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Exponential and double exponential tails for maximum of two-dimensional discrete Gaussian free field

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Abstract We study the tail behavior for the maximum of discrete Gaussian free field on a 2D box with Dirichlet boundary condition after centering by its expectation. We show that it exhibits an exponential decay for the right tail and a double exponential decay for the left tail. In particular, our result implies that the variance of the maximum is of order 1, improving an $o(\log n)$ bound by Chatterjee (Chaos, concentration, and multiple valleys, 2008) and confirming a folklore conjecture. An important ingredient for our proof is a result of Bramson and Zeitouni (Commun. Pure Appl. Math, 2010), who proved the tightness of the centered maximum together with an evaluation of the expectation up to an additive constant.

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1 Introduction

Denote by $A_n \subset \mathbb{Z}^2$ a box of side length n, i.e., $A = \{(x, y) \in \mathbb{Z}^2 : 0 \leq x, y \leq n\}$, and let $\partial A_n = \{v \in A_n : \exists u \in \mathbb{Z}^2 \backslash A_n : v \sim u\}$. The discrete Gaussian free field (GFF) $\{\eta_v : v \in A_n\}$ on A_n with Dirichlet boundary condition, is then defined to be a mean zero Gaussian process which takes value 0 on ∂A_n and satisfies the following Markov field condition for all $v \in A_n \backslash \partial A_n$: η_v is distributed as a Gaussian variable with variance 1 and mean equal to the average over the neighbors given the GFF on $A_n \backslash \{v\}$ (see later for a definition of GFF using Green functions). Throughout the

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paper, we use the notation

$$M_n = \sup_{v \in A_n} \eta_v. \tag{1}$$

We prove the following tail behavior for M_n.

Theorem 1.1 *There exist absolute constants $C, c > 0$ so that for all $n \in \mathbb{N}$ and $0 \le \lambda \le (\log n)^{2/3}$*

$$c\mathrm{e}^{-C\lambda} \le \mathbb{P}(M_n \ge \mathbb{E}M_n + \lambda) \le C\mathrm{e}^{-c\lambda}$$
$$c\mathrm{e}^{-C\mathrm{e}^{C\lambda}} \le \mathbb{P}(M_n \le \mathbb{E}M_n - \lambda) \le C\mathrm{e}^{-c\mathrm{e}^{c\lambda}}$$

The preceding theorem gives the tail behavior when the deviation is less than $(\log n)^{2/3}$. For $\lambda \ge (\log n)^{2/3}$, by isoperimetric inequality for general Gaussian processes (see, e.g., Ledoux [16, Theorem 7.1, Eq. (7.4)]) and the simple fact that $\max_v \mathrm{Var}\eta_v = 2\log n/\pi + O(1)$ (see Lemma 2.2), we have

$$\mathbb{P}(|M_n - \mathbb{E}M_n| \ge \lambda) \le 2\,\mathrm{e}^{-c\lambda^2/\log n}, \quad \text{for an absolute constant } c > 0.$$

Combined with Theorem 1.1, this immediately gives the order of the variance for M_n. Before stating the result, let us specify some conventions for notations throughout the paper. The letters c and C denote absolute positive constants, whose values might vary from line to line. By convention, we denote by C large constants and by c small constants. Other absolute constants that appeared are fixed once and for all. If there exists an absolute constant $C > 0$ such that $a_n = Cb_n$ for all $n \ge 1$, we write $a_n = O(b_n)$; we write $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ as well as $b_n = O(a_n)$; if $\limsup_{n\to\infty} a_n/b_n \to 0$, we write $a_n = o(b_n)$. We are now ready to state the corollary.

Corollary 1.2 *We have that* $\mathrm{Var}M_n = \Theta(1)$.

Corollary 1.2 improves an $o(\log n)$ bound on the variance due to Chatterjee [7], thereby confirming a folklore conjecture (see Question (4) of [7]). An important ingredient for our proof is the following result on the tightness of the maximum of the GFF on 2D box due to Bramson and Zeitouni [6].

Theorem 1.3 [6] *The sequence of random variables $M_n - \mathbb{E}M_n$ is tight and*

$$\mathbb{E}M_n = 2\sqrt{2/\pi}\left(\log n - \tfrac{3}{8\log 2}\log\log n\right) + O(1).$$

Previously to [6], Bolthausen et al. [3] proved that $(M_n - \mathbb{E}M_n)$ is tight along a deterministic subsequence $(n_k)_{k\in\mathbb{N}}$. Earlier works on the extremal values of GFF include Bolthausen et al. [2] who established the asymptotics for M_n, and Daviaud [8] who studied the extremes for the GFF.

We compare our results with tail behavior for the maximum of the GFF on a binary tree. Interestingly, in the case of tree, the maximum exhibits an exponential decay for the right tail, but a Gaussian type decay for the left tail as opposed to the double exponential decay for 2D box. This is because in the case of 2D box, the Dirichlet boundary condition decouples the GFF near the boundary such that the GFF behaves almost independently close to the boundary. The same phenomenon also occurs for the event that all the GFFs are nonnegative: for a binary tree of height n the probability is about $\mathrm{e}^{-\Theta(n^2)}$, and for a box of side length n the probability is about $\mathrm{e}^{-\Theta(n)}$ (see Deuschel [9]).

Much more was known about the maximal displacement of branching Brownian motion (BBM). In their classical paper, Kolmogorov et al. [13] studied its connection with the so-called KPP-equation, from which it could be deduced that both the right and left tails exhibit exponential types of decay. The probabilistic interpretation of KPP-equation in terms of BBM was further exploited by Bramson [4]. Then the precise asymptotic tails were computed, and in particular a polynomial prefactor for the right tail was detected (this appears to be fundamentally different from the tail of Gumble distribution, which arise from the maximum of, say, i.i.d. Gaussian variables). See, e.g., Bramson [5] and Harris [12] for the right tail, and see Arguin et al. [1] for the left tail (the argument is due to De Lellis). In addition, Lalley and Sellke [14] obtained an integral representation for the limiting law of the centered maximum.

We now give the definition of GFF using the connection with random walks (in particular, Green functions). Consider a connected graph $G = (V, E)$. For $U \subset V$, the Green function $G_U(\cdot, \cdot)$ of the discrete Laplacian is given by

$$G_U(x, y) = \mathbb{E}_x\left(\sum_{k=0}^{\tau_U - 1} \mathbf{1}\{\mathbf{S_k} = \mathbf{y}\}\right), \quad \text{for all } x, y \in V, \tag{2}$$

where τ_U is the hitting time to set U for random walk (S_k), defined by (the notation applies throughout the paper)

$$\tau_U = \min\{k \geq 0 : S_k \in U\}. \tag{3}$$

The GFF $\{\eta_v : v \in V\}$ with Dirichlet boundary on U is then defined to be a mean zero Gaussian process indexed by V such that the covariance matrix is given by Green function $(G_U(x, y))_{x,y \in V}$ (In general graph, it is typical to normalize the Green function by the degree of the target vertex y. In the case of 2D lattices, this normalization is usually dropped since the degrees are constant). It is clear to see that $\eta_v = 0$ for all $v \in U$.

2 Proofs

In this section, we prove Theorem 1.1. We start with a brief discussion on the proof strategy, and then demonstrate the upper (lower) bounds for the right (left) tails in the subsequent four subsections.

2.1 A word on proof strategy

Our proof typically employs a two-level structure which involves either a partitioning or a packing for a 2D box A_n by (slightly) smaller boxes. In all the proofs, we use Theorem 1.3 to control the behavior in small boxes, and study "typical" events on small boxes with probability strictly bounded away from 0 and 1. The large deviation bounds typically come from gluing the small boxes together to a big box, with the probability either inverse proportional to the number of small boxes or exponentially small in the number of boxes.

By Theorem 1.3, there exists a universal constant $\kappa > 0$ such that for all $n \geq 3n'$

$$2\sqrt{2/\pi}\log(n/n') - \tfrac{3\sqrt{2/\pi}}{4\log 2}\log(\log n/\log n') - \kappa \leq \mathbb{E}M_n - \mathbb{E}M_{n'} \\ \leq 2\sqrt{2/\pi}\log(n/n') + \kappa. \tag{4}$$

That is to say, in order to observe a difference of λ in the expectation for the maximum, the side length of the box has to increase (decrease) by a factor of $\exp(\Theta(\lambda))$. This suggests that the number of small boxes shall be $\exp(\Theta(\lambda))$ in our two-level structure. Depending on how the large deviation arises, this will yield a tail of either exponential or double exponential decay.

In order to construct the two-level structure, we use repeatedly the decomposition of Gaussian process: for a joint Gaussian process (X, Y), we can write X as a sum of a (linear) function of Y and an independent Gaussian process X'. Here, we used a crucial fact that Gaussian processes possess linear structures where orthogonality implies independence. Furthermore, the next well-known property specific to GFF proves to be quite useful (see Dynkin [10, Theorem 1.2.2]).

Lemma 2.1 *Let $\{\eta_v\}_{v\in V}$ be a GFF on a graph $G = (V, E)$. For $U \subset V$, define τ_U as in* (3). *Then, for $v \in V$, we have*

$$\mathbb{E}(\eta_v \mid \eta_u, u \in U) = \sum_{u\in U}\mathbb{P}_v(S_{\tau_U} = u)\cdot \eta_u.$$

2.2 Upper bound on the right tail

In this subsection, we prove that for an absolute constant $C, \lambda_0 > 0$

$$\mathbb{P}(M_n - \mathbb{E}M_n \geq \lambda) \leq C\mathrm{e}^{-\sqrt{\pi/2}\lambda}, \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \lambda \geq \lambda_0. \tag{5}$$

Note that we could choose λ_0 arbitrarily large by adjusting the constant C in Theorem 1.1. Let $N = n\lceil \mathrm{e}^{\sqrt{\pi/8}(\lambda-\kappa-\alpha)}\rceil$, where κ is from (4) and $\alpha > 0$ will be selected later. Denote by $p = p_\alpha = \mathrm{e}^{-\sqrt{\pi/2}(\lambda-\kappa-\alpha)}$ and $k = \lceil \mathrm{e}^{\sqrt{\pi/8}(\lambda-\kappa-\alpha)}\rceil$. It suffices to prove that $\mathbb{P}(M_n - \mathbb{E}M_n \geq \lambda) \leq p$, and we prove it by contradiction. To this end, we assume that

$$\mathbb{P}(M_n - \mathbb{E}M_n \geq \lambda) > p \tag{6}$$

and try to derive a contradiction.

Now, consider an $N \times N$ 2D box A_N and let $\{\eta_v : v \in A_N\}$ be a GFF on A_N with Dirichlet boundary condition. We partition A_N into k^2 boxes of side length n and denote by $\mathcal{B}$ the collection of these boxes. We abuse the notation $\partial\mathcal{B}$ to denote the union of the boundary sets of the smaller boxes in $\mathcal{B}$. For $B \in \mathcal{B}$, we let $\{g_v^B : v \in B\}$ be a GFF on B with Dirichlet boundary condition and we let $\{\{g_v^B : v \in B\}\}_{B\in\mathcal{B}}$ be independent from each other and independent from $\{\eta_v : v \in \partial\mathcal{B}\}$. Using the decomposition of Gaussian process, we can write that for every $v \in B \subseteq A_N$

$$\eta_v = g_v^B + \mathbb{E}(\eta_v \mid \{\eta_u : u \in \partial\mathcal{B}\}). \tag{7}$$

Denote by $\phi_v = \mathbb{E}(\eta_v \mid \{\eta_u : u \in \partial\mathcal{B}\})$. We note that ϕ_v is a convex combination of $\{\eta_u : u \in \partial\mathcal{B}\}$ where the linear coefficients are deterministic. Thus,

$$\{\phi_v : v \in A_N\} \text{ is independent of } \{\{g_v^B : v \in B\} : B \in \mathcal{B}\}. \tag{8}$$

Denote by $M_B = \sup_{v\in B} g_v^B$. It is clear that $\{M_B : B \in \mathcal{B}\}$ is a collection of i.i.d. random variables and each of them is distributed as M_n. Therefore, by (6), we obtain that $\mathbb{P}(M_B \geq \mathbb{E}M_n + \lambda) \geq p$. Using independence, we get

$$\mathbb{P}\left(\sup_{B\in\mathcal{B}} \sup_{v\in B} g_v^B \geq \mathbb{E}M_n + \lambda\right) = \mathbb{P}\left(\sup_{B\in\mathcal{B}} M_B \geq \mathbb{E}M_n + \lambda\right) \geq 1/2.$$

Let $\chi \in B \subseteq A_N$ such that $g_\chi^B = \sup_{B\in\mathcal{B}} \sup_{v\in B} g_v^B$. We see that χ is random (obviously) and independent of $\{\phi_v : v \in \partial\mathcal{B}\}$ by (8). Therefore, we obtain

$$\mathbb{P}\left(\sup_{v\in A_N} \eta_v \geq \mathbb{E}M_n + \lambda\right) \geq \mathbb{P}\left(g_\chi^B \geq \mathbb{E}M_n + \lambda, \phi_\chi \geq 0) \geq (1/2) \min_{v\in A_N} \mathbb{P}(\phi_v \geq 0\right) = 1/4. \tag{9}$$

Recalling (4) and our definition of N, we thus derive that

$$\mathbb{P}(M_N - \mathbb{E}M_N \geq \alpha) \geq 1/4.$$

However, Theorem 1.3 implies that there exists a universal constant $\alpha(1/4) > 0$ such that $\mathbb{P}(M_n - \mathbb{E}M_n \geq \alpha(1/4)) < 1/4$ for all $n \in \mathbb{N}$. Setting $\alpha = \alpha(1/4)$, we arrive at a contradiction and thus show that (6) cannot hold, thereby establishing (5).

2.3 Lower bound on the right tail

In this subsection, we analyze the lower bound on the right tail and aim to prove that for absolute constant $c, \lambda_0 > 0$

$$\mathbb{P}(M_n - \mathbb{E}\, M_n \geq \lambda) \geq \tfrac{c}{\lambda}\, \mathrm{e}^{-8\sqrt{2\pi}\lambda}, \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \lambda_0 \leq \lambda \leq (\log n)^{2/3}. \tag{10}$$

To prove the above lower bound, we consider a box $A_{n'}$ of side length $n' = n\mathrm{e}^{-\beta\lambda}$ in the center of A_n, where $\beta > 0$ is to be selected (note that since $\lambda \le (\log n)^{2/3}$, we have $n' \ge 1$ is well defined). Let $\{g_v : v \in A_{n'}\}$ be a Gaussian free field on $A_{n'}$ with Dirichlet boundary condition and independent from $\{\eta_v : v \in \partial A_{n'}\}$. Analogous to (7), we can write that

$$\eta_v = g_v + \phi_v, \quad \text{for all } v \in A_{n'},$$

where $\phi_v = \mathbb{E}(\eta_v \mid \{\eta_u : u \in \partial A_{n'}\})$ is a convex combination of $\{\eta_u : u \in \partial A_{n'}\}$. We wish to estimate the variance of ϕ_v. For this purpose, we need the following standard estimates on Green functions for random walks in 2D lattices. See, e.g., [15, Proposition 4.6.2, Theorem. 4.4.4] for a reference.

Lemma 2.2 *For $A \subset \mathbb{Z}^2$, consider a random walk (S_t) on $\mathbb{Z}^2$ and define $\tau_{\partial A} = \min\{j \ge 0 : S_j \in \partial A\}$ be the hitting time to ∂A. For $u, v \in A$, let $G_{\partial A}(u, v)$ be the Green function as in (2). For a certain nonnegative function $a(\cdot,\cdot)$ such that $a(x, x) = 0$ and $a(x, y) = \frac{2}{\pi}\log|x - y| + \frac{2\gamma\log 8}{\pi} + O(|x - y|^{-2})$, where γ is Euler's constant. Then, we have*

$$G_{\partial A}(u, v) = \mathbb{E}_u(a(S_{\tau_{\partial A}}, v)) - a(u, v).$$

By the preceding lemma, we infer that for any $u, w \in \partial A_{n'}$,

$$\mathrm{Cov}(\eta_u, \eta_w) = G_{\partial A_n}(u, w) \ge \tfrac{2}{\pi}\beta\lambda + O(1).$$

Since ϕ_v is a convex combination of $\{\eta_u : u \in \partial A_{n'}\}$, this implies that for all $v \in A_{n'}$

$$\mathrm{Var}\phi_v \ge \tfrac{2}{\pi}\beta\lambda + O(1). \tag{11}$$

By Theorem 1.3, there exists an absolute constant $\alpha(1/2)$ such that

$$\mathbb{P}(M_n - \mathbb{E}M_n \ge -\alpha(1/2)) \ge 1/2 \quad \text{for all } n \in \mathbb{N}. \tag{12}$$

Let $\chi \in A_{n'}$ such that $g_\chi = \sup_{v \in A_{n'}} g_v$. Recalling that $|\mathbb{E}M_n - \mathbb{E}M_{n'}| \le 2\sqrt{2/\pi}\beta\lambda + O(\log\beta\lambda) + \kappa$ and that $\lambda \ge \lambda_0$, we obtain that

$$\begin{aligned}\mathbb{P}\left(\sup_{v\in A_n} \eta_v \ge \mathbb{E}M_n + \lambda\right) &\ge \mathbb{P}(g_\chi \ge \mathbb{E}M_{n'} - \alpha(1/2), \phi_\chi \ge \alpha(1/2)\\ &\quad + \kappa + (2\sqrt{2/\pi}\beta + 1)\lambda)\\ &\ge \frac{1}{2}\frac{\pi}{\sqrt{\beta\lambda + O(1)}}\int\limits_{z \ge \alpha(1/2)+\kappa+(2\sqrt{2/\pi}\beta+1)\lambda} \mathrm{e}^{-\frac{z^2}{2\beta\lambda/\pi + O(1)}}dz\\ &\ge \frac{c}{\sqrt{\lambda}}\mathrm{e}^{-\pi(2\sqrt{2/\pi}\beta+1)^2\lambda/\beta},\end{aligned}$$

where the first inequality follows from (11) and the independence between χ and $\{\phi_v : v \in A_{n'}\}$ (analogous to (8)), and in the second inequality $c > 0$ is a small absolute constant. Setting $\beta = \sqrt{\pi/8}$, we obtain the desired estimate (10).

2.4 Upper bound on the left tail

In this subsection, we give the upper bound for the lower tail of the maximum and prove the following for absolute constants $C, c, \lambda_0 > 0$.

$$\mathbb{P}(M_n - \mathbb{E}M_n \le -\lambda) \le C\mathrm{e}^{-ce^{c\lambda}}, \quad \text{for all} \quad n \in \mathbb{N} \quad \text{and} \quad \lambda_0 \le \lambda \le (\log n)^{2/3}. \tag{13}$$

Let $\alpha = \alpha(1/2)$ be defined as in (12). Denote by $r = n\exp(-\sqrt{\pi/8}(\lambda - \alpha - \kappa - 4))$ and $\ell = n\exp(-\sqrt{\pi/8}(\lambda - \alpha - \kappa - 4)/3)$. Assume that the left bottom corner of A_n is the origin $o = (0, 0)$. Define $o_i = (i\ell, 2r)$ for $1 \le i \le m = \lfloor n/2\ell \rfloor$. Let $\mathcal{C}_i$ be a discrete ball of radius r centered at o_i and let $B_i \subset \mathcal{C}(i)$ be a box of side length $r/8$ centered at o_i. Let $\mathfrak{C} = \{\mathcal{C}_i : 1 \le i \le m\}$ and $\mathcal{B} = \{B_i : 1 \le i \le m\}$. Analogous to (7), we can write

$$\eta_v = g_v^B + \phi_v, \quad \text{for all } v \in B \subseteq \mathcal{C} \in \mathfrak{C},$$

where $\{g_v^B : v \in B\}$ is the projection of the GFF on $\mathcal{C}$ with Dirichlet boundary condition on $\partial\mathcal{C}$, and $\{\{g_v^B : v \in B\} : B \in \mathcal{B}\}$ are independent of each other and of $\{\eta_v : v \in \partial\mathfrak{C}\}$ (here $\partial\mathfrak{C} = \cup_{\mathcal{C}\in\mathfrak{C}}\partial\mathcal{C}$), and $\phi_v = \mathbb{E}(\eta_v \mid \{\eta_u : u \in \partial\mathfrak{C}\})$ is a convex combination of $\{\eta_u : u \in \partial\mathfrak{C}\}$. For every $B \in \mathcal{B}$, define $\chi_B \in B$ such that

$$g_{\chi_B}^B = \sup_{v\in B} g_v^B.$$

Recalling (4), we get that $\mathbb{E}M_n - \mathbb{E}M_{r/8} \le \lambda - \alpha$ (here we assume λ_0 is large enough such that $n > r/8$).

Using an analogous derivation of (9), we get that

$$\mathbb{P}\left(g_{\chi_B}^B \ge \mathbb{E}M_n - \lambda\right) \ge 1/4,$$

where we used definition of α in (12). Let $W = \{\chi_B : g_{\chi_B}^B \ge \mathbb{E}M_n - \lambda, B \in \mathcal{B}\}$. By independence, a standard concentration argument gives that for an absolute constant $c > 0$

$$\mathbb{P}(|W| \le \tfrac{1}{8}m) \le \mathrm{e}^{-cm}. \tag{14}$$

It remains to study the process $\{\phi_v : v \in W\}$. If there exists $v \in W$ such that $\phi_v > 0$, we have $\sup_{u\in A_n} \eta_u > \mathbb{E}M_n - \lambda$. Thanks to independence, it then suffices to prove the following lemma.

Lemma 2.3 *Let $U \subset \cup_{B \in \mathcal{B}} B$ such that $|U \cap B| \le 1$ for all $B \in \mathcal{B}$. Assume that $|U| \ge m/8$. Then, for some absolute constants $C, c > 0$*

$$\mathbb{P}(\phi_v \le 0 \; for \; all \quad v \in U) \le Ce^{-ce^{c\lambda}}.$$

To prove the preceding lemma, we need to study the correlation structure for the Gaussian process $\{\phi_v : v \in U\}$.

Lemma 2.4 [15, Lemma 6.3.7] *For all $n \ge 1$, let $\mathcal{C}(n) \subset \mathbb{Z}^2$ be a discrete ball of radius n centered at the origin. Then there exist absolute constants $c, C > 0$ such that for all $n \ge 1$ and $x \in \mathcal{C}(n/4)$ and $y \in \partial\mathcal{C}(n)$*

$$c/n \le \mathbb{P}_x(\tau_{\partial\mathcal{C}(n)} = y) \le C/n.$$

Write $a_{v,w} = \mathbb{P}_v(\tau_{\partial\mathcal{C}} = \tau_w)$. The preceding lemma implies that $c/r \le a_{v,w} \le C/r$ for all $v \in B \subset \mathcal{C}$. Combined with Lemma 2.1, it follows that

$$\phi_v = \sum_{w \in \partial\mathcal{C}} a_{v,w}\eta_w. \tag{15}$$

Therefore, we have

$$\mathrm{Var}\phi_v = \Theta(1/r^2) \sum_{u,w} \mathrm{Cov}(\eta_u, \eta_w) = \Theta(1/r^2) \sum_{u,w \in \partial\mathcal{C}} G_{\partial A_n}(u, w). \tag{16}$$

In order to estimate the sum of Green functions, one could use Lemma 2.2. Alternatively, it is computation free if we apply the next lemma.

Lemma 2.5 [15, Proposition 6.4.1] *For all $n \ge 1$, let $\mathcal{C}(n) \subset \mathbb{Z}^2$ be a discrete ball of radius n centered at the origin. Then for all $k < n$ and $x \in \mathcal{C}(n) \backslash \mathcal{C}(k)$, we have*

$$\mathbb{P}_x(\tau_{\partial\mathcal{C}(n)} < \tau_{\partial\mathcal{C}(k)}) = \frac{\log|x| - \log k + O(1/k)}{\log n - \log k}.$$

Now, write

$$p_{\min} = \min_{\mathcal{C} \in \mathfrak{C}} \min_{u \in \partial\mathcal{C}} \mathbb{P}_u(\tau_{\partial A_n} < \tau^+_{\partial\mathcal{C}}), \quad \text{and} \quad p_{\max} = \max_{\mathcal{C} \in \mathfrak{C}} \max_{u \in \partial\mathcal{C}} \mathbb{P}_u(\tau_{\partial A_n} < \tau^+_{\partial\mathcal{C}}),$$

where $\tau^+_{\partial\mathcal{C}} = \min\{k \ge 1 : S_k \in \partial\mathcal{C}\}$ is the first returning time to $\partial\mathcal{C}$. By the preceding lemma, we have

$$1/(4r\lambda) \le p_{\min} \le p_{\max} \le O(1/r) \quad \text{for all } u \in \partial\mathcal{C} \quad \text{and} \quad \mathcal{C} \in \mathfrak{C}.$$

Therefore, by Markovian property we have

$$\Theta(r) \le \frac{1}{p_{\max}} \le \sum_{w \in \partial\mathcal{C}} G_{\partial A_n}(u, w) \le 1 + \frac{1}{p_{\min}} = O(r\lambda),$$
$$\text{for all } u \in \partial\mathcal{C} \quad \text{and} \quad \mathcal{C} \in \mathfrak{C}. \tag{17}$$

Combined with (16), this implies that

$$\Theta(1) \leq \mathrm{Var}(\phi_v) = O(\lambda)\,, \quad \text{for all } v \in U.$$

We also wish to bound the covariance between ϕ_v and ϕ_u for $u, v \in U$. Assume $u \in \mathcal{C}_i$ and $v \in \mathcal{C}_j$ for $i \neq j$. By (17), we see that

$$\begin{aligned}
\mathrm{Cov}(\phi_u, \phi_v) &\leq O(1/r) \max_{x \in \mathcal{C}_i} G_{\partial A_n}(x, \partial \mathcal{C}_j) \leq O(1/r) \max_{x \in \mathcal{C}_i} \mathbb{P}_x(\tau_{\partial \mathcal{C}_j} < \tau_{\partial A_n}) \\
&\quad \times \max_{y \in \partial \mathcal{C}_j} G_{\partial A_n}(y, \partial \mathcal{C}_j) \\
&\leq O(1/r) \max_{x \in \mathcal{C}_i} \mathbb{P}_x(\tau_{\partial \mathcal{C}_j} < \tau_{\partial A_n}) \max_{y \in \partial \mathcal{C}_j} \sum_{z \in \partial \mathcal{C}_j} G_{\partial A_n}(y, z) \\
&\leq O(\lambda) \max_{x \in \mathcal{C}_i} \mathbb{P}_x(\tau_{\partial \mathcal{C}_j} < \tau_{\partial A_n}). \qquad (18)
\end{aligned}$$

We incorporate the estimate for the above hitting probability in the next lemma.

Lemma 2.6 *For any $i \neq j$ and $x \in \mathcal{C}_i$, we have*

$$\mathbb{P}_x(\tau_{\partial \mathcal{C}_j} < \tau_{\partial A_n}) \leq C\sqrt{r/\ell},$$

where $C > 0$ is a universal constant.

Proof We consider the projection of the random walk to the horizontal and vertical axes, and denote them by (X_t) and (Y_t) respectively. Define

$$T_X = \min\{t : |X_t - x| \geq \ell/2\}\,, \quad \text{and} \quad T_Y = \min\{t : Y_t = 0\}.$$

It is clear that $\tau_{\partial A_n} \leq T_Y$ and $T_X \leq \tau_{\partial \mathfrak{C} \setminus \partial \mathcal{C}_i}$. Write $t^\star = r\ell$. Since the number of steps spent on waling in the horizontal (vertical) axis is a Binomial distribution with parameter t and $1/2$, an application of CLT yields that with probability at least $1 - \exp(-ct^\star)$ (here $c > 0$ is an absolute constant) the number of such steps is at least $t^\star/3$ (and thus, at most $2t^\star/3$). Combined with standard estimates for 1-dimensional random walks (see, e.g., [18, Theorem 2.17, Lemma 2.21]), it follows that for a universal constant $C > 0$

$$\mathbb{P}(T_Y \geq t^\star) \leq C\sqrt{r/\ell}.$$

Using Markov property for random walk, we see that

$$\mathbb{P}(T_X \leq t^\star) \leq (\mathbb{P}(T_X \leq \ell^2))^{t^\star/\ell^2} \leq \varepsilon^{r/\ell},$$

where $\varepsilon < 1$ is an absolute constant. This completes the proof. □

Combining the preceding lemma and (18), we obtain that (here we assume that λ_0 is large enough)

$$\mathrm{Cov}(\phi_u, \phi_v) = O(\lambda\sqrt{r/\ell}), \quad \text{for all } u, v \in U.$$

Therefore, we have the following bounds on the correlation coefficients $\rho_{u,v}$:

$$0 \le \rho_{u,v} = O(\lambda\sqrt{r/\ell}), \quad \text{for all } u \neq v \in U. \tag{19}$$

At this point, we wish to apply Slepian's [20] comparison theorem (see also, [11,17]).

Theorem 2.7 *If $\{\xi_i : 1 \le i \le n\}$ and $\{\zeta_i : 1 \le i \le n\}$ are two mean zero Gaussian process such that*

$$Var\xi_i = Var\zeta_i, \quad and \quad Cov(\xi_i, \xi_j) \le Cov(\zeta_i, \zeta_j) \quad for\ all \quad 1 \le i, j \le n. \tag{20}$$

Then for all real numbers $\lambda_1, \ldots, \lambda_n$,

$$\mathbb{P}(\xi_i \le \lambda_i\ for\ all\ 1 \le i \le n) \le \mathbb{P}(\zeta_i \le \lambda_i\ forall \quad 1 \le i \le n).$$

The following is an immediate consequence.

Corollary 2.8 *Let $\{\xi_i : 1 \le i \le n\}$ be a mean zero Gaussian process such that the correlation coefficients satisfy $0 \le \rho_{i,j} \le \rho \le 1/2$ for all $1 \le i < j \le n$. Then,*

$$\mathbb{P}(\xi_i \le 0,\ forall\ 1 \le i \le n) \le e^{-1/(2\rho)} + (9/10)^n.$$

Proof Since we are comparing ξ_i's with zero, it allows us to assume that $\mathrm{Var}\xi_i = 1$ for all $1 \le i \le n$. Let $\zeta_i = \sqrt{\rho}X + \sqrt{1-\rho^2}Y_i$ where X and Y_i's are i.i.d. standard Gaussian variables. It is clear that our processes $\{\xi_i : 1 \le i \le n\}$ and $\{\zeta_i : 1 \le i \le n\}$ satisfy (20). By Theorem 2.7, we obtain that

$$\mathbb{P}(\xi_i \le 0 \text{ for all } 1 \le i \le n) \le \mathbb{P}(\zeta_i \le 0 \text{ for all } 1 \le i \le n).$$

Since $\{\zeta_i \le 0 \text{ for all } 1 \le i \le n\} \subseteq \{X \le -1/\sqrt{\rho}\} \cup \{Y_i \le 1/\sqrt{1-\rho^2} \text{ for all } 1 \le i \le n\}$, we have

$$\begin{aligned}\mathbb{P}(\zeta_i \le 0 \text{ for all } 1 \le i \le n) &\le \mathbb{P}(X \le -1/\sqrt{\rho}) \\ &\quad + \mathbb{P}(Y_i \le 1/\sqrt{1-\rho^2} \text{ for all } 1 \le i \le n) \\ &\le \mathrm{e}^{-1/(2\rho)} + (9/10)^n.\end{aligned}$$

Altogether, this completes the proof. □

Proof of Lemma 2.3 Recall definitions of r, ℓ and m. The desired estimate follows from an application of the preceding corollary to $\{\phi_v : v \in U\}$ and the correlation bounds (19) (here we assume that λ is large enough such that $\rho_{u,v} \le 1/2$ for all $u \neq v$). □

Combining Lemma 2.3 and (14), we finally complete the proof for the upper bound on the left tail as in (13).

2.5 Lower bound on the left tail

In this subsection, we study the lower bound for the lower tail of the maximum and show that for absolute constants $C, c, n_0, \lambda_0 > 0$

$$\mathbb{P}(M_n - \mathbb{E}M_n \le -\lambda) \ge c\mathrm{e}^{-C\mathrm{e}^{C\lambda}} \quad \text{for all} \quad n \ge n_0 \quad \text{and} \quad \lambda_0 \le \lambda \le (\log n)^{2/3}. \tag{21}$$

The proof consists of two steps: (1) We estimate the probability for $\sup_{v\in B} \eta_v \le \mathbb{E}M_n - \lambda$ for a small box B in A_n. (2) Applying FKG inequality for GFF, we bootstrap the estimate on a small box to the whole box.

By Theorem 1.3, there exists an absolute constant $\alpha^* > 0$ such that

$$\mathbb{P}(M_n \le \mathbb{E}M_n + \alpha^*) \ge 3/4 \quad \text{for all}\, n \in \mathbb{N}. \tag{22}$$

We first consider the behavior of GFF in a box of side length ℓ, where

$$\ell \triangleq n\mathrm{e}^{-10(\lambda+\kappa+\alpha^*+2)}. \tag{23}$$

Lemma 2.9 *Let $B \subseteq A_n$ be a box of side length ℓ. Then,*

$$\mathbb{P}\left(\sup_{v\in B} \eta_v \le \mathbb{E}M_n - \lambda\right) \ge 1/2.$$

In order to prove the lemma, let B' be a box of side length 2ℓ that has the same center as B, and let $\hat{B} = B' \cap A_n$. Consider the GFF $\{g_v : v \in \hat{B}\}$ on $\hat{B}$ with Dirichlet boundary condition (on $\partial\hat{B}$). We wish to compare $\{\eta_v : v \in B\}$ with $\{g_v : v \in B\}$. For $u, v \in B$, let

$$\rho_{u,v} = \frac{\mathrm{Cov}(\eta_u, \eta_v)}{\sqrt{\mathrm{Var}\eta_u\, \mathrm{Var}\eta_v}} \quad \text{and} \quad \hat{\rho}_{u,v} = \frac{\mathrm{Cov}(g_u, g_v)}{\sqrt{\mathrm{Var}g_u\, \mathrm{Var}g_v}}$$

be the correlations coefficients of two GFFs under consideration.

Lemma 2.10 *For all $u, v \in B$, we have $\rho_{u,v} \ge \hat{\rho}_{u,v}$ for all $u, v \in B$.*

Proof Since by definition $\hat{B} \subset A_n$, we see that $\tau_{\partial\hat{B}} \le \tau_{\partial A_n}$ deterministically for a random walk started from an arbitrary vertex in B. Note that

$$G_{\partial A_n}(u, v) = \mathbb{P}_u(\tau_v < \tau_{\partial A_n})G_{\partial A_n}(v, v) \quad \text{and} \quad G_{\partial\hat{B}}(u, v) = \mathbb{P}_u(\tau_v < \tau_{\partial\hat{B}})G_{\partial\hat{B}}(v, v)$$

Altogether, we obtain that

$$\rho_{u,v} = \sqrt{\mathbb{P}_u(\tau_v < \tau_{\partial A_n})\mathbb{P}_v(\tau_u < \tau_{\partial A_n})} \geq \sqrt{\mathbb{P}_u(\tau_v < \tau_{\partial \hat{B}})\mathbb{P}_v(\tau_u < \tau_{\partial \hat{B}})} = \hat{\rho}_{u,v}.$$

□

We next compare the variances for the two GFFs.

Lemma 2.11 *For all $v \in B$, we have that*

$$\mathrm{Var}\eta_v \leq \Big(1 + \frac{(1+o(1)(\log(n/\ell) + O(1))}{\log n}\Big)\mathrm{Var} g_v.$$

Proof It suffices to compare the Green functions $G_{\partial A_n}(v, v)$ and $G_{\partial \hat{B}}(v, v)$. We can decompose them in terms of the hitting points to $\partial \hat{B}$ and obtain that

$$G_{\partial A_n}(v, v) = G_{\partial \hat{B}}(v, v) + \sum_{w \in \partial \hat{B}} \mathbb{P}_v(\tau_w = \tau_{\partial \hat{B}}) G_{\partial A_n}(w, v).$$

Note that for $w \in \partial \hat{B} \cap \partial A_n$, we have $G_{\partial A_n}(w, v) = 0$. For $w \in \partial \hat{B} \backslash \partial A_n$, we see that $|v - w| \geq \ell$ by our definition of $\hat{B}$. Therefore, by Lemma 2.2, we have

$$G_{\partial A_n}(w, v) \leq \tfrac{2}{\pi} \log(n/\ell) + O(1).$$

Since $|v - w| \geq \ell$ for $w \in \partial \hat{B} \backslash \partial A_n$, Lemma 2.2 gives that

$$\begin{aligned} G_{\partial \hat{B}}(v, v) &= \sum_{w \in \partial \hat{B} \backslash \partial A_n} \mathbb{P}_v(\tau_w = \tau_{\partial \hat{B}}) \cdot a(w, v) \\ &\geq \big(\tfrac{2}{\pi} + o(1)\big) \log n \sum_{w \in \partial \hat{B} \backslash \partial A_n} \mathbb{P}_v(\tau_w = \tau_{\partial \hat{B}}), \end{aligned}$$

where we used the assumption that $\lambda \leq (\log n)^{2/3}$. Altogether, we get that

$$G_{\partial A_n}(v, v) \leq \big(1 + \tfrac{(1+o(1))(\log(n/\ell) + O(1))}{\log n}\big) G_{\partial \hat{B}}(v, v),$$

completing the proof. □

We will need the following lemma to handle some technical issues.

Lemma 2.12 *For a graph $G = (V, E)$, consider $V_1 \subset V_2 \subset V$. Let $\{\eta_v^{(1)}\}_{v \in V}$ and $\{\eta_v^{(2)}\}_{v \in V}$ be GFFs on V such that $\eta^{(1)}|_{V_1} = 0$ and $\eta^{(2)}|_{V_2} = 0$, respectively. Then for any number $t \in \mathbb{R}$*

$$\mathbb{P}\left(\sup_{v \in U} \eta_v^{(1)} \geq t\right) \geq \tfrac{1}{2} \mathbb{P}\left(\sup_{v \in U} \eta_v^{(2)} \geq t\right).$$

Proof Note that the conditional covariance matrix of $\{\eta_v^{(1)}\}_{v\in U}$ given the values of $\{\eta_v^{(1)}\}_{v\in V_2\setminus V_1}$ corresponds to the covariance matrix of $\{\eta_v^{(2)}\}_{v\in U}$. This implies that

$$\left\{\eta_v^{(1)} : v \in U\right\} \stackrel{law}{=} \left\{\eta_v^{(2)} + \mathbb{E}\left(\eta_v^{(1)} \mid \left\{\eta_u^{(1)} : u \in V_2 \setminus V_1\right\}\right) : v \in U\right\},$$

where on the right hand side $\{\eta_v^{(2)} : v \in U\}$ is independent of $\{\eta_u^{(1)} : u \in V_2\setminus V_1\}$. Write $\phi_v = \mathbb{E}(\eta_v^{(1)} \mid \{\eta_u^{(1)} : u \in V_2\setminus V_1\})$. Note that ϕ_v is a linear combination of $\{\eta_u^{(1)} : u \in V_2\setminus V_1\}$, and thus a mean zero Gaussian variable. By the above identity in law, we derive that

$$\mathbb{P}\left(\sup_{v\in U} \eta_v^{(1)} \geq t\right) \geq \mathbb{P}\left(\eta_\xi^{(2)} + \phi_\xi \geq t\right) = \tfrac{1}{2}\mathbb{P}\left(\eta_\xi^{(2)} \geq t\right) = \tfrac{1}{2}\mathbb{P}\left(\sup_{v\in U} \eta_v^{(2)} \geq t\right),$$

where we denote by $\xi \in U$ the maximizer of $\{\eta_u^{(2)} : u \in U\}$ and the second transition follows from the independence of $\{\eta_v^{(1)}\}$ and $\{\phi_v\}$. □

We are now ready to give

Proof of Lemma 2.9 Write $b_v = \sqrt{\mathrm{Var}\eta_v/\mathrm{Var}g_v}$ for every $v \in B$. By Lemma 2.11, we see that $b_v \leq 1+(1/2+o(1))(\log(n/\ell)+O(1))/\log n$ for all $v \in B$. Consider the Gaussian process defined by $\xi_v = \eta_v/b_v$. By Lemma 2.10, we see that $\{\xi_v : v \in B\}$ and $\{g_v : v \in B\}$ satisfy the assumption in Theorem 2.7, and thus

$$\mathbb{P}\left(\sup_{v\in B} \xi_v \leq \gamma\right) \geq \mathbb{P}\left(\sup_{v\in B} g_v \leq \gamma\right), \quad \text{for all } \gamma \in \mathbb{R}. \tag{24}$$

Plugging into $\gamma = \mathbb{E}M_{2\ell} + \alpha^*$ and using (22) and Lemma 2.12 (we need to use Lemma 2.12 as the box $\hat{B}$ might not be a squared box of side-length 2ℓ but a subset of that), we obtain that

$$\begin{aligned}\mathbb{P}\left(\sup_{v\in B} \xi_v \leq \mathbb{E}M_{2\ell} + \alpha^*\right) &\geq \mathbb{P}\left(\sup_{v\in B} g_v \leq \mathbb{E}M_{2\ell} + \alpha^*\right)\\ &\geq \mathbb{P}\left(\sup_{v\in \hat{B}} g_v \leq \mathbb{E}M_{2\ell} + \alpha^*\right) \geq 1/2.\end{aligned}$$

Also, By definition of ℓ and (4) as well as our assumption that $\lambda \leq (\log n)^{2/3}$, we see that

$$\mathbb{E}M_n \geq \mathbb{E}M_{2\ell} + 2\sqrt{2/\pi}\log(n/\ell) - 10.$$

Therefore, for large constants λ_0, n_0, we can deduce that

$$\begin{aligned}&(1 + (1/2 + o(1))(\log(n/\ell) + O(1))/\log n)(\mathbb{E}M_{2\ell} + \alpha^*)\\ &\quad \leq \mathbb{E}M_{2\ell} + \tfrac{2}{3}\log(n/\ell) + 1 \leq \mathbb{E}M_n - \lambda,\end{aligned}$$

where we used Theorem 1.3 and the definition of ℓ in (23). Altogether, we deduce that

$$\mathbb{P}\left(\sup_{v\in B}\eta_v \le \mathbb{E}M_n - \lambda\right) \ge 1/2.$$

□

Now, we wish to apply FKG inequality and obtain the estimate on the probability $\sup_{v\in A_n}\eta_v \le \mathbb{E}M_n - \lambda$. Pitt [19] proves that the FKG inequality holds for a Gaussian process with nonnegative covariances. Since clearly the GFF has nonnegative covariances, the FKG inequality holds for GFF.

Partition A_n into a union of boxes $\mathcal{B}$ where each of the boxes is of side length at most ℓ. We choose $\mathcal{B}$ in a way such that $|\mathcal{B}|$ is minimized. Clearly, $|\mathcal{B}| \le (\lceil n/\ell\rceil)^2$. Observing that the event $\{\sup_{v\in B}\eta_v \le \mathbb{E}M_n - \lambda\}$ is decreasing for all $B \in \mathcal{B}$, we apply FKG inequality and Lemma 2.9, and conclude that

$$\mathbb{P}\left(\sup_{v\in A_n} \le \mathbb{E}M_n - \lambda\right) \ge \prod_{B\in\mathcal{B}}\mathbb{P}\left(\sup_{v\in B} \le \mathbb{E}M_n - \lambda\right) \ge (1/2)^{|\mathcal{B}|}.$$

Recalling the definition of ℓ as in (23), this completes the proof of (21).

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References

1. Arguin, L.-P., Bovier, A., Kistler, N.: The genealogy of extremal particles of branching brownian motion. Preprint, available at http://arxiv.org/abs/1008.4386
2. Bolthausen, E., Deuschel, J.-D., Giacomin, G.: Entropic repulsion and the maximum of the two-dimensional harmonic crystal. Ann. Probab. **29**(4), 1670–1692 (2001)
3. Bolthausen, E., Deuschel, J.-D., Zeitouni, O.: Recursions and tightness for the maximum of the discrete, two dimensional gaussian free field. Electron. Commun. Probab. **16**, 114–119 (2011)
4. Bramson, M.: Maximal displacement of branching Brownian motion. Commun. Pure Appl. Math. **31**(5), 531–581 (1978)
5. Bramson, M.: Convergence of solutions of the Kolmogorov equation to travelling waves. Mem. Am. Math. Soc. **44**(285), iv+190 (1983)
6. Bramson, M., Zeitouni, O.: Tightness of the recentered maximum of the two-dimensional discrete gaussian free field. Commun. Pure Appl. Math. (2010)
7. Chatterjee, S.: Chaos, concentration, and multiple valleys. Preprint, available at http://arxiv.org/abs/0810.4221 (2008)
8. Daviaud, O.: Extremes of the discrete two-dimensional Gaussian free field. Ann. Probab. **34**(3), 962–986 (2006)
9. Deuschel, J.-D.: Entropic repulsion of the lattice free field. II. The 0-boundary case. Commun. Math. Phys. **181**(3), 647–665 (1996)
10. Dynkin, E.B.: Markov processes and random fields. Bull. Am. Math. Soc. (N.S.) **3**(3), 975–999 (1980)
11. Fernique, X.: Regularité des trajectoires des fonctions aléatoires gaussiennes. In: École d'Été de Probabilités de Saint-Flour, IV-1974, pages 1–96. Lecture Notes in Math., vol. 480. Springer, Berlin (1975)

12. Harris, S.C.: Travelling-waves for the FKPP equation via probabilistic arguments. Proc. R. Soc. Edinb. Sect. A **129**(3), 503–517 (1999)
13. Kolmogorov, A., Petrovsky, I., Piskunov, N.: Etude de l'quation de la diffusion avec croissance de la quantit de matire et son application un problme biologique. Bulletin Universit d'Etat Moscou, Bjul. Moskowskogo Gos. Univ. (1937)
14. Lalley, S.P., Sellke, T.: A conditional limit theorem for the frontier of a branching Brownian motion. Ann. Probab. **15**(3), 1052–1061 (1987)
15. Lawler, G.F., Limic, V.: Random walk: a modern introduction, volume 123 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge (2010)
16. Ledoux, M.: The concentration of measure phenomenon, volume 89 of Mathematical Surveys and Monographs. American Mathematical Society, Providence (2001)
17. Ledoux, M., Talagrand, M.: Probability in Banach spaces, volume 23 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer, Berlin (1991). (Isoperimetry and processes)
18. Levin, D.A., Peres, Y., Wilmer, E.L.: Markov chains and mixing times. American Mathematical Society, Providence (2009). (with a chapter by James G. Propp and David B. Wilson)
19. Pitt, L.D.: Positively correlated normal variables are associated. Ann. Probab. **10**(2), 496–499 (1982)
20. Slepian, D.: The one-sided barrier problem for Gaussian noise. Bell Syst. Tech. J. **41**, 463–501 (1962)