Probab. Theory Relat. Fields (2013) 156:51–74
DOI 10.1007/s00440-012-0420-9

On-diagonal oscillation of the heat kernels on post-critically finite self-similar fractals

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Received: 12 June 2011 / Revised: 30 January 2012 / Accepted: 29 February 2012 /
Published online: 23 March 2012

Abstract For the canonical heat kernels $p_t(x, y)$ associated with Dirichlet forms on post-critically finite self-similar fractals, e.g. the transition densities (heat kernels) of Brownian motion on affine nested fractals, the *non*-existence of the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x, x)$ is established for a "*generic*" (in particular, almost every) point x, where d_s denotes the spectral dimension. Furthermore the same is proved for *any* point x in the case of the d-dimensional standard Sierpinski gasket with $d \geq 2$ and the N-polygasket with $N \geq 3$ odd, e.g. the pentagasket ($N = 5$) and the heptagasket ($N = 7$).

Keywords Post-critically finite self-similar fractals · Affine nested fractals · Dirichlet form · Heat kernel · Oscillation · Short time asymptotics

Mathematics Subject Classification Primary 28A80 · 60J35; Secondary 31C25 · 58C40

1 Introduction

It is a general belief that the heat kernels on fractals should exhibit highly oscillatory behavior as opposed to the classical case of Riemannian manifolds. For example, on the Sierpinski gasket (Fig. 1), the canonical *"Brownian motion"* has been constructed

The author was supported by the Japan Society for the Promotion of Science (JSPS Research Fellow PD (20 · 6088)).

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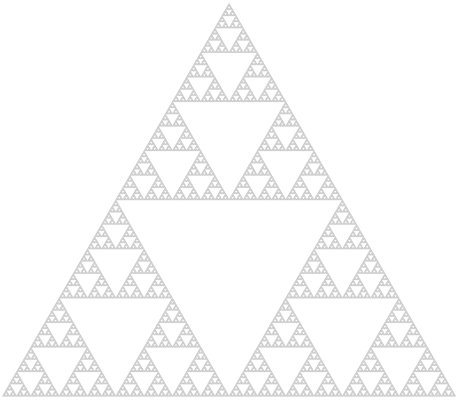

Fig. 1 Sierpinski gasket

by Goldstein [11] and Kusuoka [22], and Barlow and Perkins [3] have proved that its transition density (heat kernel) $p_t(x, y)$ is jointly continuous and subject to the following *sub-Gaussian estimate*

$$\frac{c_1}{t^{d_s/2}}\exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_w}}{c_1 t}\Bigr)^{\frac{1}{d_w-1}}\biggr) \le p_t(x,y) \le \frac{c_2}{t^{d_s/2}}\exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_w}}{c_2 t}\Bigr)^{\frac{1}{d_w-1}}\biggr) \tag{1.1}$$

for $t \in (0, 1]$; here $c_1, c_2 \in (0, \infty)$ are some constants, $d_s := 2\log_5 3$ and $d_w := \log_2 5$ are called the *spectral dimension* and the *walk dimension* of the Sierpinski gasket, respectively, and ρ is the shortest path metric in the gasket which is easily seen to be equivalent to the Euclidean metric. In particular, for any point x of the Sierpinski gasket we have

$$c_1 \le t^{d_s/2} p_t(x, x) \le c_2, \quad t \in (0, 1], \tag{1.2}$$

and Barlow and Perkins have conjectured in [3, Problem 10.5] that the limit

$$\lim_{t\downarrow 0} t^{d_s/2} p_t(x, x) \tag{1.3}$$

does *not* exist, but this problem has been open since then. The main purpose of this paper is to prove this conjecture, namely:

Theorem 1.1 *Let the heat kernel* $p_t(x, y)$ *and* $d_s = 2\log_5 3$ *be as above. Then the limit* $\lim_{t\downarrow 0} t^{d_s/2} p_t(x, x)$ *does not exist for* ***any*** *point* x *of the Sierpinski gasket.*

We can consider the same problem for a class of finitely ramified self-similar fractals, called *affine nested fractals*. (See Sect. 4 for their definition; typical examples of affine nested fractals are shown in Fig. 2, and see Figs. 3, 4 and 5 below for further examples). By the results of Fitzsimmons, Hambly and Kumagai [9], an affine nested fractal K admits a canonical Brownian motion on it, and the associated (jointly continuous) transition density $p_t(x, y)$ satisfies the two-sided sub-Gaussian bound (1.1)

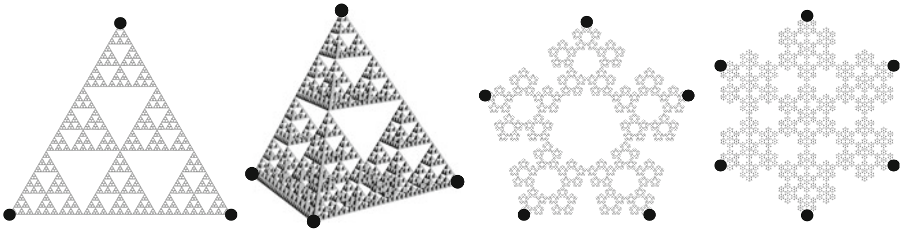

Fig. 2 Typical examples of affine nested fractals. From the *left*, two-dimensional level-3 Sierpinski gasket, three-dimensional standard (level-2) Sierpinski gasket, pentagasket (5-polygasket) and snowflake. In each fractal, the set V_0 of its boundary points is marked by *solid circles*

with certain d_s and d_w and a suitably constructed geodesic metric ρ on K. In particular, the on-diagonal estimate (1.2) holds for any $x \in K$, and then it is natural to ask whether the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ exists or not. We address this question in the present article, and the following theorem summarizes our main results. Recall that a *self-similar measure* on K is defined as the image of a Bernoulli measure on the corresponding shift space through the canonical projection; see [18, Section 1.4]. See Examples 5.1 and 5.3 for the precise definition of the d-dimensional level-l Sierpinski gasket and the N-polygasket, respectively.

Theorem 1.2 *Let V_0 be the set of boundary points of our affine nested fractal K.*

(1) *Assume $\#V_0 \geq 3$. Then the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ does not exist for any $x \in K \setminus \mathcal{S}_*$, where $\mathcal{S}_*$ is a Borel subset of K satisfying $\nu(\mathcal{S}_*) = 0$ for any self-similar measure ν on K.* ($\mathcal{S}_*$ is explicitly defined by (4.4) and (3.1) and satisfies $V_0 \subset \mathcal{S}_*$.)

(2) *The limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ does not exist for any $x \in V_0$ when K is either*
   - *the d-dimensional level-l Sierpinski gasket with $d \geq 2$, $l \geq 2$, or*
   - *the N-polygasket with $N \geq 3$, $N/4 \notin \mathbb{N}$.*

(3) *The limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ does not exist for* ***any*** *$x \in K$ when K is either*
   - *the d-dimensional* ***standard*** *(i.e. level-2) Sierpinski gasket with $d \geq 2$, or*
   - *the N-polygasket with $N \geq 3$* ***odd****.*

Remark 1.3 The above description contains some ambiguity in the choice of a *"canonical"* Brownian motion on K since an affine nested fractal may admit more than one self-similar diffusion compatible with its symmetry. For example, according to [9, Section 2, especially Proposition 2.3], on the two-dimensional level-3 Sierpinski gasket in Fig. 2 one can construct self-similar diffusions which are invariant under the symmetries of the space and have two different resistance scaling factors, one for cells containing a boundary point and the other for those containing the barycenter. In fact, Theorem 1.2-(1) is true for *any* choice of a self-similar diffusion on K (to be more precise, of a regular harmonic structure on K) that is invariant under certain symmetries of K, whereas Theorem 1.2-(2),(3) concern only the case where all cells have the same resistance scaling factor. See Sects. 4 and 6 for details.

Under a slightly more general framework than in Theorem 1.2-(1), Barlow and Kigami [2] have proved a similar oscillation in the asymptotic behavior of the eigenvalues of the associated Laplacian. The heart of their argument is to construct a *pre-localized eigenfunction* of the Laplacian (i.e. an eigenfunction of the Laplacian which satisfies both Neumann and Dirichlet boundary conditions on V_0), based only on the symmetry of the fractal and the Laplacian. We prove Theorem 1.2-(1) by modifying their argument to construct a pre-localized eigenfunction which is non-zero at a given specific point, and the construction is again based only on the symmetry.

Unfortunately, since $V_0 \subset \mathcal{S}_*$, Theorem 1.2-(1) tells us nothing about the non-existence of the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ for $x \in V_0$. Theorem 1.2-(2) asserts this non-existence in the particular cases of the d-dimensional level-l Sierpinski gasket and the N-polygasket, and its proof is based on a simple geometric argument which makes full use of the specific cell structures of these fractals.

Note that $\mathcal{S}_*$ is defined through another subset $\mathcal{S}$ of K given by (4.4), which is the set of *"points lying in some axis of symmetry of K"*. For the 2-dimensional standard Sierpinski gasket and the N-polygasket with N odd, we have $\mathcal{S} \subset V_*$, by virtue of which Theorem 1.2-(3) follows from Theorem 1.2-(1),(2). A similar argument applies also to the case of the d-dimensional standard Sierpinski gasket with $d \geq 3$ although $\mathcal{S} \not\subset V_*$ in this case (see Theorem 5.2). It is quite likely that Theorem 1.2-(3) can be generalized to other affine nested fractals, but they are beyond the reach of our method.

Similar oscillatory phenomena have been proved in [12,21,24] for the simple random walks on self-similar graphs by using the method of *"singularity analysis"*, and their results can be considered as giving sufficient conditions for the non-existence of the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ for $x \in V_0$, in view of the local limit theorem [6, Theorem 31]. Their sufficient conditions, however, require some concrete calculations of certain rational functions associated with the simple random walk and seem difficult to verify for a general d-dimensional level-l Sierpinski gasket. Also their results do not apply to fractals with *"less symmetric boundary"* such as the N-polygasket with $N \neq 3, 6, 9$. An important point of Theorem 1.2-(2) is that it has successfully treated all Sierpinski gaskets and polygaskets in a unified way without depending on concrete calculations.

In fact, we can conclude the non-existence of the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ for any point x of the fractal if the eigenvalues of the Laplacian possess a certain property, as treated in a forthcoming paper [17]. This result in particular applies to the two-dimensional level-3 Sierpinski gasket and the hexagasket (6-polygasket, see Fig. 5), which are beyond the scope of Theorem 1.2-(3). The property of the eigenvalues required there, however, again seems difficult to verify for a general d-dimensional level-l Sierpinski gasket since some concrete calculation is necessary. Moreover, the property can be verified only by the method of spectral decimation, which does not work for the N-polygasket, $N \neq 3, 6, 9$. In this sense, the method of this paper is the only way established so far to obtain Theorem 1.2-(2),(3) for the N-polygasket, $N \neq 3, 6, 9$.

This paper is organized as follows. In Sect. 2, we introduce our framework, recall basic facts about the heat kernel $p_t(x,y)$ and present our key criterion for the non-existence of the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$. Following the framework of Barlow and Kigami [2], in Sect. 3 we state and prove Theorem 3.4 which generalizes Theorem 1.2-(1), and then we verify in Sect. 4 that Theorem 3.4 actually applies to the case of affine nested fractals to imply Theorem 1.2-(1). We recall the definition of the d-dimensional level-l Sierpinski gasket and the N-polygasket in Sect. 5, and Sect. 6 is devoted to the proof of Theorem 1.2-(2),(3). In fact, in Sect. 6 we establish the assertions of Theorem 1.2-(2),(3) also for the (N, l)-*polygasket*, which is a post-critically finite self-similar fractal introduced in [5] as a generalization of the N-polygasket.

Notation In this paper, we adopt the following notation and conventions.

(1) $\mathbb{N} = \{1, 2, 3, \ldots\}$, i.e. $0 \notin \mathbb{N}$.
(2) The cardinality (the number of elements) of a set A is denoted by $\#A$.
(3) We set $\sup \emptyset := 0$, $\inf \emptyset := \infty$ and $0^0 := 1$. All functions in this paper are assumed to be $\mathbb{R}$-valued.
(4) For $d \in \mathbb{N}$, $\mathbb{R}^d$ is always equipped with the Euclidean norm $|\cdot|$, and $O(d)$ denotes the d-dimensional real orthogonal group. For $g \in O(d)$, $\det g$ denotes its determinant.
(5) Let E be a topological space. The Borel σ-field of E is denoted by $\mathcal{B}(E)$. We set $C(E) := \{u \mid u : E \to \mathbb{R}, u \text{ is continuous}\}$ and $\|u\|_\infty := \sup_{x \in E} |u(x)|$, $u \in C(E)$. For $A \subset E$, its interior in the topology of E is denoted by $\mathrm{int}_E A$. If ρ is a metric on E, we set $\mathrm{dist}_\rho(x, A) := \inf_{y \in A} \rho(x, y)$ for $x \in E$ and $A \subset E$.

2 Preliminaries

In this section, we first introduce our framework of a self-similar set and a Dirichlet form on it, and then present preliminary facts.

Let us start with the standard notions concerning self-similar sets. We refer to [18, Chapter 1] for details. Throughout this paper, we fix a compact metrizable topological space K, a finite set S with $\#S \geq 2$ and a continuous injective map $F_i : K \to K$ for each $i \in S$. We set $\mathcal{L} := (K, S, \{F_i\}_{i \in S})$. Also we arbitrarily take a metric ρ on K compatible with the topology of K and fix it throughout this paper.

Definition 2.1 (1) Let $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_m := S^m = \{w_1 \cdots w_m \mid w_i \in S \text{ for } i \in \{1, \ldots, m\}\}$ for $m \in \mathbb{N}$ and let $W_* := \bigcup_{m \in \mathbb{N} \cup \{0\}} W_m$.
(2) We set $\Sigma := S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3 \ldots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology, and define the *shift map* $\sigma : \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3\cdots) := \omega_2\omega_3\omega_4\cdots$. For $i \in S$ we define $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3\cdots) := i\omega_1\omega_2\omega_3\cdots$ and set $i^\infty := iii\ldots \in \Sigma$. Furthermore for $\omega = \omega_1\omega_2\omega_3\ldots \in \Sigma$ and $m \in \mathbb{N} \cup \{0\}$, we write $[\omega]_m := \omega_1 \cdots \omega_m \in W_m$.
(3) For $w = w_1 \cdots w_m \in W_*$, we set $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$ ($F_\emptyset := \mathrm{id}_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1} \circ \cdots \circ \sigma_{w_m}$ ($\sigma_\emptyset := \mathrm{id}_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$.

Definition 2.2 $\mathcal{L}$ is called a *self-similar structure* if and only if there exists a continuous surjective map $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for any $i \in S$. Note that such π, if exists, is unique and satisfies $\{\pi(\omega)\} = \bigcap_{m \in \mathbb{N}} K_{[\omega]_m}$ for any $\omega \in \Sigma$.

In what follows we always assume that $\mathcal{L}$ is a self-similar structure.

Definition 2.3 (1) We define the *critical set* $\mathcal{C}$ and the *post-critical set* $\mathcal{P}$ *of* $\mathcal{L}$ by

$$\mathcal{C} := \pi^{-1}\bigl(\textstyle\bigcup_{i,j \in S,\, i \neq j} K_i \cap K_j\bigr) \quad \text{and} \quad \mathcal{P} := \textstyle\bigcup_{m \in \mathbb{N}} \sigma^m(\mathcal{C}). \tag{2.1}$$

$\mathcal{L}$ is called *post-critically finite*, or *p.c.f.* for short, if and only if $\mathcal{P}$ is a finite set.

(2) We set $V_0 := \pi(\mathcal{P})$, $V_m := \bigcup_{w \in W_m} F_w(V_0)$ for $m \in \mathbb{N}$ and $V_* := \bigcup_{m \in \mathbb{N}} V_m$.

V_0 should be considered as the *"boundary"* of the self-similar set K; recall that $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$ for any $w, v \in W_*$ with $\Sigma_w \cap \Sigma_v = \emptyset$ by [18, Proposition 1.3.5-(2)]. Note that $V_{m-1} \subset V_m$ for any $m \in \mathbb{N}$ by [18, Lemma 1.3.11].

From now on our self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is always assumed to be post-critically finite with K connected, so that $\#V_0 \geq 2$ and V_* is dense in K.

Next we briefly describe the construction of a Dirichlet form on K; see [18, Chapter 3] for details. Let $D = (D_{pq})_{p,q \in V_0}$ be a real symmetric matrix of size $\#V_0$ (which we also regard as a linear operator on $\mathbb{R}^{V_0}$) such that

(D1) $\{u \in \mathbb{R}^{V_0} \mid Du = 0\} = \mathbb{R}\mathbf{1}_{V_0}$,
(D2) $D_{pq} \geq 0$ for any $p, q \in V_0$ with $p \neq q$.

We define $\mathcal{E}^{(0)}(u, v) := -\sum_{p,q \in V_0} D_{pq} u(q) v(p)$ for $u, v \in \mathbb{R}^{V_0}$, so that $(\mathcal{E}^{(0)}, \mathbb{R}^{V_0})$ is a Dirichlet form on $L^2(V_0, \#)$. Furthermore let $\mathbf{r} = (r_i)_{i \in S} \in (0, \infty)^S$ and define

$$\mathcal{E}^{(m)}(u, v) := \sum_{w \in W_m} \frac{1}{r_w} \mathcal{E}^{(0)}(u \circ F_w|_{V_0}, v \circ F_w|_{V_0}), \quad u, v \in \mathbb{R}^{V_m} \tag{2.2}$$

for each $m \in \mathbb{N}$, where $r_w := r_{w_1} r_{w_2} \cdots r_{w_m}$ for $w = w_1 w_2 \cdots w_m \in W_m$ ($r_\emptyset := 1$).

Definition 2.4 The pair $(D, \mathbf{r})$ with D and $\mathbf{r}$ as above is called a *harmonic structure on* $\mathcal{L}$ if and only if $\mathcal{E}^{(0)}(u, u) = \inf_{v \in \mathbb{R}^{V_1},\, v|_{V_0} = u} \mathcal{E}^{(1)}(v, v)$ for any $u \in \mathbb{R}^{V_0}$; note that then $\mathcal{E}^{(m)}(u, u) = \min_{v \in \mathbb{R}^{V_{m+1}},\, v|_{V_m} = u} \mathcal{E}^{(m+1)}(v, v)$ for any $m \in \mathbb{N} \cup \{0\}$ and any $u \in \mathbb{R}^{V_m}$. If $\mathbf{r} \in (0, 1)^S$ in addition, then $(D, \mathbf{r})$ is called *regular*.

In the rest of this paper, we assume that $(D, \mathbf{r})$ is a *regular* harmonic structure on $\mathcal{L}$. Let $d_H \in (0, \infty)$ be such that $\sum_{i \in S} r_i^{d_H} = 1$, and let μ be the *self-similar measure on K with weight* $(r_i^{d_H})_{i \in S}$, i.e. the unique Borel measure on K such that $\mu(K_w) = r_w^{d_H}$ for any $w \in W_*$. We set $d_s := 2d_H/(d_H + 1)$, which is called the *spectral dimension*. In this case, $\{\mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m})\}_{m \in \mathbb{N} \cup \{0\}}$ is non-decreasing and hence has the limit in $[0, \infty]$ for any $u \in C(K)$. Then we define

$$\begin{aligned} \mathcal{F} &:= \{u \in C(K) \,|\, \lim_{m \to \infty} \mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m}) < \infty\}, \\ \mathcal{E}(u, v) &:= \lim_{m \to \infty} \mathcal{E}^{(m)}(u|_{V_m}, v|_{V_m}) \in \mathbb{R}, \quad u, v \in \mathcal{F}, \end{aligned} \tag{2.3}$$

so that $(\mathcal{E}, \mathcal{F})$ possesses the following self-similarity: for any $u, v \in \mathcal{F}$,

$$u \circ F_i \in \mathcal{F} \text{ for any } i \in S \quad \text{and} \quad \mathcal{E}(u, v) = \sum_{i \in S} \frac{1}{r_i} \mathcal{E}(u \circ F_i, v \circ F_i). \tag{2.4}$$

By [18, Theorem 3.3.4], $(\mathcal{E}, \mathcal{F})$ is a resistance form on K whose resistance metric $R : K \times K \to [0, \infty)$ is compatible with the original topology of K, and then

[20, Corollary 6.4 and Theorems 9.4], (2.4) and $\mathcal{E}(\mathbf{1},\mathbf{1})=0$ together imply that $(\mathcal{E},\mathcal{F})$ is a strong local regular Dirichlet form on $L^2(K,\mu)$. See [18, Definition 2.3.1] or [20, Definition 3.1] for the definition of resistance forms and their resistance metrics, and see [10, Section 1.1] for the definition of regular Dirichlet forms and their strong locality. Furthermore by [20, Theorem 10.4] (or by [18, Section 5.1]), the Markovian semigroup $\{T_t\}_{t\in(0,\infty)}$ on $L^2(K,\mu)$ associated with $(\mathcal{E},\mathcal{F})$ admits a unique continuous function $p=p_t(x,y):(0,\infty)\times K\times K\to[0,\infty)$, called the *heat kernel of* $(K,\mu,\mathcal{E},\mathcal{F})$, such that for each $f\in L^2(K,\mu)$ and $t\in(0,\infty)$,

$$T_t f=\int_K p_t(\cdot,y)f(y)d\mu(y) \quad \mu\text{-a.e.} \tag{2.5}$$

Also by [18, Corollary 5.3.2] (or by [20, Theorem 15.10]; see the proof of Lemma 2.5 below), there exist $c_1,c_2\in(0,\infty)$ such that for any $x\in K$,

$$c_1\le t^{d_s/2}p_t(x,x)\le c_2,\quad t\in(0,1], \tag{2.6}$$

where $d_s=2d_H/(d_H+1)$ is the spectral dimension defined above.

Now we prepare several preliminary lemmas. The following lemma is standard.

Lemma 2.5 *There exist* $c_3,c_4,c_5\in(0,\infty)$ *such that for any* $(t,x,y)\in(0,1]\times K\times K$,

$$|p_t(x,x)-p_t(y,y)|\le c_3R(x,y)^{1/2}t^{-(d_s+2)/4}, \tag{2.7}$$

$$p_t(x,y)\le c_4t^{-d_s/2}\exp\left(-c_5\left(\frac{R(x,y)^{d_H+1}}{t}\right)^{1/d_H}\right). \tag{2.8}$$

Proof (2.7) is immediate from [20, (3.1) and Lemma 10.8-(2)] and (2.6) (or from [16, Lemma 5.2]). We easily see from [18, Lemmas 3.3.5 and 4.2.3] and (2.4) (see also [18, Lemma 4.2.4]) that $c_6s^{d_H}\le\mu(B_s(x,R))\le c_7s^{d_H}$ for any $(s,x)\in(0,\operatorname{diam}_R K]\times K$ for some $c_6,c_7\in(0,\infty)$, where $\operatorname{diam}_R K:=\sup_{x,y\in K}R(x,y)$ and $B_s(x,R):=\{y\in K\mid R(x,y)<s\}$. Therefore an application of [20, Theorem 15.10] yields (2.8). □

Remark 2.6 The power $1/d_H$ *in the exponential in the right-hand side of* (2.8) *is not best possible in general.* Under the same framework, Hambly and Kumagai [16] have obtained a sharp two-sided estimate of $p_t(x,y)$.

Lemma 2.7 *Let* U *be a non-empty open subset of* K *and set* $\mu|_U:=\mu|_{\mathcal{B}(U)}$, $\mathcal{F}_U:=\{u\in\mathcal{F}\mid u|_{K\setminus U}=0\}$ *and* $\mathcal{E}^U:=\mathcal{E}|_{\mathcal{F}_U\times\mathcal{F}_U}$. *Then* $(\mathcal{E}^U,\mathcal{F}_U)$ *is a strong local regular Dirichlet form on* $L^2(U,\mu|_U)$ *whose associated Markovian semigroup* $\{T_t^U\}_{t\in(0,\infty)}$ *admits a unique continuous integral kernel* $p^U=p_t^U(x,y):(0,\infty)\times U\times U\to[0,\infty)$, *called the* Dirichlet heat kernel on U, *similarly to* (2.5). *Moreover,* p^U *is extended to a continuous function on* $(0,\infty)\times K\times K$ *by setting* $p^U:=0$ *on* $(0,\infty)\times(K\times K\setminus U\times U)$, *and* $p_t^U(x,y)\le p_t(x,y)$ *for any* $(t,x,y)\in(0,\infty)\times K\times K$.

Proof This is immediate from [20, Theorem 10.4]. □

Lemma 2.8 *Let U be a non-empty open subset of K. Then for any $(t, x, y) \in (0, \infty) \times U \times U$,*

$$p_t(x, y) - p_t^U(x, y) \le \sup_{s \in [t/2, t]} \sup_{z \in \overline{U} \setminus U} p_s(x, z) + \sup_{s \in [t/2, t]} \sup_{z \in \overline{U} \setminus U} p_s(z, y). \quad (2.9)$$

Proof This is immediate from [14, Theorem 5.1] (or [13, Theorem 10.4]) and the continuity of the heat kernels $p_t(x, y)$ and $p_t^U(x, y)$. □

Finally we relate the non-existence of the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ to properties of eigenvalues and eigenfunctions of the Laplacian. Let Δ be the non-positive self-adjoint operator (*"Laplacian"*) associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ and let $\mathcal{D}[\Delta]$ be its domain. Recall that $\mathcal{D}[\Delta] \subset \mathcal{F}$ and that for $u \in \mathcal{F}$ and $f \in L^2(K, \mu)$,

$$u \in \mathcal{D}[\Delta] \text{ and } -\Delta u = f \quad \text{if and only if} \quad \mathcal{E}(u, v) = \int_K f v \, d\mu \text{ for any } v \in \mathcal{F}. \quad (2.10)$$

Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a complete orthonormal system of $L^2(K, \mu)$ such that for each $n \in \mathbb{N}$, φ_n *is an eigenfunction of* Δ, i.e. $\varphi_n \in \mathcal{D}[\Delta]$ and $-\Delta\varphi_n = \lambda_n \varphi_n$ for some $\lambda_n \in \mathbb{R}$. Such $\{\varphi_n\}_{n \in \mathbb{N}}$ exists since Δ has compact resolvent by [20, Lemma 9.7], and then necessarily $\{\lambda_n\}_{n \in \mathbb{N}} \subset [0, \infty)$ and $\lim_{n \to \infty} \lambda_n = \infty$. Therefore without loss of generality we assume that $\{\lambda_n\}_{n \in \mathbb{N}}$ is non-decreasing, and note that $\lambda_1 = 0 < \lambda_2$.

Lemma 2.9 *Let $x \in K$. Then the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ exists if and only if so does the limit*

$$\lim_{\lambda \to \infty} \frac{\sum_{n \in \mathbb{N},\, \lambda_n \le \lambda} \varphi_n(x)^2}{\lambda^{d_s/2}}. \quad (2.11)$$

Proof [20, Proof of Lemma 10.7] tells us that

$$p_t(x, y) = \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y), \quad (t, x, y) \in (0, \infty) \times K \times K, \quad (2.12)$$

where the series is uniformly absolutely convergent on $[T, \infty) \times K \times K$ for any $T \in (0, \infty)$. Let $x \in K$ and set $\mathcal{N}_x(\lambda) := \sum_{n \in \mathbb{N},\, \lambda_n \le \lambda} \varphi_n(x)^2$ for $\lambda \in \mathbb{R}$. Then $p_t(x, x) = \int_{[0,\infty)} e^{-\lambda t} d\mathcal{N}_x(\lambda)$ for any $t \in (0, \infty)$ by (2.12), and the assertion follows by Karamata's Tauberian theorem [8, p. 445, Theorem 2]; note that (2.6) and [7, Theorem 1] yield $0 < \inf_{\lambda \in [1,\infty)} \lambda^{-d_s/2} \mathcal{N}_x(\lambda) \le \sup_{\lambda \in [1,\infty)} \lambda^{-d_s/2} \mathcal{N}_x(\lambda) < \infty$. □

Lemma 2.10 *The limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ does not exist for any $x \in K$ satisfying*

$$\limsup_{n \to \infty} \frac{\varphi_n(x)^2}{\lambda_n^{d_s/2}} > 0. \quad (2.13)$$

Proof Let $x \in K$ satisfy (2.13), and for $\lambda \in \mathbb{R}$ let $\mathcal{N}_x(\lambda)$ be as in the previous proof. Then since

$$\limsup_{n\to\infty} \frac{\mathcal{N}_x(\lambda_n) - \mathcal{N}_x(\lambda_n - 1)}{\lambda_n^{d_s/2}} \geq \limsup_{n\to\infty} \frac{\varphi_n(x)^2}{\lambda_n^{d_s/2}} > 0,$$

the limit (2.11) cannot exist and hence neither does the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ by Lemma 2.9. □

Lemma 2.10 will play fundamental roles in the proofs of our main results below.

3 Symmetry group and oscillation at "generic" points

Throughout this section and the next, we follow the framework described in the previous section. Namely, $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a post-critically finite self-similar structure with K connected and $\#S \geq 2$, $(D, \mathbf{r} = (r_i)_{i\in S})$ is a regular harmonic structure on $\mathcal{L}$, and μ is the self-similar measure on K with weight $(r_i^{d_H})_{i\in S}$. Also, $(\mathcal{E}, \mathcal{F})$ is the resistance form on K associated with $(D, \mathbf{r})$ as in (2.3), $R : K \times K \to [0,\infty)$ is the resistance metric of $(\mathcal{E}, \mathcal{F})$, and $p = p_t(x, y) : (0,\infty) \times K \times K \to [0,\infty)$ is the heat kernel of $(K, \mu, \mathcal{E}, \mathcal{F})$.

In this section, we establish the non-existence of the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ for a *"generic"* point $x \in K$ under the assumption of a certain symmetry of $(K, \mu, \mathcal{E}, \mathcal{F})$, following closely the arguments in [18, Section 4.4] and [2, Sections 5 and 6].

Let us start with the following definition. Note that $\pi(A) \in \mathcal{B}(K)$ for any $A \in \mathcal{B}(\Sigma)$.

Definition 3.1 For each $Z \subset K$, we define $Z_* \in \mathcal{B}(K)$ by

$$Z_* := \{x \in K \mid \lim_{m\to\infty} \mathrm{dist}_\rho(\pi(\sigma^m(\omega)), Z) = 0 \text{ for any } \omega \in \pi^{-1}(x)\}, \quad (3.1)$$

which is independent of a particular choice of the metric ρ on K.

Then we have the following easy proposition. Note that any Borel measure on K vanishing on V_* is of the form $\nu \circ \pi^{-1}$ with ν a Borel measure on Σ, since $\pi|_{\Sigma\setminus\pi^{-1}(V_*)} : \Sigma \setminus \pi^{-1}(V_*) \to K \setminus V_*$ is a continuous bijective map with Borel measurable inverse. Recall that a Borel measure ν on Σ is called *σ-ergodic* if and only if $\nu \circ \sigma^{-1} = \nu$ and $\nu(A)\nu(\Sigma \setminus A) = 0$ for any $A \in \mathcal{B}(\Sigma)$ with $\sigma^{-1}(A) = A$.

Proposition 3.2 *Let Z be a closed subset of K. If ν is a σ-ergodic finite Borel measure on Σ and satisfies $\nu \circ \pi^{-1}(K \setminus Z) > 0$, then $\nu \circ \pi^{-1}(Z_*) = 0$.*

Proof Since Z is closed and $\nu \circ \pi^{-1}(K \setminus Z) > 0$, we can choose $\varepsilon \in (0,\infty)$ so that $\nu \circ \pi^{-1}(\{x \in K \mid \mathrm{dist}_\rho(x, Z) \geq \varepsilon\}) > 0$. Define $A \in \mathcal{B}(\Sigma)$ by

$$A := \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} \sigma^{-m}\big(\pi^{-1}\big(\{x \in K \mid \mathrm{dist}_\rho(x, Z) \geq \varepsilon\}\big)\big).$$

Then $\sigma^{-1}(A) = A$ and $\pi^{-1}(Z_*) \subset \Sigma \setminus A$. By virtue of $\nu \circ \sigma^{-1} = \nu$, a version [4, Proposition II.5.14] of the Borel–Cantelli lemma yields $\nu(A) > 0$ and hence we have $\nu \circ \pi^{-1}(Z_*) \leq \nu(\Sigma \setminus A) = 0$ by the σ-ergodicity of ν. □

The following definition is fundamental for the arguments of this section.

Definition 3.3 (1) We define the *symmetry group* $\mathcal{G}$ *of* $(\mathcal{L}, (D, \mathbf{r}), \mu)$ by

$$\mathcal{G} := \left\{ g \;\middle|\; \begin{array}{l} g \text{ is a homeomorphism from } K \text{ to itself, } g(V_0) = V_0, \mu \circ g = \mu, \\ u \circ g, u \circ g^{-1} \in \mathcal{F} \text{ and } \mathcal{E}(u \circ g, u \circ g) = \mathcal{E}(u, u) \text{ for any } u \in \mathcal{F} \end{array} \right\}, \tag{3.2}$$

which clearly forms a subgroup of the group of homeomorphisms of K.
(2) For a finite subgroup G of $\mathcal{G}$ and $h \in \mathcal{G}$, we define $\mathcal{S}(G, h)$ and $\mathcal{S}_*(G, h)$ by

$$\mathcal{S}(G, h) := \bigcup_{g \in G} \{x \in K \mid h^{-1}g(x) = x\}, \quad \mathcal{S}_*(G, h) := (\mathcal{S}(G, h) \cup V_0)_*. \tag{3.3}$$

(3) For $g \in \mathcal{G}$ and $u : K \to \mathbb{R}$, we define $T_g u := u \circ g^{-1}$, so that T_g defines a linear surjective isometry $T_g : L^2(K, \mu) \to L^2(K, \mu)$ by virtue of $\mu \circ g = \mu$.

In the situation of Definition 3.3-(2), $\mathcal{S}(G, h)$ is closed in K, $V_* \subset \mathcal{S}_*(G, h)$ since $\sigma^m(\pi^{-1}(V_m)) = \mathcal{P}$ for $m \in \mathbb{N} \cup \{0\}$ by [18, Proposition 1.3.5-(1)], and Proposition 3.2 says that $\mathcal{S}_*(G, h)$ may be considered as *"measure-theoretically small"* if $\mathcal{S}(G, h) \neq K$. Keeping this observation in mind, now we state the main theorem of this section.

Theorem 3.4 *Suppose that a finite subgroup G of $\mathcal{G}$ and $h \in \mathcal{G} \setminus G$ satisfy $\mathcal{S}(G, h) \neq K$ and $h^{-1}(q) \in \{g(q) \mid g \in G\}$ for any $q \in V_0$. Then the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ does not exist for any $x \in K \setminus \mathcal{S}_*(G, h)$. If in addition the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ does not exist for any $x \in \mathcal{S}(G, h) \setminus V_0$, then neither does it for any $x \in K \setminus V_*$.*

In view of $V_* \subset \mathcal{S}_*(G, h)$, Theorem 3.4 tells us nothing about the non-existence of the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ for $x \in V_*$, which we will establish in Sect. 6 below in the case of certain examples such as Sierpinski gaskets and polygaskets.

The rest of this section is devoted to the proof of Theorem 3.4. The essential part is the proof of the following two lemmas.

Lemma 3.5 *Suppose that a finite subgroup G of $\mathcal{G}$ and $h \in \mathcal{G} \setminus G$ satisfy $\mathcal{S}(G, h) \neq K$ and $h^{-1}(q) \in \{g(q) \mid g \in G\}$ for any $q \in V_0$. Then for each $x \in K \setminus (\mathcal{S}(G, h) \cup V_0)$, there exists an eigenfunction φ_x of Δ such that $\varphi_x|_{V_0} = 0$ and $\varphi_x(x) \neq 0$.*

Proof We follow [18, Proof of Theorem 4.4.4]. We define R_G, $R_{G,h}$, $R^*_{G,h}$ by

$$R_G := (\#G)^{-1} \textstyle\sum_{g \in G} T_g, \quad R_{G,h} := R_G T_{h^{-1}} - R_G, \quad R^*_{G,h} := T_h R_G - R_G, \tag{3.4}$$

so that $\int_K (R_{G,h} u) v d\mu = \int_K u R^*_{G,h} v d\mu$ for $u, v \in L^2(K, \mu)$, and $R_{G,h} u, R^*_{G,h} v \in \mathcal{F}$ and $\mathcal{E}(R_{G,h} u, v) = \mathcal{E}(u, R^*_{G,h} v)$ for any $u, v \in \mathcal{F}$. Moreover for $u \in C(K)$ and

$q \in V_0$, $h^{-1}(q) = g^{-1}(q)$ for some $g \in G$ and hence $R^*_{G,h}u(q) = R_G u(g^{-1}(q)) - R_G u(q) = 0$, from which it follows that $R^*_{G,h}(\mathcal{F}) \subset \mathcal{F}_{K\setminus V_0}$.

Let $x \in K \setminus (\mathcal{S}(G,h) \cup V_0)$. Since $V_0 \cup \{g(x) \mid g \in G\}$ is finite and does not contain $h(x)$, we can choose $u \in \mathcal{F}_{K\setminus V_0}$ so that $u \geq 0$, $u(h(x)) = 1$ and $u(g(x)) = 0$ for $g \in G$. Then $(\#G)R_{G,h}u(x) = \sum_{g\in G}\bigl(u(hg(x)) - u(g(x))\bigr) \geq u(h(x)) = 1$. Let $\{\varphi^0_n\}_{n\in\mathbb{N}}$ be a complete orthonormal system of $L^2(K,\mu)$ consisting of eigenfunctions of the non-positive self-adjoint operator on $L^2(K,\mu|_{K\setminus V_0})$ associated with $(\mathcal{E}^{K\setminus V_0}, \mathcal{F}_{K\setminus V_0})$; such $\{\varphi^0_n\}_{n\in\mathbb{N}}$ exists by [20, Lemma 9.7]. Then letting $u_n := \sum_{k=1}^n \bigl(\int_K u\varphi^0_k d\mu\bigr)\varphi^0_k$ for $n \in \mathbb{N}$, we see from [20, (3.1)] that $\|u - u_n\|^2_\infty \leq (\mathrm{diam}_R K)\mathcal{E}(u - u_n, u - u_n) \to 0$ as $n \to \infty$. Thus $\lim_{n\to\infty} R_{G,h}u_n(x) = R_{G,h}u(x) \geq (\#G)^{-1}$, and it follows that $R_{G,h}\varphi^0_j(x) \neq 0$ for some $j \in \mathbb{N}$. Now by using $R^*_{G,h}(\mathcal{F}) \subset \mathcal{F}_{K\setminus V_0}$ and (2.10) for $(\mathcal{E}^{K\setminus V_0}, \mathcal{F}_{K\setminus V_0})$ we can easily verify that $\varphi_x := R_{G,h}\varphi^0_j \in \mathcal{F}_{K\setminus V_0}$ is an eigenfunction of Δ with $\varphi_x(x) \neq 0$. □

Lemma 3.6 *Let* $\omega \in \Sigma$ *and* $y \in K \setminus V_0$. *If* $\liminf_{m\to\infty} \rho(\pi(\sigma^m(\omega)), y) = 0$ *and the limit* $\lim_{t\downarrow 0} t^{d_s/2}p_t(y,y)$ *does not exist, then the limit* $\lim_{t\downarrow 0} t^{d_s/2}p_t(\pi(\omega),\pi(\omega))$ *does not exist, either.*

Proof Set $x := \pi(\omega)$. By the assumption we have $\lim_{k\to\infty} R(\pi(\sigma^{m_k}(\omega)), y) = 0$ for some strictly increasing sequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$. Let $k \in \mathbb{N}$ be large enough so that $R(\pi(\sigma^{m_k}(\omega)), y) \leq \mathrm{dist}_R(y, V_0)/2 =: D_y$, and set $w_k := [\omega]_{m_k}$, $x_k := F^{-1}_{w_k}(x) = \pi(\sigma^{m_k}(\omega))$, $\tau_k := r_{w_k}^{-(d_H+1)}$ and $K^I_k := K_{w_k} \setminus F_{w_k}(V_0)$. Then K^I_k is open in K since $K \setminus K^I_k = F_{w_k}(V_0) \cup \bigcup_{w\in W_{m_k}\setminus\{w_k\}} K_w$. By [19, Theorem A.1] there exists $c_8 \in (0,1]$ such that $R(F_w(x_1), F_w(x_2)) \geq c_8 r_w R(x_1,x_2)$ for any $w \in W_*$ and $x_1, x_2 \in K$, and therefore

$$R(x, F_{w_k}(q)) \geq c_8 r_{w_k} R\bigl(x_k, q\bigr) \geq c_8 D_y r_{w_k}, \quad q \in V_0. \tag{3.5}$$

Let $t \in (0, \tau_k^{-1}]$. Then Lemmas 2.5, 2.7, 2.8 and (3.5) together yield

$$0 \leq p_t(x,x) - p_t^{K^I_k}(x,x) \leq 4c_4 t^{-d_s/2}\exp\bigl(-c_y(\tau_k t)^{-1/d_H}\bigr), \tag{3.6}$$

$$0 \leq p_{\tau_k t}(x_k,x_k) - p_{\tau_k t}^{K\setminus V_0}(x_k,x_k) \leq 4c_4(\tau_k t)^{-d_s/2}\exp\bigl(-c_y(\tau_k t)^{-1/d_H}\bigr), \tag{3.7}$$

$$\bigl|p_{\tau_k t}(x_k,x_k) - p_{\tau_k t}(y,y)\bigr| \leq c_3 R(x_k,y)^{1/2}(\tau_k t)^{-(d_s+2)/4}, \tag{3.8}$$

where $c_y := c_5(c_8 D_y)^{1+1/d_H}$. Since $t^{d_s/2}p_t^{K^I_k}(x,x) = (\tau_k t)^{d_s/2}p_{\tau_k t}^{K\setminus V_0}(x_k,x_k)$ by (2.3) and (2.4), it follows from (3.6), (3.7) and (3.8) that for any $t \in (0, \tau_k^{-1}]$,

$$\begin{aligned}\bigl|t^{d_s/2}p_t(x,x) - (\tau_k t)^{d_s/2}p_{\tau_k t}(y,y)\bigr| &\leq 4c_4\exp\bigl(-c_y(\tau_k t)^{-1/d_H}\bigr)\\ &\quad + c_3 R(x_k,y)^{1/2}(\tau_k t)^{(d_s-2)/4}.\end{aligned} \tag{3.9}$$

Set $A_y := \limsup_{t\downarrow 0} t^{d_s/2}p_t(y,y) - \liminf_{t\downarrow 0} t^{d_s/2}p_t(y,y) \in (0,\infty)$ and choose $t_y \in (0,1]$ so that $4c_4\exp\bigl(-c_y t_y^{-1/d_H}\bigr) \leq A_y/6$. The definition of A_y tells us that

$t_1^{d_s/2} p_{t_1}(y, y) - t_2^{d_s/2} p_{t_2}(y, y) \geq A_y/2$ for some $t_1, t_2 \in (0, t_y]$. Setting $t = t_1/\tau_k$ and $t = t_2/\tau_k$ in (3.9), from $\lim_{k\to\infty} R(x_k, y) = 0$ we easily see that

$$\liminf_{k\to\infty}\Bigl((t_1/\tau_k)^{d_s/2} p_{t_1/\tau_k}(x, x) - (t_2/\tau_k)^{d_s/2} p_{t_2/\tau_k}(x, x)\Bigr) \geq A_y/6 > 0,$$

in view of which the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x, x)$ cannot exist since $\tau_k^{-1} = r_{w_k}^{d_H+1} \to 0$ as $k \to \infty$ by $\mathbf{r} \in (0, 1)^S$. □

We also need the following easy lemma.

Lemma 3.7 $(V_0)_* = V_*$. (Here $(V_0)_*$ is of course given by (3.1) with $Z = V_0$).

Proof We have $V_* \subset (V_0)_*$ since $\sigma^m(\pi^{-1}(V_m)) = \mathcal{P}$ for any $m \in \mathbb{N} \cup \{0\}$ by [18, Proposition 1.3.5-(1)]. Let $x \in (V_0)_*$ and $\omega \in \pi^{-1}(x)$. Then from $\pi^{-1}(V_0) = \mathcal{P}$ and $\lim_{m\to\infty} \mathrm{dist}_\rho(\pi(\sigma^m(\omega)), V_0) = 0$ we see that $\lim_{m\to\infty} \mathrm{dist}_\delta(\sigma^m(\omega), \mathcal{P}) = 0$, where δ is a metric on Σ compatible with the product topology of Σ. Since $\mathcal{P}$ is finite and $\sigma(\mathcal{P}) \subset \mathcal{P}$, there exist $n \in \mathbb{N}$ and $w_k, v_k \in W_n$ for $k \in \{1, \dots, \#\mathcal{P}\}$ such that $\mathcal{P} = \{w_k v_k^\infty \mid k \in \{1, \dots, \#\mathcal{P}\}\}$, where $wv^\infty := wvvv\ldots \in \Sigma$ for $w, v \in W_n$ in the natural manner. Take $\varepsilon \in (0, \infty)$ such that $[\tau]_{3n} = [\kappa]_{3n}$ for any $\tau, \kappa \in \Sigma$ with $\delta(\tau, \kappa) < \varepsilon$, and choose $N \in \mathbb{N}$ so that $\mathrm{dist}_\delta(\sigma^{mn}(\omega), \mathcal{P}) < \varepsilon$ for any $m \geq N$. Then for each $m \geq N$, $\delta(\sigma^{mn}(\omega), w_{k_m} v_{k_m}^\infty) < \varepsilon$ for some $k_m \in \{1, \dots, \#\mathcal{P}\}$, hence $[\sigma^{mn}(\omega)]_{3n} = [w_{k_m} v_{k_m}^\infty]_{3n}$, and it turns out that $v_{k_m} = v_{k_{m+1}}$ for $m \geq N$. Thus $\sigma^{Nn}(\omega) = w_{k_N} v_{k_N}^\infty \in \mathcal{P}$ and $x = F_{[\omega]_{Nn}}(\pi(\sigma^{Nn}(\omega))) \in V_*$. □

Proof of Theorem 3.4 Let $x \in K \setminus \mathcal{S}_*(G, h)$, so that $x \notin V_*$, and let $\omega \in \pi^{-1}(x)$. Then $\limsup_{m\to\infty} \mathrm{dist}_\rho(\pi(\sigma^m(\omega)), \mathcal{S}(G, h) \cup V_0) > 0$, and by the compactness of K there exist $y \in K \setminus (\mathcal{S}(G, h) \cup V_0)$ and a strictly increasing sequence $\{m_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $\lim_{k\to\infty} \rho(\pi(\sigma^{m_k}(\omega)), y) = 0$. By Lemma 3.5 we can take an eigenfunction φ_y of $-\Delta$ with eigenvalue $\lambda \in (0, \infty)$ such that $\varphi_y|_{V_0} = 0$, $\varphi_y(y) > 0$ and $\int_K \varphi_y^2 d\mu = 1$. Let $k \in \mathbb{N}$ be large enough so that $\varphi_y(\pi(\sigma^{m_k}(\omega))) \geq \varphi_y(y)/2$, and define $\varphi_{x,k} \in C(K)$ by $\varphi_{x,k}|_{K_{[\omega]_{m_k}}} := r_{[\omega]_{m_k}}^{-d_H} \varphi_y \circ F_{[\omega]_{m_k}}^{-1}$ and $\varphi_{x,k}|_{K\setminus K_{[\omega]_{m_k}}} := 0$ (recall $\varphi_y|_{V_0} = 0$). Then $\int_K \varphi_{x,k}^2 d\mu = 1$, and (2.3) and (2.4) easily imply that $\varphi_{x,k}$ is an eigenfunction of $-\Delta$ with eigenvalue $\lambda/r_{[\omega]_{m_k}}^{d_H+1}$. Now since $\lim_{k\to\infty} \lambda/r_{[\omega]_{m_k}}^{d_H+1} = \infty$ and

$$\frac{\varphi_{x,k}(x)^2}{\left(\lambda/r_{[\omega]_{m_k}}^{d_H+1}\right)^{d_s/2}} = \frac{\varphi_y(\pi(\sigma^{m_k}(\omega)))^2}{\lambda^{d_s/2}} \geq \frac{\varphi_y(y)^2}{4\lambda^{d_s/2}} > 0,$$

the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x, x)$ does not exist by Lemma 2.10.

For the proof of the second assertion let $x \in \mathcal{S}_*(G, h) \setminus V_*$ and $\omega \in \pi^{-1}(x)$. By Lemma 3.7 we have $\limsup_{m\to\infty} \mathrm{dist}_\rho(\pi(\sigma^m(\omega)), V_0) > 0$, which together with the compactness of K yields $y \in K \setminus V_0$ such that $\liminf_{m\to\infty} \rho(\pi(\sigma^m(\omega)), y) = 0$. Then $y \in (\mathcal{S}(G, h) \cup V_0) \setminus V_0 = \mathcal{S}(G, h) \setminus V_0$ by $x \in \mathcal{S}_*(G, h)$, and the second assertion follows since the non-existence of the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(y, y)$ implies that of the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x, x)$ by virtue of Lemma 3.6. □

4 The case of affine nested fractals

In this section, we recall the definition of affine nested fractals and show that Theorem 3.4 is applicable to them. Throughout this section, we follow the same framework and notation as in the previous section, and furthermore we assume the following:

$$d \in \mathbb{N},\ K \text{ is a compact subset of } \mathbb{R}^d, \text{ and } F_i = f_i|_K \text{ for some contractive similitude } f_i \text{ on } \mathbb{R}^d \text{ for each } i \in S. \tag{4.1}$$

Recall that $f : \mathbb{R}^d \to \mathbb{R}^d$ is called a *contractive similitude on* $\mathbb{R}^d$ if and only if there exist $\alpha \in (0,1)$, $U \in O(d)$ and $b \in \mathbb{R}^d$ such that $f(x) = \alpha Ux + b$ for any $x \in \mathbb{R}^d$. According to [18, Theorem 1.2.3], any finite family of contractive similitudes on $\mathbb{R}^d$ actually gives rise to a self-similar structure satisfying (4.1) by taking the associated self-similar set.

Notation For $x, y \in \mathbb{R}^d$ with $x \neq y$, let $g_{xy} : \mathbb{R}^d \to \mathbb{R}^d$ denote the reflection in the hyperplane $H_{xy} := \{z \in \mathbb{R}^d \mid |z-x| = |z-y|\}$.

First we prove that Theorem 3.4 is applicable if $\#V_0 \geq 3$ and $g_{xy}|_K \in \mathcal{G}$ for any $x, y \in V_0$ with $x \neq y$, following [18, Proof of Theorem 4.4.10]; see Theorem 4.3 below. Later we will see that affine nested fractals with $\#V_0 \geq 3$ satisfy this condition.

Lemma 4.1 *Assume that $g_{xy}(V_0) = V_0$ for any $x, y \in V_0$ with $x \neq y$, and define*

$$G_0 := \{g_{x_1y_1}g_{x_2y_2}\cdots g_{x_ny_n} \mid n \in \mathbb{N},\ x_i, y_i \in V_0,\ x_i \neq y_i,\ i \in \{1,\ldots,n\}\}, \tag{4.2}$$
$$G_1 := \{g_{x_1y_1}g_{x_2y_2}\cdots g_{x_{2n}y_{2n}} \mid n \in \mathbb{N},\ x_i, y_i \in V_0,\ x_i \neq y_i,\ i\in\{1,\ldots,2n\}\}. \tag{4.3}$$

Then for $n \in \mathbb{N}$ and $x_i, y_i \in V_0$ with $x_i \neq y_i$, $i \in \{1,\ldots,n\}$, $g_{x_1y_1}g_{x_2y_2}\cdots g_{x_ny_n} \in G_0 \setminus G_1$ if and only if n is odd. Moreover, $G_0 \ni g \mapsto g|_{V_0}$ is injective and $\#G_0 \leq (\#V_0)!$.

Proof Without loss of generality assume $\sum_{p\in V_0} p = 0_{\mathbb{R}^d}$. Let $g \in G_0$ and choose $n \in \mathbb{N}$ and $x_i, y_i \in V_0$ with $x_i \neq y_i$ so that $g = g_{x_1y_1}g_{x_2y_2}\cdots g_{x_ny_n}$. Then $g \in O(d)$ by $g(V_0) = V_0$, and we have $\det g = (-1)^n$, from which the first assertion is immediate.

Next let $H_0 := \{\sum_{p\in V_0} a_p p \mid (a_p)_{p\in V_0} \in \mathbb{R}^{V_0}\}$, which is a linear subspace of $\mathbb{R}^d$. Since each $g \in G_0$ is the identity on the orthogonal complement of H_0, $G_0 \ni g \mapsto g|_{V_0}$ is injective with $g|_{V_0} : V_0 \to V_0$ bijective and hence $\#G_0 \leq (\#V_0)!$. □

Proposition 4.2 *Assume that $g_{xy}(V_0) = V_0$ for any $x, y \in V_0$ with $x \neq y$, and define*

$$\mathcal{S} := \left\{ x \in K \;\middle|\; \begin{array}{l} g_{x_1y_1}g_{x_2y_2}\cdots g_{x_{2n-1}y_{2n-1}}(x) = x \text{ for some } n \in \mathbb{N} \\ \text{and } x_i, y_i \in V_0 \text{ with } x_i \neq y_i,\ i \in \{1,2,\ldots,2n-1\} \end{array} \right\}. \tag{4.4}$$

Then we have the following statements (recall that $\mathcal{S}_*$ is given by (3.1) with $Z = \mathcal{S}$).

(1) *$\mathcal{S}$ is closed in K and $\mathrm{int}_K\, \mathcal{S} = \emptyset$. If $\#V_0 \geq 3$ then $V_0 \subset \mathcal{S}$ and $V_* \subset \mathcal{S}_*$.*
(2) *If ν is a σ-ergodic finite Borel measure on Σ and satisfies $\nu \circ \pi^{-1}(K \setminus \mathcal{S}) > 0$, then $\nu \circ \pi^{-1}(\mathcal{S}_*) = 0$.*

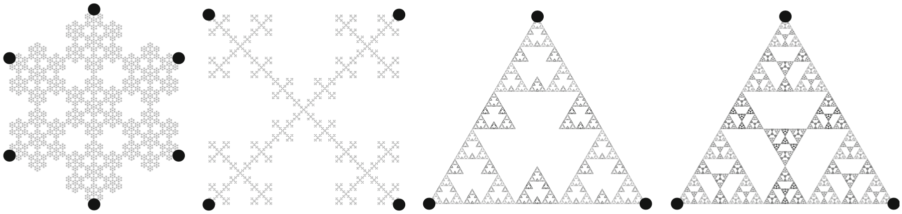

Fig. 3 Some examples of affine nested fractals. From the *left*, snowflake, the Vicsek set, and some modified Sierpinski gaskets

Proof (1) Without loss of generality assume $\sum_{p \in V_0} p = 0_{\mathbb{R}^d}$, and let H_K be the linear subspace of $\mathbb{R}^d$ generated by K. Then for any $g \in G_0 \setminus G_1$, $g|_{H_K}$ is a linear isometry of H_K with determinant -1 by Lemma 4.1, and therefore $\mathrm{int}_K\{x \in K \mid g(x) = x\} = \emptyset$ by virtue of the second assertion of [18, Lemma 4.4.5-(3)], which is in fact valid without assuming $g(K) = K$. Now since $\mathcal{S} = \bigcup_{g \in G_0 \setminus G_1}\{x \in K \mid g(x) = x\}$ and $\#G_0 < \infty$ by Lemma 4.1, $\mathcal{S}$ is closed in K and $\mathrm{int}_K\, \mathcal{S} = \emptyset$. If $\#V_0 \geq 3$, then $g_{xy}g_{yz}g_{zx}(x) = x$ for any distinct $x, y, z \in V_0$ and hence $V_0 \subset \mathcal{S}$, which easily implies $V_* \subset \mathcal{S}_*$.

(2) Since $\mathcal{S}$ is closed in K, this is a special case of Proposition 3.2. □

Now a simple application of Theorem 3.4 yields the following theorem.

Theorem 4.3 *Assume* $\#V_0 \geq 3$ *and that* $g_{xy}|_K \in \mathcal{G}$ *for any* $x, y \in V_0$ *with* $x \neq y$. *Then the limit* $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ *does not exist for any* $x \in K \setminus \mathcal{S}_*$. *If in addition the limit* $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ *does not exist for any* $x \in \mathcal{S} \setminus V_0$, *then neither does it for any* $x \in K \setminus V_*$.

Proof Set $G_1|_K := \{g|_K \mid g \in G_1\}$ and let $h \in G_0 \setminus G_1$. Then by the assumption and Lemma 4.1, $G_1|_K$ is a finite subgroup of $\mathcal{G}$, $h|_K \in \mathcal{G} \setminus G_1|_K$ and $K \neq \mathcal{S} = \bigcup_{g \in G_0 \setminus G_1}\{x \in K \mid g(x) = x\} = \mathcal{S}(G_1|_K, h|_K) \supset V_0$, whence $\mathcal{S}_* = \mathcal{S}_*(G_1|_K, h|_K)$. Moreover, $g_{yz}g_{xz}(x) = y$ and $g_{yz}g_{xz} \in G_1$ for any distinct $x, y, z \in V_0$ and therefore $\{g(q) \mid g \in G_1|_K\} = V_0$ for $q \in V_0$. Now the assertions follow from Theorem 3.4. □

Next we recall the definition of affine nested fractals and apply Theorem 4.3 to them.

Definition 4.4 (1) A homeomorphism $g : K \to K$ is called a *symmetry of* $\mathcal{L}$ if and only if, for any $m \in \mathbb{N} \cup \{0\}$, there exists an injective map $g^{(m)} : W_m \to W_m$ such that $g(F_w(V_0)) = F_{g^{(m)}(w)}(V_0)$ for any $w \in W_m$.

(2) We set $\mathcal{G}_s := \{g \mid g$ is a symmetry of $\mathcal{L}$, $g = f|_K$ for some isometry f of $\mathbb{R}^d\}$.

(3) $\mathcal{L}$ is called an *affine nested fractal* if and only if it is post-critically finite, K is connected and $g_{xy}|_K \in \mathcal{G}_s$ for any $x, y \in V_0$ with $x \neq y$.

(4) We call a real matrix $L = (L_{pq})_{p,q \in V_0}$ $\mathcal{G}_s$*-invariant* if and only if $L_{pq} = L_{g(p)g(q)}$ for any $p, q \in V_0$ and $g \in \mathcal{G}_s$. Also $\mathbf{a} = (a_i)_{i \in S} \in (0, \infty)^S$ is called $\mathcal{G}_s$*-invariant* if and only if $a_i = a_j$ for any $i, j \in S$ satisfying $g(F_i(V_0)) = F_j(V_0)$ for some $g \in \mathcal{G}_s$.

By [18, Propositions 3.8.7 and 3.8.9], if $\mathcal{L}$ is an affine nested fractal, then $L = (L_{pq})_{p,q\in V_0}$ is $\mathcal{G}_s$-invariant if and only if $L_{pq} = L_{p'q'}$ whenever $|p - q| = |p' - q'|$.

Theorem 4.5 *Assume that* $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ *is an affine nested fractal with* $\#V_0 \geq 3$ *and that both* $D = (D_{pq})_{p,q\in V_0}$ *and* $\mathbf{r} = (r_i)_{i\in S}$ *are* $\mathcal{G}_s$*-invariant. Further assume that*

$$\#(F_i(V_0) \cap F_j(V_0)) \leq 1 \qquad \textit{for any } i, j \in S \textit{ with } i \neq j. \tag{4.5}$$

Then the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x, x)$ *does not exist for any* $x \in K \setminus \mathcal{S}_*$. *If in addition the limit* $\lim_{t\downarrow 0} t^{d_s/2} p_t(x, x)$ *does not exist for any* $x \in \mathcal{S} \setminus V_0$, *then neither does it for any* $x \in K \setminus V_*$.

Proof In view of Theorem 4.3, it suffices to show $\mathcal{G}_s \subset \mathcal{G}$. Let $m \in \mathbb{N} \cup \{0\}$ and suppose $\mu \circ g(K_w) = \mu(K_w)$ for any $w \in W_m$ and any $g \in \mathcal{G}_s$. Let $i \in S$, $w \in W_m$ and $g \in \mathcal{G}_s$. Since g is a symmetry of $\mathcal{L}$, $g(F_i(V_0)) = F_j(V_0)$ for some $j \in S$, and by [18, Proposition 3.8.20] there exists $g_i \in \mathcal{G}_s$ such that $g \circ F_i = F_j \circ g_i$. Then $\mu(g(K_{iw})) = \mu \circ F_j(g_i(K_w)) = r_j^{d_H} \mu(g_i(K_w)) = r_i^{d_H} \mu(K_w) = r_i^{d_H} r_w^{d_H} = \mu(K_{iw})$. Thus for any $g \in \mathcal{G}_s$, $\mu \circ g(K_w) = \mu(K_w)$ for any $w \in W_*$ and hence $\mu \circ g = \mu$, which together with [18, Corollary 3.8.21] implies that $\mathcal{G}_s \subset \mathcal{G}$. □

Remark 4.6 (1) The following fact is known for the existence of $\mathcal{G}_s$-invariant harmonic structures (see [18, Section 3.8] and references therein for details):

If $\mathcal{L}$ *is an affine nested fractal and satisfies* (4.5), *then for each* $\mathcal{G}_s$*-invariant* $\mathbf{r} \in (0, \infty)^S$, *there exist a unique* $\lambda \in (0, \infty)$ *and a unique (up to constant multiples)* $\mathcal{G}_s$*-invariant real symmetric matrix* $D = (D_{pq})_{p,q\in V_0}$ *satisfying* (D1), (D2) *such that* $(D, \lambda\mathbf{r})$ *is a harmonic structure on* $\mathcal{L}$.

(2) It is quite unclear whether the assumption (4.5) can be removed from Theorem 4.5 (or more specifically, from [18, Proposition 3.8.20]; see the previous proof and [18, Proof of Corollary 3.8.21]), although (4.5) should be regarded as a technical assumption to avoid nonessential difficulties, as noted in [1, Remark 5.25-2.(c)] and [18, p. 118].

(3) *The non-existence of the limit* $\lim_{t\downarrow 0} t^{d_s/2} p_t(x, x)$ *may or may not occur when* $\#V_0 = 2$. Of course this limit exists for any x in the case [18, Example 3.1.4] of the unit interval $[0, 1]$ with its usual Dirichlet form. On the other hand, Example 4.7 below presents an affine nested fractal with $\#V_0 = 2$ to which Theorem 3.4 applies.

Example 4.7 Following [18, Example 4.4.9], let $S := \{1, 2, 3, 4\}$ and define $f_i : \mathbb{C} \to \mathbb{C}$ for $i \in S$ by $f_1(z) := \frac{1}{2}(z + 1)$, $f_2(z) := \frac{1}{2}(z - 1)$, $f_3(z) := \frac{\sqrt{-1}}{4}(z + 1)$ and $f_4(z) := \frac{\sqrt{-1}}{4}(z - 1)$. Let K be the *self-similar set associated with* $\{f_i\}_{i\in S}$, i.e. the unique non-empty compact subset of $\mathbb{C} \cong \mathbb{R}^2$ that satisfies $K = \bigcup_{i\in S} f_i(K)$, and set $F_i := f_i|_K$, $i \in S$. Then $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is a self-similar structure, and we have $\mathcal{P} = \{1^\infty, 2^\infty\}$ and $V_0 = \{-1, 1\}$. Defining $g, h : \mathbb{C} \to \mathbb{C}$ by $g(z) := -\overline{z}$ and $h(z) := \overline{z}$, we easily see that $g|_K, h|_K \in \mathcal{G}_s$, and thus $\mathcal{L}$ is an affine nested fractal.

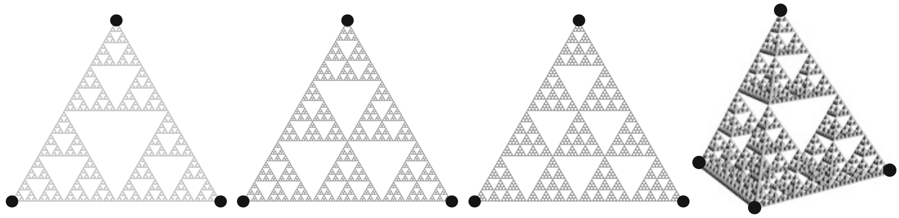

Fig. 4 Sierpinski gaskets. From the *left*, two-dimensional level-l Sierpinski gasket ($l = 2, 3, 4$) and three-dimensional level-2 Sierpinski gasket

Let $D = (D_{pq})_{p,q \in V_0} := \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$, $r \in (0, 1)$ and $\mathbf{r} = (r_i)_{i \in S} := \bigl(\frac{1}{2}, \frac{1}{2}, r, r\bigr)$. Then $(D, \mathbf{r})$ is clearly a regular harmonic structure on $\mathcal{L}$, and similarly to the proof of Theorem 4.5 we can verify $g|_K, h|_K \in \mathcal{G}$. Now since $h|_K \neq \mathrm{id}_K$, $\mathcal{S}(\{\mathrm{id}_K\}, h|_K) = \{x \in K \mid h(x) = x\} \neq K$ and $h(q) = q$ for $q \in V_0$, Theorem 3.4 implies that the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ does not exist for any $x \in K \setminus \mathcal{S}_*(\{\mathrm{id}_K\}, h|_K)$.

5 Examples

In this section, we apply Theorems 3.4 and 4.5 to basic examples. Note that by [18, Theorem 1.6.2], if $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is a self-similar structure, then K is connected if and only if any $i, j \in S$ admit $n \in \mathbb{N}$ and $\{i_k\}_{k=0}^n \subset S$ with $i_0 = i$ and $i_n = j$ such that $K_{i_{k-1}} \cap K_{i_k} \neq \emptyset$ for any $k \in \{1, \ldots, n\}$. Recall that, given a post-critically finite self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ with K connected and a regular harmonic structure $(D, \mathbf{r} = (r_i)_{i \in S})$ on $\mathcal{L}$, we always equip K with the self-similar measure μ on K with weight $(r_i^{d_H})_{i \in S}$, where $d_H \in (0, \infty)$ is such that $\sum_{i \in S} r_i^{d_H} = 1$.

5.1 Sierpinski gaskets

Example 5.1 (Sierpinski gaskets) Let $d, l \in \mathbb{N}$, $d \geq 2$, $l \geq 2$, and let $\{q_k\}_{k=0}^d \subset \mathbb{R}^d$ be the set of the vertices of a regular d-dimensional simplex. Further let $S := \{(i_k)_{k=1}^d \in (\mathbb{N} \cup \{0\})^d \mid \sum_{k=1}^d i_k \leq l - 1\}$, and for each $i = (i_k)_{k=1}^d \in S$ we set $q_i := q_0 + \sum_{k=1}^d (i_k/l)(q_k - q_0)$ and define $f_i : \mathbb{R}^d \to \mathbb{R}^d$ by $f_i(x) := q_i + l^{-1}(x - q_0)$. Let K be the self-similar set associated with $\{f_i\}_{i \in S}$ and set $F_i := f_i|_K$. Then $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is a self-similar structure, which is called the *d-dimensional level-l Sierpinski gasket* (see Fig. 4 above). This is an affine nested fractal satisfying (4.5), and we have $\mathcal{P} = \{\boldsymbol{i}_k^\infty \mid k \in \{0, 1, \ldots, d\}\}$ and $V_0 = \{q_k \mid k \in \{0, 1, \ldots, d\}\}$, where $\boldsymbol{i}_k := ((l-1)\mathbf{1}_{\{k\}}(j))_{j=1}^d \in S$. Moreover, $\mathcal{G}_s = \{g|_K \mid g \in G_0\}$ (recall (4.2)).

Define $D = (D_{pq})_{p,q \in V_0}$ by $D_{pp} := -d$ and $D_{pq} := 1$ for $p, q \in V_0$, $p \neq q$. Note that any $\mathcal{G}_s$-invariant real symmetric matrix satisfying (D1), (D2) is a constant multiple of D. By the symmetry of $\mathcal{L}$ and D, there exists a unique $r \in (0, \infty)$ such that $(D, \mathbf{r} = (r_i)_{i \in S})$ with $r_i := r$ is a harmonic structure on $\mathcal{L}$. Moreover, [18, Corollary 3.1.9] yields $r < 1$, so that $(D, \mathbf{r})$ is a regular harmonic structure on $\mathcal{L}$.

The d-dimensional level-2 Sierpinski gasket (i.e. the case of $l = 2$) is also referred to as the *d-dimensional standard Sierpinski gasket*, for which we can easily verify that $r = (d+1)/(d+3)$ and hence that $d_s = 2\log_{d+3}(d+1)$. Unfortunately, however, it seems impossible to calculate the value of r explicitly for a general d-dimensional level-l Sierpinski gasket.

For this example, the assumptions of Theorem 4.5 are clearly satisfied and hence the non-existence of the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ is assured for any $x \in K \setminus \mathcal{S}_*$. In fact, since the d-dimensional level-l Sierpinski gasket possesses a quite large group of symmetries, we can conclude a slightly stronger result as follows.

Theorem 5.2 *Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be the d-dimensional level-l Sierpinski gasket with $d \geq 2$, $l \geq 2$ and let $(D, \mathbf{r})$ be the harmonic structure on $\mathcal{L}$ as in Example 5.1. Define a closed subset $\hat{\mathcal{S}}$ of K by*

$$\hat{\mathcal{S}} := \bigcap\nolimits_{I\subset\{0,\ldots,d\},\,\#I=3} \bigcup\nolimits_{i,j\in I,\,i\neq j} \{x \in K \mid g_{q_i q_j}(x) = x\}. \tag{5.1}$$

Then the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ does not exist for any $x \in K \setminus \hat{\mathcal{S}}_*$ (recall that $\hat{\mathcal{S}}_*$ is given by (3.1) with $Z = \hat{\mathcal{S}}$). *If in addition the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ does not exist for any $x \in \hat{\mathcal{S}} \setminus V_0$, then neither does it for any $x \in K \setminus V_*$.*

Proof For each $I \subset \{0,\ldots,d\}$ with $\#I = 3$, we define $h_I := g_{q_i q_j}|_K$ and $G_I := \{\mathrm{id}_K, g_{q_i q_k} g_{q_i q_j}|_K, g_{q_i q_j} g_{q_i q_k}|_K\}$, where $I = \{i, j, k\}$, $i < j < k$, so that G_I is a subgroup of $\mathcal{G}$ and $h_I \in \mathcal{G} \setminus G_I$. Theorem 3.4 implies that the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ does not exist for any $x \in K \setminus \mathcal{S}_*(G_I, h_I)$, which yields the first assertion since

$$\bigcap\nolimits_{I\subset\{0,\ldots,d\},\,\#I=3} \mathcal{S}_*(G_I, h_I) = \Big(\bigcap\nolimits_{I\subset\{0,\ldots,d\},\,\#I=3} \mathcal{S}(G_I, h_I)\Big)_* = \hat{\mathcal{S}}_*$$

by the compactness of $\mathcal{S}(G_I, h_I)$. Similarly to the second paragraph of the proof of Theorem 3.4, the second assertion follows from Lemmas 3.6 and 3.7. □

Note that $\hat{\mathcal{S}} \subset V_*$ if and only if $l = 2$; indeed, if $l \geq 3$ then by setting $i := (\mathbf{1}_{[1,l)}(k))_{k=1}^d \in S$ we have $\pi(i^\infty) = q_0 + (l-1)^{-1} \sum_{k=1}^{\min\{l-1,d\}} (q_k - q_0) \in \hat{\mathcal{S}} \setminus V_*$, whereas we easily see $\hat{\mathcal{S}} \subset V_*$ when $l = 2$. This fact will be used in the next section to show that the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(x,x)$ does not exist for any $x \in K$ when $l = 2$.

5.2 Polygaskets

Example 5.3 (N-polygasket) Let $N \in \mathbb{N}$ satisfy $N \geq 3$ and $N/4 \notin \mathbb{N}$. Let $S := \{0, 1, \ldots, N-1\}$, and for each $i \in S$ we set $q_i := e^{2\pi(i/N)\sqrt{-1}} \in \mathbb{C} \cong \mathbb{R}^2$ and define $f_i : \mathbb{C} \to \mathbb{C}$ by $f_i(z) := q_i + \alpha_N(z - q_i)$, where

$$\alpha_N := \begin{cases} 1 - (1 + 2\sin\frac{\pi}{2N})^{-1} & \text{if } N \text{ is odd,} \\ 1 - (1 + \sin\frac{\pi}{N})^{-1} & \text{if } N \text{ is even.} \end{cases} \tag{5.2}$$

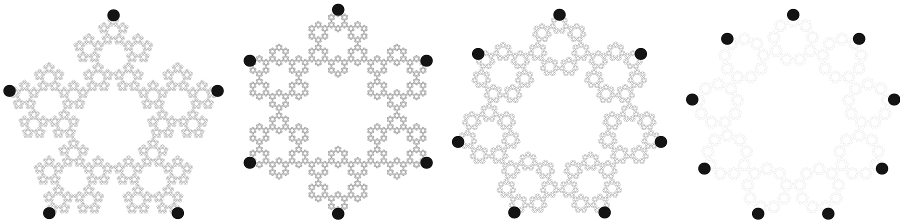

Fig. 5 N-polygasket ($N = 5, 6, 7, 9$). From the *left*, pentagasket ($N = 5$), hexagasket ($N = 6$), heptagasket ($N = 7$) and nonagasket ($N = 9$)

The self-similar structure $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$, with K the self-similar set associated with $\{f_i\}_{i \in S}$ and $F_i := f_i|_K$, is called the *N-polygasket*. The 3-polygasket is nothing but the (two-dimensional standard) Sierpinski gasket, and the N-polygasket for $N = 5, 6, 7, 9$ (Fig. 5) is called the *pentagasket*, *hexagasket*, *heptagasket* and *nonagasket*, respectively. Again $\mathcal{L}$ is an affine nested fractal satisfying (4.5), and it holds that $\mathcal{P} = \{i^\infty \mid i \in S\}$ and $V_0 = \{q_i \mid i \in S\}$. Moreover, $\mathcal{G}_s = \{g|_K \mid g \in G_0\}$.

Remark 5.4 The N-polygasket is suitably defined also for $N \in \mathbb{N}$ with $N/4 \in \mathbb{N}$, but then it satisfies $\#V_0 = \infty$, which is why we have excluded this case in this paper.

In fact, Example 5.3 is a special case of the following example adopted from [5].

Example 5.5 ((N, l)-polygasket) Let $N, l \in \mathbb{N}$, $N \geq 3$, $l < N/2$ and set $S := \{0, 1, \ldots, N-1\}$. For $k \in \mathbb{Z}$, let $[k]$ denote the unique $i \in S$ such that $(k - i)/N \in \mathbb{Z}$. Define an equivalence relation $\sim$ on $\Sigma = S^{\mathbb{N}}$ by saying $\omega \sim \tau$ if and only if either

$$\{\omega, \tau\} = \{wi[i+l]^\infty, w[i+1][i+1-l]^\infty\} \text{ for some } (w, i) \in W_* \times S \quad (5.3)$$

or $\omega = \tau$. Let $K := \Sigma/\sim$ be equipped with the quotient topology and let $\pi : \Sigma \to K$ be the quotient map. For $i \in S$, since $i\omega \sim i\tau$ whenever $\omega, \tau \in \Sigma$ and $\omega \sim \tau$, we can define a continuous injective map $F_i : K \to K$ by $F_i(\pi(\omega)) := \pi(i\omega)$, $\omega \in \Sigma$, so that $F_i \circ \pi = \pi \circ \sigma_i$. We further define $\mathcal{P}$ and V_0 as in Definition 2.3. Then $\mathcal{P} = \{i^\infty \mid i \in S\}$, $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$ for any $w, v \in W_*$ with $\Sigma_w \cap \Sigma_v = \emptyset$, and $\pi^{-1}(K_w \setminus F_w(V_0)) = \Sigma_w \setminus \sigma_w(\mathcal{P})$ for any $w \in W_*$. By using these facts, we easily see that K is a compact metrizable topological space and hence that $\mathcal{L} := (K, S, \{F_i\}_{i \in S})$ is a post-critically finite self-similar structure with K connected. We call $\mathcal{L}$ the *(N, l)-polygasket*. Let $q_i := \pi(i^\infty)$ for $i \in S$, so that $V_0 = \{q_i \mid i \in S\}$.

For $\omega = (\omega_m)_{m \in \mathbb{N}} \in \Sigma$, define $\omega^{\mathbf{1}}, \omega^- \in \Sigma$ by $\omega^{\mathbf{1}} := ([\omega_m + 1])_{m \in \mathbb{N}}$ and $\omega^- := ([-\omega_m])_{m \in \mathbb{N}}$. Then $\omega^{\mathbf{1}} \sim \tau^{\mathbf{1}}$ and $\omega^- \sim \tau^-$ for any $\omega, \tau \in \Sigma$ with $\omega \sim \tau$, and therefore we can define continuous maps $g, h : K \to K$ by $g(\pi(\omega)) := \pi(\omega^{\mathbf{1}})$ and $h(\pi(\omega)) := \pi(\omega^-)$, $\omega \in \Sigma$. Clearly $g(V_0) = h(V_0) = V_0$ and $g^N = h^2 = ghgh = \mathrm{id}_K$, and hence $\hat{G} := \{\mathrm{id}_K, g, \ldots, g^{N-1}, h, hg, \ldots, hg^{N-1}\}$ is a subgroup of the group of symmetries of $\mathcal{L}$ which is isomorphic to the dihedral group of order $2N$ (recall Definition 4.4-(1)). We set $G := \{\mathrm{id}_K, g, \ldots, g^{N-1}\}$, which is a subgroup of $\hat{G}$.

A simple calculation similar to [23, §4.3] immediately shows the existence of a unique $r \in (0, \infty)$ and a unique (up to constant multiples) real symmetric matrix $D = (D_{pq})_{p,q \in V_0}$ with (D1), (D2) and $D_{g(p)g(q)} = D_{h(p)h(q)} = D_{pq}$, $p, q \in V_0$, such that $(D, \mathbf{r} = (r_i)_{i \in S})$ with $r_i := r$ is a harmonic structure on $\mathcal{L}$. In fact,

$$r = \frac{2N}{N + 2l(N - 2l) + \sqrt{(N - 2l(N - 2l))^2 + 8l^2 N}} < 1 \tag{5.4}$$

and thus $(D, \mathbf{r})$ is a regular harmonic structure on $\mathcal{L}$. Then we also have $\hat{G} \subset \mathcal{G}$.

Theorem 3.4 clearly applies to this example to yield the non-existence of the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ for any $x \in K \setminus \mathcal{S}_*(G, h)$. We remark that $\mathcal{S}(G, h) \subset V_*$ if and only if N is odd, which will be used in the next section to show that the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ does not exist for any $x \in K$ when N is odd.

Note that for $N \in \mathbb{N}$ with $N \geq 3$ and $N/4 \notin \mathbb{N}$, the N-polygasket is nothing but the $(N, \lceil N/4 \rceil)$-polygasket, where $\lceil a \rceil := \min\{n \in \mathbb{Z} \mid n \geq a\}$, and that we have $\mathcal{G}_s = \hat{G}$, $\mathcal{S} = \mathcal{S}(G, h)$ and $\mathcal{S}_* = \mathcal{S}_*(G, h)$ in this case.

6 Further results for Sierpinski gaskets and polygaskets

The purpose of this section is to prove the following theorem.

Theorem 6.1 *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be either the d-dimensional level-l Sierpinski gasket with $d \geq 2$, $l \geq 2$ in Example 5.1 or the (N, l)-polygasket with $N \geq 3$, $l < N/2$ in Example 5.5. Also let $(D, \mathbf{r})$ be the harmonic structure on $\mathcal{L}$ described there. Then the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ does not exist for any $x \in V_*$.*

Corollary 6.2 *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be either the d-dimensional **standard** Sierpinski gasket with $d \geq 2$ in Example 5.1 or the (N, l)-polygasket in Example 5.5 with $N \geq 3$ **odd** and $l < N/2$. Also let $(D, \mathbf{r})$ be the harmonic structure on $\mathcal{L}$ described there. Then the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ does not exist for **any** $x \in K$.*

Proof This is immediate from Theorems 3.4, 5.2 and 6.1 since $\hat{\mathcal{S}} \subset V_*$ for the d-dimensional standard Sierpinski gasket and $\mathcal{S}(G, h) \subset V_*$ for the (N, l)-polygasket with N odd, where $\hat{\mathcal{S}}$ is given by (5.1) and G and h are as in Example 5.5. □

The rest of this section is devoted to the proof of Theorem 6.1. First we prove the following lemma, which reduces the proof of Theorem 6.1 to the case of $x \in V_0$.

Lemma 6.3 *Under the same framework and notation as in Sect. 3, let $q \in V_0$ and suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and that $r_i = r$ for any $i \in S$ for some $r \in (0, 1)$. Then there exist $c_9, c_{10} \in (0, \infty)$ such that for any $m \in \mathbb{N} \cup \{0\}$, any $x \in V_m$ and any $t \in (0, 1]$, with $n_{x,m} := \#\{w \in W_m \mid x \in K_w\}$,*

$$\left| n_{x,m} (r^{(d_H+1)m} t)^{d_s/2} p_{r^{(d_H+1)m} t}(x, x) - t^{d_s/2} p_t(q, q) \right| \leq c_9 \exp\left(-c_{10} t^{-1/d_H}\right). \tag{6.1}$$

Proof Let $m \in \mathbb{N} \cup \{0\}$, $x \in V_m$ and set $W_{m,x} := \{w \in W_m \mid x \in K_w\}$. We also set $U_w^x := (K_w \setminus F_w(V_0)) \cup \{x\}$ for $w \in W_{m,x}$ and $U^x := \bigcup_{w \in W_{m,x}} U_w^x$, which is open in K. For each $w \in W_{m,x}$, $x \in K_w \cap V_m = F_w(V_0)$, and hence by $\{g(q) \mid g \in \mathcal{G}\} = V_0$ we can choose $g_w \in \mathcal{G}$ so that $x = F_w(g_w(q))$. Further let $U := (K \setminus V_0) \cup \{q\}$. We claim that for $v \in W_{m,x}$ and for any $(t, y, z) \in (0, \infty) \times K \times K$,

$$p^U_{t/r^{(d_H+1)m}}(y, z) = r^{d_H m} \sum_{w \in W_{m,x}} p_t^{U^x}(F_v \circ g_v(y), F_w \circ g_w(z)), \tag{6.2}$$

which together with (2.8), Lemmas 2.7 and 2.8 easily yields the assertion. Note here that $n_{x,m} \le \#\pi^{-1}(x) \le \#\mathcal{C} \le \#S\#\mathcal{P} < \infty$ by [18, Proof of Lemma 4.2.3] and that $R(F_w(y), F_w(z)) \ge c_8 r_w R(y, z)$ for any $w \in W_*$ and $y, z \in K$ for some $c_8 \in (0, 1]$ by [19, Theorem A.1]. Thus it remains to show (6.2).

For each bijective map $\tau : W_{m,x} \to W_{m,x}$, we define $R_\tau : U^x \to U^x$ by $R_\tau|_{U_w^x} := F_{\tau(w)} \circ g_{\tau(w)} \circ g_w^{-1} \circ F_w^{-1}|_{U_w^x}$. Then R_τ is a homeomorphism with $R_\tau^{-1} = R_{\tau^{-1}}$, and $\mu|_{U^x} \circ R_\tau = \mu|_{U^x}$ since $r_i = r$ for $i \in S$. Moreover, regarding $\mathcal{F}_{U^x}$ as a linear subspace of $C(U^x)$, we have $u \circ R_\tau \in \mathcal{F}_{U^x}$ and $\mathcal{E}(u \circ R_\tau, u \circ R_\tau) = \mathcal{E}(u, u)$ for any $u \in \mathcal{F}_{U^x}$ by (2.3), (2.4) and $r_i = r$, $i \in S$. It easily follows from these facts that

$$T_t^{U^x}(u \circ R_\tau) = (T_t^{U^x} u) \circ R_\tau, \quad t \in (0, \infty),\ u \in L^2(U^x, \mu|_{U^x}). \tag{6.3}$$

On the other hand, for a Borel measurable function $u : U \to \mathbb{R}$ we define a Borel measurable function $\iota_x u : U^x \to \mathbb{R}$ by $\iota_x u|_{U_w^x} := u \circ g_w^{-1} \circ F_w^{-1}|_{U_w^x}$, $w \in W_{m,x}$. Then $\int_{U^x} (\iota_x u)^2 d\mu = n_{x,m} r^{d_H m} \int_U u^2 d\mu$, hence ι_x defines an injective linear operator $\iota_x : L^2(U, \mu|_U) \to L^2(U^x, \mu|_{U^x})$, and furthermore $\iota_x u \in \mathcal{F}_{U^x}$ and $\mathcal{E}(\iota_x u, \iota_x u) = n_{x,m} r^{-m} \mathcal{E}(u, u)$ for any $u \in \mathcal{F}_U$ by (2.3) and (2.4). Based on these facts and (6.3), we can easily verify that for any $t \in (0, \infty)$,

$$T_t^{U^x} \iota_x\big(L^2(U, \mu|_U)\big) \subset \iota_x(\mathcal{F}_U), \quad \iota_x^{-1} T_t^{U^x} \iota_x = T^U_{t/r^{(d_H+1)m}}, \tag{6.4}$$

from which (6.2) immediately follows. □

Remark 6.4 In the situation of Lemma 6.3, there exist $c_{11} \in (0, \infty)$ *and a continuous* $\log(r^{-d_H-1})$*-periodic function* $G : \mathbb{R} \to (0, \infty)$ *such that for any* $x \in V_*$,

$$p_t(x, x) = n_x^{-1} t^{-d_s/2} G(-\log t) + O\big(\exp\big(-c_{11} r^{2m_x/d_s} t^{-1/d_H}\big)\big) \quad \text{as } t \downarrow 0, \tag{6.5}$$

where $m_x := \min\{m \in \mathbb{N} \cup \{0\} \mid x \in V_m\}$ *and* $n_x := \#\{w \in W_{m_x} \mid x \in K_w\}$.

Indeed, it suffices to verify (6.5) for $x = q$ in view of (6.1). We easily see from (6.1) and (2.6) that, for each $x \in V_*$, $n_x = n_{x,m} (= \#\{w \in W_m \mid x \in K_w\})$ for any $m \in \mathbb{N} \cup \{0\}$ satisfying $x \in V_m$. In particular, $n_{q,1} = n_q = 1$, and (6.1) with $m = 1$ and $x = q$ immediately shows (6.5) for $x = q$, similarly to [15, Theorem 5.3].

The assumptions of Lemma 6.3 are clearly satisfied for the d-dimensional level-l Sierpinski gasket and for the (N, l)-polygasket. Thus it suffices to prove the non-existence of the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(x, x)$ for $x \in V_0$. We first treat the case of the

d-dimensional level-l Sierpinski gasket. The proof for the (N, l)-polygasket will be provided later.

Lemma 6.5 *Let $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ be the d-dimensional level-l Sierpinski gasket with $d \geq 2, l \geq 2$ and let $(D, \mathbf{r})$ be the harmonic structure on $\mathcal{L}$ as in Example 5.1. Then there exists an eigenfunction φ of Δ such that $\varphi(q_0) = 1 > |\varphi(q_1)|$ and $\varphi(q_k) = \varphi(q_1)$ for any $k \in \{2, \dots, d\}$* (recall $V_0 = \{q_k \mid k \in \{0, 1, \dots, d\}\}$).

Proof Let G be the subgroup of $\mathcal{G}$ generated by $\{g_{xy}|_K \mid x, y \in V_0 \setminus \{q_0\},\ x \neq y\}$, which is finite by Lemma 4.1, and let $R_G := (\#G)^{-1} \sum_{g \in G} T_g$, so that $R_G(\mathcal{F}) \subset \mathcal{F}$, $\mathcal{E}(R_G u, v) = \mathcal{E}(u, R_G v)$ for $u, v \in \mathcal{F}$ and $\int_K (R_G u) v d\mu = \int_K u R_G v d\mu$ for $u, v \in L^2(K, \mu)$. Then we easily see that $R_G u \in \mathcal{D}[\Delta]$ and $\Delta R_G u = R_G \Delta u$ for any $u \in \mathcal{D}[\Delta]$, and therefore there exist $\{\varphi_n\}_{n \in \mathbb{N}} \subset R_G(\mathcal{F})$ and $\{\psi_n\}_{n \in \mathbb{N}} \subset (T_{\mathrm{id}_K} - R_G)(\mathcal{F})$ such that $\{\varphi_n\}_{n \in \mathbb{N}} \cup \{\psi_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system of $L^2(K, \mu)$ consisting of eigenfunctions of Δ. Note that then for any $n \in \mathbb{N}$, $\varphi_n(q_k) = \varphi_n(q_1)$ for $k \in \{2, \dots, d\}$ and $\psi_n(q_0) = 0$.

Suppose that $|\varphi_n(q_0)| \leq |\varphi_n(q_1)|$ for any $n \in \mathbb{N}$. Let $t \in (0, \infty)$, and for $n \in \mathbb{N}$ let $\lambda_n, \lambda'_n \in [0, \infty)$ be such that $-\Delta\varphi_n = \lambda_n \varphi_n$ and $-\Delta\psi_n = \lambda'_n \psi_n$. Then since $p_t(g(x), g(y)) = p_t(x, y)$ for $g \in \mathcal{G}$ and $x, y \in K$, from (2.12) we get

$$
\begin{aligned}
p_t(q_0, q_0) &= \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \varphi_n(q_0)^2 \leq \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \varphi_n(q_1)^2 \\
&\leq \sum_{n \in \mathbb{N}} \bigl(e^{-\lambda_n t} \varphi_n(q_1)^2 + e^{-\lambda'_n t} \psi_n(q_1)^2 \bigr) = p_t(q_1, q_1) = p_t(q_0, q_0),
\end{aligned}
$$

which means that $\psi_n(q_1) = 0$ for any $n \in \mathbb{N}$. On the other hand, choose $u \in \mathcal{F}$ so that $u(q_1) = 1$ and $u(q_k) = 0$ for $k \in \{2, \dots, d\}$, and set $v := u - R_G u \in (T_{\mathrm{id}_K} - R_G)(\mathcal{F})$. Then $v(q_1) > 0$, but setting $v_n := \sum_{k=1}^n \bigl(\int_K v \psi_k d\mu \bigr) \psi_k$ for $n \in \mathbb{N}$, we have $\|v - v_n\|_\infty^2 \leq (\mathrm{diam}_R\, K) \mathcal{E}(v - v_n, v - v_n) \to 0$ as $n \to \infty$ by [20, (3.1)] and hence $v(q_1) = 0$. This contradiction shows that $|\varphi_j(q_0)| > |\varphi_j(q_1)|$ for some $j \in \mathbb{N}$. Now the function $\varphi := (\varphi_j(q_0))^{-1} \varphi_j$ has the desired properties. □

Proof of Theorem 6.1 for the d-dimensional level-l Sierpinski gasket We follow the same notation as in Example 5.1 during this proof. It suffices to show the assertion for $x = q_0$ by virtue of Lemma 6.3. We set

$$
\mathcal{A} := \{u \in C(K) \mid u(q_0) = 1 > |u(q_1)|,\ u(q_k) = u(q_1) \text{ for } k \in \{2, \dots, d\}\}, \tag{6.6}
$$

and for $u \in \mathcal{A}$ we define $\Phi u \in C(K)$ by

$$
\Phi u|_{K_i} := u(q_1)^{\sum_{k=1}^d i_k} u \circ F_i^{-1}, \quad i = (i_k)_{k=1}^d \in S, \tag{6.7}
$$

so that $\Phi u \in \mathcal{A}$ and $\Phi : \mathcal{A} \to \mathcal{A}$. Then $\Phi(\mathcal{F} \cap \mathcal{A}) \subset \mathcal{F} \cap \mathcal{A}$ by (2.3). Furthermore for $u \in \mathcal{A}$ we can easily verify that

$$
\int_K (\Phi^n u)^2 d\mu \leq c_u r^{d_H n} \quad \text{for any } n \in \mathbb{N}, \tag{6.8}
$$

where $c_u := \int_K u^2 d\mu \prod_{n\in\mathbb{N}\cup\{0\}} \bigl(1 + (\#S - 1)u(q_1)^{2l^n}\bigr) \in (0, \infty)$.

Now for the eigenfunction $\varphi \in \mathcal{A}$ of Δ as in Lemma 6.5, let $\lambda \in (0, \infty)$ be such that $-\Delta\varphi = \lambda\varphi$ and define $\varphi_n := \bigl(\int_K (\Phi^n\varphi)^2 d\mu\bigr)^{-1/2} \Phi^n\varphi$ for $n \in \mathbb{N}$. Then for each $n \in \mathbb{N}$, $\int_K \varphi_n^2 d\mu = 1$, φ_n is an eigenfunction of $-\Delta$ with eigenvalue $\lambda/r^{(d_H+1)n}$ by (2.4), and (6.8) yields

$$\frac{\varphi_n(q_0)^2}{(\lambda/r^{(d_H+1)n})^{d_s/2}} = \frac{r^{d_H n}}{\lambda^{d_s/2}\int_K (\Phi^n\varphi)^2 d\mu} \geq \frac{1}{c_\varphi \lambda^{d_s/2}} > 0.$$

Therefore Lemma 2.10 implies that the limit $\lim_{t\downarrow 0} t^{d_s/2} p_t(q_0, q_0)$ does not exist. □

Lemma 6.6 *Let $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ be the (N, l)-polygasket with $N \geq 3$, $l < N/2$ and let $(D, \mathbf{r})$ be the harmonic structure on $\mathcal{L}$ as in Example 5.5.* (Recall that $q_i = \pi(i^\infty)$ for $i \in S$ and that $V_0 = \{q_i \mid i \in S\}$).

(1) *If $N = 4l$, then there exists an eigenfunction φ of Δ such that $\varphi(q_l) = \varphi(q_{3l}) = 0$ and $\varphi(q_0) = -\varphi(q_{2l}) = 1$.*
(2) *If $N \neq 4l$, then there exists an eigenfunction φ of Δ such that $\varphi(q_0) = 1$, $\varphi(q_l) = \varphi(q_{N-l}) \in (-1, 1)$ and $\varphi(q_{2l}) = \varphi(q_{N-2l}) \in (-1, 1)$.*

Proof Let $g, h : K \to K$ be the homeomorphisms defined in Example 5.5. Similarly to the proof of Lemma 6.5, there exist $\{\varphi_n\}_{n\in\mathbb{N}}, \{\psi_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$ such that $\varphi_n \circ h = \varphi_n$ and $\psi_n \circ h = -\psi_n$ for any $n \in \mathbb{N}$ and $\{\varphi_n\}_{n\in\mathbb{N}} \cup \{\psi_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system of $L^2(K, \mu)$ consisting of eigenfunctions of Δ. Then in the same way as the second paragraph of the proof of Lemma 6.5, we have $|\varphi_j(q_0)| > |\varphi_j(q_l)|$ and $\psi_k(q_l) \neq 0$ for some $j, k \in \mathbb{N}$.

(1) Since $\psi_k(q_0) = \psi_k(q_{2l}) = 0$ and $\psi_k(q_{3l}) = -\psi_k(q_l)$ by $\psi_k \circ h = -\psi_k$, the function $\varphi := (\psi_k(q_l))^{-1}\psi_k \circ g^l$ has the desired properties.
(2) Let $\psi := (\varphi_j(q_0))^{-1}\varphi_j$, so that $\psi(q_0) = 1 > |\psi(q_l)|$, $\psi(q_l) = \psi(q_{N-l})$ and $\psi(q_{2l}) = \psi(q_{N-2l})$. If $N = 3l$, then it suffices to set $\varphi := \psi$ since $q_{2l} = q_{N-l}$ and $q_{N-2l} = q_l$. Thus we may assume that $N \neq 3l, 4l$, so that $q_l, q_{N-l}, q_{2l}, q_{N-2l}$ are distinct and $N \geq 5$. Define $\varphi \in C(K)$ by, for each $i \in S = \{0, 1, \ldots, N-1\}$,

$$\varphi|_{K_i} := \begin{cases} \psi \circ g^{-i} \circ F_i^{-1} & \text{if } i = 0 \text{ or } i = N/2, \\ \psi(q_l)\psi \circ g^{l-i} \circ F_i^{-1} & \text{if } 0 < i < N/2 \text{ and } i \text{ is odd}, \\ \psi(q_l)\psi \circ g^{-l-i} \circ F_i^{-1} & \text{if } 0 < i < N/2 \text{ and } i \text{ is even}, \\ \psi(q_l)\psi \circ g^{-l-i} \circ F_i^{-1} & \text{if } i > N/2 \text{ and } N - i \text{ is odd}, \\ \psi(q_l)\psi \circ g^{l-i} \circ F_i^{-1} & \text{if } i > N/2 \text{ and } N - i \text{ is even}. \end{cases} \tag{6.9}$$

Then $\varphi(q_0) = 1$, $\varphi(q_l) = \varphi(q_{N-l}) = \varphi(q_{2l}) = \varphi(q_{N-2l}) = \psi(q_l)^2 \in [0, 1)$ by $N/2 \notin \{l, N-l, 2l, N-2l\}$, and φ is an eigenfunction of Δ by (2.3) and (2.4). □

Proof of Theorem 6.1 for the (N, l)-polygasket We will use the same notation as in Example 5.5 during this proof. Again it suffices to show the assertion for $x = q_0$

by virtue of Lemma 6.3. Similarly to (6.6) and (6.7), we define $\mathcal{A} \subset C(K)$ and $\Phi : \mathcal{A} \to \mathcal{A}$ by, if $N = 4l$,

$$\begin{aligned} \mathcal{A} &:= \{u \in C(K) \mid u(q_0) = 1,\ u(q_l) = u(q_{3l}) = 0\}, \\ \Phi u|_{K_i} &:= \mathbf{1}_{\{0\}}(i) u \circ F_i^{-1}, \quad i \in S = \{0, 1, \dots, N-1\}, \end{aligned} \tag{6.10}$$

and if $N \neq 4l$,

$$\begin{aligned} \mathcal{A} &:= \left\{ u \in C(K) \,\middle|\, \begin{array}{l} u(q_0) = 1,\ u(q_l) = u(q_{N-l}) \in (-1, 1) \\ \text{and } u(q_{2l}) = u(q_{N-2l}) \in (-1, 1) \end{array} \right\}, \\ \Phi u|_{K_i} &:= \begin{cases} u \circ F_i^{-1} & \text{if } i = 0, \\ u(q_l) u(q_{2l})^{i-1} u \circ g^{l-i} \circ F_i^{-1} & \text{if } 0 < i < N/2, \\ u(q_l) u(q_{2l})^{N-i-1} u \circ g^{-l-i} \circ F_i^{-1} & \text{if } i > N/2, \\ u(q_{2l})^{i-1} u \circ g^{-i} \circ F_i^{-1} & \text{if } i = N/2 \end{cases} \end{aligned} \tag{6.11}$$

for $i \in S = \{0, 1, \dots, N-1\}$. Then we can easily show the non-existence of the limit $\lim_{t \downarrow 0} t^{d_s/2} p_t(q_0, q_0)$ by applying Lemma 2.10 to $\varphi_n := \left(\int_K (\Phi^n \varphi)^2 d\mu\right)^{-1/2} \Phi^n \varphi$, where φ is the eigenfunction of Δ given in Lemma 6.6, in exactly the same way as in the previous case of the d-dimensional level-l Sierpinski gasket. □

Acknowledgments The author would like to thank Professor Jun Kigami for fruitful discussions and helpful comments and Professor Alexander Teplyaev for information on the reference [5].

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