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# Non-extinction of a Fleming-Viot particle model

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**Abstract** We consider a branching particle model in which particles move inside a Euclidean domain according to the following rules. The particles move as independent Brownian motions until one of them hits the boundary. This particle is killed but another randomly chosen particle branches into two particles, to keep the population size constant. We prove that the particle population does not approach the boundary simultaneously in a finite time in some Lipschitz domains. This is used to prove a limit theorem for the empirical distribution of the particle family.

**Keywords** Brownian motion · Branching particle system

**Mathematics Subject Classification (2000)** 60J65 · 60J80

## 1 Introduction

The paper is concerned with a branching particle system $\mathbf{X}_t = (X_t^1, \ldots, X_t^N)$ in which individual particles $X^j$ move as $N$ independent Brownian motions and die

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when they hit the complement of a fixed domain $D \subset \mathbb{R}^d$. To keep the population size constant, whenever any particle $X^j$ dies, another one is chosen uniformly from all particles inside $D$, and the chosen particle branches into two particles. Alternatively, the death/branching event can be viewed as a jump of the $j$-th particle. See Sect. 5 for a more detailed description of the construction.

Let $\tau_k$ be the time of the $k$-th jump of $\mathbf{X}_t$. Since the distribution of the hitting time of $\partial D$ by Brownian motion has a continuous density, only one particle can hit $\partial D$ at time $\tau_k$, for every $k$, a.s. The construction of the process is elementary for all $t < \tau_\infty = \lim_{k\to\infty} \tau_k$. However, there is no obvious way to continue the process $\mathbf{X}_t$ after the time $\tau_\infty$ if $\tau_\infty < \infty$. Hence, the question of the finiteness of $\tau_\infty$ is interesting. Theorem 1.1 in [10] asserts that $\tau_\infty = \infty$, a.s., for every domain $D$. Unfortunately, the proof of that theorem contains an irreparable error (see Example 5.7 below). The cited theorem might be true but it appears to be much harder to prove than the original incorrect argument might have suggested. Example 5.7 given below shows that result cannot be generalized to arbitrary Markov processes. We will show in Remark 5.6 that the other main results in [10], i.e., Theorems 1.3 and 1.4 hold true and an argument showing that $\tau_\infty = \infty$, a.s., in domains satisfying the internal ball condition is implicit in the proof of Theorem 1.4 of [10].

In this article, we will prove that $\tau_\infty = \infty$, a.s., if the domain $D \subset \mathbb{R}^d$ is Lipschitz with a Lipschitz constant $c$ depending on $d$ and the number $N$ of particles—see Theorem 5.4 and Remark 5.5 below. In addition, we prove theorems on existence and the form of the stationary distribution of the process $\mathbf{X}_t$, generalizing those in [10]—see Sect. 7.

We use this attempt to rectify an error in an earlier paper to introduce two new techniques. In the end, these techniques may have greater interest or significance than the main theorems. The first technique, developed in Sect. 4, is the construction of a process of Brownian excursions in a cone, with all excursions starting at the vertex. Such a process exists only in cones with certain angles. The construction is combined with a coupling argument to provide a "lower bound" for $\mathbf{X}_t$, in an appropriate sense. The process constructed from Brownian excursions is simpler to analyze than $\mathbf{X}_t$.

The second technique is a new type of boundary Harnack principle (see Sect. 3). The standard boundary Harnack principle compares two functions satisfying a PDE with the same operator, for example, Laplacian, and different boundary conditions. Our new version of the boundary Harnack principle compares a harmonic function with a function $u$ satisfying $\Delta u = -1$. The reason for proving the new form of the boundary Harnack principle is that it allows one to compare certain probabilities and expectations, and then use a method of proof that goes back at least to Davis [12]. The "new boundary Harnack principle" has been proved independently by Atar et al. [3], together with a number of other interesting theorems. We include a full proof of the new boundary Harnack principle because it is different from that in [3], and ours is amenable to generalizations that will be the subject of a forthcoming article.

Both techniques mentioned above—the Brownian excursion process and the boundary Harnack principle—are limited to Lipschitz domains and, moreover, the Lipschitz constant has to satisfy a certain inequality. A natural question arises whether such special Lipschitz domains are the largest natural family of sets where our results hold. It turns out that they are not. In the last section of the paper we will show that, for the two particle process, $\tau_\infty = \infty$, a.s., in all polyhedral domains, with arbitrary angles between the faces of the boundary. Unfortunately, our method cannot be easily adapted to the multiparticle case, so we leave this generalization as an open problem.

For some related results on Fleming-Viot type models in smooth domains, see [16] and references therein. The discrete version of the model is studied in [2]; see also references in that paper.

## 2 Preliminaries

For $y = (y^1, \dots, y^d) \in \mathbb{R}^d$, let $|y|$ denote the Euclidean norm of $y$ and let $\widetilde{y} = (y^1, \dots, y^{d-1})$. We will denote the open ball with center $x$ and radius $r$ by $B(x, r)$. The closure of a set $A$ will be denoted $\overline{A}$ and its interior will be denoted $\text{Int}A$. All constants, typically denoted by $c$ with or without subscript, are assumed to be strictly positive and finite.

A function $F : \mathbb{R}^{d-1} \to \mathbb{R}$ is called Lipschitz if there exists a constant $L$ such that

$$|F(x) - F(y)| \le L|x - y|, \quad x, y \in \mathbb{R}^{d-1}.$$

Any constant $L$ satisfying the above condition will be called a Lipschitz constant of $F$.

Consider a bounded connected open set $D \subset \mathbb{R}^d$, $d \ge 2$. We will call $D$ a Lipschitz domain with Lipschitz constant $L$ if $\partial D$ can be covered by a finite number of open balls $B_1, \dots, B_n$ such that for every $i = 1, \dots, n$, there exists a Lipschitz function $F_i : \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant $L$, and an orthonormal coordinate system $CS_i$ such that

$$D \cap B_i = \left\{ (y^1, \dots, y^d) \text{ in } CS_i : y^d > F_i(\widetilde{y}) \right\} \cap B_i.$$

The following Harnack principles can be found in [5].

**Theorem 2.1** (Harnack inequality)

(a) *Suppose* $0 < r < R$. *There exists* $c = c(r, R, d)$ *such that if* $u$ *is nonnegative and harmonic in* $B(0, R) \subset \mathbb{R}^d$ *and* $x, y \in B(0, r)$, *then*

$$u(x) \le c\, u(y).$$

(b) *Suppose that* $D \subset \mathbb{R}^d$ *is a domain and* $x, y \in D$ *can be connected by a curve* $\gamma \subset D$ *such that* $\inf_{z \in \gamma} \text{dist}(z, \partial D) \ge R$. *There exists* $c = c(\gamma, R, d)$ *such that if* $u$ *is nonnegative and harmonic in* $D$, *then*

$$u(x) \le c\, u(y).$$

**Theorem 2.2** (Boundary Harnack principle) *Suppose* $D$ *is a connected Lipschitz domain. Suppose* $V$ *is open,* $M$ *is compact and* $M \subset V$. *Then there exists a constant* $c = c(M, V, D)$ *such that if* $u$ *and* $v$ *are two positive and harmonic functions* *on $D$ that both vanish continuously on $V \cap \partial D$, then*

$$\frac{u(x)}{v(x)} \leq c\,\frac{u(y)}{v(y)}, \quad x, y \in M \cap D.$$

The next theorem is a simplified version of Theorem 1 of [1].

**Theorem 2.3** *Assume that $D$ is a Lipschitz domain. Then there exist constants $r_0 = r_0(D) > 0$, $c = c(D) < \infty$ and $a = a(D) > 1$ such that if $z \in \partial D$ and $0 < r \leq r_0$ then for all functions $u$ and $v$ that are bounded, positive and harmonic on $D \cap B(z, ar)$, and vanishing continuously on $\partial D \cap B(z, ar)$, we have*

$$\frac{u(x)}{v(x)} \leq c\frac{u(y)}{v(y)}, \quad x, y \in D \cap B(z, r).$$

*Remark 2.4* Theorem 2.3 can be used to estimate the constant $c(M, V, D)$ in Theorem 2.2 as follows. Suppose that $r_0$ and $a$ are as in Theorem 2.3 and we can find balls $B_i(x_i, r_i)$, $i = 1, \dots, n$, and $B'_j(y_j, \rho)$, $j = 1, \dots, m$, $\rho > 0$, $r_i \leq r_0$, $x_i \in \partial D$, $y_j \in D$, $M \subset \bigcup_i B_i(x_i, r_i) \cup \bigcup_j B'_j(y_j, \rho)$, and $\bigcup_i B_i(x_i, ar_i) \subset V$ and $\bigcup_j B'_j(y_j, 2\rho) \subset D$. A simple chaining argument based on Theorems 2.1 and 2.3 then shows that the constant $c(M, V, D)$ in Theorem 2.2 depends only on $n$, $m$ and $D$.

Next we recall some notation and results from [11]. Fix $d \geq 2$ and $p > 0$. Let

$$h(\theta) = h_{p,d}(\theta) = F\left(-p, p + d - 2; (d-1)/2; (1 - \cos\theta)/2\right), \tag{2.1}$$

where

$$F(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} x^k, \quad |x| < 1,$$

denotes the hypergeometric function and $(a)_k = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$. The function $h$ has at least one zero in $(0, \pi)$; let $\theta_{p,d}$ denote the smallest one. The quantity $\theta_{p,d}$ is strictly decreasing in $p$ for any fixed $d \geq 2$, and strictly increasing to $\pi/2$ in $d$ for any fixed $p > 1$. In particular, if $p = 2$, then

$$h_{2,d}(\theta) = 1 - \frac{d}{d-1}\sin^2\theta,$$

$\theta_{2,d} = \arccos\frac{1}{\sqrt{d}}$ and $\cot\theta_{2,d} = \frac{1}{\sqrt{d-1}}$. Therefore $\theta_{2,2} = \pi/4$ and $p < 2$ is equivalent to $\cot\theta_{p,d} < \frac{1}{\sqrt{d-1}}$.

For $d \geq 2$ and $p > 0$ we let $\theta$ be the angle between $y$ and $(0, \dots, 0, 1)$,

$$K_{p,d} = \left\{ y \in \mathbb{R}^d : y \neq 0,\ 0 \leq \theta < \theta_{p,d} \right\},$$

and let $O$ denote the axis of $K_{p,d}$. Obviously $p < p'$ implies $K_{p',d} \subset K_{p,d}$. We will drop the subscripts $p$ and $d$ and write $K$ instead of $K_{p,d}$ whenever there is no danger of confusion.

The function $v(x) = |x|^p h(\theta)$, where $h$ is given by (2.1), is positive and harmonic inside $K$ and continuous on $\overline{K}$ with $v(x) = 0$ for $x \in \partial K$.

Let $(\mathbb{P}^x, X_t)$ be $d$-dimensional Brownian motion and for a Borel set $A \subset \mathbb{R}^d$ define

$$T_A = \inf \{t > 0 : X_t \in A\}. \tag{2.2}$$

**Lemma 2.5** *Let $F$ denote the intersection of $K = K_{p,d}$ and a hyperplane orthogonal to $O$. Let $z_0$ be the point of intersection of $O$ with $F$ and assume that $z_0 \in K$. There exists $c = c(p, d)$ such that for all $z_1, z_2 \in O$ with $|z_0| < |z_1| < |z_2|$, we have*

$$\frac{\mathbb{P}^{z_2}(T_F < T_{\partial K})}{\mathbb{P}^{z_1}(T_F < T_{\partial K})} \geq c \left( \frac{|z_2|}{|z_1|} \right)^{2-d-p}. \tag{2.3}$$

*Proof* Let $K_*$ be the unbounded component of $K \setminus F$ and

$$u(z) = \mathbb{P}^z (T_F < T_{\partial K}), \quad z \in K_*.$$

Then $u$ is positive and harmonic in $K_*$ and continuous on $\overline{K}_* \setminus (F \cap \partial K)$, with $u(z) = 0$ for $z \in \partial K \setminus F$. It is easy to see that $u(x) \to 0$ as $|x| \to \infty$.

If $I(x) = x/|x|^2$, then the function $\widetilde{u}(x) = |x|^{2-d} u(I(x))$ is positive and harmonic in $\widetilde{K} = I(K_*)$ (see Lemma 1.18 of [5]). The function $\widetilde{u}$ vanishes continuously on $\partial \widetilde{K} \setminus I(F)$. Let $K' = (1/2)\widetilde{K}$. Recall that $v(x) = |x|^p h(\theta)$ is positive and harmonic inside $K$ and continuous on $\overline{K}$ with $v(x) = 0$ for $x \in \partial K$. By the boundary Harnack principle,

$$\frac{\widetilde{u}(z)}{\widetilde{u}(z')} \geq c \frac{v(z)}{v(z')}, \tag{2.4}$$

for $z, z' \in K'$, where $c$ depends on $\widetilde{K}$ and $K'$ and does not depend on $z$ and $z'$. Note that $u(x) = |x|^{2-d} \widetilde{u}(I(x))$. Hence, for $z_1, z_2 \in O \cap I(K')$,

$$\frac{u(z_2)}{u(z_1)} = \frac{|z_2|^{2-d} \widetilde{u}(I(z_2))}{|z_1|^{2-d} \widetilde{u}(I(z_1))} \geq c \frac{|z_2|^{2-d} v(I(z_2))}{|z_1|^{2-d} v(I(z_1))} = c \frac{|z_2|^{2-d} |z_2|^{-p} h(0)}{|z_1|^{2-d} |z_1|^{-p} h(0)} = c \left( \frac{|z_2|}{|z_1|} \right)^{2-p-d}.$$

The inequality holds for all $z_1, z_2 \in O \cap I(K_*)$ (possibly with a different value of $c$) because the function $u$ is bounded below and above on $O \setminus I(K')$ by strictly positive and finite constants. This completes the proof of (2.3). ◻

We will use the following estimate in the proof of Lemma 4.1.

**Lemma 2.6** *There exists a cone $K' \subset K = K_{p,d}$ and a constant $c = c(K, K')$ such that for $x \in K'$ and $t \geq |x|^2$,*

$$c^{-1}\left(\frac{t}{|x|^2}\right)^{-\frac{p}{2}} \leq \mathbb{P}^x\left(T_{\partial K} > t\right) \leq c\left(\frac{t}{|x|^2}\right)^{-\frac{p}{2}}. \tag{2.5}$$

*Proof* See [4,6,14] or [22]. □

## 3 A boundary Harnack principle

Let $D \subset \mathbb{R}^d$, $d \geq 2$, be a bounded Lipschitz domain and let $A \subset D$ be a compact set with $\mathrm{Int}A \neq \emptyset$. For $x \in D$, define

$$f(x) = \mathbb{P}^x(T_A < T_{\partial D}),$$
$$g(x) = \mathbb{E}^x T_{\partial D}.$$

**Theorem 3.1** *Assume that the Lipschitz constant $L$ of $D$ satisfies $L < \frac{1}{\sqrt{d-1}}$. Then there exists a constant $c = c(A, D)$ such that for all $x \in D$,*

$$\frac{1}{c} \leq \frac{f(x)}{g(x)} \leq c. \tag{3.1}$$

*Remark 3.2* The condition $L < \frac{1}{\sqrt{d-1}}$ is sharp. See Example 3.3 below.

*Proof of RHS of* (3.1) Since $A$ is compact, $\inf_{x\in A} \mathrm{dist}(x, D^c) = c_1 > 0$. Therefore,

$$\inf_{x\in A} \mathbb{E}^x T_{\partial D} \geq \inf_{x\in A} \mathbb{E}^x T_{\partial B(x,c_1)} = c_2 > 0.$$

By the strong Markov property applied at $T_A$, we have for $x \in D$,

$$\mathbb{E}^x T_{\partial D} \geq c_2 \mathbb{P}^x(T_A < T_{\partial D}),$$

which implies the RHS of (3.1). □

*Proof of LHS of* (3.1) Since $D$ is a bounded Lipschitz domain with Lipschitz constant $L < \frac{1}{\sqrt{d-1}}$, it is easy to see that there exist $p \in (0, 2)$ and $\rho > 0$ with the following properties.

(i) $\mathrm{dist}(A, \partial D) > 2\rho$.
(ii) Consider any $x \in D$ with $\mathrm{dist}(x, \partial D) < \rho 2^{-5}$. Then there exists $x_0 \in \partial D$ and an orthonormal coordinate system $CS = CS_{x_0}$ with the following properties.

The origin of $CS$ is $x_0$, $K_{p,d} \cap B(x_0, 2\rho) \subset D \cap B(x_0, 2\rho)$, and $x \in O$ (that is, $x$ belongs to the axis of $K_{p,d}$). For $r > 0$ and integer $k$, let

$$E_r^* = \left\{ y \in \mathbb{R}^d \text{ in } CS : |\widetilde{y} - \widetilde{x}_0| \le r \tan(\theta_{p,d}), |y^d - x_0^d| \le r \right\},$$
$$\widetilde{E}_k = E^*_{2^{-k}}.$$

We can choose $x_0$ and $CS$ so that for some Lipschitz function $F = F_{x_0} : \mathbb{R}^{d-1} \to \mathbb{R}$ with Lipschitz constant $L$, and all $k$ such that $2^{-k} \le \rho$,

$$D \cap \widetilde{E}_k = \left\{ (y^1, \dots, y^d) \text{ in } CS : y^d > F(\widetilde{y}) \right\} \cap \widetilde{E}_k.$$

We fix $x \in D$ with $\operatorname{dist}(x, \partial D) < \rho 2^{-5}$ and the corresponding coordinate system $CS$ for the rest of the proof.

Let $E_k = \widetilde{E}_k \setminus \widetilde{E}_{k+1}$ and $C_k = \operatorname{Int}(D \cap E_k)$ for $k = N_0, \dots, N_1$, where

$$N_0 = \min\{k : 2^{-k} \le \rho\}, \qquad N_1 = \max\left\{ k : |x| = x^d \le 2^{-k-3} \right\}.$$

Also let $C_{N_0-1} = \operatorname{Int}\left(D \setminus \widetilde{E}_{N_0}\right)$ and $C_{N_1+1} = \operatorname{Int}(D \cap \widetilde{E}_{N_1+1})$.

Note that $C_i \cap C_j = \emptyset$ if $i \neq j$, and $D = \overline{C}_{N_0-1} \cup \dots \cup \overline{C}_{N_1+1}$.

Let $G(x, y)$ denote the Green function for Brownian motion killed on exiting $D$. Then

$$g(x) = \mathbb{E}^x T_{\partial D} = \int_D G(x, y)\, dy = \sum_{k=N_0-1}^{N_1+1} \int_{C_k} G(x, y)\, dy. \tag{3.2}$$

For $k = N_0, \dots, N_1$ denote by $y_k$ the midpoint of the line segment being the intersection of $C_k$ with $x^d$-axis in $CS$. In other words, $\{y_k\} = \partial E^*_{(3/4)2^{-k}} \cap O$. Fix $k$ and $j$ such that $j \ge 1$, $k \ge N_0$, $j + k \le N_1$ and consider the points $y_k$ and $y_{k+j}$.

Let

$$F_k = \overline{C}_k \cap \overline{C}_{k+1} \cap K_{p,d},$$

and

$$u(z) = \mathbb{P}^z \Big( T_{F_{k+j}} < T_{\partial K_{p,d}} \Big).$$

By Lemma 2.5,

$$u(y_k) \ge c_1 u(y_{k+j}) \left( \frac{2^{-k}}{2^{-k-j}} \right)^{2-p-d} = c_1 u(y_{k+j}) 2^{j(2-p-d)},$$

where $c_1 = c_1(p, d)$. By scaling properties of Brownian motion, $u(y_{k+j}) = c_2 = c_2(p, d)$, that is, $u(y_{k+j})$ depends only on $p$ and $d$. We obtain

$$\mathbb{P}^z\Big(T_{F_{k+j}} < T_{\partial K_{p,d}}\Big) \geq c_3 2^{-j(p+d-2)}, \tag{3.3}$$

where $c_3 = c_3(p, d)$.

Let

$$v(z) = \mathbb{P}^z\left(T_{F_{k+j}} < T_{\partial D}\right).$$

Note that $v(y_{k+j}) \leq 1$ and $v(y_k) \geq u(y_k) \geq c_3 2^{-j(p+d-2)}$, by (3.3).

We will apply Theorem 2.2 with $M = \partial E^*_{(3/4)2^{-k-j}}$ and $V = E_{k+j}$. It follows from Remark 2.4 that the constant $c_5 = c(M, V, D)$ may be chosen independent of $k$ and $j$. The boundary Harnack principle implies that

$$\frac{G(x, z)}{G(x, y_{k+j})} \geq c_5 \frac{v(z)}{v(y_{k+j})}, \tag{3.4}$$

for $z \in D \cap M$. The harmonic functions $G(x, \cdot)$ and $v$ have zero boundary values on $\partial D \setminus \overline{E}^*_{(3/4)2^{-k-j}}$, so the inequality (3.4) extends to all $z \in D \setminus E^*_{(3/4)2^{-k-j}}$, in particular, it applies to $z = y_k$. Hence,

$$\frac{G(x, y_k)}{G(x, y_{k+j})} \geq c_5 \frac{v(y_k)}{v(y_{k+j})} \geq c_5 c_3 2^{-j(p+d-2)} = c_6 2^{-j(p+d-2)}. \tag{3.5}$$

Now consider the function

$$h_m(z) = \mathbb{P}^z\Big(T_{\widetilde{E}_{m+2}} < T_{\partial D}\Big).$$

By the scaling properties of Brownian motion, $h_m(y_m) \geq c_7 > 0$ for all $m = N_0, \dots, N_1$. By the boundary Harnack principle (Theorem 2.2) applied to $u(z) = G(x, z)$, $v(z) = h_m(z)$, $M = \overline{C}_m$ and $V = \mathrm{Int}(\widetilde{E}_{m-1} \setminus E^*_{(3/4)2^{-m-1}})$, we have

$$\frac{G(x, y)}{h_m(y)} \leq c_8 \frac{G(x, y_m)}{h_m(y_m)}$$

for $y \in C_m$, where $c_8$ depends only on $D$, by Remark 2.4. Therefore, for $y \in C_m$,

$$G(x, y) \leq c_8\, G(x, y_m) \frac{h_m(y)}{h_m(y_m)} \leq c_8 \frac{1}{c_7} G(x, y_m) = c_9\, G(x, y_m). \tag{3.6}$$

This implies

$$\int_{C_{k+j}} G(x, y)\, dy \leq c_9 G(x, y_{k+j}) \operatorname{vol}(C_{k+j}) \leq c_{10}\, 2^{-d(k+j)} G(x, y_{k+j}), \tag{3.7}$$

where $c_{10}$ depends only on $D$.

On the other hand, by the usual Harnack inequality,

$$G(x, y) \geq c_{11} G(x, y_k)$$

for $y \in B_k = B(y_k, 2^{-k-1})$, because $B(y_k, 2^{-k-1}) \subset D \setminus \{x\}$. This implies that

$$\int_{C_k} G(x, y)\, dy \geq c_{11} G(x, y_k) \operatorname{vol}(B_k) = c_{12} 2^{-kd} G(x, y_k), \tag{3.8}$$

where $c_{12}$ does not depend on $k$.

Combining (3.5), (3.7) and (3.8) we have

$$\int_{C_{k+j}} G(x, y)\, dy \leq c_{13} 2^{j(p-2)} \int_{C_k} G(x, y)\, dy,$$

where $c_{13} = c_{13}(D)$. Fix $q < 1$. Since $p \in (0, 2)$, we may choose $j$ so large that $c_{13} 2^{j(p-2)} \leq q < 1$. Let $a_k = \int_{C_k} G(x, y)\, dy$, then

$$a_{k+j} \leq q a_k, \quad k = N_0, \ldots, N_1 - j.$$

Let $N_2 = \min(N_1, N_0 + j - 1)$. The last inequality implies that

$$\sum_{k=N_0}^{N_1} a_k = \sum_{k=N_0}^{N_2} \sum_{m=0}^{\infty} a_{k+mj} \mathbf{1}_{\{k+mj \leq N_1\}} \leq \sum_{k=N_0}^{N_2} \sum_{m=0}^{\infty} a_k q^m = c_{14} \sum_{k=N_0}^{N_2} a_k. \tag{3.9}$$

Recall that $G(x, \cdot)$ has zero boundary values on $\partial D$, so it is bounded by $\sup_{z \in C_{N_0}} G(x, z)$ on the set $D \setminus \widetilde{E}_{N_0}$. This and (3.6) imply that $\sup_{z \in D \setminus \widetilde{E}_{N_0}} G(x, z) \leq c_{15} G(x, y_{N_0})$. We use (3.8) to see that

$$\begin{aligned} a_{N_0-1} = \int_{C_{N_0-1}} G(x, y)\, dy &\leq c_{15} G(x, y_{N_0}) \operatorname{vol}(C_{N_0-1}) \\ &\leq c_{15} \operatorname{vol}(C_{N_0-1}) c_{12}^{-1} 2^{N_0 d} \int_{C_{N_0}} G(x, y)\, dy = c_{16} a_{N_0}. \end{aligned} \tag{3.10}$$

Recall the definition of $N_0$ to see that $c_{16}$ depends only on $D$.

The following calculation is presented in the case $d \geq 3$ only. The case $d = 2$ requires minor modifications and is left to the reader.

Let $\widetilde{G}(x, y)$ denote the Green function for Brownian motion in $\mathbb{R}^d$, and let $\overline{G}(x, y)$ be the Green function for Brownian motion in $B(x, 2^{-N_1-4})$. It is well known that for $d \geq 3$, $\widetilde{G}(x, y) = c_{17} |x - y|^{2-d}$, where $c_{17}$ depends on $d$, and $\overline{G}(x, y) =$

$\widetilde{G}(x, y) - \widetilde{G}(x, z)$, for $y \in B(x, 2^{-N_1-4})$ and $z \in \partial B(x, 2^{-N_1-4})$. It follows that for $|y - x| \leq 2^{-N_1-5}$,

$$\overline{G}(x, y) \geq c_{18}\widetilde{G}(x, y). \tag{3.11}$$

We have $G(x, y) \leq \widetilde{G}(x, y)$ for $y \in D$, and $\int_{B(x,r)} \widetilde{G}(x, y)dy = c_{19}r^2$. Therefore,

$$\begin{aligned} a_{N_1+1} &= \int_{C_{N_1+1}} G(x, y)dy \leq \int_{C_{N_1+1}} \widetilde{G}(x, y)dy \\ &\leq \int_{B(x,\mathrm{diam}(\widetilde{E}_{N_1+1}))} \widetilde{G}(x, y)dy = c_{20}2^{-2N_1}. \end{aligned} \tag{3.12}$$

Since $B(x, 2^{-N_1-4}) \subset D$,

$$G(x, y) \geq \overline{G}(x, y). \tag{3.13}$$

Put $y_{N_1+1} = (\widetilde{x}, x_d + 2^{-N_1-5})$. Then by (3.11) and (3.13),

$$G(x, y_{N_1+1}) \geq c_{18}\widetilde{G}(x, y) = c_{21}(2^{-N_1})^{2-d}.$$

Moreover, by the usual Harnack inequality,

$$G(x, y) \geq c_{22}G(x, y_{N_1+1}),$$

for $y \in B(y_{N_1}, 2^{-N_1-2})$. Therefore,

$$\begin{aligned} a_{N_1} &= \int_{C_{N_1}} G(x, y)dy \geq \int_{B(y_{N_1},2^{-N_1-2})} G(x, y)\,dy \\ &\geq c_{22}G(x, y_{N_1+1})\,\mathrm{vol}(B(y_{N_1}, 2^{-N_1-2})) \geq c_{23}(2^{-N_1})^{2-d} \cdot 2^{-N_1 d} = c_{24}2^{-2N_1}. \end{aligned} \tag{3.14}$$

Combining (3.12) and (3.14), we obtain

$$a_{N_1+1} \leq c_{25}a_{N_1}. \tag{3.15}$$

Let $C_* = C_{N_0-1} \cup \cdots \cup C_{N_2}$ and note that $A \subset C_*$. Let $\sigma_{C_*} = \int_0^{T_{\partial D}} 1_{\{X_s \in C_*\}}\,ds$. Then (3.9), (3.10) and (3.15) imply that

$$\mathbb{E}^x T_{\partial D} \leq c_{26}\mathbb{E}^x\sigma_{C_*}. \tag{3.16}$$

Since $D$ is bounded, $\sup_{z\in D} \mathbb{E}^z T_{\partial D} = c_{27} < \infty$. By the strong Markov property applied at the hitting time of $C_*$, for $z \in D$,

$$\mathbb{E}^z \sigma_{C_*} \leq c_{27}\, \mathbb{P}^z(T_{C_*} < T_{\partial D}).$$

This and (3.16) yield

$$\mathbb{E}^x T_{\partial D} \leq c_{28} \mathbb{P}^x(T_{C_*} < T_{\partial D}). \tag{3.17}$$

Consider functions

$$\xi_1(z) = \mathbb{P}^z(T_A < T_{\partial D}),$$
$$\xi_2(z) = \mathbb{P}^z(T_{C_*} < T_{\partial D}).$$

Both functions are positive and harmonic in $D \setminus \overline{C}_*$, and continuous on $\overline{D} \setminus \overline{C}_*$ with $u(z) = v(z) = 0$ for $z \in \partial D \setminus \overline{C}_*$. We apply the boundary Harnack principle with $V = D \setminus \overline{C}_*$ and $M = \widetilde{E}_{N_2+1}$ to see that

$$\frac{\xi_1(x)}{\xi_2(x)} \geq c_{29} \frac{\xi_1(y_{N_2+1})}{\xi_2(y_{N_2+1})}. \tag{3.18}$$

We use Remark 2.4 to see that $c_{29}$ may be chosen so that it depends only on $D$. It follows from the definitions of $N_0, N_2$ and $j$ that for some constant $c_{30}$, we have $\operatorname{dist}(y_{N_2+1}, \partial D) > c_{30}$. This implies that $\xi_1(y_{N_2+1}) = \mathbb{P}^{y_{N_2+1}}(T_A < T_{\partial D}) \geq c_{31}$, for some $c_{31}$ depending only on $D$. We obtain from (3.18) that $\xi_1(x)/\xi_2(x) \geq c_{29}c_{31}$, and this combined with (3.17) gives

$$\mathbb{E}^x T_{\partial D} \leq (c_{28}/c_{29}c_{31}) \mathbb{P}^z(T_A < T_{\partial D}).$$

We have proved the LHS of (3.1) for $x$ satisfying $\operatorname{dist}(x, \partial D) \leq \rho 2^{-5}$.

It is easy to check that $\inf\{f(x) : \operatorname{dist}(x, \partial D) \geq \rho 2^{-5}\} > 0$ and $\sup\{g(x) : x \in D\} < \infty$, so the LHS of (3.1) holds for all $x \in D$. □

*Example 3.3* The condition $L < \frac{1}{\sqrt{d-1}}$ in Theorem 3.1 is sharp. To see this, note that for any $L > \frac{1}{\sqrt{d-1}}$ there is a $p > 2$, such that the cone $K = K_{p,d}$ is a Lipschitz domain with the Lipschitz constant $L$. Let $r > 0$ be such that for every $x \in O$, $B(x, r|x|) \subset K$. Then $g(x) = \mathbb{E}^x T_{\partial K} \geq \mathbb{E}^x T_{\partial B(x,r|x|)} \geq c_1 r^2 |x|^2$. Recall that $f(x) = \mathbb{P}^x(T_A < T_{\partial K})$ and let $u(x) = |x|^p h_{p,d}(\theta)$. By the boundary Harnack principle applied to $f$ and $u$ in a neighborhood of 0, $f(x) \leq c_2 |x|^p$ for $x \in O$, $|x| < 1$. Since $p > 2$, we cannot have $f(x) \geq c_3 g(x)$ in a neighborhood of 0. The domain $K$ is unbounded but it is easy to extend the argument to $K \cap B(0, 1)$.

## 4 Construction of an auxiliary process from Brownian excursions

Let $\Omega$ denote the family of all functions $\omega : [0, \infty) \to \mathbb{R}^d \cup \{\delta\}$ continuous up to their lifetime $R(\omega) = \inf\{t \geq 0 : \omega(t) = \delta\}$ and constantly equal to $\delta$ for $t \geq R$, where $\delta$ denotes the coffin state outside $\mathbb{R}^d$. Let $X$ be the canonical process on $\Omega$, i.e., $X_t(\omega) = \omega(t)$ and let $\mathbb{P}^x$ denote the distribution of Brownian motion starting from $x \in \mathbb{R}^d$. As in (2.2), for a Borel set $A \subset \mathbb{R}^d$ let $T_A = \inf\{t > 0 : X_t \in A\}$. Let $K = K_{p,d}$ for some $p > 0$, and let $X'$ denote the process

$$X'_t = \begin{cases} X_t, & \text{for } t < T_{\partial K}, \\ \delta, & \text{otherwise,} \end{cases}$$

i.e., $X'$ is the process $X$ killed on exiting $K$. If $X$ has the distribution $\mathbb{P}^x$, then $X'$ is called Brownian motion in $K$ and its distribution is denoted by $\mathbb{P}^x_K$.

Let $U$ denote the family of all functions $\omega : [0, \infty) \to K \cup \{\delta\}$ such that $\omega(0) = 0$, continuous up to their lifetime $R$. Let $H^0$ denote a standard excursion law of Brownian motion in $K_{p,d}$ starting from 0. Namely, $H^0$ is a nonnegative and $\sigma$-finite measure on $\Omega$ such that $X$ is strong Markov under $H^0$ with the $\mathbb{P}_K$ transition probabilities and $H^0(\lim_{t\to 0} X_t \neq 0) = 0$. We have $H^0(\Omega \setminus U) = 0$. The existence of $H^0$ follows from results of [8] and [20].

**Lemma 4.1** *There exists $c \in (0, \infty)$ such that*

$$H^0(R > t) = c\, t^{-\frac{p}{2}}, \quad t > 0. \tag{4.1}$$

*Proof* Let $y_\varepsilon = (0, \dots, 0, \varepsilon) \in \mathbb{R}^d$ and let $G_K(x, y)$ denote the Green function for $K$. By Theorem 4.1 of [8],

$$H^0(R > t) = c_1 \lim_{\substack{z\to 0\\ z\in K}} \frac{\mathbb{P}^z(T_{\partial K} > t)}{G_K(z, y_1)}. \tag{4.2}$$

By Theorem 2.2 of [8], which is an improvement of the boundary Harnack principle, there exists $c(K, \varepsilon)$ such that for all functions $h_1$ and $h_2$ which are positive and harmonic in $K$ and vanish continuously on $\partial K$, we have

$$c(K, \varepsilon)^{-1} \frac{h_1(y)}{h_2(y)} \le \frac{h_1(x)}{h_2(x)} \le c(K, \varepsilon) \frac{h_1(y)}{h_2(y)},$$

for all $x, y \in K \cap B(0, \varepsilon)$, and $\lim_{\varepsilon\to 0} c(K, \varepsilon) = 1$. Therefore, the limit

$$\lim_{\substack{z\to 0\\ z\in K}} \frac{h_1(z)}{h_2(z)}$$

exists and belongs to $(0, \infty)$ for all functions $h_1, h_2$ satisfying the above assumptions. We apply this claim to $h_1(z) = G_K(z, y_1)$ and $h_2(z) = |z|^p h(\theta)$, to conclude that

$$\lim_{\varepsilon\to 0} \frac{G_K(y_\varepsilon, y_1)}{\varepsilon^p} = c \in (0, \infty),$$

and

$$H^0(R > t) = c \lim_{\varepsilon \to 0} \frac{\mathbb{P}^{y_\varepsilon}(T_{\partial K} > t)}{\varepsilon^p}. \tag{4.3}$$

By Lemma 2.6,

$$c^{-1} t^{-\frac{p}{2}} \le \frac{\mathbb{P}^{y_\varepsilon}(T_{\partial K} > t)}{\varepsilon^p} \le c t^{-\frac{p}{2}},$$

for $t \ge \varepsilon^2$ which implies $c^{-1} t^{-\frac{p}{2}} \le H^0(R > t) \le c t^{-\frac{p}{2}}$, for $t \ge 0$. Therefore $H^0(R > 1)$ is a positive and finite number.

Now mimicking the proof of Proposition 5.1 of [8], using (4.3) instead of (4.2), we easily see that if $\{X(t),\ t \ge 0\}$ has the distribution $H^0$, then for every $a > 0$ the scaled process $\left\{\sqrt{a} X(t/a),\ t \ge 0\right\}$ has the distribution $a^{p/2} H^0$. In particular, for every $a > 0$

$$H^0(R > t) = a^{p/2} H^0(R > at), \quad t \ge 0,$$

and putting $a = 1/t$ we obtain (4.1) with $c = H^0(R > 1)$. □

Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}_+ = [0, \infty)$ and let $\mathcal{P}$ be a Poisson point process on $\mathbb{R}_+ \times U$ with characteristic measure $\lambda \times H^0$, i.e., $\mathcal{P}$ is a random subset of $\mathbb{R}_+ \times U$ such that for every pair $A_1, A_2$ of disjoint nonrandom subsets of $\mathbb{R}_+ \times U$, $\operatorname{card}(\mathcal{P} \cap A_1)$ and $\operatorname{card}(\mathcal{P} \cap A_2)$ are independent random variables with Poisson distributions with means $(\lambda \times H^0)(A_1)$ and $(\lambda \times H^0)(A_2)$, respectively [18]. With probability 1, there are no two points with the same first coordinate, and therefore the elements of $\mathcal{P}$ may be unambiguously denoted by $(t, e_t)$. Let

$$R_t = \inf\{s > 0 : e_t(s) = \delta\}.$$

By abuse of notation, for a generic element $e$ of $U$ we will write

$$R(e) = \inf\{s > 0 : e(s) = \delta\}.$$

**Lemma 4.2** *If $p \in (0, 2)$, then for every $s > 0$,*

$$\sum_{t \le s} R_t < \infty, \quad \text{a.s.}$$

*Proof* We use Theorem 4.6 of [18]: if $\varphi : \mathbb{R}_+ \times U \to \mathbb{R}_+$ is a measurable function, then

$$\sum_t \varphi(t, e_t) < \infty, \quad \text{a.s.}$$

iff

$$\iint\limits_{\mathbb{R}_+\times U} (\varphi(t,e)\wedge 1)dt\,H^0(de) < \infty.$$

In particular, if $\varphi(t,e) = R(e)\mathbf{1}_{[0,s]}(t)$, then

$$\sum_{t\le s} R_t < \infty, \quad \text{a.s.}$$

iff

$$\iint\limits_{[0,s]\times U} (R(e)\wedge 1)dt\,H^0(de) < \infty.$$

If we let $U^- = \{e \in U : R(e) \le 1\}$ and $U^+ = \{e \in U : R(e) > 1\}$ then

$$\iint\limits_{[0,s]\times U} (R(e)\wedge 1)dt\,H^0(de) = s\int\limits_{U^-} R(e)H^0(de) + sH^0(U^+).$$

By Lemma 4.1,

$$H^0(U^+) = \int\limits_1^\infty H^0(R\in dt) = c\int\limits_1^\infty t^{-p/2-1}dt < \infty,$$

because $p > 0$, and

$$\int\limits_{U^-} R\,dH^0 = \int\limits_0^1 tH^0(R\in dt) = c\int\limits_0^1 t\cdot t^{-p/2-1}dt < \infty,$$

because $p < 2$. □

Let $\sigma_v = \sum_{s\le v} R_s$ and $\sigma_{v-} = \sum_{u<v} R_u$ for $v \ge 0$. By Lemma 4.2, if $p \in (0,2)$ then $\sigma_v < \infty$ for all $v < \infty$, a.s.

**Lemma 4.3** *The process $\sigma$ is a stable subordinator with index $p/2$.*

*Proof* The process $\sigma$ is increasing and has values in $[0,\infty)$. Its paths are right-continuous with left limits. Note that $\{(t, R(e_t))\}_{e\in\mathcal{P}}$ is a Poisson point process on $\mathbb{R}_+\times\mathbb{R}_+$ with characteristic measure $\lambda\times\Pi$, where $\Pi$ is given by

$$\Pi(dx) = H^0(R\in dx) = c\,x^{-p/2-1}dx,$$

where the last formula follows from Lemma 4.1. This implies that $\sigma$ is a process with independent and stationary increments, so $\sigma$ is a Lévy process. Moreover $\sigma$ is a subordinator, since it has values in $[0, \infty)$ only. We use calculations that can be found in Section 0.5 and on page 73 of [7] to see that the Laplace transform of $\sigma$ is

$$\begin{aligned} E\exp(-\lambda\sigma_t) &= \exp\left\{-t\int_0^\infty (1-\mathrm{e}^{-\lambda x})\Pi(dx)\right\} \\ &= \exp\left\{-ct\int_0^\infty (1-\mathrm{e}^{-\lambda x})x^{-p/2-1}dx\right\} \\ &= \exp(-ct\lambda^{p/2}). \end{aligned}$$

Therefore $\sigma$ is stable with index $p/2$. □

It is well known that for a stable subordinator $\sigma$ we have $\lim_{v\to\infty}\sigma_v = \infty$, a.s. So with probability 1, for every $t \geq 0$, the formula $r = \inf\{v \geq 0 : \sigma_v \geq t\}$ defines a unique $r \geq 0$. For $t \geq 0$ let

$$Z_t = \begin{cases} e_r(t-\sigma_{r-}), & \text{if } \ \sigma_{r-} < \sigma_r \text{ and } t \in (\sigma_{r-}, \sigma_r), \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

The process $Z$ takes values in $K \cup \{0\}$.

*Remark 4.4* The above construction is similar to the classical Itô representation of Brownian motion using the Poisson point process of excursions, see [24, Chap. XII]. The construction of a Markov process from excursions is presented in [25]. The history of the idea, related papers and results are discussed in that article. The process $Z$ is strong Markov by [25, Thm. 4.1]—it is straightforward to check that the assumptions of that theorem are satisfied in our case.

**Corollary 4.5** *Let $Z_t^1, \ldots, Z_t^N$ be jointly independent copies of $Z_t$ defined in* (4.4). *If $p < 2 - \frac{2}{N}$ and $T \in (1, \infty)$ then* $\inf_{1/T\leq t\leq T}\max_{1\leq i\leq N}|Z_t^i| > 0$, *a.s.*

*Proof* For each $i$, let $\sigma_t^i$ be a stable subordinator associated with the process $Z_t^i$ as in Lemma 4.3 and let $A_i = \{t \in [\frac{1}{T}, T] : Z_t^i = 0\}$. In other words, $A_i$ is the range of $\sigma_t^i$ over $[\frac{1}{T}, T]$. We use the following result of Hawkes [17]: The ranges of two independent stable subordinators with indices $\alpha$ and $\beta$ intersect if and only if $\alpha + \beta > 1$, in which case the intersection is stochastically equivalent to the range of a stable subordinator of index $\alpha + \beta - 1$. Therefore, by induction, $A_1 \cap \cdots \cap A_N = \emptyset$, a.s., if and only if $\frac{Np}{2} - N + 1 < 0$. This condition holds since $p < 2 - \frac{2}{N}$.

It is easy to see that $t \to |Z_t^i|$ is lower semicontinuous. Hence, $t \to \max_{1\leq i\leq N}|Z_t^i|$ is also lower semicontinuous and, therefore, it attains its infimum on $[\frac{1}{T}, T]$. It follows that $\{\inf_{1/T\leq t\leq T}\max_{1\leq i\leq N}|Z_t^i| > 0\} = \{A_1 \cap \cdots \cap A_N = \emptyset\}$. We have shown that the last event has probability one if $p < 2 - \frac{2}{N}$. □

## 5 Construction of a Fleming-Viot process

We recall the following description of a Fleming-Viot-type particle system from [10]. Consider an open set $D \subset \mathbb{R}^d$ and an integer $N \geq 2$. Let $\mathbf{X}_t = (X_t^1, \ldots, X_t^N)$ be a process with values in $D^N$ defined as follows. Let $\mathbf{X}_0 = (x^1, \ldots, x^N) \in D^N$. Then the processes $X_t^1, \ldots, X_t^N$ evolve as independent Brownian motions until the time $\tau_1$ when one of them, say, $X^j$ hits the boundary of $D$. At this time one of the remaining particles is chosen uniformly, say, $X^k$, and the process $X^j$ jumps at time $\tau_1$ to $X_{\tau_1}^k$. The processes $X_t^1, \ldots, X_t^N$ continue evolving as independent Brownian motions after time $\tau_1$ until the first time $\tau_2 > \tau_1$ when one of them hits the boundary of $D$. Again at the time $\tau_2$ the particle which approaches the boundary jumps to the current location of a particle chosen uniformly at random from amongst the ones strictly inside $D$. The subsequent evolution of $\mathbf{X}$ proceeds in the same way. The total number of jumps may be finite or infinite. The above recipe defines the process $\mathbf{X}_t$ only for $t < \tau_\infty = \lim_{k\to\infty} \tau_k$. There is no natural way to define the process $\mathbf{X}_t$ for $t \geq \tau_\infty$. Hence, we add a cemetery state $\delta$ to the state space and we let $\mathbf{X}_t = \delta$ for all $t \geq \tau_\infty$. We define $\mathbf{X}_t$ so that it is right-continuous with left limits on the interval $[0, \tau_\infty)$. We do not make any *a priori* claims about existence or non-existence of the left limit $\mathbf{X}_{\tau_\infty -}$.

*Remark 5.1* The proof of the main theorem in this section, Theorem 5.4, involves an inductive construction of the Fleming-Viot process. In particular the proof relies on a special construction of a Brownian motion started in $D$ and stopped on hitting $\partial D$. In preparation we introduce a sequence of stopped processes which may be used as an alternative construction of a Fleming-Viot process. Let $\mathbf{X}_t = (X_t^1, \ldots, X_t^N)$ be a Fleming-Viot process in $D$. For each particle $X_t^i$ and $n \in \mathbb{N}$ we name a sequence of stopping times $s_n^i = \inf\{t \geq \tau_n : X_{t-}^i \in \partial D\}$. Notice there is exactly one $j \in \{1, \ldots, N\}$ for which $s_n^j = \tau_n$. For every $i \neq j$, $s_n^i$ is the first time after $\tau_n$ that $X_t^i$ hits $\partial D$. Notice that $s_{n+1}^i = s_n^i$ for every $i \neq j$. Now let $\mathcal{Q}^n$ be the distribution of the stopped process $(X^1_{t\wedge s_n^1}, \ldots, X^N_{t\wedge s_n^N})$. Informally we allow the process to evolve until $\tau_n$, at which point all but one of the particles are in the interior of $D$. After $\tau_n$ the remaining particles continue as independent Brownian motions until they are stopped on exiting $D$. It is easy to construct a process $\mathbf{X}_t^1$ distributed as $\mathcal{Q}^1$ because trivially each particle evolves independently as a Brownian motion stopped on exiting $D$. Now suppose that we have constructed $\mathbf{X}_t^\ell = (X_t^{\ell,1}, \ldots, X_t^{\ell,N})$ distributed as $\mathcal{Q}^\ell$. There is exactly one particle, say $X^{\ell,j}$ such that $s_\ell^j = \tau_\ell$ and $X^{\ell,j}_{\tau_\ell^-} \in \partial D$. For every $i \neq j$ we have $s_{\ell+1}^i = s_\ell^i$. Therefore to construct $\mathcal{Q}^{\ell+1}$ from $\mathcal{Q}^\ell$ we need only to extend the lifetime of $X_t^j$ until $s_{\ell+1}^j$ in accordance with the rules of Fleming-Viot. So choose $\lambda_\ell \in \{1, \ldots, \mathbb{N}\} \setminus \{j\}$ uniformly and independently of $\mathbf{X}^\ell$. Set $x_\ell = X_{\tau_\ell}^{\ell,\lambda_\ell}$ and let $\widetilde{X}_t^\ell$ be a Brownian motion independent of everything else, started at $x_\ell$ and stopped on exiting $D$. We may construct $\mathbf{X}^{\ell+1}$ distributed as $\mathcal{Q}^{\ell+1}$ by setting $X_t^{\ell+1,i} = X_t^{\ell,i}$ whenever $t \leq s_\ell^i$, $i \neq j$, and $X_t^{\ell+1,j} = \widetilde{X}^\ell_{t-\tau_\ell}$ for $t \in [\tau_\ell, s_{\ell+1}^j)$. By an application of the Kolmogorov extension theorem there is a unique process $\mathbf{X}_t$ with $\mathbf{X}_t = \mathbf{X}_t^\ell$ whenever $t < \tau_\ell$ and $\mathbf{X}_t$ agrees in distribution with the construction at the start of this section.

The following lemma shows that if $\tau_\infty < \infty$ then all processes $X_t^1, \dots, X_t^N$ approach $\partial D$ at time $\tau_\infty$. This result does not require any assumptions on the smoothness or regularity of $\partial D$, unlike our main results, so it may have independent interest.

**Lemma 5.2** *Let $R_t = \max_{1 \le i \le N} \operatorname{dist}(X_t^i, \partial D)$. If $D \subset \mathbb{R}^d$ is an open set and $N \ge 2$ then*

$$\mathbb{P}\left(\{\tau_\infty = \inf\{t > 0 : R_{t-} = 0\} < \infty\} \cup \{\tau_\infty = \infty\}\right) = 1. \tag{5.1}$$

*Proof* Let $\Lambda_j$ be the closure of the set $\{t \ge 0 : X_{t-}^j \in \partial D\}$. Suppose that $\tau_\infty < \infty$ with positive probability. Then at least one of the processes $X_t^j$ must have an infinite number of jumps before $\tau_\infty$. For every $j$ with this property we have $\tau_\infty \in \Lambda_j$. We will show that there are no processes $X_t^j$ with only a finite number of jumps before $\tau_\infty$, a.s.

Let $\tau_k^j$ denote the time of the $k$-th jump of $X_t^j$ for $j \in \{1, \dots, N\}$. Let $\widehat{X}_t^j = X_t^j$ for $t \in [0, \tau_1^j)$. Then we define inductively $\widehat{X}_t^j = X_t^j + \widehat{X}_{\tau_k^j -}^j - X_{\tau_k^j}^j$ for $t \in [\tau_k^j, \tau_{k+1}^j)$, $k \ge 1$. It is easy to see that $\{\widehat{X}_t^j, 0 \le t \le \tau_k^j\}$ is a $d$-dimensional Brownian motion for every $k$. Hence, $\{\widehat{X}_t^j, 0 \le t < \tau_\infty\}$ is also a Brownian motion defined on a random time interval. Let $m_j$ be the number of jumps of $X_t^j$ before $\tau_\infty$. Suppose that $\tau_\infty < \infty$ and $m_n = \infty$ for some $n$. Assume that $\limsup_{k\to\infty} \operatorname{dist}(X_{\tau_k^n}^n, \partial D) > r$ for some $r > 0$. Then $\widehat{X}_t^n$ has an infinite number of oscillations of size greater than or equal to $r$ on every time interval of the form $(\tau_\infty - \varepsilon, \tau_\infty)$, for every $\varepsilon > 0$. This implies that $\widehat{X}_t^n$ does not have a left limit at $\tau_\infty$. The last event has zero probability, so we conclude that, for all rational $r > 0$ and all $n \in \{1, \dots, N\}$,

$$\mathbb{P}\left(\{\tau_\infty < \infty\} \cap \{m_n = \infty\} \cap \left\{\limsup_{k\to\infty} \operatorname{dist}(X_{\tau_k^n}^n, \partial D) \ge r\right\}\right) = 0.$$

Hence, for all $n \in \{1, \dots, N\}$,

$$\mathbb{P}\left(\{\tau_\infty < \infty\} \cap \{m_n = \infty\} \cap \left\{\lim_{k\to\infty} \operatorname{dist}(X_{\tau_k^n}^n, \partial D) \ne 0\right\}\right) = 0. \tag{5.2}$$

Next suppose that $m_j < \infty$ for some $j \in \{1, \dots, N\}$. By continuity of Brownian motion, the left limit $X_{\tau_\infty -}^j$ exists on the event $\{\tau_\infty < \infty, m_j < \infty\}$. We will prove that

$$\mathbb{P}\left(\{\tau_\infty < \infty\} \cap \{m_j < \infty\} \cap \left\{X_{\tau_\infty -}^j \notin \partial D\right\}\right) = 0 \tag{5.3}$$

for every $j$. Suppose to the contrary that for some $j$,

$$\mathbb{P}\left(\{\tau_\infty < \infty\} \cap \{m_j < \infty\} \cap \left\{X_{\tau_\infty -}^j \notin \partial D\right\}\right) > 0. \tag{5.4}$$

Then there exists a rational $r > 0$ such that

$$\mathbb{P}\left(\{\tau_\infty < \infty\} \cap \{m_j < \infty\} \cap \left\{X^j_{\tau_\infty -} \in D_r\right\}\right) > 0,$$

where $D_r = \{x \in D : \mathrm{dist}(x, \partial D) > r\}$. Recall that each process $X^n_t$ jumps at times $\tau^n_k$ to the location of another process $X^i$, $i \neq n$, chosen in a uniform way. Let $n$ be such that $m_n = \infty$ (there exists at least one such $n$, a.s.). There exists an infinite subsequence $\{s_k\}_{k\geq 1}$ of $\{\tau^n_k\}_{k\geq 1}$, such that $X^n_{s_k} = X^j_{s_k}$. It follows that, on the event $\{X^j_{\tau_\infty -} \in D_r\}$, it is not true that $\lim_{k\to\infty} \mathrm{dist}(X^n_{\tau^n_k}, \partial D) = 0$. This contradicts (5.2), so we conclude that (5.4) is false. This completes the proof of (5.3). The lemma follows from (5.2) and (5.3). □

*Remark 5.3* The proof of Theorem 5.4, the main result of this paper, is quite complicated so we will outline the proof of a similar result in the 1-dimensional case to help the reader follow the main argument. The structure of the proof in the 1-dimensional case is the same as in the higher dimensional case but there are fewer technical details to deal with. See [26] for a more general argument based on a similar idea.

Let $Y$ be one dimensional Brownian motion starting from $Y_0 = 0$ and $v_0 \geq 0$. It is well known ([19, Sect. 3.6 C]) that, a.s., there exist unique continuous processes $V$ and $L$ such that $V_0 = v_0$ and

$$dV_t = dY_t + dL_s, \quad \text{for } t \geq 0. \tag{5.5}$$

Here $L$ is the local time of $V$ at 0. In other words, $L$ is a non-decreasing continuous process which does not increase when $V$ is 0, i.e., $\int_0^\infty \mathbf{1}_{\{0\}}(V_t)dL_t = 0$, a.s. The process $V$ is called reflected Brownian motion driven by $Y$. The construction of $V$ is based on deterministic Skorokhod lemma [19, Lemma 3.6.14] so we have strong existence and uniqueness for (5.5).

Let $D = (0, \infty)$, $N \geq 2$, let $Y^k, k = 1, \dots, N$, be independent 1-dimensional Brownian motions and $x_1, \dots, x_N \in D$. Let $(V^k, L^k)$ be the solution to (5.5) driven by $Y^k$, with $V^k_0 = x_k$, for $k = 1, \dots, N$.

Let $\tau_1$ be the first time when one of the processes $V^k$ hits 0. Suppose that $V^j_{\tau_1} = 0$. We let $X^k_t = V^k_t$ for $t \in [0, \tau_1]$ and $k \neq j$, and $X^j_t = V^j_t$ for $t \in [0, \tau_1)$. We choose uniformly an integer $M_1$ in the set $\{1, \dots, N\} \setminus \{j\}$ and let $X^j_{\tau_1} = X^{M_1}_{\tau_1}$. Note that $X^k_t \geq V^k_t$ for $t \in [0, \tau_1]$ and all $k$. We proceed by induction. Suppose that $\tau_1, \dots, \tau_n$ have been defined and $\{X^k_t, t \in [0, \tau_n]\}$ have also been defined. Assume that $X^k_t \geq V^k_t$ for $t \in [0, \tau_n]$ and all $k$. Let $\{(V^{k,n}_t, L^{k,n}_t), t \geq \tau_n\}$ be the solution to (5.5) driven by $Y^k$, with $V^{k,n}_{\tau_n} = X^k_{\tau_n}$ for all $k$. Note that $V^{k,n}_{\tau_n} \geq V^k_{\tau_n}$ for all $k$. Then, by the strong uniqueness of solutions to (5.5), we have $V^{k,n}_t \geq V^k_t$ for all $t \geq \tau_n$ and $k$. Let $\tau_{n+1}$ be the first time $t \geq \tau_n$ when one of the processes $\{V^{k,n}_t, t \geq \tau_n\}$ hits 0. Suppose that $V^m_{\tau_{n+1}} = 0$. We let $X^k_t = V^{k,n}_t$ for $t \in (\tau_n, \tau_{n+1}]$ and $k \neq m$, and $X^m_t = V^{m,n}_t$ for $t \in (\tau_n, \tau_{n+1})$. We choose uniformly an integer $M_{n+1}$ in the set $\{1, \dots, N\} \setminus \{m\}$ and let $X^m_{\tau_{n+1}} = X^{M_{n+1}}_{\tau_{n+1}}$.

Let $\tau_\infty = \lim_{n\to\infty} \tau_n$. The process $(X_t^1, \dots, X_t^N)$ is well defined on the interval $(0, \tau_\infty)$. The process $R_t = ((V_t^1)^2 + \cdots + (V_t^N)^2)^{1/2}$ is $N$-dimensional Bessel process and, therefore, it never hits 0. On the event $\{\tau_\infty < \infty\}$ we have $R_{\tau_\infty} > 0$ and, therefore, $\limsup_{t\uparrow\tau_\infty} X_t^k > 0$ for at least one $k$. In view of Lemma 5.2, we conclude that the probability of the event $\{\tau_\infty < \infty\}$ is 0.

The above argument is more complicated in higher dimensions because it is much harder to construct a process which always lies "closer to the boundary" of $D$ than $X^k$ and has a structure that can be easily analyzed. The construction of such a process uses the process $Z$ defined in (4.4).

**Theorem 5.4** *There exists a constant $c = c(N, d)$ such that if $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain with the Lipschitz constant $L < c(N, d)$, then $\tau_\infty = \infty$, a.s. Moreover, $c(N, d)$ increases in $N$, decreases in $d$ and*

$$\lim_{N\to\infty} c(N, d) = c(d) = \frac{1}{\sqrt{d-1}}. \tag{5.6}$$

*Proof Part 1.* We start by defining $c(N, d)$ and some other constants used in the proof. Recall the definition of $\theta_{p,d}$ and $K_{p,d}$. Let $p' = 2 - 2/N$ and $c(N, d) = \cot\theta_{p',d}$, and fix a $p$ such that $L < \cot\theta_{p,d} < c(N, d)$. Recall that $D_r = \{x \in D : \mathrm{dist}(x, \partial D) > r\}$. Since $D$ is bounded and Lipschitz, there exists a small $r > 0$ for which the following is true. For every $x \in D \setminus D_r$ there exist an orthonormal coordinate system $CS_x$, $\mathcal{O}_x \in \partial D$, a Lipschitz function $F_x : \mathbb{R}^{d-1} \to \mathbb{R}$ and a cone $K_x$, such that $\mathcal{O}_x$ is the origin of $CS_x$, $K_x$ has vertex $\mathcal{O}_x$ and axis passing through $x$, $K_x$ can be described in $CS_x$ as $K_{p,d}$, and

$$\begin{aligned}
&D \cap B(\mathcal{O}_x, r) \subset \left\{ y \text{ in } CS_x : y^d > F_x(\widetilde{y}) \right\} \cap B(\mathcal{O}_x, r),\\
&K_{p,d} \cap B(\mathcal{O}_x, r) \subset D \cap B(\mathcal{O}_x, r).
\end{aligned}$$

As we have chosen $\cot\theta_{p,d} > L$, there exists $c_1 = c_1(p, D) > 0$ such that $\mathrm{dist}(y, \partial D) > c_1|y - \mathcal{O}_x|$ for every $y \in K_x$ and $x \in D \setminus D_{2r}$. We set $\mathcal{T}_x$ to be an isometry that maps $K_{p,d}$ onto $K_x$.

*Part 2.* Recall that $R_t = \max_{1\le i\le N} \mathrm{dist}(X_t^i, \partial D)$. The process $R_t$ is continuous on $[0, \tau_\infty)$ because Brownian motion is continuous and $R_t$ does not jump at any stopping time $\tau_k$.

We will split the lifetime of the process $\mathbf{X}_t$ into two phases, a 'safe' phase where $R_t$ is large and an 'unsafe' phase where $R_t$ is small. The main part of the proof will involve a special construction of $\mathbf{X}_t$ in the unsafe phase that ensures the process cannot terminate. The two phases are defined using two sequences of stopping times $V_i^r, \widehat{V}_i^r, i \ge 0$. Fix $r > 0$ as in Part 1 of the proof. Set $V_0^r = 0$ and for $i \ge 0$ let

$$\begin{aligned}
\widehat{V}_i^r &= \inf\{t > V_i^r : R_t \le \tfrac{r}{2}\},\\
V_{i+1}^r &= \inf\{t > \widehat{V}_i^r : R_t \ge r\},
\end{aligned}$$

with the convention that $\inf\emptyset = \infty$.

If for some $i$ we have $\widehat{V}_i^r = \infty$ then $\widehat{V}_j^r = \infty$ and $V_j^r = \infty$ for all $j > i$. Hence, $\lim_{i\to\infty} V_i^r = \infty$. Similarly, if $V_i^r = \infty$ for some $i$ then $\widehat{V}_j^r = \infty$ and $V_{j+1}^r = \infty$ for all $j \ge i$. In this case we also have $\lim_{i\to\infty} V_i^r = \infty$.

Next suppose that $\widehat{V}_i^r < \infty$ and $V_i^r < \infty$ for all $i$.

Recall Brownian motions $\widehat{X}_t^j$ from the proof of Lemma 5.2. On any interval $[V_i^r, \widehat{V}_i^r)$, $i \ge 1$, at least one of the processes $X_t^j$ must travel a distance of $\frac{r}{2}$. Thus, at least one of the processes $\widehat{X}_t^j$ must travel a distance of $\frac{r}{2}$ on this interval. Since Brownian motions $\widehat{X}_t^j$ cannot make infinitely many such oscillations on a finite time interval and the number $N$ of processes $\widehat{X}_t^j$ is finite, we have $\lim_{i\to\infty} V_i^r = \infty$, a.s.

In view of (5.1), the probability that $\tau_\infty < \infty$ and $\tau_\infty \in [V_i^r, \widehat{V}_i^r]$ for some $i$ is zero. So if $\tau_\infty < \infty$ with positive probability then there exists $i \ge 0$ such that $\mathbb{P}(\tau_\infty \in [\widehat{V}_i^r, V_{i+1}^r)) > 0$.

Note that $\widehat{V}_0^r = 0$ if $\mathbf{X}_0 \in (D \setminus D_{r/2})^N$. Suppose that we can show that

$$\mathbb{P}(\tau_\infty \in [\widehat{V}_i^r, V_{i+1}^r)) = 0 \tag{5.7}$$

for $i = 0$ and arbitrary $\mathbf{X}_0 \in (D \setminus D_{r/2})^N$. Then, by the strong Markov property applied at $\widehat{V}_i^r$'s, (5.7) holds for all $i \ge 0$, a.s., and this implies the theorem.

*Part 3*. We present an informal overview of the remaining part of the proof.

Our aim is to construct a coupling of a process $\mathbf{X}_t$ with a vector of independent copies of the excursion process $Z_t$ constructed in such a way that $\operatorname{dist}(X_t^j, \partial D) \ge c_2|Z_t^j|$ for some fixed constant $c_2 = c_2(N, D)$, at least up until the stopping time $V_r^1$.

The construction consists of three consecutive inductive constructions. The first inductive construction generates a coupling of a Brownian motion $Y_t$ in the cone $K_{p,d}$ and a copy of the process $Z_t$. The key point is to couple $Y$ and $Z$ in such a way that $Z$ is at the vertex of $K_{p,d}$ when $Y$ hits the boundary of $K_{p,d}$. In addition the details of the construction give the bound $|Y_t| \ge |Z_t| \cos\theta_{p,d}$.

In the second level of the inductive construction we map processes $Y_t$ from the first level of the construction into $D \setminus D_r$ using the maps $\mathcal{T}_x$ described in Part 1. As $|Z_t| = 0$ when $Y_t$ exits the cone we may concatenate a sequence of such processes to define a Brownian motion $X_t$ stopped on exiting $D \setminus D_r$, coupled with an excursion process $Z_t$. Using estimates in Parts 1 and 4 we may bound $\operatorname{dist}(X_t, \partial D)$ away from zero by $c_2|Z_t|$. If $X_t$ enters $D_r$ then a Fleming Viot process constructed using $X_t$ has survived until $V_r^1$ and is safe. The coupling with $Z_t$ is no longer needed and $X_t$ is allowed to continue independently of $Z_t$ until exiting $D$.

The second level of the construction produces a Brownian motion started anywhere in $D \setminus D_r$ coupled appropriately with a copy of $Z_t$. For the third and last level we follow the construction in Remark 5.1 using the products of the second level as building blocks. If a particle $X_t^j$ exits $D$ at time $\tau_n < V_r^1$ its sister process $Z_t^i$ must be at the origin at time $\tau_i$ and we may extend the lifetime of the particle using the construction in Part 5. The resulting construction gives us the desired coupling of $\mathbf{X}_t$ with $N$ independent copies of $Z_t$ until at least $V_r^1$. After which the processes are allowed to decouple to give us the full process $\mathbf{X}_t$ as required.

Combining this coupling with Corollary 4.5, we may bound $R_t$ away from zero up until time $V_r^1$. Therefore the process cannot terminate before $V_r^1$ and we must have $\tau_\infty = \infty$ by observations in Part 2.

*Part 4.* We now present a detailed description of the first level of construction. For $y^d > 0$ we construct a coupling of the excursion process $Z_t$ with a Brownian motion $Y_t$, started at $Y_0 = y = (0, 0, \ldots, y^d)$ and stopped on exiting the cone $K_{p,d}$. Let $Y_t$ be a Brownian motion started at $Y_0 = y$ and consider the moving cone $C_t = Y_t - K_{p,d}$. Then $0 \in C_t$ as long as $Y_t \in K_{p,d}$ so we may let $Y_t$ and $Z_t$ evolve independently until the first time $Z_t$ hits $\partial C_t$. At this point $Z_t$ 'sticks' to $\partial C_t$ and $Y$ and $Z$ evolve together with $dZ_t = dY_t$ until the next time $Z_t$ exits $K_{p,d}$, after which $Z_t$ starts again from zero evolving independently from $Y_t$. The process is repeated until $Y_t$ exits $K_{p,d}$.

More formally, let $W_t$ be a $d$-dimensional Brownian motion started at the origin and independent of $Z_t$ and define $Y_t$ inductively through two sequences of stopping times $\xi_i, \xi_i'$ with $\xi_0 = 0$ and

$$\begin{aligned} \xi_i' &= \inf\{t > \xi_i \ : \ Y_t - Z_t \in \partial K_{p,d}\}, \\ \xi_{i+1} &= \inf\{t > \xi_i \ : \ Z_t = 0\}. \end{aligned}$$

Let $Y_0 = y = (0, 0, \ldots, y^d)$ and

$$\begin{aligned} Y_t &= Y_{\xi_i} + (W_t - W_{\xi_i}) \quad \text{for } \ t \in [\xi_i, \xi_i'), \\ Y_t &= Y_{\xi_i'} + (Z_t - Z_{\xi_i'}) \quad \text{for } \ t \in [\xi_i', \xi_{i+1}). \end{aligned}$$

We stop the process $Y_t$ at $\zeta = \inf\{t \ : \ Y_t \in \partial K_{p,d}\}$. An induction on $i$ shows that $Y_t$ is well defined and adapted to $(Z_t, W_t)$ up until any stopping time of the form $\zeta \wedge \xi_i$.

By the strong Markov property of $(W, Z)$, $Y_t$ is a Brownian motion on all intervals $[\xi_i, \xi_i')$ and $[\xi_i', \xi_{i+1})$. Notice that each time $\xi_i'$ corresponds to a separate excursion of $Z_t$, and as $Y_{\xi_i'} - Z_{\xi_i'} \in \partial K_{p,d}$ we must have $|Z_{\xi_i'}| \geq \operatorname{dist}(Y_{\xi_i'}, \partial K_{p,d})$. The probability that Brownian motion hits the vertex of $K_{p,d}$ is zero. For any $a > 0$ the excursion law $H^0$ is finite on the set of excursions that exit $B(0, a)$, so as $Y_t$ is continuous there may be only finitely many times $\xi_i' < \zeta - \varepsilon$, for any fixed $\varepsilon > 0$. All these observations imply that $Y_t$ is well defined and a Brownian motion on $[0, \zeta]$. Furthermore, by construction, we have $Y_t \in Z_t + \overline{K_{p,d}}$ for all $t \in [0, \zeta]$ and so we must have $Z_\zeta = 0$. Furthermore $Z_t \in \overline{K_{p,d}}$. Hence by considering the projections of $Y_t$ and $Z_t$ onto the axis of the cone it is easy to see that $|Y_t| \geq |Z_t| \cos \theta_{p,d}$.

With probability 1, $\{Y_t, 0 \leq t \leq \zeta\}$ and $\zeta$ are unique functions of $y$ and $\{(W_t, Z_t), 0 \leq t \leq \zeta\}$. We will denote these functions $Y_t = \mathcal{U}_y(W_t, Z_t)$ and $\zeta = \zeta_y(W_t, Z_t)$ respectively.

*Part 5.* We construct a Brownian motion $X_t$, stopped on exiting $D$ by concatenating processes constructed in Part 4. To guide the construction we name some stopping times of $X_t$.

First set $\zeta_0 = 0$ and choose $x_0 \in D \setminus D_r$ arbitrarily. Then if $X_t$ is a Brownian motion started at $x_0$ we may set

$$\tau = \inf\{t \ : \ X_{t^-} \in \partial D\}, \quad \upsilon = \inf\{t \ : \ X_{t^-} \in \partial(D \setminus D_r)\}.$$

Next recall the definitions of $K_x$ and $\mathcal{O}_x$ from Part 1. Define a sequence of stopping times $\zeta_n \leq v$ inductively by setting $\zeta_{n+1} = v \wedge \inf\{t > \zeta_n : X_t \in \partial K_{x_n}\}$ where $x_n = X_{\zeta_n}$. Consider the random sequence of cones $K_{x_n}$. At each time $\zeta_n < v$ the particle $X_t$ is on the axis of the cone $K_{x_n}$ and on the boundary of the cone $K_{x_{n-1}}$. Hence we have split the lifetime of the process into a sequence of Brownian motions in cones isomorphic to $K_{p,d}$. We construct the Brownian motion $X_t$ by mapping processes $Y_t^n$ constructed as in Part 4 into the random cones $K_{x_n}$ to form a continuous process in $D \setminus D_r$. First we must argue that this construction partitions the entire lifetime of a Brownian motion in $D \setminus D_r$. That is the sequence of times $\zeta_n$ converges to $\tau \wedge v$, the first exit time of $D \setminus D_r$.

By definition $\zeta_n \leq v$, notice also that if $\zeta_n < \tau \wedge v$, then $X_t \in K_{x_n} \cap D \setminus D_r$ for $t \in [\zeta_n, \zeta_{n+1})$. As the probability that $X_t$ hits the vertex $\mathcal{O}_{x_n}$ is zero and from Part 1 $\operatorname{dist}(X_t, \partial D) \geq c_1|X_t - \mathcal{O}_{x_n}|$ in that interval we must have $\zeta_{n+1} < \tau$. So as $\tau$ is finite we may set $\zeta_\infty = \lim_{n\to\infty} \zeta_n$. By continuity of Brownian motion we have $x_n \to X_{\zeta_\infty}$ as $n \to \infty$. We have $|x_n - x_{n+1}| \geq |x_n - \mathcal{O}_{x_n}| \sin\theta_{p,d}$ whenever $\zeta_{n+1} < v$. As in addition $|x_n - \mathcal{O}_{x_n}| \geq \operatorname{dist}(x_n, \partial D)$, the sequence $x_n$ cannot converge to any point in $D$ and we have $\zeta_\infty = \tau \wedge v$ with probability one.

Let $(W_t, Z_t)$ be as in Part 4. Our aim is to construct a Brownian motion $X_t$ in such a way that $X_t$ is adapted to $(W_t, Z_t)$ and there exists a constant $c_2 = c_2(N, D)$ such that $\operatorname{dist}(X_t, \partial D) \geq c_2|Z_t|$.

Suppose we have constructed $X_t$ satisfying the above on the interval $[0, \zeta_n)$ with $\zeta_n < v$ and $Z_{\zeta_n} = 0$. This is trivial for $n = 0$. As $X_t$ is adapted to $(W_t, Z_t)$ and $Z_{\zeta_n} = 0$ we may set $W_t^n = W_{\zeta_n+t} - W_{\zeta_n}$ and $Z_t^n = Z_{\zeta_n+t}$. The pair $(W_t^n, Z_t^n)$ agrees in distribution with $(W_t, Z_t)$ and by the strong Markov property is independent of $X_{t\wedge\zeta_n}$.

Recall the maps $\mathcal{T}_x$ from Part 1. As $\zeta_n < v$ we have $x_n \in D \setminus D_r$ and we may set $y_n = \mathcal{T}_{x_n}^{-1}$. Now using the construction in Part 4 set $Y_t^n = \mathcal{U}_{y_n}(W_t^n, Z_t^n)$, $\zeta_n^\star = \zeta_{y_n}(W_t^n, Z_t^n)$ and map the process into $D$ by setting $\widetilde{X}_t^n = \mathcal{T}_{x_n}(Y_t^n)$. If $\widetilde{X}_t^n$ hits $D_r$ at some time $v' < \zeta_n^\star$ during its lifetime we set $\zeta_{n+1} = v = \zeta_n + v'$, if not we set $\zeta_{n+1} = \zeta_n + \zeta_n^\star$.

By construction $X_t^n$ is a Brownian motion started at $x_n$ and stopped on exiting $K_{x_n} \cap D \setminus D_r$ and so $\operatorname{dist}(X_t^n, \partial D) > c_1|Y_t^n|$. Recall from Part 4 that $|Y_t^n| \geq |Z_t^n| \cos\theta_{p,d}$ for $t < \zeta_n^\star$. So, let $c_2 = c_1 \cos\theta_{p,d}$ and set $X_t = X_{t-\zeta_n}^n$ for all $t \in [\zeta_n, \zeta_{n+1})$. Then $X_t$ is a concatenation of two independent Brownian motions and is a Brownian motion. Furthermore we have $\operatorname{dist}(\widetilde{X}_t^n, \partial D) \geq c_2|Z_t|$ on the interval $[\zeta_n, \zeta_{n+1})$ as required.

Now if $\zeta_{n+1} < v$ then $Z_{\zeta_{n+1}} = Z_{\zeta_n^\star}^n = 0$ by construction and we may repeat the inductive step. If $\zeta_{n+1} = v$ we stop the construction. In this case we have constructed a Brownian motion until time $v$ with $\operatorname{dist}(X_t, \partial D) \geq c_2|Z_t|$, so we need only continue $X_t$ until time $\tau$ in such a way that it is adapted to $(W_t, Z_t)$. We achieve this by setting $X_t = X_v + W_t - W_v$ on the interval $[v, \tau]$.

If the construction does not terminate then $X_t$ is a Brownian motion satisfying the above conditions until any stopping time $\zeta_n$. Therefore we may take the limit as $n \to \infty$ and $X_t$ is a Brownian motion up until $\zeta_\infty$. Arguing as above we have $\zeta_\infty = \tau$.

So we have constructed a process $X_t$ stopped on exiting $D$ at time $\tau$. As before, with probability 1, we may express $X$ and $\tau$ as functions $\{X_t, t \geq 0\} = \mathcal{V}_{x_0}(\{W_t,$

$t \geq 0\}, \{Z_t, t \geq 0\})$ and $\tau = \tau_{x_0}(\{W_t, t \geq 0\}, \{Z_t, t \geq 0\})$. We will use the following abbreviations, $X = \mathcal{V}_{x_0}(W, Z)$ and $\tau = \tau_{x_0}(W, Z)$.

*Part 6*. In the final stage we construct a coupling of a Fleming-Viot process $\mathbf{X}_t$ with a vector of independent excursion processes. Let $Z_t^1, \ldots, Z_t^N$ be independent copies of the excursion process $Z_t$ and let $W_t^1, \ldots, W_t^N$ be independent Brownian motions. We use each pair $(W_t^i, Z_t^i)$ as driving noise for a particle $X_t^i$ using the constructions in Parts 4 and 5.

Recall the stopping times $s_n^i$ and the distributions $\mathcal{Q}^n$ of stopped processes from Remark 5.1. Our strategy is to follow the method outlined in Remark 5.1 and construct a sequence of processes $\mathbf{X}_t^n = (X_t^{n,1}, \ldots, X_t^{n,N})$ with the following properties.

- $\mathbf{X}_t^n$ is distributed as $\mathcal{Q}^n$ and adapted to $\mathcal{F}_t$.
- $\mathrm{dist}(X_t^{n,j}, \partial D) \geq c_2|Z_t^j|$ for every $j \in 1, \ldots, \mathbb{N}$ and $t < V_r^1$.
- The sequence of processes is coherent in the sense that with probability 1 for every $m > n$ and $t < \tau_n$ we have $\mathbf{X}_t^n = \mathbf{X}_t^m$.

It is easy to construct a process $\mathbf{X}_t^1$ satisfying the above as $\mathcal{Q}^1$ is just the distribution of $N$ independent stopped Brownian motions. For any starting vector $\mathbf{X}_0 = (x_0^1, \ldots, x_0^N) \in (D \setminus D_r)^N$ set $X^{1,i} = \mathcal{V}_{x_0^i}(W^i, Z^i)$ for each $i \in \{1, \ldots, N\}$ and $t < s_1^i = \tau_{x_0^i}(W^i, Z^i)$. As the pairs $(W_t^i, Z_t^i)$ are independent and distributed as $(W_t, Z_t)$ in Part 5 the Brownian motions $X_t^{1,i}$ are independent and satisfy the required bound on $\mathrm{dist}(X_t^i, \partial D)$.

Now suppose after $\ell$ inductive steps we have constructed processes $\mathbf{X}_t^1, \ldots, \mathbf{X}_t^\ell$ satisfying the three conditions above. Suppose further that for each $j \in \{0, \ldots, N\}$ the shifted processes $Z_t^{j,\ell} = Z^j_{t+s_\ell^j}$ and $W_t^{j,\ell} = W^j_{t+s_\ell^j} - W^j_{s_\ell^j}$ are independent of $\mathbf{X}^\ell$. This fact is easy to check for $\ell = 1$.

Recall from Remark 5.1 that there is exactly one particle, say $X_t^{\ell,j}$ such that $s_\ell^j = \tau_\ell$. To construct $\mathbf{X}^{\ell+1}$ we must extend the lifetime of $X_t^{\ell,j}$ by adding an independent Brownian motion with an appropriately chosen starting position.

So as in Remark 5.1, set $X_t^{\ell+1,i} = X_t^{\ell,i}$ whenever $t < s_\ell^i$. Next choose, $\lambda_{\ell+1}$ uniformly from $\{1, \ldots, N\} \setminus \{j\}$ and independent of every other random variable. Then the particle $X^{\ell+1,j}$ will jump to $x_\ell = X_{\tau_\ell}^{\ell,\lambda_\ell}$ at time $\tau_\ell$.

If $\tau_\ell < V_r^1$ then we must have $Z_{\tau_\ell}^j = 0$ and $x_\ell \in D \setminus D_r$ so we may set $s_{\ell+1}^j = s_\ell^j + \tau_{x_\ell}(W^{\ell,j}, Z^{\ell,j})$ and $X^{\ell+1,j} = \mathcal{V}_{x_\ell}(W^{\ell,j}, Z^{\ell,j})$ for $t \in [s_\ell^j, s_{\ell+1}^j)$. By construction we have $\mathrm{dist}(X_t^{\ell+1,j}, \partial D) \geq c_2|Z_t^j|$ on the interval $[\tau_j, s_{\ell+1}^j \wedge v)$.

Alternatively if $\tau_\ell > V_r^1$ we do not need to couple $X_t^{\ell+1,j}$ with $Z_t^j$ after time $\tau_\ell$ so we may set $X_t^{\ell+1,j} = x_\ell + W_t^j - W^j_{s_\ell^j}$ until $s_{\ell+1}^j$, the next time $X_t^{\ell+1,j}$ exits $D$.

By assumption the pair $(W_t^\ell, Z_t^\ell)$ is independent of $\mathbf{X}_t^\ell$. In both cases above $X^{\ell+1,j}_{t-s_\ell^j}$ is a Brownian motion adapted to $(W_t^\ell, Z_t^\ell)$. Arguing as in Remark 5.1, $\mathbf{X}_t^{\ell+1}$ is distributed as $\mathcal{Q}^{\ell+1}$ and is adapted to $\mathcal{F}_t$. The independence assumption for $\mathbf{X}^{\ell+1}$ follows from the strong Markov property.

Now extend the sequence $\mathbf{X}_t^n$ to a Fleming-Viot process by setting $\mathbf{X}_t = \mathbf{X}_t^n$ whenever $t < \tau_n$.

Suppose there exists a finite deterministic time $T$ with $\mathbb{P}(1/T < \tau_\infty < T) > 0$. Let $A = \{1/T < \tau_\infty < T\}$. Then from Lemma 5.2 we must have $R_{\tau_n} \to 0$ as $n \to \infty$, on $A$. But from Corollary 4.5 the maximum process $\max_{i \in \{1,\dots,N\}} |Z_t^i|$ is bounded below on $[1/T, T]$ by a strictly positive random variable, a.s. Therefore we must have $\mathrm{dist}(X_{\tau_n}^i, \partial D) < c_2 |Z_t^i|$ for all but finitely many $n$, for all $i$, assuming $A$ holds. According to our construction, this is impossible unless $\tau_\infty > V_r^1$ on $A$. Letting $T \to \infty$, we obtain $\mathbb{P}(\tau_\infty > V_r^1) = 1$. Part 2 now implies that we must have $\mathbb{P}(\tau_\infty < \infty) = 0$. □

*Remark 5.5* Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain with the Lipschitz constant $L < \frac{1}{\sqrt{d-1}}$. Then by (5.6) we see that there exists $N_0$ so large that $L < c(N, d)$ for all $N \geq N_0$. In consequence, the Fleming-Viot-type particle process $\mathbf{X}_t$ in $D$ is well defined for all $t \geq 0$ provided it consists of $N$ particles with $N \geq N_0$.

*Remark 5.6* We will argue that Theorems 1.3 and 1.4 in [10] hold true even though we do not know whether Theorem 1.1 in that paper is true.

Theorem 1.3 is concerned with a fixed time $t > 0$. A sequence of processes $\mathbf{X}^N$ is considered and it is assumed that the initial empirical distributions $(1/N)\sum_{k=1}^N \delta_{X_0^k}$ converge weakly to a probability measure $\mu_0$ in $D$. Let $A$ be a compact subset of $D$ such that $\mu_0(A) > 0$. Since the distance $r$ from $A$ to $\partial D$ is strictly positive and Brownian motion can stay in the ball of radius $r/2$ for time $t$ with a strictly positive probability $p_1$, it follows that if $j$ Brownian motions start from points in $A$ then with probability equal to or greater than $1-(1-p_1)^j$ at least one of these Brownian motions never comes closer to $\partial D$ than $r/2$ units on the interval $[0, t]$. Fix an arbitrarily small $p_2$ and let $j$ be so large that $(1 - p_1)^j < p_2/2$. Let $N_0$ be so large that for $N \geq N_0$, the probability that there are at least $j$ processes $X_0^k$ in $A$ is equal to or greater than $1-p_2/2$. Then with probability equal to or greater than $1-p_2$ there exists a process $X^k$ which never approaches $\partial D$ closer than $r/2$ units on the interval $[0, t]$. By Lemma 5.2, this implies that $\tau_\infty \geq t$ with probability equal to or greater than $1 - p_2$. Hence, the empirical distribution $(1/N)\sum_{k=1}^N \delta_{X_t^k}$ is well defined with probability equal to or greater than $1 - p_2$. When the empirical distribution of $\mathbf{X}_t$ is not well defined at time $t$, we can define it arbitrarily to be the atom at $(x_0, x_0, \dots, x_0)$ for some $x_0 \in D$. Theorem 1.3 of [10] makes an assertion about convergence of $(1/N)\sum_{k=1}^N \delta_{X_t^k}$ in probability. Since $p_2 > 0$ is arbitrarily small, the proof given in [10] and the remarks given above show that Theorem 1.3 is true.

We will now discuss Theorems 1.4 in [10]. That theorem is concerned with domains which satisfy the internal ball condition with radius $r > 0$. The family of such domains is not contained in the class of Lipschitz domains and neither does it contain all Lipschitz domains. To see this, consider a square which is a Lipschitz domain but does not satisfy the internal ball condition. A two-dimensional example illustrating the opposite claim is $D = B((10, 0), 10) \cup B((0, 1), 1) \cup B((0, -1), 1)$. This example shows that Theorem 5.4 of the present paper cannot be applied to some domains satisfying the internal ball condition. On page 698 of [10] it is shown that $\mathrm{dist}(X_t^k, D^c) \geq r - R_t^k$, where $R^k$'s are independent $d$-dimensional Bessel processes reflected at $r$. A claim is made in [10] that this relation holds for all finite $t$, based on Theorem 1.1. Although Theorem 1.1 has incorrect proof, the argument given in the proof of

Theorem 1.4 does show that $\text{dist}(X_t^k, D^c) \geq r - R_t^k$ holds for all $t < \tau_\infty$. The process $\Gamma_t = ((r - R_t^1)^2 + \cdots + (r - R_t^N)^2)^{1/2}$ has the distribution absolutely continuous with respect to the distribution of $N$-dimensional Bessel process on every finite time interval, by the Girsanov theorem. Hence $\Gamma_t$ does not hit 0 at any finite time. We now reason as in Remark 5.3. On the event $\{\tau_\infty < \infty\}$ we have $\Gamma_{\tau_\infty} > 0$ and, therefore, $\limsup_{t\uparrow\tau_\infty} \text{dist}(X_t^k, D^c) > 0$ for at least one $k$. In view of Lemma 5.2, we conclude that the probability of the event $\{\tau_\infty < \infty\}$ is 0. This shows that the process $\mathbf{X}_t$ is well defined for all $t$ under assumptions of Theorem 1.4 in [10] and, therefore, Theorem 1.4 is true.

*Example 5.7* The proof of Theorem 1.1 in [10] contains an error. Formula (2.1) in [10] does not follow "by induction" from the previous statement. We will show that the error is irreparable in the following sense. The proof of Theorem 1.1 in [10] is based only on two properties of Brownian motion—the strong Markov property and the fact the the hitting time distribution of a compact set has no atoms (assuming that the starting point lies outside the set). Hence, if some version of that argument were true, it would apply to almost all non-trivial examples of Markov processes with continuous time, and in particular to all diffusions. However we may find a diffusion for which the analogue of Theorem 1.1 in [10] is false. Let $X_t$ be the diffusion on $[0, \infty)$, started at $X_0 = 1$ and satisfying the SDE

$$dX_t = dW_t - \frac{5}{2X_t}dt.$$

We make 0 absorbing so that it can play the role of the boundary for the domain $D = (0, \infty)$. Notice that although $X_t$ is not a Bessel process, as we have reversed the drift term, it scales in the same way. That is, for $\alpha > 0, \alpha X_{t\alpha^{-2}}$ is a diffusion satisfying the same SDE, but started at $\alpha$. Let $\mathbf{Y}_t^i = (Y_t^{i,1}, Y_t^{i,2})$, $i = 1 \ldots \infty$, be a double sequence of independent copies of $X_t$, and set

$$\sigma_i = \inf\{t > 0 : Y_t^{i,1} \wedge Y_t^{i,2} = 0\},$$
$$\alpha_i = Y_{\sigma_i}^{i,1} \vee Y_{\sigma_i}^{i,2}.$$

Now, construct a two-particle Fleming-Viot type process $\mathbf{X}_t = (X_t^1, X_t^2)$ as follows. First let $\tau_1 = \sigma_1$ and set $\mathbf{X}_t = \mathbf{Y}_t^1$ for $t \in [0, \tau_1)$. At $\tau_1$ one of the particles hits the boundary and jumps to $\xi_1 = \alpha_1$. To continue the process we use the scaling property of $\mathbf{Y}_t$ and set $\mathbf{X}_t = \xi_1 \mathbf{Y}^2_{(t-\tau_1)\xi_1^{-2}}$ for $t \in [\tau_1, \tau_2)$ where $\tau_2 = \tau_1 + {\xi_1}^2\sigma_2$. At $\tau_2$ a second particle hits the boundary and jumps, this time to $\xi_2 = \alpha_2\xi_1$, and we continue the process in the same way by setting

$$\xi_i = \prod_{j=1}^{i} \alpha_j, \quad \tau_i = \sum_{j=1}^{i} {\xi_{j-1}}^2\sigma_j,$$
$$\mathbf{X}_t = \xi_i \mathbf{Y}^i_{(t-\tau_i)\xi_i^{-2}}, \quad \text{for } t \in [\tau_i, \tau_{i+1}).$$

Then $\mathbf{X}_t$ evolves as two independent copies of $X_t$ with Fleming-Viot type jumps when a particle hits the boundary. The process $\mathbf{X}_t$ is well defined up until $\tau_\infty$ and if the analogue of [10, Theorem 1.1] were to hold for this process we would have $\tau_\infty = \infty$ almost surely. In fact the opposite is true. We will show now that $\mathbb{E}\tau_\infty < \infty$ and hence $\tau_\infty < \infty$ almost surely. To do this it will be sufficient to show $\mathbb{E}({\alpha_1}^2) < 1$ and $\mathbb{E}\sigma_1 < \infty$. Let $f(x, y) = x^4 + y^4 - x^2y^2$ and notice $f(x, x) = f(x, 0) = f(0, x) = x^4$. We may check using Ito's formula that $f(Y^{i,1}_{t\wedge\sigma_i}, Y^{i,2}_{t\wedge\sigma_i})$ is a positive local martingale and hence a supermartingale. By the optional stopping theorem

$$\mathbb{E}\left({\alpha_1}^4\right) = \mathbb{E}f\left(Y^{1,1}_{\sigma_1}, Y^{1,2}_{\sigma_1}\right) \le \mathbb{E}f\left(Y^{1,1}_0, Y^{1,2}_0\right) = 1.$$

Furthermore, $\alpha_1$ is not almost surely constant and so by Jensen's inequality

$$\mathbb{E}({\alpha_1}^2) < \sqrt{\mathbb{E}({\alpha_1}^4)} = 1.$$

We may use Ito's formula again to show that ${X_t}^2 + 4t$ is a local martingale and so by the optional stopping theorem again we have that $\mathbb{E}(\sigma_1) \le \frac{1}{4}$.

By independence of the $\mathbf{Y}^i$ processes we have that $\mathbb{E}({\xi_i}^2) = \mathbb{E}({\alpha_1}^2)^i$ and so

$$\mathbb{E}\tau_\infty = \sum_{j=1}^{\infty} \mathbb{E}\left({\xi_{j-1}}^2\sigma_j\right) \le \frac{1}{4}\sum_{j=0}^{\infty} \mathbb{E}\left({\alpha_1}^2\right)^j < \infty.$$

## 6 Hitting probabilities of compact sets

This section is devoted to a technical estimate needed in the proof of Theorem 7.1. Recall definitions of $D_r$ and $\mathbf{X}_t = (X^1_t, \ldots, X^N_t)$.

**Lemma 6.1** *Fix $N \ge 2$ and let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain with the Lipschitz constant $L < c(N, d)$.*

(i) *For any fixed $k \in \{1, \ldots, N\}$, and for every $r > 0$ such that* $\mathrm{Int} D_r \ne \emptyset$, *there exist $c > 0$ and $t > 0$ such that for all $\mathbf{x} \in D^N$,*

$$\mathbb{P}^{\mathbf{x}}\left(X^k_t \in D_r\right) \ge c.$$

(ii) *For every $r > 0$ such that* $\mathrm{Int} D_r \ne \emptyset$, *there exist $c > 0$ and $t > 0$ such that for all $\mathbf{x} \in D^N$,*

$$\mathbb{P}^{\mathbf{x}}\left(\mathbf{X}_t \in D^N_r\right) \ge c.$$

*Proof* (i) Fix $r > 0$ such that $\mathrm{Int} D_r \ne \emptyset$. Recall that notation such as $T_{D_r}$, $T_{\partial D}$, etc. refers to hitting times by Brownian motion. By Theorem 3.1 there exists $c_0 = c_0(r)$ such that for all $x \in D$,

$$\mathbb{P}^x\left(T_{D_r} < T_{\partial D}\right) \ge c_0 \mathbb{E}^x T_{\partial D}. \tag{6.1}$$

Fix $k$ and let $T_{D_r}^{X^k} = \inf\left\{t \geq 0 : X_t^k \in D_r\right\}$, and

$$Y_t = X^k(t \wedge T_{D_r}^{X^k}).$$

Define $T_0 = 0$ and

$$T_{n+1} = \inf\left\{t > T_n : \lim_{s \to t^-} Y_s \in \partial D\right\} \wedge T_{D_r}^{X^k}.$$

Let $M_0 = 0$ and

$$M_n = \frac{1}{c_0}\mathbf{1}_{\{Y(T_n) \in D_r\}} - T_n, \quad n \geq 1,$$

and

$$\mathcal{F}_n = \sigma(\mathbf{X}_t, t \leq T_n).$$

It is easy to see that $ET_n < \infty$ so $E|M_n| < \infty$. For $\mathbf{x} = (x_1, x_2, \ldots, x_N) \in D^N$ with $x_k \notin D_r$,

$$\begin{aligned}
\mathbb{E}^{\mathbf{x}}\left(M_{n+1} - M_n \mid \mathcal{F}_n\right) &= \frac{1}{c_0}\mathbb{E}^{\mathbf{x}}\Big(\mathbf{1}_{\{Y(T_{n+1}) \in D_r\}}(\mathbf{1}_{\{Y(T_n) \notin D_r\}} + \mathbf{1}_{\{Y(T_n) \in D_r\}}) \\
&\quad -\mathbf{1}_{\{Y(T_n) \in D_r\}} \mid \mathcal{F}_n\Big) - \mathbb{E}^{\mathbf{x}}\left(T_{n+1} - T_n \mid \mathcal{F}_n\right) \\
&= \frac{1}{c_0}\mathbb{E}^{\mathbf{x}}\Big(\mathbf{1}_{\{Y(T_{n+1}) \in D_r\}}\mathbf{1}_{\{Y(T_n) \notin D_r\}} \\
&\quad +\mathbf{1}_{\{Y(T_n) \in D_r\}} - \mathbf{1}_{\{Y(T_n) \in D_r\}} \mid \mathcal{F}_n\Big) - \mathbb{E}^{\mathbf{x}}\left(T_{n+1} - T_n \mid \mathcal{F}_n\right) \\
&= \frac{1}{c_0}\mathbf{1}_{\{Y(T_n) \notin D_r\}}\mathbb{P}^{\mathbf{x}}\left(Y(T_{n+1}) \in D_r \mid \mathcal{F}_n\right) \\
&\quad -\mathbb{E}^{\mathbf{x}}\left(T_{n+1} - T_n \mid \mathcal{F}_n\right).
\end{aligned}$$

We have on the event $\{Y(T_n) \notin D_r\}$,

$$\mathbb{E}^{\mathbf{x}}\left(M_{n+1} - M_n \mid \mathcal{F}_n\right) \geq \frac{1}{c_0}\mathbb{P}^{X^k(T_n)}\left(T(D_r) < T_{\partial D}\right) - \mathbb{E}^{X^k(T_n)}T_{\partial D} \geq 0,$$

by (6.1). On the event $\{Y(T_n) \in D_r\}$, we have $T_{n+1} = T_n$, $Y_{T_{n+1}} \in D_r$, and so

$$\mathbb{E}^{\mathbf{x}}(M_{n+1} - M_n \mid \mathcal{F}_n) = 0.$$

Combining the last two formulas, we conclude that $\{M_n\}$ is a submartingale with respect to $\{\mathcal{F}_n\}$.

Define

$$S = \inf\left\{j : T_j \geq 1\right\} \wedge \inf\left\{j : Y_{T_j} \in D_r\right\}.$$

Fix an $\mathbf{x} \in D^N$ and consider two cases. First, we may have

$$\mathbb{P}^{\mathbf{x}}\left(S = \inf\left\{j : Y_{T_j} \in D_r\right\}\right) \geq 1/2.$$

In this case,

$$\mathbb{P}^{\mathbf{x}}(T_{D_r}^{X^k} \leq 1) \geq 1/2. \tag{6.2}$$

The second case is when

$$\mathbb{P}^{\mathbf{x}}\left(S = \inf\left\{j : Y_{T_j} \in D_r\right\}\right) < 1/2.$$

In this case, $\mathbb{P}^{\mathbf{x}}(S \geq 1) \geq 1/2$, so $\mathbb{E}^{\mathbf{x}} T_S \geq 1/2$. The submartingale $M_n$ is bounded above by $1/c_0$ so we can apply the optional stopping theorem to obtain

$$\mathbb{E}^{\mathbf{x}} M_S \geq \mathbb{E}^{\mathbf{x}} M_0 = 0.$$

Hence

$$\mathbb{P}^{\mathbf{x}}\left(Y_{T_S} \in D_r\right) \geq c_0 \mathbb{E}^{\mathbf{x}} T_S \geq c_0/2. \tag{6.3}$$

We will show that for some $t_0$,

$$\mathbb{P}^{\mathbf{x}}\left(T_{D_r}^{X^k} \leq t_0\right) \geq c_0/4. \tag{6.4}$$

If $T_S > s_0$ for some $s_0 > 1$ then $X_t^k$ must not hit $D_r \cup \partial D$ for $t \in (1, s_0)$. The probability of this event is bounded above by the probability of the event that Brownian motion starting from $X_1^k$ will not leave the ball $B(X_1^k, 2\,\mathrm{diam}(D))$ for $s_0 - 1$ units of time. The last probability is $c_1 < 1$, depending on $s_0 > 1$, but not depending on $X_1^k$. By the Markov property,

$$\sup_{\mathbf{x} \in D^N} \mathbb{P}^{\mathbf{x}}\left(T_S > s_0\right) \leq c_1 < 1.$$

Applying the Markov property repeatedly at times $s_0, 2s_0, \ldots,$ we obtain for any $\mathbf{x} \in D^N$,

$$\mathbb{P}^{\mathbf{x}}(T_S > n s_0) \leq c_1^n.$$

We choose $n$ so large that $c_1^n \leq c_0/4$ and let $t_0 = n s_0$. Then for $\mathbf{x} \in D^N$,

$$\mathbb{P}^{\mathbf{x}}(T_S > t_0) \leq c_0/4. \tag{6.5}$$

We use (6.3) and (6.5) to see that

$$
\begin{aligned}
c_0/2 &\le \mathbb{P}^{\mathbf{x}}\left(Y_{T_S} \in D_r\right) \\
&= \mathbb{P}^{\mathbf{x}}\left(Y_{T_S} \in D_r, T_S > t_0\right) + \mathbb{P}^{\mathbf{x}}\left(Y_{T_S} \in D_r, T_S \le t_0\right) \\
&\le \mathbb{P}^{\mathbf{x}}(T_S > t_0) + \mathbb{P}^{\mathbf{x}}\left(T^{X^k}_{D_r} \le t_0\right) \\
&\le c_0/4 + \mathbb{P}^{\mathbf{x}}\left(T^{X^k}_{D_r} \le t_0\right).
\end{aligned}
$$

This implies (6.4). We combine the two cases, that is, (6.2) and (6.4), to see that for some $t_1 < \infty$ and $c_2$, for all $\mathbf{x} \in D^N$,

$$\mathbb{P}^{\mathbf{x}}\left(T^{X^k}_{D_r} \le t_1\right) \ge c_2. \tag{6.6}$$

Let $r_1$ be such that $0 < r < r_1$ and $\mathrm{Int} D_{r_1} \ne \emptyset$. Let $t_2$ and $c_3$ be such that (6.6) holds with $r_1, t_2$ and $c_3$ in place of $r, t_1$ and $c_2$, i.e.,

$$\mathbb{P}^{\mathbf{x}}\left(T^{X^k}_{D_{r_1}} \le t_2\right) \ge c_3. \tag{6.7}$$

Let $r_2 = (r_1 - r)/2$ and $p_1 = \mathbb{P}^0(T_{\partial B(0,r_2)} \ge t_2) > 0$. By translation invariance of Brownian motion, $p_1 = \mathbb{P}^y(T_{\partial B(y,r_2)} \ge t_2)$ for every $y$. If the process $X^k$ hits $D_{r_1}$ before time $t_2$ and then stays in the ball $B(X^k(T^{X^k}_{D_{r_1}}), r_2)$ for at least $t_2$ units of time then $X^k$ will be inside $D_r$ at time $t_2$. By the strong Markov property applied at the stopping time $T^{X^k}_{D_{r_1}}$, we obtain, using (6.7), for all $\mathbf{x} \in D^N$,

$$\mathbb{P}^{\mathbf{x}}(X^k_{t_2} \in D_r) \ge p_1 \mathbb{P}^{\mathbf{x}}\left(T^{X^k}_{D_{r_1}} \le t_2\right) \ge p_1 c_3 > 0. \tag{6.8}$$

This completes the proof of part (i) of the lemma.

(ii) Recall that $r > 0$ is fixed and such that $\mathrm{Int} D_r \ne \emptyset$. Let $r_3$ and $r_4$ be such that $0 < r < r_3 < r_4$ and $\mathrm{Int} D_{r_4} \ne \emptyset$. Let $r_5 = \min(r_3 - r, r_4 - r_3)/2$. We choose $t_3$ and $c_4$ so that (6.8) can be applied with $r_4$ in place of $r$,

$$\mathbb{P}^{\mathbf{x}}(X^k_{t_3} \in D_{r_4}) \ge c_4 > 0.$$

Let $p_2 = \inf_{y \in D} \mathbb{P}^y(T_{\partial D} \le t_3)$ and note that $p_2 > 0$. Let $p_3 = \mathbb{P}^y(T_{\partial B(y,r_5)} \ge 2t_3) > 0$ and note that $p_3$ does not depend on $y$.

Let $A$ be the intersection of the following events.

(a) The process $X^1$ is in $D_{r_4}$ at time $t_3$, and it stays in $B(X^1_{t_3}, r_5)$ for all $t \in [t_3, 3t_3]$.

(b) For every $j = 2, \dots, N$, the process $X^j$ jumps at a time $s_j \in [t_3, 2t_3]$ to $X^1_{s_j}$, and then stays in the ball $B(X^j_{s_j}, r_5) = B(X^1_{s_j}, r_5)$ for all $t \in [s_j, s_j + 2t_3]$.

By the strong Markov property and the definition of the process $\mathbf{X}$, the probability of $A$ is bounded below by $c_5 = c_4 p_3 (p_2(1/(N-1)) p_3)^{N-1}$. If $A$ occurs then $\mathbf{X}_{3t_3} \in D_r^N$. Hence, for every $\mathbf{x} \in D^N$,

$$\mathbb{P}^{\mathbf{x}} \left( \mathbf{X}_{3t_3} \in D_r^N \right) \geq c_5 > 0.$$

This proves part (ii) of the lemma. □

## 7 Stationary distribution for the particle system

The two theorems proved in this section generalize the analogous results in [10], where the proofs were given only for domains satisfying the internal ball condition.

**Theorem 7.1** *Suppose that $D \subset \mathbb{R}^d$ is a bounded Lipschitz domain with the Lipschitz constant $L < c(N, d)$, where $c(N, d)$ is as in Theorem 5.4. Then there exists a unique stationary probability distribution $\mathcal{M}^N$ for $\mathbf{X}_t$. The process $\mathbf{X}_t$ converges to its stationary distribution exponentially fast, i.e., there exists $\lambda > 0$ such that for every $A \subset D^N$,*

$$\lim_{t \to \infty} \mathrm{e}^{\lambda t} \sup_{\mathbf{x} \in D^N} \left| \mathbb{P}^{\mathbf{x}} (\mathbf{X}_t \in A) - \mathcal{M}^N (A) \right| = 0. \tag{7.1}$$

*Proof* We have shown in Lemma 6.1 (ii) that for any $r > 0$, with probability higher than $p_0 = p_0(r) > 0$, the process $\mathbf{X}_t$ can reach the compact set $D_r^N$ within $t_0 > 0$ units of time. This and the strong Markov property applied at times $2t_0, 4t_0, 6t_0, \ldots$ show that the hitting time of $D_r^N$ is stochastically bounded by an exponential random variable with the expectation independent of the starting point of $\mathbf{X}_t$. Since the transition densities $p_t^{\mathbf{X}}(\mathbf{x}, \mathbf{y})$ for $\mathbf{X}_t$ are bounded below by the densities for the Brownian motion killed at the exit time from $D^N$, we see that $p_t^{\mathbf{X}}(\mathbf{x}, \mathbf{y}) > c_1 > 0$ for $\mathbf{x}, \mathbf{y} \in D_r^N$. Fix arbitrarily small $s > 0$ and consider the "skeleton" $\{\mathbf{X}_{ns}\}_{n \geq 0}$. The properties listed in this paragraph imply that the skeleton has a stationary probability distribution and that it converges to that distribution exponentially fast, i.e., (7.1) holds for the skeleton, by Theorem 2.1 in [15] or Theorem 16.0.2 (ii) and (vi) of [21]. See the proof of Proposition 1.2 in [9] for an argument showing how to pass from the the statement of uniform ergodicity for the skeleton to the analogous statement for the continuous process $t \to \mathbf{X}_t$. We sketch this argument here. Take any $\varepsilon > 0$ and find $t_1 = n_1 s$ such that

$$\mathrm{e}^{\lambda t} \sup_{\mathbf{x} \in D^N} \left| \mathbb{P}^{\mathbf{x}} (\mathbf{X}_t \in A) - \mathcal{M}^N (A) \right| \leq \varepsilon \tag{7.2}$$

holds for $t \geq t_1$ of the form $t = ns$. Consider an arbitrary $t_2 > t_1$, not necessarily of the form $ns$. Let $m$ be the integer part of $t_2/s$ and let $u = t_2 - ms$. Note that $m \geq n_1$. Since (7.2) holds for $t = ms$, the semigroup property applied at time $u$ shows that (7.2) holds also at time $t_2$. □

**Theorem 7.2** *Suppose that $D$ is a bounded Lipschitz domain with the Lipschitz constant $L < \frac{1}{\sqrt{d-1}}$. For $N \geq N_0$ (see Remark 5.5) let $\mathcal{X}^N_{\mathcal{M}}$ be the stationary empirical measure. Let $\varphi$ be the first eigenfunction for Laplacian in $D$ with the Dirichlet boundary conditions, normalized so that $\int_D \varphi = 1$. Then the sequence of random measures $\mathcal{X}^N_{\mathcal{M}}$, $N \geq N_0$, converges as $N \to \infty$ to the (non-random) measure with the density $\varphi$, in the sense of weak convergence of random measures.*

*Proof* Recall processes $Y^j$ defined in the proof of Theorem 5.4. By construction, we have $\operatorname{dist}(Y^j_t, \partial D) \leq \operatorname{dist}(X^j_t, \partial D)$, for all $j$ and $t$.

It is elementary to see that the process $Z$ constructed in Sect. 4 has the property that

$$\lim_{r \downarrow 0} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{\operatorname{dist}(Z_s, \partial D) \leq r\}} ds = 0, \text{ a.s.}$$

In view of the construction of $Y^j$ from independent copies of $Z$, we also have, for every $j$,

$$\lim_{r \downarrow 0} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{\operatorname{dist}(Y^j_s, \partial D) \leq r\}} ds = 0, \text{ a.s.}$$

Hence, for every $j$,

$$\lim_{r \downarrow 0} \limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{1}_{\{\operatorname{dist}(X^j_s, \partial D) \leq r\}} ds = 0, \text{ a.s.}$$

This implies that for every $p_1 > 0$, one can find $r > 0$ so small that if $\mathbf{X}$ has the stationary measure $\mathcal{M}^N$ then for every $t$, $\mathbb{P}(X^j_t \notin D_r) \leq p_1$. It follows that for any $N$, the mean measure $E\mathcal{X}^N_{\mathcal{M}}$ of the compact set $D_r$ is not less than $1 - p_1$. Hence, the mean measures $E\mathcal{X}^N_{\mathcal{M}}$ are tight in $D$. Lemma 3.2.7, p. 32, of [13] implies that the sequence of random measures $\mathcal{X}^N_{\mathcal{M}}$ is tight and so it contains a convergent subsequence.

One can complete the proof of the claim that the random measures $\mathcal{X}^N_{\mathcal{M}}$ converge as $N \to \infty$ to the measure with the density $\varphi$ exactly as in the proof of Theorem 1.4 in [10], starting on line 9 of page 699. □

## 8 Polyhedral domains

In this section we show that the Lipschitz constant $c(N, d)$ in Theorem 5.4 is not sharp, that is, $\tau_\infty = \infty$, a.s., in some Lipschtz domains with arbitrarily large Lipschitz constant. Specifically, we will demonstrate the existence of the two particle process for all times in arbitrary polyhedral domains. Unfortunately, our method cannot be easily adapted to the multiparticle case, so we leave this generalization as an open problem.

**Definition 8.1** We say an open set $D \subset \mathbb{R}^d$ is a *polyhedral domain* if there exist simplicial complexes $\mathcal{K} \supset \partial\mathcal{K}$ such that $\overline{D} = |\mathcal{K}|$ and $\partial D = |\partial\mathcal{K}|$.

For the remainder of this section we will assume that $D = \mathrm{Int}|\mathcal{K}|$ is a polyhedral domain. Let $\mathbf{X}_t = (X_t^1, X_t^2)$ be a Fleming-Viot process in $D$ and define jump times $\tau_i$ as before. We will show:

**Theorem 8.2** *If $D$ is a polyhedral domain and $\mathbf{X}_t = (X_t^1, X_t^2)$ is a Fleming-Viot process with jump times $\tau_i$ then $\tau_i \to \infty$ as $i \to \infty$ almost surely.*

As $\mathbf{X}_t$ is a CÀDLÀG process we have $X_{\tau_i}^1 = X_{\tau_i}^2$ for each $i \in \mathbb{N}$, so we may define a sequence of *jump points* $\xi_i = X_{\tau_i}^1 = X_{\tau_i}^2$. Since $\overline{D}$ is compact, $\xi_i$ has at least one limit point in $\overline{D}$. To prove Theorem 8.2 we will examine the behavior of $\mathbf{X}_t$ when both particles are close to a limit point of $\xi_i$ and, assuming that $\tau_\infty < \infty$, arrive at a contradiction.

First we will show that if $t \in [\tau_i, \tau_{i+1})$ then $\mathbf{X}_t$ cannot stray too far from $(\xi_i, \xi_i)$.

**Lemma 8.3** *Set $V_t^1 = \|X_t^1 - \xi_i\|$, $V_t^2 = \|X_t^2 - \xi_i\|$ for $t \in [\tau_i, \tau_{i+1})$. If $\tau_\infty < \infty$ then $V_t^1 \to 0$ and $V_t^2 \to 0$ as $t \to \tau_\infty$.*

*Proof* It suffices to consider only $V_t^1$. Notice that $V_t^1$ is a $d$-dimensional Bessel process (Bes($d$), for short), reset to 0 at each $\tau_i$. So setting $\Delta V_i^1 = V_{\tau_i^-}^1$ we may extract a Brownian motion

$$W_t = V_t^1 + \sum_{\{i \in \mathbb{N} : \tau_i \le t\}} \Delta V_i^1 - \int_0^t \frac{d-1}{2V_t^1} dt.$$

Consider $\varepsilon > 0$. We will count the number of upcrossings of the interval $[\frac{\varepsilon}{2}, \varepsilon]$ within a short time interval $[t, t+\delta]$, where $\delta = \varepsilon^2/(4(d-1))$. Consider times $t < s' < s < t+\delta$ where $V_s^1 \ge \varepsilon$ and $s' = \sup\{\tilde{s} < s : V_{\tilde{s}}^1 = \frac{\varepsilon}{2}\}$. Notice as $V^1$ only jumps downwards there is no $i \in \mathbb{N}$ such that $s' < \tau_i \le s$. We have

$$\begin{aligned} W_s - W_{s'} &= V_s^1 - V_{s'}^1 - \int_{s'}^{s} \frac{d-1}{2V_t^1} dt \\ &\ge \frac{\varepsilon}{2} - (s-s')\frac{d-1}{\varepsilon} \\ &\ge \frac{\varepsilon}{2} - \frac{\varepsilon^2}{4(d-1)} \frac{d-1}{\varepsilon} = \frac{\varepsilon}{4}. \end{aligned}$$

So on a short time interval, each upcrossing of $[\frac{\varepsilon}{2}, \varepsilon]$ by $V^1$ corresponds to an oscillation of $\frac{\varepsilon}{4}$ by $W$. As $W$ is a Brownian motion, with probability 1, $V^1$ makes only finitely many upcrossings of $[\frac{\varepsilon}{2}, \varepsilon]$ in a given time interval $[t, t+\delta]$. If $\tau_\infty < \infty$, we may find $n \in \mathbb{N}$ with $\tau_n \ge \tau_\infty - \delta$. So if $V_t^1 > \varepsilon$ for some $\tau_n < \tau_i < t < \tau_{i+1}$ then as $V^1$ is reset to 0 at $\tau_i$ there must be an upcrossing of $[\frac{\varepsilon}{2}, \varepsilon]$ in the interval $[\tau_i, \tau_{i+1}) \subset [\tau_n, \tau_n + \delta]$.

So $V_t^1 > \varepsilon$ in only finitely many intervals $[\tau_i, \tau_{i+1})$ and, as $\varepsilon$ is arbitrary, $V_t^1 \to 0$ as $t \to \tau_\infty$. □

**Corollary 8.4** *If $\tau_\infty < \infty$ then the sequence $\xi_i$ has no limit point $\xi_\infty \in D$.*

*Proof* Fix $x \in D$. As $D$ is open, there exists some $\varepsilon$ with $B(x, 2\varepsilon) \subset D$. If $\xi_i \in B(x, \varepsilon)$ then, as both particles follow continuous paths until one exits $D$, we must have $V_t^1 \vee V_t^2 > \varepsilon$ for some $t \in [\tau_i, \tau_{i+1})$. So if $V_t^1, V_t^2 \to 0$ as $t \to \tau_\infty$ then $\xi_i \in B(x, \varepsilon)$ for only finitely many $i$. As $x$ is arbitrary we see that so long as $V_t^1, V_t^2 \to 0$ as $t \to \tau_\infty$, $\xi_i$ can have no limit point in $D$. □

It is convenient at this point to introduce some notation that will allow us to consider the behavior of $\mathbf{X}_t$ when it is close to the boundary of a simplicial complex. Let $\sigma$ be a $k$-simplex with vertices $\{v_0, \dots, v_k\}$, that is

$$\sigma = \left\{ \sum_{i=0}^{k} \lambda_i v_i \lambda_0, \dots, \lambda_k \geq 0, \quad \sum_{i=0}^{k} \lambda_i = 1 \right\}.$$

Then define the *interior* of $\sigma$

$$\mathring{\sigma} = \left\{ \sum_{i=0}^{k} \lambda_i v_i \lambda_0, \dots, \lambda_k > 0, \quad \sum_{i=0}^{k} \lambda_i = 1 \right\}$$

and the *span* of $\sigma$ to be the subspace

$$\mathcal{S}_\sigma = \left\{ \sum_{i=0}^{k} \lambda_i v_i \sum_{i=0}^{k} \lambda_i = 0 \right\}.$$

For two simplices $\sigma_1, \sigma_2 \in \mathcal{K}$ we write $\sigma_1 \leq \sigma_2$ if $\sigma_1$ is a face of $\sigma_2$ and $\sigma_1 < \sigma_2$ if $\sigma_1$ is a proper face of $\sigma_2$. We name the *star* of a simplex $\sigma$ to be the set

$$St(\sigma) = \{\sigma_1 \in \mathcal{K} : \sigma_1 \geq \sigma\}$$

and define the *neighborhood* of $\sigma$ as

$$\mathcal{N}(\sigma) = \{x \in \overline{D} : x \in \mathring{\sigma}_1 \text{ for some } \sigma_1 \geq \sigma\}.$$

Given simplices $\sigma \leq \sigma_1$ name the vertices of $\sigma$ and $\sigma_1$, $\{v_0, \dots, v_k\}$ and $\{v_0, \dots, v_n\}$ respectively. Define the *wedges*

$$\mathcal{W}(\sigma, \sigma_1) = \left\{ \sum_{i=0}^{n} \lambda_i v_i \lambda_{k+1}, \dots, \lambda_n > 0, \quad \sum_{i=0}^{n} \lambda_i = 1 \right\},$$

$$\mathcal{W}(\sigma) = \bigcup_{\sigma_1 \in St(\sigma)} \mathcal{W}(\sigma, \sigma_1).$$

Notice that $\mathcal{N}(\sigma) \subset \overline{\mathcal{W}(\sigma)}$ and that $\mathcal{N}(\sigma)$ is open with respect to the subspace topology of $\overline{D}$. Notice also that $\mathcal{W}(\sigma)$ is a product space

$$\mathcal{W}(\sigma) = \mathcal{C}(\sigma) \times \mathcal{S}_\sigma,$$

where the cone $\mathcal{C}(\sigma)$ is the projection of $\mathcal{W}(\sigma)$ onto $\mathcal{S}_\sigma^\perp$.

Now, consider $\sigma \in \partial\mathcal{K}$ and suppose there exists a subsequence $\xi_{i_n} \to \xi_\infty \in \mathring{\sigma}$. As $\xi_\infty \in \mathring{\sigma} \subset \mathcal{N}(\sigma)$ and $\mathcal{N}(\sigma)$ is open in $\overline{D}$ we may assume without loss of generality that $\xi_{i_n} \in \mathcal{N}(\sigma)$ for each $n$. So consider $\mathbf{X}_t$ started at $(\xi_{i_n}, \xi_{i_n})$ at time $\tau_{i_n}$ and stopped at the first time $T > \tau_{i_n}$ where one of $X_t^1, X_t^2$ exits $\mathcal{N}(\sigma)$. Of course, as $\mathcal{N}(\sigma) \subset \mathcal{W}(\sigma) \cap \overline{D}$, this has the same distribution as a Fleming-Viot process in $\mathcal{W}(\sigma)$ started and stopped in the same way.

So, let $\mathbb{P}_\sigma^x$ and $\mathbb{E}_\sigma^x$ be the probability measure and expectation operator associated with a Fleming-Viot process in $\mathcal{W}(\sigma)$ started at $\mathbf{X}_0 = (x, x)$. The $\mathcal{S}_\sigma$ and $\mathcal{S}_\sigma^\perp$ components are not quite independent as they have the same jumps, but $\mathbb{P}_\sigma$ allows a partial factorization as follows.

**Lemma 8.5** *If $\mathbf{X}_t$ is a Fleming-Viot process in $\mathcal{W}(\sigma)$ then there is a well defined decomposition $\mathbf{X}_t = \mathbf{Y}_t + \mathbf{Z}_t$ with $\mathbf{Y}_t = (Y_t^1, Y_t^2) \in \mathcal{C}(\sigma)^2$, $\mathbf{Z}_t = (Z_t^1, Z_t^2) \in \mathcal{S}_\sigma^2$ with the following properties*

- *$\mathbf{Y}_t$ is a Fleming-Viot process in $\mathcal{C}(\sigma)$;*
- *there exists a Brownian motion $\tilde{Z}_t$ in $\mathcal{S}_\sigma$ (not adapted to the filtration of $\mathbf{X}_t$), independent of $\mathbf{Y}_t$, such that for each $i \in \mathbb{N}$ we have $\tilde{Z}_{\tau_i} = \zeta_i$ with $\zeta_i = Z_{\tau_i}^1 = Z_{\tau_i}^2$.*

*Proof* Obviously, as $\mathcal{C}(\sigma) \subset \mathcal{S}_\sigma^\perp$, the factorization $\mathbf{X}_t = \mathbf{Y}_t + \mathbf{Z}_t$ is unique. Further, on each interval $[\tau_i, \tau_{i+1})$, the processes $Y_t^1, Y_t^2, Z_t^1$ and $Z_t^2$ evolve as independent Brownian motions on $\mathcal{S}_\sigma^\perp$ and $\mathcal{S}_\sigma$ respectively. So as $\mathcal{S}_\sigma$ is a subspace and has no boundary, $X_t^j$ jumps when and only when $Y_t^j$ hits $\partial\mathcal{C}(\sigma)$, and so $\mathbf{Y}_t$ is indeed a Fleming-Viot process on $\mathcal{C}(\sigma)$.

Now for each $i \in \mathbb{N}$ only one of $X_t^1, X_t^2$ has a discontinuity at $\tau_{i+1}$, so there is a well defined sequence of random variables $J_i \in \{1, 2\}$ such that $X_t^{J_i}$ is continuous on the closed interval $[\tau_i, \tau_{i+1}]$ and we may define a continuous process

$$\tilde{Z}_t = Z_t^{J_i}, \quad t \in \big[\tau_i, \tau_{i+1}\big].$$

Then $\tilde{Z}_{\tau_i} = \zeta_i$ for every $i$ and it remains to show that $\tilde{Z}_t$ is a Brownian motion independent of $\mathbf{Y}_t$. Of course $\tilde{Z}_t$ is only defined up to $\tau_\infty$. But we may continue $\tilde{Z}_t$ after $\tau_\infty$ with an independent Brownian motion if necessary.

Now as $\tilde{Z}_t$ follows either $Z_t^1$ or $Z_t^2$ then the quadratic variation $\langle \tilde{Z} \rangle_t = t$ and, by Lévy's characterization, we need only check that $\tilde{Z}_t$ is a martingale with respect to its own natural filtration and is independent of $\mathbf{Y}_t$. Furthermore, although $\tilde{Z}_t$ is not adapted to $\mathbf{X}_t$, for each $\tau_i$, the path $\tilde{Z}|_{[0,\tau_i]}$ is measurable with respect to $\mathbf{X}|_{[0,\tau_i]}$. Therefore, by the strong Markov property, it is sufficient to consider only intervals $[\tau_i, \tau_{i+1})$.

In fact it suffices to consider only the first time interval $[0, \tau_1)$. Let $\mathbf{X}_t$ be a Fleming-Viot process started at $\xi_0 \in \mathcal{W}(\sigma)$ and stopped at $\tau_1$. Then the left limit process is a pair of independent Brownian motions stopped at $\tau = \tau_1^-$. Set $J = J_0$ and we have $\xi_1 = X_\tau^J \in \mathcal{W}(\sigma)$ and $X_\tau^{3-J} \in \partial\mathcal{W}(\sigma)$.

So set $\mathbf{X}_t = \mathbf{Y}_t + \mathbf{Z}_t$ as in the statement of the lemma and let $\mathcal{F}_t^{\mathbf{Y}}$, $\mathcal{F}_t^{\mathbf{Z}}$ and $\mathcal{F}_t^{\tilde{Z}}$ be the natural filtrations of $\mathbf{Y}$, $\mathbf{Z}$ and $\tilde{Z}$ respectively. Set $\zeta_0 = Z_0^1$, $\zeta_1 = Z_\tau^J$ to be the $\mathcal{F}_\tau^{\mathbf{X}}$-measurable $\mathbf{Z}$-components of $\xi_0$ and $\xi_1$, respectively. Thus, $\tau$ is a stopping time of $\mathcal{F}_t^{\mathbf{Y}}$ and $J$ is measurable with respect to $\mathcal{F}_\tau^{\mathbf{Y}}$. Now crucially $\mathbf{Y}$ and $\mathbf{Z}$ are independent processes so for $t < \tau$ we have

$$\mathbb{E}_\sigma^{\xi_0}\left(\zeta_1 \,\middle|\, \mathcal{F}_\tau^{\mathbf{Y}} \vee \mathcal{F}_t^{\mathbf{Z}}\right) = \mathbb{E}_\sigma^{\xi_0}\left(Z_\tau^J \,\middle|\, \mathcal{F}_\tau^{\mathbf{Y}} \vee \mathcal{F}_t^{\mathbf{Z}}\right) = \tilde{Z}_t.$$

Thus $\tilde{Z}$ is a martingale, and hence a Brownian motion, with respect to the filtration $\mathcal{G}_t = \mathcal{F}_\tau^{\mathbf{Y}} \vee \mathcal{F}_t^{\mathbf{Z}}$. Therefore $\tilde{Z}$ is independent of $\mathcal{F}_\tau^{\mathbf{Y}} \subset \mathcal{G}_0$ and is a Brownian motion with respect to its own natural filtration $\mathcal{F}_t^{\tilde{Z}} \subset \mathcal{G}_t$. □

Now $\mathbf{Y}_t$ is a process in a cone and if $\xi_i$ converges to some point in $\mathring{\sigma}$ then $\mathbf{Y}_t$ must converge to the apex of $\mathcal{C}(\sigma)$. Our next step is to show that this cannot be the case.

**Lemma 8.6** *If $\mathbf{Y}_t$ is a Fleming-Viot process in a cone $C \subset \mathbb{R}^d$ then, with probability one, $\mathbf{Y}_t$ does not converge to* $(\underline{0}, \underline{0})$.

To prove this we will need to consider the angular components, $\Phi_t^j = \frac{Y_t^j}{\|Y_t^j\|}$, of $\mathbf{Y}$. We will recall briefly some facts about spherical Brownian motion. We will omit details, which can be found in [23, Chapter 8], particularly Example 8.5.8.

Let $B_t$ be a Brownian motion on $\mathbb{R}^d$, let the unit sphere be denoted

$$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d \;:\; \|x\| = 1\},$$

and define the map $\phi : \mathbb{R}^d \backslash \{\underline{0}\} \to \mathbb{S}^{d-1}$ by $\phi(x) = \frac{x}{\|x\|}$.

Now let $\Phi_t = \phi(B_t)$. Applying Ito's formula,

$$d\Phi_t = \frac{1}{\|B_t\|}\left(I - \Phi_t \Phi_t^\top\right) dB_t - \frac{d-1}{2\|B_t\|^2}\Phi_t\, dt.$$

Note we are interpreting $\Phi_t$ as a column vector so $\Phi_t \Phi_t^\top$ is a square matrix. Now define a differential operator $A : \mathcal{C}^2(\mathbb{S}^{d-1}, \mathbb{R}) \to \mathcal{C}^0(\mathbb{S}^{d-1}, \mathbb{R})$ by

$$Af(x) = \frac{1}{2}\left(\Delta f(x) - \sum_{i,j} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j}\right) - \frac{d-1}{2}\sum_i x_i \frac{\partial f}{\partial x_i}.$$

Applying Ito's formula again, we see that $f(\Phi_t) - \int_0^t \frac{Af(\Phi_t)}{\|B_t\|^2} dt$ is a local martingale for each $f \in \mathcal{C}^2(\mathbb{S}^{d-1}, \mathbb{R})$.

We may extend this to functions of two Brownian motions by defining $\mathcal{A}^1, \mathcal{A}^2$ by

$$\mathcal{A}^1 f(x,y) = A(f(\cdot,y))(x),$$
$$\mathcal{A}^2 f(x,y) = A(f(x,\cdot))(y).$$

Then by a similar application of Ito's formula, if $B_t^1$ and $B_t^2$ are independent Brownian motions and $\Phi_t^1 = \phi(B_t^1)$, $\Phi_t^2 = \phi(B_t^2)$, $\Phi_t = (\Phi_t^1, \Phi_t^2)$, then

$$N_t^f = f\left(\Phi_t^1, \Phi_t^2\right) - \int_0^t \left(\frac{\mathcal{A}^1 f(\Phi_t)}{\|B_t^1\|^2} + \frac{\mathcal{A}^2 f(\Phi_t)}{\|B_t^2\|^2}\right) dt \tag{8.1}$$

is a local martingale.

Now apply a time change to $\Phi_t$ as follows. If $\alpha(t) = \inf\{s \in \mathbb{R}^+ : \int_0^s \|B_{\tilde{s}}\|^{-2} d\tilde{s} \geq t\}$, then $\Theta_t = \Phi_{\alpha(t)}$ is a Markov diffusion on $\mathbb{S}^{d-1}$ with generator $A$. Let $\mathbb{P}_{\mathbb{S}}^{\theta_1,\theta_2}$ and $\mathbb{E}_{\mathbb{S}}^{\theta_1,\theta_2}$ be the probability measure and expectation operator associated with two independent copies of $\Theta_t$ started at $\theta_1$ and $\theta_2 \in \mathbb{S}^{d-1}$ respectively.

**Lemma 8.7** *Let $U$ be an open subset of $\mathbb{S}^{d-1}$ and set*

$$T_1^U = \inf\{t \in \mathbb{R} : \Theta_t^1 \in \partial U\},$$
$$T_2^U = \inf\{t \in \mathbb{R} : \Theta_t^2 \in \partial U\},$$
$$h^U(\theta_1,\theta_2) = \mathbb{P}_{\mathbb{S}}^{\theta_1,\theta_2}[T_1^U < T_2^U].$$

*Then $h^U \in \mathcal{C}^2(U^2, \mathbb{R})$ and $\mathcal{A}^1 h^U = -\mathcal{A}^2 h^U \geq 0$.*

*Proof* The process $(\Theta_t^1, \Theta_t^2)$ is a Markov diffusion with generator $\mathcal{A}^1 + \mathcal{A}^2$, so by Dynkin's formula $\mathcal{A}^1 h^U + \mathcal{A}^2 h^U = 0$ and it remains to show that $\mathcal{A}^1 h^U \geq 0$.

By definition of the Markov generator

$$\begin{aligned}\mathcal{A}^1 h^U(\theta_1,\theta_2) &= \lim_{t\to 0} \frac{1}{t}\left(\mathbb{E}_{\mathbb{S}}^{\theta_1,\theta_2}\left(h^U\left(\Theta_t^1,\theta_2\right)\right) - h^U(\theta_1,\theta_2)\right)\\ &= \lim_{t\to 0}\frac{1}{t}\left(\mathbb{E}_{\mathbb{S}}^{\theta_1,\theta_2}\left(\mathbb{P}_{\mathbb{S}}^{\Theta_t^1,\theta_2}\left[T_1^U < T_2^U\right]\right) - \mathbb{P}_{\mathbb{S}}^{\theta_1,\theta_2}\left[T_1^U < T_2^U\right]\right).\end{aligned}$$

But $\mathbb{E}_{\mathbb{S}}^{\theta_1,\theta_2}(\mathbb{P}_{\mathbb{S}}^{\Theta_t^1,\theta_2}(\cdot))$ is the probability measure associated with the process $(\Theta_{s+t}^1, \Theta_s^2)$, $s > 0$, obtained by giving $\Theta_1$ a headstart. So we have

$$\begin{aligned}\mathbb{E}_{\mathbb{S}}^{\theta_1,\theta_2}\left(\mathbb{P}_{\mathbb{S}}^{\Theta_t^1,\theta_2}\left[T_1^U < T_2^U\right]\right) &\geq \mathbb{P}_{\mathbb{S}}^{\theta_1,\theta_2}\left[T_1^U - t < T_2^U\right] - \mathbb{P}_{\mathbb{S}}^{\theta_1,\theta_2}\left[T_1^U < t\right]\\ &\geq \mathbb{P}_{\mathbb{S}}^{\theta_1,\theta_2}\left[T_1^U < T_2^U\right] - \mathbb{P}_{\mathbb{S}}^{\theta_1,\theta_2}\left[T_1^U < t\right]\end{aligned}$$

and, since $\frac{1}{t}\mathbb{P}_{\mathbb{S}}^{\theta_1,\theta_2}(T_1^U < t) \to 0$ as $t \to 0$, we may pass to the limit, and we see that $\mathcal{A}^1 h^U(\theta^1,\theta^2) \geq 0$. $\square$

We are ready to prove Lemma 8.6.

*Proof of Lemma 8.6* Set $C = \{\lambda u : \lambda \in \mathbb{R}^+, u \in U\}$ for some open subset $U \subset \mathbb{S}^{d-1}$ and let $\mathbf{Y}_t$ be a Fleming-Viot process in $C$.

We deal first with the special case when $d = 1$, in which case either $C = \mathbb{R}$ and there is nothing to prove or $C = \mathbb{R}^+$. If $C = \mathbb{R}^+$ then $\mathbf{Y}_t$ is a 2-dimensional Brownian motion in the quarter plane with jumps $(y, 0) \mapsto (y, y)$ or $(0, y) \mapsto (y, y)$ whenever the process exits the first quadrant. As these jumps only increase $\|\mathbf{Y}_t\|$ then $\|\mathbf{Y}_t\|$ dominates a Bes(2) process and $\mathbf{Y}_t$ does not converge to 0.

For $d \geq 2$ define a function

$$\mu(x) = \begin{cases} \log \|x\|, & \text{if } d = 2, \\ \frac{\|x\|^{2-d}}{2-d} & \text{if } d \geq 3, \end{cases}$$

and define processes

$$\begin{aligned} \Phi_t^1 &= \phi(Y_t^1), & M_t^1 &= \mu(Y_t^1), \\ \Phi_t^2 &= \phi(Y_t^2), & M_t^2 &= \mu(Y_t^2), \\ H_t &= h^U(\Phi_t^1, \Phi_t^2), & S_t &= M_t^1 + (M_t^2 - M_t^1) H_t. \end{aligned}$$

Now, $\mu$ is harmonic on $\mathbb{R}^d$ and it will be key to our argument that $M_t^1$ and $M_t^2$ are both local martingales except when $\mathbf{Y}_t$ jumps. We say a $\mathbf{Y}_t$-adatapted process $R_t$ is a *martingale between jumps* if $R_t - \sum_{\{i \in \mathbb{N} : \tau_i \leq t\}} (R_{\tau_i} - R_{\tau_i^-})$ is a continuous local martingale. The process $S_t$ is a convex combination of $M_t^1$ and $M_t^2$, so if both $Y_t^1$ and $Y_t^2$ converge to the origin, then $S_t$ converges to $-\infty$. Notice also that if $Y^1$ approaches $\partial C$ then $H_t \to 1$ and so $S_t \to M_t^2$. Similarly, if $Y_t^2$ approaches the boundary then $S_t \to M_t^1$. So $S_t$ is continuous.

Set

$$N_s = H_s - \int_0^s \left( \frac{\mathcal{A}^1 h^U(\Phi_t)}{\|B_t^1\|^2} + \frac{\mathcal{A}^2 h^U(\Phi_t)}{\|B_t^2\|^2} \right) dt.$$

By (8.1) $N_t$ is a martingale between jumps. We may check that the cross variation terms $\langle M^1, \Phi^1 \rangle_t = \langle M_t^2, \Phi_t^2 \rangle = 0$ and so, as $H_t$ is a $\mathcal{C}^2$ function of $\Phi_t^1$ and $\Phi_t^2$, we have $\langle M^1, H \rangle_t = \langle M_t^2, H \rangle_t = 0$ and for $s \in [\tau_i, \tau_{i+1})$ we may calculate

$$\begin{aligned} S_s &= S_{\tau_i} + \int_{\tau_i}^s (1 - H_t)\, dM_t^1 + \int_{\tau_i}^s H_t\, dM_t^2 + \int_{\tau_i}^s (M_t^2 - M_t^1)\, dH_t \\ &= S_{\tau_i} + \int_{\tau_i}^s (1 - H_t)\, dM_t^1 + \int_{\tau_i}^s H_t\, dM_t^2 + \int_{\tau_i}^s (M_t^2 - M_t^1)\, dN_t \\ &\quad + \int_{\tau_i}^s \left(M_t^2 - M_t^1\right) \left( \frac{\mathcal{A}^1 h^U(\Phi_t)}{\|B_t^1\|^2} + \frac{\mathcal{A}^2 h^U(\Phi_t)}{\|B_t^2\|^2} \right) dt. \end{aligned}$$

Therefore $S_s - \int_{\tau_i}^{s}(M_t^2 - M_t^1)\left(\frac{\mathcal{A}^1 h^U(\Phi_t)}{\|B_t^1\|^2} + \frac{\mathcal{A}^2 h^U(\Phi_t)}{\|B_t^2\|^2}\right)dt$ is a martingale between jumps.

Now from Lemma 8.7 we have $\mathcal{A}^1 h^U = -\mathcal{A}^2 h^U \geq 0$ and so

$$\frac{\mathcal{A}^1 h^U(\Phi_t)}{\|B_t^1\|^2} + \frac{\mathcal{A}^2 h^U(\Phi_t)}{\|B_t^2\|^2} = \mathcal{A}^1 h^U(\Phi_t)\left(\|B_t^1\|^{-2} - \|B_t^2\|^{-2}\right).$$

But $\mu$ is an increasing function of the norm $\|\cdot\|$, so for $\tau_i \leq s_1 \leq s_2 < \tau_{i+1}$,

$$\int_{s_1}^{s_2}\left(M_t^2 - M_t^1\right)\left(\frac{\mathcal{A}^1 h^U(\Phi_t)}{\|B_t^1\|^2} + \frac{\mathcal{A}^2 h^U(\Phi_t)}{\|B_t^2\|^2}\right)dt \geq 0.$$

Therefore $S_t$ is a continuous local submartingale and it cannot converge to $-\infty$. Thus $\mathbf{Y}_t$ does not converge to $(0, 0)$. □

**Corollary 8.8** *If $\mathbf{X}_t$ is a Fleming-Viot process in a polyhedral domain $D$ then with probability one the sequence of jump points $\xi_i$ does not converge to any $\xi_\infty \in \partial D$ as $i \to \infty$.*

*Proof* First, for $\sigma \in \partial\mathcal{K}$, let $F^\sigma$ be the event that $\xi_i \to \xi_\infty$ for some $\xi_\infty \in \mathring{\sigma}$ and assume without loss of generality that $\underline{0} \in \sigma$. Set

$$F_i^\sigma = F^\sigma \cap \left[X_t^j \in \mathcal{N}(\sigma);\ t \geq \tau_i, \quad j = 1, 2\right].$$

Then, as $\mathcal{N}(\sigma)$ is open in $\overline{D}$, from Lemma 8.3, $F_i^\sigma$ increases to $F^\sigma$ up to an event of probability 0. By the strong Markov property and Lemma 8.6,

$$\mathbb{P}(F_i^\sigma) = \mathbb{P}_\sigma^{\xi_i}\left(\mathbf{Y}_t \to (\underline{0}, \underline{0}) \cap \left[X_t^j \in \mathcal{N}(\sigma);\ t \geq \tau_i, \quad j = 1, 2\right]\right) = 0.$$

So as $\partial\mathcal{K}$ is a finite set of simplices we have $\mathbb{P}[\exists\xi_\infty \in \partial D \text{ s.t. } \xi_i \to \xi_\infty \text{ as } i \to \infty] = 0$. □

To complete the proof of Theorem 8.2 we consider the set

$$L = \{\sigma \in \mathcal{K} : \text{there exists a subsequence } \xi_{i_n} \to \xi \in \mathring{\sigma} \text{ as } n \to \infty\}.$$

It is easy to check that the event $\{\sigma \in L\}$ is $\mathbf{X}$-measurable. We say $\sigma$ is a *local maximum* of $L$ if $L \cap St(\sigma) = \{\sigma\}$. Of course any non-empty subset of a finite lattice contains at least one local maximum, and $L$ is non empty by compactness of $\overline{D}$. We will prove Theorem 8.2 by showing that for each $\sigma \in \mathcal{K}$ the event that $\tau_\infty < \infty$ and $\sigma$ is a local maximum of $L$ has probability 0.

*Proof of Theorem 8.2* Fix $\sigma \in \partial\mathcal{K}$, and note that $\mathcal{N}(\sigma)\backslash\sigma$ is non empty. We show first that if $\xi_i$ has a limit point in $\mathring{\sigma}$ and $\tau_\infty < \infty$, then $\xi_i$ has a second limit point in $\mathcal{N}(\sigma)\backslash\sigma$.

First suppose that $\sigma = \{v\}$ is a vertex of $\mathcal{K}$ and $v$ is a limit point of $\xi_i$. By Corollary 8.8, the sequence $\xi_i$ does not converge to $v$ as $i \to \infty$, so we may choose $\varepsilon > 0$ such that $B(v,\varepsilon) \cap \overline{D} \subset \mathcal{N}(\sigma)$ and that $\|\xi_i - v\| > \varepsilon$ infinitely often. If this is the case then there are infinitely many pairs $(\xi_{i_n}, \xi_{i_{n+1}})$ such that $\xi_{i_n} \in B(v,\varepsilon)$ and $\xi_{i_{n+1}} \notin B(v,\varepsilon)$. But from Lemma 8.3 we have $\|\xi_i - \xi_{i+1}\| \to 0$ as $i \to \infty$ hence $\|\xi_{i_n} - v\| \to \varepsilon$ as $i \to \infty$. Therefore, as $\partial B(v,\varepsilon)$ is compact, $\xi_i$ must have some limit point in $\partial B(v,\varepsilon) \cap \overline{D} \subset \mathcal{N}(\sigma) \backslash \{v\}$.

If $\sigma$ is a $k$-simplex for $0 < k < d$ then for each $x \in \overset{\circ}{\sigma}$ there exists $\varepsilon > 0$ such that $B(x, 2\varepsilon) \cap \overline{D} \subset \mathcal{N}(\sigma)$. We will consider upcrossings of the interval $[\varepsilon, 2\varepsilon]$ by $\|\xi_i - x\|$. Define sequences $i_n, j_n \in \mathbb{N} \cup \{\infty\}$ and $T_n, \eta_n \in \mathbb{R} \cup \{\infty\}$ by: $j_0 = 0$,

$$\begin{aligned} i_{n+1} &= \inf\{i > j_n \ : \ \xi_i \in B(x,\varepsilon)\}, \\ j_n &= \inf\{j > i_n \ : \ \xi_j \notin B(x, 2\varepsilon)\}, \\ T_n &= \inf\{t > \tau_{i_n} \ : \ X_t^1 \notin B(x,2\varepsilon) \text{ or } X_t^2 \notin B(x,2\varepsilon)\}, \\ \eta_n &= \sup_i\{\tau_i \ : \ \tau_i < T_n\}. \end{aligned}$$

Then we put $N = \sup\{n \in \mathbb{N} \ : \ j_n < \infty\}$ to be the number of upcrossings.

Note that $B(x, 2\varepsilon) \cap \overline{D} \subset \mathcal{N}(\sigma)$ and so $\mathbf{X}_{(t+\tau_{i_n}) \wedge T_n}$ is a Fleming-Viot process in $\mathcal{W}(\sigma)$ started at $(\xi_{i_n}, \xi_{i_n})$ and stopped on exiting $B(x, 2\varepsilon)$. So we may consider $\mathbb{P}_\sigma^{\xi_{i_n}}$ and factorize $\mathbf{X}_t = \mathbf{Y}_t + \mathbf{Z}_t$ as in Lemma 8.5. For $t \in [\tau_{i_n}, \eta_n]$, the process $\tilde{Z}_t$ is measurable with respect to $\mathbf{X}|_{[\tau_{i_n}, T_n]}$ which is distributed according to $\mathbb{P}_\sigma^{\xi_{i_n}}$. Hence $\tilde{Z}|_{[\tau_{i_n}, \eta_n]}$ is a Brownian motion in $\mathcal{S}_\sigma$ with respect to its own natural filtration.

Recall $\mathbf{Z}_{\tau_i} = (\zeta_i, \zeta_i)$ and set

$$\tilde{V}_t = \begin{cases} \left\| \tilde{Z}_t - \zeta_{i_n} \right\|, & \text{if } t \in [\tau_{i_n}, \eta_n], \\ 0, & \text{otherwise.} \end{cases}$$

Then $\tilde{V}_t$ is dominated by a $\mathrm{Bes}(d)$ process reset to zero at times $\tau_{i_n}$. So arguing as in the proof of Lemma 8.3, if $\tau_\infty < \infty$ and the number of upcrossings $N = \infty$, then $\tau_{i_n} < \tau_\infty < \infty$ for each $n \in \mathbb{N}$, and $\tilde{V}_t \to 0$ as $\tau_i \to \infty$. But $\eta_n = \sup_i\{\tau_i \ : \ \tau_i < T_n\}$, hence $\mathbf{X}_{\eta_n} = (\xi_{k_n}, \xi_{k_n})$ for some $k_n \in \mathbb{N}$ and either $\|x - X^1_{T_n}\| = 2\varepsilon$ or $\|x - X^2_{T_n}\| = 2\varepsilon$. So if $\tau_\infty < \infty$ and $N = \infty$, we must have $\|\xi_{k_n} - x\| \to 2\varepsilon$ as $n \to \infty$ and so $\xi_{k_n}$ has a limit point $\xi_\infty \in \partial B(x, 2\varepsilon) \cap \overline{D} \subset \mathcal{N}(\sigma)$. But $\tilde{V}_t \to 0$ as $t \to \tau_\infty$ with probability one, so we cannot have $\xi_\infty \in \sigma$ and we must have $\xi_\infty \in \mathcal{N}(\sigma) \backslash \sigma$.

Now let $Q^\sigma$ be a countably dense subset of $\overset{\circ}{\sigma}$ and suppose $\xi_i$ has some limit point $x \in \overset{\circ}{\sigma}$. By Corollary 8.8, $\xi_i$ does not converge to $x$ as $i \to \infty$ and we may choose some rational $\varepsilon > 0$ such that $B(x, 3\varepsilon) \cap \overline{D} \subset \mathcal{N}(\sigma)$ and $\|\xi_i - x\| > 3\varepsilon$ infinitely often. Now choose $q \in Q^\sigma \cap B(x, \varepsilon)$ and notice that $\|\xi_i - q\|$ makes infinitely many upcrossings of the interval $[\varepsilon, 2\varepsilon]$. If $\tau_\infty < \infty$ then as $Q^\sigma$ is countable, with probability one we may find some limit point $\xi_\infty \in \mathcal{N}(\sigma) \backslash \sigma$.

Recall the definition of the set $L$. As $L$ is nonempty there must exist some local maximum $\sigma$. However if $\tau_\infty < \infty$ then, by Corollary 8.4, we have $L \subseteq \partial\mathcal{K}$. We have just shown that if $\tau_\infty < \infty$ then $L$ has no local maximum in $\partial\mathcal{K}$. Hence we must have $\tau_\infty = \infty$. □

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