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Lower bounds on fluctuations for internal DLA

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Abstract We consider internal diffusion limited aggregation in dimension two or more. This is a random cluster growth model, where random walks start at the origin of the lattice, one at a time, and stop moving when reaching a site that is not occupied by previous walks. When n random walks are sent from the origin, we establish a lower bound for the inner and outer errors fluctuations of order square root of the logarithm of n. When dimension is three or more, this lower bound matches the upper bound recently obtained in independent works of Asselah and Gaudillière (Ann Prob arXiv:1011.4592, 2010) and Jerison et al. (Internal DLA and the Gaussian free field arXiv:1012.3453, 2010). Also, we produce as a corollary of our proof of Asselah and Gaudillière (Ann Prob arXiv:1011.4592, 2010), an upper bound for the fluctuation of the inner error in a specified direction.

Keywords Internal diffusion limited aggregation · Cluster growth · Random walk · Shape theorem · Logarithmic fluctuations

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1 Introduction

We establish lower bounds for the inner and outer errors in internal diffusion limited aggregation (internal DLA). Internal DLA models a discrete cluster growth, and is defined as follows. Let Λ be a subset of $\mathbb{Z}^d$ which represents the explored region at time 0. Let N be an integer, and $\xi = (\xi_1, \ldots, \xi_N)$ be the initial positions of N independent simple random walks $S_1, \ldots, S_N$ on $\mathbb{Z}^d$. The cluster that we denote $A(\Lambda, \xi)$ is obtained inductively. First, set $\tilde{A}(0) = \Lambda$, and assume $\tilde{A}(k-1)$ is obtained. Define

$$\tau_k = \inf\{t \geq 0 : \ S_k(t) \notin \tilde{A}(k-1)\}, \quad \text{and} \quad \tilde{A}(k) = \tilde{A}(k-1) \cup \{S_k(\tau_k)\}. \tag{1.1}$$

The internal DLA cluster is $A(\Lambda, \xi) = \tilde{A}(N)$. We call explorers the random walks obeying the aggregation rule (1.1). We say that the explorer k settles at time τ_k. If at time 0, the N walks start at the origin, with an empty explored region, we denote $A(\emptyset, \xi)$ by $A(N)$.

In dimension two or more, Lawler et al. [7] consider $A(N)$, and prove that in order that $A(N)$ covers, without hole, a sphere of radius n, we need N of order of the number of sites of $\mathbb{Z}^d$ in this sphere. In other words, the asymptotic shape of the cluster is a sphere. Then, Lawler in [8] shows a subdiffusive upper bound for the worse fluctuation to the spherical shape. More precisely, the latter result is formulated in terms of inner and outer errors, which we now introduce with some notation. We denote with $\|\cdot\|$ the euclidean norm on $\mathbb{R}^d$. For any x in $\mathbb{R}^d$ and r in $\mathbb{R}^+$, set

$$B(x, r) = \{y \in \mathbb{R}^d : \ \|y - x\| < r\} \quad \text{and} \quad \mathbb{B}(x, r) = B(x, r) \cap \mathbb{Z}^d. \tag{1.2}$$

For $\Lambda \subset \mathbb{Z}^d$, $|\Lambda|$ denotes the number of sites in Λ. The inner error $\delta_I(n)$ is such that

$$n - \delta_I(n) = \sup\{r \geq 0 : \ \mathbb{B}(0, r) \subset A(|\mathbb{B}(0, n)|)\}. \tag{1.3}$$

Also, the outer error $\delta_O(n)$ is such that

$$n + \delta_O(n) = \inf\{r \geq 0 : \ A(|\mathbb{B}(0, n)|) \subset \mathbb{B}(0, r)\}. \tag{1.4}$$

Note that δ_I and δ_O are associated with the worse fluctuations to the spherical shape.

In 2010, Asselah–Gaudillière in [1,2], and Jerison et al. in [4,5], independently showed the following upper bound.

Theorem 1.1 *When dimension $d \geq 3$, there are constants $\{\beta_d, \ d \geq 3\}$ such that with probability 1,*

$$\limsup_{n\to\infty} \frac{\delta_I(n)}{\sqrt{\log(n)}} \leq \beta_d, \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\delta_O(n)}{\sqrt{\log(n)}} \leq \beta_d. \tag{1.5}$$

When dimension is 2*, there is a constant* $\{\beta_2\}$ *such that with probability 1,*

$$\limsup_{n\to\infty} \frac{\delta_I(n)}{\log(n)} \le \beta_2, \quad \textit{and} \quad \limsup_{n\to\infty} \frac{\delta_O(n)}{\log(n)} \le \beta_2. \tag{1.6}$$

All these studies are concerned with upper bounds on δ_I and δ_O, and the matter of showing that they were indeed realized remained untouched. We present now two results on lower bounds. The first one is independent from previous results. Morally, it says that if there are no *deep* hole, then *long* tentacles form. The result is optimal in $d \ge 3$.

Proposition 1.2 *When dimension is three or more, and* $h(k) = \sqrt{\log(k)}$*, there is* $\alpha_d > 0$ *such that*

$$\lim_{n\to\infty} P(\exists k > n,\ \delta_O(k) \ge \alpha_d h(k) \ \mid\ \delta_I(n) < \alpha_d h(n)) = 1. \tag{1.7}$$

When dimension is two, set $h(k) = \sqrt{\log(k)\log(\log(k))}$*. There is* $\alpha_2 > 0$ *such that*

$$\lim_{n\to\infty} P(\exists k > n,\ \delta_O(k) \ge \alpha_2 h(k) \ \mid\ \delta_I(n) < \alpha_2 h(n)) = 1. \tag{1.8}$$

Remark 1.3 Note that as a consequence of (1.7), we have α_d positive such that

$$P(\delta_O(n) \ge \alpha_d\sqrt{\log(n)} \text{ or } \delta_I(n) \ge \alpha_d\sqrt{\log(n)},\ \text{i.o.}) = 1, \tag{1.9}$$

The second result uses the upper bound of Theorem 1.1. It says that *deep* holes do form.

Theorem 1.4 *There are positive constants* $\{\alpha_d, d \ge 2\}$ *such that when* $d \ge 2$*,*

$$P(\delta_I(n) > \sqrt{\alpha_d \log(n)} \quad i.o.) = 1 \tag{1.10}$$

Remark 1.5 We believe that in dimension 2, for some $\alpha_2 > 0$, almost surely both $\delta_I(n)$ and $\delta_O(n)$ are larger than $\alpha_2 \log(n)$ infinitely often. However, the way of realizing this event is probably different from what we describe in $d \ge 3$.

We present now a side result dealing with fluctuation in a given direction. The results on fluctuations for internal DLA [1,2,4,5,7,8] all focus on inner and outer errors. The paper [6] is still of a different nature: it addresses averaged error, in a sense where inner and outer errors can cancel each other out.

Though interesting, fluctuations in a given direction remained untreated. This is presented as an open problem in Section 3 of [5], and we note that the proof of [2] yields the following bound for the inner error in a given direction.

Proposition 1.6 *There are positive constants* $\{\kappa_d, d \ge 2\}$*, such that for* $z \in \mathbb{Z}^d$ *with* $\|z\| < n$*, we have*

$$P(z \notin A(|\mathbb{B}(0,n)|)) \le \begin{cases} \exp\left(-\kappa_2 \frac{(n-\|z\|)^2}{\log(n)}\right) & for \quad d = 2, \\ \exp(-\kappa_d (n - \|z\|)^2) & for \quad d \ge 3. \end{cases} \tag{1.11}$$

Let us explain briefly why Proposition 1.6 follows from [2]. In [7], the authors consider the number of explorers exiting $\mathbb{B}(0, r)$ from a site z in the boundary of $\mathbb{B}(0, R)$ (actually they consider z inside $\mathbb{B}(0, R)$). This quantity is denoted $W_R(z)$ when the explorers are initially at the origin. Lawler et al. in [7] express $W_R(z)$ as a difference of two sums of independent Bernoulli random variables, say for simplicity $M_R(z)$ and $M'_R(z)$. The structure of the relation is that $W_R(z) + M'_R(z) \geq M_R(z)$. $M_R(z)$ (resp. $M'_R(z)$) counts the number of *random walks* starting at the origin (resp. starting on each site of $\mathbb{B}(0, r)$), and which exit $\mathbb{B}(0, R)$ on site z. Now, if $\mu_R(z)$ is the expectation of $W_R(\eta, z)$, we have obtained in [1] that for a positive constant κ

$$P(W_R(z) = 0) \leq \exp\left(-\kappa \frac{\mu_R^2(z)}{\mu_R(z) + \sum_{y \in \mathbb{B}(0,R)} h_z^2(y)}\right), \tag{1.12}$$

where $h_z(y)$ is the probability to exit from $\mathbb{B}(0, R)$ on site z, when the initial position of the walk is y. Now, $\mu_R(z)$ and $y \mapsto h_z(y)$ are estimated in [2].

The rest of the paper is organized as follows. In Sect. 2, we set notation and recall useful results. In Sect. 3, we deal with the outer error and prove Proposition 1.2. In Sect. 4, we deal with the inner error, and prove Theorem 1.4. Finally, in Sect. 5, we explain how to read Proposition 1.6 as a corollary of our previous work [2].

2 Notation and prerequisite

Let $S : \mathbb{N} \to \mathbb{Z}^d$ denotes a simple random walk on $\mathbb{Z}^d$. When the initial condition is $S(0) = z \in \mathbb{Z}^d$, its law is denoted $\mathbb{P}_z$. The first time S hits a domain $\Lambda \subset \mathbb{Z}^d$, is denoted $H(\Lambda)$, or $H(S; \Lambda)$ to emphasize that S is the walk.

For a positive γ, we denote by $\rho(\gamma)$ the radius of the largest ball centered at 0 whose volume is less than γ. In other words,

$$\rho(\gamma) = \sup\{n \geq 0 : |\mathbb{B}(0, n)| \leq \gamma\}. \tag{2.1}$$

The abbreviation $b(n) = |\mathbb{B}(0, n)|$ is also handy. We denote with $\|\cdot\|$ the euclidean norm on $\mathbb{R}^d$. For a subset Λ of $\mathbb{Z}^d$, we denote by $\partial\Lambda = \{z \notin \Lambda : \exists y \in \Lambda, \|y - z\| = 1\}$.

We now consider a configuration $\eta \in \mathbb{N}^{\mathbb{Z}^d}$ with a finite number of particles

$$|\eta| := \sum_{z \in \mathbb{Z}^d} \eta(z) < \infty. \tag{2.2}$$

For $R > 0$, $\Lambda \subset \mathbb{Z}^d$, and $z \in \partial\mathbb{B}(0, R)$, we denote by $W_R(\Lambda; \eta, z)$ the number of explorers visiting z *at the moment* they reach $\partial\mathbb{B}(0, R)$, when the explorers start on η with an explored region Λ. The final positions of explorers which settle before they exit $\mathbb{B}(0, R)$ is denoted $A_R(\Lambda; \eta)$, and is a subset of $\mathbb{B}(0, R)$.

When we build the internal DLA cluster from n independent random walks, we also consider the unrestricted trajectories. This gives a natural coupling between explorers and independent random walks. Note that if $\mathbb{B}(0, R) \subset \Lambda$ and $z \in \partial\mathbb{B}(0, R)$, then

$$W_R(\Lambda; \eta, z) = W_R(\mathbb{B}(0, R); \eta, z),$$

and this corresponds to the number of independent random walks which exit $\mathbb{B}(0, R)$ at site z, and this latter number is denoted $M_R(\eta, z)$.

2.1 The abelian property

Diaconis and Fulton [3] allow explorers to start on distinct sites, and show that the law of the cluster is invariant under permutation of the order in which explorers are launched. This invariance is named *the abelian property*. As a consequence, one can realize the cluster by sending many *exploration waves*. Let us illustrate this observation by building $A(\emptyset, (n+m)\delta_0)$ in three waves, since we need later this very example. The first wave consists in launching n explorers, and they settle in $A_1 = A(\emptyset, n\delta_0)$, which we call the cluster *after the first wave*. Then, we launch m explorers that we color *green* for simplicity, and if they reach $\partial\mathbb{B}(0, R)$ before settling, then we stop them on $\partial\mathbb{B}(0, R)$. The settled green explorers make up the cluster $A_R(A_1; m\delta_0)$. The cluster after the second wave is then

$$A_2 = A_1 \cup A_R(A_1; m\delta_0).$$

For $z \in \partial\mathbb{B}(0, R)$, we call $\zeta_R(z) = W_R(A_1; m\delta_0, z)$. This is the configuration of the green explorers stopped on $\partial\mathbb{B}(0, R)$. Finally, the cluster after the third wave is obtain as we launch the stopped green explorers, and

$$A_3 = A_2 \cup A(A_2; \zeta_R).$$

The abelian properties implies that $A(\emptyset, (n+m)\delta_0)$ equals in law to A_3. It is convenient to think of the growing cluster as evolving in discrete time, where time counts the number of exploration waves.

2.2 On the harmonic measure.

We first recall a well known property of Poisson variables.

Lemma 2.1 *Let $\{U_n, n \in \mathbb{N}\}$ be an i.i.d. sequence with values in a set E, and $\{E_1, \ldots, E_n\}$ a partition of E. Then, if X is an independent Poisson random variable of parameter λ, and if*

$$X_i = \sum_{n \le X} \mathbb{1}_{U_n \in E_i},$$

then $\{X_i,\ i = 1, \ldots, n\}$ are independent Poisson variables with $E[X_i] = \lambda \times P(U \in E_i)$.

Now, for $z \in \mathbb{Z}^d \backslash \{0\}$, let

$$\Sigma(z) = \partial\mathbb{B}(0, \|z\|), \tag{2.3}$$

and note that z belongs to $\Sigma(z)$ since there is $z' \in \mathbb{B}(0, \|z\|)$ and $\|z - z'\| = 1$ (see Lemma 2.1 of [1]). We start independent random walks with initial configuration η, and for $z \neq 0$, $h > 0$, and $\Lambda \subset \Sigma(z)$, we denote by $N_z(\eta, \Lambda, h)$ the number of these walks which exit $\Sigma(z)$ on Λ, and visit z before exiting $\mathbb{B}(0, \|z\| + h)$. As a consequence of Lemma 2.1, we have

Corollary 2.2 *Let X be a Poisson variable of parameter λ, and $\eta = X\delta_0$. For any $z \neq 0$, $h, h' > 0$, and $\Lambda, \Lambda' \in \Sigma(z)$ with $\Lambda \cap \Lambda' = \emptyset$, we have that $N_z(\eta, \Lambda, h)$ and $N_z(\eta, \Lambda', h')$ are independent Poisson variable, and*

$$
\begin{aligned}
E[N_z(\eta, \Lambda, h)] &= \lambda \times \mathbb{P}_0(S(H(\Sigma(z))) \in \Lambda,\ H(z) < H(\mathbb{B}^c(0, \|z\| + h))) \\
&= \lambda \sum_{y \in \Lambda} \mathbb{P}_0(S(H(\Sigma(z)) = y))\mathbb{P}_y(\ H(z) < H(\mathbb{B}^c(0, \|z\| + h))).
\end{aligned} \tag{2.4}
$$

By combining well known asymptotics of the harmonic measure with Corollary 2.2, we obtain the following lemma.

Lemma 2.3 *Assume that dimension is two or more. Let η be as in Corollary 2.2. For $z \neq 0$, and $R > 0$ let $\Lambda = \mathbb{B}(z, R) \cap \Sigma(z)$, and $\Lambda' = \Sigma(z) \backslash \Lambda$. There is $\kappa > 0$, independent of z, and R such that*

$$
P(N_z(\eta, \Lambda, \infty) = 0,\ N_z(\eta, \Lambda', R) = 0) \geq \exp\left(-\kappa \frac{\lambda R}{\|z\|^{d-1}}\right). \tag{2.5}
$$

Proof Since $N_z(\eta, \Lambda, \infty)$ and $N_z(\eta, \Lambda', R)$ are independent Poisson variables, we have

$$
\begin{aligned}
&P(N_z(\eta, \Lambda, \infty) = 0,\ N_z(\eta, \Lambda', R) = 0) \\
&\quad = \exp(-E[N_z(\eta, \Lambda, \infty)] - E[N_z(\eta, \Lambda', R)]).
\end{aligned} \tag{2.6}
$$

It remains to compute expected values. To estimate $E[N_z(\eta, \Lambda, \infty)]$, we recall that there is a constant c_d such that

$$
\mathbb{P}_y(H(z) < \infty) \leq \frac{c_d}{1 + \|y - z\|^{d-2}}. \tag{2.7}
$$

Using (2.4), there is a constant c

$$E[N_z(\eta,\Lambda,\infty)] \le \sum_{y\in\Lambda} \lambda \mathbb{P}_0(S(H(\Sigma(z)))=y) \times \frac{c_d}{1+\|y-z\|^{d-2}}$$
$$\le \frac{c}{2}\frac{\lambda}{\|z\|^{d-1}}\left(1+\sum_{k=1}^{R}\frac{k^{d-2}}{1+k^{d-2}}\right) \le c\frac{\lambda R}{\|z\|^{d-1}}. \tag{2.8}$$

Now, to estimate $E[N_z(\eta,\Lambda',R)]$, we recall Lemma 5(b) of [4], which states that for some constant c'_d, for $y\in\Sigma(z)$

$$\mathbb{P}_y(H(z) < H(\mathbb{B}^c(0,\|z\|+R))) \le \frac{c'_d R^2}{\|z-y\|^d} \tag{2.9}$$

Using (2.4) and (2.9), there is a constant c' such that

$$E[N_z(\eta,\Lambda',R)] \le \sum_{y\in\Sigma(z)\setminus\mathbb{B}(z,R)} \lambda\mathbb{P}_0(S(H(\Sigma(z)))=y)\frac{c'_d R^2}{\|z-y\|^d}$$
$$\le \frac{c'}{2}\frac{\lambda R^2}{\|z\|^{d-1}}\sum_{k=R}^{2\|z\|+2}\frac{k^{d-2}}{k^d} \le c'\frac{\lambda R}{\|z\|^{d-1}} \tag{2.10}$$

Combining (2.8), and (2.10), we obtain the desired result.

3 The outer error

In this section, we prove Proposition 1.2. Let us explain the proof in dimension three or more, and explain, in Remark 3.1 of Step 2, how we adapt the proof to dimension two.

For positive reals α and γ, to be chosen later, we set $h(n)=\alpha\sqrt{\log(n)}$ and $\underline{L}(n)=\gamma\sqrt{\log(n)}$ for estimates on the outer and inner fluctuations. Even though we eventually take $h(n)=\underline{L}(n)$, it is useful to keep in mind their distinct nature. The limit (1.9) follows if for some small $\gamma=\alpha$, we have

$$\lim_{n\to\infty} P(\exists k\ge n,\ \delta_O(k)\ge h(k)\mid \delta_I(n)<\underline{L}(n))=1. \tag{3.1}$$

Indeed,

$$\begin{aligned}
&P(\exists k\ge n,\ \delta_O(k)\ge h(k)\quad \text{or}\quad \delta_I(k)\ge\underline{L}(k))\\
&\ge P(\delta_I(n)\ge\underline{L}(n),\ \text{or}\ \exists k\ge n,\ \delta_O(k)\ge h(k))\\
&\ge P(\delta_I(n)\ge\underline{L}(n))+P(\exists k\ge n,\ \delta_O(k)\ge h(k)\mid\delta_I(n)<\underline{L}(n))P(\delta_I(n)<\underline{L}(n))\\
&\ge 1-(1-P(\exists k\ge n,\ \delta_O(k)\ge h(k)\mid\delta_I(n)<\underline{L}(n)))P(\delta_I(n)<\underline{L}(n)).
\end{aligned} \tag{3.2}$$

We now prove 3.1. For an integer n, assume that $A(b(n)\delta_0)$ is realized. Let X_n be a Poisson random variable with parameter $\lambda_n = |\mathbb{B}(0, n+h(n))\backslash\mathbb{B}(0, n)|$. We realize the cluster $A((b(n) + X_n)\delta_0)$ through three exploration waves, as explained in Sect. 2.1. After the first wave with $b(n)$ explorers, we launch X_n explorers, the green ones, and stop them on $\Sigma := \partial\mathbb{B}(0, n - \underline{L}(n))$. Recall that $W_{n-\underline{L}(n)}(A(b(n)\delta_0); X_n\delta_0, z)$ corresponds to the number of explorers, out of X_n initially at 0, which exit the *explored region* $A(b(n)\delta_0)$ at site z. Under the event $\{\delta_I(n) < \underline{L}(n)\}$, we have for $z \in \Sigma$

$$W_{n-\underline{L}(n)}(A(b(n)\delta_0); X_n\delta_0, z) = M_{n-\underline{L}(n)}(X_n\delta_0, z).$$

The configuration of the *random walks* (associated with the green explorers) stopped on Σ is denoted ζ'. Note that ζ' is independent of $A(b(n)\delta_0)$ and that on the event $\{\delta_I(n) < \underline{L}(n)\}$ we have $\zeta' = \zeta_{n-\underline{L}(n)}$ (with the notation of Sect. 2.1). The key observation now is that $\{\zeta'(z),\ z \in \Sigma\}$ are independent Poisson variables which are also independent of $\delta_I(n)$. Indeed, $\{\zeta'(z),\ z \in \Sigma\}$ deals with the walks associated with the green explorers, whereas $\delta_I(n)$ depends on the $b(n)$ explorers which we launch first. Moreover, the expected value of $\zeta'(z)$ is easy to estimate. Note that $E[\zeta'(z)] = E[X_n] \times \mathbb{P}_0(S(H(\Sigma)) = z)$, and that $|\Sigma|$ is of order n^{d-1}, so that there are two positive constants $\bar{c}$, $\underline{c}$ such that

$$\underline{c}h(n) \le E[\zeta'(z)] \le \bar{c}h(n). \tag{3.3}$$

Also, since $\zeta'(z)$ is a Poisson variable, for $A \ge 1$, we mention an obvious tail estimate.

$$P(\zeta'(z) \ge AE[\zeta'(z)]) \ge \frac{\exp(-\log(A)AE[\zeta'(z)])}{\sqrt{\pi AE[\zeta'(z)]}}. \tag{3.4}$$

For $z \in \Sigma$, we call $\mathrm{cov}(z)$ the event that $\zeta'(z)$ explorers starting on z produce a cluster $A(\emptyset, \zeta'(z)\delta_z)$ which satisfies

$$A(\emptyset, \zeta'(z)\delta_z) \cap \mathbb{B}^c(0, n + 4h(n)) \neq \emptyset. \tag{3.5}$$

Note that the explorers contributing to $\mathrm{cov}(z)$ start on the positions of the *random walks* stopped on $z \in \Sigma$, which are associated with the green explorers. Assume, for a moment, that when $\mathrm{cov}(z)$ happens, there is a tentacle of $A((b(n) + X_n)\delta_0)$ which protrudes $\mathbb{B}(0, n + 4h(n))$. Assume also that under condition on X_n, we have

$$n + 4h(n) \ge R_n + h(R_n), \quad \text{where} \quad R_n = \rho(b(n) + X_n). \tag{3.6}$$

We would deduce that $\delta_O(R_n) \ge h(R_n)$.

We now proceed through four steps. First, we show that (3.5) implies that the final cluster is not inside $\mathbb{B}(0, n + 4h(n))$. Secondly, we estimate the cost of producing a tentacle realizing $\mathrm{cov}(z)$. Then, we establish conditions ensuring (3.6). Finally, we show that for an appropriate choice of α, one event $\mathrm{cov}(z)$ realizes for some $z \in \Sigma$.

Step 1: Coupling By coupling, it is easy to see that for any subset Λ, and $z \in \Sigma$

$$A(\emptyset; \zeta'(z)\delta_z) \subset \Lambda \cup A(\Lambda; \zeta'(z)\delta_z) \subset \Lambda \cup A(\Lambda; \zeta'). \tag{3.7}$$

If we denote by A_2 the cluster after the second exploration wave (see Sect. 2.1), then we have with an equality in law

$$A_2 \cup A(A_2; \zeta') = A((b(n) + X_n)\delta_0). \tag{3.8}$$

Now, using (3.7), (3.8), and (3.5), we conclude that the final cluster is not in $\mathbb{B}(0, n+4h(n))$.

Step 2: Long tentacles To produce $\text{cov}(z)$, we first bring a number of explorers at z proportional to $h(n)$, and force them to make a tentacle normal to Σ at z, with a height $4h(n) + \underline{L}(n)$. More precisely, draw unit cubes centered on the points of the sequence

$$x_n = (\|z\| + n)\frac{z}{\|z\|} \in \mathbb{R}^d \quad (\text{and} \quad \|x_n\| = \|z\| + n).$$

Each such cube contains at least a site of $\mathbb{Z}^d$, say z_n. Note that $\|z_n - z_{n-1}\| \le 2\sqrt{d}+1$, so that we can exhibit a sequence $\{z = y_1, y_2, \dots, y_N\}$ of nearest neighbors in $\mathbb{Z}^d$ such that $\|y_N - z\| \ge 4h(n) + \underline{L}(n)$, with $N \le c(4h(n) + \underline{L}(n))$ for some constant c independent of n. Now, if $\zeta'(z) \ge N$, and if we launch the green explorers stopped on z and force the first N of them them to walk along the sequence $\{y_1, y_2, \dots, y_N\}$, with the k-th explorer settling on y_k, then we realize $\text{cov}(z)$, and

$$\begin{aligned} P(\text{cov}(z)) &\ge P(\zeta'(z) \ge N) \times \left(\frac{1}{2d}\right)^{\sum_{k=1}^N k} \ge P(\zeta'(z) \ge N) \\ &\quad \times \exp(-c(4h(n) + \underline{L}(n))^2). \end{aligned} \tag{3.9}$$

Remark 3.1 In dimension 2, there is a better strategy to build a tentacle. Since we believe that it yields an estimate which is not optimal, we do not give the full proof, but give enough steps of the construction so that the interested reader can easily fill the details.

We first bring a larger number of green explorers at z, about $\frac{h(n)}{\log^2(h(n))} \times h(n)$ of them. The probability of so doing is larger than

$$\exp\left(-C\frac{h(n)}{\log^2(h(n))}\log\left(\frac{h(n)}{\log^2(h(n))}\right) \times h(n)\right) \ge \exp\left(-c\frac{h^2(n)}{\log(h(n))}\right), \tag{3.10}$$

for two positive constants C, c. Then, the explorers are forced to fill sequentially cylindrical compartments of a telescope-like domain that we now describe. Let R be the integer part of $4h(n) + \underline{L}(n)$, and divide $B(z, R)$ into R shells of length $h_1, \dots, h_R$ with for $i = 1, \dots, R$

$$h_i = \frac{R}{i \times \log(R)}, \quad \text{and} \quad H_i = \sum_{j=1}^{i} h_j. \tag{3.11}$$

Choose the sequence of R points of $\mathbb{R}^d$

$$\forall i \in \{1, \ldots, R\} \quad x_i = (\|z\| + H_i)\frac{z}{\|z\|}$$

There is $z_i \in \partial\mathbb{B}(z, H_i)$ with $\|z_i - x_i\| \le 2$. Now, for a constant A_0 appearing in Lemma 1.3 of [2] (and which is independent of R), we bring $n_i := A_0 \pi h_{i+1}^2$ explorers in $\mathbb{B}(z_i, h_i/4)$. Then, with a probability larger than $1 - 1/h_i^2$, they cover the ball $\mathbb{B}(z_i, h_{i+1})$ by Lemma 1.3 of [2]. Now, if we bring additional explorers in $\mathbb{B}(z_i, h_i/4)$, they can reach $\mathbb{B}(z_{i+1}, h_{i+1}/4)$ with a positive probability, say $\exp(-\kappa)$: indeed, they only need to escape a square-like domain centered on z_i, of side-length $h_{i+1}/4$ on the side which separate z_i from z_{i+1}. The cost of the scenario, for which we only described the i-th step, is therefore of order

$$\prod_{i=1}^{R-1}\left(1 - \frac{1}{h_i^2}\right) \times e^{-\kappa \sum_{i=2}^{R} n_i} \times e^{-\kappa \sum_{i=3}^{R} n_i} \ldots \times e^{-\kappa n_R} \ge C \exp\left(-\kappa \sum_{i=2}^{R}(i-1)n_i\right), \tag{3.12}$$

for positive constants κ, C. Note that with the choice of h_i in (3.11), and $n_i = A_0 \pi h_{i+1}^2$, we have with $\underline{L}(n) = h(n)$ and for a positive constant κ'

$$\sum_{i=2}^{R}(i-1)n_i \ge \kappa' \frac{h^2(n)}{\log(h(n))}.$$

Thus, the probability of bringing $h^2(n)/\log^2(n)$ explorers in z, and the probability of building a tentacles of height $5h(n)$ is larger than

$$\exp\left(-\kappa_2 \frac{h^2(n)}{\log(h(n))}\right), \tag{3.13}$$

for some positive constant κ_2.

Step 3: Bounding X_n We impose that $X_n \le 2\lambda_n$.

$$\begin{aligned} X_n &\le 2|\mathbb{B}(0, n + h(n)) \backslash \mathbb{B}(0, n)| \\ &\le |\mathbb{B}(0, n + 2h(n)) \backslash \mathbb{B}(0, n)| \Longrightarrow X_n + b(n) \\ &\le |\mathbb{B}(0, n + 2h(n))| \Longrightarrow R_n \le n + 2h(n). \end{aligned} \tag{3.14}$$

The conclusion of (3.14) implies also, for n large enough, that $h(R_n) \leq 2h(n)$, and this implies (3.6). Note that since X_n is a Poisson variable of mean λ_n, there is a constant c, such that

$$P(X_n > 2\lambda_n) \leq \exp(-ch(n)n^{d-1}). \tag{3.15}$$

Step 4: Many possible tentacles We summarize Step 1 to Step 3, as establishing that

$$\begin{aligned}\{X_n \leq 2\lambda_n,\ \exists z \in \Sigma\ \mathrm{cov}(z),\ \delta_I(n) < \underline{L}(n)\} &\subset \{\delta_O(R_n) \geq h(R_n),\ \delta_I(n) < \underline{L}(n)\}\\ &\subset \{\exists k > n, \delta_O(k) \geq h(k),\ \delta_I(n) < \underline{L}(n)\}.\end{aligned} \tag{3.16}$$

Taking probability on both sides of (3.16), and dividing by $P(\delta_I(n) < \underline{L}(n))$, we obtain

$$P(\exists k > n, \delta_O(k) \geq h(k) | \delta_I(n) < \underline{L}(n)) \geq P(X_n \leq 2\lambda_n,\ \exists z \in \Sigma\quad \mathrm{cov}(z)). \tag{3.17}$$

Thus,

$$P(\exists k > n, \delta_O(k) \geq h(k) | \delta_I(n) < \underline{L}(n)) \geq P\left(\bigcup_{z\in\Sigma} \mathrm{cov}(z)\right) - P(X_n > 2\lambda_n). \tag{3.18}$$

Using now (3.9), (3.3), the lower bound (3.4), and that $|\Sigma|$ is of order n^{d-1}, there are some positive constants κ, κ' such that

$$\begin{aligned}P\left(\bigcup_{z\in\Sigma} \mathrm{cov}(z)\right) &\geq\ 1 - \prod_{z\in\Sigma}(1 - P(\mathrm{cov}(z) \cap\ \{\zeta'(z) > c(4h(n) + \underline{L}(n))\}))\\ &\geq\ 1 - \exp(-\kappa' n^{d-1} \exp(-\kappa(4h(n) + \underline{L}(n))^2)).\end{aligned} \tag{3.19}$$

Now, for $\alpha > 0$ small enough, and $\underline{L}(n) = h(n) = \alpha\sqrt{\log(n)}$, we have

$$\lim_{n\to\infty} n^{d-1} \exp(-\kappa(h^2(n) + h(n))) = \infty. \tag{3.20}$$

The proof is now completed. In dimension 2, the estimate (3.19) has to be replaced with (3.13).

4 The inner error

In this section, we prove Theorem 1.4. We show that in the process of going from a cluster of volume $b(n)$ to one of volume $b(2n)$, chances tend to one as n tends to infinity, that there appears a cluster A whose inner error is larger than $\alpha\sqrt{\log(\rho(|A|))}$

for some positive α independent of n. To do so, we launch many exploration waves, each one is made up of a Poisson number of explorers.

We proceed inductively. For positive reals α, β, to be chosen later, we set $h(n) = \alpha\sqrt{\log(n)}$, and $\bar{L}(n) = \beta\sqrt{\log(n)}$. $\bar{L}(n)$ will refer to an outer radius. Let $\{\mathcal{G}_n,\ n \geq 0\}$ denotes the natural filtration associated with the evolution by waves.

First, we launch $b(n)$ explorers. Assume that explorers of wave $k-1$ have been launched, and are settled. Knowing $\mathcal{G}_{k-1}$, the size of the k-th wave, denoted X_k, is a Poisson variable of parameter

$$\begin{aligned} \lambda(k) = |\mathbb{B}(0, R_{k-1} + 2h(R_{k-1}))\backslash\mathbb{B}(0, R_{k-1}))|, \quad \text{where} \\ R_{k-1} = \rho(b(n) + X_1 + \cdots + X_{k-1}). \end{aligned} \tag{4.1}$$

Since R_{k-1} is of order n, each wave fills approximately a peel of width $2h(n)$, and $n/2h(n)$ waves fill approximately $\mathbb{B}(0, 2n)$. We prove in this section that for an appropriate α

$$\lim_{n\to\infty} P\left(\bigcup_{1\leq k<n/2h(n)} \{\delta_I(R_k) > \alpha\sqrt{\log(R_k)}\}\right) = 1. \tag{4.2}$$

We now proceed with estimating the probability of observing a *deep* hole after each exploration waves. We set $\mathcal{A}_k = \{\delta_I(R_k) > \alpha\sqrt{\log(R_k)}\}$.

On the holes left after wave $k-1$ Observe that by definition, on $\mathcal{A}_{k-1}^c$

$$\mathbb{B}(0, R_{k-1} - h(R_{k-1})) \subset A(b(n) + X_1 + \cdots + X_{k-1}),$$

which implies that

$$\begin{aligned} (\mathbb{B}(0, R_{k-1})\backslash\mathbb{B}(0, R_{k-1} - h(R_{k-1})) \cup \partial\mathbb{B}(0, R_{k-1})) \\ \cap A(b(n) + X_1 + \cdots + X_{k-1}) \neq \emptyset. \end{aligned} \tag{4.3}$$

Choose any Z_k in the intersection of the non-empty set of (4.3), and note that

$$R_{k-1} - h(R_{k-1}) \leq \|Z_k\| \leq R_{k-1} + 1. \tag{4.4}$$

Recall that we have defined $\Sigma(Z_k) := \partial\mathbb{B}(0, \|Z_k\|)$. We launch the X_k explorers, that we name the *green explorers*, and we stop them as they reach $\Sigma(Z_k)$. The green explorers which settle before reaching $\Sigma(Z_k)$ play no role here, and we bound the number of green explorers stopped on some region $\Lambda \in \Sigma(Z_k)$, by the number of corresponding random walks exiting $\Sigma(Z_k)$ on Λ. Thus, if we choose

$$\Lambda_k = \mathbb{B}(Z_k, \bar{L}(R_k)) \cap \Sigma(Z_k), \quad \text{and} \quad \Lambda'_k = \Sigma(Z_k)\backslash\Lambda_k, \tag{4.5}$$

and if we denote

$$I_k = \{N_{Z_k}(X_k\delta_0, \Lambda_k, \infty) = 0, \quad N_{Z_k}(X_k\delta_0, \Lambda'_k, 7\bar{L}(R_{k-1})) = 0\} \tag{4.6}$$

then, on the event I_k, green explorers either exit a ball of radius $\|Z_k\| + 7\bar{L}(R_{k-1})$, or do not visit Z_k. In other words,

$$I_k \subset \{R_k + \delta_O(R_k) \geq \|Z_k\| + 7\bar{L}(R_{k-1})\} \cup \{\delta_I(R_k) > R_k - \|Z_k\|\}.$$

In order to conclude that $\{\delta_I(R_k) \geq h(R_k)\}$ or $\{\delta_O(R_k) \geq \bar{L}(R_k)\}$, we need to find conditions on X_k that guarantee that

$$R_k - \|Z_k\| \geq h(R_k), \quad \text{and} \quad \|Z_k\| - R_k + 7\bar{L}(R_{k-1}) \geq \bar{L}(R_k). \tag{4.7}$$

Conditions on X_k fulfilling (4.7) We call

$$\mathcal{C}_k = \{2\lambda(k) \geq X_k\} \cap \left\{X_k \geq \frac{2}{3}\lambda(k)\right\}.$$

On the one hand, if $X_k \geq \frac{2}{3}\lambda(k)$, then

$$\begin{aligned} X_k &\geq \frac{2}{3}\left|\mathbb{B}(0, R_{k-1} + 2h(R_{k-1}))\backslash\mathbb{B}(0, R_{k-1})\right| \\ &\geq \left|\mathbb{B}(0, R_{k-1} + \frac{4}{3}h(R_{k-1}))\backslash\mathbb{B}(0, R_{k-1})\right| \Longrightarrow R_k \geq R_{k-1} + \frac{4}{3}h(R_{k-1}). \end{aligned} \tag{4.8}$$

On the other hand, if $X_k \leq 2\lambda_k$, then

$$\begin{aligned} X_k &\leq 2\left|\mathbb{B}(0, R_{k-1} + 2h(R_{k-1}))\backslash\mathbb{B}(0, R_{k-1})\right| \\ &\leq \left|\mathbb{B}(0, R_{k-1} + 4h(R_{k-1}))\backslash\mathbb{B}(0, R_{k-1})\right| \Longrightarrow R_k \leq R_{k-1} + 4h(R_{k-1}). \end{aligned} \tag{4.9}$$

Now, for x large enough, the following implication is obvious

$$x \leq y + 4h(y) \Longrightarrow h(x) \leq \frac{4}{3}h(y) - 1 \quad \text{and} \quad \bar{L}(x) \leq 2\bar{L}(y). \tag{4.10}$$

If n is large enough, (4.10) and (4.9) imply that $h(R_k) \leq \frac{4}{3}h(R_{k-1}) - 1$, which in turn, with (4.8), yields

$$R_k \geq R_{k-1} + 1 + h(R_k). \tag{4.11}$$

Also, $\bar{L}(R_k) \leq 2\bar{L}(R_{k-1})$, and when α is small enough, then $\bar{L}(R_{k-1}) \geq h(R_{k-1})$. Recalling (4.9) we have

$$R_{k-1} - h(R_{k-1}) + 7\bar{L}(R_{k-1}) - \bar{L}(R_k) \geq R_{k-1} + 4h(R_{k-1}) \geq R_k. \tag{4.12}$$

Thus, if $\mathcal{C}_k \cap \mathcal{A}_{k-1}^c$ holds, then (4.4), (4.11) and (4.12) imply that conditions (4.7) holds.

On a deep hole in one shell We choose an integer $k < n/2h(n)$. We have seen that

$$\mathcal{A}_{k-1}^c \cap I_k \cap \mathcal{C}_k \subset \mathcal{A}_{k-1}^c \cap (\mathcal{A}_k \cup \{\delta_0(R_k) \geq \bar{L}(R_k)\}). \tag{4.13}$$

Taking conditional probabilities on both sides of (4.13), we obtain,

$$\begin{aligned} &\mathbb{1}_{\mathcal{A}_{k-1}^c} P(\mathcal{A}_k^c \cap \mathcal{C}_k \,|\mathcal{G}_{k-1}) \\ &\quad = \mathbb{1}_{\mathcal{A}_{k-1}^c}(P(\mathcal{C}_k) - P(\mathcal{A}_k \cap \mathcal{C}_k|\mathcal{G}_{k-1})) \\ &\quad \leq \mathbb{1}_{\mathcal{A}_{k-1}^c}(P(\mathcal{C}_k) - P(I_k \cap \mathcal{C}_k|\mathcal{G}_{k-1}) + P(\{\delta_0(R_k) \geq \bar{L}(R_k)\}|\mathcal{G}_{k-1})) \\ &\quad \leq P(\{\delta_0(R_k) \geq \bar{L}(R_k)\}|\mathcal{G}_{k-1}) + \mathbb{1}_{\mathcal{A}_{k-1}^c}(1 - P(I_k|\mathcal{G}_{k-1})). \end{aligned} \tag{4.14}$$

Now, we invoke Lemma 2.3 with $\|z\|$, R and λ respectively of order n, $\bar{L}(n)$, and $h(n)n^{d-1}$. As a consequence, we have on $\mathcal{C}_k \cap \mathcal{A}_{k-1}^c$ for a constant κ,

$$\inf_{k \leq n/2h(n)} P(I_k \mid \mathcal{G}_{k-1}) \geq \exp(-\kappa h(n)\bar{L}(n)). \tag{4.15}$$

If we denote by N the integer part of $n/2h(n)$, and proceed inductively, we obtain

$$\begin{aligned} &P\left(\cup_{k\leq N}\mathcal{A}_k\right) - P\left(\cap_{k\leq N}\mathcal{C}_k\right) \\ &\quad \geq -P(\forall k \leq N,\ \mathcal{A}_k^c \cap \mathcal{C}_k) \\ &\quad \geq -E[\mathbb{1}]_{\forall k<N,\ \mathcal{A}_k^c\cap\mathcal{C}_k} P(\mathcal{A}_N^c \cap \mathcal{C}_N \,|\mathcal{G}_{N-1}) \\ &\quad \geq -E[\mathbb{1}_{\forall k<N,\ \mathcal{A}_k^c\cap\mathcal{C}_k}](P(\{\delta_0(R_N) \geq \bar{L}(R_N)\}|\mathcal{G}_{N-1}) + 1 - P(I_N|\mathcal{G}_{N-1})) \\ &\quad \geq -P(\{\delta_0(R_N) \geq \bar{L}(R_N)\}) - (1 - \exp(-\kappa h(n)\bar{L}(n)))P(\forall k < N,\ \mathcal{A}_k^c \cap \mathcal{C}_k) \\ &\quad \geq -\sum_{k\leq N} P(\{\delta_0(R_k) \geq \bar{L}(R_k)\}) - (1 - \exp(-\kappa h(n)\bar{L}(n)))^N. \end{aligned}$$

Thus,

$$\begin{aligned} P(\cup_{k\leq N}\mathcal{A}_k) \ &\geq 1 - \sum_{k\leq N}(P(\{\delta_0(R_k) \geq \bar{L}(R_k)\}) \\ &\qquad + P(\mathcal{C}_k^c)) - (1 - \exp(-\kappa h(n)\bar{L}(n)))^N. \end{aligned} \tag{4.16}$$

Now, we have established in [2], that for β large enough, the probability of $\{\delta_0(R_k) \geq \bar{L}(R_k)\}$ decays faster than any power in n, whereas the fact that X_k is Poisson implies that for some constant c, we have $P(\mathcal{C}_k^c) \leq \exp(-ch(n)n^{d-1})$. The last term on the last display of (4.16) tends to 0 if

$$\lim_{n\to\infty} \frac{n}{2h(n)} \exp(-\kappa h(n)\bar{L}(n)) = \infty. \tag{4.17}$$

In dimension 2 or more, (4.17) holds for α small enough.

5 Proof of Proposition 1.6

The proof is a direct corollary of formula (3.11) of [2]. We consider actually *tiles* of size 1, that is site of $\mathbb{Z}^d$. Inequality (3.8) of [2] shows that for some constant c_d (depending only on dimension) and $R = \|z\| < n$, we have

$$E[W_R(\emptyset, b(n)\delta_0, z)] \geq c_d(n - \|z\|). \tag{5.1}$$

Inequality (3.10) of [2] is written a little differently as

$$P(W_R(\emptyset, b(n)\delta_0, z) = 0) \leq \begin{cases} \exp(-\lambda\kappa_2(n - \|z\|) + \lambda^2 c_2' \log(n)) & \text{for} \quad d = 2, \\ \exp(-\lambda\kappa_d(n - \|z\|) + \lambda^2 c_d') & \text{for} \quad d \geq 3. \end{cases} \tag{5.2}$$

As we optimize (5.2) in $\lambda > 0$, we obtain (1.11).

References

1. Asselah, A., Gaudillière, A.: From logarithmic to subdiffusive polynomial fluctuations for internal DLA and related growth models. Ann. Probab. (2010). arXiv:1009.2838
2. Asselah, A., Gaudillière, A.: Sub-logarithmic fluctuations for internal DLA. Ann. Prob. (2010). arXiv:1011.4592
3. Diaconis, P., Fulton, W.: A growth model, a game, an algebra, Lagrange inversion, and characteristic classes. Rend. Sem. Mat. Univ. Politec. Torino **49**(1991), 95–119 (1993)
4. Jerison, D., Levine, L.: Sheffield, S.: Logarithmic fluctuations for internal DLA (2010). arXiv:1010.2483
5. Jerison, D., Levine, L., Sheffield, S.: Internal DLA in higher dimensions (2010). arXiv:1012.3453
6. Jerison, D., Levine, L., Sheffield, S.: Internal DLA and the Gaussian free field (2011). arXiv:1101.0596
7. Lawler, G., Bramson, M., Griffeath, D.: Internal diffusion limited aggregation. Ann. Probab. **20**(4), 2117–2140 (1992)
8. Lawler, G.: Subdiffusive fluctuations for internal diffusion limited aggregation. Ann. Probab. **23**(1), 71–86 (1995)