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Immortal particle for a catalytic branching process

Ilie Grigorescu · Min Kang

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Abstract We study the existence and asymptotic properties of a conservative branching particle system driven by a diffusion with smooth coefficients for which birth and death are triggered by contact with a set. Sufficient conditions for the process to be non-explosive are given. In the Brownian motions case the domain of evolution can be non-smooth, including Lipschitz, with integrable Martin kernel. The results are valid for an arbitrary number of particles and non-uniform redistribution after branching. Additionally, with probability one, it is shown that only one ancestry line survives. In special cases, the evolution of the surviving particle is studied and for a two particle system on a half line we derive explicitly the transition function of a chain representing the position at successive branching times.

Keywords Fleming–Viot branching · Immortal particle · Martin kernel · Doeblin condition · Jump diffusion process

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I. Grigorescu (✉)
Department of Mathematics, University of Miami, 1365 Memorial Drive,
Coral Gables, FL 33124-4250, USA
e-mail: igrigore@math.miami.edu

M. Kang
Department of Mathematics, North Carolina State University,
SAS Hall, 2311 Stinson Dr., Box 8205, Raleigh, NC 27695, USA
e-mail: kang@math.ncsu.edu

1 Introduction

This paper is the second part of an effort to characterize the non-explosiveness and ergodic properties of a class of stochastic processes built by piecing together countably many consecutive episodes of a driving process killed upon contact with a set (catalyst), which is restarted at a random point of the state space by a redistribution probability measure, to be prescribed according to the particular evolution model. The first part [16] looks at a number of models that need a finite number of jumps before entering a certain center of the state space (i.e. a set away from the boundary and a small set in the sense of Doeblin theory). The results in the present paper do not depend on [16]. We are now focussing on the harder example of the N particle system with Fleming–Viot dynamics introduced in [8] for Brownian motions. Similarly to the Wright–Fisher model, a killed particle is replaced by having one of the surviving particles branch. This can be interpreted as a jump to the location of one of the survivors, chosen according to a (possibly non-uniform) distribution, as in Definition 1. Perturbations of the diffusions driving the process between jumps and of the redistribution probabilities appear naturally; for example, when considering large deviations estimates from the hydrodynamic limit [14] of the model with uniform redistribution, the Brownian motions acquire a drift and the redistribution measures become biased accordingly.

The main results are Theorem 1, which proves that the system is non-explosive on domains with regularity prescribed in Definition 2; Theorem 2, which proves geometric ergodicity using a comparison with a process without jumps obtained by coupling; Theorem 3, which gives the sharpest conditions for non-explosion for non-smooth domains, and Theorem 4, establishing the existence of a unique infinite continuous path, or ancestry line—the immortal particle in the sense of [10, 11]. They are valid for all $N \geq 2$, general diffusions and non-uniform redistribution probabilities, and in the Brownian motion case for non-smooth domains (including Lipschitz) with integrable Martin kernel.

Theorem 1 solves a long standing open problem posed in [8]. We refer the reader to [6] for a discussion on why the arguments in [8] were not sufficient to prove non-explosion. A second attempt was Theorem 7 in [20], which states a conjecture on how particles approach the boundary, needed to prove the non-explosiveness result for Brownian motions in smooth domains. The proof has several errors, but the most important is to ignore that all calculations considered must take place for times $t < \tau^*$, the time of explosion, i.e. the transition probabilities are defective, in similar fashion like for an absorbed process. In equation (8.7) the author works with stopping times exceeding $S > 0$, on an event B where $S \geq \tau^*$. The conclusion is therefore trivial. In the smooth boundary case and Brownian motions with drift, there is a third attempt in Theorem 2.1 in the preprint [22].

Our interest in the model was motivated by the scaling properties of the F–V branching model [15]. The hydrodynamic limit (law of large numbers for the empirical measures as objects on the Skorohod space) has been explored in [5, 12] as a tool to study the quasi-invariant measures of a killed process, providing an important application of the Fleming–Viot mean-field redistribution dynamics.

Let D be an open connected set in $\mathbb{R}^d$ with regular boundary ∂D and $((\tilde{x}(t))_{t\geq 0}$ a diffusion on D absorbed at the boundary, generated by the second order strictly elliptic operator $\mathcal{L}$. We shall assume that the diffusion coefficients are smooth up to the boundary, i.e. belong to the $C^\infty(\bar{D})$. Naturally, lower regularity may be considered but the problems considered are difficult enough for the Laplacian. This setup can accommodate with minor changes the case of a diffusion with some boundary conditions (i.e. reflecting) on a subset of its topological boundary. In that case ∂D will denote without loss of generality, the *absorbing boundary*, where the process is killed upon arrival. Under these assumptions $P_x(\tau^D > 0) = 1$ for all $x \in D$, where $\tau^D = \inf\{t > 0 \mid x(t) \in D^c\}$ is the hitting time of D^c, the complement of D, and the transition probabilities $P^D(t, x, dy)$ will have a density

$$P_x(\tilde{x}(t) \in dy, \tau^D > t) = P^D(t, x, dy) = p^D(t, x, y)dy. \tag{1.1}$$

We note that the harmonic measures $P_x(x(\tau^D-) \in d\xi)$ are absolutely continuous with respect to the Lebesgue measure on the boundary $\lambda_0(d\xi)$, $\xi \in \partial D$.

In addition, for any $\xi \in \partial D$ we have a probability measure $\nu(\xi, dx)$ on D such that $\xi \to \nu(\xi, dx)$ is measurable with respect to the Borel σ-algebras of ∂D and of $M_1(D)$, where $M_1(D)$ denotes the space of probability measures on D with the topology of convergence in distribution.

Constructively, we define a Markov process $(x(t))_{t\geq 0}$, starting at $x \in D$, as follows. We set $x_0 := x$ and $\tau_0 := 0$. The process follows the diffusion P^D starting at x_0 up to $\tau_1 := \tau_1^D$, which means $x(t) := \tilde{x}(t)$ for $0 \leq t < \tau_1$. As soon as it reaches ∂D at $\xi_0 = x(\tau_1-)$ it instantaneously jumps to a random point $x_1 \in D$, independent of the process $x(t)$, with distribution $\nu(\xi_0, dx)$. We continue the motion according to the diffusion p^D starting at x_1 until $\tau_2 = \inf\{t > \tau_1 \mid x(t) \in D^c\}$. We set $x(t) = \tilde{x}(t - \tau_1)$ on $\tau_1 \leq t < \tau_2$, where $\tilde{x}(\cdot)$ is an independent version of the killed process, this time starting at x_1. Evidently $\tau_2 - \tau_1 = \tau_2^D$ and we continue indefinitely. Since $P_x(\tau^D > 0) = 1$ for all $x \in D$ we have that τ_l is strictly increasing in $l \geq 0$. It is possible that $\tau_{l'} = \infty$ for a given l', in which case $\tau_l \equiv \infty$ for all $l \geq l'$. Without loss of generality, let $l' = \inf\{l \geq 1 | \tau_l = \infty\}$ and we denote l^* the total number of jumps; obviously $l^* = l' - 1$. We denote $\tau^* = \lim_{l\to\infty} \tau_l \leq \infty$.

In the following, for a sufficiently small $\delta > 0$, we denote $D_\delta = \{x \in D \mid d(x, \partial D) > \delta\}$. An open set $V \subseteq D$ is said a *vicinity of the boundary* if there exists $\delta > 0$ such that $D \backslash V \subseteq \bar{D}_\delta$. The complement of V is said an *interior* set.

The underlying diffusion will be assumed to satisfy the uniform bound on the exit time from a vicinity of the boundary D_δ^c, trivial for a bounded D,

$$\lim_{t\to\infty} \sup_{x\in \bar{D}_\delta^c} P_x(\tau^{D_\delta^c} > t) = 0. \tag{1.2}$$

We are interested in conditions guaranteeing that $x(t)$ is *non-explosive* or, equivalently, does not finish in finite time with positive probability (1.3)

$$\forall x \in D, \quad P_x\left(\lim_{l\to\infty} \tau_l = \infty\right) = 1. \tag{1.3}$$

Lemma 1 contains the key element in the proof of non-explosiveness exhibited by the function $\ln \Phi(x)$, where $\Phi(x)$ is intuitively emulating the distance to a subset A of ∂D, with properties $\Phi(x) > 0$ in D and $\Phi(x) = 0$ on A. The reader may want to think of $D' \subseteq D$ as a subset of $D \backslash \bar{D}_\delta$ representing the "worst case scenario" for survival because $\partial D' \supseteq A$, in other words a set where the process may have the highest chance of extinction - see also the remark following Lemma 1.

Besides technical assumptions contained in (i), properties (ii) and (iii) from Lemma 1 guarantee that $\ln \Phi(x(t))$ is a (local) semi-martingale that experiences a strictly positive jump (iii) on the boundary, implying that the process pays a "price" (1.11) for each jump.

To clarify the notation, the random times τ^D, $\tau^{D_\delta^c}$ from (1.1) to (1.2) refer to hitting times of the process $(\tilde{x}(t))$. Similarly, we use $(\tau_l)_{l\geq 0}, \alpha(D_1)$ for the sequence of boundary hits and, respectively, the first hitting time of a set $D_1 \subseteq D$ by the process $((x(t))$, i.e. $\alpha(D_1) = \inf\{t > 0 \,|\, x(t) \in \bar{D}_1\}$, with $\alpha(D_1) = +\infty$ if $x(t)$ never hits $\bar{D}_1$. The number of jumps up to time $t \geq 0$ will be denoted $J(t)$. Let $l(D_1) = J(\alpha(D_1))$, the number of jumps until the process hits $\bar{D}_1$, i.e. $l(D_1) = \max\{l \,|\, \tau_l \leq \alpha(D_1)\}$.

We give a summary of the paper. Section 1 proves two essential lemmas. Lemma 1 establishes an upper bound for the expected value of the number of jumps before entering an interior set in terms of properties of a test function Φ. One of the terms in the bound is the expected value of the time to enter the interior set. It is Lemma 2 that gives an upper bound for this time, in terms of another test function Ψ. To apply the two lemmas in the case of the Fleming–Viot particle system on the set G described in Sect. 2, we need tests functions ψ, ϕ on G corresponding to Ψ, respectively Φ. It is ψ that requires (C1) (Definition 2, Sect. 2) and ϕ that requires (C2) (Definition 3, Sect. 2). While (C1) is nontrivial, (C2) is satisfied when G has the exterior cone condition (Proposition 1). As a consequence (C1) is essentially the only requirement for non-explosion (Theorem 1, Sect. 2) and geometric ergodicity (Theorem 2, Sect. 3), the latter using ϕ in a coupling argument (Proposition 3). Section 4 proves that (C1) is satisfied for bounded Lipschitz domains when the Martin kernel is integrable (Theorem 3). Propositions 7 and 6 give sufficient conditions for (C1) in terms of the Green function and the solution of the eikonal equation. Section 5 proves the almost sure existence of a unique infinite ancestry line (the immortal particle) and Section 6 calculates explicitly a law of large numbers for the Markov chain, given by the meeting points after jump in the two-particles case, in dimension one.

Lemma 1 *Assume there exists a (possibly empty) closed subset A of the boundary ∂D with $\lambda_0(A) = 0$, an open subset $D' \subseteq D$ and a bounded real function $\Phi \in C^2(D') \cap C(\bar{D})$ with the properties* (i) $\Phi(x) > 0$ *on* $\bar{D}\backslash A$ *and* $\Phi(x) = 0$ *on* A; (ii) *there exists a constant $q(\Phi) \geq 0$ with $\mathcal{L} \ln \Phi(x) \geq -q(\Phi)$ for all $x \in D'$ and* (iii) $U = \inf_{\xi\in(\partial D\cap\partial D')\backslash A} U(\xi) > 0$, *where*

$$U(\xi) = \int_D \ln \Phi(x) \nu(\xi, dx) - \ln \Phi(\xi). \tag{1.4}$$

Then, for all $x \in D'$,

$$E_x[J(\tau^* \wedge \alpha(D\backslash D'))] \leq U^{-1} \times \left[[\sup_{x' \in \bar{D}} \{\ln \Phi(x')\} - \ln \Phi(x)] + q(\Phi) E_x[\tau^* \wedge \alpha(D\backslash D')] \right]. \tag{1.5}$$

If either $q(\Phi) = 0$ or $q(\Phi) > 0$ and $E_x[\tau^* \wedge \alpha(D\backslash D')] < \infty$ for all $x \in D'$, then $P_x(\alpha(D\backslash D') < \tau^*) = 1$ for all $x \in D'$, with the understanding that if $\tau^* = \infty$, then $\alpha(D\backslash D')$ is finite with probability one.

Remark In the F-V particle system from Sect. 2 we have $D = G^N$, where G is the underlying domain for each of the N particles and A will be the *corners* of ∂D, i.e. where all particles are at the boundary and for G' a vicinity of the boundary of G, $D' = (G')^N$.

Proof Step 1. The plan of the proof is as follows. We apply Ito's formula to show that $\ln \Phi(x(t \wedge \alpha(D\backslash D'))), t \geq 0, x(0) = x \in D'$ is a local $(\mathcal{F}_t)$ semi-martingale. Condition (ii) shows that $\ln \Phi(x(t))$ changes in time, being controlled by the lower bound $-q(\Phi)$ as long as $x(t) \in D'$ and (iii) shows that it has strictly positive jumps at the boundary of D' shared with ∂D. This proves the statement up to the first hitting time of $D\backslash D'$.

Since $\Phi(x) = 0$ on A, we create a localizing sequence on $\bar{D}\backslash A$. Due to $\lambda_0(A) = 0$, there exists a nested sequence of open sets $B_k \subseteq \mathbb{R}^d$, $B_k \supseteq A$, such that for all $k \geq 0, d(y, A) < 1/k$ when $y \in B_k$. We may assume without loss of generality that $x \notin B_0$ and $B_0 \subseteq D\backslash D_\delta$. We claim that if $\tau(B_k^c) = \inf\{t > 0 \,|\, x(t) \in \bar{B}_k\}$ and we denote the limit $\tau(B_\infty^c) = \lim_{k\to\infty} \tau(B_k^c)$, then $P_x(\tau(B_\infty^c) \geq \tau^*) = 1$ for all $x \in D\backslash B_0$. Assume $\tau(B_\infty^c) < \tau^*$. The sequence $\tau(B_k^c)$ is non-decreasing, but we want to show that it cannot be constant from a certain rank on. If this would be the case, $\tau(B_k^c) = \tau(B_{k_0}^c)$ for all $k \geq k_0$ and there exists l such that $\tau(B_{k_0}^c) \in [\tau_{l-1}, \tau_l)$. Consequently $x(\tau(B_{k_0}^c)) \in D$ yet $d(x(\tau(B_{k_0}^c)), A) \leq 1/k$ for all $k \geq k_0$, thus $x(\tau(B_{k_0}^c)) \in A$, a contradiction. Without loss of generality, we assume that the sequence $\tau(B_k^c)$ is *strictly increasing*. There are two possibilities: Either $(\tau(B_k^c)), k \geq 0$ has only finitely many points in each episode $[\tau_{l-1}, \tau_l)$, $l \geq 1$, or there exists $l_A < \infty$ with infinitely many $\tau(B_k^c)$ in $[\tau_{l_A-1}, \tau_{l_A})$. In the first case $\tau(B_\infty^c) \geq \tau^*$, and we are done. In the second case, $\tau(B_\infty^c) \neq \tau_{l_A-1}$, so there are two scenarios: Either $\tau(B_\infty^c) \in (\tau_{l_A-1}, \tau_{l_A})$, or $\tau(B_\infty^c) = \tau_{l_A}$. In both, the process $x(t)$ has continuous paths on $(\tau_{l_A-1}, \tau_{l_A})$ and $d(x(\tau(B_k^c)), A) \leq 1/k$ for an infinite subsequence, which implies that the path of the diffusion killed at the boundary has a limit point on A. This event has zero probability on any episode and there are countably many episodes. By choosing the localizing sequence $\tau(B_k^c) \wedge \alpha(D\backslash D')$, $k \geq 0$ we proved *Step 1.*

Step 2. Fix $x(0) = x \in D'$. Denote $M(\Phi) = \sup_{x' \in \bar{D}}\{\ln \Phi(x')\}$, let m be a positive integer, $T > 0$ and put $\tau_j' = \tau_{j\wedge m} \wedge (\tau(B_k^c) \wedge \alpha(D\backslash D')) \wedge T$ for all $j \geq 0$ and $\tau(B_k^c)$, k fixed at the moment, as in *Step 1*. With this notation, the summations below are finite, and we can write

$$M(\Phi) - \ln \Phi(x) \geq E_x[\ln \Phi(x(\tau_{l(D\backslash D')\wedge m} \wedge (\tau(B_k^c) \wedge \alpha(D\backslash D')) \wedge T)) - \ln \Phi(x(0))] \tag{1.6}$$

$$= E_x\left[\sum_{j=1}^{l(D\backslash D')} \ln\Phi(x(\tau'_j)) - \ln\Phi(x(\tau'_{j-1}))\right] \tag{1.7}$$

$$\begin{aligned} &= E_x\left[\sum_{j=1}^{l(D\backslash D')} [\ln\Phi(x(\tau'_j)) - \ln\Phi(x(\tau'_j-))]\right.\\ &\quad \left. + \sum_{j=1}^{l(D\backslash D')} [\ln\Phi(x(\tau'_j-)) - \ln\Phi(x(\tau'_{j-1}))\right] \end{aligned} \tag{1.8}$$

The second term of (1.8) representing the sum over diffusive time intervals $[\tau'_{j-1}, \tau'_j-)$ is bounded below by $-q(\Phi)E_x[\tau'_{l(D\backslash D')}]$ by applying Ito's formula on the intervals between jumps. The first term, representing the jump at τ'_j is bounded below by

$$E_x\left[\sum_{j=1}^{m} E_x[\ln\Phi(x(\tau'_j)) - \ln\Phi(x(\tau'_j-))|\mathcal{F}_{\tau'_j-}]\right] \tag{1.9}$$

$$= \sum_{j=1}^{m} E_x[E_x[\ln\Phi(x(\tau'_j)) - \ln\Phi(x(\tau'_j-))\,\big|\,\mathbf{x}(\tau'_j-)]], \tag{1.10}$$

where we used the strong Markov property. Due to the choice of the times τ'_j, the sequence τ'_j becomes constant for $j \geq m$ (or possibly earlier on). Let $\eta(s)$, $s > 0$ be equal to one if s is an actual jump time of the process $x(s) - x(s-) \neq 0$ and to zero if it is a continuity point. With $J(t)$ denoting the number of jumps up to time t,

$$E_x[\ln\Phi(x(\tau'_j)) - \ln\Phi(x(\tau'_j-))\,\big|\,x(\tau'_j-)] \geq U\eta(\tau'_j) \tag{1.11}$$

leading to the lower bound $UE_x[J((\tau(B_k^c) \wedge \alpha(D\backslash D')) \wedge T) \wedge m]$ for line (1.10). Moving the lower bound $E_x[\tau'_{l(D\backslash D')}]$ of the second term in (1.8) to the left hand side of (1.6), we have shown

$$E_x[J((\tau(B_k^c) \wedge \alpha(D\backslash D')) \wedge T) \wedge m] \leq U^{-1}\left[(M(\Phi) - \ln\Phi(x)) + q(\Phi)E_x[\tau'_{l(D\backslash D')}]\right], \tag{1.12}$$

with the first term on the right hand side not depending on T, k, and m. We let $m \to \infty$, then $T \to \infty$ and finally $k \to \infty$ to obtain

$$E[J(\tau^* \wedge \alpha(D\backslash D'))] \leq U^{-1}\left[(M(\Phi) - \ln\Phi(x)) + q(\Phi)E_x[\tau^* \wedge \alpha(D\backslash D')]\right]. \tag{1.13}$$

By hypothesis, when either $q(\Phi) = 0$, or $q(\Phi) > 0$ and $E_x[\tau^* \wedge \alpha(D\backslash D')] < \infty$, the right hand side is finite, showing that $J(\tau^* \wedge \alpha(D\backslash D')) < \infty$ almost surely. Since

$J(\tau^*) = \infty$, it is not possible that $\alpha(D\backslash D') \geq \tau^*$, therefore $\alpha(D\backslash D') < \tau^*$ with probability one. In particular, if $\tau^* = \infty$, then $\alpha(D\backslash D')$ is finite. □

The next result gives conditions under which $E_x[\tau^* \wedge \alpha(D\backslash D')] < \infty$.

Lemma 2 *Let D, D' be as in Lemma 1 and let Ψ satisfy property* (i) *from the lemma;* (ii) *there exists a constant $q_1(\Psi) > 0$ with $\mathcal{L}\Psi(x) \geq q_1(\Psi)$ for all $x \in D'$ and* (iii) $U_1 = \inf_{\xi\in(\partial D\cap\partial D')\backslash A} U_1(\xi) > 1$, *where*

$$U_1(\xi) = \Psi(\xi)^{-1} \int\limits_D \Psi(x)\nu(\xi, dx). \tag{1.14}$$

Then, for all $x \in D'$,

$$E_x[\tau^* \wedge \alpha(D\backslash D')] < \infty, \qquad E_x\left[\sum_{j=1}^{J(\tau^*\wedge\alpha(D\backslash D'))} \Psi(x((\tau_l \wedge \alpha(D\backslash D')-))\right] < \infty. \tag{1.15}$$

As a consequence, if D' is a vicinity of the boundary and $\alpha(D\backslash D') = +\infty$, then the configurations $(x(\tau_l-))$ at jump times $l \geq 0$ converge to the subset A of the boundary with probability one.

Remark When applied to the F-V model from Sect. 2, the lemma proves the following dichotomy: *Either the particles enter an interior set before τ^*, or they simultaneously converge to the boundary of the set in finite time.*

Proof The proof is almost identical to the proof of Lemma 1. Since $\int_D \Psi(x)\nu(\xi, dx) \geq U_1\Psi(x)$ is (trivially) satisfied even when $\Psi(x) = 0$, we do not have to use the localization sequence (B_k^c)). The process $(\Psi(x(t \wedge \alpha(D\backslash D')))$ is a sub-martingale. We obtain

$$\Psi(x(t \wedge \alpha(D\backslash D'))) = \Psi(x) + (I) + (II) + (III) \tag{1.16}$$

where

$$(I) = \Psi(x(t \wedge \alpha(D\backslash D'))) - \Psi(x(\tau_{l(D\backslash D')} \wedge \alpha(D\backslash D'))) \tag{1.17}$$

$$(II) = \sum_{l=1}^{J(t\wedge\alpha(D\backslash D'))} \Psi(x(\tau_l \wedge \alpha(D\backslash D')-)) - \Psi(x(\tau_{l-1} \wedge \alpha(D\backslash D')) \tag{1.18}$$

$$(III) = \sum_{l=1}^{J(t\wedge\alpha(D\backslash D'))} \Psi(x(\tau_l \wedge \alpha(D\backslash D'))) - \Psi(x(\tau_l \wedge \alpha(D\backslash D')-)) \tag{1.19}$$

For (i) and (II) we apply Ito's formula in the intervals between jumps. For (III) we follow the same steps as in (1.9)–(1.11) to obtain

$$\begin{aligned} &E_x[\Psi(x(\tau_l \wedge \alpha(D\backslash D'))) - \Psi(x(\tau_l \wedge \alpha(D\backslash D')-)) \,\big|\, x(\tau_l-)] \\ &\quad \geq \eta(U_1 - 1)\Psi(x(\tau_l \wedge \alpha(D\backslash D')-)), \end{aligned} \tag{1.20}$$

where η, as in (1.11) is equal to one if there was an actual jump at $\tau_l \wedge \alpha(D\backslash D')$ and zero otherwise. Summing up, with $C(\Psi, x) = \sup_{x\in\bar{D}'} |\Psi(x)| - \Psi(x)$, we have

$$\begin{aligned} C(\Psi, x) \geq E_x \Bigg[& q_1(\Psi)(t \wedge \alpha(D\backslash D')) + (U_1 - 1) \\ & \times \sum_{j=1}^{J(t\wedge\alpha(D\backslash D'))} \Psi(x((\tau_l \wedge \alpha(D\backslash D')-)) \Bigg] \end{aligned} \tag{1.21}$$

and, after letting $t \to \infty$, we obtain (1.15). The last claim is a consequence of the fact that the series on the right hand side of (1.15) is convergent it this case and $\Psi(x)$ is continuous and reaches zero only on the $A \subseteq \partial D$. □

2 The Fleming–Viot redistribution case

In this setup, $N \geq 2$ is a positive integer, the domain $D = G^N$, with G a domain in $\mathbb{R}^q$, $d = Nq$ with *regular boundary* ∂G. The process $\{\mathbf{x}(t)\}_{t\geq 0}$ has components $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))$ (called particles), each $\{x_i(t)\}_{t\geq 0}$, $1 \leq i \leq N$ evolving in G as a q - dimensional diffusion with jumps at the boundary ∂G to be described in the following. As before, the process $\{\mathbf{x}(t)\}_{t\geq 0}$ is adapted to a right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$. For $\xi \in \partial D$ we write $I(\xi) = \{i \mid \xi_i \in \partial G\}$ and $\xi^{ij} \in G^N$ denotes the vector with the same components as ξ with the exception of ξ_i which is replaced by ξ_j.

When a particle x_i reaches ∂G at τ, it jumps instantaneously to the location of one of the remaining particles x_j, $1 \leq j \leq N$, $j \neq i$ (there are no simultaneous boundary visits a.s.) with probabilities $p(\mathbf{x}(\tau-), j)$, $1 \leq j \leq N$, having only the restriction $p(\mathbf{x}(\tau-), i) = 0$. It is obviously possible to allow positive probabilities for stopping at the boundary, a standard construction being to allow an exponential time before attempting a new jump. However we do not pursue this approach here since it rather obscures the natural question of non-explosiveness. There is no real ambiguity concerning points on the "edges" of the boundary (i.e. when at least two components are on ∂G, or $|I(\xi)| \geq 2$) since the underlying diffusion does not visit a.s. sets of co-dimension greater than two as soon as it starts at points $\mathbf{x} \in D$. The state space is only the open set D, so we shall not start the process on the boundary. However we may define without loss of generality $\nu_\xi(d\mathbf{x})$ for all $\xi \in \partial D$ as in (2.1). More precisely, there exist measurable functions $\partial G^N \ni \xi \to p_{ij}(\xi) \in [0, 1]$, indexed by $1 \leq i, j \leq N$ such that $p_{ij}(\xi) = 0$ whenever $i = j$ and $\sum_j p_{ij}(\xi) = 1$ such that

$$\forall \xi \in \partial G^N, \qquad \nu(\xi, d\mathbf{x}) = \frac{1}{|I(\xi)|} \sum_{i\in I(\xi)} \sum_{j=1}^{N} p_{ij}(\xi)\delta_{\xi^{ij}}(d\mathbf{x}). \tag{2.1}$$

Definition 1 We shall say that the redistribution probabilities $p_{ij}(\xi)$ are *non-degenerate* if they are bounded away from zero uniformly; i.e. there exists $p_0 > 0$ independent of $\xi \in \partial G^N$, such that $p_{ij}(\xi) \geq p_0$, $1 \leq i, j \leq N, i \neq j$.

Remark (1) Except on the edges of $D = G^N$, formula (2.1) does not have a proper average over $i \in I(\xi)$. The definition is consistent over all $\xi \in \partial D$.

(2) The most common choice of $p_{ij}(\xi)$ is uniform $p_{ij}(\xi) = (N-1)^{-1}$, $j \neq i, \xi \in \partial G$. In that case $p_0 = (N-1)^{-1}$.

(3) The definition (2.1) is *not necessarily continuous* as a function in ξ into $M_1(\bar{D})$ with the topology of weak convergence of measures; the reader may check the case $N = 3, d = 1$ with the redistribution measures from 2).

(4) Assume D is bounded. Then $\bar{D}$ is compact, and the family of measures $(\nu_\xi(d\mathbf{x}))_{\xi \in \partial D}$ is tight. Nonetheless, limit points might be concentrated on ∂D, which raise the possibility that the process be explosive.

(5) Definition 1 can be relaxed, with proper care for the regularity of the domain, as follows. It is only the $p_{ij}(\xi)$ corresponding to the j with *maximum distance* from the boundary that needs a lower bound.

We shall further assume that the particles $x_i(t)$ evolve *independently* between jumps, each following a diffusion with generator L on $\mathbb{R}^q$ killed at the boundary ∂G. More specifically

$$Lu(x) = \sum_{1\leq\alpha\leq q} b^\alpha(x)\frac{\partial u}{\partial x^\alpha}(x) + \frac{1}{2}\sum_{1\leq\alpha,\beta\leq q} a^{\alpha,\beta}(x)\frac{\partial^2 u}{\partial x^\alpha \partial x^\beta}(x), \quad u \in C_0(\mathbb{R}^q) \cap C^2(\mathbb{R}^q), \tag{2.2}$$

with coefficients $\{b^\alpha(x)\}_\alpha$, $\{a^{\alpha,\beta}(x)\}_{\alpha,\beta}$ in $C^\infty(\mathbb{R}^q)$. With the notation $\sigma(x)\sigma^*(x) = a(x)$ (the star stands for the matrix transposition), the coefficients are uniformly bounded, with L strictly elliptic

$$|b^\alpha(x)| \leq ||b||, \qquad 0 < \sigma_0^2||v||^2 \leq ||\langle\sigma(x)\sigma^*(x)v, v\rangle|| \leq ||\sigma||^2||v||^2, \quad v \in \mathbb{R}^q, \tag{2.3}$$

where $||b||, \sigma_0, ||\sigma||$ do not depend on x, α, β. Under these conditions, there exists a family of Brownian motions $\{w_i^\beta(t)\}_{1\leq\beta\leq q}$, mutually independent in i as well as β, adapted to $(\mathcal{F}_t)$, such that between successive jumps, the N components $x_i(t) = (x_i^1(t), \ldots, x_i^q(t)) \in G$, where $(x_i^\alpha(t))_{1\leq\alpha\leq q}$ are solutions to the stochastic differential equations

$$dx_i^\alpha(t) = b^\alpha(x_i(t))dt + \sum_{1\leq\beta\leq q} \sigma^{\alpha,\beta}(x_i(t))dw_i^\beta(t), \quad 1 \leq \alpha, \beta \leq q\,, \quad x_i(0) = x_{i0} \in G, \tag{2.4}$$

for all $1 \leq i \leq N$.

2.1 Domain regularity

Until this point we only required that ∂G be regular. For any regular domain G and any one-particle diffusion with smooth coefficients (2.2), if U, V are two open subsets of G with $U \subseteq V \subseteq G$, $T > 0$, we denote by $p_{\pm}(T, U, V)$ the supremum, respectively infimum over $x \in U$ of $P_x(\tau^V > T)$, where τ^V denotes the first hitting time of ∂V. We start with the following remark. If $U \subset\subset V$ such that $0 < d_- \le d(\partial U, \partial V) \le d_+ < \infty$, then there exist constants $p_{\pm}(T, U, V)$ such that for all $x \in \bar{U}$

$$0 < p_-(T, U, V) \le P_x(\tau^V > T) \le p_+(T, U, V) < 1. \tag{2.5}$$

To check (2.5), we set $w(T, x) = P_x(\tau^V > T)$ on $x \in V$ and note that $(\partial_T - L)w(T, x) = 0, 0 \le w(T, x) \le 1$ and $w(T, x) = 0$ on ∂V. The lower bound is guaranteed by the maximum principle applied to $w(T, x)$ and the upper bound by applying it to $1 - w(T, x)$.

We remind the reader the definition of a vicinity of the boundary ∂G is given right before Lemma 1.

Definition 2 Condition (C1). We shall say that G satisfies (C1) if there exists a vicinity of the boundary G' and there exists a function ψ such that (i) $\psi \in C^2(G') \cap C(\bar{G}')$; (ii) $\psi(x) > 0, x \in G'$; (iii) $\psi(x) = 0, x \in \partial G$; (iv) there exists a constant q_- depending on G' and ψ only, such that $L\psi(x) \ge q_- > 0$.

Remark In case G is a bounded C^2 domain there are many choices of ψ satisfying Definition 2. The most natural is $\psi(x) = d^2(x, \partial D)$, the square of the distance to the boundary. It is known [19] that ϕ solves the *eikonal equation* $||\nabla\psi(x)||^2 = 1$ and coincides with the classical smooth solution when D is smooth; more examples are discussed in Sect. 4.

Definition 3 Condition (C2). We shall say that G satisfies (C2) if there exists a vicinity of the boundary G' and there exists a function $\phi \in C(\bar{G}') \cap C^2(G')$ such that (i) $L\phi(x) \ge 0$, (ii) $\phi(x) > 0$ on G', (iii) $\phi(x) = 0$ on ∂G and (iv) there exists a constant $c_2 > 0$ such that $||\nabla\phi(x)||^2 \le c_2\phi(x)$ for any $x \in G'$.

Proposition 1 *Assume $G \subseteq \mathbb{R}^q$ is a bounded domain satisfying the* uniform exterior cone condition *(*[13]*, page* 205*). Then condition (C2) is satisfied.*

Proof Without loss of generality, we may take $\partial G'$ smooth so we shall be concerned with boundary estimates at ∂G only. Let $u(x)$ be the solution of the Dirichlet problem $Lu = 0$, $u(x) = 0$ on ∂G and $u(x) = 1$ on $\partial G' \backslash \partial G$. The goal is to show that $\phi(x) = u^m(x)$, for a power $m \ge 2$ to be chosen later on, satisfies the requirements of the proposition.

(i) Directly we obtain

$$L\phi(x) = \frac{1}{2}m(m-1)u^{m-2}(x)||\sigma^*(x)\nabla u(x)||^2 + mu^{m-1}(x)Lu(x) \ge 0 \tag{2.6}$$

while (ii) is a consequence of the maximum principle. Property (iii) is true by construction. The only difficult part is to ensure (iv) is true.

(iv) From Theorem 8.29 in [13] we know that $u \in C^{\beta'}(\bar{G}')$, $\beta' \in (0, 1)$, where $C^{\beta'}(\bar{G}')$ is the space of Hölder continuous functions with exponent β'. Since u vanishes on ∂G, we have $0 \leq u(x) \leq c_3 d(x, \partial G)^{\beta'}$. On the other hand, we know (again [13], page 38, eq. 3.16) that $d(x, \partial G)||\nabla u(x)|| \leq c_4$, where c_4 depends on the set G and boundary values. It follows that

$$
\begin{aligned}
||\nabla \phi(x)||^2 &= m^2 u(x)^{2(m-1)} ||\nabla u(x)||^2 \\
&\leq u^m(x)[d(x, \partial G)^2 ||\nabla u(x)||^2][m^2 d(x, \partial G)^{-2} u^{m-2}(x)] \\
&\leq \phi(x) c_4^2 m^2 c_3^{m-2} d(x, \partial G)^{(m-2)\beta' - 2},
\end{aligned}
$$

where $d(G) = \sup_{x, x' \in G} d(x, x') < \infty$, proving condition (iv) for any $m \geq 2\beta'^{-1} + 2$ with $c_2 = c_4^2 m^2 c_3^{m-2} d(G')^{(m-2)\beta' - 2}$. □

The following propositions shows that we can apply Lemma 2.

Proposition 2 *On the set $D = G^N$, the function $\Psi(\mathbf{x}) = \sum_{i=1}^N \psi(x_i)$, with ψ as in (C1), $D' = (G')^N$, $A = (\partial G)^N$, satisfies Lemma 2 with $U_1 = 1 + p_0$, where $p_0 > 0$ is the constant from Definition 1 and $q_1(\Psi) = N q_-$.*

Proof Without loss of generality, ψ can be extended to all G so that $\psi(x)$ remains bounded away from zero on $G \backslash G'$ and $\Psi(\mathbf{x}) = \sum_{i=1}^N \psi(x_i)$ satisfies (i) in Lemma 2. The only thing to verify is the lower bound for (1.14) which is evident due to the lower bound $p_{ij}(\xi) \geq p_0 > 0$ in Definition 1, for all $\xi \in (\partial G^N) \backslash (\partial G)^N$. □

We are ready to state the main result.

Theorem 1 *Assume that G satisfies (C1) and (C2) and the relocation probabilities satisfy the condition in Definition 1. Then, for any $N \geq 2$, the process is non-explosive in the sense of (1.3).*

Proof Proposition 2 shows that Lemma 2 is applicable to the process $(\mathbf{x}(t))$. The plan is to prove the theorem in two steps. *Step 1* will apply Lemma 1 to $D = G^N$ with $D' = (G')^N$, where $G \backslash G' \subseteq \bar{G}_{2^N \delta}$ for some suitably small but fixed $\delta > 0$ and the set $A = \{\xi \in \partial G^N \mid I(\xi) = N\}$ will be the *vertices* of the domain, i.e. the part of the boundary ∂G^N with all components in ∂G. *Step 1* will conclude that the process $\mathbf{x}(t)$ exits in finite time D', with probability one. In *Step 2* we show that once in $D \backslash D'$, the process will hit the set $(\bar{G}_\delta)^N$ in *a finite number of jumps* with probability one. From that point on we apply Lemma 4 and we are done.

Step 1. As in the proof of Proposition 2 without loss of generality we may extend ϕ to all G with $\phi(x)$ bounded away from zero on $G \backslash G'$. Let $(\mathbf{y}(t))$ be the process with one-dimensional components $y_i(t) := \phi(x_i(t))$, $t \geq 0$, where ϕ is the function in (C2). We are interested in the logarithm of the radial process $(r(t))$

$$
r(t) = \Phi(x(t)), \quad \Phi(\mathbf{x}) = \left(\sum_{i=1}^N \phi^2(x_i) \right)^{\frac{1}{2}}. \tag{2.7}
$$

Using Ito's lemma, the N-dimensional process $(\mathbf{y}(t))$ satisfies the stochastic differential equations

$$dy_i(t) = \tilde{b}_i(t)dt + \tilde{\sigma}_i(t)d\tilde{w}_i(t), \quad y_i(0) = \phi(x_{i0}), \tag{2.8}$$

where $\{\tilde{w}_i(t)\}_{1\leq i\leq N}$ are Brownian motions adapted to $(\mathcal{F}_t)$ obtained from (2.4) by the representation theorem for continuous martingales. Concretely, $\tilde{b}(t) = (\tilde{b}_i(t))_{1\leq i\leq N}$, $(\tilde{\sigma}_i(t))_{1\leq i\leq N}$ have components

$$\tilde{b}_i(t) = L\phi(x_i(t)), \quad \tilde{\sigma}_i(t) = ||\sigma^*(x_i(t))\nabla\phi(x_i(t))|| \tag{2.9}$$

with the inequalities

$$0 < \sigma_0^2||\nabla\phi(x_i(t))||^2 \leq \tilde{\sigma}_i^2(t) \leq ||\sigma||^2||\nabla\phi(x_i(t))||^2 \tag{2.10}$$

due to (2.3). By construction, $\Phi(\mathbf{x}) = 0$ if and only if all $\phi(x_i) = 0$. In D', this means only on A. The only conditions on Φ from Lemma 1 that have to be verified are (ii) and (iii).

Between jumps $r(t)$ satisfies

$$dr(t) = B(t)dt + S(t)dW(t), \quad r(0) = ||\phi(x(0))||, \tag{2.11}$$

where $(W(t))$ is a one - dimensional Brownian motion adapted to $(\mathcal{F}_t)$, while the drift $B(t)$ and variance matrix $S(t)$ are given by (here $Tr(A)$ is the trace of the $N \times N$ matrix A)

$$B(t) = \frac{1}{2r(t)}\left(2\langle\mathbf{y}(t), \tilde{b}(t)\rangle + Tr(\tilde{\sigma}(t)\tilde{\sigma}^*(t)) - \frac{||\tilde{\sigma}^*(t)\mathbf{y}(t)||^2}{r^2(t)}\right) \tag{2.12}$$

$$S(t) = \frac{||\tilde{\sigma}^*(t)\mathbf{y}(t)||}{r(t)}. \tag{2.13}$$

In the formula above $\tilde{\sigma}^*(t)$ is the $N \times N$ diagonal matric with entries $\tilde{\sigma}_i(t)$ from (2.9).

Relations (2.12)–(2.13) show that in order to verify the conditions of Lemma 1 we have to prove $(2r(t))^{-1}(2r(t)B(t) - S^2(t)) \geq -q(\Phi), q(\Phi) > 0$. This is equivalent to

$$\frac{1}{2r(t)}\left(-2\langle\mathbf{y}(t), \tilde{b}(t)\rangle - Tr(\tilde{\sigma}(t)\tilde{\sigma}^*(t)) + 2\frac{||\tilde{\sigma}^*(t)\mathbf{y}(t)||^2}{r^2(t)}\right) \leq q(\Phi). \tag{2.14}$$

Since $\phi(x) \geq 0$ and $L\phi(x) \geq 0$ we only have to check if the last term is uniformly bounded above. Multiplying by $(2r(t))^{-1}$ and using the bound (iv) in (C2), Definition 3,

$$\frac{||\tilde{\sigma}^*(t)\mathbf{y}(t)||^2}{r^3(t)} \leq c_2||\sigma||^2\frac{\sum_{i=1}^N y_i^3(t)}{r(t)^3} \leq Nc_2||\sigma||^2 = q(\Phi).$$

We verify (iii) from Lemma 1. We shall prove (iii) for boundary points ξ with $|I(\xi)| \leq N-1$, which includes the set $(\partial D \cap \partial D')\backslash A$. We note that, with probability one, only boundary points ξ with $I(\xi) = 1$ are visited. Abusing notation, we write $I(\xi) = i$ for the component located on the boundary ∂G. The process $\mathbf{y}(t)$ jumps if and only if a component reaches zero, which is equivalent to $\mathbf{x}(t)$ reaching ∂G^N at some point ξ (here we make use of the condition that $\phi(x) > 0$ except on A). To simplify notation, let $p_{Ij} = p_{ij}(\xi)$ denote the corresponding relocation probabilities.

Due to the condition in Definition 1 we have the non-random lower bound away from zero, uniformly in N:

$$\int_{G^N} \ln \Phi(\mathbf{x})\nu(\xi, d\mathbf{x}) - \ln \Phi(\xi) = \sum_{j\neq I} \frac{p_{Ij}}{2} \ln\left(1 + \frac{\phi^2(x_j)}{\sum_{k\neq I}\phi^2(x_k)}\right)$$
$$\geq \frac{p_0}{2}\ln\left(\frac{N}{N-1}\right) > 0, \tag{2.15}$$

which shows (1.4) with $U = \frac{p_0}{2}\ln\left(\frac{N}{N-1}\right)$. With the notation of Lemma 1, we have

$$\forall \mathbf{x} \in D \qquad P_{\mathbf{x}}(l(D\backslash D') < \infty) = 1, \quad P_{\mathbf{x}}(\alpha(D\backslash D') < \tau^*) = 1. \tag{2.16}$$

This concludes the proof of Step 1.

Step 2. For a $\delta > 0$ fixed, let F_k be the set of configurations with exactly $N-k$ particles in $\bar{G}_{2^k\delta}$ (or exactly k in the vicinity of the boundary $G\backslash\bar{G}_{2^k\delta}$). For a small $a > 0$,

$$F_k(a) = \left\{\mathbf{x} \in \bar{G}^N \mid \sum_{i=1}^{N} \mathbf{1}_{G\backslash\bar{G}_a}(x_i) = k\right\}, \quad A_k(a) = \cup_{j=0}^{k} F_j(a). \tag{2.17}$$

Let $F_k = F_k(2^k\delta)$ for $a = 2^k\delta$ and $A_k = \cup_{j=0}^{k} F_j$. We notice that $F_0 = (G_\delta)^N \subseteq \bar{D}_\delta$. Set $D' = F_N = (G\backslash\bar{G}_{2^N\delta})^N$, with $\alpha(D\backslash D')$ the first hitting time of $D\backslash D'$, as in Lemma 1. We have shown in *Step 1* that the lemma applies to the process $(\mathbf{x}(t))_{t\geq 0}$ and the open set D' and thus $P_{\mathbf{x}}(\alpha(D\backslash D') < \infty) = 1$ for all $\mathbf{x} \in D'$. In other words, if α_k is the first hitting time of A_k for all $k = 0, \ldots, N-1$, then $\alpha_{N-1} \leq \alpha(D\backslash D')$ is finite with probability one. To verify this inequality, we show that $\mathbf{x}(\alpha(D\backslash D')) \in A_{N-1}$. Since $\mathbf{x}(\alpha(D\backslash D')) \in F_N^c$ we only have to check that $F_N^c \subseteq A_{N-1}$.

$$F_N^c \subseteq A_{N-1}(2^N\delta) \subseteq A_{N-1}(2^{N-1}\delta) = \cup_{j=0}^{N-1} F_j(2^{N-1}\delta) \subseteq \cup_{j=0}^{N-1} A_j(2^j\delta) = A_{N-1}.$$

For all $k \geq 1$ and all $\mathbf{x} \in F_k$, $d(\mathbf{x}, F_0) \leq N2^N\delta$, $d(\mathbf{x}, \partial D) \leq 2^N\delta$, and thus $d(x, \partial(D\backslash F_0)) \leq N2^N\delta$, which implies that for any $\mathbf{x} \in F_k$, the time to reach either the interior set F_0 or the boundary ∂D is finite with probability one.

Let $\tau_0(D') = \alpha(D\backslash D')$ and $\tau_k(D')$, $k = 1, 2, \ldots, N-1$ be the first $N-1$ jump times coming right after $\alpha(D\backslash D')$. Starting with A_{N-1}, we want to reach $A_{N-2}, \ldots A_0$ with positive probability in each step. We proceed to show that for each $1 \leq k \leq N$ (in the proof k runs in decreasing order from $k = N$ to $k = 1$), the probability of the event $\mathcal{E} = \{\alpha_{k-1} \le \tau_{N-k}(D')\}$ of reaching A_{k-1} at the time of the $(N-k)$-th jump or before has a lower bound away from zero, independent of the starting point in F_k. The fact that we reach the set at jump time is important, since we want to reach A_{k-1} at a time $\alpha_{k-1} < \tau^*$. Note first that $k = N$ is satisfied by *Step 1*. For other k, denote τ' the first time when one of the $N-k$ particles situated at time $t = 0$ in $G_{2^k\delta}$ reaches $G_{2^{k-1}\delta}$, $\mathcal{E}'$ the event that the first jump is onto one of these $N-k$ particles and τ'' the first time when one of the k particles in $G\backslash\bar{G}_{2^k\delta}$ at time $t = 0$ reaches ∂G. Then, for a fixed $T_0 > 0$,

$$\mathcal{E} \supseteq \{\tau' > T_0, \tau'' \le T_0\} \cap \mathcal{E}'. \tag{2.18}$$

Under the event from the right-hand side of (2.18) we have $\tau^D = \tau'' \le T_0$, which implies that we may analyze all N particles independently up to τ^D-. At the same time, the jump is independent of the past. The uniform lower bound for the probability of $\mathcal{E}$ is based on the bounds on the exit probability, respectively the redistribution probability ν_ξ when $k \le N-1$

$$\inf_{\mathbf{x}\in F_k} P_{\mathbf{x}}(\mathcal{E}) \ge \inf_{\mathbf{x}\in F_k} P_{\mathbf{x}}\left(\tau' > T_0\right) \inf_{\mathbf{x}\in F_k} P_{\mathbf{x}}\left(\tau'' \le T_0\right) \inf_{\mathbf{x}\in\partial D\cap F_k} \nu_\xi(F_{k-1}) \tag{2.19}$$

$$\ge p_-(T_0, G_{2^k\delta}, G_{2^{k-1}\delta})^{N-k}\left[1-(p_+(T_0,G,G))^k\right] p_0 = p_{0,k} \tag{2.20}$$

where p_0 is the lower bound from Definition 1 and $p_\pm$ are defined in (2.5). Summarizing the information from (2.19) to (2.20), the probability to reach F_0 after the $N-1$ jumps following $\alpha(D\backslash D')$ when starting at an arbitrary $\mathbf{x} \in D\backslash D'$ has a positive lower bound $p = \Pi_{k=1}^{N-1} p_{0,N-k}$ independent of $\mathbf{x}$. With the notation $l(D_\delta)$ for the number of jumps until reaching the set $\overline{D_\delta}$, we have shown

$$\inf_{\mathbf{x}\in D\backslash D'} P_{\mathbf{x}}(l(D_\delta) \le N-1) \ge p > 0. \tag{2.21}$$

We shall use this and (2.16) to complete the proof.

Let $(X_n)_{n\ge 0}$ be the *interior chain* on D generated by $(\mathbf{x}(t))$—see [16] for more details—displaying the consecutive positions of the process $(\mathbf{x}(t))$ at jumps times. In other words, $X_n = \mathbf{x}(\tau_n)$, $n \ge 0$. In discrete time $n = 0, 1, \dots$ we denote $\alpha_X(B) = \inf\{n \ge 0 | X_n \in B\}$, B a Borel subset of D. We now apply Lemma 3 to $F = A_{N-1} \supseteq D\backslash F_N$, $\tau_X = \alpha_X(F_0)$, $m = N-1$ to show that $P_{\mathbf{x}}(\alpha_X(F_0) < \infty) = 1$ for all $\mathbf{x} \in D$. This shows that the number of jumps $l(\delta)$ until reaching $\bar{D}_\delta$ satisfies $P_{\mathbf{x}}(l(D_\delta) < \infty) = 1$, which implies that $P_{\mathbf{x}}(\alpha(D_\delta) < \tau^*) = 1$. Based on Lemma 4 we have that $\tau^* = \infty$ almost surely. □

Lemma 3 *Let $(X_n)_{n\ge 0}$ be a Markov chain on D, $F \subseteq D$ be a closed subset of D and τ_X a stopping time. If $P_{\mathbf{x}}(\alpha_X(F) < \infty) = 1$ for all $\mathbf{x} \in D$ and there exists an integer $m > 0$ and a number $p > 0$ independent of m such that $P_{\mathbf{x}}(\tau_X \le m) \ge p$ uniformly in $\mathbf{x} \in F$, then $P_{\mathbf{x}}(\tau_X < \infty) = 1$ for all $\mathbf{x} \in D$.*

Proof Let $\xi_0 = 0$, $\alpha_{X,1} = \inf\{n > \xi_0 \,|\, X_n \in F\}$, $\xi_1 = \alpha_{X,1} + m$ and inductively

$$\alpha_{X,l} = \inf\{n > \xi_{l-1} \,|\, X_n \in F\}, \quad \xi_l = \alpha_{X,l} + m, \quad l \ge 2. \tag{2.22}$$

By construction, the stopping times ξ_l satisfy $P_{\mathbf{x}}(\xi_l < \infty)$ for all $\mathbf{x} \in D$ and $l = 1, 2, \ldots$, and $P_{\mathbf{x}}(\lim_{l\to\infty} \xi_l = \infty)$. Set k a positive integer. Successive applications of the strong Markov property on the intervals $[\xi_{l-1}, \alpha_l]$, $[\alpha_l, \xi_l]$, $l \geq 1$ give

$$P_{\mathbf{x}}(\tau_X > \xi_k) \leq E_{\mathbf{x}}[\Pi_{l=1}^{k} P_{X_{\alpha_{X,l}}}(\tau > m)] \leq (1-p)^k, \tag{2.23}$$

where the first inequality is obtained by neglecting the intervals $[\xi_{l-1}, \alpha_{X,l}]$. Since k is arbitrary, we proved that $P_{\mathbf{x}}(\tau_X < \infty) = 1$. □

The following lemma formalizes the idea that if the interior set $\bar{D}_\delta$ is reached a.s. before τ^*, then the process has to cross the region $D\backslash\bar{D}_\delta$ infinitely many times. Since the duration of such a crossing is uniformly bounded away from zero (in some appropriate sense), the process cannot end in finite time.

Lemma 4 *Let $F \subseteq \bar{D}_\delta$ for some $\delta > 0$. If for any $\mathbf{x} \in D$ we have $P_{\mathbf{x}}(\alpha(F) < \tau^*) = 1$, then for any $\mathbf{x} \in D$ we have $P_{\mathbf{x}}(\tau^* = \infty) = 1$.*

Proof In view of the hypothesis, $\tau^* > \alpha(F) \geq \alpha(\bar{D}_\delta)$. It is then sufficient to prove the Lemma for $F = \bar{D}_\delta$. Let $S < \infty$ be a positive deterministic time; we want to show that $P_{\mathbf{x}}(\tau^* \leq S) = 0$. Since the first jump time satisfies $\tau_1 < \infty$ a.s., we define $\alpha_1 = \inf\{t > \tau_1 \,|\, \mathbf{x}(t) \in \bar{D}_\delta\}$. An application of the strong Markov property to τ_1 together with the hypothesis imply that $P_{\mathbf{x}}(\alpha_1 < \tau^*) = 1$ for any $\mathbf{x} \in D$. We note that this also implies that $\alpha_1 < \infty$ with probability one. Put $u(S) = \sup_{\mathbf{x}\in\bar{D}_\delta} P_{\mathbf{x}}(\tau^* \leq S)$. Applying the strong Markov property to the stopping time α_1 in the second inequality below, we obtain

$$\begin{aligned}
P_{\mathbf{x}}(\tau^* \leq S) &= P_{\mathbf{x}}(\tau^* \leq S\,,\, \alpha_1 < \tau^*) \\
&\leq P_{\mathbf{x}}(\tau^* \leq S\,,\, \alpha_1 < S) = \int_0^S P_{\mathbf{x}}(\tau^* \leq S \,|\, \alpha_1 = s) P_{\mathbf{x}}(\alpha_1 \in ds) \\
&\leq \int_0^S E_{\mathbf{x}}[P_{\mathbf{x}(\alpha_1)}(\tau^* \leq S - \alpha_1) \,|\, \alpha_1 = s]\, P_{\mathbf{x}}(\alpha_1 \in ds) \leq u(S) P_{\mathbf{x}}(\alpha_1 \leq S).
\end{aligned}$$

The last inequality was obtained by taking the supremum over $\mathbf{x}(\alpha_1) \in \bar{D}_\delta$. The supremum over $\mathbf{x} \in \bar{D}_\delta$ on both sides of the inequality, as well as the fact that $\alpha_1 \geq \tau^D$ give

$$0 \geq u(S)(1 - \sup_{\mathbf{x}\in\bar{D}_\delta} P_{\mathbf{x}}(\alpha_1 \leq S)) \geq u(S) \inf_{\mathbf{x}\in\bar{D}_\delta} P_{\mathbf{x}}(\tau^D > S).$$

Our claim is proved since $\inf_{\mathbf{x}\in\bar{D}_\delta} P_{\mathbf{x}}(\tau^D > S) > 0$ for any $S > 0$. □

3 Geometric ergodicity

In this section G is assumed bounded and regular.

Proposition 3 *Assume there exists a function ϕ satisfying (C1) form Definition 2. Fix an index i, $1 \leq i \leq N$ and recall that $x_i(t)$ denotes the i - th component of $\mathbf{x}(t)$. If we denote by α_1 the first hitting time of the set $G \backslash G'$ by the process $(x_i(t))$, then there exist $\theta > 0$, $C_0 > 0$ independent of $\mathbf{x} \in G'$ such that $E_{\mathbf{x}}[\exp(\theta\alpha_1)] \leq C_0$.*

Remark We do not need the uniform exterior cone condition because we are not interested in the upper bound of the gradient.

Proof Without loss of generality we may assume that $0 < \phi(x) < 1$ on G' and $\phi(x) = 1$ on $\partial G'$ by choosing the vicinity of the boundary to be the connected component containing ∂G of the set $\{x \in G|\phi(x) < \frac{1}{m}\}$ and noticing that $m\phi(x)$ must satisfy the property for some sufficiently large $m \in \mathbb{Z}_+$.

Denote $y_i = \phi(x_i)$, where $\mathbf{x}(0) = \mathbf{x}$ has components x_i, $1 \leq i \leq N$ and the process $(\mathbf{y}(t))$ with components $y_i(t) = \phi(x_i(t))$, $t \geq 0$. In the following the particle index i is not important and we denote y_i simply by y and similarly x_i by x. Denote by β_1 the first hitting time of the point $y = 1$ by the process $(y(t))$. We have the almost sure inequality $\alpha_1 \leq \beta_1$.

The process $(y(t))$ evolves in $[0, 1]$ undergoing jumps at a subset of the jump times (τ_l) for the process $(\mathbf{x}(t))$. To simplify notation, we shall still denote these jumps by τ_l, $l \geq 1$, $\tau_0 = 0$. Due to the properties of ϕ, with probability one, at each time τ_l, the jump pushes the one-dimensional process $y(t)$ to the right, from $y(\tau_l-) = 0$ to $y(\tau_l) > 0$. We shall construct by coupling a new process $z(t)$ evolving on $(-\infty, 1]$ with a monotonicity property. At start, the processes $z(t)$ and $y(t)$ coincide - until τ_1. At τ_1, $z(t)$ suppresses the jump, but continues to diffuse being driven by the same stochastic differential equation as $y(t)$. Based on (2.4), we construct inductively for $l \geq 0$ a sequence $z_{0,l}$, by setting $z_{0,0} = y_0 = \phi(x_0)$, and a process

$$dz(t) = \tilde{b}_i(t)dt + \tilde{\sigma}_i(t)dW(t), \quad \tau_l \leq t < \tau_{l+1}, \quad z(\tau_l) = z_{0,l}, \tag{3.1}$$

where the coefficients are defined in (2.9). At each step, we update $z_{0,l+1} := z(\tau_{l+1}-)$. Due to the pathwise coupling (3.1), $z(t) \leq y(t)$ almost surely when $z(\tau_l) \leq y(\tau_l)$, which is true by construction. Denoting with γ_1 the first hitting time of the point one by $(z(t))$, we see that $\beta_1 \leq \gamma_1$ with probability one. Let $\mu = q_-$ from (C1) and $\theta < \mu^2/(2||\sigma||^2)$ and $u(\theta, z) = \exp((z-1)||\sigma||^{-2}[-\mu + \sqrt{\mu^2 - 2\theta||\sigma||^2}])$ be the solution on $z \in (-\infty, 1)$ of $\theta u + \mu u' + \frac{1}{2}||\sigma||^2 u'' = 0$. We note that u is a natural choice, as it is the moment generating function of the hitting time of the boundary $z = 1$ when starting on $(-\infty, 1)$ for the Brownian motion with diffusion coefficient $||\sigma||^2$ and drift μ. By checking that the expression $\exp(\theta t)u(\theta, z(t))$ is a local supermartingale and comparing its expected values at both $t = 0$ and $t \uparrow \gamma_1$, we have shown that if $z \in [0, 1]$ is the starting point $z = \phi(x_i)$ of the process corresponding to $\mathbf{x}(t)$ starting at $\mathbf{x}$ with i-th component equal to x_i, then

$$E_{\mathbf{x}}[\exp(\theta\gamma_1)] \leq u(\theta, z). \tag{3.2}$$

By taking the supremum over $z \in [0, 1]$ we obtain the desired bound. □

We remind the reader the definition of the interior chain (X_n) from the paragraph right before Lemma 3.

Proposition 4 *Under the same conditions as in Theorem 1, the interior chain $(X_n)_{n\geq 0}$ has a unique invariant measure μ_X.*

Proof We have seen at the end of the proof of Theorem 1 how (2.21) and (2.16) imply that $D_\delta = (G_\delta)^N$ is reached with probability one. We shall prove that the interior set $\bar{D}_\delta$ is a Doeblin set. In other words, there exists a probability measure $\gamma(d\mathbf{x})$ on $\bar{D}_\delta$ and a constant $c_X \in (0, 1)$ such that $P(X_1 \in B|X_0 = \mathbf{x}) = P_{\mathbf{x}}(X_1 \in B) \geq c_X\gamma(B)$ for any B a Borel set on $\bar{D}_\delta$ and any $\mathbf{x} \in \bar{D}_\delta$. Pick a time $S > 0$, $B = \Pi_{i=1}^N B_i$, B_i Borel sets in G. Let $\mathcal{A} = \{\tau_1^G \leq S, \tau_2^G > S, \ldots, \tau_2^G > S\}$ and $\mathcal{C}$ the event that particle #1 jumps onto particle #2. Define $\gamma(dy_1, dy_2, \ldots, dy_N) = c(\gamma, \delta)\delta(y_1 - y_1)dy_1dy_2\ldots dy_N$ where $c(\gamma, \delta)$ is the normalizing constant to make γ a probability measure on $\bar{D}_\delta$. As in (1.1), we write $P^G(s, x, dy)$, $p^G(s, x, y)$ for the transition probabilities, respectively densities of the diffusion on G killed at the boundary and note that $p^G(S, x, y)$ is bounded away from zero for $x, y \in \bar{G}_\delta$. Then, for a suitably chosen c_X independent of $\mathbf{x}$,

$$
\begin{aligned}
P_{\mathbf{x}}(X_1 \in B) &\geq P_{\mathbf{x}}(\mathbf{x}(\tau^D) \in B, \mathcal{A} \cap \mathcal{C}) \\
&\geq p_0 \inf_{x_1 \in \bar{G}_\delta} \{1 - P^G(S, x_1, G)\} \int_{B_1 \times B_2} \delta(y_1 - y_2)dy_1dy_2 \\
&\quad \times \Pi_{i=3}^N P^G(S, x_i, y_i \in B_i) \geq c_X\gamma(B).
\end{aligned} \tag{3.3}
$$

□

We denote by $K(x, x')$ the Green function of L (2.2) on G with zero boundary conditions.

Theorem 2 *Assume G satisfies the conditions of Theorem 1. Then $(\mathbf{x}(t))$ is geometrically ergodic. The invariant probability measure has a density with respect to the Lebesgue measure equal to $Z^{-1}\int_G K(x, x')\mu_X(dx')$, where Z is a normalizing constant.*

Remark We refer the reader to Theorem 3 in [16] for more details on the invariant measure. In the context of the Fleming–Viot particle process, obtaining (3.5) needs the intermediate step from Proposition 3.

Proof The set $\overline{D_\delta}$ is a *small* set for the process due to the fact that $(\mathbf{x}(t))$ has a density bounded below by the density function of the process killed at the boundary; in its turn, this density function has a uniform lower bound on $\overline{D_\delta}$ for any $t > 0$. Exponential ergodicity is guaranteed [9] by the sufficient condition (3.5) that there exists an exponential moment of the time to reach $\overline{D_\delta}$, uniformly over all $\mathbf{x} \in D = G^N$.

Most of the proof is contained in Theorem 3 in [16]. We prove the part that is new to the context of the Fleming–Viot redistribution function. Recall that $D = G^N$, $D' =$

$(G\backslash\overline{G_{2^N\delta}})^N$ and $\alpha(D\backslash D')$ is the first exit time from D', i.e. the hitting time of the set of configurations with at least one particle at distance larger than $2^N\delta$ from the boundary. Proposition 3 shows that there exists $\theta > 0$ such that

$$\sup_{\mathbf{x}\in D} E_{\mathbf{x}}[e^{\theta\alpha(D\backslash D')}] < \infty \tag{3.4}$$

due to the uniform bound and a Markov property inductive argument similar to the one in Lemma 3. We want a similar uniform bound on $\alpha(D_\delta)$. This is guaranteed by the Step 2 of the proof of Theorem 1, where it is shown that once in $D\backslash D'$, the probability to reach $\overline{D_\delta}$ in $N-1$ consecutive jumps in time at most T (for a fixed but arbitrary T) is bounded away from zero uniformly on the configuration in $D\backslash D'$. Another iteration of the argument from Lemma 3 in continuous time setting (there is virtually no modification needed) gives

$$\sup_{\mathbf{x}\in D} E_{\mathbf{x}}[e^{\theta\alpha(D_\delta)}] < \infty \tag{3.5}$$

concluding the proof of exponential ergodicity. To verify the formula for the density of the invariant measure, we use Theorem 3 in [16] and the fact that there exists an invariant measure μ_X of the interior chain, a fact proven in Proposition 4. $\square$

4 Examples of sets satisfying the regularity conditions

The set G is assumed to have *regular* boundary, guaranteed, for instance, by the exterior cone condition. We remind the reader that an open set G' is said a *vicinity of the boundary* if there exists $\delta' > 0$ such that $G\backslash G' \subseteq \bar{G}_{\delta'}$ and $K(x, x')$, $K^0(x, x')$ will denote the Green functions of L, respectively $\frac{1}{2}\Delta$ on G with zero boundary conditions.

Let u_j, $j = 1, 2$ be solutions to the Poisson equation $Lu_j = f_j$ with zero boundary conditions at ∂G, where f_j, $j = 1, 2$ are smooth with $f_1(x) = 0$ and $f_2(x) = -1$ on some vicinity of the boundary G'. It is easy to see that as soon as $\sup_{x\in G'} \frac{u_2(x)}{u_1(x)} < \infty$, condition (C1) is satisfied with $\psi(x) = u_1(x) - \epsilon u_2(x)$ for sufficiently small $\epsilon > 0$. For a specific choice of $u_1(x) = P_x(x(\tau^{G\backslash\bar{G}'}) \in \partial G')$ and $u_2(x) = E_x[\tau^{G\backslash\bar{G}'}]$ we obtain a probabilistic interpretation of the condition.

Considering the *Martin kernel* of the set G with reference point $x' \in G\backslash G'$, i.e. $M(x_0, y) = \lim_{x\to x_0} K^0(x, y)/K^0(x, x')$, where $x_0 \in \partial G$, we can state the following result. Here the limit is taken in the Martin topology, as $x \in G$ approaches x_0 in the Martin boundary of the domain G. For a Lipschitz domain G, it is known that the Martin topology coincides with the regular Euclidean topology of the domain and the Martin boundary coincides with the actual Euclidean boundary, ∂G (see [4], Sect. 8.8, page 269 and Theorem 8.8.4), but the same is true for more general domains, like *uniform* domains - see Remark 1 after the theorem. In the following, a k-Lipschitz domain is a Lipschitz domain with Lipschitz constant k, in other words, a domain G whose boundary ∂G can be given locally by a Lipschitz function whose Lipschitz constant is less than or equal to k. We say that G satisfies the *interior cone condition* *with aperture* A, $A \in (0, \frac{\pi}{2})$, if for each point $x \in G$ there is a truncated cone with vertex at x, aperture A and constant radius included in G.

Theorem 3 *Assume* $L = \frac{1}{2}\Delta$ *and* $G \subseteq \mathbb{R}^d$ *is a bounded Lipschitz domain.* (i) *(C1) is satisfied whenever* $\int_G M(x_0, y)dy < \infty$ *for all* $x_0 \in \partial G$. *The Martin kernel is integrable* (ii) *if the Lipschitz constant* k *satisfies* $k < K_d$ *for some dimension-dependent constant* K_d, *or* (iii) G *satisfies the interior cone condition with aperture* A *such that* $\cos A > 1/\sqrt{d}$.

Remark 1 The theorem is actually true for more general G than Lipschitz. If G is a bounded *uniform* domain, its Martin boundary coincides with ∂G and all boundary points are minimal (Theorem 3 and Corollary 3 in [2]). The integrability is verified for bounded John domains with John constant $c_J \geq 1 - 2^{-d-1}$ (Theorem 1 in [3]). The non-smooth domains are related to each other: *Lipschitz* $\subsetneq$ *Uniform* $\subsetneq$ *John* and *Interior cone* $\subseteq$ *John*.

Remark 2 The constant K_d is obtained in [21] and then [1] shows that $K_d = (d-1)^{-1/2}$. Since $k = 0$ when G is a C^1 domain, property (C1) is automatically satisfied in any dimension in this case.

Proof Put $u_1(x) = K^0(x, x')$, $u_2(x) = \int_G K^0(x, y)g(y)dy$ where $x' \in G \backslash \bar{G}'$ and $0 \leq g(x) \leq 1$ on G, $g(x) = 1$ on G'. We want to show that $\sup_{x \in G'} \frac{u_2(x)}{u_1(x)} < \infty$.

If that were not true, let (x_n) be a sequence of points such that $\lim_{n\to\infty} \frac{u_2(x_n)}{u_1(x_n)} = +\infty$; since G is bounded, the sequence has a convergent subsequence with limit x_0. By continuity, $x_0 \in \partial G$. Without loss of generality, we consider $x_n \to x_0$.

(i) We use the notations in [4]. First note that if G is Lipschitz, then G is not minimally thin at any point $x_0 \in \partial G$ and also all the boundary points of G are minimal, in other words, the set of all minimal boundary points Δ_1 is equal to ∂G. Now Using Theorem 9.2.7 in [4] with $\Omega = G$, $E = G$, $\mu'(dz) = g(z)dz$ and $y = x_0 \in \partial G = \Delta_1$, we see that as soon as $\int_G M(x_0, z)dz < \infty$, we have $\lim_{n\to\infty} \frac{u_2(x_n)}{u_1(x_n)} < \infty$ with $x_n \to x_0$ which concludes (i) by contradiction.

(ii) and (iii). Part (i) shows that when G is a bounded Lipschitz domain, it is sufficient to show that the Martin kernel is integrable. The Martin kernel $M(x_0, y)$ is a kernel function (again in [4]), thus $M(x_0, \cdot)$ is positive harmonic for every fixed $x_0 \in \partial G$. Corollary 9 in [21] shows that a sufficient condition for a positive superharmonic function to be integrable is that the domain G be k-Lipschitz with Lipschitz constant $k < K_d$ for some dimension-dependent K_d, exactly computable as the solution to the equation $p_d(K_d) = 1$. For a detailed expression of p_d, we refer to [21], the discussion in [1], page 112 and Remark 1 and 2 from above. The integrability in the interior cone case is proven in Theorem 2 from [3]. □

4.1 Conditions based on the distance to the boundary

Proposition 5 gives an easier to verify criterion for (C1). This and especially Proposition 6 indicate that the function we are looking is, in essence, the distance from the boundary. In the following proposition, G may be unbounded.

Proposition 5 *If there exists a vicinity G' of the boundary of G and $\phi(x)$ that satisfies* (i), (ii), (iii) *from (C1) plus condition (v), i.e. there exists a positive constant c_- such that $||\nabla\phi(x)|| \geq c_-$ and $L\phi(x) \geq -c_-$ for all $x \in G'$, then G satisfies $(C1)$.*

Remark Any exterior C^2 domain (i.e. whose complement is bounded) immediately satisfies the conditions in the proposition with $\phi(x) = d(x, \partial G)$. One does not need a compact boundary though, only a uniform $\delta > 0$ such that $G\backslash G' \subset \bar{G}_\delta$ where the properties are satisfied.

Proof We define $\psi(x) = \phi^2(x)$, where ϕ is the function in the hypothesis of the proposition. Then (2.6) implies that $L\psi(x) \geq \sigma_0^2 c_-^2 - 2c_-\phi(x)$. Since $c(\phi, \delta') = \sup_{x\in G\backslash G_{\delta'}} \phi(x)$ converges to zero as $\delta' \to 0$, the lower bound of $L\psi(x)$ can be made strictly positive for sufficiently small δ'. Condition (C1) is satisfied with $G' \mapsto G\backslash G_{\delta'}$. □

The connections between (C1) and the distance to the boundary is explored in the following proposition.

Proposition 6 *Suppose G is bounded and regular and there exists $\epsilon > 0$ such that the solution $u_\epsilon(x)$ of the viscous equation $-\epsilon\Delta u_\epsilon + |\nabla u_\epsilon|^2 = 1$, $u_\epsilon|_{\partial G} = 0$ verifies condition (v) from Proposition 5, then G satisfies (C1). When G has C^2 boundary then we may take $\epsilon = 0$; the eikonal equation $||\nabla u||^2 = 1$, $u|_{\partial G} = 0$ has a classical solution $u \in C^2(\overline{G'})$ equal to $d(x, \partial G)$ on G', which satisfies the conditions from Proposition 5.*

Remark 1 The second part of (v) is trivial in this case. To ensure the first part, it is sufficient to either have a lower bound of the gradient directly, or prove a lower bound on Δu_ϵ independently of ϵ.

Remark 2 In the C^2 case, the passage from the half-Laplacian to L is easy assuming (2.3). Under the same conditions on G, we replace $||\nabla\phi(x)||$ with $||\sigma^*(x)\nabla\phi(x)||$ from (2.9) and solve the generalized eikonal equation in the Riemannian metric [10] given by $(a^{\alpha,\beta}(x))$ from (2.2). The theorem extends immediately to an *exterior domain* and, more generally, to any domain, possibly unbounded, where the eikonal equation has a solution on a vicinity of the boundary.

Proof The solution $u_\epsilon(x)$ belongs to $C(\bar{G}) \cap C^2(G)$ and is positive by the maximum principle. In addition, we do not need a bound on Lu_ϵ simply by writing $\Delta u_\epsilon = \epsilon^{-1}(||\nabla u_\epsilon(x)||^2 - 1)$. When G has C^2 boundary, the direct proof based on the method of characteristics can be found in [19] in Chapter 1. Since Δu_ϵ has a lower bound independent of ϵ (Chapter 2 in [19]), condition (v) is immediate. □

We conclude with some less general sufficient conditions for (C1), yet easier to verify in many special cases.

Proposition 7 (i) *A sufficient condition for $(C1)$ is that there exists G' a vicinity of the boundary and $x' \in G\backslash\overline{G'}$ such that $\phi(x) = K(x, x')$ satisfies (v) from Proposition 5.*

(ii) *The same statement as* (i) *holds for* $\phi(x)$ *equal to the first eigenfunction of* L *on* G *with zero boundary conditions.*
(iii) *Any domain with the interior sphere condition and* $\phi(x) = K(x, x') \in C^1(\bar{G})$ *will satisfy* (i).
(iv) *properties* (i), (ii) *and* (iii) *are satisfied if* $\partial G \in C^2$.

Proof (i) Pick $x' \in G$ and $\delta < d(x', \partial G)$. The Green function is continuous except at x', positive in $G\backslash\{x'\}$. We set $G' = \{x \in G | K(x, x') < \delta\}$. The function $\phi(x)$ satisfies $LK(x, x') = 0$ in G', is positive in G', vanishes on ∂G and thus satisfies (C1). To adjust for (C3), we only have to normalize $\phi(x) \to \delta^{-1}\phi(x)$.

(ii) We notice that $L\phi(x) = -\lambda_0\phi(x)$ and thus $L\phi(x)$ is uniformly bounded up to the boundary.

(iii) The Hopf maximum principle [13] shows that $\langle \nabla\phi(x), n\rangle < 0$ on ∂G, where n is the outward normal to ∂G. From the boundedness of the domain, G and ∂G are compact, and from the continuity up to the boundary we have that $||\nabla\phi(x)||$ is bounded away from zero in a neighborhood of the boundary (otherwise it would reach zero on ∂G). For sufficiently small δ we obtain all conditions required.

(iv) $\partial G \in C^2$ implies the interior sphere condition and the smoothness up to the boundary of the solutions of elliptic equations - for example, in [10]. □

5 The immortal particle

This section investigates the particle ancestry. The realization of the process is a tree with continuous branches, representing diffusive episodes performed by the particles. Reaching the boundary ends a certain branch, that will never be revived. Branching at a given location allows the continuation of the tree, provided non-extinction (Theorem 4), ad infinitum. The goal is to prove that, almost surely, there exists a unique infinite continuous path on the tree, in the sense of Theorem 4 (iv). This is, informally, the immortal particle. It is not a proper tagged particle because it changes its label infinitely many times.

The reader is reminded that $x_i(t)$ represents the particle of *index* $i \in \{1, \ldots, N\}$ and that the indices are fixed forever; also, $(\tau_l)_{l\geq 0}$, $\tau_0 = 0$ denote the increasing sequence of times when particles hit the boundary. At time $t = 0$, each particle is given a label (or color). The label is preserved as long as the particle is alive; when it is killed, the particle that replaces it will acquire the label of the particle it jumps to. Or, in a different but equivalent interpretation, the particle is killed and the newly born particle will have the same label as its parent. We want to show that, with probability one, exactly one label survives. Ultimately, all particles at time t can be traced to only one original ancestor, all other lineages (to be defined precisely) dying in finite time.

5.1 The multi-color process

Formally we shall consider a Markov process with state space $(G \times \mathcal{C})^N$, where $\mathcal{C}$ is a finite set of labels (colors). One example is $\mathcal{C} = \{1, \ldots, N\}$ and another important one is when $\mathcal{C} = \{0, 1\}$. It will be shown that the two-color model is sufficient to trace ancestry. An element in the state space is a vector with N components $(x_i, \mathcal{C}(x_i))$, $1 \le i \le N$ designating the position x_i of particle i and its color. We used $\mathcal{C}(x_i) \in \mathcal{C}$ for the color of particle to avoid more complicated notation.

The particles $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t)) \in G^N$ follow exactly the branching mechanism from Sect. 2 with redistribution measure (2.1). At the same time, the labels follow the rule that they remain constant until the particle hits the boundary, at which time it instantaneously and always adopts the label of the particle it jumped to; equivalently, the particle reaching the boundary is killed and a new particle is born from a surviving one, with the same label as the parent. Naturally the latest interpretation is more relevant to our investigation. It is easy to see that the joint process (particle-label) is Markovian.

Proposition 8 *Assuming the unlabeled process is non-explosive, with probability one, all but one label have finite lifetime.*

Remark (1) Once only one color has been achieved, it is evident that the process follows the unlabeled branching mechanism and continues its evolution forever (as long as the process is not explosive).

(2) Considering a discrete space and time version of the process, the reader may see why the proposition is true, since all multi-colored states are transient. It is sufficient to observe that one color can be *forced* to hit the boundary while all other colors are not reaching the boundary and upon killing only the other colors are allowed to branch (a small but positive probability event).

Proof The proof follows a different idea than described in Remark 2), better suited to the context of diffusions. First, we notice that it is enough to prove the proposition for two colors (zero and one) in the sense that the time for one color to disappear will be shown to be finite almost surely. At time zero we re-label particles of a type with one and all the others with zero. Inductively, it will follow that the number of colors is reduced to exactly one in finite time. Denote τ_L the first time when the number of labels has been reduced to one, with the usual convention that $\tau_L = \infty$ if the event does not happen in finite time.

Let $\delta > 0$ be such that $\bar{G}_{2\delta} \subset G$ (the reason why we use 2δ becomes apparent immediately). On the one hand, we know that from any initial position $\mathbf{x}$, the particle system will reach the complement $F_{2\delta}$ of $(G \backslash G_{2\delta})^N$ a.s., that is, at least one particle will be within $\bar{G}_{2\delta}$. On the other hand, for $T > 0$ fixed and $\mathbf{x} \in F_{2\delta}$, we shall obtain a lower bound $p_0 > 0$ of $P_{\mathbf{x}}(\tau_L \le T)$, uniformly over $\mathbf{x} \in F_{2\delta}$. Starting with an arbitrary $\mathbf{x}$, the system will have an infinite number of attempts to reach a one-label configuration. Since the failure probability is $1 - p_0 < 1$ in each episode, it follows that $\tau_L < \infty$ with probability one.

Part 1. Let $\mathbf{x} \in F_{2\delta}$. Without loss of generality we assume that $x_1 \in \bar{G}_{2\delta}$. Let $K = \{\tau^{\bar{G}_\delta,1} > T\}$, where $\tau^{\bar{G}_\delta,1}$ is the first time when the particle #1 hits $G \backslash G_\delta$, $\tau_1^{G,j}$, $\tau_2^{G,j}$ the first, respectively second boundary hit of particle #j, $1 \le j \le N$. Denote A_j, B_j, C_j the events pertaining to particles #j, $2 \le j \le N$

$$A_j = \{\tau_1^{G,j} \le T\}, \quad B_j = \{x_j(\tau_1^{G,j}) = x_1(\tau_1^{G,j})\}, \quad C_j = \{\tau_2^{G,j} > T\} \quad (5.1)$$

with $A = \cap_{j=2}^{N} A_j$, $B = \cap_{j=2}^{N} B_j$ and $C = \cap_{j=2}^{N} C_j$. In other words, K means that x_1 will not exit G_δ before time T; A_j that x_j hits the boundary in $[0, T]$; B_j that x_j jumps to the location of x_1 at its first boundary hit, and C_j that x_j will not jump again before time T. With the observation that $\{\tau_L \le T\} \supseteq A \cap B \cap C \cap K$, it is sufficient to prove $P_{\mathbf{x}}(A \cap B \cap C \cap K) \ge p_0 > 0$ with p_0 independent of $\mathbf{x} \in F_{2\delta}$. Two particles are independent until they meet, i.e. there is a jump/birth involving the two. Consequently, conditional on K, the events $(A_j \cap B_j \cap C_j)_{2 \le j \le N}$ are mutually independent with

$$\begin{aligned} P_{\mathbf{x}}(A \cap B \cap C \cap K) &= P_{\mathbf{x}}(A \cap B \cap C \mid K) P_{\mathbf{x}}(K) \\ &= \prod_{j=2}^{N} P_{\mathbf{x}}(A_j \cap B_j \cap C_j \mid K) P_{x_1}(\tau^{\bar{G}_\delta, 1} > T) \qquad (5.2) \\ &\ge \prod_{j=2}^{N} P_{\mathbf{x}}(A_j \cap B_j \cap C_j \mid K) p_-(T, G_{2\delta}, G_\delta), \qquad (5.3) \end{aligned}$$

where $p_\pm$ are defined in (2.5). We write

$$P_{\mathbf{x}}(A_j \cap B_j \cap C_j \mid K) = P_{\mathbf{x}}(C_j \mid A_j \cap B_j \cap K) P_{\mathbf{x}}(A_j \cap B_j \mid K) \qquad (5.4)$$

and see that the first factor is bounded below (by introducing $\tau_2^{G,j} > T + \tau_1^{G,j}$ instead of $\tau_2^{G,j} > T$) by

$$\begin{aligned} P_{\mathbf{x}}(C_j \mid A_j \cap B_j \cap K) &\ge \int_G P_x(\tau^G > T) P_{\mathbf{x}}(x_j(\tau_1^{G,j}) \in dx \mid A_j \cap B_j \cap K) \\ &\ge p_-(T, G_\delta, G) \qquad (5.5) \end{aligned}$$

(note that the position of the jump is on the trajectory of x_1 that stays in G_δ). At the same time A_j, B_j and K are independent with $P_{\mathbf{x}}(A_j|K) = P_{\mathbf{x}}(A_j) \ge 1 - p_+(T, G_{2\delta}, G)$ and $P_{\mathbf{x}}(B_j|K) = (N-1)^{-1}$. Putting all together, the probability from (5.2) is bounded below by

$$p_0 = \Big[p_-(T, G_\delta, G)(1 - p_+(T, G_{2\delta}, G))(N-1)^{-1} \Big]^{N-1} p_-(T, G_{2\delta}, G_\delta) > 0. \qquad (5.6)$$

Part 2. We shall apply Lemma 3 with $F = F_{2\delta}$, $\tau = \tau_L$ to obtain the conclusion of the theorem. □

Let $l : [0, \infty) \to \{1, 2, \ldots, N\}$ and $\eta : [0, \infty) \to \bar{G}$ be random processes adapted to $(\mathcal{F}_t)_{t \ge 0}$ such that (i) $l(t)$ is piecewise constant and $\eta(t) = x_{l(t)}(t)$ on intervals $[\tau_{k-1}, \tau_k)$, $k \ge 1$ and (ii) η continuous with $\eta(t) \equiv \eta(\tau_k-)$ for all $t \ge \tau_k$ if $\eta(\tau_k-) \in \partial G$. A pair $(l(\cdot), \eta(\cdot))$ is said a *lineage*. The stopping time τ_k when (ii) happens is said the *lifetime* of the lineage and is denoted by $\tau(\eta)$.

For $t_1 < t_2$, i_1, i_2 two of the N labels, we say that $x_{i_1}(t_1)$ is an *ancestor* of $x_{i_2}(t_2)$ (or there exists a lineage from $x_{i_1}(t_1)$ to $x_{i_2}(t_2)$) and we write $(t_1, i_1) \preceq (t_2, i_2)$ if there exists a lineage $(l(\cdot), \eta(\cdot))$ with $\tau(\eta) \geq t_2$ such that $l(t_1) = i_1$, $\eta(t_1) = x_{i_1}(t_1)$ and $l(t_2) = i_2$, $\eta(t_2) = x_{i_2}(t_2)$. On the set of pairs (t, i), the lineage introduces a relation of partial order.

Theorem 4 *Assume G is a regular bounded domain and the process is non-explosive. Let $t_1 < t_2$ and i_1, i_2 two of the N labels. If $(t_1, i_1) \preceq (t_2, i_2)$, then*

(i) *the lineage they belong to is unique up to time $t = t_2$;*
(ii) *the labels/colors are identical at both endpoints, $\mathcal{C}(x_{i_1}(t_1)) = \mathcal{C}(x_{i_2}(t_2))$ and as a consequence, a lineage will never change label;*
(iii) *For any $t \geq 0$ and any index i, there exists an index i_0 such that $(0, i_0) \preceq (t, i)$;*
(iv) *There exists a unique lineage with infinite lifetime.*

Proof (i) Assume $(l'(\cdot), \eta'(\cdot))$, $(l''(\cdot), \eta''(\cdot))$ are two lineages going from (t_1, i_1) to (t_2, i_2). Lineages may intersect in two ways: either on open intervals (τ_{k-1}, τ_k) as diffusion paths (with zero probability except in dimension one), or at branching times τ_k. Only intersections of the second type are *proper* because the particles do not interact during the diffusive episodes. Two lineages will properly intersect at time t only if they coincide on $[0, t]$; otherwise, they will have to intersect in the open set G, which is impossible by construction. Evidently, lineages may diverge after t.

(ii) The colors may change only at times τ_k. At jump time, the particle performing the jump from the boundary adopts the label of the one in G, whose label coincides with the label of the lineage. Again by construction, at a branching point the label is preserved for all offspring, so the lineage does not change label, having $\mathcal{C}(x_{l(\tau_k-)}) = \mathcal{C}(x_{l(\tau_k)})$.

(iii) Theorem 1 shows that $0 = \tau_0 < \tau_1 < \tau_2 < \ldots$ and $\lim_{k\to\infty} \tau_k = +\infty$ a.s. Let $k(t)$ be the integer $k \geq 1$ such that $\tau_{k-1} \leq t < \tau_k$; then one can verify (iii) by induction over k.

(iv) At time $t = 0$ we label $\mathcal{C}(x_i(0)) = i$ for all indexes i. We know from Proposition 8 that $\tau_L < \infty$ a.s., which implies due to (ii) that at time $t = \tau_L$ only one lineage, starting at $(0, i_0)$ is still alive (did not reach the boundary). Due to (iii), we deduce that at time $t \geq \tau_L$, all particles have lineages all the way to $(0, i_0)$. Let τ_L^k, $k \geq 1$ be defined inductively by setting $\tau_L = \tau_L^1$ and re-labeling the particles at time τ_L by $\mathcal{C}(x_i(\tau_L)) = i$ with $\tau_L^2 > \tau_L^1$ being exactly the time after τ_L^1 when all labels become identical once again. Due to the strong Markov property and again Proposition 8, $\tau_L^2 < \infty$ a.s. and we re-apply (ii)-(iii) to see that only one index i_1 survives, making (τ_L^1, i_1) the only ancestor of all (τ_L^2, i), $1 \leq i \leq N$. Since $\tau_L \geq \tau_1$ we immediately have τ_L^k bounded below by a subsequence of $(\tau_{j_k})_{k\geq 1}$ of the boundary hits. Then $\lim_{k\to\infty} \tau_L^k = +\infty$ with probability one, implying that the construction can be done for any $t > 0$. The uniqueness is a consequence of (i). □

6 The two particle case

When $N = 2$, the jump re-distribution measures (2.1) are delta functions, i.e. deterministic; the two particles start each diffusive episode from the same point $x(\tau_l-)$ (the meeting point). This allows some explicit calculations, which are of interest, especially for $L = \frac{1}{2}\Delta$ in $d = 1$, where we obtain a law of large numbers on the logarithmic scale for the Markov chain of configurations at the meeting point.

We start by deriving the transition function of the surviving particle. Denote X the position of the surviving particle at the time of the first boundary visit. If the particles start at x_1 and x_2 respectively, then

$$P_{(x_1,x_2)}(X \in dy) = P_{x_1}(x_1(\tau_2) \in dy, \tau_1 > \tau_2) + P_{x_2}(x_2(\tau_1) \in dy, \tau_2 > \tau_1) \quad (6.1)$$

$$= \int_0^\infty P_{x_1}(x_1(t) \in dy, \tau_1 > t) P_{x_2}(\tau_2 \in dt) + \int_0^\infty P_{x_2}(x_2(t) \in dy, \tau_2 > t) P_{x_1}(\tau_1 \in dt). \quad (6.2)$$

When $x_1 = x_2 = x$ we obtain the transition probability $S(x, dy)$ of the interior Markov chain tracing the locations $X_k = x_1(\tau_k)$, $k \geq 1$ *right after a jump*. It is

$$S(x, dy) = P(X_1 \in dy \,|\, X_0 = x) = P_x(X \in dy) = 2 \int_0^\infty P^G(t, x, dy) P_x(\tau^G \in dt), \quad (6.3)$$

where

$$P_x(\tau^G > t) = \int_G p^G(t, x, y) dy. \quad (6.4)$$

Combining (6.3) and (6.4) and integrating by parts we can write the alternative formula (not used in this paper)

$$P_x(X \in dy) = 2\delta_x(dy) + 2 \int_0^\infty P^G(\tau^G > t) \partial_t p^G(t, x, dy) dt. \quad (6.5)$$

Due to independence,

$$P_x(\tau_1 \wedge \tau_2 > t) = (P_x(\tau^G > t))^2, \quad E_x[\tau_1 \wedge \tau_2] = \int_0^\infty (P_x(\tau^G > t))^2 dt. \quad (6.6)$$

6.1 Two particles on the half-line

Assume $D = (0, \infty)$, $N = 2$ and each particle follows $x_i(t) = x_i - \mu t + w_i(t)$, $i = 1, 2$, where $w_i(t)$ are independent Brownian motions. The density function of the Brownian motion on the positive half-line with drift $-\mu$ killed at the origin is

$$p^G(t,x,y) = \frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(y-x)^2}{2t}} - e^{-\frac{(y+x)^2}{2t}}\right)e^{-\mu(y-x)-\frac{1}{2}\mu^2}, \tag{6.7}$$

as can be seen by applying Girsanov's formula or directly by verification of the Kolmogorov equations. Starting with (6.4) and noticing that the adjoint of L is $L_y^* = \frac{1}{2}\frac{d^2}{dy^2} + \mu\frac{d}{dy}$ with Dirichlet b.c. at zero, the density of τ^G, in this case, is

$$\frac{d}{dt}P_x(\tau^G \in dt) = -\int_G \frac{d}{dt}p^G(t,x,y)dy = -\int_G L_y^* p^G(t,x,y)dy \tag{6.8}$$

$$= \frac{1}{2}\partial_y p^G(t,x,0). \tag{6.9}$$

The transition probability (6.3) reads

$$P_x(X \in dy) = \int_0^\infty P^G(t,x,dy)\partial_y p^G(t,x,0)dt. \tag{6.10}$$

Proposition 9 *The following estimates are satisfied*

$$2E_x[\tau_1 \wedge \tau_2] = E_x[X^2] \sim o(x), \quad \lim_{x\to 0}\frac{E_x[X]}{x} = 2. \tag{6.11}$$

Proof Observing that $-\mu < 0$, then $\tau^G < \infty$ and even more so $\tau_1 \wedge \tau_2 \le \tau^G < \infty$ with probability one, the optional stopping theorem (at $t = \tau_1 \wedge \tau_2$) applied to the martingales $M_1(t) = x_1(t) + x_2(t) + 2\mu t$ and $M_2(t) = x_1^2(t) + x_2^2(t) - 2x_1(t)x_2(t) - 2t$ shows that

$$E_x[X] + 2\mu E_x[\tau_1 \wedge \tau_2] = 2x, \quad E_x[X^2] - 2E_x[\tau_1 \wedge \tau_2] = 0. \tag{6.12}$$

We want to prove the two limits (the second is a consequence of the first)

$$\lim_{x\to 0}\frac{2E_x[\tau_1 \wedge \tau_2]}{x} = \lim_{x\to 0}\frac{E_x[X^2]}{x} = 0, \quad \lim_{x\to 0}\frac{E_x[X]}{x} = 2. \tag{6.13}$$

Since we calculate the limit as $x \to 0$, we may assume $0 < x \le 1$. Using (6.6), we shall prove directly the first limit in (6.13)

$$\lim_{x\to 0}\frac{\int_0^\infty (P_x(\tau^G > t))^2 dt}{x} = \lim_{x\to 0}\left(2\int_0^\infty P_x(\tau^G > t)\frac{d}{dx}P_x(\tau^G > t)dt\right) = 0. \tag{6.14}$$

To have (6.14), we use L'Hospital's rule; it is necessary to justify the differentiation under the integral and the limits as $x \to 0$.

From (6.7) we derive

$$P_x(\tau^G > t) = \Phi\left(\frac{x-\mu t}{\sqrt{t}}\right) - e^{2\mu x}\left(1 - \Phi\left(\frac{x+\mu t}{\sqrt{t}}\right)\right), \tag{6.15}$$

where $\Phi'(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$. This is evidently in the interval $[0, 1]$ and thus bounded and has limit zero at $x = 0$. It remains to show that the absolute value of the derivative has an upper bound, uniformly in $x \in [0, 1]$ that is integrable in $t \in (0, \infty)$. The derivative is

$$\begin{aligned}\frac{d}{dx}P_x(\tau^G > t) = \frac{1}{\sqrt{t}}\left(\Phi'\left(\frac{x-\mu t}{\sqrt{t}}\right) + e^{2\mu x}\Phi'\left(\frac{x+\mu t}{\sqrt{t}}\right)\right)\\ -2\mu e^{2\mu x}\left(1 - \Phi\left(\frac{x+\mu t}{\sqrt{t}}\right)\right).\end{aligned} \tag{6.16}$$

We break down (6.16) in the term containing $\frac{1}{\sqrt{t}}\Phi'(\frac{x-\mu t}{\sqrt{t}})$; the term containing $\frac{e^{2\mu x}}{\sqrt{t}}\Phi'(\frac{x+\mu t}{\sqrt{t}})$, both bounded above by $\frac{e^{\mu}}{\sqrt{t}}\Phi'(\mu\sqrt{t})$, which is integrable in t on $(0, \infty)$; and the third part, with absolute value bounded above by $2\mu e^{2\mu}(1 - \Phi(\mu\sqrt{t}))$, which is also integrable

$$\int_0^\infty 1 - \Phi(\mu\sqrt{t})dt \le \left(1 + \sqrt{\frac{2}{\pi}}\right)\frac{1}{\mu^2} < \infty.$$

The last inequality comes from the estimate on the error function

$$1 - \Phi(\mu\sqrt{t}) = \int_{\mu\sqrt{t}}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \le \int_{\mu\sqrt{t}}^\infty z\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz = \frac{1}{\sqrt{2\pi}}e^{-\frac{\mu^2 t}{2}}$$

when $\mu\sqrt{t} \ge 1$. □

6.2 Brownian motion without drift

Proposition 10 *When $\mu = 0$, the distribution of $V = X/x$ is independent of the starting point x having density*

$$f_V(v) = \frac{8v}{\pi[(v-1)^2 + 1][(v+1)^2 + 1]}. \tag{6.17}$$

Since $f_V(v) \sim O(v)$ at $v = 0$ and $f_V(v) \sim O(v^{-3})$ at $v = +\infty$, the random variable V has moments $E[V^a]$ up to $a < 2$, with $\mu_V = 2$, $\sigma_V^2 = \infty$ and $E[\ln V] > 0$.

Proof The cumulative distribution function of the hitting time τ^G, based on (6.4) applied to (6.7) is $2\big(1-\Phi(\frac{x}{\sqrt{t}})\big)$ and the density is

$$-\frac{d}{dt}P_x(\tau^G > t) = \frac{x}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}} \tag{6.18}$$

so (6.3) reads

$$\frac{P_x(X\in dy)}{dy} = \int_0^\infty \frac{x}{\pi t^2}\left(e^{-\frac{(y-x)^2+x^2}{2t}} - e^{-\frac{(y+x)^2+x^2}{2t}}\right)dt \tag{6.19}$$

$$= \frac{x}{\pi}\left(\frac{2}{(y-x)^2+x^2} - \frac{2}{(y+x)^2+x^2}\right) = \frac{1}{x}f_V\left(\frac{y}{x}\right). \tag{6.20}$$

In the last equality we identified the alternative formula

$$f_V(v) = \frac{2}{\pi}\left(\frac{1}{(v-1)^2+1} - \frac{1}{(v+1)^2+1}\right) \tag{6.21}$$

with

$$F_V(v) = P(V\le v) = 1 - \frac{2}{\pi}\left(\arctan(v+1) - \arctan(v-1)\right). \tag{6.22}$$

One can calculate explicitly

$$E[V]=\left[\frac{1}{\pi}\ln(\frac{1+(v-1)^2}{1+(v+1)^2})+\frac{2}{\pi}(\arctan(v-1)+\arctan(v+1))\right]\Big|_0^\infty=2. \tag{6.23}$$

The logarithm $\ln V$ is integrable and we can determine numerically that $E[\ln V] \approx 0.34$. □

The interior chain (X_n) satisfies $\ln X_n = \ln x_0 + \sum_{k=1}^n \ln V_k$ where V_k are i.i.d. with distribution (6.17). By the law of large numbers, we have $\frac{\ln X_n}{n} \to E[\ln V] > 0$ as $n\to\infty$ with probability one so $P_{x_0}(\lim_{n\to\infty} X_n = \infty) = 1$.

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