Probab. Theory Relat. Fields (2014) 158:785–808
DOI 10.1007/s00440-013-0496-x

Filtrations at the threshold of standardness

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Received: 13 July 2012 / Revised: 19 February 2013 / Published online: 10 March 2013

Abstract A. Vershik discovered that filtrations indexed by the non-positive integers may have a paradoxical asymptotic behaviour near the time $-\infty$, called non-standardness. For example, two dyadic filtrations with trivial tail σ-field are not necessarily isomorphic. Yet, from any essentially separable filtration indexed by the non-positive integers, one can extract a subsequence which is a standard filtration. In this paper, we focus on the non-standard filtrations which become standard if (and only if) infinitely many integers are skipped. We call them filtrations at the threshold of standardness, since they are as close to standardardness as they can be although they are non-standard. Two classes of filtrations are studied, first the filtrations of the split-words processes, second some filtrations inspired by an unpublished example of B. Tsirelson. They provide examples which disprove some naive intuitions. For example, it is possible to have a standard filtration extracted from a non-standard one with no intermediate (for extraction) filtration at the threshold of standardness. It is also possible to have a filtration which provides a standard filtration on the even times but a non-standard filtration on the odd times.

Keywords Filtrations · Standardness · Split-words processes

Mathematical Subject Classification (2010) 60G05 · 60J10

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1 Introduction

The notion of standardness has been introduced by Vershik [10] in the context of decreasing sequences of measurable partitions indexed by the non-negative integers. Vershik's definition and characterizations of standardness have been translated from their original ergodic theoretic formulation into a probabilistic language by Émery and Schachermayer [2]. In this framework, the objects of focus are the filtrations indexed by non-positive integers. These are the non-decreasing sequences $(\mathcal{F}_n)_{n\le 0}$ of sub-σ-fields of a probability space $(\Omega, \mathcal{A}, \mathbb{P})$.

All the sub-σ-fields of $\mathcal{A}$ that we will consider are assumed to be complete and essentially separable with respect to $\mathbb{P}$. By definition, a sub-σ-field of $(\Omega, \mathcal{A}, \mathbb{P})$ is separable if it can be generated as a complete σ-field by a sequence of events, or equivalently, by some real random variable. One can check that a sub-σ-field $\mathcal{B} \subset \mathcal{A}$ is separable if and only if the Hilbert space $L^2(\Omega, \mathcal{B}, \mathbb{P})$ is separable.

Almost all filtrations that we will consider in this study have the following property: for each n, $\mathcal{F}_n$ is generated by $\mathcal{F}_{n-1}$ and by some random variable U_n which is independent of $\mathcal{F}_{n-1}$ and uniformly distributed on some finite set with r_n elements, for some sequence $(r_n)_{n\le 0}$ of positive integers. Such filtrations are called $(r_n)_{n\le 0}$-adic.

For such filtrations, as shown by Vershik [10], standardness turns out to be tantamount to a simpler, much more intuitive property: an (r_n)-adic filtration $\mathcal{F}$ is standard if and only if $\mathcal{F}$ is of product type, that is, $\mathcal{F}$ is the natural filtration of some process $V = (V_n)_{n\le 0}$ where the V_n are independent random variables; in this case, it is easy to see that the process V can be chosen with the same law as $U = (U_n)_{n\le 0}$. So, at first reading, 'standard' can be replaced with 'of product type' in this introduction.

Although intuitive, the notion of product-type filtrations is not as simple as one could believe. For example, the assumption that the tail σ-field $\mathcal{F}_{-\infty} = \bigcap_{n\le 0} \mathcal{F}_n$ is trivial, and the property $\mathcal{F}_n = \mathcal{F}_{n-1} \vee \sigma(U_n)$ for every $n \le 0$ do not ensure that $(\mathcal{F}_n)_{n\le 0}$ is generated by $(U_n)_{n\le 0}$. In the standard case, $(\mathcal{F}_n)_{n\le 0}$ can be generated by some other sequence $(V_n)_{n\le 0}$ of independent random variables which has the same law as $(U_n)_{n\le 0}$. In the non-standard case, no sequence of independent random variables can generate the filtration $(\mathcal{F}_n)_{n\le 0}$.

The first examples of such a situation were given by Vershik [10]. By modifying and generalizing one of these examples, Smorodinsky [8] and Émery and Schachermayer [2] introduced the split-words processes.

The law of a split-words process depends on an alphabet A, endowed with some probability measure, and a decreasing sequence $(\ell_n)_{n\le 0}$ of positive integers (the lengths of the words) such that $\ell_0 = 1$ and the ratios $r_n = \ell_{n-1}/\ell_n$ are integers. For the sake of simplicity, we consider here only finite alphabets endowed with the uniform measure.

A split-words process is an inhomogeneous Markov process $((X_n, U_n))_{n\le 0}$ such that for every $n \le 0$:

- (X_n, U_n) is uniform on $A^{\ell_n} \times [\![1, r_n]\!]$.
- U_n is independent of $\mathcal{F}^{(X,U)}_{n-1}$.
- if one splits the word X_{n-1} (of length $\ell_{n-1} = r_n \ell_n$) into r_n subwords of lengths ℓ_n, then X_n is the U_n-th subword of X_{n-1}.

Such a process is well-defined since the sequence of uniform laws on the sets $A^{\ell_n} \times [\![1, r_n]\!]$ is an entrance law for the transition probabilities given above. By construction, the natural filtration $\mathcal{F}^{X,U}$ of $((X_n, U_n))_{n\le 0}$ is $(r_n)_{n\le 0}$-adic. One can check that the tail σ-field $\mathcal{F}^{X,U}_{-\infty}$ is trivial. Thus, it is natural to ask whether $\mathcal{F}^{X,U}$ is standard or not.

Whether a split-words process with lengths $(\ell_n)_{n\le 0}$ generates a standard filtration or not is completely characterised: the filtration is non-standard if and only if

$$\sum_n \frac{\log r_n}{\ell_n} < +\infty \quad (\Delta).$$

Note that this condition does not depend on the alphabet A.

In this statement, the 'if' part and a partial converse have been proved by Vershik [10] (in a very similar framework) and by Laurent [5]. The 'only if' part has been proved by Heicklen [4] (in Vershik's framework) and by Ceillier [1]. The generalization to arbitrary alphabets has been performed by Laurent in [7]: the characterisations and all the results below still hold are when the alphabet is a Polish space endowed with some probability measure.

Although these examples are rather simple to construct, proving the non-standardness requires sharp tools like Vershik's standardness criterion [2,10]. One can also use the I-cosiness criterion of Émery and Schachermayer [2] which may be seen as more intuitive by probabilists. Actually, Laurent proved directly that both criteria are equivalent. Moreover, applying these criteria to the examples above leads to rather technical estimations.

Another question concerns what happens to a filtration when time is accelerated by extracting a subsequence. Clearly, every subsequence of a standard filtration is still standard. But Vershik's lacunary isomorphism theorem [10] states that from *any* filtration $(\mathcal{F}_n)_{n\le 0}$ such that $\mathcal{F}_0$ is essentially separable and $\mathcal{F}_{-\infty}$ is trivial, one can extract a filtration $(\mathcal{F}_{\phi(n)})_{n\le 0}$ which is standard. This striking fact is mind-boggling for anyone who is interested by the boundary between standardness and non-standardness. A natural question arises:

when $(\mathcal{F}_n)_{n\le 0}$ is not standard, how close to identity the increasing map ϕ (from $\mathbb{Z}_-$ to $\mathbb{Z}_-$) provided by the lacunary isomorphism theorem can be?

Of course, as standardness is an asymptotic property, the extracting map ϕ has to skip an infinity of times integers (equivalently, $\phi(n) - n \to -\infty$ as $n \to -\infty$).

In [10], Vershik provides an example of a non-standard dyadic filtration $(\mathcal{F}_n)_{n\le 0}$ such that $(\mathcal{F}_{2n})_{n\le 0}$ is standard. Gorbulsky also provides such an example in [3].

Using the fact that the family of split-words filtrations is stable by extracting subsequences, Ceillier exhibits in [1] an example of a non-standard filtration $(\mathcal{F}_n)_{n\le 0}$ which is as close to standardness as it can be: every subsequence $(\mathcal{F}_{\phi(n)})_{n\le 0}$ is standard as soon as ϕ skips an infinity of integers.

This paper is devoted to the filtrations sharing this property. We call them filtrations *at the threshold of standardness*.

1.1 Main results and organization of the paper

Some definitions and classical facts used in the paper are recalled in an Appendix, at the end of the paper. In the Sects. 2 and 3 which are the core of the paper, two classes of filtrations are studied, first the filtrations of the split-words processes, second some filtrations inspired by an unpublished example of Tsirelson.

The case of split-words filtrations The first part deals with split-words filtrations.

First, we characterise the filtrations at the threshold of standardness among the split-words filtrations.

Proposition 1 *A split-words filtration with lengths $(\ell_n)_{n\le 0}$ is at the threshold of standardness if and only if*

$$\sum_{n\le 0}\frac{\log r_n}{\ell_n} < +\infty \quad (\Delta)$$

and

$$\inf_{n\le 0}\frac{\log r_{n-1}}{\ell_n} > 0 \quad (\star).$$

Next, we give a sufficient condition which ensures that a standard split-words filtration cannot be extracted from any split-words filtration at the threshold of standardness.

Proposition 2 *If*

$$\sum_{n\le 0}\frac{\log r_n}{\ell_n} = +\infty \quad (\neg\Delta)$$

and

$$\lim_{n\to-\infty}\frac{\log r_n}{\ell_n} = 0 \quad (\square),$$

then any split-words filtration with lengths $(\ell_n)_{n\le 0}$ is standard but cannot be extracted from a split-words filtration at the threshold of standardness.

One could think that the threshold of standardness is a kind of boundary between standardness and non-standardness. Yet, the situation is not so simple. Indeed, Proposition 2 provides an example (Example 3) of two split-words filtrations, where

- the first one is non-standard,
- the second one is standard,
- the second one is extracted from the first one,
- yet, no intermediate filtration (for extraction) is at the threshold of standardness.

Furthermore, we provide an example of a non-standard split-words filtration from which no filtration at the threshold of standardness can be extracted (Example 9). The proof relies on Theorem A below.

Recall that, given any filtration $(\mathcal{F}_n)_{n\le 0}$ and an infinite subset B of $\mathbb{Z}_-$, the extracted filtration $(\mathcal{F}_n)_{n\in B}$ is standard if and only if the complement $B^c = \mathbb{Z}_- \setminus B$ is large enough in a certain way. Here, the meaning of "large enough" depends on the filtration $\mathcal{F}$ considered. When $\mathcal{F}$ is at the threshold of standardness, "large enough" means exactly "infinite". But various types of transition from non-standardness to standardness are possible, and the next theorem provides some other possible conditions.

Theorem A *Let $(\alpha_n)_{n\le 0}$ be any sequence of non-negative real numbers. There exists a split-words filtration $(\mathcal{F}_n)_{n\le 0}$ such that for every infinite subset B of $\mathbb{Z}_-$, the extracted filtration $(\mathcal{F}_n)_{n\in B}$ is standard if and only if*

$$\sum_{n\in B^c} \alpha_n = +\infty \quad or \quad \sum_{n\le 0} \mathbf{1}_{[n\notin B,\, n+1\notin B]} = +\infty.$$

In other words, the extracted filtration $(\mathcal{F}_n)_{n\in B}$ is standard if and only if B skips infinitely many pairs of consecutive integers or if B skips sufficiently many integers to make the series $\sum_{n\in B^c} \alpha_n$ diverge. The smaller the α_n are, the most time must be accelerated to get a standard filtration. In contrast, if the sequence $(\alpha_n)_{n\le 0}$ is bounded away from 0, the extraction has only to skip infinitely many integers, therefore $(\mathcal{F}_n)_{n\le 0}$ is at the threshold of standardness.

Theorem A immediately provides other interesting examples. For example, it may happen that $(\mathcal{F}_{2n})_{n\le 0}$ is standard while $(\mathcal{F}_{2n-1})_{n\le 0}$ is not, or vice versa. When this phenomenon occurs, we will say that the filtration $(\mathcal{F}_n)_{n\le 0}$ "interlinks" standardness and non-standardness.

Repeated interlinking is possible. By slowing time suitably in a filtration at the threshold of standardness (Example 11), one gets a filtration $(\mathcal{F}_n)_{n\le 0}$ such that $(\mathcal{F}_{2n})_{n\le 0}$, $(\mathcal{F}_{4n})_{n\le 0}$, $(\mathcal{F}_{8n})_{n\le 0}, \ldots$ are non-standard, whereas $(\mathcal{F}_{2n-1})_{n\le 0}$, $(\mathcal{F}_{4n-2})_{n\le 0}$, $(\mathcal{F}_{8n-4})_{n\le 0}, \ldots$ are standard.

Improving on an example of Tsirelson In a second part, we study another type of filtrations inspired by a construction of Tsirelson in unpublished notes [9].

Tsirelson has constructed an inhomogeneous discrete Markov process $(Z_n)_{n\le 0}$ such that the random variables $(Z_{2n})_{n\le 0}$ are independent and such that the natural filtration $(\mathcal{F}^Z_n)_{n\le 0}$ is non-standard although its tail σ-field is trivial. This example is illuminating since "simple" reasons explain why the standardness criteria do not hold and no technical estimates are required. Tsirelson's construction relies on a particular structure of the triples $(Z_{2n-2}, Z_{2n-1}, Z_{2n})$ that we explain. We call "bricks" these triples.

In this paper, we give a modified and simpler construction which provides stronger results by requiring more on the bricks: in our construction, for every $n \le 0$, Z_{2n-2} is a deterministic function of Z_{2n-1} and Z_{2n-1} is a deterministic function of (Z_{2n-2}, Z_{2n}), hence the filtration $(\mathcal{F}^Z_{2n})_{n\le 0}$ is generated by the sequence $(Z_{2n})_{n\le 0}$ of independent random variables. Yet, $(\mathcal{F}^Z_{2n-1})_{n\le 0}$ is not standard. Thus the filtration $\mathcal{F}^Z$ "interlinks" standardness and non-standardness. Actually, we have a complete characterisation of the standard filtrations among the filtrations extracted from $\mathcal{F}^Z$.

Theorem B *There exists a Markov process $(Z_n)_{n\le 0}$ such that*

- *for each $n \le 0$, Z_n takes its values in some finite set F_n,*
- *the random variables $(Z_{2n})_{n\le 0}$ are independent,*
- *for each $n \le 0$, Z_{2n-1} is a deterministic function of (Z_{2n-2}, Z_{2n}),*
- *the filtration $(Z_n)_{n\le 0}$ is $(r_n)_{n\le 0}$-adic for some sequence $(r_n)_{n\le 0}$,*
- *for any infinite subset D of $\mathbb{Z}_-$, the filtration $(\mathcal{F}^Z_n)_{n\in D}$ is standard if and only if $2n-1 \notin D$ for infinitely many $n \le 0$.*

In particular, the filtration $(\mathcal{F}^Z_{2n-1})_{n\le 0}$ is at the threshold of standardness.

In this theorem, the statement that $(\mathcal{F}^Z_{2n-1})_{n\leq 0}$ is at the threshold of standardness cannot be deduced from the standardness of $(\mathcal{F}^Z_{2n})_{n\leq 0}$ and the non-standardness of $(\mathcal{F}^Z_{2n-1})_{n\leq 0}$ only. Indeed, the example of repeated interlinking mentioned above (see Example 11 in Sect. 2) provides a counterexample (modulo a time-translation). The proof that $(\mathcal{F}^Z_{2n-1})_{n\leq 0}$ is at the threshold of standardness actually uses the fact that $(Z_n)_{n\leq 0}$ is an inhomogeneous Markov process.

2 The case of split-words filtrations

In the whole section, excepted in Sect. 2.5, $\mathcal{F} = (\mathcal{F}_n)_{n\leq 0}$ denotes a split-words filtration associated to a finite alphabet A (endowed with the uniform measure) and a decreasing sequence $(\ell_n)_{n\leq 0}$ of positive integers (the lengths) such that $\ell_0 = 1$ and the ratios $r_n = \ell_{n-1}/\ell_n$ are integers.

First, we prove the characterisation at the threshold of standardness among the split-words filtrations stated in Proposition 1.

2.1 Proof of Proposition 1

Preliminary observations: let B be an infinite subset of $\mathbb{Z}_-$ such that B^c is infinite. Then $(\mathcal{F}_n)_{n\in B}$ is a split-words filtration with lengths $(\ell_n)_{n\in B}$. The ratios between successive lengths are the integers $(R_n)_{n\in B}$ given by

$$R_n = \ell_{m(n)}/\ell_n \quad \text{where } m(n) = \sup\{k < n : k \in B\}.$$

Set $B_1 = B \cap (1 + B)$ and $B_2 = B\backslash(1 + B)$. Then B_2 is infinite and

- for $n \in B_1$, $R_n = r_n$,
- for $n \in B_2$, $R_n \geq r_n r_{n-1} \geq r_{n-1}$.

Furthermore, if B^c does not contain two consecutive integers, then for any $n \in B_2$, one has $n - 2 \in B$ since $n - 1 \notin B$, thus $m(n) = n - 2$ and $R_n = r_n r_{n-1}$.

Proof of the "if" part: assume that (Δ) and $(\star)$ hold:

$$\sum_{n\leq 0} \frac{\log(r_n)}{\ell_n} < +\infty \quad \text{and} \quad \inf_{n\leq 0} \frac{\log r_{n-1}}{\ell_n} > 0.$$

Then $\mathcal{F}$ is not standard, and for every infinite subset B of $\mathbb{Z}_-$ such that B^c is infinite,

$$\sum_{n\in B} \frac{\log(R_n)}{\ell_n} \geq \sum_{n\in B_2} \frac{\log(R_n)}{\ell_n} \geq \sum_{n\in B_2} \frac{\log r_{n-1}}{\ell_n} = +\infty.$$

since $B_2 = B\backslash(1 + B)$ is infinite. Thus, $(\mathcal{F}_n)_{n\in B}$ is standard since the sequence of lengths $(\ell_n)_{n\in B}$ fulfills condition $\neg(\Delta)$. Therefore $\mathcal{F}$ is at the threshold of standardness.

Proof of the "only if" part: condition (Δ), which is equivalent to the non-standardness of $\mathcal{F}$, is necessary for $\mathcal{F}$ to be at the threshold of standardness. Let us show that if (Δ) and $\neg(\star)$ hold, then $\mathcal{F}$ is not at the threshold of standardness. By (Δ), $\log(r_n)/\ell_n \to 0$ as $n \to -\infty$, hence

$$\liminf_{n\to-\infty} \frac{\log(r_n r_{n-1})}{\ell_n} = \liminf_{n\to-\infty} \frac{\log r_{n-1}}{\ell_n} = 0,$$

since 0 is the greatest lower bound of the positive real numbers $\log r_{n-1}/\ell_n$. Thus one can find a subsequence such that

$$\forall n \in \mathbb{Z}_-, \ \frac{\log(r_{\phi(n)} r_{\phi(n)-1})}{\ell_{\phi(n)}} \leq 2^n \quad \text{and} \quad \phi(n-1) \leq \phi(n) - 2.$$

Set $B = (\phi(\mathbb{Z}_-) - 1)^c$. Let us show that the filtration $(\mathcal{F}_n)_{n\in B}$ is not standard. By construction, $\phi(\mathbb{Z}_-)$ is infinite and does not contain two consecutive integers. Hence B and B^c are both infinite and $B_2 = B\backslash(B+1) = \phi(\mathbb{Z}_-)$. Moreover, according to the preliminary observations, $R_n = r_n$ for every $n \in B_1$ and $R_n = r_n r_{n-1}$ for every $n \in B_2$ since B^c does not contain two consecutive integers. Thus

$$\begin{aligned}\sum_{n\in B} \frac{\log(R_n)}{\ell_n} &= \sum_{n\in B_1} \frac{\log(r_n)}{\ell_n} + \sum_{n\in\phi(\mathbb{Z}_-)} \frac{\log(r_n r_{n-1})}{\ell_n} \\ &\leq \sum_{n\leq 0} \frac{\log(r_n)}{\ell_n} + \sum_{m\leq 0} 2^m \\ &< +\infty.\end{aligned}$$

Therefore $(\mathcal{F}_n)_{n\in B}$ is not standard. Thus $\mathcal{F}$ is not at the threshold of standardness.

2.2 Proof of Proposition 2 and example

Proof Assume that $(\neg\Delta)$ and $(\Box)$ hold and that $\mathcal{F}$ is extracted from some split-words filtration $\mathcal{H}$ with lengths $(\ell'_n)_{n\leq 0}$, namely $\mathcal{F}_n = \mathcal{H}_{\phi(n)}$ for every $n \leq 0$, for some increasing map ϕ from $\mathbb{Z}_-$ to $\mathbb{Z}_-$. Then for every $n \leq 0$, $\ell_n = \ell'_{\phi(n)}$ and $r_n = r'_{\phi(n-1)+1} \cdots r'_{\phi(n)}$ where $r'_k = \ell'_{k-1}/\ell'_k$. Let us show that $\mathcal{H}$ cannot be at the threshold of standardness.

Condition $(\neg\Delta)$ ensures that $\mathcal{F}$ is standard. If ϕ skips only finitely many integers, then $\mathcal{H}$ is standard and the conclusion holds. Otherwise, $\phi(n-1) \leq \phi(n) - 2$ for infinitely many n, and for those n,

$$\frac{\log\left(r'_{\phi(n)-1}\right)}{\ell'_{\phi(n)}} \leq \frac{\log\left(r'_{\phi(n-1)+1} \cdots r'_{\phi(n)}\right)}{\ell'_{\phi(n)}} = \frac{\log r_n}{\ell_n}.$$

Thus, ($\square$) implies that

$$\inf_{k\leq 0} \frac{\log(r'_{k-1})}{\ell'_k} = 0.$$

Since the sequence $(r'_n)_{n\leq 0}$ does not fulfill condition ($\star$), $\mathcal{H}$ is not at the threshold of standardness. $\square$

Example 3 Define the sequence of lengths $(\ell_n)_{n\leq 0}$ by $\ell_0 = 1$, $\ell_{-1} = 2$ and, for every $n \leq -1$,

$$\ell_{n-1} = \ell_n 2^{\lfloor \ell_n/|n| \rfloor},$$

where $\lfloor x \rfloor$ denotes the integer part of x.

A recursion shows that for every $n \leq 0$, ℓ_n is a power of 2, and that $\ell_n \geq 2^{|n|} \geq |n|$, hence $r_n = \ell_{n-1}/\ell_n \geq 2$.

Moreover, for every $n \leq -1$,

$$\frac{\log_2(r_n)}{\ell_n} = \frac{\lfloor \ell_n/|n| \rfloor}{\ell_n} \in \left[\frac{1}{2|n|}, \frac{1}{|n|}\right].$$

Therefore, ($\neg\Delta$) and ($\square$) hold, hence $\mathcal{F}$ is standard but cannot be extracted from any split-words filtration at the threshold of standardness.

Yet, since each ℓ_n is a power of 2, $\mathcal{F}$ is extracted from the dyadic split-words filtration $\mathcal{H}$, which is not standard. Since every filtration extracted from $\mathcal{H}$ is a split-words filtration, one can deduce that no intermediate filtration (for extraction) between $\mathcal{H}$ and $\mathcal{F}$ is at the threshold of standardness.

Remark There are trivial examples of standard split-words filtrations which cannot be extracted from any split-words filtration at the threshold of standardness. For example, consider any split-words filtrations such that $\neg(\Delta)$ holds and such that r_n is a prime number for every $n \leq 0$. The last condition prevents the filtration from being extracted from *any other* split-words filtration. Yet, it still could be extracted from some filtration at the threshold of standardness which is not a split-words filtration.

2.3 Proof of Theorem A

Replacing α_n by $(\alpha_n \vee |n+2|^{-2}) \wedge 1$ for $n \leq -3$ does not change the nature of the series $\sum_{k\in B^c} \alpha_k$, hence we may assume that for $n \leq -3$,

$$|n+2|^{-2} \leq \alpha_n \leq 1.$$

Set $\ell_0 = 1$, $\ell_{-1} = 2$, $\ell_{-2} = 8$, $\ell_{-3} = 64$, $\ell_{-4} = 2^{11} = 2{,}048$ and $\ell_{n-2} = 2^{\lfloor \alpha_{n-1}\ell_n \rfloor}$ for every $n \leq -3$, where $\lfloor x \rfloor$ denotes the integer part of x. We begin with two technical lemmas.

Lemma 4 *For every $n \leq -1$, $\ell_n \geq |n|^3$ and $\ell_n \geq 2|n+1|^2\ell_{n+1}$.*

Proof of Lemma 4 The proof of Lemma 4 is done by induction. One checks that the above inequalities hold for $-4 \leq n \leq -1$.

Fix some $n \leq -3$. Assume that the inequalities hold for $n+1$, n and $n-1$. Then

$$\begin{aligned}
\log_2 \ell_{n-2} - \log_2 \ell_{n-1} &= \lfloor \alpha_{n-1}\ell_n \rfloor - \lfloor \alpha_n \ell_{n+1} \rfloor \\
&\geq \alpha_{n-1}\ell_n - 1 - \alpha_n \ell_{n+1} \\
&\geq |n+1|^{-2}\ell_n - \ell_{n+1} - 1 \\
&\geq \ell_{n+1} - 1 \text{ (since } \ell_n \geq 2|n+1|^2 \ell_{n+1}) \\
&\geq |n+1|^3 - 1 \text{ (since } \ell_{n+1} \geq |n+1|^3),
\end{aligned}$$

hence

$$\ell_{n-2}/\ell_{n-1} \geq 2^{|n+1|^3-1} \geq 2|n-1|^2 \text{ (since } n \leq -3).$$

Since $\ell_{n-1} \geq |n-1|^3$, one has

$$\ell_{n-2} \geq 2|n-1|^2 \ell_{n-1} \geq 2|n-1|^5 \geq |n-2|^3 \text{ (since } n \leq -3).$$

Thus the inequalities hold for $n-2$. The proof is complete. □

Lemma 5 *For every $n \leq -4$,*

$$\begin{aligned}
\frac{\log_2 \ell_{n-1}}{\ell_n} &\leq \frac{1}{2|n+1|^2}, \\
\frac{\alpha_{n-1}}{2} \leq \frac{\log_2 \ell_{n-2}}{\ell_n} &\leq \alpha_{n-1}, \\
\frac{\log_2 \ell_{n-3}}{\ell_n} &\geq 1.
\end{aligned}$$

Proof of Lemma 5 For $n \leq -3$, the assumptions made on the sequence $(\alpha_k)_{k \leq 0}$, and Lemma 4 entail $\ell_n \alpha_{n-1} \geq |n|^3/|n+1|^2 \geq 1$, thus $\alpha_{n-1}\ell_n/2 \leq \lfloor \alpha_{n-1}\ell_n \rfloor \leq \alpha_{n-1}\ell_n$. Thus, the recursion formula $\ell_{n-2} = 2^{\lfloor \alpha_{n-1}\ell_n \rfloor}$ yields

$$\frac{\alpha_{n-1}}{2} \leq \frac{\log_2 \ell_{n-2}}{\ell_n} \leq \alpha_{n-1}.$$

When $n \leq -4$, the same inequalities hold for $n+1$ and $n-1$, hence by Lemma 4

$$\frac{\log_2 \ell_{n-1}}{\ell_n} \leq \alpha_n \frac{\ell_{n+1}}{\ell_n} \leq \frac{\ell_{n+1}}{\ell_n} \leq \frac{1}{2|n+1|^2},$$

and

$$\frac{\log_2 \ell_{n-3}}{\ell_n} \geq \frac{\alpha_{n-2}}{2} \frac{\ell_{n-1}}{\ell_n} \geq \frac{1}{2|n|^2} 2|n|^2 = 1.$$

The proof is complete. □

We now prove Theorem A.

Let us check that the split-words filtration associated to the to the lengths $(\ell_n)_{n\le 0}$ fulfills the properties of the previous proposition.

Let B be an infinite subset of $\mathbb{Z}_-$. Since replacing B by $B \setminus \{-2,-1,0\}$ does not change the nature of the filtration $(\mathcal{F}_n)_{n\in B}$, one may assume that $B \subset]-\infty,-3]$.

Set $m(n) = \sup\{k < n : k \in B\}$ for every $n \le 0$. Then $(\ell_{m(n)}/\ell_n)_{n\in B}$ is the sequence of ratios associated to the lengths $(\ell_n)_{n\in B}$. Since (Δ) characterises standardness of split-words filtrations,

$$(\mathcal{F}_n)_{n\in B} \text{ is standard } \iff \sum_{n\in B} \frac{\log_2(\ell_{m(n)}/\ell_n)}{\ell_n} = +\infty \iff \sum_{n\in B} \frac{\log_2 \ell_{m(n)}}{\ell_n} = +\infty,$$

where the last equivalence follows from the convergence of the series $\sum_n \log_2 \ell_n/\ell_n$ since $\ell_n \ge 2^{|n|}$ for every $n \le 0$.

Let us split B into three subsets:

- $B_1 = \{n \in B : m(n) = n-1\}$,
- $B_2 = \{n \in B : m(n) = n-2\}$,
- $B_3 = \{n \in B : m(n) \le n-3\}$.

Then

$$\sum_{n\in B} \frac{\log_2 \ell_{m(n)}}{\ell_n} = \sum_{n\in B_1} \frac{\log_2 \ell_{n-1}}{\ell_n} + \sum_{n\in B_2} \frac{\log_2 \ell_{n-2}}{\ell_n} + \sum_{n\in B_3} \frac{\log_2 \ell_{m(n)}}{\ell_n}.$$

The inequality $\ell_{m(n)} \ge \ell_{n-3}$ for $n \in B_3$ and Lemma 5 show that in the right-hand side,

- the first sum (over B_1) is always finite,
- the middle sum (over B_2) has the same nature as $\sum_{n\in B_2} \alpha_{n-1}$,
- the last sum (over B_3) is finite if and only if B_3 is finite.

When B_3 is finite, any pair of consecutive integers excepted a finite number of them contain at least one element of B. Hence, $(B_2 - 1)$ only differs from B^c by a finite set of integers. Thus the sum $\sum_{n\in B_2} \alpha_{n-1} = \sum_{n\in B_2-1} \alpha_n$ has the same nature as $\sum_{n\in B^c} \alpha_n$. Theorem A follows.

2.4 Some applications of Theorem A

Choosing particular sequences $(\alpha_n)_{n\le 0}$ in Theorem A provides interesting examples of non-standard filtrations. In what follows, $\mathcal{F}$ denotes the filtration associated to the sequence $(\alpha_n)_{n\le 0}$ given by Theorem A.

Example 6 If α_n is bounded away from 0, then $\mathcal{F}$ is at the threshold of standardness.

Example 7 If $\alpha_n = 0$ for every even n and $\alpha_n = 1$ for every odd n, then $(\mathcal{F}_{2n})_{n\le 0}$ is standard whereas $(\mathcal{F}_{2n-1})_{n\le 0}$ is not.

Example 8 If the series $\sum \alpha_n$ converges, then for every infinite subset B of $\mathbb{Z}_-$, the extracted filtration $(\mathcal{F}_n)_{n \in B}$ is standard if and only if $(B \cup (B-1))^c$ is infinite. In particular, the filtrations $(\mathcal{F}_{2n})_{n \le 0}$ and $(\mathcal{F}_{2n-1})_{n \le 0}$ are at the threshold of standardness.

Example 9 If $\alpha_n \sim 1/|n|$ as n goes to $-\infty$, then $\mathcal{F}$ is not standard and no filtration at the threshold of standardness can be extracted from $\mathcal{F}$.

Proof of Example 9 The non-standardness of $\mathcal{F}$ is immediate by Theorem A.

Call μ the non-finite positive measure on $\mathbb{Z}_-$ defined by

$$\mu(B) = \sum_{n \in B} \alpha_n \quad \text{for } B \subset \mathbb{Z}_-.$$

Let $(\mathcal{F}_n)_{n \in B}$ be any non-standard filtration extracted from $\mathcal{F}$. We show that $(\mathcal{F}_n)_{n \in B}$ cannot be at the threshold of standardness by constructing a subset B' of B such that $(\mathcal{F}_n)_{n \in B'}$ is not standard although $B \setminus B'$ is infinite.

By the Theorem A, we know that $\mu(B^c) < +\infty$ and

$$n \notin B \text{ and } n+1 \notin B \text{ only for finitely many } n \in \mathbb{Z}_-.$$

Since $\mu(B^c)$ is finite, the elements of B^c get rarer and rarer as $n \to -\infty$. In particular, the set $A = (B-1) \cap B \cap (B+1)$ is infinite.

We get B' from B by removing a "small" infinite subset of A. Namely, we set $B' = B \backslash A'$ where A' is an infinite subset of A which does not contain two consecutive integers and chosen such that $\mu(A') < +\infty$. By construction, $B \backslash B' = A'$ is infinite and $\mu((B')^c) < +\infty$ since $(B')^c = B^c \cup A'$. Thus B' is an infinite subset of B.

Using the definition of A and the fact that A' does not contain two consecutive integers and by construction of A, one checks that $(B' \cup (B'-1)) = (B \cup (B-1))$, therefore $(B' \cup (B'-1))^c$ is infinite.

Thus $(\mathcal{F}_n)_{n \in B'}$ is not standard, which shows that $(\mathcal{F}_n)_{n \in B}$ is not at the threshold of standardness. □

2.5 Interlinking standardness and non standardness

In this subsection, we show that any filtration at the threshold of standardness (not necessarily a split-words filtration) provides a filtration which interlinks repeatedly standardness and non-standardness by a suitable slowing-down of time.

Given any filtration $(\mathcal{F}_n)_{n \le 0}$, a simple way to get a slowed down filtration is to repeat each $\mathcal{F}_n$ some finite number of times, which may depend of n. We now show that this procedure does not change the nature of the filtration.

Lemma 10 *Let $(\mathcal{F}_n)_{n \le 0}$ be any filtration and ϕ an increasing map from $\mathbb{Z}_-$ to $\mathbb{Z}_-$ such that $\phi(0) = 0$. For every $n \le 0$, set $\mathcal{G}_n = \mathcal{F}_k$ if $\phi(k-1)+1 \le n \le \phi(k)$. Then:*

- *$(\mathcal{G}_n)_{n \le 0}$ is a filtration,*
- *$(\mathcal{F}_n)_{n \le 0}$ is extracted from $(\mathcal{G}_n)_{n \le 0}$,*
- *$(\mathcal{G}_n)_{n \le 0}$ is standard if and only if $(\mathcal{F}_n)_{n \le 0}$ is standard.*

Proof of Lemma 10 By construction, $\mathcal{G}_{\phi(k)} = \mathcal{F}_k$ for every $k \le 0$ and the sequence $(\mathcal{G}_n)_{n\le 0}$ is constant on every interval $[\![\phi(k-1)+1, \phi(k)]\!]$. The first two points follow.

The "only if" part of the third point is immediate since $\mathcal{F}$ is extracted from $\mathcal{G}$.

Assume that $\mathcal{F}$ is standard. Then, up to a enlargement of the probability space, one may assume that $\mathcal{F}$ is immersed in some product-type filtration $\mathcal{H}$. Define a slowed-down filtration by $\mathcal{K}_n = \mathcal{H}_k$ if $\phi(k-1)+1 \le n \le \phi(k)$. Then $\mathcal{K}$ is still a product-type filtration To prove that $\mathcal{G}$ is immersed in $\mathcal{K}$, we have to check that for every $n \le -1$, $\mathcal{G}_{n+1}$ and $\mathcal{K}_n$ are independent conditionally on $\mathcal{G}_n$. This holds in any case since:

- when $\phi(k-1)+1 \le n \le \phi(k)-1$, $\mathcal{G}_{n+1} = \mathcal{F}_k$, $\mathcal{K}_n = \mathcal{H}_k$ and $\mathcal{G}_n = \mathcal{F}_k$;
- when $n = \phi(k)$, $\mathcal{G}_{n+1} = \mathcal{F}_{k+1}$, $\mathcal{K}_n = \mathcal{H}_k$ and $\mathcal{G}_n = \mathcal{F}_k$.

Hence $\mathcal{G}$ is standard. □

Example 11 Assume that $(\mathcal{F}_n)_{n\le 0}$ is at the threshold of standardness. Set $\phi(0) = 0$, $\phi(-1) = -1$ and, for every $k \le -1$, $\phi(2k) = -2^{|k|}$ and $\phi(2k-1) = -2^{|k|} - 1$. Let $\mathcal{G}$ be the slowed-down filtration obtained from $\mathcal{F}$ as above. Then for any $d \ge 1$, the filtration $(\mathcal{G}_{2^d n})_{n\le 0}$ is not standard, whereas the filtration $(\mathcal{G}_{2^d n - 2^{d-1}})_{n\le 0}$ is standard.

Proof of Example 11 Fix $d \ge 1$. The filtrations $(\mathcal{G}_{2^d n})_{n\le -2}$ and $(\mathcal{G}_{2^d n - 2^{d-1}})_{n\le -1}$ can be obtained from $(\mathcal{F}_n)_{n\le -2d-2}$ and $(\mathcal{F}_{2n-1})_{n\le -d}$ by time-translations and by the slowing-down procedure just introduced. And truncations, time-translations and slowing-down procedure preserve the nature of the filtrations. □

3 Improving on an example of Tsirelson

3.1 A construction of Tsirelson

In some non-published notes, Tsirelson gives a method to construct an inhomogeneous Markov process $(Z_n)_{n\le 0}$ whose natural filtration is easily proved to be non-standard, although the tail σ-field $\mathcal{F}^Z_{-\infty}$ is trivial and the random variables $(Z_{2n})_{n\le 0}$ are independent. This construction relies on a particular structure of each triple $(Z_{2n-2}, Z_{2n-1}, Z_{2n})$. We will call *bricks* these triples, since the sequence $(Z_n)_{n\le 0}$ is obtained by gluing the triples $(Z_{2n-2}, Z_{2n-1}, Z_{2n})$ in a Markovian way. We now give a formal definition which is directly inspired by Tsirelson's construction.

Definition 12 Fix $\alpha \in]0,1[$. Let F_0, F_1, F_2 be finite sets. We will say that a triple (Z_0, Z_1, Z_2) of uniform random variables with values in F_0, F_1, F_2 is an α-brick if

- the triple (Z_0, Z_1, Z_2) is Markov.
- Z_0 and Z_2 are independent.
- for any non-anticipative coupling of two copies (Z'_0, Z'_1, Z'_2) and (Z''_0, Z''_1, Z''_2) of (Z_0, Z_1, Z_2), defined on some probability space $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbb{P}})$,

$$\bar{\mathbb{P}}\left[Z'_2 \ne Z''_2 \middle| \sigma\left(Z'_0, Z''_0\right)\right] \ge 1-\alpha \quad \text{on the event } \left[Z'_0 \ne Z''_0\right].$$

Here, the expression "non-anticipative" means that the filtrations generated by the three-time processes Z' and Z'' are immersed in the natural filtration of (Z', Z'').

In particular, Z_1' and Z_0'' are independent conditionally on Z_0' (the couple (Z_0', Z_0'') gives no more information on Z_1' than Z_0' does). Similarly, Z_2' and (Z_0'', Z_1'') are independent conditionally on (Z_0', Z_1'). And the same holds when the roles of Z' and Z'' are exchanged.

The last condition helps to negate the I-cosiness criterion recalled in Sect. 3.3 of the present paper. Actually, Tsirelson used another formulation involving iterated Kantorovitch–Rubinstein metrics to negate Vershik's criterion.

The next example is a slight simplification of the example provided by Tsirelson, which is enlightening.

Example 13 Let p be a prime number, and $\mathbb{Z}_p$ be the finite field with p elements. Note F_0 the set of all two-dimensional linear subspaces of $(\mathbb{Z}_p)^5$, F_1 the set of all two-dimensional affine subspaces of $(\mathbb{Z}_p)^5$ and $F_2 = (\mathbb{Z}_p)^5$. Construct a Markovian triple (Z_0, Z_1, Z_2) as follows:

- choose uniformly Z_0 in F_0;
- given Z_0, choose uniformly Z_1 among the affine planes with direction Z_0;
- given Z_0 and Z_1, choose uniformly Z_2 on the affine plane Z_1.

Then (Z_0, Z_1, Z_2) is an α-brick with $\alpha = 1/p$.

Indeed, one checks that Z_1 is uniform on F_1, Z_2 is uniform on F_2 and that Z_0 and Z_2 are independent. Now, let (Z_0', Z_1', Z_2') and (Z_0'', Z_1'', Z_2'') be any non-anticipative coupling of two copies of (Z_0, Z_1, Z_2), defined on some probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$.

Conditionally on $(Z_0', Z_0'', Z_1', Z_1'')$, Z_2' is uniform on the affine plane Z_1' and Z_2'' is uniform on the affine plane Z_1''. Since two affine planes with different directions are distinct and since two distinct planes have at most p common points, one has

$$\bar{\mathbb{P}}\left[Z_2' \neq Z_2'' \middle| \sigma\left(Z_0', Z_0'', Z_1', Z_1''\right)\right] \geq \frac{p^2 - p}{p^2} \mathbf{1}_{[Z_1' \neq Z_1'']} \geq \frac{p-1}{p} \mathbf{1}_{[Z_0' \neq Z_0'']}.$$

Thus

$$\bar{\mathbb{P}}\left[Z_2' \neq Z_2'' \middle| \sigma\left(Z_0', Z_0''\right)\right] \geq (1 - 1/p)\, \mathbf{1}_{[Z_0' \neq Z_0'']}.$$

Remark In Tsirelson's original example, Z_1 was an affine *line* uniformly chosen among the lines whose direction are included in the linear plane Z_0, and (Z_0, Z_1, Z_2) was an α-brick with $\alpha = 2/(p+1)$.

Note that the size of F_2, namely $|F_2| = p^5$, can be as large as one wants, and that

$$|F_0| = \frac{\left(p^5 - 1\right)\left(p^5 - p\right)}{\left(p^2 - 1\right)\left(p^2 - p\right)} = \left(p^4 + p^3 + p^2 + p + 1\right)\left(p^2 + 1\right)$$

is much larger. But $|F_0|$ has at least two prime divisors since the greatest common divisor of $p^4 + p^3 + p^2 + p + 1$ and $p^2 + 1$ is 1, hence $|F_0|$ cannot be a power of a prime number. Thus, such bricks cannot be glued together. Fortunately, a slight modification solves this problem, and we shall see examples of this later on.

Up to the detail just mentioned, the next theorem achieves Tsirelson's construction.

Theorem C *Let $(Z_n)_{n\le 0}$ be a sequence of uniform random variables with values in finite sets $(F_n)_{n\le 0}$ and $(\alpha_n)_{n\le 0}$ be an $]0,1[$-valued sequence such that the series $\sum_n \alpha_n$ converges. Assume that*

- *the sets F_{2n} are not singles,*
- *$(Z_n)_{n\le 0}$ is a non-homogeneous Markov process,*
- *for each $n \le 0$, the subprocess $(Z_{2n-2}, Z_{2n-1}, Z_{2n})$ is an α_n-brick.*

Then the natural filtration $\mathcal{F}^X$ is not standard. Moreover, if the tail σ-field $\mathcal{F}^Z_{-\infty}$ is trivial, then $|F_{2n}| \to +\infty$ as $n \to -\infty$.

In one excepts the necessary condition $|F_{2n}| \to +\infty$ as $n \to -\infty$ for $\mathcal{F}^Z_{-\infty}$ to be trivial, which is due to the authors of the present paper, the proof is adapted from Tsirelson's notes.

Proof of Theorem C First, we show that Z_0 does not fulfills the I-cosiness criterion (see Sect. 3.3). Indeed, set

$$c = \prod_{k\le 0}(1-\alpha_k) > 0$$

and consider any non-anticipative coupling $(Z'_n)_{n\le 0}$ and $(Z''_n)_{n\le 0}$ of the process $(Z_n)_{n\le 0}$, defined on some probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$. By assumption, for every $n \le 0$,

$$\bar{\mathbb{P}}\left[Z'_{2n} \neq Z''_{2n} \middle| \sigma\left(Z'_{2n-2}, Z''_{2n-2}\right)\right] \ge (1-\alpha_n)\,\mathbf{1}_{[Z'_{2n-2}\neq Z''_{2n-2}]}.$$

By induction, for every $n \le 0$,

$$\bar{\mathbb{P}}\left[Z'_0 \neq Z''_0 \middle| \sigma\left(Z'_{2n}, Z''_{2n}\right)\right] \ge \left(\prod_{k=n+1}^{0}(1-\alpha_k)\right)\mathbf{1}_{[Z'_{2n}\neq Z''_{2n}]} \ge c\mathbf{1}_{[Z'_{2n}\neq Z''_{2n}]}$$

If, for some $N \le 0$, the σ-fields $\mathcal{F}^{Z'}_{2N}$ and $\mathcal{F}^{Z''}_{2N}$ are independent, then

$$\bar{\mathbb{P}}\left[Z'_0 \neq Z''_0\right] \ge c\bar{\mathbb{P}}\left[Z'_{2N} \neq Z''_{2N}\right] = c\left(1-|F_{2N}|^{-1}\right) \ge c/2.$$

Hence $\bar{\mathbb{P}}\left[Z'_0 \neq Z''_0\right]$ is bounded away from 0, which negates the I-cosiness criterion. The non-standardness of $\mathcal{F}^X$ follows.

The second part of the theorem directly follows from the next proposition, applied to the sequence $(Y_n)_{n\le 0} = (Z_{2n})_{n\le 0}$. □

Proposition 14 *Let $(\gamma_n)_{n\le 0}$ be a sequence of real numbers in* $[0,1]$ *such that*

$$\prod_{n\le 0}\gamma_n > 0.$$

Let $(Y_n)_{n\le 0}$ *be a family of random variables which are uniformly distributed on finite sets* $(E_n)_{n\le 0}$. *Let* $(Y'_n)_{n\le 0}$ *and* $(Y''_n)_{n\le 0}$ *be independent copies of the process* $(Y_n)_{n\le 0}$, *defined on some probability space* $(\bar\Omega, \bar{\mathcal{A}}, \bar{\mathbb{P}})$. *Assume that* $\mathcal{F}^Y_{-\infty}$ *is trivial and that for every* $n \le 0$,

$$\bar{\mathbb{P}}\left[Y'_n \neq Y''_n | \sigma\left(Y'_{n-1}, Y''_{n-1}\right)\right] \ge \gamma_n \mathbf{1}_{[Y'_{n-1}\neq Y''_{n-1}]}.$$

Then $|E_n| \to 1$ *or* $|E_n| \to +\infty$ *as* $n \to -\infty$.

Proof of Proposition 14 By the independence of $(Y'_n)_{n\le 0}$ and $(Y''_n)_{n\le 0}$, the following exchange properties apply (see [11])

$$\begin{aligned}
\bigcap_{m\le 0}\bigcap_{n\le 0}\left(\mathcal{F}^{Y'}_m \vee \mathcal{F}^{Y''}_n\right) &= \bigcap_{m\le 0}\left(\mathcal{F}^{Y'}_m \vee \left(\bigcap_{n\le 0}\mathcal{F}^{Y''}_n\right)\right)\\
&= \bigcap_{m\le 0}\left(\mathcal{F}^{Y'}_m \vee \mathcal{F}^{Y''}_{-\infty}\right)\\
&= \left(\bigcap_{m\le 0}\mathcal{F}^{Y'}_m\right) \vee \mathcal{F}^{Y''}_{-\infty}\\
&= \mathcal{F}^{Y'}_{-\infty} \vee \mathcal{F}^{Y''}_{-\infty}.
\end{aligned}$$

But, using that $\mathcal{F}^{Y'}_m \vee \mathcal{F}^{Y''}_n$ is non-decreasing with respect to m and n, one gets

$$(\mathcal{F}^{Y'} \vee \mathcal{F}^{Y''})_{-\infty} = \bigcap_{n\le 0}\left(\mathcal{F}^{Y'}_n \vee \mathcal{F}^{Y''}_n\right) = \bigcap_{m\le 0}\bigcap_{n\le 0}\left(\mathcal{F}^{Y'}_m \vee \mathcal{F}^{Y''}_n\right).$$

Hence the tail σ-field $(\mathcal{F}^{Y'} \vee \mathcal{F}^{Y''})_{-\infty}$ is trivial. Thus the asymptotic event

$$\liminf_{n\to-\infty}\left[Y'_n \neq Y''_n\right]$$

has probability 0 or 1.

But a recursion shows that for every $n \le 0$

$$\bar{\mathbb{P}}\left(\bigcap_{n\le k\le 0}\left[Y'_k \neq Y''_k\right]\Big|\sigma\left(Y'_n, Y''_n\right)\right) \ge \left(\prod_{n+1\le k\le 0}\gamma_k\right)\mathbf{1}_{[Y'_n\neq Y''_n]}.$$

By taking expectations,

$$\bar{\mathbb{P}}\left(\bigcap_{n\le k\le 0}\left[Y'_k \neq Y''_k\right]\right) \ge \left(1 - |E_n|^{-1}\right)\prod_{n+1\le k\le 0}\gamma_k.$$

If $|E_n| \geq 2$ for infinitely many $n \leq 0$, then

$$\bar{\mathbb{P}}\left(\bigcap_{k\leq 0}\left[Y'_k \neq Y''_k\right]\right) \geq \frac{1}{2}\prod_{k\leq 0}\gamma_k > 0.$$

Thus $|E_n| \geq 2$ for every $n \leq 0$ and

$$\bar{\mathbb{P}}\left(\liminf_{n\to-\infty}\left[Y'_n \neq Y''_n\right]\right) = 1.$$

But by Fatou's lemma,

$$\bar{\mathbb{P}}\left(\liminf_{n\to-\infty}\left[Y'_n \neq Y''_n\right]\right) \leq \liminf_{n\to-\infty}\bar{\mathbb{P}}\left[Y'_n \neq Y''_n\right].$$

Hence $1 - |E_n|^{-1} = \bar{\mathbb{P}}\left[Y'_n \neq Y''_n\right] \to 1$ thus $|E_n| \to +\infty$ as $n \to -\infty$. □

3.2 Strong bricks

The slight variation we made on Tsirelson's example of a brick when replacing the affine *lines* with the affine *planes* provides two additional properties: Z_0 becomes a deterministic function of Z_1 and Z_1 becomes a deterministic function of (Z_0, Z_2). As we shall see, these properties allow to simplify the construction and the proofs, and enable us to get stronger results. The next definition exploits this idea and requires large family of partitions, each with a fixed number of blocks, each block having a fixed size, such that any two blocks chosen in any two different partitions have a small intersection.

Definition 15 (*Strong bricks*) Fix $\alpha \in]0, 1[$ and two positive integers r_1, r_2. Let F_0, F_1, F_2 be finite sets such that $|F_1| = |F_0|r_1$ and $|F_2| = r_1 r_2$. Let h be a bijection from $F_0 \times [\![1, r_1]\!]$ to F_1. Let $(\Pi_z)_{z\in F_0}$ be a family of partitions of F_2 indexed by F_0 such that

- each partition Π_z has r_1 blocks $S_{z,1}, \dots, S_{z,r_1}$;
- each block has r_2 elements.
- for any distinct (z', i') and (z'', i'') in $F_0 \times [\![1, r_1]\!]$, $|S_{z',i'} \cap S_{z'',i''}| \leq \alpha r_2$.

We will say that a triple (Z_0, Z_1, Z_2) of random variables with values in F_0, F_1, F_2 is a strong (r_1, r_2)-adic α-brick, associated to the family $(\Pi_z)_{z\in F_0}$, if

- Z_0 and Z_2 are independent random variables, uniformly distributed in F_0 and F_2.
- $Z_1 = h(Z_0, J)$, where J is the index of the only block of Π_{Z_0} which contains Z_2 (that is to say $Z_2 \in S_{Z_0,J}$).

The next lemma shows that strong bricks are bricks with some additional properties.

Lemma 16 *Let (Z_0, Z_1, Z_2) be strong (r_1, r_2)-adic α-brick. Then*

- *Z_0 is a deterministic function of Z_1;*
- *Z_1 is a deterministic function of (Z_0, Z_2);*
- *the conditional law of Z_1 given Z_0 is uniform on some finite random set of size r_1;*
- *the conditional law of Z_2 given Z_1 is uniform on some finite random set of size r_2;*
- *for all different z_1' and z_1'' of Z_1,*

$$\sum_{z\in F_2} \mathbb{P}\big[Z_2 = z\big|Z_1 = z_1'\big] \wedge \mathbb{P}\big[Z_2 = z\big|Z_1 = z_1''\big] \leq \alpha; \tag{1}$$

- *if (Z_0', Z_1', Z_2') and (Z_0'', Z_1'', Z_2'') is any non-anticipative coupling of (Z_0, Z_1, Z_2), defined on some probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$, then*

$$\mathbb{P}\big[Z_2' \neq Z_2''|\mathcal{G}\big] \geq (1-\alpha)\mathbf{1}_{[Z_1'\neq Z_1'']} \geq (1-\alpha)\mathbf{1}_{[Z_0'\neq Z_0'']};$$

- *(Z_0, Z_1, Z_2) is an α-brick.*

Proof of Lemma 16 Keep the notations of Definition 15.

The first two statements directly follow from the definition, and the triple (Z_0, Z_1, Z_2) is Markov since Z_0 is a function of Z_1.

For every $z_0 \in F_0$, $j \in [\![1, r_1]\!]$ and $z_2 \in F_2$,

$$\begin{aligned}\mathbb{P}[Z_0 = z_0; J = j; Z_2 = z_2] &= \mathbf{1}_{[z_2\in S_{z_0,j}]}\,\mathbb{P}[Z_0 = z_0; Z_2 = z_2] \\ &= \mathbf{1}_{[z_2\in S_{z_0,j}]} \times \frac{1}{|F_0|} \times \frac{1}{r_1 r_2}.\end{aligned}$$

Summing over z_2 yields

$$\mathbb{P}[Z_0 = z_0; J = j] = \frac{1}{|F_0|} \times \frac{1}{r_1},$$

and, by division,

$$\mathbb{P}\big[Z_2 = z_2\big|Z_0 = z_0; J = j\big] = \mathbf{1}_{[z_2\in S_{z_0,j}]} \times \frac{1}{r_2}.$$

Thus (Z_0, J) is uniform on $F_0 \times [\![1, r_1]\!]$, and given (Z_0, J), Z_2 is uniform on the block $S_{Z_0,J}$. Using the equality $Z_1 = h(Z_0, J)$, one gets the third and the fourth statements and the fact that Z_1 is uniform on F_1.

Let z_1' and z_1'' be distinct elements in F_1. Since the conditional laws $\mathcal{L}(Z_2|Z_1 = z_1')$ and $\mathcal{L}(Z_2|Z_1 = z_1'')$ are uniform on the blocks $S_{h^{-1}(z_1')}$ and $S_{h^{-1}(z_1'')}$, one has

$$\sum_{z\in F_2} \mathbb{P}\big[Z_2 = z\big|Z_1 = z_1'\big] \wedge \mathbb{P}\big[Z_2 = z\big|Z_1 = z_1''\big] = \sum_{z\in S_{h^{-1}(z_1')}\cap S_{h^{-1}(z_1'')}} \frac{1}{r_2} \leq \alpha,$$

which is the the fifth statement.

Let (Z'_0, Z'_1, Z'_2) and (Z''_0, Z''_1, Z''_2) be any non-anticipative coupling of (Z_0, Z_1, Z_2), defined on some probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$. Set $\mathcal{G} = \sigma(Z'_0, Z'_1, Z''_0, Z''_1)$. By the non-anticipative and the Markov properties,

$$\mathcal{L}\left(Z'_2 \middle| \mathcal{G}\right) = \mathcal{L}\left(Z'_2 \middle| \sigma\left(Z'_0, Z'_1\right)\right) = \mathcal{L}\left(Z'_2 \middle| \sigma\left(Z'_1\right)\right)$$

and the same holds with Z''.

Thus for all different z', z'' in F_1, one has, on the event $[Z'_1 = z'; Z''_1 = z'']$,

$$\begin{aligned}
\mathbb{P}\left[Z'_2 = Z''_2 \middle| \mathcal{G}\right] &= \sum_{z \in F_2} \mathbb{P}\left[Z'_2 = z; Z''_2 = z \middle| \mathcal{G}\right] \\
&\leq \sum_{z \in F_2} \mathbb{P}\left[Z'_2 = z \middle| \mathcal{G}\right] \wedge \mathbb{P}\left[Z''_2 = z \middle| \mathcal{G}\right] \\
&= \sum_{z \in F_2} \mathbb{P}\left[Z'_2 = z \middle| Z'_1 = z'\right] \wedge \mathbb{P}\left[Z''_2 = z \middle| Z''_1 = z''\right] \\
&= \sum_{z \in F_2} \mathbb{P}\left[Z_2 = z \middle| Z_1 = z'\right] \wedge \mathbb{P}\left[Z_2 = z \middle| Z_1 = z''\right] \\
&\leq \alpha.
\end{aligned}$$

Hence

$$\mathbb{P}\left[Z'_2 \neq Z''_2 \middle| \mathcal{G}\right] \geq (1-\alpha)\mathbf{1}_{[Z'_1 \neq Z''_1]} \geq (1-\alpha)\mathbf{1}_{[Z'_0 \neq Z''_0]},$$

since $[Z'_0 \neq Z''_0] \subset [Z'_1 \neq Z''_1]$, which shows the sixth statement.

Conditioning by $\sigma(Z'_0, Z''_0)$ completes the proof of the last statement. □

Examples of strong bricks

One checks that Example 13 provides a (p^3, p^2)-adic $1/p$-brick.

More generally, any finite family of partitions $(\Pi_z)_{z \in F_0}$ on a finite set F_2 satisfying the conditions of Definition 15 provides a strong brick: simply take two independent random variables Z_0 and Z_2, uniformly distributed in F_0 and F_2, set $Z_1 = (Z_0, J)$ and $F_1 = F_0 \times [\![1, r_1]\!]$, where J is the index of the only block of Π_{Z_0} which contains Z_2.

Algebra helps us to construct large such families of partitions. Here are two examples using a finite field K with q elements, where q can be any power of a prime number.

Example 17 We set $r_1 = r_2 = q^4$, $F_0 = K^{16}$ identified with the set $\mathcal{M}_4(K)$ of all 4×4 matrices with entries in K and $F_2 = K^8$ identified with $K^4 \times K^4$.

To each matrix $A \in \mathcal{M}_4(K)$, one can associate the partition of K^8 given by all four-dimensional affine subspaces of K^8 with equations $y = Ax + b$ where b ranges over K^4. Each of these subspaces has size q^4. But two subspaces of equations $y = A'x + b'$ and $y = A''x + b''$ intersect in at most q^3 points (a three dimensional affine subspace) when $A' \neq A''$. Hence these partitions provide a (q^4, q^4)-adic $1/q$-brick.

Example 18 We set $r_1 = r_2 = q$, $F_0 = K^4$ and $F_2 = K^2$.

To each quadruple $(a, b, c, d) \in K^4$, one can associate the partition of K^2 given by the q graphs of equations $y = ax^4 + bx^3 + cx^3 + dx + e$ where e ranges over K. Each of these graphs has size q. But two such graphs with different $(a, b, c, d) \in K^4$ intersect in at most 4 points. Hence, if $p \geq 5$ these partitions provide a (q, q)-adic $4/q$-brick.

The bricks provided by these two examples can be glued together since the sets K^{16} and K^4 can be identified by bijective maps with L^8 and L^2, where L denotes the field with q^2 elements. For each $n \leq 0$, call K_n the field with $q_n = q^{2^{|n|}}$ elements and set

$$\forall n \leq 0, \quad F_{2n} = K_n^8,\ r_{2n-1} = r_{2n} = q_n^4 \quad \text{and} \quad \alpha_n = 1/q_n,$$

or

$$\forall n \leq 0, \quad F_{2n} = K_n^2,\ r_{2n-1} = r_{2n} = q_n \quad \text{and} \quad \alpha_n = 4/q_n.$$

Start with a sequence $(Z_{2n})_{n\leq 0}$ of independent random variables uniformly distributed in the $(F_{2n})_{n\leq 0}$. For each $n \leq 0$, one can construct a random variable Z_{2n-1} with values in $F_{2n-1} = F_{2n-2} \times [\![1, r_{2n-1}]\!]$, using Z_{2n-2} and Z_{2n} and the family of partitions as above, to get an (r_{2n-1}, r_{2n})-adic α_n-brick $(Z_{2n-2}, Z_{2n-1}, Z_{2n})$. The process $(Z_n)_{n\leq 0}$ thus defined provides an example which proves the existence stated in Theorem B.

3.3 Proof of Theorem B

Indeed, Theorem B directly follows from the construction above and the theorem below.

Theorem D *Let $(\alpha_n)_{n\leq 0}$ be a sequence of reals in $]0, 1[$ such that the series $\sum_n \alpha_n$ converges. Let $(Z_n)_{n\leq 0}$ be any sequence of random variables taking values in some finite sets $(F_n)_{n\leq 0}$ of size ≥ 2. Assume that*

- *the random variables $(Z_{2n})_{n\leq 0}$ are independent;*
- *for each $n \leq 0$, $(Z_{2n-2}, Z_{2n-1}, Z_{2n})$ is a strong (r_{2n-1}, r_{2n})-adic α_n-brick.*

Then

- *$(Z_n)_{n\leq 0}$ is a Markov process which generates a (r_n)-adic filtration;*
- *for every infinite subset D of $\mathbb{Z}_-$, $(\mathcal{F}_n^Z)_{n\in D}$ is standard if and only if $2n - 1 \notin D$ for infinitely many $n \leq 0$.*

In particular, the filtration $(\mathcal{F}_{2n-1}^Z)_{n\leq 0}$ is at the threshold of standardness.

Proof of Theorem D We now prove the statements.

Proof that $(Z_n)_{n\leq 0}$ is a Markov process and generates a (r_n)-adic filtration

First, note that the filtration $(\mathcal{F}_{2n}^Z)_{n\leq 0}$ is generated by the independent random variables $(Z_{2n})_{n\leq 0}$ since for every $n \leq 0$, Z_{2n-1} is a deterministic function of

(Z_{2n-2}, Z_{2n}). Hence, for every $n \le 0$,

$$\mathcal{F}^Z_{2n-2} = \sigma\left(Z_{2n-2}\right) \vee \mathcal{F}^Z_{2n-4},$$

Moreover, since Z_{2n-2} is a deterministic function of Z_{2n-1},

$$\mathcal{F}^Z_{2n-1} = \sigma(Z_{2n-1}) \vee \mathcal{F}^Z_{2n-2} = \sigma(Z_{2n-1}) \vee \mathcal{F}^Z_{2n-4}.$$

By independence of $(Z_{2n-2}, Z_{2n-1}, Z_{2n})$ and $\mathcal{F}^Z_{2n-4}$, we get

$$\mathcal{L}\left(Z_{2n-1}\middle|\mathcal{F}^Z_{2n-2}\right) = \mathcal{L}\left(Z_{2n-1}\middle|\sigma\left(Z_{2n-2}\right) \vee \mathcal{F}^Z_{2n-4}\right) = \mathcal{L}\left(Z_{2n-1}\middle|\sigma\left(Z_{2n-2}\right)\right),$$
$$\mathcal{L}\left(Z_{2n}\middle|\mathcal{F}^Z_{2n-1}\right) = \mathcal{L}\left(Z_{2n}\middle|\sigma\left(Z_{2n-1}\right) \vee \mathcal{F}^Z_{2n-4}\right) = \mathcal{L}\left(Z_{2n}\middle|\sigma\left(Z_{2n-1}\right)\right).$$

The Markov property follows. But for every $n \le 0, (Z_{2n-2}, Z_{2n-1}, Z_{2n})$ is an (r_{2n-1}, r_{2n})-adic α_n-brick. The (r_n)-adic character of $\mathcal{F}^Z$ follows.

Proof that $(\mathcal{F}^Z_n)_{n\in D}$ is not standard when D contains all but finitely many odd negative integers

First, we show that $(\mathcal{F}^Z_{2n-1})_{n\le 0}$ is not standard. To do this, we check that the random variable Z_{-1} does not satisfy the I-cosiness criterion. Note that $(\mathcal{F}^Z_{2n-1})_{n\le 0}$ is the natural filtration of $(Z_{2n-1})_{n\le 0}$ only since for every $n \le 0$, Z_{2n-2} is some deterministic function f_n of Z_{2n-1}.

Let $(Z'_{2n-1})_{n\le 0}$ and $(Z''_{2n-1})_{n\le 0}$ be two copies of the process $(Z_{2n-1})_{n\le 0}$, defined on some probability space $(\bar{\Omega}, \bar{\mathcal{A}}, \bar{\mathbb{P}})$. Set $Z'_{2n-2} = f_n(Z'_{2n-1})$ and $Z''_{2n-2} = f_n(Z''_{2n-1})$ for every $n \le 0$. Then $(Z'_n)_{n\le 0}$ and $(Z''_n)_{n\le 0}$ are copies of the process $(Z_n)_{n\le 0}$. Moreover, $(\mathcal{F}^{Z'}_{2n-1})_{n\le 0}$ and $(\mathcal{F}^{Z''}_{2n-1})_{n\le 0}$ are the natural filtrations of $(Z'_{2n-1})_{n\le 0}$ and $(Z''_{2n-1})_{n\le 0}$.

Assume that these filtrations are immersed in some filtration $(\mathcal{G}_{2n-1})_{n\le 0}$.

Then, for every $n \le -1$,

$$\mathcal{L}\left(Z'_{2n+1}\middle|\mathcal{G}_{2n-1}\right) = \mathcal{L}\left(Z'_{2n+1}\middle|\mathcal{F}^{Z'}_{2n-1}\right) = \mathcal{L}\left(Z'_{2n+1}\middle|\sigma\left(Z'_{2n-1}\right)\right),$$

and since Z'_{2n} is a deterministic function of Z'_{2n+1},

$$\mathcal{L}\left(Z'_{2n}\middle|\mathcal{G}_{2n-1}\right) = \mathcal{L}\left(Z'_{2n}\middle|\sigma\left(Z'_{2n-1}\right)\right).$$

The same holds with the process Z''.

Hence, since $[Z'_{2n+1} = Z''_{2n+1}] \subset [Z'_{2n} = Z''_{2n}]$, the same proof as in Lemma 16 yields

$$\bar{\mathbb{P}}\left[Z'_{2n+1} \ne Z''_{2n+1}\middle|\mathcal{G}_{2n-1}\right] \ge \bar{\mathbb{P}}\left[Z'_{2n} \ne Z''_{2n}\middle|\mathcal{G}_{2n-1}\right] \ge (1-\alpha_n)\mathbf{1}_{[Z'_{2n-1}\ne Z''_{2n-1}]}.$$

By recursion,

$$\mathbb{P}\left[Z'_{-1} \neq Z''_{-1} \middle| \mathcal{G}_{2n-1}\right] \geq \prod_{n \leq k \leq -1} (1 - \alpha_k) \mathbf{1}_{[Z'_{2n-1} \neq Z''_{2n-1}]}.$$

Taking the expectations, one gets

$$\mathbb{P}\left[Z'_{-1} \neq Z''_{-1}\right] \geq \prod_{n \leq k \leq -1} (1 - \alpha_k) \mathbb{P}\left[Z'_{2n-1} \neq Z''_{2n-1}\right].$$

If, for some $N > -\infty$, the σ-fields $\mathcal{F}'_{2N-1}$ and $\mathcal{F}''_{2N-1}$ are independent, then

$$\mathbb{P}\left[Z'_{2N-1} \neq Z''_{2N-1}\right] = 1 - \frac{1}{|F_{2N-1}|} \geq \frac{1}{2},$$

since Z'_{2N-1} and Z''_{2N-1} are independent and uniform on F_{2N-1}, and

$$\mathbb{P}\left[Z'_{-1} \neq Z''_{-1}\right] \geq \frac{1}{2} \prod_{N \leq k \leq -1} (1 - \alpha_k) \geq \frac{1}{2} \prod_{k \leq -1} (1 - \alpha_k) > 0,$$

which shows that Z_{-1} does not satisfy the I-cosiness criterion.

Thus $(\mathcal{F}^Z_{2n-1})_{n \leq 0}$ is not standard. Thus, if D is any subset of $\mathbb{Z}_-$ which contains all odd negative integers, the filtration $(\mathcal{F}^Z_n)_{n \in D}$ is not standard (since standardness is preserved by extraction). This conclusion still holds when D contains all but finitely many odd negative integers (since standardness is an asymptotic property).

Proof that $(\mathcal{F}^Z_n)_{n \in D}$ is standard when D skips infinitely many odd negative integers

Since standardness is preserved by extraction, one only needs to consider the case where D not only skips infinitely many odd negative integers, but also contains all even ones. In this case, the filtration $(\mathcal{F}^Z_n)_{n \in D}$ is generated by $(Z_n)_{n \in D}$ only. Indeed, if n is any integer in $\mathbb{Z}_- \setminus D$, then n is odd, hence $n - 1 \in D$, $n + 1 \in D$ and Z_n is a function of (Z_{n-1}, Z_{n+1}).

For each $n \leq 0$, the conditional law $\mathcal{L}(Z_n | \mathcal{F}^Z_{n-1}) = \mathcal{L}(Z_n | Z_{n-1})$ is (almost surely) uniform on some random subset of F_n with r_n elements. By fixing a total order on the set F_n, one can construct a uniform random variable U_n on $[\![1, r_n]\!]$, independent of $\mathcal{F}^Z_{n-1}$, such that Z_n is a function of Z_{n-1} and U_n. Set $Y_n = Z_n$ if $n - 1 \notin D$ (which may happen only for even n) and $Y_n = U_n$ otherwise. Then Y_n is $\mathcal{F}^Z_n$-measurable. This shows that $\mathcal{F}^Y_n \subset \mathcal{F}^Z_n$ for every $n \in D$.

Let us prove the reverse inclusion. Fix $n \in D$, and call $m \leq n$ the integer such that $m - 1 \notin D$ but $k \in D$ for all $k \in [\![m, n]\!]$. Then Z_n is $\mathcal{F}^Y_n$-measurable as a function of $Y_m = Z_m, Y_{m+1} = U_{m+1}, \ldots, Y_n = U_n$.

Last, for every $n \in D$, Y_n is independent of $\mathcal{F}^Z_{n-1}$ if $n-1 \in D$ and Y_n is independent of $\mathcal{F}^Z_{n-2}$ otherwise. This shows the independence of the random variables $(Y_n)_{n \in D}$. Hence the filtration $(\mathcal{F}^Z_n)_{n \in D}$ is of product type, which completes the proof. □

Appendix A: Some basic facts on standardness

We summarize here the main definitions and results used in this paper. A complete exposition can be found in [2]. Recall that we work with filtrations indexed by the non-positive integers on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and that all the sub-$\sigma$-fields of $\mathcal{A}$ that we consider here are assumed to be complete and essentially separable with respect to $\mathbb{P}$. The role of the probability measure $\mathbb{P}$ is important althought it is often implicit. Actually, the true object of study are filtered probability spaces $(\Omega, \mathcal{A}, \mathbb{P}, (\mathcal{F}_n)_{n\le 0})$.

A.1 Isomorphisms of filtered probability spaces

Let $\mathcal{F} = (\mathcal{F}_n)_{n\le 0}$ and $\mathcal{F}' = (\mathcal{F}'_n)_{n\le 0}$ be filtrations on $(\Omega, \mathcal{A}, \mathbb{P})$ and $(\Omega', \mathcal{A}', \mathbb{P}')$.

Definition 19 An isomorphism of filtered probability spaces from $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ into $(\Omega', \mathcal{A}', \mathbb{P}', \mathcal{F}')$ is a bijective application from the space $\mathbf{L}^0(\Omega, \mathcal{F}_\infty, \mathbb{P})$ of the real random variables on $(\Omega, \mathcal{F}_\infty, \mathbb{P})$ into $\mathbf{L}^0(\Omega', \mathcal{F}'_\infty, \mathbb{P}')$ which preserves the laws of the random variables, commutes with Borelian applications, and sends $\mathcal{F}$ on $\mathcal{F}'$.

By definition, saying that an isomorphism Ψ sends $\mathcal{F}$ on $\mathcal{F}'$ means that for every $n \le 0$, the random variables $\Psi(X)$ for $X \in \mathbf{L}^0(\Omega, \mathcal{F}_n, \mathbb{P})$ generate $\mathcal{F}'_n$. Saying that Ψ commutes with Borelian applications means that for every sequence $(X_n)_{n\ge 1}$ of real random variables on $(\Omega, \mathcal{A}, \mathbb{P})$, and every Borelian application $F : \mathbb{R}^\infty \to \mathbb{R}$,

$$\Psi\left(F \circ (X_n)_{n\ge 1}\right) = F \circ (\Psi(X_n))_{n\ge 1}.$$

The case where $F((x_n)_{n\ge 1}) = \alpha x_1 + x_2$ with $\alpha \in \mathbb{R}$ shows that Ψ must be linear.

Any bimeasurable application ψ from $(\Omega, \mathcal{F}_\infty)$ to $(\Omega', \mathcal{F}'_\infty)$ which sends $\mathbb{P}$ on $\mathbb{P}'$ induces an isomorphism Ψ from $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ into $(\Omega', \mathcal{A}', \mathbb{P}', \mathcal{F}')$, defined by $\Psi(X) = X \circ \psi^{-1}$. Yet, an isomorphism from $(\Omega, \mathcal{A}, \mathbb{P}, \mathcal{F})$ into $(\Omega', \mathcal{A}', \mathbb{P}', \mathcal{F}')$ is not necessarily associated to some bimeasurable application from Ω to Ω' which sends $\mathbb{P}$ on $\mathbb{P}'$.

Note that for any sequence $(X_n)_{n\le 0}$ of random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$, the filtrations which are isomorphic to the natural filtration of $(X_n)_{n\le 0}$ are exactly the filtrations of the copies of $(X_n)_{n\le 0}$ on arbitrary probability spaces.

A.2 Immersion, immersibility and standardness of filtrations

Let $\mathcal{F} = (\mathcal{F}_n)_{n\le 0}$ and $\mathcal{G} = (\mathcal{G}_n)_{n\le 0}$ be filtrations on $(\Omega, \mathcal{A}, \mathbb{P})$.

Definition 20 One says that $\mathcal{F}$ is immersed into $\mathcal{G}$, if, for every $n \le 0$, $\mathcal{F}_n \subset \mathcal{G}_n$ and $\mathcal{F}_n$ is independent of $\mathcal{G}_{n-1}$ conditionally on $\mathcal{F}_{n-1}$. Equivalently, $\mathcal{F}$ is immersed into $\mathcal{G}$ if and only if every martingale in $\mathcal{F}$ is still a martingale in $\mathcal{G}$.

Immersion is stronger than mere inclusion. If $\mathcal{F}$ is immersed into $\mathcal{G}$, the additional information contained in $\mathcal{G}$ cannot give information on $\mathcal{F}$ in advance: intuitively, the independence of $\mathcal{F}_n$ and $\mathcal{G}_{n-1}$ conditionally on $\mathcal{F}_{n-1}$ means that $\mathcal{G}_{n-1}$ gives no more information on $\mathcal{F}_n$ than $\mathcal{F}_{n-1}$ does.

The notion of immersion can be weakened to provide a notion invariant by isomorphism.

Definition 21 Let $\mathcal{F} = (\mathcal{F}_n)_{n\le 0}$ and $\mathcal{G}' = (\mathcal{G}'_n)_{n\le 0}$ be filtrations on $(\Omega, \mathcal{A}, \mathbb{P})$ and $(\Omega', \mathcal{A}', \mathbb{P}')$. One says that $\mathcal{F}$ is immersible into $\mathcal{G}'$ if there exists a filtration $\mathcal{F}'$ on $(\Omega', \mathcal{A}', \mathbb{P}')$, isomorphic to $\mathcal{F}$, such that $\mathcal{F}'$ is immersed into $\mathcal{G}'$.

We can now define the standardness of filtrations.

Definition 22 A filtration is standard if it is immersible into a product-type filtration.

By Kolmogorov's 0–1 law, any filtration must have a trivial tail σ-field in order to be standard, but this necessary condition is not sufficient. In [10], Vershik established two different characterisations of standardness in the context of decreasing sequences of measurable partitions. Émery and Schachermayer [2] extended and reformulated them into a probabilistic language and called them Vershik's "first level" and "second level" criteria. They also introduced a new standardness criterion, namely the I-cosiness criterion. The I stands for independence, to distinguish I-cosiness from other variants of cosiness.

A.3 I-cosiness criterion

Let $\mathcal{F} = (\mathcal{F}_n)_{n\le 0}$ be a filtration on $(\Omega, \mathcal{A}, \mathbb{P})$.

Definition 23 Let R be any $\mathcal{F}_0$-measurable real random variable R. One says that R satisfies I-cosiness criterion for $(\mathcal{F}_n)_{n\le 0}$ (to abbreviate, we say that I(R) holds) if for any positive real number δ, there exists a probability space $(\overline{\Omega}, \overline{\mathcal{A}}, \overline{\mathbb{P}})$ supplied with two filtrations $\mathcal{F}'$ and $\mathcal{F}''$ such that:

- the filtrations $\mathcal{F}'$ and $\mathcal{F}''$ are isomorphic to the filtration $\mathcal{F}$;
- the filtrations $\mathcal{F}'$ and $\mathcal{F}''$ are immersed into $\mathcal{F}' \vee \mathcal{F}''$;
- there exists an integer $n_0 < 0$ such that the σ-fields $\mathcal{F}'_{n_0}$ and $\mathcal{F}''_{n_0}$ are independent;
- the copies R' and R'' of R given by the isomorphisms of the first condition are such that $\overline{\mathbb{P}}[|R' - R''| \ge \delta] \le \delta$.

One says that $\mathcal{F}$ is I-cosy when I(R) holds for every $R \in L^0(\Omega, \mathcal{F}_0, \mathbb{P})$.

I-cosiness was implicitly used by Smorodinsky [8] to prove that the dyadic split-words filtration has no "generating parametrization". Intuitively, condition I(R) means that one can couple two copies of $\mathcal{F}$ in a non-anticipative way so that old enough independent initial conditions have weak influence on the final value of R.

Laurent noticed that if $I(R)$ holds, then $I(\phi(R))$ holds for every Borel function ϕ from $\mathbb{R}$ to $\mathbb{R}$. Hence, to prove that $\mathcal{F}$ is I-cosy, it is sufficient to check that $I(R)$ for *one* real random variable generating $\mathcal{F}_0$.

It is also sufficient and sometimes handful to check $I(R)$ for all random variables with values in an arbitrary finite set, with the discrete distance $\mathbf{1}_{[R'\neq R'']}$ replacing $|R' - R''|$ in the definition of $I(R)$.

I-cosiness provides a standardness criterion.

Theorem E (Émery and Schachermayer [2]) *$\mathcal{F}$ is standard if and only if $\mathcal{F}$ is I-cosy.*

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