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Exact value of the resistance exponent for four dimensional random walk trace

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Abstract Let S be a simple random walk starting at the origin in $\mathbb{Z}^4$. We consider $\mathcal{G} = S[0, \infty)$ to be a random subgraph of the integer lattice and assume that a resistance of unit 1 is put on each edge of the graph $\mathcal{G}$. Let $R_{\mathcal{G}}(0, S_n)$ be the effective resistance between the origin and S_n. We derive the exact value of the resistance exponent; more precisely, we prove that $n^{-1} E(R_{\mathcal{G}}(0, S_n)) \approx (\log n)^{-\frac{1}{2}}$. As an application, we obtain sharp heat kernel estimates for random walk on $\mathcal{G}$ at the quenched level. These results give the answer to the problem raised by Burdzy and Lawler (J Phys A Math Gen 23(1):L23–L28, 1990) in four dimensions.

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1 Introduction and main results

1.1 Introduction

Let S be the simple random walk starting at the origin on $\mathbb{Z}^d$. We consider $S[0, \infty)$ to be a random subgraph of the integer lattice; namely, we let $\mathcal{G} = (V(\mathcal{G}), B(\mathcal{G}))$ be the graph with

$$V(\mathcal{G}) = \{S_k : k \geq 0\} \quad B(\mathcal{G}) = \{\{S_k, S_{k+1}\} : k \geq 0\}.$$

The fractal nature of the graph $\mathcal{G}$ has been studied in a number of papers in both the physics and mathematical literature (see [1,8] and reference therein). One particular quantity of interest has been the effective resistance for $\mathcal{G}$ assuming a unit resistor on

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each edge in $B(\mathcal{G})$. Let $R_{\mathcal{G}}(0, S_n)$ be the effective resistance between 0 and S_n. It was shown in [1] that there exists a constant $c = c_d > 0$ depending only on dimension such that

$$\begin{aligned} & E(R_{\mathcal{G}}(0, S_n)) \sim cn \quad \text{for } d \geq 5 \\ & cn(\log n)^{-\frac{1}{2}} \leq E(R_{\mathcal{G}}(0, S_n)) \leq \frac{1}{c} n(\log n)^{-\frac{1}{3}} \quad \text{for } d = 4 \\ & \limsup_{n\to\infty} \frac{\log E(R_{\mathcal{G}}(0, S_n))}{\log n} \leq \frac{5}{6} \quad \text{for } d = 3, \end{aligned} \tag{1.1}$$

where we use E to denote expectation, and we write $a_n \sim b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. Thus, $d = 4$ is a critical dimension for the effective resistance on $\mathcal{G}$. For the remainder of this paper, we focus our attention on the four dimensional case.

For $d = 4$, it was conjectured in [1] that there exists a $\rho > 0$ such that

$$\psi(n) := \frac{E(R_{\mathcal{G}}(0, S_n))}{n} \approx (\log n)^{-\rho},$$

and they called ρ the resistance exponent where we write $a_n \approx b_n$ if $\log a_n \sim \log b_n$. By (1.1), we know that $\frac{1}{3} \leq \rho \leq \frac{1}{2}$ if it exists, where $\frac{1}{3}$ is the loop-erasing exponent and $\frac{1}{2}$ is the exponent for cut-times (see [1]). One of the main result in this paper is Theorem 1.2.3, which shows $\rho = \frac{1}{2}$. This gives the answer to the problem raised in [1] in four dimensions. We give a heuristic reason why $\rho = \frac{1}{2}$ here. Let $T_1 < T_2 < \cdots$ be the sequence of cut-times. Since the expected number of cut-times up to n is of order $n(\log n)^{-\frac{1}{2}}$ (see [6]), we have to estimate the following quantity;

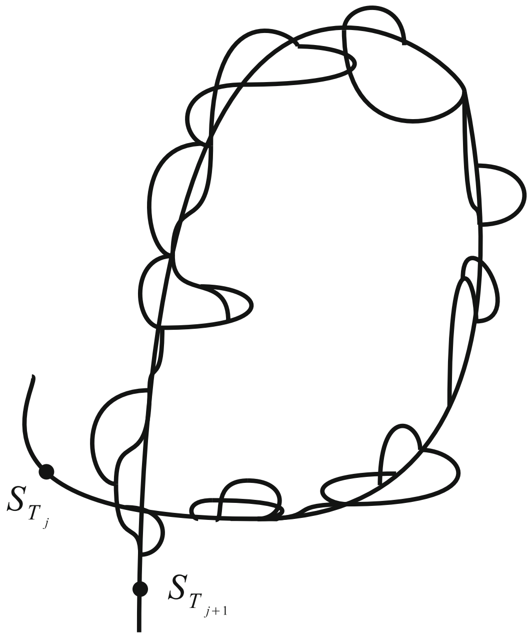
$$\sum_{j=1}^{a_n} R_{\mathcal{G}}\left(S_{T_j}, S_{T_{j+1}}\right),$$

where $a_n = \lfloor n(\log n)^{-\frac{1}{2}} \rfloor$. It is clear that $R_{\mathcal{G}}(S_{T_j}, S_{T_{j+1}}) \geq 1$ for each j. Thus, in order to prove $\rho = \frac{1}{2}$, all we need is to show $R_{\mathcal{G}}(S_{T_j}, S_{T_{j+1}})$ is not large even if $T_{j+1} - T_j$ is large. Indeed there is j such that $T_{j+1} - T_j$ is of order $(\log n)^{\frac{1}{2}}$. For the purpose, we study the shape of the random walk trace between such successive cut-times T_j and T_{j+1} that are far apart and show that the random walk trace near S_{T_j} and $S_{T_{j+1}}$ intersects typically, namely the "long range intersection" occurs between them (see Fig. 1). By this intersection, we can find a path on the trace connecting S_{T_j} and $S_{T_{j+1}}$ whose length is not long and so we conclude $R_{\mathcal{G}}(S_{T_j}, S_{T_{j+1}})$ is not large.

The effective resistance for $\mathcal{G}$ is strongly related to the heat kernel of X, where X is the simple random walk on $\mathcal{G}$ (see [4], for example). By using Theorem 1.2.3, we are able to obtain a sharp estimate for the heat kernel of X. It was shown in [9] that for $d = 4$, there exists a $c > 0$ such that for all $\delta \in (0, 1)$,

$$n^{-\frac{1}{2}}(\log n)^{-\frac{3}{2}-\delta} \leq p_{2n}^{\mathcal{G}(\omega)}(0, 0) \leq cn^{-\frac{1}{2}}(\log n)^{-\frac{1}{6}}, \quad \text{for large } n \text{ almost surely,} \tag{1.2}$$

Fig. 1 A shape of $S[T_j, T_{j+1}]$ when $T_{j+1} - T_j$ is large



where $p_n^{\mathcal{G}(\omega)}(x, y)$ denotes the quenched heat kernel of X (see Sect. 1.2) for a definition of X and its heat kernel). As we can see, the power of the logarithm for the upper bound on $p_{2n}^{\mathcal{G}(\omega)}(0, 0)$ is different from one for the lower bound in (1.2), and the exact power of this logarithmic correction was not known. In this paper, we show that this logarithmic correction is equal to $\psi(n)^{\frac{1}{2}}$ (Theorem 1.2.2). Combining Theorem 1.2.2 and Theorem 1.2.3, we conclude that the exact power of the logarithm of the heat kernel of X is $-\frac{1}{4}$ at the quenched level, improving (1.2).

The organization of the paper is as follows. In Sect. 2, we study asymptotic behavior of $R_{\mathcal{G}}(0, S_n)$ in order to obtain Theorem 1.2.1. It is worth emphasizing that the resistance has oscillations of order $(\log \log n)^{-\frac{1}{2}}$ almost surely as in (1.7). This is due to the fact that $\mathcal{G}$ has the above mentioned long range intersections infinitely often. In Sect. 3, we prove Theorem 1.2.2. To do this, we study the connectivity of $\mathcal{G}$ around the long range intersection point. We show that once the long range intersection occurs, then the trace near the intersection point is relatively sparse. Thus, although the trace contains large scale fluctuations, they give no effect on the asymptotic behavior of X. Finally, we show $\rho = \frac{1}{2}$ (Theorem 1.2.3) in Sect. 4.

Throughout the paper, we write $a_n = O(b_n)$ if $a_n \le cb_n$ for some constant $c > 0$. If we wish to imply that the constant may depend on another quantity, say ϵ, we write $O_\epsilon(b_n)$. We use $c, \tilde{c}, c_1, \dots$ to denote arbitrary positive constants which may change from line to line. If a constant is to depend on some other quantity, this will be made explicit. For example, if c depends on ϵ, we write c_ϵ. We write $a_n \asymp b_n$ if there exist constants c_1, c_2 such that

$$c_1 b_n \le a_n \le c_2 b_n.$$

1.2 Framework and main results

Let $S = (S_n)_{n \geq 0}$ be the simple random walk on $\mathbb{Z}^4$ starting from 0, built on underlying probability space $(\Omega, \mathcal{F}, P)$. Define the range of the random walk $S(\omega)$ to be the graph $\mathcal{G}(\omega) = (V(\mathcal{G}(\omega)), B(\mathcal{G}(\omega)))$ with vertex set

$$V(\mathcal{G}(\omega)) := \{S_n(\omega) : n \geq 0\},$$

and edge set

$$B(\mathcal{G}(\omega)) := \{\{S_n(\omega), S_{n+1}(\omega)\} : n \geq 0\},$$

where ω is an element of Ω. (For simplicity, we often omit ω.)

We define a quadratic form $\mathcal{E}$ by

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_{\substack{x,y \in V(\mathcal{G}),\\ \{x,y\} \in B(\mathcal{G})}} (f(x) - f(y))(g(x) - g(y)).$$

If we regard $\mathcal{G}$ as an electrical network with a unit resistor on each edge in $B(\mathcal{G})$, then $\mathcal{E}(f, f)$ is the energy dissipation when the vertices of $V(\mathcal{G})$ are at a potential f. Set

$$H^2 = \{f \in \mathbb{R}^{V(\mathcal{G})} : \mathcal{E}(f, f) < \infty\}.$$

Let A, B be disjoint subsets of $V(\mathcal{G})$. The effective resistance between A and B is defined by

$$R_{\mathcal{G}}(A, B)^{-1} = \inf\{\mathcal{E}(f, f) : f \in H^2, f|_A = 1, f|_B = 0\}. \tag{1.3}$$

Let $R_{\mathcal{G}}(x, y) = R_{\mathcal{G}}(\{x\}, \{y\})$.

In this article, the main object of study will be the following function ψ;

$$\psi(n) = \frac{E\left(R_{\mathcal{G}}(0, S_n)\right)}{n}. \tag{1.4}$$

Let $\mu_{\mathcal{G}}(x)$ be the number of bonds that contain x, i.e.,

$$\mu_{\mathcal{G}}(x) = \sharp\{\{x, y\} \in B(\mathcal{G})\}.$$

We extend $\mu_{\mathcal{G}}$ to a measure on $\mathcal{G}$ by setting $\mu_{\mathcal{G}}(A) = \sum_{x \in A} \mu_{\mathcal{G}}(x)$ for $A \subset \mathcal{G}$.

We denote the simple random walk on $\mathcal{G}(\omega)$ by

$$X = \left((X_n)_{n \geq 0}, P_x^{\mathcal{G}(\omega)}, x \in V(\mathcal{G}(\omega))\right),$$

and its heat kernel (transition density) with respect to $\mu_{\mathcal{G}(\omega)}$ by $p_n^{\mathcal{G}(\omega)}(x, y)$, i.e.,

$$p_n^{\mathcal{G}(\omega)}(x, y) = P_x^{\mathcal{G}(\omega)}(X_n = y)\frac{1}{\mu_{\mathcal{G}(\omega)}(y)}.$$

To define X we introduce a second measure space $(\overline{\Omega}, \overline{\mathcal{F}})$, and define X on the product $\Omega \times \overline{\Omega}$. We write $\overline{\omega}$ to denote elements of $\overline{\Omega}$.

The following theorems are our main results in this paper.

Theorem 1.2.1 *ψ is slowly varying and*

$$\frac{R_{\mathcal{G}}(0, S_n)}{n\psi(n)} \to 1 \quad \textit{in probability}. \tag{1.5}$$

Furthermore, there exists $c > 0$ such that for P-a.s. realization of $\mathcal{G}$,

$$R_{\mathcal{G}}(0, S_n)(\omega) \le cn\psi(n) \quad \textit{for large } n, \tag{1.6}$$

$$R_{\mathcal{G}}(0, S_n)(\omega) \le cn\psi(n)(\log\log n)^{-\frac{1}{2}} \quad \textit{for infinitely many } n, \tag{1.7}$$

$$R_{\mathcal{G}}(0, S_n)(\omega) \ge n\psi(n)(\log\log n)^{-7} \quad \textit{for large } n. \tag{1.8}$$

Theorem 1.2.2 *There exist $c_1, c_2 > 0$ such that for P-a.s. realization of $\mathcal{G}$,*

$$c_1 n^{-\frac{1}{2}} (\psi(n))^{\frac{1}{2}} \le p_{2n}^{\mathcal{G}(\omega)}(0, 0) \le c_2 n^{-\frac{1}{2}} (\psi(n))^{\frac{1}{2}}, \tag{1.9}$$

for large n.

Theorem 1.2.3

$$\psi(n) \approx (\log n)^{-\frac{1}{2}}. \tag{1.10}$$

We will give the proofs of Theorems 1.2.1, 1.2.2 and 1.2.3 in Sects. 2, 3 and 4 respectively.

For the convenience of the reader, we list the notations we will use and in which subsection they can be found.

Notation	*Meaning*	*Subsection*
$\mathcal{G}_{k,l}$	Random walk trace between k and l	2.1
$\overline{\psi}(n)$	$\frac{E\left(R_{\mathcal{G}_{0,n}}(0,S_n)\right)}{n}$	2.1
$\psi(n)$	$\frac{E(R_{\mathcal{G}}(0,S_n))}{n}$	1.2
$a_{n,r}$	$\lfloor n(\log n)^r \rfloor$	2.1
$b_{n,r}$	$\lfloor n(\log\log n)^r \rfloor$	2.1
$\overline{b}_{j,n}$	$b_{n,-j-j\delta}$	2.3
$\tilde{Y}_k$	$\mathbf{1}\{R_{\mathcal{G}}(0, S_k) \leq \epsilon^2 n\psi(n)\}$	3.1
$\tilde{Y}$	$\sum_{k=0}^{b_{n,7}} \tilde{Y}_k$	3.2
$\bar{Y}$	$\sum_{k=n}^{b_{n,7}} \tilde{Y}_k$	3.2
d_n	$\lfloor \epsilon^2 n \rfloor$	3.2
$\phi(n)$	$\frac{(\log\log n)^2}{\log n}$	3.2
$\bar{a}_n$	$\lfloor \frac{1}{3} n \rfloor$	3.2
Z	$\sharp\{1 \leq i \leq C : S[0, \bar{a}_n] \cap S[t_i - \bar{a}_n, t_i + \bar{a}_n] \neq \emptyset\}$	3.2
$\tilde{F}_i$	$\{S[0, \bar{a}_n] \cap S[t_i - \bar{a}_n, t_i + \bar{a}_n] \neq \emptyset\}$	3.2
$\tilde{K}_l$	$\mathbf{1}\{l$ is a local cut-time between $\bar{a}_n$ and $t_C - \bar{a}_n\}$	3.2
$\tilde{A}_i$	$\left\{ \sum_{l=t_i-3a_{n,-1}}^{t_i} \tilde{K}_l \geq 1, \quad \sum_{l=t_i}^{t_i+3a_{n,-1}} \tilde{K}_l \geq 1 \right\}$	3.2
R_n^1	$R_{\mathcal{G}_{0,n}}(0, S_n)$	4.2
R_n^2	$R_{\mathcal{G}_{n,2n}}(S_n, S_{2n})$	4.2

2 Proof of Theorem 1.2.1

2.1 Approximation of the resistance

We first give some notations that are used in this paper. For $0 \leq k \leq l < \infty$, let $\mathcal{G}_{k,l} = \left(V(\mathcal{G}_{k,l}), B(\mathcal{G}_{k,l})\right)$ be the graph with

$$V(\mathcal{G}_{k,l}) = \{S_j : k \leq j \leq l\} \quad B(\mathcal{G}_{k,l}) = \{\{S_j, S_{j+1}\} : k \leq j < l\}.$$

(We use $\mathcal{G}_{k,\infty}$ when we consider $S[k,\infty)$ as a graph.) We write $R_{\mathcal{G}_{k,l}}(\cdot,\cdot)$ when we consider the effective resistance on the graph $\mathcal{G}_{k,l}$, where a unit resistance is put on each edge of the graph $\mathcal{G}_{k,l}$. Let

$$\overline{\psi}(n) = \frac{E\left(R_{\mathcal{G}_{0,n}}(0, S_n)\right)}{n}. \tag{2.1}$$

Throughout this paper, we use key tools called cut-times which we will explain below in order to divide an electrical circuit into two disjoint ones.

We call a time k a global cut-time (for S) if $S[0,k] \cap S(k,\infty) = \emptyset$. Let $0 \leq j \leq k \leq l < \infty$. We call a time k a local cut-time between j and l if $S[j,k] \cap S(k,l] = \emptyset$.

(We allow the case in which j and l depend on k.) We call k a local cut-time between j and ∞ if $S[j,k] \cap S(k,\infty) = \emptyset$.

We write

$$a_{n,r} = \lfloor n(\log n)^r \rfloor, \quad \text{for } n \in \mathbb{N}, r \in \mathbb{R} \tag{2.2}$$

and

$$b_{n,r} = \lfloor n(\log\log n)^r \rfloor, \quad \text{for } n \in \mathbb{N}, r \in \mathbb{R}. \tag{2.3}$$

Lemma 2.1.1

$$\psi(n) \sim \overline{\psi}(n).$$

Proof Let $I_n = I(n)$ be the indicator function of the event $\{n \text{ is a global cut-time}\}$ and B_n be the event

$$B_n = \{I_k = 0 \quad \text{for all } k \in [n - a_{n,-6}, n]\}.$$

If there exists a global cut-time $T \in [n - a_{n,-6}, n]$, then by definition, we have

$$\begin{aligned} |R_{\mathcal{G}_{0,n}}(0,S_n) - R_{\mathcal{G}}(0,S_n)| &= |R_{\mathcal{G}_{0,T}}(0,S_T) + R_{\mathcal{G}_{T,n}}(S_T,S_n) - R_{\mathcal{G}_{0,T}}(0,S_T) \\ -R_{\mathcal{G}_{T,\infty}}(S_T,S_n)| &= |R_{\mathcal{G}_{T,n}}(S_T,S_n) - R_{\mathcal{G}_{T,\infty}}(S_T,S_n)| \\ &\le 2n(\log n)^{-6}. \end{aligned}$$

Therefore,

$$\begin{aligned} &|E\left(R_{\mathcal{G}_{0,n}}(0,S_n) - R_{\mathcal{G}}(0,S_n)\right)| \\ &\quad \le |E\left(R_{\mathcal{G}_{0,n}}(0,S_n) - R_{\mathcal{G}}(0,S_n); B_n\right)| + |E\left(R_{\mathcal{G}_{0,n}}(0,S_n) - R_{\mathcal{G}}(0,S_n); B_n^c\right)| \\ &\quad \le nP(B_n) + 2n(\log n)^{-6}. \end{aligned}$$

Since $P(B_n) = O\left(\frac{\log\log n}{\log n}\right)$, (see, [6, Lemma 7.7.4]) we have

$$|\psi(n) - \overline{\psi}(n)| = O\left(\frac{\log\log n}{\log n}\right).$$

By the fact that

$$\psi(n) \ge c(\log n)^{-\frac{1}{2}} \tag{2.4}$$

(see the proof of Lemma 2.2.2 in [9], for example), we obtain the lemma. □

Cut-times fill the role of separating the random walk trace in the proof of Lemma 2.1.1. By using this technique, we will show (1.5).

Proof of Theorem 1.2.1 (1.5) Let $N = \lfloor \frac{n}{a_{n,-2}} \rfloor$. For $i \in \{1, \ldots, N\}$, let J_i be the indicator function of the event

$$A_i = \{I_k = 0, \text{ for all } k = (i-1)a_{n,-2}, \ldots, (i-1)a_{n,-2} + a_{n,-6}\} \\ \cup \{I_k = 0, \text{ for all } k = ia_{n,-2} - a_{n,-6}, \ldots, ia_{n,-2}\}.$$

Then we have

$$|R_{\mathcal{G}}(0, S_n) - \sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2}, ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}})| \\ \leq n(\log n)^{-4} + n(\log n)^{-2} \sum_{i=1}^{N} J_i. \tag{2.5}$$

However, it is known (see, [6, Lemma 7.7.4]) that

$$E\left(\sum_{i=1}^{N} J_i\right) = O\left((\log n)(\log\log n)\right).$$

Therefore, it follows from (2.4) that

$$E\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2}, ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}})\right) \sim n\psi(n), \tag{2.6}$$

and for any $\epsilon > 0$,

$$P\left(n(\log n)^{-2} \sum_{i=1}^{N} J_i \geq \tfrac{\epsilon}{4} n\psi(n)\right) \leq P\left(\sum_{i=1}^{N} J_i \geq c\tfrac{\epsilon}{4}(\log n)^{\frac{3}{2}}\right) \\ \leq c_\epsilon (\log n)^{-\frac{3}{2}} E\left(\sum_{i=1}^{N} J_i\right) \\ \leq c_\epsilon (\log n)^{-\frac{1}{2}} \log\log n. \tag{2.7}$$

On the other hand, by independence and (2.4),

$$\mathrm{Var}\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2}, ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}})\right) \\ = \sum_{i=1}^{N} \mathrm{Var}\left(R_{\mathcal{G}_{(i-1)a_{n,-2}, ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}})\right) \\ \leq \sum_{i=1}^{N} E\left(\left(R_{\mathcal{G}_{(i-1)a_{n,-2}, ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}})\right)^2\right) \\ \leq cn^2(\log n)^{-2}\psi(n), \tag{2.8}$$

where we use $R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}) \le n(\log n)^{-2}$ and (2.6) in the third inequality. Thus, for any fixed $\epsilon > 0$ it follows that for large n,

$$
\begin{aligned}
&P\left(|R_{\mathcal{G}}(0,S_n) - n\psi(n)| \ge \epsilon n\psi(n)\right)\\
&\le P\left(n(\log n)^{-2}\sum_{i=1}^{N} J_i \ge \tfrac{\epsilon}{4} n\psi(n)\right)\\
&\quad + P\left(|\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}) - n\psi(n)| \ge \tfrac{\epsilon}{2} n\psi(n)\right)\\
&\le c_\epsilon (\log n)^{-\frac{1}{2}}(\log\log n) + c_\epsilon \frac{\mathrm{Var}\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}})\right)}{(n\psi(n))^2}\\
&\le c_\epsilon (\log n)^{-\frac{1}{2}}(\log\log n),
\end{aligned}
$$

where we use (2.4) in the third inequality. This implies (1.5). □

Proposition 2.1.2 *ψ is slowly varying.*

Proof By Lemma 2.1.1, it suffices to prove the result for $\overline{\psi}$. What we have to show is

$$
\lim_{n\to\infty} \frac{\overline{\psi}(\lfloor rn\rfloor)}{\overline{\psi}(n)} = 1 \quad \text{for all } r \in (0,1). \tag{2.9}
$$

We first show (2.9) when $r \in \mathbb{Q}\cap(0,1)$. Let $r = \frac{q}{p} \in \mathbb{Q}\cap(0,1)$, where $p, q \in \mathbb{N}$ satisfy $1 \le q \le p$ and $\gcd(p,q) = 1$.

By modifying the proof of Lemma 2.1.1, we have

$$
\overline{\psi}(n) \sim \overline{\psi}(k), \overline{\psi}(\lfloor rn\rfloor) \sim \overline{\psi}(\lfloor rk\rfloor)
$$

for all $n - p \le k \le n$. Hence we may assume that $\frac{n}{p} =: N \in \mathbb{N}$.

Let

$$
j_i = \frac{in}{p} \quad i = 0, \ldots, p.
$$

By the similar argument as in (2.5), we know that

$$
\left|E\left(R_{\mathcal{G}_{0,j_q}}(0, S_{j_q})\right) - E\left(\sum_{i=0}^{q} R_{\mathcal{G}_{j_{i-1},j_i}}(S_{j_{i-1}}, S_{j_i})\right)\right| = nO\left(\frac{\log\log n}{\log n}\right),
$$
$$
\left|E\left(R_{\mathcal{G}_{0,n}}(0, S_n)\right) - E\left(\sum_{i=0}^{p} R_{\mathcal{G}_{j_{i-1},j_i}}(S_{j_{i-1}}, S_{j_i})\right)\right| = nO\left(\frac{\log\log n}{\log n}\right).
$$

Therefore,

$$\overline{\psi}(j_q) = \overline{\psi}(j_1) + O\left(\frac{\log\log n}{\log n}\right),$$
$$\overline{\psi}(n) = \overline{\psi}(j_1) + O\left(\frac{\log\log n}{\log n}\right).$$

Since $\psi(n) \geq c(\log n)^{-\frac{1}{2}}$ and $\psi(n) \sim \overline{\psi}(n)$, we have

$$\overline{\psi}(j_q) \sim \overline{\psi}(n).$$

For $r \notin \mathbb{Q}$, it follows that for all $\epsilon > 0$, there exist $r_1, r_2 \in \mathbb{Q}$ such that

$$1 - \epsilon \leq \frac{r_1}{r} \leq 1 \leq \frac{r_2}{r} \leq 1 + \epsilon.$$

Let F_n be the event

$$F_n = \{I_k = 0 \quad \text{for all } k \in [\lfloor rn \rfloor - a_{n,-6}, \lfloor rn \rfloor]\}.$$

Then,

$$\begin{aligned} E\left(R_{\mathcal{G}_{0,\lfloor rn \rfloor}}(0, S_{\lfloor rn \rfloor})\right) &= E\left(R_{\mathcal{G}_{0,\lfloor rn \rfloor}}(0, S_{\lfloor rn \rfloor}); F_n\right) + E\left(R_{\mathcal{G}_{0,\lfloor rn \rfloor}}(0, S_{\lfloor rn \rfloor}); F_n^c\right) \\ &= nO\left(\frac{\log\log n}{\log n}\right) + E\left(R_{\mathcal{G}_{0,\lfloor rn \rfloor}}(0, S_{\lfloor rn \rfloor}); F_n^c\right) \\ &\leq nO\left(\frac{\log\log n}{\log n}\right) + a_{n,-6} + E\left(R_{\mathcal{G}_{0,\lfloor r_2 n \rfloor}}(0, S_{\lfloor r_2 n \rfloor})\right). \end{aligned}$$

Therefore,

$$\begin{aligned} \overline{\psi}(\lfloor rn \rfloor) &\leq \frac{r_2}{r}\overline{\psi}(\lfloor r_2 n \rfloor) + O\left(\frac{\log\log n}{\log n}\right) \\ &\leq (1 + 2\epsilon)\overline{\psi}(\lfloor r_2 n \rfloor). \end{aligned}$$

Similarly, we have

$$(1 - 2\epsilon)\overline{\psi}(\lfloor r_1 n \rfloor) \leq \overline{\psi}(\lfloor rn \rfloor).$$

Since (2.9) holds for $r = r_1, r_2$, we have

$$1 - 2\epsilon \leq \liminf_{n\to\infty} \frac{\overline{\psi}(\lfloor rn \rfloor)}{\overline{\psi}(n)} \leq \limsup_{n\to\infty} \frac{\overline{\psi}(\lfloor rn \rfloor)}{\overline{\psi}(n)} \leq 1 + 2\epsilon,$$

and hence we have (2.9). □

Remark 2.1.3 Note that for $r \in (0, 4)$, we have

$$\overline{\psi}(n) \sim \overline{\psi}\left(a_{n,-r}\right). \tag{2.10}$$

Indeed, by modifying the argument in (2.5),

$$\left| E\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-r},ia_{n,-r}}}\left(S_{(i-1)a_{n,-r}}, S_{ia_{n,-r}}\right)\right) - E\left(R_{\mathcal{G}_{0,n}}(0, S_n)\right)\right| = nO\left(\frac{\log\log n}{\log n}\right),$$

where $N = \lfloor \frac{n}{a_{n,-r}} \rfloor$. Hence,

$$|Na_{n,-r}\overline{\psi}(a_{n,-r}) - n\overline{\psi}(n)| = nO\left(\frac{\log\log n}{\log n}\right).$$

Since $\overline{\psi}(n) \geq c(\log n)^{-\frac{1}{2}}$, we have

$$\frac{Na_{n,-r}}{n}\frac{\overline{\psi}(a_{n,-r})}{\overline{\psi}(n)} = 1 + O\left(\frac{\log\log n}{(\log n)^{\frac{1}{2}}}\right).$$

Therefore, (2.10) holds.

Proposition 2.1.4 *For all $\epsilon > 0$,*

$$P\left(R_{\mathcal{G}_{0,n}}(0, S_n) \leq (1+\epsilon)n\overline{\psi}(n)\right) = 1 - O_\epsilon\left((\log n)^{-\frac{3}{2}}\right).$$

Proof Let $N = \lfloor \frac{n}{a_{n,-2}} \rfloor$. We have

$$\begin{aligned} R_{\mathcal{G}_{0,n}}(0, S_n) &\leq \sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}\left(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}\right) + R_{\mathcal{G}_{Na_{n,-2},n}}\left(S_{Na_{n,-2}}, S_n\right) \\ &\leq \sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}\left(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}\right) + n(\log n)^{-2}. \end{aligned} \tag{2.11}$$

Therefore, by (2.4),

$$\begin{aligned} &P\left(R_{\mathcal{G}_{0,n}}(0, S_n) \geq (1+\epsilon)n\overline{\psi}(n)\right) \\ &\leq P\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}) \geq (1+\tfrac{2}{3}\epsilon)n\overline{\psi}(n)\right). \end{aligned} \tag{2.12}$$

It follows from (2.6) that

$$E\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}})\right) \leq \left(1+\frac{\epsilon}{3}\right)n\overline{\psi}(n),$$

for large n. So the right-hand side of (2.12) is bounded above by

$$P\left(|\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}) \right.$$
$$\left. -E\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}})\right)| \geq \frac{\epsilon}{3}n\overline{\psi}(n)\right)$$
$$\leq \frac{9}{\epsilon^2}\frac{\mathrm{Var}\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}})\right)}{(n\overline{\psi}(n))^2} \leq c_\epsilon(\log n)^{-\frac{3}{2}}, \quad (2.13)$$

where we use (2.4) and (2.8) in the second inequality. □

Remark 2.1.5 By Proposition 2.1.4 and the Borel–Cantelli lemma, it follows that there exists $c > 0$ such that for P-a.s. ω,

$$R_{\mathcal{G}_{0,n}}(0, S_n)(\omega) \leq cn\overline{\psi}(n), \quad (2.14)$$

for large n. Indeed, let $M_n := \max_{0\leq k\leq l\leq n} R_{\mathcal{G}_{k,l}}(S_k, S_l)$. Then we have

$$M_n \leq 2n(\log n)^{-2} + \sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}).$$

Therefore, it follows from (2.13) that

$$P\left(M_n \leq 2n\overline{\psi}(n)\right) = 1 - O\left((\log n)^{-\frac{3}{2}}\right).$$

By the Borel–Cantelli lemma, for P-a.s. ω, $M_{2^k}(\omega) \leq 2^{k+1}\overline{\psi}(2^k)$ for large k. Now (2.14) can be shown by using the monotonicity of M_n.(See also the proof of Theorem 1.2.1 in [9].)

2.2 Oscillations of the effective resistance

In this subsection, we give the proof of (1.7). Comparing (1.7) with (1.5), we know the effective resistance $R_{\mathcal{G}}(0, S_n)$ has at least oscillations of order $(\log\log n)^{-\frac{1}{2}}$ at the quenched level. These oscillations are due to the fact that the random walk trace has "long range intersections" infinitely often (see Lemma 2.2.3 below). We now begin with several lemmas.

Lemma 2.2.1 *There exists $c > 0$ such that for P-a.s. ω,*

$$\max_{0\leq k\leq l\leq b_{n,-\frac{1}{2}}} R_{\mathcal{G}_{k,l}}(S_k, S_l)(\omega) \leq cb_{n,-\frac{1}{2}}\overline{\psi}(n), \quad (2.15)$$

$$\max_{n-b_{n,-\frac{1}{2}}\leq k\leq l\leq n} R_{\mathcal{G}_{k,l}}(S_k, S_l)(\omega) \leq cb_{n,-\frac{1}{2}}\overline{\psi}(n). \quad (2.16)$$

Proof The proof of (2.15) and (2.16) are similar, we will only prove (2.15). Let $N = \lfloor \frac{b_{n,-\frac{1}{2}}}{a_{n,-2}} \rfloor$. Let $\epsilon > 0$ be an arbitrary positive number. Assume that

$$\max_{0 \le k \le l \le b_{n,-\frac{1}{2}}} R_{\mathcal{G}_{k,l}}(S_k, S_l) \ge (1+\epsilon) b_{n,-\frac{1}{2}} \overline{\psi}(n).$$

Recall that $\overline{\psi}(n) \sim \overline{\psi}(b_{n,-\frac{1}{2}})$ (see Remark 2.1.3), and hence we have

$$\max_{0 \le k \le l \le b_{n,-\frac{1}{2}}} R_{\mathcal{G}_{k,l}}(S_k, S_l) \ge \left(1+\frac{\epsilon}{2}\right) b_{n,-\frac{1}{2}} \overline{\psi}\left(b_{n,-\frac{1}{2}}\right),$$

for large n (depending on ϵ). This implies that there exist $0 \le k_0 \le l_0 \le b_{n,-\frac{1}{2}}$ such that

$$R_{\mathcal{G}_{k_0,l_0}}\left(S_{k_0}, S_{l_0}\right) \ge \left(1+\frac{\epsilon}{2}\right) b_{n,-\frac{1}{2}} \overline{\psi}\left(b_{n,-\frac{1}{2}}\right).$$

However, we know

$$R_{\mathcal{G}_{k_0,l_0}}(S_{k_0}, S_{l_0}) \le \sum_{j=1}^{N} R_{\mathcal{G}_{i_{j-1},i_j}}(S_{i_{j-1}}, S_{i_j}) + 2n(\log n)^{-2},$$

where $i_j := j a_{n,-2}$ for $j = 1, 2, \ldots, N$. Therefore, for large n,

$$\sum_{j=1}^{N} R_{\mathcal{G}_{i_{j-1},i_j}}\left(S_{i_{j-1}}, S_{i_j}\right) \ge \left(1+\tfrac{\epsilon}{4}\right) b_{n,-\frac{1}{2}} \overline{\psi}\left(b_{n,-\frac{1}{2}}\right).$$

So, by the similar argument as in (2.13), we have

$$\begin{aligned}
&P\left(\max_{0 \le k \le l \le b_{n,-\frac{1}{2}}} R_{\mathcal{G}_{k,l}}(S_k, S_l) \ge (1+\epsilon) b_{n,-\frac{1}{2}} \overline{\psi}(n)\right) \\
&\le P\left(\sum_{j=1}^{N} R_{\mathcal{G}_{i_{j-1},i_j}}\left(S_{i_{j-1}}, S_{i_j}\right) \ge \left(1+\tfrac{\epsilon}{4}\right) b_{n,-\frac{1}{2}} \overline{\psi}\left(b_{n,-\frac{1}{2}}\right)\right) \\
&\le P\left(\left|\sum_{j=1}^{N} R_{\mathcal{G}_{i_{j-1},i_j}}\left(S_{i_{j-1}}, S_{i_j}\right) - E\left(\sum_{j=1}^{N} R_{\mathcal{G}_{i_{j-1},i_j}}\left(S_{i_{j-1}}, S_{i_j}\right)\right)\right| \ge \frac{\epsilon}{8} b_{n,-\frac{1}{2}} \overline{\psi}\left(b_{n,-\frac{1}{2}}\right)\right) \\
&\le c_\epsilon \frac{b_{n,-\frac{1}{2}} a_{n,-2} \overline{\psi}\left(b_{n,-\frac{1}{2}}\right)}{\left(b_{n,-\frac{1}{2}} \overline{\psi}\left(b_{n,-\frac{1}{2}}\right)\right)^2} \\
&\le c_\epsilon (\log n)^{-\frac{3}{2}} (\log\log n)^{\frac{1}{2}}.
\end{aligned}$$

Using the Borel–Cantelli lemma (see the proof of Theorem 1.2.1 in [9]), we get (2.15). □

Remark 2.2.2 Similar arguments as above give

$$P\left(\max_{0\le k\le l\le n} R_{\mathcal{G}_{k,l}}(S_k,S_l)\ge(1+\epsilon)n\overline{\psi}(n)\right)\le c_\epsilon(\log n)^{-\frac{3}{2}}, \tag{2.17}$$

for all $\epsilon>0$.

Lemma 2.2.3

$$P\left(\left\{S\left[0,b_{n,-\frac{1}{2}}\right]\cap S\left[n-b_{n,-\frac{1}{2}},n\right]\neq\emptyset\right\}\quad i.o.\right)=1 \tag{2.18}$$

Proof Let S_1, S_2 be independent simple random walk in $\mathbb{Z}^4$ starting from the origin. Noting that the time-reversal of $S[b_{n,-\frac{1}{2}}/2,b_{n,-\frac{1}{2}}]$ and $S[b_{n,-\frac{1}{2}},n]$ are two independent simple random walks from $S_{b_{n,-\frac{1}{2}}}$. By the translation invariance,

$$\begin{aligned}&P\left(S\left[b_{n,-\frac{1}{2}}/2,b_{n,-\frac{1}{2}}\right]\cap S\left[n-b_{n,-\frac{1}{2}},n\right]\neq\emptyset\right)\\&=P\left(S^1\left[0,b_{n,-\frac{1}{2}}/2\right]\cap S^2\left[n-2b_{n,-\frac{1}{2}},n-b_{n,-\frac{1}{2}}\right]\neq\emptyset\right).\end{aligned}$$

Using [6], Theorem 4.3.6, we have

$$\begin{aligned}&P\left(S^1\left[0,b_{n,-\frac{1}{2}}/2\right]\cap S^2\left[n-2b_{n,-\frac{1}{2}},n-b_{n,-\frac{1}{2}}\right]\neq\emptyset\right)\\&\sim\frac{\pi^2}{8}(\log n)^{-1}\sum_{j=0}^{b_{n,-\frac{1}{2}}/2}\sum_{k=n-2b_{n,-\frac{1}{2}}}^{n-b_{n,-\frac{1}{2}}}P\left(S^1_j=S^2_k\right).\end{aligned} \tag{2.19}$$

By the local central limit theorem, (see, [6]. Theorem 1.2.1) it follows that

$$\sum_{j=0}^{b_{n,-\frac{1}{2}}/2}\sum_{k=n-2b_{n,-\frac{1}{2}}}^{n-b_{n,-\frac{1}{2}}}P\left(S^1_j=S^2_k\right)\ge c(\log\log n)^{-1},$$

so

$$P\left(S\left[b_{n,-\frac{1}{2}}/2,b_{n,-\frac{1}{2}}\right]\cap S\left[n-b_{n,-\frac{1}{2}},n\right]\neq\emptyset\right)\ge c(\log n)^{-1}(\log\log n)^{-1}.$$

By the second Borel–Cantelli lemma, we get the result. □

Proof of Theorem 1.2.1 (1.7). By Lemma 2.2.1 and 2.2.3, for P-a.s. ω,

$$\begin{aligned}&\max_{0\le k\le l\le b_{n,-\frac{1}{2}}} R_{\mathcal{G}_{k,l}}(S_k,S_l)(\omega)\le cb_{n,-\frac{1}{2}}\overline{\psi}(n),\quad \textit{for large } n\\&\max_{n-b_{n,-\frac{1}{2}}\le k\le l\le n} R_{\mathcal{G}_{k,l}}(S_k,S_l)(\omega)\le cb_{n,-\frac{1}{2}}\overline{\psi}(n)\quad \textit{for large } n\\&S[0,b_{n,-\frac{1}{2}}]\cap S[n-b_{n,-\frac{1}{2}},n]\neq\emptyset\quad \textit{for infinitely many } n.\end{aligned}$$

This implies that

$$R_{\mathcal{G}_{0,n}}(0, S_n)(\omega) \leq 2cb_{n,-\frac{1}{2}}\overline{\psi}(n) \quad \textit{for infinitely many } n,$$

and we get the result. □

2.3 Lower bound of the effective resistance

In this subsection, we will show (1.8). To do this, we first show that there exist three global cut-times $T^{(j)} (j = 1, 2, 3)$ such that each $T^{(j)}$ is in $[0, n]$ and $T^{(j)} - T^{(j-1)} \geq b_{n,-7}$. Then we divide $\mathcal{G}_{0,T^{(3)}}$ into three disjoint parts and give an appropriate lower bound for the sum of three i.i.d. random variables instead of $R_{\mathcal{G}}(0, S_n)$ (Proposition 2.3.4). We now begin with several lemmas.

Lemma 2.3.1 *For any* $\epsilon > 0$*, it follows that*

$$P\left(R_{\mathcal{G}}(0, S_n) \leq (1-\epsilon)n\psi(n)\right) = O_\epsilon\left(\frac{\log\log n}{(\log n)^{\frac{1}{2}}}\right). \tag{2.20}$$

Proof Let $N = \lfloor \frac{n}{a_{n,-2}} \rfloor$. By the similar argument as in the proof of Theorem 1.2.1 (1.5), we have

$$\begin{aligned}
&P\left(R_{\mathcal{G}}(0, S_n) \leq (1-\epsilon)n\psi(n)\right)\\
&\leq P\Bigg(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}\left(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}\right) \leq (1-\epsilon)n\psi(n) + n(\log n)^{-4} + n(\log n)^{-2}\sum_{i=1}^{N} J_i\Bigg)\\
&\leq P\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}\left(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}\right) \leq \left(1-\tfrac{\epsilon}{2}\right)n\psi(n)\right)\\
&\quad + P\left(n(\log n)^{-2}\sum_{i=1}^{N} J_i \geq \tfrac{\epsilon}{4}n\psi(n)\right)\\
&\leq P\left(\sum_{i=1}^{N} R_{\mathcal{G}_{(i-1)a_{n,-2},ia_{n,-2}}}\left(S_{(i-1)a_{n,-2}}, S_{ia_{n,-2}}\right) \leq \left(1-\tfrac{\epsilon}{2}\right)n\psi(n)\right) + c_\epsilon(\log n)^{-\frac{1}{2}}\log\log n\\
&\leq c_\epsilon(\log n)^{-\frac{1}{2}}\log\log n.
\end{aligned}$$

This implies (2.20). □

Let $\delta > 0$. Define the events

$$\begin{aligned}
B_1 &= \left\{S\left[0, 3b_{n,-1-\delta}\right] \cap S\left[n - a_{n,-6}, \infty\right) = \emptyset\right\}\\
B_2 &= \left\{S\left[0, b_{n,-2-2\delta} + a_{n,-6}\right] \cap S\left[2b_{n,-1-\delta}, \infty\right) = \emptyset\right\}\\
B_3 &= \left\{S\left[0, 3b_{n,-3-3\delta}\right] \cap S\left[b_{n,-2-2\delta}, \infty\right) = \emptyset\right\}\\
B_4 &= \left\{S\left[0, b_{n,-4-4\delta} + a_{n,-6}\right] \cap S\left[2b_{n,-3-3\delta}, \infty\right) = \emptyset\right\}\\
B_5 &= \left\{S\left[0, 3b_{n,-5-5\delta}\right] \cap S\left[b_{n,-4-4\delta}, \infty\right) = \emptyset\right\}\\
B_6 &= \left\{S\left[0, b_{n,-6-6\delta} + a_{n,-6}\right] \cap S\left[2b_{n,-5-5\delta}, \infty\right) = \emptyset\right\}.
\end{aligned}$$

(For the simplicity, we write $\overline{b}_{j,n} = b_{n,-j-j\delta}$ for $j = 1, \ldots, 6$).

Lemma 2.3.2 *There exists $c > 0$ such that*

$$P(B_j^c) \le c(\log n)^{-1}(\log\log n)^{-1-\delta} \quad j = 1, \cdots, 6. \tag{2.21}$$

Proof We will prove (2.21) for $j = 1$, the other cases are proved similarly. Let S^1, S^2 be be independent simple random walk in $\mathbb{Z}^4$ starting from the origin. Then,

$$P(B_1^c) = P\left(S^1[0, 3\overline{b}_{1,n}] \cap S^2[n - a_{n,-6} - 3\overline{b}_{1,n}, \infty) \neq \emptyset\right).$$

Using [6] Theorem 4.3.6, we have

$$P\left(S^1\left[0, 3\overline{b}_{1,n}\right] \cap S^2\left[n - a_{n,-6} - 3\overline{b}_{1,n}, \infty\right) \neq \emptyset\right)$$
$$\le c(\log n)^{-1} \sum_{j=0}^{3\overline{b}_{1,n}} \sum_{k=n-a_{n,-6}-3\overline{b}_{1,n}}^{\infty} P\left(S_j^1 = S_k^2\right),$$

for some $c > 0$.

It follows from [6] Theorem 1.2.1 that

$$\sum_{j=0}^{3\overline{b}_{1,n}} \sum_{k=n-a_{n,-6}-3\overline{b}_{1,n}}^{\infty} P\left(S_j^1 = S_k^2\right) \le \sum_{j=0}^{3\overline{b}_{1,n}} \sum_{k=n-a_{n,-6}-3\overline{b}_{1,n}}^{\infty} c' \frac{1}{(j+k)^2} \le c(\log\log n)^{-1-\delta}.$$

Hence, the result is proved. □

Define the indicator function

$$Y_k = \begin{cases} \mathbf{1}\{k \text{ is a local cut-time between } 0 \text{ and } 2\overline{b}_{5,n}\}, & \overline{b}_{6,n} \le k \le \overline{b}_{6,n} + a_{n,-6}, \\ \mathbf{1}\{k \text{ is a local cut-time between } 3\overline{b}_{5,n} \text{ and } 2\overline{b}_{3,n}\}, & \overline{b}_{4,n} \le k \le \overline{b}_{4,n} + a_{n,-6}, \\ \mathbf{1}\{k \text{ is a local cut-time between } 3\overline{b}_{3,n} \text{ and } 2\overline{b}_{1,n}\}, & \overline{b}_{2,n} \le k \le \overline{b}_{2,n} + a_{n,-6}, \\ \mathbf{1}\{k \text{ is a local cut-time between } 3\overline{b}_{1,n} \text{ and } \infty\}, & n - a_{n,-6} \le k \le n. \end{cases} \tag{2.22}$$

It is easy to see that on the event $B_1 \cap B_2 \cap \cdots \cap B_6$,

$$Y_k = I_k. \tag{2.23}$$

Let

$$\tilde{J}_1 = \mathbf{1}\left\{Y_k = 0 \text{ for all } k \in I^{(1)} := \left[\overline{b}_{6,n}, \overline{b}_{6,n} + a_{n,-6}\right]\right\} \tag{2.24}$$

$$\tilde{J}_2 = \mathbf{1}\left\{Y_k = 0 \text{ for all } k \in I^{(2)} := \left[\overline{b}_{4,n}, \overline{b}_{4,n} + a_{n,-6}\right]\right\} \tag{2.25}$$

$$\tilde{J}_3 = \mathbf{1}\left\{Y_k = 0 \text{ for all } k \in I^{(3)} := \left[\overline{b}_{2,n}, \overline{b}_{2,n} + a_{n,-6}\right]\right\} \tag{2.26}$$

$$\tilde{J}_4 = \mathbf{1}\left\{Y_k = 0 \text{ for all } k \in I^{(4)} := \left[n - a_{n,-6}, n\right]\right\}. \tag{2.27}$$

Lemma 2.3.3 *There exists $c > 0$ such that*

$$P\left(\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 \geq 2\right) \leq c\frac{(\log\log n)^2}{(\log n)^2}. \tag{2.28}$$

Proof By definition of $\tilde{J}_i$, it follows that $\tilde{J}_1, \ldots, \tilde{J}_4$ are independent. Also we know

$$E\left(\tilde{J}_i\right) \leq c\frac{\log\log n}{\log n}.$$

(See [6, Lemma 7.7.4]). Therefore,

$$\begin{aligned} P\left(\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 \geq 2\right) &= P\left(\tilde{J}_i = \tilde{J}_j = 1 \text{ for some } 1 \leq i < j \leq 4\right) \\ &\leq \sum_{1 \leq i < j \leq 4} P\left(\tilde{J}_i = \tilde{J}_j = 1\right) \\ &= \sum_{1 \leq i < j \leq 4} P\left(\tilde{J}_i = 1\right) P\left(\tilde{J}_j = 1\right) \\ &\leq c\frac{(\log\log n)^2}{(\log n)^2}. \end{aligned}$$

□

Let

$$T_n^{\max} = \begin{cases} \max\{0 \leq k \leq n : k \text{ is a global cut-time}\} & \text{if } \{\} \neq \emptyset \\ 0 & \text{if } \{\} = \emptyset. \end{cases}$$

Proposition 2.3.4

$$P\left(R_{\mathcal{G}}(0, S_{T_n^{\max}}) \leq \frac{1}{3}b_{6,n}\psi(n)\right) \leq c(\log n)^{-1}(\log\log n)^{-1-\delta}. \tag{2.29}$$

Proof By Lemma 2.3.2 and 2.3.3,

$$\begin{aligned} &P\left(R_{\mathcal{G}}\left(0, S_{T_n^{\max}}\right) \leq \frac{1}{3}\overline{b}_{6,n}\psi(n)\right) \\ &\leq P\left(\left\{R_{\mathcal{G}}\left(0, S_{T_n^{\max}}\right) \leq \frac{1}{3}\overline{b}_{6,n}\psi(n)\right\} \cap B_1 \cap B_2 \cap \cdots \cap B_6 \cap \left\{\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 \leq 1\right\}\right) \\ &\quad + c(\log n)^{-1}(\log\log n)^{-1-\delta}. \end{aligned}$$

Assume that $B_1, \ldots, B_6$ and $\tilde{J}_1 + \ldots + \tilde{J}_4 \leq 1$ hold. Since $Y_k = I_k$ on the event $B_1 \cap B_2 \cap \ldots \cap B_6$, there are at least three "good" intervals, namely, there are at least three intervals of $I^{(1)}, \ldots, I^{(4)}$ which contain global cut-time. Hence,

$$P\left(\left\{R_{\mathcal{G}}\left(0, S_{T_n^{\max}}\right) \leq \frac{1}{3}\overline{b}_{6,n}\psi(n)\right\} \cap B_1 \cap B_2 \cap \cdots \cap B_6 \cap \left\{\tilde{J}_1 + \tilde{J}_2 + \tilde{J}_3 + \tilde{J}_4 \leq 1\right\}\right)$$
$$\leq P\left(\left\{R_{\mathcal{G}}\left(0, S_{T_n^{\max}}\right) \leq \frac{1}{3}\overline{b}_{6,n}\psi(n)\right\} \cap \left\{\text{there are at least three "good" intervals of } I^{(1)}, \ldots, I^{(4)}\right\}\right).$$

Assume that there are at least three "good" intervals of $I^{(1)}, \ldots, I^{(4)}$. Without loss of generality, we may assume $I^{(1)}$, $I^{(2)}$ and $I^{(3)}$ are good intervals. Then there exist $T^{(j)} \in I^{(j)}$ for $j = 1, 2, 3$ such that $T^{(j)}$ is a global cut-time. If

$$R_{\mathcal{G}}\left(0, S_{T_n^{\max}}\right) \leq \frac{1}{3}\overline{b}_{6,n}\psi(n),$$

then

$$R_{\mathcal{G}_{0,T^{(1)}}}(0, S_{T^{(1)}}) + R_{\mathcal{G}_{T^{(1)},T^{(2)}}}(S_{T^{(1)}}, S_{T^{(2)}}) + R_{\mathcal{G}_{T^{(2)},T^{(3)}}}(S_{T^{(2)}}, S_{T^{(3)}}) \leq \frac{1}{3}\overline{b}_{6,n}\psi(n).$$

Since $|T^{(j)} - \overline{b}_{8-2j,n}| \leq n(\log n)^{-6}$ for $j = 1, 2, 3$, it follows from (2.4) that

$$R_{\mathcal{G}_{0,\overline{b}_{6,n}}}\left(0, S_{\overline{b}_{6,n}}\right) + R_{\mathcal{G}_{\overline{b}_{6,n},\overline{b}_{4,n}}}\left(S_{\overline{b}_{6,n}}, S_{\overline{b}_{4,n}}\right) + R_{\mathcal{G}_{\overline{b}_{4,n},\overline{b}_{2,n}}}\left(S_{\overline{b}_{4,n}}, S_{\overline{b}_{2,n}}\right) \leq \frac{1}{2}\overline{b}_{6,n}\psi(n),$$

for large n. By (2.10), Lemma 2.1.1, Lemma 2.3.1 and independence,

$$\begin{aligned}
&P\left(R_{\mathcal{G}_{0,\overline{b}_{6,n}}}\left(0, S_{\overline{b}_{6,n}}\right) + R_{\mathcal{G}_{\overline{b}_{6,n},\overline{b}_{4,n}}}\left(S_{\overline{b}_{6,n}}, S_{\overline{b}_{4,n}}\right) + R_{\mathcal{G}_{\overline{b}_{4,n},\overline{b}_{2,n}}}\left(S_{\overline{b}_{4,n}}, S_{\overline{b}_{2,n}}\right) \leq \frac{1}{2}\overline{b}_{6,n}\psi(n)\right)\\
&\leq P\left(R_{\mathcal{G}_{0,\overline{b}_{6,n}}}\left(0, S_{\overline{b}_{6,n}}\right) \leq \frac{1}{2}\overline{b}_{6,n}\psi(n)\right)\\
&\quad \times P\left(R_{\mathcal{G}_{\overline{b}_{6,n},\overline{b}_{4,n}}}\left(S_{\overline{b}_{6,n}}, S_{\overline{b}_{4,n}}\right) \leq \frac{1}{2}\overline{b}_{6,n}\psi(n)\right)\\
&\quad \times P\left(R_{\mathcal{G}_{\overline{b}_{4,n},\overline{b}_{2,n}}}\left(S_{\overline{b}_{4,n}}, S_{\overline{b}_{2,n}}\right) \leq \frac{1}{2}\overline{b}_{6,n}\psi(n)\right)\\
&\leq c\frac{(\log\log n)^3}{(\log n)^{\frac{3}{2}}}.
\end{aligned}$$

This implies (2.29). □

Proof of Theorem 1.2.1 (1.8) Since for any $\delta > 0$,

$$\sum_{k=2}^{\infty} \frac{1}{k(\log k)^{1+\delta}} < \infty,$$

it follows from Proposition 2.3.4 and the argument using the Borel–Cantelli lemma that for P-a.s. ω and large n,

$$R_{\mathcal{G}}(0, S_{T_n^{\max}})(\omega) \geq n\psi(n)(\log\log n)^{-7}.$$

However, we know

$$R_{\mathcal{G}}(0, S_{T_n^{\max}})(\omega) \leq R_{\mathcal{G}}(0, S_n)(\omega).$$

Hence, the result is proved. □

3 Proof of Theorem 1.2.2

3.1 Heat kernel estimate w.r.t the resistance metric

In this section, we will prove Theorem 1.2.2. It is known that the effective resistance $R_{\mathcal{G}}(\cdot, \cdot)$ is a metric on $\mathcal{G}$ (see, for example [4] and the references therein). We write

$$B_{\mathcal{G}}(x, R) = \left\{y \in \mathcal{G} : R_{\mathcal{G}}(x, y) < R\right\}, \quad V_{\mathcal{G}}(x, R) = \mu_{\mathcal{G}}\left(B_{\mathcal{G}}(x, R)\right), \quad R > 0.$$

For $\epsilon > 0$, we set

$$\tilde{Y}_k = \mathbf{1}\left\{R_{\mathcal{G}}(0, S_k) \leq \epsilon^2 n\psi(n)\right\},$$
$$\tilde{Y} = \tilde{Y}^{(n)} = \sum_{k=0}^{2b_{n,7}} \tilde{Y}_k.$$

We now state the key proposition. The proof will be given in the next subsection.

Proposition 3.1.1 *There exist $\epsilon \in (0, 1)$ and $c > 0$ such that*

$$P\left(\tilde{Y} \geq cn\right) = O\left((\log n)^{-\frac{3}{2}}(\log\log n)^{\alpha}\right), \tag{3.1}$$

for some $\alpha > 0$.

Remark 3.1.2 It follows from (3.1) and the Borel–Cantelli lemma that for P-a.s. ω,

$$\tilde{Y}(\omega) \leq cn, \tag{3.2}$$

for large n. On the other hand, it follows from (1.8) that for P-a.s. ω,

$$R_{\mathcal{G}}(0, S_k)(\omega) > \epsilon^2 n\psi(n) \quad \text{for all } k \geq 2n(\log\log n)^7. \tag{3.3}$$

Indeed, by (1.8), for P-a.s. ω,

$$R_{\mathcal{G}}(0, S_k)(\omega) \geq k\psi(k)(\log\log k)^{-7} \quad \text{for large } k.$$

Assume $k \geq 2n(\log\log n)^7$ and n is sufficiently large so that $\frac{k}{(\log\log k)^7} \geq n$. It follows from (2.10) that

$$k\psi(k)(\log\log k)^{-7} \geq \frac{1}{2}\frac{k}{(\log\log k)^7}\psi\left(\frac{k}{(\log\log k)^7}\right).$$

Recall that $B_n = \{I_k = 0 \text{ for all } k \in [n - a_{n,-6}, n]\}$ in the proof of Lemma 2.1.1. Then it follows from $P(B_n) = O\left(\frac{\log\log n}{\log n}\right)$, (see [6, Lemma 7.7.4]) and (2.4) that for large $n \leq l$,

$$\begin{aligned} l\psi(l) &= E\left(R_{\mathcal{G}}(0, S_l)\right) \\ &\geq E\left(R_{\mathcal{G}}(0, S_l);\, B_n^c\right) \\ &\geq E\left(R_{\mathcal{G}}(0, S_n);\, B_n^c\right) - n(\log n)^{-6} \\ &= E\left(R_{\mathcal{G}}(0, S_n)\right) - E\left(R_{\mathcal{G}}(0, S_n);\, B_n\right) - n(\log n)^{-6} \\ &\geq n\psi(n) - cn\frac{\log\log n}{\log n} - n(\log n)^{-6} \\ &\geq \frac{1}{2}n\psi(n). \end{aligned}$$

Therefore, if $k \geq 2n(\log\log n)^7$, then for P-a.s. ω,

$$R_{\mathcal{G}}(0, S_k)(\omega) \geq \frac{1}{2}\frac{k}{(\log\log k)^7}\psi\left(\frac{k}{(\log\log k)^7}\right) \geq \frac{1}{4}n\psi(n)$$

which gives (3.3) when $\epsilon < \frac{1}{2}$. Therefore, for P-a.s. ω,

$$\sum_{k=0}^{\infty} \tilde{Y}_k(\omega) \leq cn \tag{3.4}$$

for large n.

We now give a proof of Theorem 1.2.2, assuming the above proposition.

Proof of Theorem 1.2.2. By Remark 2.2.2, it follows that for P-a.s. ω,

$$\max_{0\leq k\leq n} R_{\mathcal{G}}(0, S_k)(\omega) \leq cn\psi(n),$$

for some $c > 0$. Therefore,

$$\{S_k : 0 \leq k \leq n\} \subset B_{\mathcal{G}}\left(0, cn\psi(n)\right).$$

However, it is known (see [2] (2.22) and [3] (4.1), for example) that for P-a.s. ω,

$$\sharp\{S_k(\omega) : 0 \leq k \leq n\} \asymp n.$$

Hence,

$$\tilde{c}n \leq V_{\mathcal{G}}\left(0, cn\psi(n)\right)(\omega),$$

for some $\tilde{c} > 0$. By a simple reparameterisation,

$$\tilde{c}_1 n\left(\psi(n)\right)^{-1} \leq V_{\mathcal{G}}(0, n)(\omega), \tag{3.5}$$

for some $\tilde{c}_1 > 0$. (Here we use the fact ψ is slowly varying. See Remark 2.1.3.)

For the upper bound, we have

$$B_{\mathcal{G}}\left(0, \epsilon^2 n\psi(n)\right) = \{S_k : R_{\mathcal{G}}(0, S_k) < \epsilon^2 n\psi(n)\},$$

and

$$\begin{aligned} V_{\mathcal{G}}\left(0, \epsilon^2 n\psi(n)\right) &\leq 8\sharp\left\{S_k : R_{\mathcal{G}}(0, S_k) < \epsilon^2 n\psi(n)\right\} \\ &\leq 8\sharp\left\{0 \leq k < \infty \ : \ R_{\mathcal{G}}(0, S_k) < \epsilon^2 n\psi(n)\right\} \\ &\leq 8\sum_{k=0}^{\infty} \tilde{Y}_k. \end{aligned}$$

Therefore, it follows from (3.4) that for P-a.s. ω,

$$V_{\mathcal{G}}\left(0, \epsilon^2 n\psi(n)\right)(\omega) \leq cn \quad \text{for large } n.$$

Hence, by a simple reparameterisation,

$$V_{\mathcal{G}}(0, n)(\omega) \leq \tilde{c}_2 n\left(\psi(n)\right)^{-1}, \quad \text{for } P\text{-a.s. } \omega. \tag{3.6}$$

By Proposition 3.1 and Proposition 3.2 in [4], we can conclude that for P-a.s. ω,

$$c_1 n^{-\frac{1}{2}}\left(\psi(n)\right)^{\frac{1}{2}} \leq p_{2n}^{\mathcal{G}(\omega)}(0, 0) \leq c_2 n^{-\frac{1}{2}}\left(\psi(n)\right)^{\frac{1}{2}} \quad \text{for large } n, \tag{3.7}$$

for some $c_1, c_2 > 0$. (Note that since we consider the resistance metric on the graph $\mathcal{G}(\omega)$, we can apply the results in [4] as $v(R) = R\left(\psi(R)\right)^{-1}$, $r(R) = R$. See [4] for details.) □

3.2 Proof of Proposition 3.1.1

In this subsection, we will give the proof of Proposition 3.1.1. In order to prove this proposition, we now make some preparations. Let

$$\bar{Y} = \bar{Y}^{(n)} = \sum_{k=n}^{2b_{n,7}} \tilde{Y}_k.$$

To prove Proposition 3.1.1, it suffices to show that there exist $\epsilon \in (0, 1)$ and $C > 0$ such that

$$P\left(\bar{Y} \geq Cn\right) = O\left(\frac{(\log\log n)^{\alpha}}{(\log n)^{\frac{3}{2}}}\right), \tag{3.8}$$

for some $\alpha > 0$.

Before we proceed the proof of (3.8), we give a guideline intuitively. Since the random walk trace has long range intersections as in (2.18), it is possible that there exists a $k \in [n, b_{n,7}]$ such that $\tilde{Y}_k = 1$. So it is not trivial to show that the number of such times are of order $O(n)$. To do this, we study the connectivity of the trace near the long range intersection point and show that the trace is relatively sparse around the intersection point. The key fact is Proposition 4.3 in [7] which gives a uniform estimate for the probability that the simple random walk escapes from the recurrent set satisfying a certain condition, called slowly recurrent set in [7]. This proposition enables to analyze the shape of the trace around the long range intersection point, and we can obtain (3.8).

Let $\epsilon \in (0, 1)$ (the exact values of this number will be determined later) and $d_n = \lfloor \epsilon^2 n \rfloor$. We write

$$N = \lfloor \frac{2b_{n,7} - n}{d_n} \rfloor.$$

Then

$$N \leq \frac{2}{\epsilon^2} (\log\log n)^7.$$

Let

$$I'_j = [n + (j-1)d_n, n + jd_n] \quad \text{for } 1 \leq j \leq N, \tag{3.9}$$

$$I'_{N+1} = [n + Nd_n, 2b_{n,7}]. \tag{3.10}$$

Then

$$[n, 2b_{n,7}] \subset \bigcup_{j=1}^{N+1} I'_j.$$

We call a time k "bad" if $\tilde{Y}_k = 1$ and a interval I'_j "bad" if there exists $k \in I'_j$ such that k is bad. Let

$$L = \sharp \left\{1 \leq j \leq N+1 : I'_j \text{ is bad}\right\}. \tag{3.11}$$

Assume that $\bar{Y} \geq 2Cn$. (Again the exact value of C will be determined later.) Then

$$L \geq \frac{2C}{\epsilon^2}. \tag{3.12}$$

So assume that $L \geq \frac{2C}{\epsilon^2}$. Under this assumption, we choose the intervals $I'_{j_1}, I'_{j_2}, \ldots, I'_{j_C}$ as follows:

$$j_1 = \min \left\{ 1 \leq j \leq N+1 : I'_j \text{ is bad} \right\}. \tag{3.13}$$

For $i \geq 2$, define

$$j_i = \min \left\{ k > j_{i-1} : \ \mathrm{dist}\left(I'_{j_{i-1}}, I'_k \right) \geq n, I'_k \text{ is bad} \right\}, \tag{3.14}$$

where $\mathrm{dist}(I'_k, I'_l)$ denotes the distance between I'_k and I'_l with respect to the Euclidean distance. Since $L \geq \frac{2C}{\epsilon^2}$, we can define j_i at least for $i = 1, \ldots, C$. Let

$$\begin{aligned} \mathcal{I} = \{(j_1, \ldots, j_C) : 1 \leq j_1 < j_2 < \cdots < j_C \leq N+1, \\ \mathrm{dist}\left(I'_{j_{i-1}}, I'_{j_i} \right) \geq n \quad \text{for all } i = 2, \cdots, C \right\}. \end{aligned} \tag{3.15}$$

Then, we have shown the following lemma.

Lemma 3.2.1

$$P\left(\bar{Y} \geq 2Cn\right) \leq \sum_{(j_1,\ldots,j_C)\in\mathcal{I}} P\left(I'_{j_i} \textit{ is bad for all } 1 \leq i \leq C \right). \tag{3.16}$$

We now estimate the right-hand side of (3.16). Fix $(j_1, \ldots, j_C) \in \mathcal{I}$ and let

$$t_i = n + (j_i - 1)d_n \quad \text{for } i = 1, \ldots, C. \tag{3.17}$$

Then

$$I'_{j_i} = [t_i, t_i + d_n] \quad i = 1, \ldots, C.$$

Lemma 3.2.2 *Assume that $\epsilon < \frac{1}{3}$. Then,*

$$\begin{aligned} & P\left(I'_{j_i} \textit{ is bad for all } 1 \leq i \leq C \right) \\ & \quad \leq P\left(R_{\mathcal{G}}(0, S_{t_i}) \leq \epsilon n \psi(n) \textit{ for all } 1 \leq i \leq C \right) + \tilde{c}(\log n)^{-\frac{3}{2}}, \end{aligned}$$

for some $\tilde{c} > 0$.

Proof By Lemma 2.1.1, Proposition 2.1.2 and Remark 2.2.2, we have

$$P\left(\max_{0\le k<l\le d_n} R_{\mathcal{G}_{k,l}}(S_k, S_l) \ge 2d_n\psi(n)\right) \le \bar{c}(\log n)^{-\frac{3}{2}}.$$

Therefore, if $\epsilon < \frac{1}{3}$, then

$$\begin{aligned} &P\left(I'_{j_i} \text{ is bad for all } 1 \le i \le C\right) \\ &\le P\left(I'_{j_i} \text{ is bad and } \max_{t_i\le k<l\le t_i+d_n} R_{\mathcal{G}}(S_k, S_l) \le 2d_n\psi(n) \text{ for all } 1 \le i \le C\right) \\ &\quad + P\left(\max_{t_i\le k<l\le t_i+d_n} R_{\mathcal{G}}(S_k, S_l) \ge 2d_n\psi(n) \text{ for some } 1 \le i \le C\right) \\ &\le P\left(R_{\mathcal{G}}(0, S_{t_i}) \le \epsilon n\psi(n) \text{ for all } 1 \le i \le C\right) + \tilde{c}(\log n)^{-\frac{3}{2}}, \end{aligned}$$

for some $\tilde{c} > 0$. □

By Lemma 3.2.2, all we need is to estimate

$$P\left(R_{\mathcal{G}}(0, S_{t_i}) \le \epsilon n\psi(n) \text{ for all } 1 \le i \le C\right).$$

Let

$$\tilde{I}'_i = [t_i - \bar{a}_n, t_i + \bar{a}_n] \quad i = 1, \ldots, C \tag{3.18}$$

and

$$Z = \sharp\{1 \le i \le C : S[0, \bar{a}_n] \cap S[t_i - \bar{a}_n, t_i + \bar{a}_n] \ne \emptyset\}, \tag{3.19}$$

where $\bar{a}_n = \lfloor \frac{1}{3} n \rfloor$. [Note that $\tilde{I}'_i$ are disjoint for $i = 1, \ldots, C$. See (3.15).] We write

$$\phi(n) = \frac{(\log\log n)^2}{\log n}. \tag{3.20}$$

Lemma 3.2.3

$$P(Z \ge 2) = O\left((\phi(n))^2\right). \tag{3.21}$$

Proof By definition,

$$\begin{aligned} &P(Z \ge 2) \\ &\le \sum_{1\le i<k\le C} P\left(S[0, \bar{a}_n] \cap S[t_i - \bar{a}_n, t_i + \bar{a}_n] \ne \emptyset, S[0, \bar{a}_n] \cap S[t_k - \bar{a}_n, t_k + \bar{a}_n] \ne \emptyset\right). \end{aligned}$$

So it suffices to prove

$$
\begin{aligned}
&P\left(S[0,\bar{a}_n]\cap S[t_i-\bar{a}_n,t_i+\bar{a}_n]\neq\emptyset,\ S[0,\bar{a}_n]\cap S[t_k-\bar{a}_n,t_k+\bar{a}_n]\neq\emptyset\right)\\
&\quad=O\left((\phi(n))^2\right),
\end{aligned}
\tag{3.22}
$$

for each $1\leq i<k\leq C$.

Let $1\leq i<k\leq C$. Let S^1, S^2 denote independent simple random walks in $\mathbb{Z}^4$. We write P_i^x to denote the probability law of S^i assuming $S^i(0)=x$. We use E_i^x for expectation with respect to P_i^x. If the x is missing then it is assumed that $S^i(0)=0$. We have

$$
\begin{aligned}
&P\left(S[0,\bar{a}_n]\cap S[t_i-\bar{a}_n,t_i+\bar{a}_n]\neq\emptyset,\ \ S[0,\bar{a}_n]\cap S[t_k-\bar{a}_n,t_k+\bar{a}_n]\neq\emptyset\right)\\
&\quad=P\left(S^1[0,\bar{a}_n]\cap S^2[t_i-2\bar{a}_n,t_i]\neq\emptyset,\ \ S^1[0,\bar{a}_n]\cap S^2[t_k-2\bar{a}_n,t_k]\neq\emptyset\right).
\end{aligned}
$$

Define the event

$$
D=\left\{\begin{array}{c}
S^1[0,\bar{a}_n]\subset C_{\sqrt{n}\log n}\\
S^1[0,\infty)\cap C_{\sqrt{n}\log n}\in\mathcal{A}_{\sqrt{n}\log n}\\
\sqrt{n}(\log n)^{-1}\leq|S^2_{t_k-2\bar{a}_n}|\leq\frac{1}{2}\sqrt{n}\log n\\
\operatorname{dist}\left(S^2_{t_k-2\bar{a}_n},S^1[0,\infty)\right)\geq\sqrt{n}(\log n)^{-3}
\end{array}\right\},
$$

where $\mathcal{A}_n$ is defined in [7] Proposition 4.1 (we omit the definition of $\mathcal{A}_n$ since we do not need the exact shape of it) and

$$
C_n(x)=\left\{y\in\mathbb{Z}^4:|x-y|<n\right\},\quad C_n=C_n(0).
$$

($|\cdot|$ denotes the Euclidean distance.) Lemma 1.5.1 in [6] gives that

$$
P\left(S^1[0,\bar{a}_n]\subset C_{\sqrt{n}\log n}\right)=1-O\left(\frac{1}{n}\right).
$$

By Proposition 4.1 in [7],

$$
P\left(S^1[0,\infty)\cap C_{\sqrt{n}\log n}\in\mathcal{A}_{\sqrt{n}\log n}\right)=1-O\left((\log n)^{-6}\right).
$$

It follows from Theorem 1.2.1 and Lemma 1.5.1 in [6] that

$$
P\left(\sqrt{n}(\log n)^{-1}\leq|S^2_{t_k-2\bar{a}_n}|\leq\frac{1}{2}\sqrt{n}\log n\right)=1-O\left((\log n)^{-4}\right).
$$

By Proposition 1.5.10 in [6],

$$P\left(\operatorname{dist}\left(x, S^1[0,\infty)\right) \geq \sqrt{n}(\log n)^{-3}\right) = 1 - O\left((\log n)^{-4}\right),$$

for all $x \in \mathbb{Z}^4$ with $\sqrt{n}(\log n)^{-1} \leq |x| \leq \frac{1}{2}\sqrt{n}\log n$. Hence,

$$P(D^c) = O\left((\log n)^{-4}\right). \tag{3.23}$$

Therefore, it suffices to estimate

$$P\left(\bar{F}_i \cap \bar{F}_k \cap D\right),$$

where $\bar{F}_i = \{S^1[0,\bar{a}_n] \cap S^2[t_i - 2\bar{a}_n, t_i] \neq \emptyset\}$. By the Markov property,

$$\begin{aligned} &P\left(\bar{F}_i \cap \bar{F}_k \cap D\right) \\ &\quad \leq E_1\left(E_2\left(\mathbf{1}_{\{\bar{F}_i \cap D\}} P_2^{S^2_{t_k - 2\bar{a}_n}}\left(S^1[0,\infty) \cap C_{\sqrt{n}\log n} \cap S^2[0, 2\bar{a}_n]\right.\right.\right. \\ &\quad \left.\left.\left.\cap \left(C_{\sqrt{n}(\log n)^{-3}}(S^2_0)\right)^c \neq \emptyset\right)\right)\right). \end{aligned} \tag{3.24}$$

It follows from Proposition 4.3 in [7] that

$$P_2^{S^2_{t_k - 2\bar{a}_n}}\left(S^1[0,\infty) \cap C_{\sqrt{n}\log n} \cap S^2[0, 2\bar{a}_n] \cap \left(C_{\sqrt{n}(\log n)^{-3}}(S^2_0)\right)^c \neq \emptyset\right) \leq c\phi(n), \tag{3.25}$$

on the event D. Hence,

$$P\left(\bar{F}_i \cap \bar{F}_k \cap D\right) \leq c\phi(n) P\left(\bar{F}_i\right).$$

If we repeat the same arguments as above, we have

$$P\left(\bar{F}_i\right) \leq c\phi(n). \tag{3.26}$$

This gives the lemma. □

By Lemma 3.2.3, all we need is to estimate

$$\begin{aligned} &P\left(R_{\mathcal{G}}(0, S_{t_i}) \leq \epsilon n\psi(n) \text{ for all } 1 \leq i \leq C, Z = 1\right) \\ &\quad + P\left(R_{\mathcal{G}}(0, S_{t_i}) \leq \epsilon n\psi(n) \text{ for all } 1 \leq i \leq C, Z = 0\right). \end{aligned} \tag{3.27}$$

First we will consider the first term of (3.27). Note that

$$\begin{aligned} &P\left(R_{\mathcal{G}}(0,S_{t_j})\leq \epsilon n\psi(n) \text{ for all } 1\leq j\leq C, Z=1\right)\\ &\leq \sum_{i=1}^{C-2} P\left(R_{\mathcal{G}}(0,S_{t_j})\leq \epsilon n\psi(n) \text{ for all } 1\leq j\leq C, \tilde{F}_i\right) \qquad (3.28)\\ &+P\left(R_{\mathcal{G}}(0,S_{t_j})\leq \epsilon n\psi(n) \text{ for all } 1\leq j\leq C, \tilde{F}_{C-1}\right) \qquad (3.29)\\ &+P\left(R_{\mathcal{G}}(0,S_{t_j})\leq \epsilon n\psi(n) \text{ for all } 1\leq j\leq C, \tilde{F}_{C}\right), \qquad (3.30)\end{aligned}$$

where $\tilde{F}_i=\{S[0,\bar{a}_n]\cap S[t_i-\bar{a}_n,t_i+\bar{a}_n]\neq\emptyset\}$.

Lemma 3.2.4 *Let* $1\leq i\leq C-2$ *and* $\epsilon\in(0,\frac{1}{4})$. *Then*

$$P\left(R_{\mathcal{G}}(0,S_{t_j})\leq \epsilon n\psi(n) \textit{ for all } 1\leq j\leq C, \tilde{F}_i\right)\leq \tilde{c}\phi(n)(\log n)^{-\frac{1}{2}}\log\log n, \qquad (3.31)$$

for some $\tilde{c}>0$.

Proof Let $1\leq i\leq C-2$. We write

$$K_l=\begin{cases}\mathbf{1}\{l \text{ is a local cut-time between } 0 \text{ and } l+\bar{a}_n\} & \text{if } t_i+\bar{a}_n\leq l\leq t_i+\bar{a}_n+a_{n,-6}\\ \mathbf{1}\{l \text{ is a local cut-time between } l-\bar{a}_n \text{ and } \infty\} & \text{if } t_{i+1}-a_{n,-6}\leq l\leq t_{i+1}\end{cases}, \qquad (3.32)$$

and

$$K=\sum_{l=t_i+\bar{a}_n}^{t_i+\bar{a}_n+a_{n,-6}} K_l,\quad \bar{K}=\sum_{l=t_{i+1}-a_{n,-6}}^{t_{i+1}} K_l. \qquad (3.33)$$

By [6], Lemma 7.7.4 and independence,

$$P\left(K=\bar{K}=0\right)=O\left(\phi(n)^2\right).$$

Therefore, for $1\leq i\leq C-2$,

$$\begin{aligned} &P\left(R_{\mathcal{G}}(0,S_{t_j})\leq \epsilon n\psi(n) \text{ for all } 1\leq j\leq C, \tilde{F}_i\right)\\ &\leq P\left(R_{\mathcal{G}}(0,S_{t_j})\leq \epsilon n\psi(n) \text{ for all } 1\leq j\leq C, \tilde{F}_i, K\geq 1\right)\\ &+P\left(R_{\mathcal{G}}(0,S_{t_j})\leq \epsilon n\psi(n) \text{ for all } 1\leq j\leq C, \tilde{F}_i, \bar{K}\geq 1\right)\\ &+\tilde{c}(\phi(n))^2.\end{aligned}$$

Assume $K\geq 1$. (The case $\bar{K}\geq 1$ can be dealt with similarly.) If there is no global cut-time in $[t_i+\bar{a}_n,t_i+\bar{a}_n+a_{n,-6}]$, then it is easy to see that

$$S[0,t_i+\bar{a}_n+a_{n,-6}]\cap S[t_i+2\bar{a}_n,\infty)\neq\emptyset.$$

However, the similar argument to that in the proof of Lemma 3.2.3 gives that

$$P\left(S[0,t_i+\bar{a}_n+a_{n,-6}]\cap S[t_i+2\bar{a}_n,\infty)\neq\emptyset,\ \tilde{F}_i\right)=O\left((\phi(n))^2\right).$$

Hence, it suffices to consider

$$P\Big(R_{\mathcal{G}}(0,S_{t_j})\le \epsilon n\psi(n) \text{ for all } 1\le j\le C, \tilde{F}_i, K\ge 1, \\ [t_i+\bar{a}_n, t_i+\bar{a}_n+a_{n,-6}] \text{ has a global cut-time}\Big). \tag{3.34}$$

If

$$R_{\mathcal{G}}(0,S_{t_j})\le \epsilon n\psi(n) \text{ for all } 1\le j\le C,$$

then we have

$$R_{\mathcal{G}}(S_{t_{i+1}},S_{t_{i+2}})\le 2\epsilon n\psi(n).$$

Assume that $[t_i+\bar{a}_n, t_i+\bar{a}_n+a_{n,-6}]$ has a global cut-time and let $T\in[t_i+\bar{a}_n, t_i+\bar{a}_n+a_{n,-6}]$ be the cut-time. Then

$$\begin{aligned}R_{\mathcal{G}}(S_{t_{i+1}},S_{t_{i+2}})&=R_{\mathcal{G}_{T,\infty}}(S_{t_{i+1}},S_{t_{i+2}})\\&=R_{\mathcal{G}_{t_i+\bar{a}_n,\infty}}(S_{t_{i+1}},S_{t_{i+2}}).\end{aligned}$$

Therefore, if $\epsilon\in(0,\frac{1}{4})$, then by Lemma 2.3.1, (3.26) and independence,

$$\begin{aligned}&P\Big(R_{\mathcal{G}}(0,S_{t_j})\le \epsilon n\psi(n) \text{ for all } 1\le j\le C, \tilde{F}_i, K\ge 1,\\ &\quad [t_i+\bar{a}_n, t_i+\bar{a}_n+a_{n,-6}] \text{ has a global cut-time}\Big)\\ &\quad\le P\left(\tilde{F}_i, R_{\mathcal{G}_{t_i+\bar{a}_n,\infty}}(S_{t_{i+1}},S_{t_{i+2}})\le 2\epsilon n\psi(n)\right)\\ &\quad\le \tilde{c}\phi(n)(\log n)^{-\frac{1}{2}}\log\log n,\end{aligned}$$

for some $\tilde{c}>0$. □

To complete the estimate for the first term of (3.27), it remains to give the bounds of (3.29) and (3.30). Since the estimates for (3.29) and (3.30) are similar, we will only consider (3.30).

Let

$$\tilde{K}_l=\mathbf{1}\{l \text{ is a local cut-time between } \bar{a}_n \text{ and } t_C-\bar{a}_n\} \quad \text{for } \bar{a}_n\le l\le t_C-\bar{a}_n, \tag{3.35}$$

and for $1\le i\le C-2$, define

$$\tilde{A}_i=\left\{\sum_{l=t_i-3a_{n,-1}}^{t_i}\tilde{K}_l\ge 1,\quad \sum_{l=t_i}^{t_i+3a_{n,-1}}\tilde{K}_l\ge 1\right\}. \tag{3.36}$$

Lemma 3.2.5

$$P\left(\tilde{F}_C \cap \tilde{A}_i^c\right) = O\left((\phi(n))^2\right) \quad \textit{for } i = 1, 2, \cdots, C-2. \tag{3.37}$$

Proof Let $1 \le i \le C-2$. It suffices to show that

$$P\left(\tilde{F}_C, \sum_{l=t_i}^{t_i+3a_{n,-1}} \tilde{K}_l = 0\right) = O\left((\phi(n))^2\right).$$

We write

$$K_l' = \mathbf{1}\{l \text{ is a local cut-time between } l - a_{n,-1} \text{ and } l + a_{n,-1}\}.$$

By [6], Lemma 7.7.4 and independence,

$$P\left(\sum_{l=t_i}^{t_i+a_{n,-6}} K_l' = \sum_{l=t_i+3a_{n,-1}-a_{n,-6}}^{t_i+3a_{n,-1}} K_l' = 0\right) = O\left((\phi(n))^2\right).$$

Hence, all we need is to estimate

$$P\left(\tilde{F}_C, \sum_{l=t_i}^{t_i+3a_{n,-1}} \tilde{K}_l = 0, \sum_{l=t_i}^{t_i+a_{n,-6}} K_l' \ge 1\right) \tag{3.38}$$

$$+P\left(\tilde{F}_C, \sum_{l=t_i}^{t_i+3a_{n,-1}} \tilde{K}_l = 0, \sum_{l=t_i+3a_{n,-1}-a_{n,-6}}^{t_i+3a_{n,-1}} K_l' \ge 1\right). \tag{3.39}$$

We will only consider (3.38). Assume that

$$\sum_{l=t_i}^{t_i+3a_{n,-1}} \tilde{K}_l = 0, \quad \sum_{l=t_i}^{t_i+a_{n,-6}} K_l' \ge 1.$$

Then by definition of $\tilde{K}_l$ and K_l', it is easy to see that the following event occurs:

$$\begin{aligned} B_1 \cup B_2 \cup B_3 := &\{S[t_i - a_{n,-1}, t_i + a_{n,-6}] \cap S[t_i + a_{n,-1}, t_C - \bar{a}_n] \neq \emptyset\} \\ &\cup\{S[\bar{a}_n, t_i + a_{n,-6} - a_{n,-1}] \cap S[t_i, t_C - \bar{a}_n] \neq \emptyset\} \\ &\cup\{S[\bar{a}_n, t_i + a_{n,-6}] \cap S[t_i + a_{n,-1}, t_C - \bar{a}_n] \neq \emptyset\}. \end{aligned} \tag{3.40}$$

However, it follows from the similar arguments as in (3.22) that

$$P\left(\tilde{F}_C, \quad (3.40)\right) \le \sum_{k=1}^{3} P(\tilde{F}_C \cap B_k) = O\left((\phi(n))^2\right).$$

This gives the lemma. □

We are now in a position to estimate (3.30).

Lemma 3.2.6 *Suppose That* $\epsilon \in (0, \frac{1}{10})$ *and* $C > 120$. *Then we have*

$$P\left(R_{\mathcal{G}}(0, S_{t_i}) \leq \epsilon n\psi(n) \text{ for all } 1 \leq i \leq C, \tilde{F}_C\right) = O\left(\phi(n)(\log n)^{-\frac{1}{2}} \log\log n\right). \tag{3.41}$$

Proof By Lemma 3.2.5, in order to derive (3.41), we have only to show

$$P\left(R_{\mathcal{G}}(0, S_{t_i}) \leq \epsilon n\psi(n) \text{ for all } 1 \leq i \leq C, \tilde{F}_C, \tilde{A}_i \text{ for all } 1 \leq i \leq C-1\right)$$
$$= O\left(\phi(n)(\log n)^{-\frac{1}{2}} \log\log n\right). \tag{3.42}$$

Choose $i_0 \in \{2, 3, \dots, C-3\}$ be a number satisfying that

$$t_{i_0+1} - t_{i_0} = \min_{2 \leq i \leq C-3} (t_{i+1} - t_i).$$

By definition of $\mathcal{I}$ [see (3.15)], we have $r := t_{i_0+1} - t_{i_0} \geq n$. Assume that $\tilde{A}_i$ holds for all $1 \leq i \leq C-1$. Then there exist

$$\begin{aligned} T^1 &\in [t_1, t_1 + 3a_{n,-1}] \\ T^2 &\in [t_{i_0}, t_{i_0} + 3a_{n,-1}] \\ T^3 &\in [t_{i_0+1} - 3a_{n,-1}, t_{i_0+1}] \\ T^4 &\in [t_{C-1} - 3a_{n,-1}, t_{C-1}] \end{aligned}$$

such that $T^1, \dots, T^4$ are local cut-times between $\bar{a}_n$ and $t_C - \bar{a}_n$. Furthermore, by modifying the proof of Lemma 3.2.3, we have

$$P\left(\tilde{F}_C, S[t_{i_0}, t_{i_0+1}] \cap (S[0, \bar{a}_n] \cup S[t_C - \bar{a}_n, \infty)) \neq \emptyset\right) = O\left((\phi(n))^2\right). \tag{3.43}$$

Therefore, if we set

$$\mathcal{G}^+ = \mathcal{G}_{T^2, T^3} \quad \mathcal{G}^- = \mathcal{G}_{0, T^2} \cup \mathcal{G}_{T^3, \infty},$$

then by (3.43), we may assume that

$$\mathcal{G}^+ \cap \mathcal{G}^- = \{S_{T^2}, S_{T^3}\}. \tag{3.44}$$

Hence, by the parallel law for electrical resistance,

$$R_{\mathcal{G}}(S_{T^2}, S_{T^3}) = \frac{R_{\mathcal{G}^+}(S_{T^2}, S_{T^3}) R_{\mathcal{G}^-}(S_{T^2}, S_{T^3})}{R_{\mathcal{G}^+}(S_{T^2}, S_{T^3}) + R_{\mathcal{G}^-}(S_{T^2}, S_{T^3})}. \tag{3.45}$$

Note that a resistance of unit 1 is put on each edge of the graph $\mathcal{G}^+$ and $\mathcal{G}^-$. By definition of t_{i_0}, we have

$$(T^2 - T^1) + (T^4 - T^3) \geq (t_{i_0} - t_2) + (t_{C-2} - t_{i_0+1}) \geq (C - 5)r.$$

Hence, because of the assumption $C \geq 15$,

$$(T^2 - T^1) \vee (T^4 - T^3) \geq \frac{C}{3} r.$$

Without loss of generality, we may assume $(T^2 - T^1) \geq \frac{C}{3} r$. We are now in a position to derive (3.42). By the above consideration, all we need is to show

$$\begin{aligned} P\Big(&R_{\mathcal{G}}(0, S_{t_i}) \leq \epsilon n \psi(n) \text{ for all } 1 \leq i \leq C, \tilde{F}_C, \tilde{A}_i \text{ for all } 1 \\ &\leq i \leq C-1, \mathcal{G}^+ \cap \mathcal{G}^- = \{S_{T^2}, S_{T^3}\}\Big) \\ &= O\left(\phi(n)(\log n)^{-\frac{1}{2}} \log\log n\right). \end{aligned} \tag{3.46}$$

Set

$$\bar{A} = \{R_{\mathcal{G}^+}(S_{T^2}, S_{T^3}) \leq 2r\psi(r), \quad R_{\mathcal{G}^-}(S_{T^1}, S_{T^2}) \geq \frac{C}{6} r\psi(r)\}.$$

Then, on $\bar{A}$, because of assumption $C > 120$,

$$R_{\mathcal{G}^+}(S_{T^2}, S_{T^3}) \leq \frac{1}{10} R_{\mathcal{G}^-}(S_{T^2}, S_{T^3}).$$

Here we use the fact that S_{T^1} is a pivotal point on the graph $\mathcal{G}^-$, i.e.,

$$R_{\mathcal{G}^-}(S_{T^2}, S_{T^3}) = R_{\mathcal{G}^-}(S_{T^2}, S_{T^1}) + R_{\mathcal{G}^-}(S_{T^1}, S_{T^3}).$$

Hence,

$$\begin{aligned} R_{\mathcal{G}}(S_{T^2}, S_{T^3}) &= \frac{R_{\mathcal{G}^+}(S_{T^2}, S_{T^3}) R_{\mathcal{G}^-}(S_{T^2}, S_{T^3})}{R_{\mathcal{G}^+}(S_{T^2}, S_{T^3}) + R_{\mathcal{G}^-}(S_{T^2}, S_{T^3})} \\ &\geq \frac{R_{\mathcal{G}^+}(S_{T^2}, S_{T^3}) R_{\mathcal{G}^-}(S_{T^2}, S_{T^3})}{\frac{11}{10} R_{\mathcal{G}^-}(S_{T^2}, S_{T^3})} \\ &= \frac{10}{11} R_{\mathcal{G}^+}(S_{T^2}, S_{T^3}). \end{aligned}$$

However, we are assuming

$$R_{\mathcal{G}}(0, S_{t_{i_0}}) \leq \epsilon n \psi(n) \quad R_{\mathcal{G}}(0, S_{t_{i_0+1}}) \leq \epsilon n \psi(n).$$

Therefore, it follows from $|T^2 - t_{i_0}| \le 3a_{n,-1}$ and $|T^3 - t_{i_0+1}| \le 3a_{n,-1}$ that

$$R_{\mathcal{G}}(S_{T^2}, S_{T^3}) \le 3\epsilon n\psi(n),$$

and

$$R_{\mathcal{G}^+}(S_{T^2}, S_{T^3}) \le \frac{33}{10}\epsilon n\psi(n) \le 4\epsilon n\psi(n).$$

Again it follows from $|T^2 - t_{i_0}| \le 3a_{n,-1}$ and $|T^3 - t_{i_0+1}| \le 3a_{n,-1}$ that

$$R_{\mathcal{G}_{t_{i_0}, t_{i_0+1}}}(S_{t_{i_0}}, S_{t_{i_0+1}}) \le 5\epsilon n\psi(n). \tag{3.47}$$

If $\epsilon < \frac{1}{10}$, then by (3.47), Lemma 2.3.1 and the similar arguments to that in the proof of Lemma 3.2.3, we have

$$\begin{aligned} &P\left(R_{\mathcal{G}}(0, S_{t_i}) \le \epsilon n\psi(n) \text{ for all } 1 \le i \le C, \tilde{F}_C, \tilde{A}_i \text{ for all } 1 \le i \le C-1, \bar{A}\right) \\ &\le P\left(\tilde{F}_C, \quad R_{\mathcal{G}_{t_{i_0}, t_{i_0+1}}}(S_{t_{i_0}}, S_{t_{i_0+1}}) \le 5\epsilon n\psi(n)\right) \\ &= O\left(\phi(n)(\log n)^{-\frac{1}{2}} \log\log n\right). \end{aligned} \tag{3.48}$$

Similar argument as in (3.22) gives that

$$\begin{aligned} &P\left(R_{\mathcal{G}}(0, S_{t_i}) \le \epsilon n\psi(n) \text{ for all } 1 \le i \le C, \tilde{F}_C, \tilde{A}_i \text{ for all } 1 \le i \le C-1, \bar{A}^c\right) \\ &= O\left(\phi(n)(\log n)^{-\frac{1}{2}} \log\log n\right). \end{aligned} \tag{3.49}$$

So, we get (3.46). □

By Lemma 3.2.4 and Lemma 3.2.6, we can conclude that if $\epsilon \in (0, \frac{1}{10})$ and $C > 120$ then

$$(\text{the first term of } (3.27)) = O\left(\phi(n)(\log n)^{-\frac{1}{2}} \log\log n\right). \tag{3.50}$$

Hence, it remains to estimate the second term of (3.27).

Lemma 3.2.7 *Let* $\epsilon \in \left(0, \frac{1}{12}\right)$ *and* $C > 120$. *Then we have*

$$P\left(R_{\mathcal{G}}(0, S_{t_i}) \le \epsilon n\psi(n) \text{ for all } 1 \le i \le C, Z = 0\right) = O\left(\phi(n)(\log n)^{-\frac{1}{2}} \log\log n\right). \tag{3.51}$$

Proof Let $\epsilon \in (0, \frac{1}{12})$ and $C > 120$. Assume that $Z = 0$. Then

$$S[0, \bar{a}_n] \cap S[t_i - \bar{a}_n, t_i + \bar{a}_n] = \emptyset \quad \text{for all } i = 1, 2, \ldots, C.$$

If $S[0,\bar{a}_n]\cap S[t_i+\bar{a}_n, t_{i+1}-\bar{a}_n]\neq\emptyset$ for some i, then the situation boils down to the same case as $Z=1$. Therefore,

$$\begin{aligned}
&P\left(R_{\mathcal{G}}(0,S_{t_i})\le \epsilon n\psi(n) \text{ for all } 1\le i\le C, Z=0\right)\\
&\le P\left(R_{\mathcal{G}}(0,S_{t_i})\le \epsilon n\psi(n) \text{ for all } 1\le i\le C,\ S[0,\bar{a}_n]\cap S[t_1,\infty)=\emptyset\right)\\
&+ P\left(R_{\mathcal{G}}(0,S_{t_i})\le \epsilon n\psi(n) \text{ for all } 1\le i\le C,\ S[0,\bar{a}_n]\cap S[t_i+\bar{a}_n,t_{i+1}-\bar{a}_n]\neq\emptyset \text{ for some } i\right)\\
&= P\left(R_{\mathcal{G}}(0,S_{t_i})\le \epsilon n\psi(n) \text{ for all } 1\le i\le C, S[0,\bar{a}_n]\cap S[t_1,\infty)=\emptyset\right)\\
&+ O\left(\phi(n)(\log n)^{-\frac{1}{2}}\log\log n\right).
\end{aligned}$$

Let

$$I'_l=\begin{cases}\mathbf{1}\{l \text{ is a local cut-time between } 0 \text{ and } \frac{3\bar{a}_n+t_1}{4}\} & \text{if } l\in[\bar{a}_n,\bar{a}_n+a_{n,-6}]\\ \mathbf{1}\{l \text{ is a local cut-time between } \frac{3\bar{a}_n+t_1}{4} \text{ and } \frac{\bar{a}_n+3t_1}{4}\} & \text{if } l\in[\frac{\bar{a}_n+t_1}{2}-a_{n,-6},\frac{\bar{a}_n+t_1}{2}]\\ \mathbf{1}\{l \text{ is a local cut-time between } \frac{\bar{a}_n+3t_1}{4} \text{ and } \infty\} & \text{if } l\in[t_1-a_{n,-6},t_1],\end{cases}$$

and

$$I'_{(1)}=\sum_{l=\bar{a}_n}^{\bar{a}_n+a_{n,-6}}I'_l,\quad I'_{(2)}=\sum_{l=\frac{\bar{a}_n+t_1}{2}-a_{n,-6}}^{\frac{\bar{a}_n+t_1}{2}}I'_l,\quad I'_{(3)}=\sum_{l=t_1-a_{n,-6}}^{t_1}I'_l.$$

By [6], Lemma 7.7.4 and independence,

$$P\left(I'_{(i)}=I'_{(j)}=0 \text{ for some } 1\le i<j\le 3\right)=O\left((\phi(n))^2\right).$$

Hence, we have only to estimate

$$\begin{aligned}
&P\left(R_{\mathcal{G}}(0,S_{t_i})\le \epsilon n\psi(n) \text{ for all } 1\le i\le C, S[0,\bar{a}_n]\cap S[t_1,\infty)=\emptyset, I'_{(1)}\ge 1, I'_{(2)}\ge 1\right)\\
&\quad +P\left(R_{\mathcal{G}}(0,S_{t_i})\le \epsilon n\psi(n) \text{ for all } 1\le i\le C, S[0,\bar{a}_n]\cap S[t_1,\infty)=\emptyset, I'_{(2)}\ge 1, I'_{(3)}\ge 1\right)\\
&\quad +P\left(R_{\mathcal{G}}(0,S_{t_i})\le \epsilon n\psi(n) \text{ for all } 1\le i\le C, S[0,\bar{a}_n]\cap S[t_1,\infty)=\emptyset, I'_{(3)}\ge 1, I'_{(1)}\ge 1\right).
\end{aligned}$$

We will only consider the first term of the above. (The other terms are estimated similarly.) Assume $I'_{(1)}\ge 1$, $I'_{(2)}\ge 1$. If there is no global cut-time in $[\bar{a}_n,\bar{a}_n+a_{n,-6}]$, then it is easy to see that

$$S[0,\bar{a}_n]\cap S\left[\frac{3\bar{a}_n+t_1}{4},\infty\right)\neq\emptyset.$$

However, by the same proof as that of the case for $Z=1$, we have the following.

$$\begin{aligned}
&P\left(R_{\mathcal{G}}\left(0,S_{t_i}\right)\le \epsilon n\psi(n) \text{ for all } 1\le i\le C, S\left[0,\bar{a}_n\right]\cap S\left[\frac{3\bar{a}_n+t_1}{4},\infty\right)\neq\emptyset.\right)\\
&= O\left(\phi(n)(\log n)^{-\frac{1}{2}}\log\log n\right).
\end{aligned}$$

Hence, we can assume that there is a global cut-time in both $[\bar{a}_n, \bar{a}_n - a_{n,-6}]$ and $\left[\frac{\bar{a}_n+t_1}{2} - a_{n,-6}, \frac{\bar{a}_n+t_1}{2}\right]$. (Note that if there is no global cut-time in $\left[\frac{\bar{a}_n+t_1}{2} - a_{n,-6}, \frac{\bar{a}_n+t_1}{2}\right]$, then we return to the case $Z = 1$ again.) So let

$$T \in \left[\bar{a}_n, \bar{a}_n + a_{n,-6}\right]$$
$$T' \in \left[\frac{\bar{a}_n + t_1}{2} - a_{n,-6}, \frac{\bar{a}_n + t_1}{2}\right]$$

be global cut-times. Since

$$R_{\mathcal{G}}(0, S_{t_1}) \leq \epsilon n \psi(n),$$

we have

$$R_{\mathcal{G}_{0,T}}(0, S_T) + R_{\mathcal{G}_{T,T'}}(S_T, S_{T'}) + R_{\mathcal{G}_{T',\infty}}\left(S_{T'}, S_{t_1}\right) \leq \epsilon n \psi(n).$$

It follows from $|T - \bar{a}_n| \leq a_{n,-6}$ and $|T' - \frac{\bar{a}_n+t_1}{2}| \leq b_{n,-6}$ that

$$R_{\mathcal{G}_{0,\bar{a}_n}}\left(0, S_{\bar{a}_n}\right) + R_{\mathcal{G}_{\bar{a}_n, \frac{\bar{a}_n+t_1}{2}}}\left(S_{\bar{a}_n}, S_{\frac{\bar{a}_n+t_1}{2}}\right) + R_{\mathcal{G}_{\frac{\bar{a}_n+t_1}{2},\infty}}\left(S_{\frac{\bar{a}_n+t_1}{2}}, S_{t_1}\right) \leq 2\epsilon n \psi(n).$$

Therefore, if we choose $\epsilon \in \left(0, \frac{1}{12}\right)$, then by Lemma 2.3.1 and independence, we can conclude

$$\begin{aligned}
&P\left(R_{\mathcal{G}}(0, S_{t_i}) \leq \epsilon n \psi(n) \text{ for all } 1 \leq i \leq C, S[0, \bar{a}_n] \cap S[t_1, \infty) = \emptyset, I'_{(1)} \geq 1, I'_{(2)} \geq 1\right) \\
&\leq P\left(R_{\mathcal{G}_{0,\bar{a}_n}}(0, S_{\bar{a}_n}) + R_{\mathcal{G}_{\bar{a}_n, \frac{\bar{a}_n+t_1}{2}}}(S_{\bar{a}_n}, S_{\frac{\bar{a}_n+t_1}{2}}) + R_{\mathcal{G}_{\frac{\bar{a}_n+t_1}{2},\infty}}(S_{\frac{\bar{a}_n+t_1}{2}}, S_{t_1}) \leq 2\epsilon n \psi(n)\right) \\
&+P\Big(R_{\mathcal{G}}(0, S_{t_i}) \leq \epsilon n \psi(n) \text{ for all } 1 \leq i \leq C,\ S[0, \bar{a}_n] \cap S[t_1, \infty) = \emptyset, I'_{(1)} \geq 1, I'_{(2)} \geq 1, \\
&\qquad \left[\bar{a}_n, \bar{a}_n + a_{n,-6}\right] \text{ does not have a global cut-time}\Big) \\
&+P\Big(R_{\mathcal{G}}(0, S_{t_i}) \leq \epsilon n \psi(n) \text{ for all } 1 \leq i \leq C, S[0, \bar{a}_n] \cap S[t_1, \infty) = \emptyset, I'_{(1)} \geq 1, I'_{(2)} \geq 1, \\
&\qquad \left[\frac{\bar{a}_n + t_1}{2} - a_{n,-6}, \frac{\bar{a}_n + t_1}{2}\right] \text{ does not have a global cut-time}\Big) \\
&= O\left(\phi(n)(\log n)^{-\frac{1}{2}} \log\log n\right).
\end{aligned}$$

This gives the result. □

Proof of Proposition 3.1.1 By Lemma 3.2.1, Lemma 3.2.2, Lemma 3.2.3, (3.50) and Lemma 3.2.7, if we fix $\epsilon \in (0, \frac{1}{12})$ and $C > 120$, then

$$P\left(\bar{Y} \geq 2Cn\right) = O\left((\log n)^{-\frac{3}{2}}(\log\log n)^{\alpha}\right),$$

for some $\alpha > 0$. This gives the proposition. □

4 Proof of Theorem 1.2.3

4.1 Preparations

We will prove Theorem 1.2.3 in this section. To do this, we first show the following easy lemma.

Lemma 4.1.1 *Assume that* ψ *satisfies the following condition: for all* $\epsilon \in (0, 1)$*, there exists* $N = N_\epsilon \in \mathbb{N}$ *such that*

$$\psi(2n) \leq \psi(n)\left(1 - \frac{\log 2}{2\log n}(1-\epsilon)\right) \quad \textit{for all } n \geq N. \tag{4.1}$$

Then it follows that for all $\epsilon \in (0, 1)$*, we have*

$$\limsup_{n\to\infty} (\log n)^{\frac{1}{2}-\epsilon}\psi(n) = 0. \tag{4.2}$$

Proof Fix $\epsilon \in (0, 1)$ and assume (4.1). Let $g(k) = \log \psi(2^k N)$. Then by (4.1),

$$g(k+1) \leq g(k) + \log\left(1 - \frac{1-\epsilon}{2(k+\log_2 N)}\right).$$

Therefore,

$$g(k) \leq g(1) + \sum_{j=1}^{k-1} \log\left(1 - \frac{1-\epsilon}{2(j+\log_2 N)}\right).$$

If we choose $M \in \mathbb{N}$ satisfying that

$$\log\left(1 - \frac{1-\frac{\epsilon}{2}}{2(j+\log_2 N)}\right) \leq -\frac{1-\frac{2\epsilon}{3}}{2j} \quad \text{for all } j \geq M,$$

then we have

$$g(k) \leq -\frac{1}{2}\left(1-\frac{2\epsilon}{3}\right)(\log k - \log M) + g(1) \quad \text{for all } k > M.$$

Hence, it follows that for large k,

$$(\log k)^{-1} g(k) \leq -\frac{1}{2}\left(1 - \frac{3\epsilon}{4}\right),$$

and

$$\psi(2^k N) \leq k^{-\frac{1}{2}+\frac{3\epsilon}{8}} \leq \left(\log 2^k N\right)^{-\frac{1}{2}+\frac{3\epsilon}{8}} \quad \text{for all } k \geq K,$$

for some $K \in \mathbb{N}$. Let $n \geq 2^K N$. Then there exists $k \geq K$ such that $2^k N \leq n < 2^{k+1} N$. It follows from the fact ψ is slowly varying that

$$\psi(n) \leq c\psi(2^k N) \leq \tilde{c}(\log n)^{-\frac{1}{2}+\frac{3\epsilon}{8}}.$$

This implies (4.2). □

Remark 4.1.2 Note that Lemma 4.1.1 is obtained by modifying Proposition 4.4.2 in [6]. If we get (4.2) for all $\epsilon \in (0, 1)$, then by (2.4), we can conclude that

$$\psi(n) \approx (\log n)^{-\frac{1}{2}}.$$

Hence, in this section, we will derive (4.1) for all $\epsilon \in (0, 1)$.

4.2 Derivation of the resistance exponent

Fix $\epsilon \in (0, 1)$. In this subsection, we will show (4.1). Since $\psi(n) \sim \overline{\psi}(n)$ (see Lemma 2.1.1), it suffices to show that there exists $N \in \mathbb{N}$ such that

$$\overline{\psi}(2n) \leq \overline{\psi}(n)\left(1 - \frac{\log 2}{2\log n}(1-\epsilon)\right) \quad \text{for all } n \geq N. \tag{4.3}$$

(Indeed, once we get (4.3), the same argument as the proof of Lemma 4.1.1 gives that $\overline{\psi}(n) \approx (\log n)^{-\frac{1}{2}}$ and we have the theorem.) Note that

$$\overline{\psi}(n) - \overline{\psi}(2n) = \frac{E\left(R_{\mathcal{G}_{0,n}}(0, S_n) + R_{\mathcal{G}_{n,2n}}(S_n, S_{2n}) - R_{\mathcal{G}_{0,2n}}(0, S_{2n})\right)}{2n}. \tag{4.4}$$

So, to get (4.3), we have to give a sharp lower bound of the right-hand side of (4.4). Let

$$R_n^1 = R_{\mathcal{G}_{0,n}}(0, S_n), \quad R_n^2 = R_{\mathcal{G}_{n,2n}}(S_n, S_{2n}).$$

Let

$$N := \lfloor \frac{n}{a_{n,-2}} \rfloor + 1.$$

We set

$$\begin{aligned}
I_j^1 &= \left[n - ja_{n,-2}, n - (j-1)a_{n,-2}\right], \quad \text{for } j = 1, \cdots, N-1 \\
I_N^1 &= \left[0, n - (N-1)a_{n,-2}\right], \\
I_j^2 &= \left[n + (j-1)a_{n,-2}, n + ja_{n,-2}\right], \quad \text{for } j = 1, \cdots, N-1 \\
I_N^2 &= \left[n + (N-1)a_{n,-2}, 2n\right].
\end{aligned}$$

We say j and k intersect if

$$S[n - ja_{n,-2}, n - (j-1)a_{n,-2}] \cap S[n + (k-1)a_{n,-2}, n + ka_{n,-2}] \neq \emptyset.$$

Let

$$L = \max_{\substack{1 \le j,k \le N, \\ j \text{ and } k \text{ intersect}}} (j + k),$$

$2 \le l \le 2N$, and $1 \le j \le l - 1$. We define the following events.

$$\begin{aligned}
A^1 &= A^1_l = \{j \text{ and } l - j \text{ intersect}\} \\
A^2 &= A^2_l = \{L = l\} \\
A^3 &= A^3_l = \{\text{there is only one pair of } (j, k) \text{ which attains the maximum of } L\} \\
A^4 &= A^4_l = \left\{\text{there is a local cut-time between 0 and } 2n \text{ in } I^1_{j+1}\right\} \\
A^5 &= A^5_l = \left\{\text{ there is a local cut-time between 0 and } 2n \text{ in } I^2_{l-j+1}\right\} \\
A^6 &= A^6_l = \left\{\max_{i,\tilde{i} \in I^1_{j-1} \cup I^1_j \cup I^1_{j+1}} R_{\mathcal{G}_{i,\tilde{i}}}(S_i, S_{\tilde{i}}) \le 4a_{n,-2}\overline{\psi}(n)\right\} \\
A^7 &= A^7_l = \left\{\max_{i,\tilde{i} \in I^2_{l-j-1} \cup I^2_{l-j} \cup I^2_{l-j+1}} R_{\mathcal{G}_{i,\tilde{i}}}(S_i, S_{\tilde{i}}) \le 4a_{n,-2}\overline{\psi}(n)\right\} \\
A^8 &= A^8_l = \left\{\text{ there is a local cut-time between 0 and } n \text{ in } I^1_{j-1}\right\} \\
A^9 &= A^9_l = \left\{\text{there is a local cut-time between } n \text{ and } 2n \text{ in } I^2_{l-j-1}\right\} \\
A^{10} &= A^{10}_l = \left\{R_{\mathcal{G}_{t^1_j,n}}\left(S_{t^1_j}, S_n\right) \ge (1-\epsilon) j a_{n,-2}\overline{\psi}(n)\right\} \\
A^{11} &= A^{11}_l = \left\{R_{\mathcal{G}_{n,t^2_{l-j}}}\left(S_n, S_{t^2_{l-j}}\right) \ge (1-\epsilon)(l-j) a_{n,-2}\overline{\psi}(n)\right\},
\end{aligned}$$

where $t^1_j = n - (j-1)a_{n,-2}$ and $t^2_{l-j} = n + (l - j - 1)a_{n,-2}$.

Lemma 4.2.1 *Fix l and j. Then, on the event $A^1 \cap A^2 \cap \cdots \cap A^{11}$,*

$$R^1_n + R^2_n - R^1_{2n} \ge \{(1-\epsilon)l - 16\}a_{n,-2}\overline{\psi}(n). \tag{4.5}$$

Proof Assume $A^1 \cap A^2 \cap \cdots \cap A^{11}$ and let

$$T^1 \in I^1_{j+1}, T^2 \in I^1_{j-1}, T^3 \in I^2_{l-j-1}, T^4 \in I^2_{l-j+1}$$

be local cut-times in the events A^4, A^8, A^9 and A^5, respectively. Then

$$\begin{aligned}
&R_n^1 + R_n^2 - R_{2n}^1 \\
&\quad = R_{\mathcal{G}_{T^2,n}}\left(S_{T^2}, S_n\right) + R_{\mathcal{G}_{n,T^3}}\left(S_n, S_{T^3}\right) \\
&\qquad + R_{\mathcal{G}_{T^1,T^2}}\left(S_{T^1}, S_{T^2}\right) + R_{\mathcal{G}_{T^3,T^4}}\left(S_{T^3}, S_{T^4}\right) - R_{\mathcal{G}_{T^1,T^4}}\left(S_{T^1}, S_{T^4}\right) \\
&\quad \geq R_{\mathcal{G}_{T^2,n}}\left(S_{T^2}, S_n\right) + R_{\mathcal{G}_{n,T^3}}\left(S_n, S_{T^3}\right) - R_{\mathcal{G}_{T^1,T^4}}\left(S_{T^1}, S_{T^4}\right) \\
&\quad \geq R_{\mathcal{G}_{T^2,n}}\left(S_{T^2}, S_n\right) + R_{\mathcal{G}_{n,T^3}}\left(S_n, S_{T^3}\right) - 8a_{n,-2}\overline{\psi}(n) \\
&\quad = R_{\mathcal{G}_{t_j^1,n}}\left(S_{t_j^1}, S_n\right) + R_{\mathcal{G}_{n,t_{l-j}^2}}\left(S_n, S_{t_{l-j}^2}\right) \\
&\qquad - R_{\mathcal{G}_{t_j^1,T^2}}\left(S_{t_j^1}, S_{T^2}\right) - R_{\mathcal{G}_{T^3,t_{l-j}^2}}\left(S_{T^3}, S_{t_{l-j}^2}\right) - 8a_{n,-2}\overline{\psi}(n) \\
&\quad \geq R_{\mathcal{G}_{t_j^1,n}}\left(S_{t_j^1}, S_n\right) + R_{\mathcal{G}_{n,t_{l-j}^2}}\left(S_n, S_{t_{l-j}^2}\right) - 16a_{n,-2}\overline{\psi}(n) \\
&\quad \geq \{(1-\epsilon)l - 16\}a_{n,-2}\overline{\psi}(n).
\end{aligned}$$

□

By Lemma 4.2.1,

$$\begin{aligned}
&E\left(R_n^1 + R_n^2 - R_{2n}^1\right) \\
&\quad = \sum_{l=2}^{2N} E\left(R_n^1 + R_n^2 - R_{2n}^1; A^2\right) \\
&\quad \geq \sum_{l=2}^{2N} E\left(R_n^1 + R_n^2 - R_{2n}^1; A^2 \cap A^3\right) \\
&\quad \geq \sum_{l=3}^{N} \sum_{j=1}^{l-1} E\left(R_n^1 + R_n^2 - R_{2n}^1; A^1 \cap A^2 \cap A^3\right) \\
&\qquad + \sum_{l=N+1}^{2N} \sum_{j=l-N}^{N} E\left(R_n^1 + R_n^2 - R_{2n}^1; A^1 \cap A^2 \cap A^3\right) \\
&\quad \geq \sum_{l=3}^{N} \sum_{j=\lfloor \epsilon l \rfloor}^{\lfloor (1-\epsilon) l \rfloor} E\left(R_n^1 + R_n^2 - R_{2n}^1; A^1 \cap A^2 \cap A^3\right) \\
&\qquad + \sum_{l=\lfloor (1+\epsilon) N \rfloor}^{2N} \sum_{j=l-N}^{N} E\left(R_n^1 + R_n^2 - R_{2n}^1; A^1 \cap A^2 \cap A^3\right) \\
&\quad \geq \sum_{l=3}^{N} \sum_{j=\lfloor \epsilon l \rfloor}^{\lfloor (1-\epsilon) l \rfloor} \{(1-\epsilon)l - 16\}a_{n,-2}\overline{\psi}(n) P\left(A^1 \cap \cdots \cap A^{11}\right) \\
&\qquad + \sum_{l=\lfloor (1+\epsilon) N \rfloor}^{2N} \sum_{j=l-N}^{N} \{(1-\epsilon)l - 16\}a_{n,-2}\overline{\psi}(n) P\left(A^1 \cap \cdots \cap A^{11}\right). \qquad (4.6)
\end{aligned}$$

Lemma 4.2.2 *There exists* $c = c_\epsilon > 0$ *such that*

$$P\left(A^1 \cap \cdots \cap A^{11}\right) \geq P\left(A^1\right)\left(1 - c(\log n)^{-\frac{1}{2}}(\log\log n)^{\alpha}\right)$$
$$\text{for } 3 \leq l \leq N,\ \lfloor \epsilon l \rfloor \leq j \leq \lfloor (1-\epsilon) l \rfloor, \tag{4.7}$$

and

$$P\left(A^1 \cap \cdots \cap A^{11}\right) \geq P\left(A^1\right)\left(1 - c(\log n)^{-\frac{1}{2}}(\log\log n)^{\alpha}\right)$$
$$\text{for } \lfloor (1+\epsilon) N \rfloor \leq l \leq 2N,\ l - N \leq j \leq N, \tag{4.8}$$

for some $\alpha > 0$.

Proof We will only prove (4.7). ((4.8) is proved similarly.) It is known that

$$P\left(A^1\right) \sim \frac{1}{2}(\log n)^{-1} l^{-2} \tag{4.9}$$

if $(j, l-j) \neq (1, 1)$. (Note that the asymptotic convergence is uniform for l. See the proof of Theorem 4.1 (a) in [5], for example.) Let $3 \leq l \leq N$, $\lfloor \epsilon l \rfloor \leq j \leq \lfloor (1-\epsilon) l \rfloor$, and $k = l - j$. To prove (4.7), we will show

$$P\left(A^1 \cap \cdots \cap A^i\right) \geq P\left(A^1 \cap \cdots \cap A^{i-1}\right) - cP\left(A^1\right)(\log n)^{-\frac{1}{2}}(\log\log n)^{\alpha}. \tag{4.10}$$

for all $i = 2, 3, \ldots, 11$. We will only consider (4.10) for $i = 11$ since the case $i = 11$ is the most complicated and the other cases can be proved similarly. Let S^1, S^2 be independent simple random walks starting from the origin in $\mathbb{Z}^4$. Note that

$$P\left(A^1 \cap \left(A^{11}\right)^c\right) = P\Big(S^1[0, a_{n,-2}] \cap S^2\left[(l-2)a_{n,-2}, (l-1)a_{n,-2}\right] \neq \emptyset,$$
$$R_{\mathcal{G}^2_{0,(k-1)a_{n,-2}}}(0, S^2_{(k-1)a_{n,-2}}) < (1-\epsilon) k a_{n,-2} \overline{\psi}(n)\Big),$$

where $\mathcal{G}^2$ denotes the random walk trace for S^2. Define the event

$$\bar{B} = \left\{ \begin{array}{c} \left|S^1_m\right| \leq \frac{1}{2}\sqrt{a_{n,-2}}(\log\log n)^2, \quad \text{for all } m = 0, 1, \cdots, a_{n,-2} \\ |S^2_{m+(l-2)a_{n,-2}} - S^2_{(l-2)a_{n,-2}}| \leq \frac{1}{2}\sqrt{a_{n,-2}}(\log\log n)^2, \quad \text{for all } m = 0, 1, \cdots, a_{n,-2} \end{array} \right\}.$$

By the large deviation estimate [see [6, Lemma 1.5.1], for example), we have $P(\bar{B}^c) \leq (\log n)^{-10}$. So

$$
\begin{aligned}
&P\Big(S^1\left[0, a_{n,-2}\right] \cap S^2\left[(l-2)a_{n,-2}, (l-1)a_{n,-2}\right] \neq \emptyset,\\
&R_{\mathcal{G}^2_{0,(k-1)a_{n,-2}}}\left(0, S^2_{(k-1)a_{n,-2}}\right) < (1-\epsilon)ka_{n,-2}\overline{\psi}(n)\Big)\\
&\leq P\Big(S^1\left[0, a_{n,-2}\right] \cap S^2\left[(l-2)a_{n,-2}, (l-1)a_{n,-2}\right] \neq \emptyset,\\
&R_{\mathcal{G}^2_{0,(k-1)a_{n,-2}}}\left(0, S^2_{(k-1)a_{n,-2}}\right) < (1-\epsilon)ka_{n,-2}\overline{\psi}(n), \bar{B}\Big) + (\log n)^{-10}\\
&:= I + (\log n)^{-10}.
\end{aligned}
$$

Assume that

$$
S^1\left[0, a_{n,-2}\right] \cap S^2\left[(l-2)a_{n,-2}, (l-1)a_{n,-2}\right] \neq \emptyset \text{ and } \bar{B}.
$$

Then we have

$$
|S^2_{(l-2)a_{n,-2}}| \leq \sqrt{a_{n,-2}}(\log\log n)^2.
$$

So,

$$
\begin{aligned}
&I \leq P\Big(S^1\left[0, a_{n,-2}\right] \cap S^2\left[(l-2)a_{n,-2}, (l-1)a_{n,-2}\right] \neq \emptyset,\\
&\quad R_{\mathcal{G}^2_{0,(k-1)a_{n,-2}}}\left(0, S^2_{(k-1)a_{n,-2}}\right) < (1-\epsilon)ka_{n,-2}\overline{\psi}(n), |S^2_{(l-2)a_{n,-2}}| \leq \sqrt{a_{n,-2}}(\log\log n)^2\Big)\\
&\leq c\phi(n)P\Big(R_{\mathcal{G}^2_{0,(k-1)a_{n,-2}}}(0, S^2_{(k-1)a_{n,-2}}) < (1-\epsilon)ka_{n,-2}\overline{\psi}(n), |S^2_{(l-2)a_{n,-2}}| \leq \sqrt{a_{n,-2}}(\log\log n)^2\Big)\\
&+ c(\log n)^{-6},
\end{aligned}
$$

where the last inequality is obtained by the same argument as in (3.22).

Note that for any $x \in \mathbb{Z}^4$,

$$
\begin{aligned}
P^x\left(|S_{(j-1)a_{n,-2}}| \leq \sqrt{a_{n,-2}}(\log\log n)^2\right) &\leq c\frac{1}{((j-1)a_{n,-2})^2}(a_{n,-2})^2(\log\log n)^8\\
&\leq c\frac{1}{j^2}(\log\log n)^8.
\end{aligned}
$$

Hence, by the Markov property and Lemma 2.3.1,

$$
\begin{aligned}
&P\left(R_{\mathcal{G}^2_{0,(k-1)a_{n,-2}}}\left(0, S^2_{(k-1)a_{n,-2}}\right) < (1-\epsilon)ka_{n,-2}\overline{\psi}(n), |S^2_{(l-2)a_{n,-2}}| \leq \sqrt{a_{n,-2}}(\log\log n)^2\right)\\
&\leq P\left(R_{\mathcal{G}^2_{0,(k-1)a_{n,-2}}}(0, S^2_{(k-1)a_{n,-2}}) < (1-\epsilon)ka_{n,-2}\overline{\psi}(n)\right) c\frac{1}{j^2}(\log\log n)^8
\end{aligned}
$$

$$\le c_\epsilon \frac{1}{j^2}(\log n)^{-\frac{1}{2}}(\log\log n)^9$$
$$\le c_\epsilon \frac{1}{l^2}(\log n)^{-\frac{1}{2}}(\log\log n)^9,$$

where we used $j \ge \lfloor \epsilon l \rfloor$ in the last inequality. Therefore, by (4.9),

$$\begin{aligned} P\left(A^1 \cap \cdots \cap A^{11}\right) &\ge P\left(A^1 \cap \cdots \cap A^{10}\right) - P\left(A^1 \cap (A^{11})^c\right) \\ &\ge P\left(A^1 \cap \cdots \cap A^{10}\right) - c_\epsilon \phi(n) \frac{1}{l^2}(\log n)^{-\frac{1}{2}}(\log\log n)^9 \\ &\quad -c(\log n)^{-6} - (\log n)^{-10} \ge P\left(A^1 \cap \cdots \cap A^{10}\right) \\ &\quad -\bar{c}_\epsilon P\left(A^1\right)(\log n)^{-\frac{1}{2}}(\log\log n)^{11}. \end{aligned}$$

This gives (4.10) for $i = 11$. □

Proof of Theorem 1.2.3. As we mentioned before, we have only to show (4.3). By (4.6), Lemma 4.2.2, (4.9) and $N \sim (\log n)^2$,

$$\begin{aligned} &E\left(R_n^1 + R_n^2 - R_{2n}^1\right) \\ &\ge \sum_{l=3}^{N} \sum_{j=\lfloor \epsilon l \rfloor}^{\lfloor (1-\epsilon) l \rfloor} \{(1-\epsilon)l - 16\} a_{n,-2} \overline{\psi}(n)(1-\epsilon)\tfrac{1}{2}(\log n)^{-1} l^{-2}\left(1 - c(\log n)^{-\frac{1}{2}}(\log\log n)^\alpha\right) \\ &\quad + \sum_{l=\lfloor (1+\epsilon)N \rfloor}^{2N} \sum_{j=l-N}^{N} \{(1-\epsilon)l - 16\} a_{n,-2} \overline{\psi}(n)(1-\epsilon)\tfrac{1}{2}(\log n)^{-1} l^{-2}\left(1 - c(\log n)^{-\frac{1}{2}}(\log\log n)^\alpha\right) \\ &\ge \sum_{l=3}^{N}(1-6\epsilon)\tfrac{1}{2}(\log n)^{-1} a_{n,-2}\overline{\psi}(n) + \sum_{l=\lfloor (1+\epsilon)N \rfloor}^{2N}(1-4\epsilon)(2N-l)l^{-1}\tfrac{1}{2}(\log n)^{-1} a_{n,-2}\overline{\psi}(n) \\ &\ge (1-7\epsilon)\frac{1}{2}(\log n)^{-1} n\overline{\psi}(n) + (1-6\epsilon)n(\log n)^{-1}\overline{\psi}(n)(\log 2) - \frac{1}{2}(\log n)^{-1} n\overline{\psi}(n) \\ &= (1-6\epsilon)n(\log n)^{-1}\overline{\psi}(n)(\log 2) - 7\epsilon\frac{1}{2}(\log n)^{-1} n\overline{\psi}(n) \\ &\ge (1-20\epsilon)n(\log n)^{-1}\overline{\psi}(n)\log 2, \end{aligned}$$

for large n. Hence, by (4.4),

$$\overline{\psi}(n) - \overline{\psi}(2n) \ge (1-20\epsilon)(\log n)^{-1}\overline{\psi}(n)\frac{\log 2}{2},$$

for large n. This implies (4.3). □

Remark 4.2.3 By modifying the above arguments, it is not difficult to show that there exists $c > 0$ such that

$$\overline{\psi}(2n) \ge \overline{\psi}(n)\left(1 - c(\log n)^{-1}\right), \tag{4.11}$$

and it follows from (4.1) and (4.11) that

$$\overline{\psi}(n^2) \asymp \overline{\psi}(n). \tag{4.12}$$

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