k_1=1}^\infty \cdots \sum_{k_\ell = k_{\ell-1}+1}^\infty \mathbb{P}[S_\ell(k_1, \dots, k_l) \cap \{\tau_x^\alpha = \infty\}],\end{aligned}$$

where $S_\ell(k_1, \dots, k_\ell) = \mathcal{N}_{K_1}^c \cap \dots \mathcal{N}_{K_\ell}^c \cap \{K_1 = k_1, \dots, K_\ell = k_\ell\}$. Notice that $S_\ell(k_1, \dots, k_\ell) \in \mathscr{F}_{(k_\ell+1)T}$, hence by the Markov property,

$$\begin{aligned}&\mathbb{P}[S_\ell(k_1, \dots, k_l) \cap \{\tau_x^\alpha > (k_\ell + 1)T\}] \\ &\quad = \mathbb{E}\big[\mathbb{1}_{S_{\ell-1}(k_1,\dots,k_{\ell-1})} \mathbb{1}_{\{\tau_x^\alpha > (k_{\ell-1}+1)T\}} \mathbb{1}_{\{K_\ell = k_\ell\}} \mathbb{P}[\mathcal{N}_{k_\ell}^c \cap \{\tau_{X(k_\ell T;x)}^\alpha > T\} | \mathscr{F}_{k_\ell T}]\big] \\ &\quad \leq (1-p)\mathbb{P}[S_{\ell-1}(k_1, \dots, k_{\ell-1}) \cap \{\tau_x^\alpha > (k_{\ell-1}+1)T\} \cap \{K_\ell = k_\ell\}].\end{aligned}$$

By summing up over k_ℓ, we have

$$\begin{aligned}&\sum_{k_\ell = k_{\ell-1}+1}^\infty \mathbb{P}[S_\ell(k_1, \dots, k_\ell) \cap \{\tau_x^\alpha > (k_\ell + 1)T\}] \\ &\qquad\qquad \leq (1-p)\mathbb{P}[S_{\ell-1}(k_1, \dots, k_{\ell-1}) \cap \{\tau_x^\alpha > (k_{\ell-1}+1)T\}].\end{aligned}$$

By iteration, $\mathbb{P}[L_0 > \ell, \tau_x^\alpha = \infty] \leq (1-p)^\ell$ and (6.6) follows. □

Proof of Theorem 6.2 Fix $\alpha \in (\beta - 2, 1 + \alpha_0)$, $\bar{p} \in (0, \beta - 3)$ and $\bar{a}_0 \in (0, \frac{1}{4}]$. Let $\bar{p}_0 > 0$, and $\bar{M}_0 > 0$ be the values given by Theorem 6.1. In view of Corollary 5.5, it suffices to prove that the (sampled) arrival time to $B_\infty(\alpha, \bar{p}, \bar{a}_0, \bar{M}_0)$ is finite on $\{\tau_x^\alpha = \infty\}$, for all $x \in V_\alpha$. By virtue of Lemma 6.6, it is sufficient to prove that there are $M, T_c, T_e, c_0 > 0$ and ψ such that $\mathrm{e}^{-\nu\lambda^2 T_c} + \frac{c_{6.6}}{M^2} < 1$ and

$$\left.\begin{array}{r}\|X(t_0; x)\|_H \leq M \\ \mathcal{N}(t_0; c_0, T_c, T_e, \psi)\end{array}\right\} \Rightarrow X(t_0 + T_c + T_e; x) \in B_\infty(\alpha, \bar{p}, \bar{a}_0, \bar{M}_0). \tag{6.7}$$

Indeed, the left-hand side of the above implication happens almost surely on $\{\tau_x^\alpha < \infty\}$ for some integer k such that $t_0 = k(T_c + T_e)$. Hence the right-hand side happens with probability one as well and $\mathbb{P}[\sigma_{B_\infty}^{x, T_c+T_e} = \infty, \tau_x^\infty = \infty] = 0$.

We finally prove (6.7). We first notice that in Lemma 6.4, the larger we choose M, the larger is the time T_c. Hence we apply Lemma 6.4 with $a_0 = \bar{a}_0/8$, $c_0 < \min\{\bar{a}_0/8, (4(1+\lambda^{\beta-3}))^{-1}\}$ and $M > 0$ large enough so that the time T_c satisfies $\mathrm{e}^{-\nu\lambda^2 T_c} + \frac{c_{6.6}}{M^2} < 1$. Moreover we know that

- $\inf_{n\geq 1}\big(\lambda_{n-1}^{\beta-2} X_n(t_0+T_c)\big) \geq -(a_0+c_0)\nu \geq -\frac{1}{4}\bar{a}_0\nu$ on $\{\tau_x^\alpha = \infty\} \cap \mathcal{N}_c(c_0, t_0, T_c)$,
- $\|X(t_0 + T_c)\|_H \leq \|X(t_0)\|_H + c_{6.3}\nu \leq M + c_{6.3}\nu$.

The second statement follows from Lemma 6.3. By Lemma 6.5 with $M_1 = M + c_{6.3}\nu$, $M_2 = \bar{M}_0$, $a_0 = \bar{a}_0/4$, $a_0' = 2a_0$ and c_0 as above, there is $T_e > 0$ such that

- $\inf_{n\geq 1}\big(\lambda_{n-1}^{\beta-2} X_n(t_0 + T_c + T_e)\big) \geq -\bar{a}_0\nu$ on $\{\tau_x^\alpha = \infty\} \cap \mathcal{N}_e(c_0, t_0 + T_c, T_e, \psi)$,

- $\|X(t_0 + T_c + T_e)\|_{\bar{p}} \geq \lambda^{\bar{p}} \|X(t_0 + T_c + T_e)\|_H \geq \bar{M}_0$,

that is $X(t_0 + T_c + T_e) \in B_\infty(\alpha, \bar{p}, \bar{a}_0, \bar{M}_0)$. □

Acknowledgments The author wish to thank the kind hospitality of the *Institut Élie Cartan*, Nancy, where part of this work has been done. Warmest thanks to David Barbato, Franco Flandoli and Jonathan Mattingly for useful conversations and insights on the subject. This work is dedicated to Martina, she has arrived whilst the paper was being completed.

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