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On unbiased stochastic Navier–Stokes equations

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Abstract A random perturbation of a deterministic Navier–Stokes equation is considered in the form of an SPDE with Wick type nonlinearity. The nonlinear term of the perturbation can be characterized as the highest stochastic order approximation of the original nonlinear term $u\nabla u$. This perturbation is unbiased in that the expectation of a solution of the perturbed equation solves the deterministic Navier–Stokes equation. The perturbed equation is solved in the space of generalized stochastic processes using the Cameron–Martin version of the Wiener chaos expansion. It is shown that the generalized solution is a Markov process and scales effectively by Catalan numbers.

Keywords Stochastic Navier–Stokes · Unbiased perturbation · Second quantization · Skorokhod integral · Wick product · Kondratiev spaces · Catalan numbers

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1 Introduction

In this paper we will consider a deterministic Navier–Stokes equation[1]

$$\begin{aligned}\partial_t \mathbf{u}_0(t,x) &= \partial_i \left(a^{ij}(t,x)\,\partial_j \mathbf{u}_0(t,x)\right)\\ &\quad - u_0^k(t,x)\,\partial_k \mathbf{u}_0(t,x) + \nabla P_0(t,x) + \mathbf{f}(t,x),\\ \mathbf{u}_0(0,x) &= \mathbf{w}(x),\ \mathrm{div}\ \mathbf{u}_0 = 0,\end{aligned} \tag{1.1}$$

and its stochastic perturbations:

$$\begin{aligned}\partial_t \mathbf{v}(t,x) &= \partial_i \left(a^{ij}(t,x)\,\partial_j \mathbf{v}(t,x)\right) - v^k(t,x)\,\partial_k \mathbf{v}(t,x)\\ &\quad + \nabla P(t,x) + \mathbf{f}(t,x)\\ &\quad + [\sigma^i(t,x)\partial_i \mathbf{v}(t,x) + \mathbf{g}(t,x) - \nabla \tilde{P}(t,x)]\dot{W}_t,\\ \mathbf{v}(0,x) &= \mathbf{w}(x),\ \mathrm{div}\ \mathbf{v} = 0,\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}\partial_t \mathbf{u}(t,x) &= \partial_i \left(a^{ij}(t,x)\,\partial_j \mathbf{u}(t,x)\right) - u^k(t,x)\,\Diamond\partial_k \mathbf{u}(t,x)\\ &\quad + \nabla P(t,x) + \mathbf{f}(t,x)\\ &\quad [\sigma^i(t,x)\partial_i \mathbf{u}(t,x) + \mathbf{g}(t,x) - \nabla \tilde{P}(t,x)]\dot{W}_t,\\ \mathbf{u}(0,x) &= \mathbf{w}(x),\ \mathrm{div}\ \mathbf{u} = 0,\end{aligned} \tag{1.3}$$

where $0 \le t \le T$, $x \in \mathbf{R}^d$, $d \ge 2$, and W_t is a cylindrical Wiener process in a separable Hilbert space Y. The coefficients a^{ij}, σ^i and the functions $\mathbf{f}$, $\mathbf{g}$ are deterministic, σ^i and $\mathbf{g}$ are Y-valued. Symbol $\Diamond$ stands for Wick product (see Sect. 2.2.1 and references [4,27]). Wick product is a stochastic convolution. It could be interpreted as a generalized Malliavin divergence operator with respect to Gaussian measure associated with white noise $\dot{W}$ (see [16]).

Stochastic PDEs involving Wick product type nonlinearity were originally discussed in the literature related to the Parisi–Wu program (see [3,12,25] and also [19, Section 6]). In these papers Wick product was defined by (Gaussian) invariant measures for the related PDEs. Other related papers include: [2,5,10,13–16,29], etc.).

Equations (1.2) and (1.3) are *stochastic perturbations* of the deterministic Navier–Stokes equation (1.1). It is shown in Sect. 3 that the generalized mean, i.e. the zero-order coefficient in the Wiener chaos expansion (1.8) of the solution of Eq. (1.3), is a solution of Eq. (1.1), i.e.

$$\mathbf{E}\mathbf{u}(t,x) = \mathbf{u}_0(t,x), \tag{1.4}$$

[1] Here and below we assume summation over repeating indices in products.

where $\mathbf{u}_0(t,x)$ is a solution of Eq. (1.1). In other words, the solution of stochastic Navier–Stokes equation (1.3) is a mean preserving (*unbiased*) random perturbation of deterministic Navier–Stokes equation (1.1).

Obviously, this nice property does not hold for Eq. (1.2) or other standard stochastic perturbations of Navier–Stokes equation (e.g. random initial conditions, random forcing, etc.)

In fact, Eq. (1.3) could be viewed as an *approximation* of stochastic Navier–Stokes equation (1.2). Indeed, under certain natural assumptions, the following equality holds:

$$\mathbf{v}\nabla\mathbf{v} = \sum_{n=0}^{\infty} \frac{\mathcal{D}^n \mathbf{v} \Diamond \mathcal{D}^n \nabla \mathbf{v}}{n!} \tag{1.5}$$

where $\mathcal{D}^n$ is the nth power of Malliavin derivative $\mathcal{D}$. Taking into account expansion (1.5), $\mathbf{v}\Diamond\nabla\mathbf{v}$ could be viewed as an approximation of the product $\mathbf{v}\nabla\mathbf{v}$. In fact, $\mathbf{v}\Diamond\nabla\mathbf{v}$ is the highest stochastic order approximation of $\mathbf{v}\nabla\mathbf{v}$ (see Appendix I, Proposition 4 and Remark 11).

Stochastic Navier–Stokes equation (1.2) is reasonably well understood and there exists substantial literature on its analytical properties as well as its derivation from the first principles (see e.g. [22,23] and the references therein). In this paper we will be focusing mostly on Eq. (1.3).

Burger's equation with Wick product was considered in [6,8,9], see also the references therein.

It was shown in [23] that under reasonable assumptions stochastic Navier–Stokes equation (1.2) has a square integrable solution. Moreover, this solution can be formally written in the Wiener chaos expansion form:

$$\mathbf{v}(t,x) = \sum_{\alpha} \mathbf{v}_\alpha(t,x)\,\xi_\alpha,$$

where $\{\xi_\alpha, \alpha \in J\}$ is the Cameron–Martin basis generated by $\dot{W}_t$, $\mathbf{v}_\alpha(t,x) = \mathbf{E}(\mathbf{v}(t,x)\,\xi_\alpha)$, and J is the set of multiindices $\alpha = \{\alpha_k, k \geq 1\}$ such that for every k, $\alpha_k \in \mathbf{N}_0(\mathbf{N}_0 = \{0,1,2,\ldots\})$ and $|\alpha| = \sum_k \alpha_k < \infty$.

It was shown in [23] that the Wiener chaos coefficients $\mathbf{v}_\alpha(t,x)$ satisfy the propagator equation:

$$\begin{aligned}
\partial_t \mathbf{v}_\alpha(t,x) &= \partial_i\left(a^{ij}\partial_j \mathbf{v}_\alpha(t,x)\right) - \nabla P(t,x) + \mathbf{f}(t,x)\, I_{\{|\alpha|=0\}} \\
&\quad - \sum_p \sum_{0\leq\beta\leq\alpha} c(\alpha,\beta,p)\left(\mathbf{v}_{\beta+p},\nabla\right)\mathbf{v}_{\alpha+p-\beta}(t,x) \\
&\quad \sum_k \sqrt{\alpha_k}[(\sigma^i, e_k)_Y \partial_i \mathbf{v}_{\alpha(k)}(t,x) + I_{\{|\alpha|=1\}}(\mathbf{g}, e_k)_Y]; \\
\mathbf{v}_\alpha(0,x) &= \mathbf{w}_\alpha(x), \text{ div } \mathbf{v}_\alpha = 0,
\end{aligned} \tag{1.6}$$

where $\alpha(k) = (\alpha_1, \alpha_2, \ldots\alpha_{k-1}, \alpha_k - 1, \alpha_{k+1}, \ldots)$ and

$$c(\alpha, \beta, p) = \left[\binom{\alpha}{\beta} \binom{\beta + p}{p} \binom{\alpha + p - \beta}{p} \right]^{1/2}.$$

One advantage of the Wiener chaos representation is that it provides convenient explicit formulae for computing statistical moments of the random field $\mathbf{u}(t, x)$ via Wiener chaos coefficients (see [22,23]). For example,

$$\mathbf{E}u^i(t,x) = u_0^i(t,x),\ \mathbf{E}\left(u^i(t,x)u^j(t,y)\right) = \sum_{|\alpha|<\infty} u_\alpha^i(t,x)u_\alpha^j(t,y),$$

In this paper, we prove that the WCE coefficients of a solution of the unbiased stochastic Navier–Stokes Eq. (1.3) are given by

$$\begin{aligned} \partial_t \mathbf{u}_\alpha(t,x) &= \partial_i \left(a^{ij}\partial_j \mathbf{u}_\alpha(t,x)\right) - \nabla P(t,x) + \mathbf{f}(t,x) I_{\{|\alpha|=0\}} \\ &\quad - \sum_{0\le\beta\le\alpha} \sqrt{\binom{\alpha}{\beta}} \left(\mathbf{u}_{\alpha-\beta}, \nabla\right) \mathbf{u}_\beta(t,x) \\ &\quad + \sum_k \sqrt{\alpha_k}[\left(\sigma^j(t,x), e_k(t)\right)_Y \partial_j \mathbf{u}_{\alpha(k)}(t,x) \\ &\quad + \left(\mathbf{g}(t,x), e_k(t)\right)_Y 1_{|\alpha|=1}] \\ \mathbf{u}_\alpha(0,x) &= \mathbf{w}(x), \operatorname{div} \mathbf{u}_\alpha = 0. \end{aligned} \tag{1.7}$$

Clearly, this system of equations is much simpler than Eq. (1.6).

If $\alpha = 0$, then $\mathbf{u}_\alpha(t, x)$ is a solution of deterministic Navier–Stokes equation (1.1). The remaining components are governed by Stokes equations and could be solved sequentially. From the computational point of view this is a substantial advantage. Indeed, the propagator for Eq. (1.2) is a full nonlinear system while Eq. (1.3) is a lower triangular system and only the first equation of this system is nonlinear.

An important feature of Eq. (1.3) is that

$$\left(u^k(t) \lozenge \partial_k \mathbf{u}(t), \mathbf{u}(t)\right)_{L_2(\mathbf{R}^d)} \neq 0.$$

Therefore, one could not expect a solution of (1.3) to be square integrable. This effect is not specific to stochastic Navier–Stokes equation. In fact, it is common for a large class of stochastic bilinear PDEs (see e.g. [14,15]).

In this paper we consider Eq. (1.3) in the class of formal Wiener chaos expansions and show that a formal series

$$\mathbf{u}(t,x) = \sum_{\alpha\in J} \mathbf{u}_\alpha(t,x)\,\xi_\alpha \tag{1.8}$$

solves (1.3) if and only if $u_\alpha(t, x)$ are given by Eq. (1.7). To make this solution square integrable, we rescale it using second quantization operators (see Appendix I, 5.1). It is shown that $u(t, x)$ is the limit of square integrable solutions of the rescaled equations (see Proposition 2).

Convergence of this solution is determined by a system of positive weights $\{r_\alpha\}_{|\alpha|<\infty}$ such that

$$\|\mathbf{u}\|_{\mathcal{R}}^2 := \sum_{|\alpha|<\infty} r_\alpha^2 \|\mathbf{u}_\alpha(t)\|_{L_2((0,T);\mathbf{R}^d)}^2 < \infty. \tag{1.9}$$

It turned out, that Catalan numbers (see [9,28]) are critical for an appropriate choice of the weights r_α in (1.9) (see Proposition 1).

In addition, it was shown that a solution of Eq. (1.3) belongs to the intersection of Sobolev spaces $\mathbb{H}_2^2\left(\mathbf{R}^d\right) \cap \mathbb{H}_p^2\left(\mathbf{R}^d\right)$ for $p > d$. We have also demonstrated that uniqueness of a solution of Eq. (1.3) holds under the same assumptions that guarantee uniqueness for the related deterministic Navier–Stokes equation. Although ξ_α in (1.8) are not $(\mathcal{F}_t^W)$-adapted, we prove that the generalized solution is $(\mathcal{F}_t^W)$-adapted and Markov (see Theorem 2 and Corollary 5).

It is not clear how, if at all, the unbiased Navier–Stokes equation fits into classical fluid mechanics. Nevertheless, Eq. (1.3) is "physical" in that it could be derived from the second Newton law (under appropriate assumptions on the velocity field), much the same way as the classical Navier–Stokes equation (see Appendix I, 5.2). Also, it was shown recently (see [11]) that, after Catalan type rescaling, finite dimensional projections of unbiased Navier–Stokes equation present an accurate and numerically inexpensive approximation of stochastic Navier–Stokes Eq. (1.2)

We conclude this section with an outline of some notations that will be used in the paper.

1.1 Notation

Let us fix a separable Hilbert space Y. The scalar product of $x, y \in Y$ will be denoted by $(x, y)_Y$.

If u is a function on $\mathbf{R}^d$, the following notational conventions will be used for its partial derivatives: $\partial_i u = \partial u/\partial x_i$, $\partial_{ij}^2 = \partial^2 u/\partial x_i \partial x_j$, $\partial_t u = \partial u/\partial t$, and $\nabla u = \partial u = (\partial_1 u, \ldots, \partial_d u)$, and $\partial^2 u = (\partial_{ij}^2 u)$ denotes the Hessian matrix of second derivatives. Let $\alpha = (\alpha_1, ..., \alpha_d)$ be a multi-index, $\alpha_i \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$, $i = 1, \ldots, d$, then $\partial_x^\alpha = \Pi_{i=1}^d \partial_{x_i}^{\alpha_i}$.

Vector fields on $\mathbf{R}^d$ are denoted by boldface letters. This convention also applies if the entries of the vector field are taking values in a Hilbert space.

We denote $\mathbf{N} = \{1, 2, \ldots\}$.

For a Banach space E, we denote $C([0, T], E)$ the space of continuous E-valued functions.

$C_0^\infty = C_0^\infty(\mathbf{R}^d)$ denotes the set of all infinitely differentiable functions on $\mathbf{R}^d$ with compact support.

For $s \in (-\infty, \infty)$, write $\Lambda^s = \Lambda_x^s = \left(1 - \sum_{i=1}^d \partial^2/\partial x_i^2\right)^{s/2}$. For $p \in [1, \infty)$ and $s \in (-\infty, \infty)$, we define the space $H_p^s = H_p^s(\mathbf{R}^d)$ as the space of generalized real valued functions u with the finite norm

$$|u|_{s,p} = |\Lambda^s u|_p,$$

where $|\cdot|_p$ is the L_p norm. Obviously, $H_p^0 = L_p$. Note that if $s \geq 0$ is an integer, the space H_p^s coincides with the Sobolev space $W_p^s = W_p^s(\mathbf{R}^d)$.

The spaces $C_0^\infty(\mathbf{R}^d)$, $H_p^s\left(\mathbf{R}^d\right)$ can be extended to vector functions (denoted by bold-faced letters). For example, the space of all vector functions $\mathbf{u} = (u^1, \ldots, u^d)$ such that $\Lambda^s u^l \in L_p$, $l = 1, \ldots, d$, with the finite norm

$$|\mathbf{u}|_{s,p} = \left(\sum_l |u^l|_{s,p}^p\right)^{1/p},$$

is denoted by $\mathbb{H}_p^s = \mathbb{H}_p^s(\mathbf{R}^d)$. Similarly, we denote by $\mathbb{H}_p^s(Y) = \mathbb{H}_p^s(\mathbf{R}^d, Y)$ the space of all vector functions $\mathbf{g} = (g^l)_{1\leq l\leq d}$, with Y-valued components g^l, $1 \leq l \leq d$, so that $||\mathbf{g}||_{s,p} = (\sum_l |g^l|_{s,p}^p)^{1/p} < \infty$. Also, for brevity, the norm $||\mathbf{g}||_{0,p}$ is denoted by $||\mathbf{g}||_p$.

When $s = 0$, $\mathbb{H}_p^s(Y) = \mathbb{L}_p(Y) = \mathbb{L}_p(\mathbf{R}^d, Y)$. To forcefully distinguish L_p-norms in spaces of Y-valued functions, we write $||\cdot||_p$, while in all other cases a norm is denoted by $|\cdot|_p$. The duality $\langle\cdot,\cdot\rangle_s$ between $\mathbb{H}_q^s\left(\mathbf{R}^d\right)$, and $\mathbb{H}_p^{-s}\left(\mathbf{R}^d\right)$ where $p \geq 2$ and $q = p/(p-1)$ is defined by

$$\langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle_s = \langle \boldsymbol{\phi}, \boldsymbol{\psi}\rangle_{s,p} = \sum_{i=1}^d \int_{\mathbf{R}^d} (\Lambda^s \phi^i)(x)\,(\Lambda^{-s}\psi^i)(x)\,dx,\ \phi \in \mathbb{H}_q^s,\ \psi \in \mathbb{H}_p^{-s}.$$

2 Generalized random variables and processes

2.1 Wiener chaos

To begin with, we shall introduce some basic notation and recall a few fundamental facts of infinite-dimensional stochastic calculus. Let us fix a separable Hilbert spaces Y and $\mathbf{H} = L_2([0,T], Y)$. Let $\{\ell_i, i \geq 1\}$ be a complete orthonormal basis (CONS) in Y and $\{m_i, i \geq 1\}$ be a CONS in $L_2([0,T])$. Denote by $\mathcal{B}$ the class of all CONS in $\mathbf{H}$ of the form $\left\{e_k = e_k(s) = m_{k_1}(s)\ell_{k_2}\right\}$ and such that for each k, $\sup_{0\leq s\leq T} |m_k(s)| < \infty$. Obviously, for each k, $\sup_{0\leq s\leq T} |e_k(s)|_Y < \infty$. Let us fix a CONS $\mathbf{b} = \{e_k, k\geq 1\} \in \mathcal{B}$.

Let $(\Omega, \mathcal{F}^W, \mathbf{P})$ be a probability space with a cylindrical Brownian motion W_t in Y and $\mathcal{F}^W$ be the σ-algebra generated by W. Let $\mathbb{F}^W$ be the right continuous filtration of σ-algebras $(\mathcal{F}_t^W)_{t\geq 0}$ generated by W_t. All the σ-algebras are assumed to be $\mathbf{P}$-completed. Hence

$$W_t = \sum_{k=1}^\infty w_t^k \ell_k,$$

where $\left\{w_t^k, k \geq 1\right\}$ is a sequence of independent standard one-dimensional Brownian motions in $(\Omega, \mathcal{F}, \mathbf{P})$. We write $W(e_k) = \int_0^T e_k(t)dW_t$. For $e_k = e_k(s) = m_{k_1}(s)\ell_{k_2}$,

$$W(e_k) = \int_0^T e_k(t)dW_t = \int_0^T m_{k_1}(t)dw_t^{k_2},$$

and

$$W_t = \sum_{k=1}^{\infty} \left(\int_0^t e_k(s)ds \right) W(e_k), \quad 0 \leq t \leq T. \tag{2.1}$$

Let $\alpha = \{\alpha_k, k \geq 1\}$ be a multiindex, i.e. for every k, $\alpha_k \in \mathbf{N}_0 = \{0, 1, 2, \ldots\}$. We shall consider only such α that $|\alpha| = \sum_k \alpha_k < \infty$, i.e., only a finite number of α_k is non-zero, and we denote by J the set of all such multiindices. For $\alpha, \beta \in J$, we define

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots), \quad \alpha! = \prod_{k \geq 1} \alpha_k!.$$

By ε_k we denote the multi-index α with $\alpha_k = 1$ and $\alpha_j = 0$ for $j \neq k$. Write

$$\alpha\,(k) = \alpha - \varepsilon_k \tag{2.2}$$

For $\alpha \in J$, write $H_\alpha := \prod_{k=1}^{\infty} H_{\alpha_k}(W\,(e_k))$, where H_n is the nth Hermite polynomial defined by $H_n(x) = (-1)^N \left(d^n e^{-x^2/2}/dx^n\right) e^{x^2/2}$.

Let $\xi_\alpha = H_\alpha/\sqrt{a!}$.

Theorem 1 (Cameron and Martin [1]) *The set* $\Xi = \{\xi_\alpha = \xi_\alpha(\mathbf{b}), \alpha \in J\}$ *is an orthonormal basis in* $L_2\left(\Omega, \mathcal{F}^W, \mathbf{P}\right)$*, where* $\mathcal{F}^W$ *is the* σ*-algebra generated by* W*. If* E *is a Hilbert space,* $\eta \in L_2\left(\Omega, \mathcal{F}^W, \mathbf{P}; E\right)$ *and* $\eta_\alpha = \mathbf{E}(\eta\xi_\alpha)$*, then* $\eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha$ *and* $\mathbf{E}|\eta|_E^2 = \sum_{\alpha \in \mathcal{J}} |\eta_\alpha|_E^2$.

The expansion $\eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha$ is often referred to as Wiener chaos expansion.

Remark 1 The basis $\xi_\alpha, \alpha \in J$, can be obtained by differentiating stochastic exponent. Let $\mathcal{Z}$ be the set of all real-valued sequences $z = (z_k)$ such that only finite number of z_k is not zero. For $\alpha \in J$, denote $\partial_z^\alpha = \Pi_k \partial^{\alpha_k} / \left(\partial z_k\right)^{\alpha_k}$ and let

$$e_z = e_z(t) = \sum_k z_k e_k(t), \quad 0 \leq t \leq T,$$

$$p_t(z) = p_t(e_z) = p_t(z, \mathbf{b}) = \exp\left\{ \int_0^t e_z(s)dW_s - \frac{1}{2}\int_0^t |e_z(s)|_Y^2 ds \right\}, \tag{2.3}$$
$$p(z) = p_T(z), \quad z \in \mathcal{Z}, 0 \leq t \leq T.$$

It is a standard fact (see, for example, [20]) that $H_\alpha = \partial_z^\alpha p(z)|_{z=0}, \xi_\alpha = H_\alpha/\sqrt{\alpha!}$. Since $p(z)$ is analytic, it follows by (4.2),

$$p(z) = p(z, \mathbf{b}) = \sum_\alpha \frac{H_\alpha}{\alpha!} z^\alpha = \sum_\alpha \frac{z^\alpha}{\sqrt{\alpha!}} \xi_\alpha. \tag{2.4}$$

2.2 Generalized random variables and processes

Let $\mathbf{b} \in \mathcal{B}$, and $\xi_\alpha = \xi_\alpha(\mathbf{b}), \alpha \in J$. Let

$$\begin{aligned} \mathcal{D} &= \mathcal{D}(\mathbf{b}) \\ &= \left\{ v = \sum_\alpha v_\alpha \xi_\alpha : \ v_\alpha \in \mathbf{R} \text{ and only finite number of } v_\alpha \text{ are not zero} \right\}. \end{aligned}$$

Definition 1 A generalized $\mathcal{D}$-random variable with values in a convex topological vector (linear) space E with Borel σ-algebra is a formal series $u = \sum_\alpha u_\alpha \xi_\alpha$, where $u_\alpha \in E$, $\xi_\alpha = \xi_\alpha(\mathbf{b})$, and $\mathbf{b} = \{e_k, k \geq 1\} \in \mathcal{B}$ is a CONS in $\mathbf{H} = L_2([0, T], Y)$.

Denote the vector space of all generalized $\mathcal{D}$-random variables by $\mathcal{D}' = \mathcal{D}'(\mathbf{b}) = \mathcal{D}'(\mathbf{b}; E)$. The elements of $\mathcal{D}$ are the test random variables for $\mathcal{D}'$. We define the action of a generalized random variable u on the test random variable v by $\langle u, v \rangle = \sum_\alpha v_\alpha u_\alpha$.

For a sequence $u^n \in \mathcal{D}'$ and $u \in \mathcal{D}'$, we say that $u^n \to u$, if for every $v \in \mathcal{D}$, $\langle u, v^n \rangle \to \langle u, v \rangle$. This implies that $u^n = \sum_\alpha u_\alpha^n \xi_\alpha \to u = \sum_\alpha u_\alpha \xi_\alpha$ if and only if $u_\alpha^n \to u_\alpha$ as $n \to \infty$ for all α.

Remark 2 Obviously, if $u = \sum_\alpha u_\alpha \xi_\alpha \in \mathcal{D}'(\mathbf{b}; E)$, F is a vector space and $f : E \to F$ is a linear map, then

$$f(u) = \sum_\alpha f(u_\alpha)\xi_\alpha \in \mathcal{D}'(\mathbf{b}; F).$$

Definition 2 An E-valued generalized $\mathcal{D}$- process $u(t)$ in $[0, T]$ is a $\mathcal{D}'(\mathbf{b}; E)$-valued function on $[0, T]$ such that for each $t \in [0, T]$

$$u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in \mathcal{D}'(\mathbf{b}; E);$$

and $u_\alpha(t)$ are deterministic measurable E-valued functions on $[0, T]$. We denote the linear space of all such processes by $\mathcal{D}'(\mathbf{b}; [0, T], E)$. If E is a topological vector space and a generalized $\mathcal{D}$-process $u(t)$ is continuous we write $u \in C\mathcal{D}'([0, T], \mathbf{b}, E)$ (note that $u(t)$ is continuous if and only if all coefficient functions u_α are continuous in E.

If there is no room for confusion, we will often say $\mathcal{D}$-process ($\mathcal{D}$-random variable) instead of generalized $\mathcal{D}$-process (generalized $\mathcal{D}$-random variable).

If E is a normed vector space, we denote

$$L_1(\mathcal{D}'(\mathbf{b};[0,T],E)) = \left\{ u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in \mathcal{D}'(\mathbf{b};[0,T],E) : \int_0^T |u_\alpha(t)|_E dt < \infty, \alpha \in J \right\}.$$

For $u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in L_1(\mathcal{D}'(\mathbf{b};[0,T],E))$ we define $\int_0^t u(s)ds, 0 \le t \le T$, in $\mathcal{D}'(\mathbf{b};[0,T],E)$ by

$$\int_0^t u(s)ds = \sum_\alpha \left(\int_0^t u_\alpha(s)ds \right) \xi_\alpha, 0 \le t \le T.$$

If $u(t) = \sum_\alpha u_\alpha(t)\xi_\alpha \in \mathcal{D}'(\mathbf{b};[0,T],E)$, then $u(t)$ is differentiable in t if and only if $u_\alpha(t)$ are differentiable in t. In that case,

$$\frac{d}{dt}u(t) = \dot{u}(t) = \sum_\alpha \dot{u}_\alpha(t)\xi_\alpha \in \mathcal{D}'([0,T],\mathbf{b},E).$$

Example 1 A cylindrical Wiener process $W_t, 0 \le t \le T$, in a Hilbert space Y, and its derivative $dW_t/dt = \dot{W}_t$ are generalized Y-valued stochastic processes. Indeed, by (2.1),

$$W_t = \sum_k \int_0^t e_k(s)ds\xi_{\varepsilon_k}, \quad 0 \le t \le T,$$

and $W_t = \int_0^t \dot{W}_s ds$, where $\dot{W}_t = \sum_k e_k(t)\xi_{\varepsilon_k}, 0 \le t \le T$.

2.2.1 Wick Product and Skorokhod Integral

Definition 3 For ξ_α, ξ_β from Ξ, define the Wick product

$$\xi_\alpha \diamondsuit \xi_\beta := \sqrt{\left(\frac{(\alpha+\beta)!}{\alpha!\beta!} \right)} \xi_{\alpha+\beta}. \tag{2.5}$$

In particular, taking in (2.5) $\alpha = k\varepsilon_i$ and $\beta = n\varepsilon_i$ we get

$$H_k(\xi_i) \diamondsuit H_n(\xi_i) = H_{k+n}(\xi_i). \tag{2.6}$$

For a Hilbert space E and arbitrary $v = \sum_\alpha v_\alpha \xi_\alpha$ and $u = \sum_\alpha u_\alpha \xi_\alpha$ in $\mathcal{D}'(\mathbf{b}; E)$, we define their Wick product as a $\mathcal{D}$-generalized real valued random variable given by

$$v \Diamond u = \sum_\alpha \sum_{\beta \le \alpha} (u_\beta, v_{\alpha-\beta})_E \sqrt{\frac{\alpha!}{\beta!(\alpha-\beta)!}} \xi_\alpha \in \mathcal{D}'(\mathbf{b}; \mathbf{R}). \tag{2.7}$$

Definition 4 Skorokhod integral (Maliavin divergence operator) of $v \in L_1(\mathcal{D}'(\mathbf{b}; [0,T], Y))$ is a generalized random variable (element of $\mathcal{D}'$ $(\mathbf{b}; \mathbf{R})$) such that

$$\delta(v) = \int_0^T v(s) dW_s = \sum_\alpha \delta(v)_\alpha \xi_\alpha,$$

with

$$\delta(v)_\alpha = \sum_k \sqrt{\alpha_k} \int_0^T \left(v_{\alpha(k)}(t), e_k(t) \right)_Y dt. \tag{2.8}$$

and $\alpha(k)$ is given by (2.2).

If $v \in L_1(\mathcal{D}'(\mathbf{b}; [0,T], Y))$, then $\delta_t(v) = \int_0^t v(s) dW_s = \delta\left(v 1_{[0,t]}\right)$, $0 \le t \le T$, is a process in $\mathcal{D}'$ $(\mathbf{b}; [0,T], Y)$. We have

$$\delta_t(v)_\alpha = \sum_k \sqrt{\alpha_k} \int_0^t \left(v_{\alpha(k)}(s), e_k(s) \right)_Y ds.$$

Since $\dot{W}_t = \sum_k e_k(t) \xi_{\varepsilon_k}$, it follows by (2.7) that

$$v_t \Diamond \dot{W}_t = \sum_\alpha \sum_k (v_{\alpha(k)}(t), e_k(t))_Y \sqrt{\alpha_k} \xi_\alpha,$$

and

$$\delta(v) = \int_0^T v_t \Diamond \dot{W}_t \, dt, \;\; \delta_t(v) = \int_0^t v(s) \Diamond \dot{W}_s ds, \quad 0 \le t \le T.$$

Remark 3 Skorokhod integral is an extension of the Itô integral[2]; (2.9) below motivates the definition of the Skorokhod integral.

[2] Of course, this statement is well known. However, the proof given here is short and straightforward.

If $u(t)=\sum_\alpha u_\alpha(t)\xi_\alpha$ is $\mathbb{F}^W$-adapted Y-valued such that

$$\mathbf{E}\int_0^T |u(t)|_H^2 dt < \infty,$$

then $v=\int_0^T u(t)dW(t)=\sum_\alpha v_\alpha\xi_\alpha$ is square integrable. By Ito formula for the product of $\int_0^t u(s)dW(s)$ and stochastic exponent $p_t(z)$ from Remark 1, we obtain

$$\mathbf{E}vp(z)=\mathbf{E}\int_0^T u(t)dW(t)p_T(z)=\int_0^T \mathbf{E}[p_t(z)(u(t),e_z(t))_Y]dt$$
$$=\int_0^T \mathbf{E}[p(z)(u(t),e_z(t))_Y]dt,\quad z\in\mathcal{Z}.$$

So,

$$\frac{\partial^{|\alpha|}\mathbf{E}vp(z)}{\partial z^\alpha}=\sum_k \alpha_k\int_0^T \frac{\partial^{|\alpha(k)|}}{\partial z^{\alpha(k)}}(\mathbf{E}p(z)u(t),e_k(t))_Y]dt,$$
$$v_\alpha=(\sqrt{\alpha!})^{-1}\frac{\partial^{|\alpha|}\mathbf{E}vp(z)}{\partial z^\alpha}|_{z=0}=\sum_k\int_0^T\sqrt{\alpha_k}(u_{\alpha(k)}(t),e_k(t))_Y]dt. \tag{2.9}$$

Comparing (2.9) and (2.8), we see that Ito and Skorokhod integrals are equal in this case.

3 Wick product Navier–Stokes equation

For $T>r\geq 0$, let us consider the following Navier–Stokes equation:

$$\begin{aligned}\partial_t\mathbf{u}(t,x)&=\partial_i\left(a^{ij}(t,x)\,\partial_j\mathbf{u}(t,x)\right)+b^i(t,x)\partial_i\mathbf{u}(t,x)\\&\quad-u^k(t,x)\lozenge\partial_k\mathbf{u}(t,x)+\nabla P(t,x)+\mathbf{f}(t,x)\\&\quad[\sigma^i(t,x)\partial_i\mathbf{u}(t,x)+\mathbf{g}(t,x)-\nabla\tilde{P}(t,x)]\lozenge\dot{W}_t,\\\mathbf{u}(r,x)&=\boldsymbol{w}(x),\ \operatorname{div}\mathbf{u}=0.\end{aligned}\tag{3.1}$$

The unknowns in the Eq. (3.1) are the functions $\mathbf{u}=\left(u^l\right)_{1\leq l\leq d}$, P, $\tilde{P}$. It is assumed that $a^{ij}, b^i, \mathbf{f}=\left(f^i\right)$, are measurable deterministic functions on $[0,\infty)\times\mathbf{R}^d$, and the matrix $\left(a^{ij}\right)$ is symmetric. Let us assume also that σ^i, $\mathbf{g}=\left(g^i\right)$ be Y-valued measurable deterministic functions on $[0,\infty)\times\mathbf{R}^d$. Let $\mathbf{w}$ be a random initial velocity field.

In addition, we will need the following assumptions.

A1. For all $t \geq 0, x \in \mathbf{R}^d, \lambda \in \mathbf{R}^d$,

$$K|\lambda|^2 \geq a^{ij}(t,x)\lambda_i\lambda_j \geq \delta|\lambda|^2,$$

where K, δ are fixed strictly positive constants.

A2. For all $t \geq 0, x$,

$$\max_{|\alpha|\leq 2} |\partial^\alpha a^{ij}(t,x)| + \max_{|\alpha|\leq 1}(|\partial^\alpha b^i(t,x)| + |\partial^\alpha \sigma^i(t,x)|_Y) \leq K.$$

A3. The functions $\mathbf{f}(t,x)$ and $\mathbf{g}(t,x)$ are measurable deterministic, $p > d$, and for all $t > 0$,

$$\int_0^t \sum_{l=2,p} [|\mathbf{f}(r)|_{1,l}^l + ||\mathbf{g}(r)||_{1,l}^{2l}]dr < \infty$$

(recall $|\mathbf{f}(r)|_{1,l}$, $||\mathbf{g}(r)||_{1,l}$ are $\mathbb{H}_p^1(\mathbf{R}^d)$ and $\mathbb{H}_p^1(\mathbf{R}^d, Y)$-norms respectively).

We will seek a solution to (3.1) in the form

$$\mathbf{u}(t) = \sum_\alpha \mathbf{u}_\alpha(t)\xi_\alpha \in \mathcal{D}'(\mathbf{b}; [0,T], \mathbb{H}_p^2), \quad p \geq 2.$$

In this case, denoting by $\mathcal{P}(\mathbf{v})$ the solenoidal projection of the vector field $\mathbf{v}$, we can rewrite (3.1) in the following equivalent form:

$$\begin{aligned} \partial_t \mathbf{u}(t) = \mathcal{P}\Big[\partial_i\Big(a^{ij}(t)\,\partial_j\mathbf{u}(t)\Big) + b^i(t)\partial_i\mathbf{u}(t) \\ -u^k(t)\,\partial_k\mathbf{u}(t) + \mathbf{f}(t)\Big] + \mathcal{P}[\sigma^i(t)\partial_i\mathbf{u}(t) + \mathbf{g}(t)]\lozenge \dot{W}_t, \qquad (3.2) \\ \mathbf{u}(r) = \boldsymbol{w}, \operatorname{div}\mathbf{u}(t) = 0, t \in [r,T]. \end{aligned}$$

If $\eta = \sum_\alpha \eta_\alpha\xi_\alpha$ with $\eta_\alpha \in \mathbb{H}_p^k$, then (see Remark 2) $\mathcal{P}(\eta) = \sum_\alpha \mathcal{P}(\eta_\alpha)\xi_\alpha$.

We start our analysis of Eq. (3.2) by introducing the definition of a solution in the "weak sense".

Definition 5 We say that a generalized $\mathcal{D}$-process $\mathbf{u}(t) = \sum_\alpha \mathbf{u}_\alpha(t)\xi_\alpha \in C\mathcal{D}'(\mathbf{b}; [r,T], \mathbb{H}_p^k)$ is $\mathcal{D} - \mathbb{H}_p^k$ solution of Eq. (3.1) in $[r,T]$, if the equality

$$\begin{aligned} \mathbf{u}(t) = \mathbf{w} + \int_r^t \mathcal{P}[-u^i(s)\lozenge\partial_i\mathbf{u}(s) + \partial_i(a^{ij}(s)\partial_j\mathbf{u}(s)) \\ +b^i(s)\partial_i\mathbf{u}(s) + \mathbf{f}(s)]ds \qquad (3.3) \\ \int_r^t \mathcal{P}[\sigma^k(s)\partial_k\mathbf{u}(s) + \mathbf{g}(s)]\lozenge\dot{W}_s ds \end{aligned}$$

holds in $\mathcal{D}(\mathbf{b}; \mathbb{H}_p^{k-2}(\mathbf{R}^d))$ for every $r \le t \le T$. If an $\mathcal{D} - \mathbb{H}_p^k$-solution in $[r, T]$ is also $\mathcal{D} - \mathbb{H}_q^{k'}$-solution in $[r, T]$, we call it $\mathcal{D} - \mathbb{H}_p^k \cap \mathbb{H}_q^{k'}$-solution in $[r, T]$.

In the future, we simply say $\mathcal{D}$-solution if there is no risk of confusion.

Remark 4 1. Assume A1-A3 hold, $p \ge 2$, $\mathbf{w} = \sum_\alpha \mathbf{w}_\alpha \xi_\alpha \in \mathcal{D}'(\mathbf{b}, \mathbb{H}_p^k)$. Applying Remark 2 and definition of the Wick product we see that $\mathbf{u}(t) = \sum_\alpha \mathbf{u}_\alpha(t)\xi_\alpha \in C\mathcal{D}'([r, T], \mathbf{b}, \mathbb{H}_p^k)$ is an $\mathcal{D} - \mathbb{H}_p^k$ solution in $[r, T]$ if and only if for each α, $\mathbf{u}_\alpha \in C([0, T], \mathbb{H}_p^k)$ and for $t \in [[r, T]$ the following equality holds in $\mathbb{H}_p^{k-2}$:

$$\begin{aligned}\mathbf{u}_\alpha(t) = \mathbf{w}_\alpha + \int_r^t &\mathcal{P}\{\partial_i \left(a^{ij}(s)\,\partial_j \mathbf{u}_\alpha(s)\right) + b^i(s)\partial_i \mathbf{u}_\alpha(s)\} \\ &- \sum_{\gamma \le \alpha} \sqrt{\binom{\alpha}{\gamma}} u^k_{\alpha-\gamma}(s)\,\partial_k \mathbf{u}_\gamma(s) + \mathbf{f}(s)\,1_{\alpha=0} \\ &+ \sum_k \sqrt{\alpha_k} \Big[\left(\sigma^i(s), e_k(s)\right)_Y \partial_i \mathbf{u}_{\alpha(k)}(s) \\ &+ (\mathbf{g}(s), e_k(s))_Y\, 1_{|\alpha|=1} \Big] ds. \end{aligned} \tag{3.4}$$

2. If $\alpha = 0$, the zero term $\mathbf{u}_\alpha(t, x) = \mathbf{u}_0(t, x)$ of an $\mathcal{D} - \mathbb{H}_p^k$ solution in $[r, T]$ satisfies Navier–Stokes equation:

$$\begin{aligned}\mathbf{u}_0(t) = \mathbf{w}_0 + \int_r^t &\mathcal{P}[\partial_i \left(a^{ij}(s)\,\partial_j \mathbf{u}_0(s)\right) + b^i(s)\partial_i \mathbf{u}_0(s) \\ &- u_0^k(s)\,\partial_k \mathbf{u}_0(s) + \mathbf{f}(s)]ds. \end{aligned} \tag{3.5}$$

For the remaining components we have to solve Stokes equations. For $|\alpha| \ge 1$, we can rewrite (3.4) as

$$\begin{aligned}\mathbf{u}_\alpha(t) = \mathbf{w}_\alpha + \int_r^t &\mathcal{P}[\partial_i \left(a^{ij}(s)\,\partial_j \mathbf{u}_\alpha(s)\right) + \mathbf{F}_\alpha(s) \\ &+ [b^i(s) - u_0^i(s)]\partial_i \mathbf{u}_\alpha(s) - u_\alpha^k(s)\,\partial_k \mathbf{u}_0(s)]ds, \end{aligned} \tag{3.6}$$

with

$$\begin{aligned}\mathbf{F}_\alpha(s) = &\sum_{\gamma \le \alpha, |\alpha|-1 \ge |\gamma| \ge 1} \sqrt{\binom{\alpha}{\gamma}} u^k_{\alpha-\gamma}(s)\,\partial_k \mathbf{u}_\gamma(s) \\ &+ \sum_k \sqrt{\alpha_k} \Big[\left(\sigma^i(s), e_k(s)\right)_Y \partial_i \mathbf{u}_{\alpha(k)}(s) + (\mathbf{g}(s), e_k(s))_Y\, 1_{|\alpha|=1} \Big]. \end{aligned} \tag{3.7}$$

Since for $|\alpha| \geq 1, \mathbf{E}\xi_\alpha = 0$, Eq. (3.2) (or (3.3)) can be regarded as a random perturbation of the deterministic Navier–Stokes Eq. (3.5).

Lemma 1 *Let A1–A3 hold,* $|\mathbf{w}_0|_{2,p}+|\mathbf{w}_0|_{2,2} < \infty$*. Then there is* $T_1 > 0$ *and a unique* $\mathbf{u}_0 \in C([0,T_1), \mathbb{H}_2^2 \cap \mathbb{H}_p^2)$ *solving* (3.5) *in* $[0,T_1)$.

Proof According to Theorem 3 in [22], there is $T_1 > 0$ and a unique $\mathbf{u}_0 \in C\left([0,T_1), \mathbb{H}_p^1 \cap \mathbb{H}_2^1\right)$ such that for each $t < T_1$

$$\sup_{0\leq s\leq t} |\mathbf{u}_0(s)|_{1,l}^l + \int_r^t |\partial^2 \mathbf{u}_0(r)|_l^l dr < \infty, \quad l = 2, p,$$

and (3.5) holds in $\mathbb{H}_l^{-1}$, $l = 2, p$. By Sobolev embedding theorem, for all $t < T_1$,

$$\sup_{x,s\leq t} |\mathbf{u}_0(s,x)| + \int_0^t \sup_x |\nabla \mathbf{u}_0(s,x)|^p ds < \infty.$$

and there is a constant C such that for all $s \in [0, T_1)$,

$$|u_0^k(s)\partial_k \mathbf{u}_0(s)|_{1,p} \leq C|u_0^k(s)|_{1,p}|\partial_k \mathbf{u}_0(s)|_{1,p}.$$

So, for each $t < T_1$,

$$\int_0^t |u_0^k(s)\partial_k \mathbf{u}_0(s)|_{1,p}^p ds \leq C \int_0^t |u_0^k(s)|_{1,p}^p |\partial_k \mathbf{u}_0(s)|_{1,p}^p ds$$

$$\leq C \sup_{s\leq t} |\mathbf{u}_0(s)|_{1,p} \int_0^t |\partial_k \mathbf{u}_0(s)|_{1,p}^p ds < \infty.$$

Also,

$$\int_0^t |u_0^k(s)\partial_k \mathbf{u}_0(s)|_{1,2}^2 ds < \infty.$$

Indeed,

$$\int_0^t |u_0^k(s)\partial_k \mathbf{u}_0(s)|_2^2 ds \leq \sup_{s\leq t,x} |u_0^k(s,x)|^2 \int_0^t |\nabla \mathbf{u}_0(s)|_2^2 ds < \infty,$$

and also

$$\int_0^t \left|\nabla\left(u_0^k(s)\partial_k \mathbf{u}_0(s)\right)\right|_2^2 ds = I_1 + I_2$$

$$= \int_0^t |\nabla \mathbf{u}_0(s)|_4^4 ds + \int_0^t |u_0^k(s)\partial_k \nabla \mathbf{u}_0(s)|_2^2 ds$$

with

$$I_1 \le \int_0^t \sup_x |\nabla \mathbf{u}_0(s,x)|^2 |\nabla \mathbf{u}_0(s)|_2^2 ds$$

$$\le \sup_{s\le t} |\nabla \mathbf{u}_0(s)|_2^2 \int_0^t |\mathbf{u}_0(s)|_{2,p}^2 < \infty,$$

and

$$I_2 \le \sup_{s\le t, x} |\mathbf{u}_0(s,x)|^2 \int_0^t |\partial^2 \mathbf{u}_0(s)|_2^2 ds < \infty.$$

By Proposition 5 in Appendix II, $\mathbf{u}_0 \in C\left([0,T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2\right)$ for every $T < T_1$ and (3.5) holds in $\mathbb{L}_l$, $l = 2, p$. □

Now we fix an arbitrary $T < T_1 (T_1$ comes from Lemma 1) and prove the existence and uniqueness of $\mathcal{D}$-solutions to (3.2) in $[r, T]$, $r < T$.

Lemma 2 *Assume that A1–A3 hold,* $\mathbf{b} \in \mathcal{B}$, $\mathbf{w} = \sum_\alpha \mathbf{w}_\alpha \xi_\alpha \in \mathcal{D}'(\mathbf{b}; \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$.

Then for each $r \le T < T_1$ *there is a unique* $\mathcal{D} - \mathbb{H}_p^2 \cap \mathbb{H}_2^2$*-solution* $\mathbf{u}(t) = \sum_\alpha \mathbf{u}_\alpha(t)\xi_\alpha \in C\mathcal{D}'\left(\mathbf{b}; [0,T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2\right)$ *of* (3.1) *in* $[r, T]$.

Equivalently, for each α, $\mathbf{u}_\alpha \in C\left([0,T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2\right)$ *and* (3.5)–(3.6) *hold in* $\mathbb{L}_l$, $l = 2, p$.

Proof According to Remark 4, it suffice to prove the existence and uniqueness of a solution for the deterministic system (3.4). For $\alpha = 0$, the existence and uniqueness of a solution to (3.5) follows from Lemma 1. We proceed by induction. Assume there are unique $\mathbf{u}_\alpha \in C([0,T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$, $|\alpha| \le n$, such that (3.4) holds in $\mathbb{L}_l$, $l = 2, p$. By Sobolev embedding theorem, it implies that

$$\sup_{x, r\le s\le T} |\mathbf{u}_\alpha(s,x)| + \int_r^T \sup_x |\partial \mathbf{u}_\alpha(s,x)|^p ds < \infty, \tag{3.8}$$

if $|\alpha| \le n$. Then for $|\alpha| = n+1$, the Eq. (3.6) for $\mathbf{u}_\alpha$ is Stokes and it is readily checked (see (3.6)) that

$$\int_r^T |\mathbf{F}_\alpha(s)|_{1,l}^l ds < \infty, l = 2, p.$$

According to Proposition 5, there is a unique $\mathbf{u}_\alpha \in C\left([0,T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2\right)$ so that (3.6) holds in $\mathbb{L}_l$, $l = 2, p$. □

Because of the uniqueness, the $\mathcal{D}$-solution has a restarting property. More specifically, the following statement holds.

Corollary 1 *Assume that A1–A3 hold,* $\mathbf{b} \in \mathcal{B}$, $\mathbf{w} = \sum_\alpha \mathbf{w}_\alpha \xi_\alpha \in \mathcal{D}'(\mathbf{b}; \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$. *Let* $\mathbf{u}^{r,\mathbf{w}}(t)$ *be the unique* $\mathcal{D} - \mathbb{H}_p^2 \cap \mathbb{H}_2^2$ *solution to* (3.1)*in* $[r, T]$, $T < T_1$, *starting at* $\mathbf{w}$. *Let* $r \leq r' \leq t \leq T$. *Then*

$$\mathbf{u}^{r,\mathbf{w}}(t) = \mathbf{u}^{r',\mathbf{u}(r')}(t).$$

Proof Indeed for $u(t) = u^{r,\mathbf{w}}(t)$, and $r \leq r' \leq t \leq T$, we have for $t \in [r, T]$

$$\begin{aligned}\mathbf{u}(t) = \mathbf{u}(r') + \int_{r'}^t \mathcal{P}\Big\{&\partial_i\left(a^{ij}(s)\,\partial_j\mathbf{u}(s)\right) + b^i(s)\partial_i\mathbf{u}(s)\\ &-u^k(s)\lozenge\partial_k\mathbf{u}(s) + \mathbf{f}(s)\\ &+[\sigma^i(s)\partial_i\mathbf{u}(s) + \mathbf{g}(s) - \nabla\tilde{P}(s)]\lozenge\dot{W}_s\Big\}\,ds,\end{aligned}$$

and the statement follows by Lemma 2. □

3.1 Rescaling and approximation of the generalized solution

To begin with, we will derive more precise estimates for $\mathcal{D} - \mathbb{H}_p^2 \cap \mathbb{H}_2^2$ solutions of Eq. (3.2). One could hardly expect that the $\mathcal{D} - \mathbb{H}_p^2 \cap \mathbb{H}_2^2$ solution of unbiased Navier–Stokes equation has finite variance, i.e. $\sum_\alpha |\mathbf{u}_\alpha(t)|^2 < \infty$. However, in this subsection we will show that the solution could be obtained as the limit of square integrable solutions of the equations rescaled in a special way using a second quantization operator (see [27], and Appendix I, 5.1).

Fix $\mathbf{b} = \{e_n\} \in \mathcal{B}$ and define an unbounded operator

$$Ae_k = 2ke_k, k \geq 1.$$

Obviously, the projective limit of the domains $\mathbf{H}_n \subseteq \mathbf{H} = L_2([0,T], Y)$ of A^n with the norm

$$||y||_{\mathbf{H}_n} = ||A^n y|| = \left(\sum_k (2k)^{2n} y_k^2\right)^{1/2}, \quad y = \sum_k y_k e_k \in \mathbf{H}_n,$$

is a nuclear space denoted $\mathcal{N} = \mathcal{N}(\mathbf{b})$. For $n \in \mathbf{N}$, let $\mathbf{H}_{-n}$ be the completion of $\mathbf{H}$ with respect to the norm

$$||y||_{\mathbf{H}_{-n}} = ||A^{-n}y|| = \left(\sum_k (2k)^{-2n}\, y_k^2\right)^{1/2}, \quad y = \sum_k y_k e_k \in \mathbf{H}_{-n}.$$

The inductive limit $\mathcal{N}' = \mathcal{N}'(\mathbf{b})$ of $\mathbf{H}_{-n}$ is the dual of $\mathcal{N}$. For a Banach space E and $\rho \in [0, 1]$, let $\mathcal{S}_\rho(E) = \mathcal{S}_\rho(\mathbf{b}; E)$ be the space of all $\eta = \sum_\alpha a_\alpha \xi_\alpha \in \mathcal{D}'(\mathbf{b}, E)$ such that

$$||\eta||_{\mathcal{S}_{\rho,q}} = ||\eta||_{\rho,q} = \left(\sum_\alpha (|\alpha|!)^\rho (2\mathbf{N})^{2q\alpha} |a_\alpha|_E^2\right)^{1/2} < \infty \quad \text{for every } q \geq 0,$$

where

$$(2\mathbf{N})^{2q\alpha} = \prod_{k=1}^{\infty} (2k)^{2q\alpha_k} = 2^{2q|\alpha|} \prod_{k=1}^{\infty} k^{2q\alpha_k}$$

(see [8]). We consider $\mathcal{S}_\rho(E) = \mathcal{S}_\rho(\mathbf{b}, E)$ with a family of seminorms $||\eta||_{\mathcal{S}_{\rho;q}}$. Similarly, for $\rho \in [0, 1]$, let $\mathcal{S}_{-\rho}(E)$ be the space of all $\eta \in \mathcal{D}'(\mathbf{b}; E)$ such that $||\eta||_{\mathcal{S}_{-\rho,-q}} = ||\eta||_{-\rho,-q} < \infty$ for some $q > 0$. It is dual of $\mathcal{S}_{-\rho}(E)$ if E is Hilbert. If $E = \mathbf{R}$, Kondratiev test function space $(\mathcal{N})^1 = \mathcal{S}_1(\mathbf{R})$ and Kondratiev distribution space $(\mathcal{N})^{-1} = \mathcal{S}_{-1}(\mathbf{R})$ (see [6,8, p. 39]).

We will show that the solution found in Lemma 2 belongs to the Kondratiev space $S_{-1}(\mathbb{H}_p^2 \cap \mathbb{H}_2^2) = (\mathcal{N})^{-1} \otimes (\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$.

Proposition 1 *Let A1–A3 hold,* $\sup_{s,k} |e_k(s)|_Y < \infty$ *and* $\mathbf{w} \in \mathbb{H}_p^2 \cap \mathbb{H}_2^2$ *be deterministic. Assume that*

$$\mathbf{u}(t) = \sum_\alpha \mathbf{u}_\alpha(t)\xi_\alpha \in C\mathcal{D}'(\mathbf{b}; [0, T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$$

solve (3.2) *in* $\mathbb{L}_l$, $l = 2, p$. *Denote*

$$\tilde{L}_\alpha = \sup_{t \leq T} |\mathbf{u}_\alpha(t)|_{2,p} + \sup_{t \leq T} |\mathbf{u}_\alpha(t)|_{2,2}, \quad \alpha \in J.$$

Then there is a constant B_0 *such that*

$$\tilde{L}_\alpha \leq \sqrt{\alpha!}\, C_{|\alpha|-1} \binom{|\alpha|}{\alpha} B_0^{|\alpha|-1} K^{|\alpha|}, \quad |\alpha| \geq 2,$$

where $K = 1 + \sup_i \tilde{L}_{\varepsilon_i}$, *and*

$$C_{|\alpha|-1} = \frac{1}{|\alpha| - 1} \binom{2(|\alpha| - 1)}{|\alpha| - 1}, \quad |\alpha| \geq 2$$

are the Catalan numbers (see e.g. [9,28]). *Moreover, there is a number* $q > 1$ *so that*

$$\sup_{r\leq t\leq T} ||u(t)||^2_{\mathcal{S}_{-1,-q}} \leq \sum_{\alpha} \frac{(2\mathbf{N})^{-2q\alpha}\tilde{L}^2_{\alpha}}{|\alpha|!} < \infty,$$

i.e., the solution $\mathbf{u}(t)$ *belongs to the Kondratiev space of generalized random functions* $S_{-1}(\mathbb{H}^2_p \cap \mathbb{H}^2_2) = (\mathcal{N})^{-1} \otimes (\mathbb{H}^2_p \cap \mathbb{H}^2_2)$.

Proof For $|\alpha| \geq 1$, $\mathbf{u}_{\alpha} \in C\left([0,T], \mathbb{H}^2_p \cap \mathbb{H}^2_2\right)$, are solutions to the Stokes equations (3.6):

$$\mathbf{u}_{\alpha}(t) = \mathbf{w}_{\alpha} + \int_r^t \mathcal{P}[\partial_i\left(a^{ij}(s)\,\partial_j\mathbf{u}_{\alpha}(s)\right) + \mathbf{F}_{\alpha}(s)$$
$$+[b^i(s) - u^i_0(s)]\partial_i\mathbf{u}_{\alpha}(s) - u^k_{\alpha}(s)\,\partial_k\mathbf{u}_0(s)]ds,$$

with $\mathbf{F}_{\alpha}(s)$ defined by (3.7). Finally, by Proposition 5 in Appendix II,

$$\tilde{L}_{\alpha} \leq C\sum_{l=2,p}\left(\int_0^T |\mathbf{F}_{\alpha}(s)|^l_{1,l}ds\right)^{1/l}.$$

Since

$$|u^k_{\alpha-\gamma}(s)\,\partial_k\mathbf{u}_{\gamma}(s)|_{1,l} \leq C|\mathbf{u}_{\alpha-\gamma}(s)|_{2,p}|\nabla\mathbf{u}_{\gamma}(s)|_{1,l},$$

it follows that

$$|\mathbf{F}_{\alpha}(s)|_{1,l} \leq C\left[\sum_{\gamma\leq\alpha,|\alpha|-1\geq|\gamma|\geq 1}\sqrt{\binom{\alpha}{\gamma}}\tilde{L}_{\alpha-\gamma}\tilde{L}_{\gamma}\right.$$
$$\left.+\sum_k 1_{\sigma\neq 0}\sqrt{\alpha_k}\tilde{L}_{\alpha(k)} + 1_{|\alpha|=1}|\mathbf{g}(s)|_{1,l}\right].$$

For $|\alpha| \geq 2$

$$\tilde{L}_{\alpha} \leq C\left[\sum_{\gamma\leq\alpha,1\leq|\gamma|\leq|\alpha|-1}\sqrt{\binom{\alpha}{\gamma}}\tilde{L}_{\alpha-\gamma}\tilde{L}_{\gamma} + 1_{\sigma\neq 0}\sum_k\sqrt{\alpha_k}\tilde{L}_{\alpha(k)}\right].$$

So, there is a constant B_0 so that for $|\alpha| = n \geq 2$, $\hat{L}_{\alpha} = (\alpha!)^{-1/2}\tilde{L}_{\alpha}$, $\hat{L}_{\varepsilon_i} = \tilde{L}_{\varepsilon_i}$ we have

$$\hat{L}_\alpha \le B_0 \left(\sum_{\gamma \le \alpha, 1 \le |\gamma| \le |\alpha|-1} \hat{L}_{\alpha-\gamma} \hat{L}_\gamma + 1_{\sigma \ne 0} \sum_k \hat{L}_{\alpha(k)} 1_{\alpha_k \ne 0} \right).$$

Denoting $L_\alpha = \hat{L}_\alpha$ if $|\alpha| > 1$, $L_\alpha = 1 + \hat{L}_\alpha$ if $|\alpha| = 1$, we get

$$L_\alpha \le B_0 \sum_{\gamma \le \alpha, 1 \le |\gamma| \le |\alpha|-1} L_{\alpha-\gamma} L_\gamma$$

and by [9] for $|\alpha| \ge 2$

$$\begin{aligned} L_\alpha &\le C_{|\alpha|-1} B_0^{|\alpha|-1} \binom{|\alpha|}{\alpha} \prod_i (1 + \tilde{L}_{\varepsilon_i})^{\alpha_i} \\ &\le C_{|\alpha|-1} \binom{|\alpha|}{\alpha} B_0^{|\alpha|-1} K^{|\alpha|} \end{aligned}$$

with

$$K = 1 + C \left[\tilde{L}_0 + \sum_{l=2,p} \left(\int_0^T ||\mathbf{g}(s)||_{1,l}^l ds \right)^{1/l} \right].$$

So,

$$\begin{aligned} \tilde{L}_\alpha &\le \sqrt{\alpha!} C_{|\alpha|-1} \binom{|\alpha|}{\alpha} B_0^{|\alpha|-1} K^{|\alpha|}, \\ \tilde{L}_\alpha^2 &\le \alpha! C_{|\alpha|-1}^2 \binom{|\alpha|}{\alpha} (2\mathbf{N})^\alpha B_0^{2(|\alpha|-1)} K^{2|\alpha|} \end{aligned}$$

and

$$\frac{r^\alpha \tilde{L}_\alpha^2}{\alpha!} \le C_{|\alpha|-1}^2 \binom{|\alpha|}{\alpha} (2\mathbf{N}r)^\alpha B_0^{2(|\alpha|-1)} K^{2|\alpha|}.$$

Therefore with $r = (r_i), r_i = (2i)^{-2q}, q > 1$,

$$\begin{aligned} \sum_{|\alpha|=n} \frac{r^\alpha \tilde{L}_\alpha^2}{\alpha!} &= C_{n-1}^2 B_0^{2(n-1)} K^{2n} \sum_{|\alpha|=n} \binom{|\alpha|}{\alpha} (2\mathbf{N}r)^\alpha \\ &\le C_{n-1}^2 B_0^{2(n-1)} K^{2n} 2^n 2^{-2qn} \left(\sum_{i=1}^\infty \frac{1}{i^{2q-1}} \right)^n. \end{aligned}$$

For large n, the Catalan numbers

$$C_{n-1} \approx \frac{4^{n-1}}{\sqrt{\pi}(n-1)^{3/2}}$$

and there is a number $q > 1$ such that

$$\sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{r^{\alpha} \tilde{L}_{\alpha}^{2}}{\alpha!} < \infty.$$

Therefore, the solution $\mathbf{u}(t)$ belongs to Kondratiev's space $S_{-1}(\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$. □

Remark 5 If $\mathbf{u}(t)$ is the solution (3.2) with a deterministic $\mathbf{w} \in \mathbb{H}_p^2 \cap \mathbb{H}_2^2$, and $q > 1$ is the number in Proposition 1, then the action of $\mathbf{u}(t) = \sum_{\alpha} \mathbf{u}_{\alpha}(t)\xi_{\alpha}$ can be extended from $\mathcal{D}(\mathbf{b})$ to

$$\mathcal{S}_{1,q}(\mathbf{R}) = \left\{ \eta = \sum_{\alpha} a_{\alpha}\xi_{\alpha} \in \mathcal{S}_1(\mathbf{R}) : ||\eta||_{\mathcal{S}_{1,q}} < \infty \right\}$$

as

$$\langle \mathbf{u}(t), \eta \rangle = \sum_{\alpha} \mathbf{u}_{\alpha}(t) a_{\alpha}, \eta = \sum_{\alpha} a_{\alpha}\xi_{\alpha} \in \mathcal{S}_{1,q}(\mathbf{R}).$$

Note that the stochastic exponent $p(z) = p(e_z)$ (see Remark 1) belongs to $\mathcal{S}_{1,q}(\mathbf{R})$ in (a) provided

$$|A^q e_z|_Y^2 = \sum_k (z_k 2^q k^q)^2 < 1.$$

For $\varepsilon > 0$ define a self-adjoint positive operator D_{ε} on H such that $D_{\varepsilon} e_k = 2^{-\varepsilon k} e_k$ and a sequence of positive numbers $\kappa_{\varepsilon,n} = e^{-\varepsilon e^n}$. Set $C_{\varepsilon} = \sum_{n=0}^{\infty} \kappa_{\varepsilon,n} D_{\varepsilon}^{\otimes n}$. It is a second quantization operator in the Fock space $\mathcal{H} = \sum_n \mathcal{H}_n, \mathcal{H}_n = \mathbf{H}^{\hat{\otimes} n}$ (see Appendix I, 5.1). Clearly,

$$C_{\varepsilon} e_{\alpha} = \kappa_{\varepsilon,|\alpha|} D_{\varepsilon}^{\otimes n} e_{\alpha} = \kappa_{\varepsilon,|\alpha|} \left(2^{-\varepsilon \mathbf{N}}\right)^{\alpha} e_{\alpha}, \tag{3.9}$$

where

$$\left(2^{-\varepsilon \mathbf{N}}\right)^{\alpha} = \prod_{k=1}^{\infty} 2^{-\varepsilon k \alpha_k}.$$

Proposition 2 *Assume A1–A3 hold and* $\sup_{s,k} |e_k(s)|_Y < \infty$. *Let* $\mathbf{u}(t) = \sum_{\alpha} \mathbf{u}_{\alpha}(t)\xi_{\alpha} \in C\mathcal{D}'([0,T], \mathbf{b}, \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$ *be a generalized* $\mathcal{D} - \mathbb{H}_p^2 \cap \mathbb{H}_2^2$*-solution of Eq.* (3.1) *in* $[0, T]$ *with a deterministic* $\mathbf{w} \in \mathbb{H}_p^2 \cap \mathbb{H}_2^2$ *and*

$$\mathbf{u}_{\varepsilon}(t) = C_{\varepsilon}\mathbf{u}(t) = \sum_{n=0}^{\infty} \kappa_{\varepsilon,n} \sum_{|\alpha|=n} \mathbf{u}_{\alpha}(t)(2^{-\varepsilon \mathbf{N}})^{\alpha}\xi_{\alpha},$$

where $C_\varepsilon \mathbf{u}$ is rescaling based on the second quantization operator C_ε (see Appendix I, 5.1).

Then $\mathbf{u}_\varepsilon(t)$ is $\mathbb{H}_2^2$-valued square integrable process satisfying the equation

$$\begin{aligned}\partial_t \mathbf{u}_\varepsilon(t) = \mathcal{P}\Big\{&\partial_i\left(a^{ij}(t)\,\partial_j \mathbf{u}_\varepsilon(t)\right) + b^i(t)\partial_i \mathbf{u}_\varepsilon(t)\\ &- C_\varepsilon(C_\varepsilon{}^{-1}u_\varepsilon^k(t))\Diamond(C_\varepsilon{}^{-1}\partial_k \mathbf{u}_\varepsilon(t)) + \mathbf{f}(t)\\ &+C_\varepsilon[(\sigma^i(t)C_\varepsilon{}^{-1}\partial_i \mathbf{u}_\varepsilon(t) + \mathbf{g}(t))\Diamond(C_\varepsilon{}^{-1}\dot{W}_t^\varepsilon)]\Big\},\\ \mathbf{u}_\varepsilon(0) = \mathbf{w}&, \operatorname{div}\mathbf{u}_\varepsilon = 0.\end{aligned} \tag{3.10}$$

Moreover, $\mathbf{u}_\varepsilon(t) \in \mathcal{S}_1(\mathbb{H}_p^2 \cap \mathbb{H}_2^2) = (\mathcal{N})^1 \otimes (\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$, $t \in [0,T]$, $\mathbf{u}(t) = C_\varepsilon{}^{-1}\mathbf{u}_\varepsilon(t)$ and

$$\sup_{t\le T} ||\mathbf{u}_\varepsilon(t) - \mathbf{u}(t)||_{S_{-1,-q}} \to 0$$

as $\varepsilon \to 0$, where q is a number in Proposition 1.

Proof Let $\tilde{L}_\alpha = \sup_{t\le T}|\mathbf{u}_\alpha(t)|_{2,p} + \sup_{t\le T}|\mathbf{u}_\alpha(t)|_{2,2}$. Since $\mathbf{u}_{\varepsilon,\alpha}(t) = \kappa_{\varepsilon,|\alpha|}\left(2^{-\varepsilon\mathbf{N}}\right)^\alpha$ $\mathbf{u}_\alpha(t)$,

$$\tilde{L}_{\varepsilon,\alpha} = \sup_{t\le T}|\mathbf{u}_{\varepsilon,\alpha}(t)|_{2,p} + \sup_{t\le T}|\mathbf{u}_{\varepsilon,\alpha}(t)|_{2,2} = \kappa_{\varepsilon,|\alpha|}\left(2^{-\varepsilon\mathbf{N}}\right)^\alpha \tilde{L}_\alpha.$$

Since for each $q' \ge 0$, there is a constant $C(\varepsilon, q', q)$ independent of α so that

$$\begin{aligned}&(|\alpha|!)^2\,(2\mathbf{N})^{2(q'+q)}\,\kappa_{\varepsilon,|\alpha|}^2\left(2^{-2\varepsilon\mathbf{N}}\right)^\alpha\\ &\le \sum_k (2k)^{2(q'+q)}\,2^{-2\varepsilon k})^{|\alpha|}e^{-2\varepsilon e^{|\alpha|}}(|\alpha|!)^2\\ &\le C(\varepsilon, q', q) < \infty,\end{aligned}$$

it follows by Proposition 1 that

$$||\mathbf{u}_\varepsilon(t)||_{\mathcal{S}_{1,q'}}^2 \le C(\varepsilon, q', q)\sum_\alpha \frac{(2\mathbf{N})^{-2q\alpha}\tilde{L}_\alpha^2}{|\alpha|!} < \infty.$$

So, $\mathbf{u}_\varepsilon(t) \in S_1(\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$, $t \in [0,T]$. In particular,

$$\mathbf{E}|\mathbf{u}_\varepsilon(t)|_{2,2}^2 = \sum_\alpha |\mathbf{u}_{\varepsilon,\alpha}(t)|_{2,2}^2 < \infty, t \in [0,T].$$

Therefore $\mathbf{u}_\varepsilon(t)$ is $\mathbb{H}_2^2$-valued square integrable process and (3.10) follows by Remark 10 in Appendix I, 5.1. Obviously, $\mathbf{u}(t) = (C_\varepsilon)^{-1}\mathbf{u}_\varepsilon(t)$. Since

$$\mathbf{u}_\varepsilon(t) - \mathbf{u}(t) = \sum_\alpha [1 - \kappa_{\varepsilon,|\alpha|}\left(2^{-\varepsilon\mathbf{N}}\right)^\alpha]\mathbf{u}_\alpha(t,x)\xi_\alpha,$$

it follows that

$$||\mathbf{u}_\varepsilon(t) - \mathbf{u}(t)||^2_{\mathcal{S}_{-1,-q}} \le \sum_\alpha |1 - \kappa_{\varepsilon,|\alpha|}\left(2^{-\varepsilon\mathbf{N}}\right)^\alpha|^2 \frac{(2\mathbf{N})^{-2q\alpha}\tilde{L}_\alpha^2}{|\alpha|!} \to 0$$

as $\varepsilon \to 0$ by Lebesgue's dominated convergence theorem uniformly in t. □

Remark 6 The solution **u** in Proposition 1 depends on a fixed uniformly bounded basis **b** in **H**. It belongs to the Kondratiev space

$$S_{-1}(\mathbb{H}^2_p \cap \mathbb{H}^2_2) = (\mathcal{N})^{-1} \otimes (\mathbb{H}^2_p \cap \mathbb{H}^2_2)$$

constructed using a Gelfand triple $\mathcal{N} \subseteq \mathbf{H} = L_2([0,T], Y) \subseteq \mathcal{N}'$ which depends on **b**.

In [7], a class $\mathcal{G}^{-1} \subseteq (\mathcal{N})^{-1}$ of regular generalized functions was introduced that does not depend on a fixed Gelfand triple or a basis in **H**. Unfortunately, the estimates in Proposition 1 (because of the factor $\mathbf{N}^{-q\alpha} = \Pi_k k^{-q\alpha_k}$) do not imply that **u** is a regular generalized function of class $\mathcal{G}^{-1}$. Also, the space

$$\mathcal{S}_{-1,-q}(\mathbb{H}^2_p \cap \mathbb{H}^2_2) = \left\{\eta \in \mathcal{S}_{-1}(\mathbb{H}^2_p \cap \mathbb{H}^2_2) : ||\eta||_{\mathcal{S}_{-1,-q}} < \infty\right\},$$

with some $q > 1$, to which the solution in Proposition 1 belongs, cannot be embedded into any space with weights depending only on $|\alpha|$ (for example, into the spaces, like $\mathcal{G}^{-1,-q}$ in [7], that do not dependent on a fixed basis or Gelfand triple in **H**): $\inf_{|\alpha|=n}(\mathbf{N})^{-q\alpha} = 0$.

4 Markov property and independence of basis

In this Section we will show that a generalized $\mathcal{D} - \mathbb{H}^2_p \cap \mathbb{H}^2_2$-solution of Eq. (3.2) has the following properties: it is adapted with respect to the filtration $(\mathcal{F}^W_t)$ generated by the Wiener process W_t; it is independent of the choice of the basis **b**, and it is a generalized Markov process.

4.1 Equivalent characterization of $\mathcal{D}$-generalized processes

A more convenient characterization of $\mathcal{D}$-solution to (1.2) (see Definition 5) is based on another (equivalent) description of $\mathcal{D}(\mathbf{b})$. It allows to introduce the notion of an adapted solution and extend it from $\mathcal{D}(\mathbf{b})$ to a space of test functions that is independent of $\mathbf{b} \in \mathcal{B}$.

4.1.1 Equivalent description of test function space

Often it is convenient to use the exponents $p(z), z \in \mathcal{Z}$, defined in Remark 1 to describe the test function space $\mathcal{D}(\mathbf{b})$.

To each multi-index α of length n we relate a set K_α whose elements are positive integers $k_i, i = 1, \ldots, n$, such that each k is represented there by α_k-copies. An ordered n-tuple $K_\alpha = \{k_1, \ldots, k_n\}$ with $k_1 \leq k_2 \leq \cdots \leq k_n$ characterizes the locations and the values of the non-zero components of α. For example, k_1 is the index of the first non-zero element of α, followed by $\max\left(0, \alpha_{k_1} - 1\right)$ of entries with the same value (see [20]).

For an orthonormal basis $\{e_k, k \geq 1\}$ in $L_2([0, T], Y)$ and $\alpha \in I$ with $K_\alpha = \{k_1, \ldots, k_n\}$, we denote

$$E_\alpha = \sum_{\sigma \in G^n} e_{k_{\sigma(1)}} \otimes \cdots \otimes e_{k_{\sigma(n)}}, \alpha \in J,$$

where G^n is a permutation group of $\{1, \ldots, n\}$. The set

$$\left\{ e_\alpha = \frac{E_\alpha}{\sqrt{\alpha!|\alpha|!}}, \alpha \in J \right\} \tag{4.1}$$

is a CONS for the symmetric part $\mathbf{H}^{\hat{\otimes} n}$ of $\mathbf{H}^{\otimes n}$.

For $|\alpha| = n$,

$$\xi_\alpha = \sqrt{|\alpha|!} W(e_\alpha), \tag{4.2}$$

where

$$W(e_\alpha) = \int_0^T \int_0^{s_n} \ldots \int_0^{s_2} e_\alpha(s_1, \ldots, s_n) dW_{s_1} \ldots dW_{s_n}. \tag{4.3}$$

If $\alpha = \varepsilon_k$, then $W(e_{\varepsilon_k}) = W(e_k) = \int_0^T e_k(t) dW_t$.

According to (2.4),

$$\begin{aligned} p(z) = p(e_z) = p(z, \mathbf{b}) &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} z^\alpha \sqrt{\frac{|\alpha|!}{\alpha!}} W(e_\alpha) \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \frac{z^\alpha}{\sqrt{\alpha!}} \xi_\alpha. \end{aligned}$$

Denote

$$p_n(z) = p_n(e_z) = \sum_{|\alpha|=n} \frac{z^\alpha}{\sqrt{\alpha!}} \xi_\alpha$$

$$= \int_0^T \int_0^{s_{n-1}} \ldots \int_0^{s_2} e_z(s_1) \ldots e_z(s_n) dW_{s_1} \ldots dW_{s_n}, \quad n \geq 2, \tag{4.4}$$

$$p_1(z) = p_1(e_z) = \int_0^T e_z(s_1) dW_{s_1},\ p_0(z) = p_0(e_z) = 1.$$

Lemma 3 *For* $\mathbf{b} \in \mathcal{B}$, *let* $\mathcal{V} = \mathcal{V}(\mathbf{b})$ *be the linear space of random variables that consists of all finite linear combinations of* $p_n(z), z \in \mathcal{Z}, n \geq 0$. *Then* $\mathcal{V}(\mathbf{b}) = \mathcal{D}(\mathbf{b})$ *(in particular,* $\xi_\alpha \in \mathcal{V}(\mathbf{b})$*).*

Proof Obviously, $\mathcal{V} = \mathcal{V}(\mathbf{b}) \subseteq \mathcal{D} = \mathcal{D}(\mathbf{b})$. For $\alpha \in \mathcal{I}$, denote $\kappa(\alpha) = \max\{k : \alpha_k \neq 0\}$. Fix N, n and $\alpha = (\alpha_k) \in \mathcal{I}$ such that $|\alpha| = N, \kappa(\alpha) = n$. Consider a finite dimensional Hilbert space

$$G = \left\{ \sum_{|\alpha|=N, \kappa(\alpha) \leq n} v_\alpha \xi_\alpha : v_\alpha \in \mathbf{R} \right\}.$$

with inner product

$$\left(\sum_a v_\alpha \xi_\alpha, \sum_a v'_\alpha \xi_\alpha \right)_G = \sum_\alpha v_\alpha v'_\alpha.$$

Let $\tilde{G}$ be a vector subspace of G generated by $p_N(z), z = (z_1, \ldots, z_n, 0, \ldots) \in \mathcal{Z}$. It is enough to show that $\tilde{G} = G$. Indeed, the subspace $\tilde{G}$ is finite-dimensional and obviously closed. Assume there is a vector $\sum_\alpha v_\alpha \xi_\alpha \in G$ which is orthogonal to $\tilde{G}$. So, for all $z = (z_1, \ldots, z_n, 0, \ldots) \in \mathcal{Z}$,

$$\left(\sum_\alpha v_\alpha \xi_\alpha, p_N(z) \right)_G = \sum_\alpha v_\alpha \frac{z^\alpha}{\sqrt{\alpha!}} = 0$$

which implies that all $v_\alpha = 0$. Therefore $\tilde{G} = G$. This completes the proof. □

Due to Lemma 3, we can characterize convergence in $\mathcal{D}' = \mathcal{D}'(\mathbf{b})$ by test functions of the form $p_m(z)$. Indeed, for $z \in \mathcal{Z}, v \in \mathcal{D}$, and $m \geq 0$, we have

$$\langle p_m(z), v \rangle = \sum_{|\alpha|=m} v_\alpha \frac{z^\alpha}{\sqrt{\alpha!}}. \tag{4.5}$$

Therefore we have the following necessary and sufficient condition:

Corollary 2 *A sequence* $v^n \to v$ *in* $\mathcal{D}'$ *if and only if for all* $z \in \mathcal{Z}$ *and all* $m \geq 0$

$$\langle p_m(z), v^n \rangle \to \langle p_m(z), v \rangle.$$

4.1.2 Action of a Skorokhod integral on $p_M(z)$

Consider $v(t) = \sum_\alpha v_\alpha(t)\xi_\alpha \in \mathcal{D}'(\mathbf{b}; [0, T], Y)$ such that for all α, k, $\int_0^T |(v_\alpha(s), e_k(s))_Y| ds < \infty$. Recall that the Skorokhod integral assigns to such v a generalized random process

$$\delta_t(v) = \int_0^t v(s) dW_s = \delta\left(v 1_{[0,t]}\right) = \sum_\alpha \delta_t(v)_\alpha \xi_\alpha, \quad 0 \le t \le T,$$

$$\delta_t(v)_\alpha = \sum_k \sqrt{\alpha_k} \int_0^t \left(v_{\alpha(k)}(s), e_k(s)\right)_Y ds.$$

Remark 7 For $p_M(z) \in \mathcal{D}(\mathbf{b})$, $z \in Z$, $M \ge 1$, it is easy to show that

$$\langle p_M(z), \delta_t(v)\rangle = \int_0^t (\langle p_{M-1}(z), v(s)\rangle, e_z(s))_Y \, ds.$$

4.1.3 Action of a Wick product on $p_M(z)$

Recall that for a Hilbert space E and arbitrary $v = \sum_\alpha v_\alpha \xi_\alpha$ and $u = \sum_\alpha u_\alpha \xi_\alpha$ in $\mathcal{D}'(\mathbf{b}, E)$, we define

$$v \Diamond u = \sum_\alpha \sum_{\beta \le \alpha} (u_\beta, v_{\alpha-\beta})_E \sqrt{\frac{\alpha!}{\beta!(\alpha-\beta)!}} \xi_\alpha \in \mathcal{D}'(\mathbf{b}, \mathbf{R}).$$

In particular,

$$\xi_\alpha \Diamond \xi_\beta = \xi_{\alpha+\beta} \sqrt{\frac{(\alpha+\beta)!}{\beta!\alpha!}}.$$

The following statement holds.

Lemma 4 *For a Hilbert space E, arbitrary elements $v = \sum_\alpha v_\alpha \xi_\alpha$ and $u = \sum_\alpha u_\alpha \xi_\alpha$ from $\mathcal{D}'(\mathbf{b}, E)$, and $z \in \mathcal{Z}$, $M \ge 0$,*

$$\langle p_M(z), v \Diamond u\rangle = \sum_{K+L=M} \langle p_K(z), v\rangle \langle p_L(z), u\rangle;$$

In particular,

$$\langle 1, v \Diamond u\rangle = (\langle 1, v\rangle, \langle 1, u\rangle)_E$$

(*the generalized expected value of $v \Diamond u$ is the product of expected values*).

Proof According to (4.5),

$$\begin{aligned}\langle p_M(z), v\Diamond u\rangle &= \sum_{|\alpha|=M}\sum_{\beta\le\alpha}\Big(\frac{z^\beta}{\sqrt{\beta!}}u_\beta, \frac{z^{\alpha-\beta}}{\sqrt{(\alpha-\beta)!}}v_{\alpha-\beta}\Big)_E \\ &= \left(\sum_{K+L=M}\sum_{|\beta|=K}\frac{z^\beta}{\sqrt{\beta!}}u_\beta, \sum_{|\gamma|=L}\frac{z^\gamma}{\sqrt{\gamma!}}u_\gamma\right)_E \\ &= \left(\langle p(z), v\rangle, \langle p(z), u\rangle\right)_E .\end{aligned}$$

□

4.1.4 An equivalent characterization of the solution

Now, we will characterize the solution of Eq. (3.1) by its action on test functions $p_M(z)$, $z\in\mathcal{Z}$, $M\ge 0$. The following statement holds.

Remark 8 Assume A1-A3 hold, $\mathbf{w}=\sum_\alpha \mathbf{w}_\alpha\xi_\alpha\in\mathcal{D}'(\mathbf{b},\mathbb{H}^2_p\cap\mathbb{H}^2_2)$, $\operatorname{div}\mathbf{w}=0$ and $\mathbf{u}(t)=\sum_\alpha\mathbf{u}_\alpha(t)\xi_\alpha\in C\mathcal{D}'(\mathbf{b};[r,T],\mathbb{H}^2_p\cap\mathbb{H}^2_2)$. Then $\mathbf{u}(t)$ is $\mathcal{D}-\mathbb{H}^2_p\cap\mathbb{H}^2_2$ solution of (3.1) in $[r,T]$ if and only if for all $z\in\mathcal{Z}$ and $M\ge 0$,

$$\mathbf{u}^{M,z}(t)=\langle\mathbf{u}(t), p_M(z)\rangle=\sum_{|\alpha|=M}\frac{\mathbf{u}_\alpha(t)z^\alpha}{\sqrt{\alpha!}}$$

is continuous in $\mathbb{H}^2_p\cap\mathbb{H}^2_2$ and the following equality holds in $\mathbb{L}_l$, $l=2,p$,

$$\begin{aligned}\mathbf{u}^{M,z}(t) &= \mathbf{w}^{M,z}+\int_r^t \mathcal{P}[\partial_i\left(a^{ij}(s)\,\partial_j\mathbf{u}^{M,z}(s)\right) \\ &\quad + b^i(s)\partial_i\mathbf{u}^{M,z}(s)-\sum_{K+L=M}u^{k,K,,z}(s)\,\partial_k\mathbf{u}^{L,z}(s) \\ &\quad + 1_{M=0}\mathbf{f}(s)+\left(\sigma^i(s), e_z(s)\right)_Y\partial_i\mathbf{u}^{M-1,z}(s) \\ &\quad + 1_{M=1}(\mathbf{g}(s), e_z(s))_Y]ds,\end{aligned} \tag{4.6}$$

where $M\ge 0$, $\mathbf{w}^{M,z}(x)=\langle\mathbf{w}(x), p_M(z)\rangle=\sum_{|\alpha|=M}\mathbf{w}_\alpha(x)z^\alpha/\sqrt{\alpha!}$, and $\mathbf{u}^{-1,z}(t,x)=0$. If $M\ge 1$, Eq. (4.6) is Stokes equation; if $M=0$, it is Navier–Stokes equation. Indeed, we obtain (4.6) by multiplying both sides of (3.4) by $z^\alpha/\sqrt{\alpha!}$ and adding.

4.2 Adapted and independent of basis generalized processes

Let $L_\infty([0,T],Y)$ be the space of measurable Y-valued bounded functions on $[0,T]$. For $h\in L_\infty([0,T],Y)$, $M\ge 0$, we denote

$$p_{M,t}(h)=\int_0^t\int_0^{s_M}\ldots\int_0^{s_2}h(s_1)\ldots h(s_M)dW_{s_1}\ldots dW_{s_M},\quad 0\le t\le T.$$

By (4.4), $p_M(z)=p_M(e_z)=p_{M,T}(e_z),z\in\mathcal{Z}$.

Lemma 5 (i) *If* $\{m_k,k\ge 1\}$ *is a CONS in* $L_2(0,T)$, *and* $\{\ell_k,k\ge 1\}$ *is a CONS in* Y, $h\in L_\infty([0,T],Y)$, *then for each* $n,n'\ge 1$, *there is* $z\in\mathcal{Z}$ *such that*

$$h_{n,n'}(t)=\sum_{i=1}^{n'}\sum_{k=1}^{n}\int_0^T(h(s),\ell_k)m_i(s)ds\ell_k m_i(t)=e_z(t),$$

$0\le t\le T$. *Obviously,* $p_{M,T}(h_{n,n'})\in\mathcal{D}(\mathbf{b}),\mathbf{b}=\left\{e_k=m_{i_k}\ell_{j_k},k\ge 1\right\}$,

$$h_{n,n'}\to h \text{ in } L_2([0,T],Y),$$
$$p_{M,T}(h_{n,n'})\to p_{M,T}(h)\text{ in } L_2(\Omega,\mathbf{P}),$$

as $n,n'\to\infty$.

(ii) *Assume* (m_i) *is trigonometric basis or unconditional* $L_p([0,T])$*-basis (for example, Haar basis, see* [18]*),* $h\in L_\infty([0,T],Y)$. *Then there is a sequence* $z(n)\in\mathcal{Z}$ *such that* $e_{z(n)}\to h$ *in* $L_p([0,T],Y)$ *for all* $p\ge 2$, *as* $n\to\infty$.

Proof We prove the second part of the statement. Let

$$h_n(s)=\sum_{k=1}^{n}(h(s),\ell_k)_Y\ell_k,n\ge 1.$$

Then $|h_n(s)|_Y\le|h(s)|_Y$ and for all $p\ge 2$,

$$\int_0^T|h_n(s)-h(s)|_Y^p ds\to 0$$

as $n\to\infty$. If (m_i) is trigonometric basis or unconditional $L_p([0,T])$-basis (for example, Haar basis), then for each n

$$\int_0^T|h_{n,n'}(s)-h_n(s)|_Y^p ds\to 0$$

as $n'\to\infty$. So, there is a subsequence l_n such that

$$\int_0^T|h_{n,l_n}(s)-h(s)|_Y^p ds\to 0$$

as $n\to\infty$, and (ii) follows according to part (i) of this remark. □

Let $\mathcal{T}$ be the space of all linear combinations of $p_{M,T}(h)$, $h \in L_\infty([0,T],Y)$, $M \geq 0$. Obviously, $\cup_{\mathbf{b}\in\mathcal{B}}\mathcal{D}(\mathbf{b}) \subseteq \mathcal{T}$, and $\mathcal{T}$ does not depend on any particular $\mathbf{b} \in \mathcal{B}$.

We say that $h_n \to h$ in $L_\infty([0,T],Y)$ if $h_n \to h$ in $L_p([0,T],Y)$ for all $p \geq 2$.

Definition 6 Given a Banach space E, let $\mathcal{T}' = \mathcal{T}'(E)$ be the space of all linear E-valued functions v on $\mathcal{T}$ such that $v(p_{M,T}(h_n)) \to v(p_{M,T}(h))$ for all $M \geq 0$ if $h_n \to h$ in $L_\infty([0,T],Y)$. We say $v \in \mathcal{T}'(E)$ is a generalized E-valued r.v.

Denote $\mathcal{T}'(\mathbf{b}) = \mathcal{T}'(\mathbf{b},E)$ the set of all $v \in \mathcal{D}'(\mathbf{b},E)$ such that for each $h \in L_\infty([0,T],Y)$ and any sequence $e_{z(n)} \to h$ in $L_\infty([0,T],Y)$, the limit $\lim_{n\to\infty}\left\langle p_{M,T}\left(z(n)\right), v\right\rangle$ exists in E for all $M \geq 0$, and does not depend on a particular sequence $z(n)$ such that $e_{z(n)} \to h$ in $L_\infty([0,T],Y)$. We define

$$\left\langle p_{M,T}\left(h\right), v\right\rangle = \lim_{n\to\infty}\langle p_M\left(z(n)\right), v\rangle.$$

Lemma 6 *Let* $\mathbf{b} \in \mathcal{B}$. *Then*

(a) $\mathcal{T}'(\mathbf{b}) \subseteq \mathcal{T}'(E)$.
(b) *For* $v \in \mathcal{T}'(E)$, *there are* $a_\alpha \in E$ *such that the restriction*

$$v|_{\mathcal{D}(\mathbf{b})} = \sum_\alpha a_\alpha \xi_\alpha(\mathbf{b}) \in \mathcal{T}'(\mathbf{b}, E).$$

Proof (a) Let $v \in \mathcal{T}'(\mathbf{b})$, $h_n \to h$ in $L_\infty([0,T],Y)$. For each n there is a sequence $z_k(n)$ such that $e_{z_k(n)} \to h_n$ in $L_\infty[0,T]$ and $v(z_k(n)) \to v(h_n)$ as $k \to \infty$. Therefore there is a subsequence $z_{k_n}(n)$ such that

$$|e_{z_{k_n}(n)} - h_n|_{L_n([0,T],Y)} + |v(e_{z_{k_n}(n)}) - v(h_n)| \leq 1/n.$$

Then for each $p \geq 2, n \geq p$,

$$\begin{aligned}
&|e_{z_{k_n}(n)} - h|_{L_p([0,T],Y)}\\
&\quad\leq |e_{z_{k_n}(n)} - h_n|_{L_p([0,T],Y)} + |h_n - h|_{L_p([0,T],Y)}\\
&\quad\leq T^{\frac{1}{p}-\frac{1}{n}}|e_{z_{k_n}(n)} - h_n|_{L_n([0,T],Y)} + |h_n - h|_{L_p([0,T],Y)}
\end{aligned}$$

So, $e_{z_{k_n}(n)} \to h$ in $L_\infty\left([0,T],Y\right)$ and

$$\begin{aligned}
&|v(h_n) - v(h)|\\
&\quad\leq |v(h_n) - v(e_{z_{k_n(n)}})| + |v(e_{z_{k_n(n)}}) - v(h)| \to 0
\end{aligned}$$

as $n \to \infty$.

(b) Let $v \in \mathcal{T}'(E)$, $\mathbf{b} = \{e_n\}$, $\xi_\alpha = \xi_\alpha(\mathbf{b})$, $N \geq 1$,

$$p_N(z) = p_N(z,\mathbf{b}) = p_N(e_z) = \sum_{|\alpha|=N}\frac{z^\alpha}{\sqrt{\alpha!}}\xi_\alpha.$$

By Lemma 3, $\xi_\alpha \in \mathcal{T}$ and by linearity

$$v(p_N(z)) = \sum_{|\alpha|=N} \frac{z^\alpha}{\sqrt{\alpha!}} v(\xi_\alpha).$$

Denoting $v_\alpha = v(\xi_\alpha)$ set $\bar{v} = \sum_\alpha v_a \xi_\alpha \in \mathcal{D}'(\mathbf{b})$.Obviously,

$$v(p_N(z)) = \sum_{|\alpha|=N} \frac{z^\alpha}{\sqrt{\alpha!}} v(\xi_\alpha) = \bar{v}(p_N(z))$$

and (b) holds. □

Let $\mathcal{T}'(\mathbf{b}; [r, T]) = \mathcal{T}'(\mathbf{b}; [r, T], E)$ be the space of all $v \in \mathcal{D}'(\mathbf{b}; [r, T], E)$ such that $v(t) \in \mathcal{T}'(\mathbf{b}, E), r \leq t \leq T$.

Definition 7 Given a Banach space E, let $\mathcal{T}' = \mathcal{T}'([r, T], E)$ be the space of all $\mathcal{T}'(E)$-valued functions $v(t)$ on $[r, T]$. We say $v \in \mathcal{T}'([r, T], E)$ is a generalized E-valued stochastic process. We denote $C\mathcal{T}'([r, T], E)$ the set of all continuous $u \in \mathcal{T}'([r, T], E)$.

The following obvious consequence of Lemma 6 holds.

Corollary 3 *Let* $\mathbf{b} \in \mathcal{B}, r < T$. *Then*

(a) $\mathcal{T}'(\mathbf{b}; [r, T], E) \subseteq \mathcal{T}'([r, T], E)$.
(b) *For* $v \in \mathcal{T}'([r, T], E)$*, there are* E*-valued functions* $a_\alpha(t), 0 \leq t \leq T$, *such that the restriction*

$$v|_{\mathcal{D}(\mathbf{b})} = \sum_\alpha a_\alpha(t)\xi_\alpha(\mathbf{b}) \in \mathcal{T}'(\mathbf{b}; [r, T], E).$$

Now, we introduce the notion of an adapted generalized process.

Definition 8 (a) We say $v \in \mathcal{T}'(E)$ is $\mathcal{F}^W_{t_0}$-measurable if

$$\langle p_{M,T}(h), v \rangle = \langle p_{M,t_0}(h), v \rangle$$

for all $h \in L_\infty([0, T], Y)$, $M \geq 0$.

(b) We say $v \in \mathcal{T}'([0, T], E)$ is $\mathbb{F}^W$-adapted if $v(t)$ is $\mathcal{F}^W_t$-measurable for each t.

Example 2 Let W_t be a cylindrical Wiener process in a Hilbert space Y and $\dot{W}_t = \frac{d}{dt} W_t$. Then (see Example 1 as well) W_t and $\dot{W}_t$ are generalized Y-valued adapted stochastic processes. For any $h \in L_\infty([0, T], Y)$,

$$\langle W_t, p_{M,T}(h) \rangle = \int_0^t h(s)ds = \langle W_t, p_{M,t}(h) \rangle,$$

$$\langle \dot{W}_t, p_{M,T}(h) \rangle = h(t) = \langle \dot{W}_t, p_{M,t}(h) \rangle$$

if $M = 1$, and $\langle W_t, p^M_s(h) \rangle = \langle \dot{W}_t, p^M_s(h) \rangle = 0, 0 \leq s \leq T$, otherwise.

4.3 Independence of basis and Markov property of the solution

Remark 8 suggests the following definition of a generalized solution to (3.2).

Definition 9 Given $\mathbf{w} \in \mathcal{T}'(\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$, $\operatorname{div} \mathbf{w} = 0$, $T \geq r$, a generalized process $\mathbf{u} \in C\mathcal{T}'([r, T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$ is called $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$-solution of Eq. (3.1) in $[r, T]$, if for each $h \in L_\infty([0, T], Y)$, $M \geq 0$, the function $\mathbf{u}^{M,h}(t, x) = \langle p_{M,T}(h), \mathbf{u}(t, x)\rangle$ is an $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$-valued continuous functions satisfying in $\mathbb{L}_l$, $l = 2, p$, Stokes ($M \geq 1$) or Navier–Stokes ($M = 0$) equation

$$\begin{aligned}\mathbf{u}^{M,h}(t) = \mathbf{w}^{M,h} + \int_r^t \mathcal{P}[\partial_i \left(a^{ij}(s)\, \partial_j \mathbf{u}^{M,h}(s)\right) \\ +b^i(s)\partial_i \mathbf{u}^{M,h}(s) - \sum_{K+L=M} u^{k,K,,h}(s)\, \partial_k \mathbf{u}^{L,h}(s) \\ +1_{M=0}\mathbf{f}(s) + \left(\sigma^i(s), h(s)\right)_Y \partial_i \mathbf{u}^{M-1,h}(s) \\ +1_{M=1}(\mathbf{g}(s), h(s))_Y]ds, \end{aligned} \tag{4.7}$$

where $M \geq 0$, $\mathbf{w}^{M,h}(x) = \langle p_{M,T}(h), \mathbf{w}(x)\rangle$, and $\mathbf{u}^{-1,h}(t, x) = 0$.

Obviously, a generalized solution is a $\mathcal{D}$-solution. Now we are in a position to prove the main result.

Theorem 2 *Assume that A1–A3 hold,* $\mathbf{w} \in \mathcal{T}'(\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$. *Then for each* $r < T < T_1$ *there is a unique* $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$*-solution* $\mathbf{u} \in C\mathcal{T}'([r, T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$ *of* (3.2) *in* $[0, T]$. *Moreover, if* $\mathbf{w}$ *is* $\mathcal{F}_r^W$*-measurable, then the solution is* $\mathbb{F}^W = \left(\mathcal{F}_t^W\right)_{t\geq r}$*-adapted and it extends all* $\mathcal{D}(\mathbf{b})$*-solutions,* $\mathbf{b} \in \mathcal{B}$.

Before proceeding with the proof of Theorem 2 we shall prove the following auxiliary statement.

Lemma 7 *Let A1–A3 hold,* $\mathbf{w} \in \mathcal{T}'(\mathbb{H}_p^2 \cap \mathbb{H}_2^2)$, $\operatorname{div} \mathbf{w} = 0$, $h \in L_\infty([0, T], Y)$. *Then the infinite system* (4.7) *with* $\mathbf{w}^{M,h} = \langle \mathbf{w}, p_{M,T}(h)\rangle$ *has a unique solution* $\mathbf{v}^{M,h} \in C\left([r, T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2\right)$, $M \geq 0$. *Moreover, for each* $N \geq 1$ *there is a constant* C *independent of* h *so that*

$$\begin{aligned}R_N \leq C \Bigg\{ \sum_{\substack{K+L=N,\\ K,L\leq N-1}} R_K R_L + \sum_{l=2,p} \Bigg[|\mathbf{w}^{N,h}|_{2,l} + R_{N-1}\Big(\int_r^T |h(s)|^l ds\Big)^{1/l} \\ +1_{N=1}\left(\int_r^T |h(s)|^{2l} ds\right)^{1/2l} \left(\int_r^T ||\mathbf{g}(s)||_{1,l}^{2l} ds\right)^{1/2l}\Bigg]\Bigg\}, \end{aligned} \tag{4.8}$$

where $R_N = \sup_{r\leq s\leq T}\left[|\mathbf{v}^{N,h}(t)|_{2,p} + |\mathbf{v}^{N,h}(t)|_{2,2}\right]$.

Proof If $M=0$, then (4.7) is Navier–Stokes equation (3.5) and, by Lemma 1 there is a unique $\mathbb{H}_p^2\cap\mathbb{H}_2^2$-valued continuous solution $\mathbf{v}^0=\mathbf{u}_0$. We proceed by induction. Assume there are unique $\mathbf{v}^M=\mathbf{v}^{M,h}\in C([0,T],\mathbb{H}_p^2\cap\mathbb{H}_2^2)$ solving the system for $M=0,\ldots,N-1$. Consider the equation for $\mathbf{v}^N$:

$$\mathbf{v}^N(t)=\mathbf{w}^{N,h}+\int_r^t\mathcal{P}[\partial_i\left(a^{ij}(s)\,\partial_j\mathbf{v}^N(s)\right)+b^i(s)\partial_i\mathbf{v}^N(s)$$
$$-v^{0,k}(s)\,\partial_k\mathbf{v}^N(s)-v^{N,k}(s)\,\partial_k\mathbf{v}^0(s)+\mathbf{f}_N^h(s)]ds \tag{4.9}$$

with

$$\mathbf{f}_N^h(s)=-\sum_{\substack{K+L=N,\\K,L\le N-1}}v^{K,k}(s)\,\partial_k\mathbf{v}^L(s)+1_{M=1}(\mathbf{g}(s),h(s))_Y$$
$$+\left(\sigma^i(s),h(s)\right)_Y\partial_i\mathbf{v}^{N-1}(s)$$

Since

$$\int_r^T|\mathbf{f}_N^h(s)|_{1,l}^l ds<\infty,\ l=2,p,$$

the existence and uniqueness follows by Proposition 5. Also, by Proposition 5, for each $N\ge 1$ there is a constant C independent of h such that, denoting $R_N^h=\sup_{s\le T}\left[|\mathbf{v}^{M,h}(s)|_{2,p}+|\mathbf{v}^{M,h}(s)|_{2,2}\right]$. Since for $l=2,p$,

$$|\mathbf{f}_N^h(s)|_{1,l}\le C\left[\sum_{\substack{K+L=N,\\K,L\le N-1}}|\mathbf{v}^K(s)|_{2,l}|\mathbf{v}^L(s)|_{2,p}+|h(s)|_Y|\mathbf{v}^{M-1}(s)|_{2,l}\right.$$
$$\left.+|h(s)|_Y||\mathbf{g}(s)||_{1,l}\right],$$

the inequality (4.8) follows by Cauchy–Scwarz inequality. □

4.3.1 Proof of Theorem 2

Fix $T<T_1$ and choose a special CONS $\mathbf{b}=\{e_n\}\in\mathcal{B}$ such that for each $h\in L_\infty([0,T],Y)$ there is a sequence $z(n)\in\mathcal{Z}$ (see Lemma 5) for which $e_{z(n)}\to h$ in $L_p([0,T],Y)$, for all $p\ge 2$, as $n\to\infty$ (for example, taking in $L_2([0,T])$

a trigonometric basis or unconditional $L_p([0,T])$-basis (Haar basis), see [18]). By Lemma 6 with $\xi_\alpha = \xi_\alpha(\mathbf{b})$,

$$\mathbf{w}|_{\mathcal{D}(\mathbf{b})} = \sum_\alpha \mathbf{w}_\alpha \xi_\alpha.$$

According to Lemma 2, there is a unique $\mathcal{D} - \mathbb{H}_p^2 \cap \mathbb{H}_2^2$-solution

$$\mathbf{u}(t,x) = \sum_\alpha \mathbf{u}_\alpha(t,x)\xi_\alpha \in \; C\mathcal{D}'\left(\mathbf{b}; [r,T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2\right)$$

of (3.2) in $[0,T]$. The coefficients $\mathbf{u}_\alpha(t,x)$ satisfy (3.4) and, by Remark 8 (4.6) holds for all $M \geq 0$, $z \in \mathcal{Z}$. Fix $h \in L_\infty([0,T], Y)$ and consider an arbitrary $e_{z(n)} \to h$ in $L_p([0,T], Y)$, for all $p \geq 2$, as $n \to \infty$.

Then

$$\begin{aligned}\mathbf{w}^{M,z(n)} &= \langle \mathbf{w}, p_M(z(n))\rangle \\ &= \sum_{|\alpha|=M} \mathbf{w}_\alpha(t,x) z(n)^\alpha/\sqrt{\alpha!} \in \mathbb{H}_p^2 \cap \mathbb{H}_2^2,\end{aligned}$$

and

$$\begin{aligned}\mathbf{u}^{M,z(n)}(t,x) &= \langle \mathbf{u}(t,x), p_M(z(n))\rangle \\ &= \sum_{|\alpha|=M} \mathbf{u}_\alpha(t,x) z(n)^\alpha/\sqrt{\alpha!} \in C\left([0,T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2\right),\end{aligned}$$

$M \geq 0$, is the unique solution to the system (4.7) corresponding to $h = e_{z(n)}$ and $\mathbf{w}^{M,h} = \mathbf{w}^{M,z(n)}$. Recall, $\mathbf{u}^{-1,z(n)} = \mathbf{0}$ and $\mathbf{u}^{0,z(n)}$ coincides with the solution of Navier–Stokes equation $\mathbf{u}_0$ in Lemma 1. By Lemma 7, there is a unique $\mathbf{v}^{M,h} \in C\left([r,T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2\right)$, $M \geq 0$, solving (4.7 with $\mathbf{w}^{M,h} = \langle \mathbf{w}, p_M(h)\rangle \in \mathbb{H}_p^2 \cap \mathbb{H}_p^2$. We have $\mathbf{v}^{-1,h}(t) = 0$, and $\mathbf{v}^{0,h}$ coincides with the solution of Navier–Stokes equation $\mathbf{u}_0$ in Lemma 1. By Lemma 7 (see (4.8)), for every $M \geq 1$ there is a constant C_0 independent of n such that

$$\sup_{r\leq t\leq T} \sum_{l=2,p} [|\mathbf{u}^{M,z(n)}(t)|_{2,l} + |\mathbf{v}^{M,h}(t)|_{2,l}] \leq C_0. \tag{4.10}$$

For $\mathbf{V}_n^M = \mathbf{v}^{M,h} - \mathbf{u}^{M,z(n)}$, $M \geq 1$, the following equation holds ($M \geq 1$):

$$\begin{aligned}\mathbf{V}_n^M(t) = \mathbf{w}^M - \mathbf{w}^{M,z(n)} &+ \int_r^t \mathcal{P}[\partial_i\left(a^{ij}(s)\,\partial_j \mathbf{V}_n^M(s)\right) \\ &+ [b^i(s) - u_0^i(s)]\partial_i \mathbf{V}_n^M(s) + V_n^i(s)\partial_i \mathbf{u}_0(s) + \mathbf{G}_n(s)]ds,\end{aligned}$$

where $\mathbf{G}_n(s) = \mathbf{G}_n^1(s) + \mathbf{G}_n^2(s)$ with

$$\mathbf{G}_n^1(s) = -\sum_{\substack{K+L=M,\\ K,L\geq 1}} [V_n^{K,k}(s)\partial_k \mathbf{u}^{L,z(n)}(s) + u^{K,z(n),k}(s)\,\partial_k \mathbf{V}_n^L(s)]$$

and

$$\mathbf{G}_n^2(h,s) = 1_{M=1}(\mathbf{g}(s), h_n(s))_Y + \left(\sigma^i(s), h(s)\right)_Y \partial_i \mathbf{V}_n^{M-1,h}(s) + \left(\sigma^i(s), h_n(s)\right)_Y \partial_i \mathbf{u}^{M-1,z(n)}(s).$$

By Proposition 5 in Appendix II, for $L_M^n = \sup_{s\leq T}\left[|\mathbf{V}_n^M(t)|_{2,p} + |\mathbf{V}_n^M(t)|_{2,2}\right]$, with $M\geq 1$,

$$L_M^n \leq C\left[A_n + \sum_{l=2,p}\left(\int_r^T |\mathbf{G}_n(s)|_{1,l}^l ds\right)^{1/l}\right],$$

where $A_n = \sum_{l=2,p}|\mathbf{w}^M - \mathbf{w}^{M,z(n)}|_{2,l}$. We estimate

$$|\mathbf{G}_n^1(s)|_{1,l} \leq CC_0 \sum_{1\leq K\leq M-1} L_K^n,$$
$$|\mathbf{G}_n^2(h,s)|_{1,l} \leq C[1_{M=1}|h_n(s)|_Y|\mathbf{g}(s)|_{1,l} + L_{M-1}^n + C_0|h_n(s)|],$$

where $h_n = h - e_{z(n)}$. So, for each $M\geq 1$ there is a constant independent of n such that

$$L_M^n \leq C\left\{A_n + \sum_{1\leq K\leq M-1} L_K^n + \sum_{l=2,p}\left[\int_0^T |h_n(s)|^l ds)^{1/l} + \left(\int_0^T |h_n(s)|^{2l} ds\right)^{1/2l}\left(\int |\mathbf{g}(s)|_{1,l}^{2l} ds\right)^{1/2l}\right]\right\}.$$

Starting with $M=0$, $L_0^n = 0$ for all n, it follows by induction that

$$L_M^n = \sup_{r\leq s\leq T}\sum_{l=2,p}|\mathbf{v}^{M,h}(s) - \mathbf{u}^{M,z(n)}(s)|_{2,l} \to 0$$

as $n\to\infty$, $M\geq 1$.

Since $h \in L_\infty([0, T], Y)$ is arbitrary,

$$\mathbf{u}(t) = \sum_\alpha \mathbf{u}_\alpha(t)\xi_\alpha \in C\mathcal{T}'(\mathbf{b}; [r, T], \mathbb{H}^2_p \cap \mathbb{H}^2_2) \subseteq C\mathcal{T}'([r, T], \mathbb{H}^2_p \cap \mathbb{H}^2_2)$$

is the unique solution to (3.1). Obviously,

$$\mathbf{v}^{M,h}(t, x) = \left\langle p_{M,T}(h), \mathbf{u}(t, x)\right\rangle, h \in L_\infty([0, T], Y)$$

satisfies (4.7). Since for each CONS $\mathbf{b}' = (e'_k) \in \mathcal{B}$ any linear combination of e'_k belongs to $L_\infty([0, T], Y)$, the generalized solution $\mathbf{u}(t)$ extends any $\mathcal{D}$-solution.

Now we will prove that the unique generalized $\mathbb{H}^2_p \cap \mathbb{H}^2_2$-solution of (3.1) in $[0, T]$ is $\mathbb{F}^W$-adapted. We fix $t^* \in (0, T), r \le t^*$ and consider a special basis $\bar{\mathbf{b}} \in \mathcal{B}$ with $\bar{m}_i(t)$ in $L_2((0, T))$ so that each $\bar{m}_i$ is supported either in $[0, t^*]$ or in $[t^*, T]$ and such that for each $h \in L_\infty([0, T], Y)$ there is a sequence $z(N) \in \mathcal{Z}$ (see Lemma 5) for which $e_{z(N)} \to h$ in $L_p([0, T], Y)$, for all $p \ge 2$, as $N \to \infty$ (for example, $(\bar{m}_k)$ is a combination of two trigonometric or unconditional $L_p([0, T])$-basis (Haar basis) on $(0, t^*)$ and (t^*, T)). Let $\bar{\xi}_\alpha = \bar{\xi}_\alpha(\bar{\mathbf{b}}),\ \alpha \in \mathcal{J}$, the corresponding orthonormal basis in $L_2(\mathbb{F}^W_T)$. Let

$$\mathbf{u}(t) = \sum_\alpha \bar{\mathbf{u}}_\alpha(t)\bar{\xi}_\alpha \in C\mathcal{T}'(\bar{\mathbf{b}}; [0, T], \mathbb{H}^2_p \cap \mathbb{H}^2_2)$$

be the unique solution to (3.1) constructed using the representation

$$\mathbf{w}|_{\mathcal{D}(\bar{\mathbf{b}})} = \sum_\alpha \bar{\mathbf{w}}_\alpha\bar{\xi}_\alpha \in \mathcal{T}'(\bar{\mathbf{b}}; \mathbb{H}^2_p \cap \mathbb{H}^2_2).$$

So, $\bar{\mathbf{u}}_\alpha \in C\left([0, T], \mathbb{H}^2_p \cap \mathbb{H}^2_2\right)$ satisfy (3.4) in $\mathbb{L}_l, l = 2, p,$ with $\mathbf{w}_\alpha = \bar{\mathbf{w}}_\alpha, \alpha \in J$. Let $J' = \alpha \in J : \alpha$ has a non zero component corresponding to $\bar{m}_k$ whose support is in (t^*, T). Since $\mathbf{w}$ is $\mathcal{F}^W_r$-measurable, $\bar{\mathbf{w}}_\alpha = 0$ if $\alpha \in J'$. Indeed, if $\alpha \in J'$, there are $c_i \in \mathbf{R}, z_i \in \mathcal{Z}, i, = 1 \ldots, n$, so that

$$\xi_\alpha = \sum_{i=1}^n c_i p_T(z_i).$$

Then

$$\begin{aligned}\bar{\mathbf{w}}_\alpha &= \left\langle \mathbf{w}, \sum_{i=1}^n c_i p_T(z_i)\right\rangle = \sum_{i=1}^n c_i \left\langle \mathbf{w}, p_T(z_i)\right\rangle \\ &= \sum_{i=1}^n c_i \left\langle \mathbf{w}, p_r(z_i)\right\rangle = \left\langle \mathbf{w}, \sum_{i=1}^n c_i p_r(z_i)\right\rangle = 0,\end{aligned}$$

because, by (4.3),

$$\sum_{i=1}^{n} c_i p_r(z_i) = \mathbf{E}[\sum_{i=1}^{n} c_i p_T(z_i)|\mathcal{F}_r^W] = \mathbf{E}\left[\xi_\alpha|\mathcal{F}_r^W\right] = 0.$$

We claim that, similarly, for $t \in (0, t^*)$, $\bar{\mathbf{u}}_\alpha(t) = 0$ if $\alpha \in J'$. Indeed if $\alpha \in J'$ and $|\alpha| = 1$, we have $\bar{\mathbf{w}}_\alpha = 0$ and in the Eq. (3.6) for $\bar{\mathbf{u}}_\alpha$ we have the input function $\mathbf{F}_\alpha(t) = 0$ for $t \in (0, t^*)$. Therefore the unique solution $\bar{\mathbf{u}}_\alpha(t) = 0$ if $t \in (0, t^*)$. Then we simply apply induction on $|\alpha| = n$ and use (3.6) (note that if $|\alpha| = n + 1$, then $\alpha = \tilde{\alpha} + \varepsilon_k$ for some k and without any loss of generality we can assume that $\tilde{\alpha} \in J'$). As a result,

$$\mathbf{u}(t) = \sum_{\alpha\in\mathcal{J}} \bar{\mathbf{u}}_\alpha(t)\bar{\xi}_\alpha = \sum_{\alpha\notin\mathcal{J}'} \bar{\mathbf{u}}_\alpha(t)\bar{\xi}_\alpha, \quad t \in [r, t^*].$$

Obviously, $\bar{\xi}_\alpha$ are $\mathcal{F}_{t^*}^W$-measurable for $\alpha \notin J'$. Also, for any $z \in \mathcal{Z}$, $M \geq 1$, $t \leq t^*$,

$$\begin{aligned}\langle p_M(z), \mathbf{u}(t)\rangle &= \sum_{\alpha\notin\mathcal{J}'} \bar{\mathbf{u}}_\alpha(t)\frac{z^\alpha}{\sqrt{\alpha!}} = \left\langle p_{M,t^*}(z), \mathbf{u}(t)\right\rangle \\ &= \left\langle p_M(e_z 1_{(0,t^*)}), \mathbf{u}(t)\right\rangle\end{aligned}$$

(note that $e_z = \sum_k z_k e_k$, $e_z 1_{(0,t^*)} = \sum_{k\notin G} z_k e_k$, where G is the set of all k such that $\bar{m}_{j_k}$ in $e_k = \bar{m}_{j_k} l_{j_k}$ has its support in (t^*, T)). The statement of Theorem 2 is proved.

The solution above has the restarting property as well. By the same arguments as in Corollary 1 we have

Corollary 4 *Let* $\mathbf{w} \in \mathcal{T}'(\mathbb{H}_2^2 \cap \mathbb{H}_2^2)$ *and A1–A3 hold. Let* $u^{r,\mathbf{w}}(t)$ *be* $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$*-solution to* (3.1)*in* $[r, T]$, $T < T_1$, *and* $r \leq r' \leq t \leq T$. *Then*

$$\mathbf{u}^{r,\mathbf{w}}(t) = \mathbf{u}^{r',\mathbf{u}(r')}(t). \tag{4.11}$$

Corollary 5 (Markov Property) *Assume that the assumptions of Corollary 4 hold true and, in addition,* $\mathbf{w}$ *is* $\mathcal{F}_r^W$*−measurable, then* $u^{r,\mathbf{w}}(t)$ *is* $(\mathcal{F}_t^W)_{t\geq r}$*-adapted. This together with* (4.11) *can be interpreted as Markov property.*

Let choose a uniformly bounded basis $\mathbf{b} = \{e_n\}$ ($\sup_{k,s} |e_k(s)|_Y < \infty$) and rescale the solution in Theorem 2

$$\mathbf{u}|_{\mathcal{D}(\mathbf{b})} = \sum_\alpha \mathbf{u}_\alpha(t)\xi_\alpha,$$

using the second quantization C_ε operator in (3.9) in the Fock space $\mathcal{H} = \sum_n \mathcal{H}_n$ ($\mathcal{H}_n = \mathbf{H}^{\hat{\otimes} n}$, see Appendix I, 5.1). Recall,

$$C_\varepsilon = \sum_{n=0}^{\infty} \kappa_{\varepsilon,n} D_\varepsilon^{\otimes n},$$

where $D_\varepsilon e_k = 2^{-\varepsilon k} e_k$ and $\kappa_{\varepsilon,n} = e^{-\varepsilon e^n}$. According to Proposition 2,

$$\mathbf{u}^\varepsilon(t) = C_\varepsilon \mathbf{u}(t) = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \kappa_{\varepsilon,n} (2^{-\varepsilon \mathbf{N}})^\alpha \mathbf{u}_\alpha(t) \xi_\alpha \tag{4.12}$$

is $\mathbb{H}_2^2$-valued continuous. The following statement holds.

Proposition 3 *Let* $\mathbf{b} = \{e_n\}$ *be uniformly bounded and A1–A3 hold. Let* $\mathbf{u} \in \mathcal{T}'([0,T], \mathbb{H}_p^2 \cap \mathbb{H}_2^2)$ *be the solution to* (3.2) *in* $[0,T]$ *with the deterministic initial* $\mathbf{w} \in \mathbb{H}_p^2 \cap \mathbb{H}_2^2$. *Then the rescaled* $\mathbb{H}_2^2$*-valued square integrable continuous process* $\mathbf{u}^\varepsilon(t)$ (*defined by* (4.12), *see Proposition* 2) *is adapted and Markov* (*in a standard, rather than generalized, sense*).

Proof For each $M \geq 1, z \in \mathcal{Z}$,

$$\begin{aligned}
\mathbf{E} p_{T,M}(z) \mathbf{u}^\varepsilon(t) &= \langle p_{T,M}(z), \mathbf{u}^\varepsilon(t) \rangle = \kappa_{\varepsilon,M} \langle p_{T,M}(z^\varepsilon), \mathbf{u}(t) \rangle \\
&= \kappa_{\varepsilon,M} \langle p_{t,M}(z^\varepsilon), \mathbf{u}(t) \rangle = \langle p_{t,M}(z), \mathbf{u}^\varepsilon(t) \rangle \\
&= \mathbf{E} p_{t,M}(z) \mathbf{u}^\varepsilon(t).
\end{aligned}$$

Since $\mathbf{u}^\varepsilon(t)$ is square integrable, $\mathbf{E}\left[\mathbf{u}^\varepsilon(t) | \mathcal{F}_t^W\right] = \mathbf{u}^\varepsilon(t)$. So $\mathbf{u}^\varepsilon(t)$ is adapted in a standard sense.

For any $0 \leq s \leq t$, by Corollary 4, $\mathbf{u}(t) = \mathbf{u}^{s,\mathbf{u}(s)}(t)$. Therefore,

$$\mathbf{u}^\varepsilon(t) = C_\varepsilon \mathbf{u}(t) = C_\varepsilon(\mathbf{u}^{s,\mathbf{u}(s)}(t)) = (\mathbf{u}^\varepsilon)^{s,\mathbf{u}^\varepsilon(s)}(t)$$

and the standard Markov property follows. □

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5 Appendix I. Wiener chaos

In the first part of Appendix we present some facts of white noise analysis.

5.1 Rescaling of Wiener chaos by second quantization operator

Consider a generalized random variable $u = \sum_\alpha u_\alpha \xi_\alpha = \sum_\alpha u_\alpha \sqrt{|\alpha|!} W(e_\alpha) \in \mathcal{D}'(\mathbf{b})$, $\mathbf{b} = \{e_k, k \geq 1\} \in B$, where

$$W(e_\alpha) = W^{\otimes n}(e_\alpha) = \int_0^T \int_0^{s_n} \dots \int_0^{s_2} e_\alpha(s_1, \dots, s_n) dW_{s_1} \dots dW_{s_n},$$

if $|\alpha| = n$ and $\{e_\alpha, \alpha \in J\}$ defined by (4.1) is a CONS of the symmetric part $\mathcal{H}_n = \mathbf{H}^{\hat{\otimes} n}$ of $\mathbf{H}^{\otimes n}$ (recall $\mathbf{H} = L_2(0, T]) \times Y)$. We can interpret

$$u = \sum_\alpha u_\alpha \xi_\alpha = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} u_\alpha \sqrt{n!} W^{\otimes n}(e_\alpha)$$

as a result of the noise W acting on an element of the Fock space: $W(\hat{u}) = u$ with

$$\begin{aligned} \hat{u} &= \sum_\alpha u_\alpha \sqrt{|\alpha|!} e_\alpha = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} u_\alpha \sqrt{n!} e_\alpha \\ &= \sum_{n=0}^{\infty} \hat{u}^{(n)} \in \mathcal{H} = \sum_{n=0}^{\infty} \mathcal{H}_n = \sum_{n=0}^{\infty} \mathbf{H}^{\hat{\otimes} n}. \end{aligned}$$

Here $\mathbf{H}^{\hat{\otimes} 0} = \mathbf{R}$ and the norm in the Fock space $\mathcal{H}$ is defined as

$$||\hat{u}||_{\mathcal{H}}^2 = \left\| \sum_{n=0}^{\infty} \hat{u}^{(n)} \right\|_{\mathcal{H}}^2 = \sum_{n=0}^{\infty} \frac{||\hat{u}^{(n)}||_{\mathcal{H}_n}^2}{n!}.$$

Obviously, $\mathbf{E}[W(\hat{u})^2] = ||\hat{u}||_{\mathcal{H}}^2$. Let $A = (A_n)_{n \geq 0.}$ be a self-adjoint positive operator in $\mathcal{H}$ such that $A_n e_\alpha = \lambda(\alpha) e_\alpha$, where $|\alpha| = n$ and $\lambda(\alpha), \alpha \in J$, are positive numbers.

Remark 9 The operator A in the Fock space $\mathcal{H}$ can be used to rescale a generalized r.v. For

$$u = \sum_\alpha u_\alpha \xi_\alpha = W(\hat{u}) \in \mathcal{D}'(\mathbf{b}),$$

we define $Au = u^A \in \mathcal{D}'(\mathbf{b})$ by

$$\begin{aligned} Au = u^A = W(A\hat{u}) &= \sum_\alpha u_\alpha \sqrt{|\alpha|!} W(Ae_\alpha) \\ &= \sum_\alpha u_\alpha \lambda(\alpha) \sqrt{|\alpha|!} W(e_\alpha) = \sum_\alpha u_\alpha \lambda(\alpha) \xi_\alpha. \end{aligned}$$

Definition 10 Since $\lambda(\alpha) > 0$, we can define

$$A^{-1}u = u^{A^{-1}} = \sum_\alpha u_\alpha \lambda(\alpha)^{-1} \xi_\alpha.$$

Example 3 1. (Second quantization operator in space-time) Consider a self-adjoint positive operator B in $\mathbf{H}$ such that $Be_k = \lambda_k e_k$. The second quantization operator $A = \Gamma(B) = \left(B^{\otimes n}\right)$ in $\mathcal{H}$ is defined as

$$Ae_\alpha = \Gamma(B)e_\alpha = B^{\otimes n} e_\alpha = \lambda^\alpha e_\alpha, |\alpha| = n,$$

where $\lambda = (\lambda_k)$ and $\lambda^\alpha = \Pi_k \lambda_k^{\alpha_k}$. We have

$$\Gamma(B)u = \sum_\alpha u_\alpha \lambda^\alpha \xi_\alpha.$$

2. (Second quantization operator in space) Consider a self-adjoint positive operator B on Y such that the sequence of its eigenvectors $\left(\ell_p\right)_{p\geq 1}$ $(B\ell_p = \lambda_p \ell_p, \lambda_p > 0)$ is a CONS in Y. Let $b = \{e_k, k \geq 1\}$, where $e_k(s) = m_{i_k}(s)\ell_{j_k}$. We extend B to $\mathbf{H}$ by

$$Be_k = B(m_{i_k}\ell_{j_k}) = m_{i_k} B\ell_{j_k} = \lambda_{j_k} e_k$$

and rescale in space-time using $A = \left(B^{\otimes n}\right)$. For $u = \sum_\alpha u_\alpha \xi_\alpha$ we have

$$\Gamma(B)u = \sum_\alpha u_\alpha \lambda^\alpha \xi_\alpha,$$

where $\lambda^\alpha = \Pi_k \lambda_{j_k}^{\alpha_k}$.

3. Consider a self-adjoint positive operator B on H such that $Be_k = \lambda_k e_k$ and a sequence of positive numbers q_n. Let $A = \sum_{n=0}^\infty q_n B^{\otimes n}$. Then $Ae_\alpha = q_n B^{\otimes n} e_\alpha = q_n \lambda^\alpha e_\alpha, |\alpha| = n$. In this case,

$$Au = \sum_{n=0}^\infty q_n \sum_{|\alpha|=n} u_\alpha \lambda^\alpha \xi_\alpha.$$

For the Wick product we have the following obvious statement.

Remark 10 1. Assume $A = (A_n)$ is a self-adjoint positive operator on $\mathcal{H}$ such that $Ae_\alpha = A_n e_\alpha = \lambda(\alpha)e_\alpha, |\alpha| = n$ and $\lambda(\alpha),\alpha \in I$, are positive numbers, $u, v \in \mathcal{D}'(\mathbf{b})$. Then, denoting $Au = u^A$, $Av = v^A$, we have

$$A(u\Diamond v) = A\left(A^{-1}u^A \Diamond A^{-1}v^A\right) = \sum_\alpha c_\alpha \xi_\alpha,$$

where

$$c_\alpha = \sum_{\beta\leq\alpha} \frac{\lambda(\alpha)}{\lambda(\beta)\lambda(\alpha-\beta)} u_\beta^A v_{\alpha-\beta}^A \sqrt{\frac{\alpha!}{\beta!(\alpha-\beta)!}}.$$

In particular, if $\Gamma(B)$ is the second quantization operator in space-time, then

$$\frac{\lambda(\alpha)}{\lambda(\beta)\lambda(\alpha-\beta)} = \frac{\lambda^{\alpha}}{\lambda^{\beta}\lambda^{\alpha-\beta}} = 1$$

and $\Gamma(B)(u\Diamond v) = u^B \Diamond v^B$.

2. For the Skorokhod stochastic integral, we have

$$A\left(\delta(u)\right) = A\int_0^T u_s \Diamond \dot{W}_s ds = \int_0^T A\left(A^{-1}u_s^A \Diamond A^{-1}\dot{W}_s^A\right) ds$$
$$= \int_0^T A\left(A^{-1}u_s^A \Diamond \dot{W}_s\right) ds;$$

for the coefficients

$$(A\delta(u))_\alpha = \sum_k \int_0^T (u^A_{\alpha(k)}(t), \lambda(\varepsilon_k)e_k(t))_Y dt \sqrt{\alpha_k}\frac{\lambda(\alpha)}{\lambda(\alpha(k))\lambda(\varepsilon_k)}\xi_\alpha$$
$$= \sum_k \int_0^T (u^A_{\alpha(k)}(t), e_k(t))_Y dt \sqrt{\alpha_k}\frac{\lambda(\alpha)}{\lambda(\alpha(k))}\xi_\alpha,$$
$$\left(A(u(t)\Diamond \dot{W}_t)\right)_\alpha = \sum_k (v^A_{\alpha(k)}(t), e_k(t))_Y \sqrt{\alpha_k}\frac{\lambda(\alpha)}{\lambda(\alpha(k))}\xi_\alpha.$$

5.2 Product, Wick product, and Malliavin derivatives

Consider Hilbert space $\mathbf{H} = L_2([0,T],Y)$, CONS $\mathbf{b} = \{e_k, k \geq 1\} \in \mathcal{B}$, cylindrical Brownian motion W_t, and Cameron–Martin basis $\{\xi_\alpha\}_{\alpha\in J}$ introduced in Sect. 2.1. The *Malliavin derivative* $\mathcal{D}$ (see e.g. [26]) is defined in $\mathcal{D}(\mathbf{b})$ as follows (it assigns to ξ_α an element of $\mathcal{D}(\mathbf{b};\mathbf{H})$):

$$\mathcal{D}(\xi_{\boldsymbol{\mu}}) = \sum_{k\geq 1}\sqrt{\mu_k}\,\xi_{\boldsymbol{\mu}-\boldsymbol{\varepsilon}_k}\,e_k = \sum_\alpha \sum_{\mu=\alpha+\varepsilon_k}\sqrt{\mu_k}e_k\xi_\alpha. \tag{5.1}$$

By induction,

$$\mathcal{D}^n(\xi_\mu) = \sum_\alpha\left(\sum_{|p|=n} 1_{p+\alpha=\mu}\sqrt{\frac{(\alpha+p)!}{\alpha!}}u_p\right)\xi_\alpha, \tag{5.2}$$

where $u_p = \sum_{k_1,\ldots,k_n}\sum_{\varepsilon_{k_1}+\cdots+\varepsilon_{k_n}=p} e_{k_1}\otimes\cdots\otimes e_{k_n} \in \mathbf{H}^{\otimes n}$.

Proposition 4 *Let ξ_θ and ξ_κ be elements of the Cameron–Martin basis. Then, with probability* 1*,*

$$\xi_\theta \xi_\kappa = \sum_{n=0}^{\infty} \frac{\mathcal{D}^n \xi_\theta \diamondsuit \mathcal{D}^n \xi_\kappa}{n!}. \tag{5.3}$$

Proof It is a standard fact (see e.g. [17]) that

$$\xi_\theta \xi_\kappa = \sum_{p \le \theta \wedge \kappa} \sqrt{\binom{\theta}{p}\binom{k}{p}\binom{\theta+\kappa-2p}{\kappa-p}}\, p! \xi_{\theta+\kappa-2p}.$$

Let us rewrite this expression as follows:

$$\xi_\theta \xi_\kappa = \sum_{\substack{p,\beta,\gamma: \\ p+\gamma=\kappa, p+\beta=\theta}} \frac{\sqrt{\theta!\kappa!(\beta+\gamma)!}}{p!(\beta)!(\gamma)!} \xi_{\beta+\gamma}$$

where the summation goes over all triples $(p, \beta, \gamma) \in J \times J \times J$ such that $p + \gamma = \kappa$, $p + \beta = \theta$. Changing variables (1-to-1 mapping that assigns to (p, β, γ) the vector (p, β, α) with $\alpha \ge \beta$) of summation by $p = p$, $\beta = \beta$, $\gamma + \beta = \alpha$, we get

$$\begin{aligned} \xi_\theta \xi_\kappa &= \sum_{\substack{p,\, \beta \le \alpha: \\ p+\alpha-\beta=\kappa, p+\beta=\theta}} \frac{\sqrt{\theta!\kappa!\alpha!}}{p!(\beta)!(\alpha-\beta)!} \xi_\alpha \\ &= \sum_{\alpha} \sum_{\beta \le \alpha} \sum_{n=0}^{\infty} \sum_{|p|=n} 1_{p+\alpha-\beta=\kappa} 1_{p+\beta=\theta} \frac{\sqrt{\theta!\kappa!\alpha!}}{p!(\beta)!(\alpha-\beta)!} \xi_\alpha \end{aligned} \tag{5.4}$$

By definition of the Wick product and (5.2), and taking into account that $|p|!/p!$ is the number of different orthogonal unit vectors in $u_p \in \mathbf{H}^{\otimes n}$, we arrive at

$$\begin{aligned} &\mathcal{D}^n \xi_\theta \diamondsuit \mathcal{D}^n \xi_\kappa \\ &= \sum_{\alpha} \sum_{\beta \le \alpha} n! \sum_{|p|=n} 1_{p+\alpha-\beta=\kappa} 1_{p+\beta=\theta} \frac{\sqrt{\theta!\kappa!\alpha!}}{p!(\beta)!(\alpha-\beta)!} \xi_\alpha. \end{aligned} \tag{5.5}$$

Comparing (5.4) with (5.5), we get (5.3). □

Remark 11 Proposition 4 implies that $\xi_\theta \xi_\kappa = \xi_\theta \diamondsuit \xi_k + \sum_{\gamma < \theta+\kappa} c_\gamma \xi_\gamma$. In other words, $\xi_\theta \diamondsuit \xi_k = \xi_{\theta+k}$ is the *highest stochastic order* component of the Wiener chaos expansion of $\xi_\theta \xi_\kappa$.

By linearity, the statement of the Proposition could be extended to

$$XY = \sum_{n=0}^{\infty} \frac{(\mathcal{D}^n X) \diamondsuit (\mathcal{D}^n Y)}{n!}, \tag{5.6}$$

where X and Y are finite linear combinations of elements of Cameron–Martin basis. If X and Y finite second moments, relation (5.6) could be derived from the former case by passing to the limit in $L_{1.}$By linearity, the statement of Proposition could be extended to

$$XY = \sum_{n=0}^{\infty} \frac{(\mathcal{D}^n X)\Diamond(\mathcal{D}^n Y)}{n!},$$

where X and Y are finite linear combinations of elements of Cameron–Martin basis.

5.3 Derivation of unbiased Navier–Stokes equation

To simplify discussion, we will consider a velocity field which depends only on one standard Gaussian random variable $\eta \sim N(0, 1)$, rather than a trajectory of the Wiener process W_t. An interested reader would have little difficulties extending the arguments below to the setting with Wiener process.

Consider a velocity field

$$\mathbf{u}(t, x) = \sum_{n=0}^{\infty} \mathbf{u}_n(t, x)\, \xi_n(\eta).$$

Note that in our setting the Cameron–Martin expansion (see Theorem 1) is indexed by integers rather than multi-indexes. Assume that for every n, u_n is analytic in x in that it could be written as

$$\mathbf{u}_n(t, x) = \sum_{\gamma \in \mathbf{N}^d} \mathbf{c}_{n,\gamma}(t)\, x^{\gamma}.$$

Let $Z = (Z_1, \ldots, Z_d)$ be a $\mathcal{F}^{\eta}$-measurable. Then by substituting Z into u we get

$$\mathbf{u}(t, Z) := \sum_n \left(\sum_{\gamma} \mathbf{c}_{n,\gamma}(t)\, Z^{\gamma} \right) \xi_n(\eta). \tag{5.7}$$

Now, let us introduce the Wick-powers of Z : $Z^{\Diamond\gamma} := Z_1^{\Diamond\gamma_1} \Diamond \cdots \Diamond Z_d^{\gamma_d}, \gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbf{N}^d$.

Next we will replace the standard algebra in (5.7) by the Wick algebra:

$$\mathbf{u}_n^{\Diamond}(t, Z) := \sum_{\gamma} \mathbf{c}_{n,\gamma}(t)\, Z^{\Diamond\gamma}$$

Consider now the following random field

$$\mathbf{u}^{\Diamond}(t, Z) := \sum_{n \geq 0} \mathbf{u}_n^{\Diamond}(t, Z) \Diamond \xi_n(\eta)$$

Remark 12 Note that Wick algebra on nonrandom elements reduces to the standard deterministic algebra.

Let $X_t = (X_t^1, \ldots, X_t^d)$ be a solution of the following dynamic equation

$$\dot{X}_t = \mathbf{u}^{\Diamond}(t, X_t).$$

Then by the Wick chain rule

$$\begin{aligned}\ddot{X}_t &= \frac{d}{dt}\mathbf{u}^{\Diamond}(t, X_t) = \partial_t \mathbf{u}^{\Diamond}(t, X_t) + \nabla \mathbf{u}^{\Diamond}(t, X_t) \Diamond \dot{X}_t, \\ &= \partial_t \mathbf{u}^{\Diamond}(t, X_t) + \mathbf{u}^{\Diamond}(t, X_t) \nabla \Diamond \mathbf{u}^{\Diamond}(t, X_t).\end{aligned}$$

If $\mathbf{F} = \mathbf{F}(t, x)$ is an acting force, this yields (Wick) Euler equation

$$\partial_t \mathbf{u}^{\Diamond}(t, x) = -\mathbf{u}^{\Diamond}(t, x) \nabla \Diamond \mathbf{u}^{\Diamond}(t, x) + \mathbf{F}(t, x)$$

If there is no randomness, due to Remark 12, this equation reduces to the standard Euler equation:

$$\partial_t \mathbf{u}(t, x) = -\mathbf{u}(t, x) \nabla \mathbf{u}(t, x) + \mathbf{F}(t, x).$$

Now, by taking $\mathbf{F} = \Delta \mathbf{u} - \nabla P$, where P stands for pressure, we get the unbiased Navier–Stokes equation

$$\partial_t \mathbf{u}^{\Diamond}(t, x) = \Delta \mathbf{u} - \mathbf{u}^{\Diamond}(t, x) \nabla \Diamond \mathbf{u}^{\Diamond}(t, x) - \nabla P + \mathbf{F}(t, x).$$

6 Appendix II. Stokes equation

Consider a deterministic Stokes equation for $\mathbf{u} = (u^l)_{1 \le l \le d}$, and scalar functions P,

$$\begin{aligned}\partial_t \mathbf{u}(t, x) &= \partial_i \left(a^{ij}(t, x)\, \partial_j \mathbf{u}(t, x)\right) + b^i(t, x)\partial_i \mathbf{u}(t, x) \\ +\mathbf{G}(t, x)\mathbf{u}(t, x) &+ \mathbf{f}(t, x) + \nabla P(t, x), \operatorname{div} \mathbf{u}(t) = 0, \\ \mathbf{u}(r, x) &= \mathbf{w}(x), x \in \mathbf{R}^d, r \le t \le T.\end{aligned}$$

equivalently,

$$\begin{aligned}\partial_t \mathbf{u}(t) &= \mathcal{S}[\partial_i \left(a^{ij}(t)\, \partial_j \mathbf{u}(t)\right) + b^i(t)\partial_i \mathbf{u}(t) + \mathbf{G}(t)\mathbf{u}(t) + \mathbf{f}(t)] \\ \mathbf{u}(r) &= \mathbf{w}, t \in [r, T],\end{aligned} \tag{6.1}$$

where $\mathcal{S}$ is the solenoidal projection of the vector fields,

$$\mathbf{a}(t, x) = \left(a^{ij}(t, x)\right)_{1 \le i, j \le d}, \mathbf{b}(t, x) = \left(b^i(t, x)\right)_{1 \le i < d}, \mathbf{G}(t, x) = \left(g^{ij}(t, x)\right)_{1 \le i, j \le d}$$

are measurable bounded functions. The matrix a is symmetric and positive.

We will need the following assumption.

B. For all $t \geq 0,\ x, \lambda \in R^d$, $K|\lambda|^2 \geq a^{ij}(t,x)\lambda^i\lambda^j \geq \delta|\lambda|^2$,where K, δ are fixed strictly positive constants. Also, for all $(t,x) \in [r,T] \times \mathbf{R}^d$,

$$\max_{t,x,|\alpha|\leq 2} |\partial_x^\alpha \mathbf{a}(t,x)| + \max_{t,x,|\alpha|\leq 1} |\partial_x^\alpha \mathbf{b}(t,x)| + \sup_{t,x} |\mathbf{G}(t,x)| \leq K.$$

Definition 11 A function $\mathbf{u} \in C([r,T], \mathbb{H}_p^s(\mathbf{R}^d))$ is an $\mathbb{H}_p^s$-solution of (6.1) if the equality

$$\mathbf{u}(t) = \mathbf{w} + \int_r^t \mathcal{S}[\partial_i(a^{ij}(s)\partial_j \mathbf{u}(s)) + b^i(s)\partial_i \mathbf{u}(s) + \mathbf{f}(s)]ds, \tag{6.2}$$

holds in $\mathbb{H}_p^{s-2}(\mathbf{R}^d)$ for every $t \in [r,T]$.

Proposition 5 *Let $p > d$, assumption* **B** *hold,* $\mathbf{w} \in \mathbb{H}_p^2 \cap \mathbb{H}_2^2$,

$$\int_r^T |\mathbf{f}(s)|_{1,l}^l ds + \int_r^T |\nabla \mathbf{G}(s)|_l^l ds < \infty, l = 2, p.$$

Then there is a unique $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$-valued continuous solution to (6.1)*. Moreover, there is a constant C independent of* $\mathbf{f}, \mathbf{w}, \mathbf{u}$ *so that*

$$\sup_{r\leq t\leq T} [|\mathbf{u}(t)|_{2,2} + |\mathbf{u}(t)|_{2,p}]$$
$$\leq C\left(|\mathbf{w}|_{2,2} + |\mathbf{w}|_{2,p} + \left(\int_r^T |\mathbf{f}(s)|_{1,p}^p ds\right)^{\frac{1}{p}} + \left(\int_r^T |\mathbf{f}(s)|_{1,2}^2 ds\right)^{1/2}\right). \tag{6.3}$$

Proof Let $\mathbf{w} \in \mathbb{H}_p^3 \cap \mathbb{H}_2^2$. By Proposition 4.7 and Corollary 4.6 in [21] (applied for $s = 0$), there is a unique $\mathbb{H}_p^1 \cap \mathbb{H}_2^1$-valued continuous solution $\mathbf{u}$ of (6.1), and

$$\int_r^T |\mathbf{u}(s)|_{2,l}^l ds < \infty, \quad l = 2, p.$$

Moreover,

$$\sup_{s\leq T} |\mathbf{u}(s)|_{1,l}^l \leq C[|\mathbf{w}|_{2,l}^l + \int_r^T |\mathbf{f}(s)|_l^l ds], \quad l = 2, p, \tag{6.4}$$

Consider Stokes equation

$$\xi(t) = \mathbf{w}_\alpha + \int_r^t \mathcal{P}[\partial_i \left(a^{ij}(s)\, \partial_j \xi(s)\right) + \mathbf{F}(s)]ds$$
$$\text{div } \xi(t) = 0, t \in [r, T], \tag{6.5}$$

where

$$\mathbf{F}(s) = b^i(s)\partial_i \mathbf{u}(s) + \mathbf{G}(s)\mathbf{u}(s) + \mathbf{f}(s).$$

It is readily checked (using Sobolev embedding theorem) that

$$|\mathbf{F}(s)|_{1,l} \leq C[K(|\mathbf{u}(s)|_{2,l} + |\nabla \mathbf{G}(s)|_l |\mathbf{u}(s)|_{1,p} + |\mathbf{f}(s)|_{1,l}], l = 2, p. \tag{6.6}$$

and

$$\int_r^T |\mathbf{F}(s)|_{1,l}^l ds < \infty, \quad l = 2, p.$$

By Corollary 4.6 and Proposition 4.7 in [21] (applied to (6.5) with $s = 1$), there is a unique $\mathbb{H}_2^2 \cap \mathbb{H}_p^2$-valued continuous solution of (6.5) $\xi = \mathbf{u}$ (by uniqueness) such that

$$\int_r^T |\mathbf{u}(s)|_{3,l}^l ds < \infty, \quad l = 2, p.$$

Let α be a multiindex such that $|\alpha| \leq 2$. Then $\mathbf{u}_\alpha = \partial^\alpha \mathbf{u}$ is $\mathbb{L}_p \cap \mathbb{L}_2$-valued continuous and satisfies the equation

$$\partial_t \mathbf{u}_\alpha(t) = \mathcal{S}\{\partial^\alpha[\partial_i(a^{ij}(t)\partial_j \mathbf{u}(t)) + \mathbf{F}(t)],$$
$$\mathbf{u}_\alpha(0) = \partial^\alpha \mathbf{w}.$$

Differentiating the product, we obtain

$$\partial^\alpha \partial_i(a^{ij}(t)\partial_j \mathbf{u}(t)) = \partial_i(a^{ij}(t)\partial_j \mathbf{u}_\alpha(t)) + \partial_i \mathbf{D}_\alpha(t) \tag{6.7}$$

with

$$|\mathbf{D}_\alpha(t)|_l \leq C|\mathbf{u}(t)|_{2,l}, \quad l = 2, p. \tag{6.8}$$

By Lemma 3 in [24], $y_{l,\alpha}(t) = |\mathbf{u}_\alpha(t)|_l^l$, $l = 2, p$, is differentiable:

$$y_{l,\alpha}(t) = y_{l,\alpha}(r) + \int_r^t h_{l,\alpha}(s)\,ds,$$

with

$$\begin{aligned} h_{l,\alpha}(s) = l\{&\langle|\mathbf{u}_\alpha(s)|^{l-2}\mathbf{u}_\alpha(s), \partial^\alpha\mathbf{F}(s)]\rangle_{1,l} \\ &-\int a^{ij}(s)\partial_i(|\mathbf{u}_\alpha(s)|^{l-2}u^k_\alpha(s))\partial_j u^k_\alpha(s)\,dx \\ &-\int \partial_i(|\mathbf{u}_\alpha(s)|^{l-2}u^k_\alpha(s))D^k_\alpha(s)\,dx\}. \end{aligned}$$

Notice $\partial^\alpha\mathbf{F}(s) \in \mathbb{H}_{-1,l}$ and, by our assumptions, there is a constant C so that for all $s \in [r, T]$

$$|\partial^\alpha\mathbf{F}(s)|_{-1,l} \le C|\mathbf{F}(s)|_{1,l}, \quad l = 2, p.$$

We have $h_{l,\alpha}(s) = lh^1_{l,\alpha}(s) + lh^2_{l,\alpha}(s)$, where

$$h^1_{l,\alpha}(s) = -\int a^{ij}(s)\partial_i(|\mathbf{u}_\alpha(s)|^{l-2}u^k_\alpha(s))\partial_j u^k_\alpha(s)\,dx.$$

Then

$$h^1_{l,\alpha}(s) \le -\delta\int |\mathbf{u}_\alpha(s)|^{l-2}|\nabla\mathbf{u}(s)|^2\,dx,$$

and for each $\varepsilon > 0$ there is a constant C_ε such that

$$\begin{aligned} |h^2_{l,\alpha}(s)| \le \varepsilon &\int |\mathbf{u}_\alpha(s)|^{l-2}|\nabla\mathbf{u}_\alpha(s)|^2\,dx \\ &+C_\varepsilon\int[|\mathbf{u}_\alpha(s)|^{l-2}(|\nabla\mathbf{F}(s)|^2 + |\mathbf{D}_\alpha(s)|^2) \\ &+|\mathbf{u}_\alpha(s)|^{l-1}|\mathbf{F}(s)|]dx, \end{aligned}$$

So, we obtain that

$$y_l(t) = \sum_{|\alpha|\le 2} y_{l,\alpha}(t) = y_l(r) + \int_r^t h_l(s)\,ds$$

with

$$h_l(s) = \sum_{|\alpha|\le 2} h_{l,\alpha}(s) \le C(y_l(s) + f_l(s)),$$

where $f_l(s) = |\mathbf{F}(s)|_{1,l}^l + |\mathbf{D}_\alpha(s)|_l^l$. Therefore, by (6.8), (6.6),

$$h_l(s) \leq C[y_l(s) + |\mathbf{f}(s)|_{1,l}^l + |\nabla \mathbf{G}(s)|_l^l |\mathbf{u}(s)|_{1,p}^l.$$

By Gronwall's inequality,

$$\sup_{r \leq s \leq T} y_p(s) \leq C \left[y_p(r) + \int_r^T |\mathbf{f}(s)|_{1,p}^p ds \right],$$

and

$$\sup_{r \leq t \leq T} |\mathbf{u}(t)|_{2,p}^p \leq C \left(|\mathbf{w}|_{2,p}^p + \int_r^T |\mathbf{f}(s)|_{1,p}^p ds \right), \tag{6.9}$$

where C is independent of $\mathbf{w}$ and $\mathbf{f}$. Similarly, by Gronwall's inequality

$$\sup_{r \leq s \leq T} y_2(s) \leq C[y_2(r) + \sup_{s \leq T} |\mathbf{u}(s)|_{1,p}^2 \int_r^T |\nabla \mathbf{G}(s)|_2^2 ds + \int_r^T |\mathbf{f}(s)|_{1,2}^2 ds],$$

and (see (6.4))

$$\sup_{r \leq t \leq T} |\mathbf{u}(t)|_{2,2}^2 \leq C \left(|\mathbf{w}|_{2,2}^2 + |\mathbf{w}|_{2,p}^2 + (\int_r^T |\mathbf{f}(s)|_p^p ds)^{\frac{2}{p}} + \int_r^T |\mathbf{f}(s)|_{1,2}^2 ds \right), \tag{6.10}$$

where C is independent of $\mathbf{w}$, $\mathbf{f}$ and $\mathbf{u}$. Combining (6.9) and (6.10) we have (6.3) with C is independent of $\mathbf{w}$, $\mathbf{f}$ and $\mathbf{u}$.

Given $\mathbf{w} \in \mathbb{H}_p^2 \cap \mathbb{H}_2^2$, there is a sequence $\mathbf{w}_n \in \mathbb{H}_p^3 \cap \mathbb{H}_2^2$ so that $\mathbf{w}_n \to \mathbf{w}$ in $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$. For every n there is a unique $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$-valued continuous solution $\mathbf{u}_n$ of (6.2) with the initial condition $\mathbf{u}(r) = \mathbf{w}_n$. By (6.3)

$$\sum_{l=2,p} \sup_{r \leq t \leq T} |\mathbf{u}_n(t) - \mathbf{u}_m(t)|_{2,l} \leq C \sum_{l=2,p} |\mathbf{w}_n - \mathbf{w}_m|_{2,l} \to 0$$

as $n, m \to \infty$. There is a continuous $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$-valued $\mathbf{u}(t)$ such that

$$\sum_{l=2,p} \sup_{r \leq t \leq T} |\mathbf{u}_n(t) - \mathbf{u}(t)|_{2,l} \to 0$$

as $n \to \infty$. Obviously, $\mathbf{u}$ is $\mathbb{H}_p^2 \cap \mathbb{H}_2^2$-valued continuous solution of (6.2) with initial condition $\mathbf{u}(r) = \mathbf{w}$ and (6.3) holds. □

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