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On the existence and position of the farthest peaks of a family of stochastic heat and wave equations

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Abstract We study the stochastic heat equation $\partial_t u = \mathscr{L} u + \sigma(u)\dot{W}$ in $(1+1)$ dimensions, where $\dot{W}$ is space-time white noise, $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous, and $\mathscr{L}$ is the generator of a symmetric Lévy process that has finite exponential moments, and u_0 has exponential decay at $\pm\infty$. We prove that under natural conditions on σ: (i) The νth absolute moment of the solution to our stochastic heat equation grows exponentially with time; and (ii) The distances to the origin of the farthest high peaks of those moments grow exactly linearly with time. Very little else seems to be known about the location of the high peaks of the solution to the stochastic heat equation under the present setting (see, however, Gärtner et al. in Probab Theory Relat Fields 111:17–55, 1998; Gärtner et al. in Ann Probab 35:439–499, 2007 for the analysis of the location of the peaks in a different model). Finally, we show that these results extend to the stochastic wave equation driven by Laplacian.

Keywords Stochastic PDEs · Stochastic heat equation · Intermittence

Mathematics Subject Classification (2000) Primary 60H15; Secondary 35R60

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1 Introduction

We study the nonlinear stochastic heat equation

$$\frac{\partial}{\partial t}u_t(x) = (\mathscr{L}u_t)(x) + \sigma(u_t(x))\frac{\partial^2}{\partial t\partial x}W(t\,,x) \qquad \text{for } t>0,\ x\in\mathbf{R}, \tag{1.1}$$

where: (i) $\mathscr{L}$ is the generator of a real-valued symmetric Lévy process $\{X_t\}_{t\geq 0}$ with Lévy exponent Ψ;[1] (ii) $\sigma:\mathbf{R}\to\mathbf{R}$ is Lipschitz continuous with Lipschitz constant Lip_σ; (iii) W is two-parameter Brownian sheet, indexed by $(t\,,x)\in\mathbf{R}_+\times\mathbf{R}$; and (iv) the initial function $u_0:\mathbf{R}\to\mathbf{R}_+$ is in $L^\infty(\mathbf{R})$. Equation (1.1) arises for several reasons that include its connections to the stochastic Burger's equation (see Gyöngy and Nualart [21]) and the parabolic Anderson model (see Carmona and Molchanov [6]).

According to the theory of Dalang [10], (1.1) has a unique solution when

$$\Upsilon(\beta) := \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\beta+2\Psi(\xi)} < \infty \quad \text{for some, hence all, } \beta>0. \tag{1.2}$$

Moreover, under various conditions on σ, (1.2) is necessary for the existence of a solution [10,26].

Foondun and Khoshnevisan [17] have shown that:

$$\overline{\gamma}(\nu) := \limsup_{t\to\infty}\frac{1}{t}\sup_{x\in\mathbf{R}}\ln \mathrm{E}\left(|u_t(x)|^\nu\right) < \infty \qquad \text{for every } \nu\geq 2; \tag{1.3}$$

and that

$$\limsup_{t\to\infty}\frac{1}{t}\inf_{x\in\mathbf{R}}\ln \mathrm{E}\left(|u_t(x)|^\nu\right) > 0 \qquad \text{for every } \nu\geq 2, \tag{1.4}$$

provided that: (a) $\inf_x|\sigma(x)/x|>0$; and (b) $\inf_x u_0(x)>0$.[2] Together these results show that if u_0 is bounded away from 0 and σ is sublinear, then the solution to (1.1) is "weakly intermittent" [that is, highly peaked for large t]. Rather than describe why this is a noteworthy property, we refer the interested reader to the extensive bibliography of [17], which contains several pointers to the literature in mathematical physics that motivate [weak] intermittency.

The case that u_0 has compact support arises equally naturally in mathematical physics, but little is known rigorously about when, why, or if the solution to (1.1) is weakly intermittent when u_0 has compact support. In fact, we know of only one article [16], which considers the special case $\mathscr{L}=\partial^2/\partial x^2$, $\sigma(0)=0$, and u_0 smooth and compactly supported. In that article it is shown that $\overline{\gamma}(2)\in(0\,,\infty)$, but the arguments

[1] Recall that Ψ is defined by $\mathrm{E}\exp(i\xi X_1)=\exp(-\Psi(\xi))\,[\xi\in\mathbf{R}]$. Because of the symmetry of $\{X_t\}_{t\geq 0}$, $\Psi(\xi)=\Psi(-\xi)\geq 0$ for all $\xi\in\mathbf{R}$.

[2] In fact, these results do not require that $\{X_t\}_{t\geq 0}$ is a symmetric process provided that we replace Ψ with $\mathrm{Re}\Psi$ in (1.2).

of [16] rely critically on several special properties of the Laplacian. A closely-related case ($u_0 := \delta_0$) appears in Bertini and Cancrini [1].

Presently, we show that weak intermittency follows in some cases from a "stochastic weighted Young inequality" (Proposition 2.5). Such an inequality is likely to have other applications as well. And more significantly, we describe quite precisely the location of the high peaks that are farthest away from the origin.

From now on, let us assume further that

$$\sigma(0) = 0 \quad \text{and} \quad \mathrm{L}_\sigma := \inf_{x\in\mathbf{R}} |\sigma(x)/x| > 0. \tag{1.5}$$

And we define two *growth indices*:

$$\overline{\lambda}(\nu) := \inf\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \ln \mathrm{E}\left(|u_t(x)|^\nu\right) < 0\right\}; \tag{1.6}$$

where $\inf\varnothing := \infty$; and

$$\underline{\lambda}(\nu) := \sup\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq\alpha t} \ln \mathrm{E}\left(|u_t(x)|^\nu\right) > 0\right\}; \tag{1.7}$$

where $\sup\varnothing := 0$.

One can check directly that $0 \leq \underline{\lambda}(\nu) \leq \overline{\lambda}(\nu) \leq \infty$. Our goal is to identify several instances when $0 < \underline{\lambda}(\nu) \leq \overline{\lambda}(\nu) < \infty$. In those instances, it follows that: (i) The solution to (1.1) has very high peaks as $t \to \infty$ ["weak intermittency"]; and (ii) The distances between the origin and the farthest high peaks grow exactly linearly in t. This seems to be the first concrete piece of information on the location of the high peaks of the solution to (1.1) when u_0 has compact support.

Let $\mathscr{D}_{exp}$ denote the collection of all bounded lower semicontinuous functions $h : \mathbf{R} \to \mathbf{R}_+$ for which there exists $\rho > 0$ such that $h(x) = O(\mathrm{e}^{-\rho|x|})$ as $|x| \to \infty$.

Theorem 1.1 *If there exists $c > 0$ such that $\mathrm{E}[e^{cX_1}] < \infty$ and $u_0 \in \mathscr{D}_{exp}$ is strictly positive on a set of positive measure, then $0 < \underline{\lambda}(\nu) \leq \overline{\lambda}(\nu) < \infty$ for all $\nu \in [2, \infty)$.*

Remark 1.2 Theorem 1.1 applies to many Lévy processes other than Brownian motion. Here we mention a simple family of examples. First, let us recall the Lévy–Khintchine formula for Ψ: There exists $\sigma \in \mathbf{R}$ and a symmetric Borel measure m on $\mathbf{R}$ such that $m(\{0\}) = 0$, $\int_{-\infty}^{\infty} (1 \wedge \xi^2)\, m(\mathrm{d}\xi) < \infty$, and for all $\xi \in \mathbf{R}$,

$$\Psi(\xi) = \sigma^2\xi^2 + 2\int_{-\infty}^{\infty} [1 - \cos(\xi z)]\, m(\mathrm{d}z). \tag{1.8}$$

It is well known that for all rapidly-decreasing functions $f : \mathbf{R} \to \mathbf{R}$,

$$(\mathscr{L} f)(x) = \sigma^2 f''(x) + \int_0^\infty [f(x+z) + f(x-z) - 2f(x)]\, m(\mathrm{d}z). \tag{1.9}$$

It is possible to show that the conditions of Theorem 1.1 are met, for example, if the support of m is bounded. One can frequently verify Dalang's condition (1.2) in such examples, as well. For instance, let us consider the particular case that X is a "truncated symmetric stable" process. That is the case when $\sigma := 0$ and the Lévy measure satisfies $m(\mathrm{d}z) = |z|^{-(1+\alpha)} \mathbf{1}_{(-1,1)}(z)\, dz$ with $1 < \alpha < 2$. In this case,

$$(\mathscr{L} f)(x) = \int_0^1 \left[\frac{f(x+z) - f(x-z) - 2f(x)}{z^{1+\alpha}} \right] \mathrm{d}z, \tag{1.10}$$

and (1.2) holds because

$$\Psi(\xi) = 2 \int_0^1 \frac{1 - \cos(\xi z)}{z^{1+\alpha}}\, \mathrm{d}z = (2 + o(1)) \int_0^\infty \frac{1 - \cos r}{r^{1+\alpha}}\, \mathrm{d}r \cdot |\xi|^\alpha, \tag{1.11}$$

as $|\xi| \to \infty$. More interesting examples can be found within the constructions of Rosiński [28] and Houdré and Kawai [22].

There are concrete instances where one can improve the results of Theorem 1.1, thereby establish quite good estimates for $\underline{\lambda}(2)$ and $\overline{\lambda}(2)$. The following typifies a good example, in which $\mathscr{L}$ is a constant multiple of the Laplacian.

Theorem 1.3 *If $\mathscr{L} f = \frac{\kappa}{2} f''$ and u_0 is lower semicontinuous and has a compact support of positive measure, then Theorem 1.1 holds. In addition,*

$$\frac{\mathrm{L}_\sigma^2}{2\pi} \le \underline{\lambda}(2) \le \overline{\lambda}(2) \le \frac{\mathrm{Lip}_\sigma^2}{2} \quad \textit{for all } \kappa > 0. \tag{1.12}$$

In the case of the Parabolic Anderson Model [$\sigma(u) := \lambda u$], (1.12) tells us that $\lambda^2/2\pi \le \underline{\lambda}(2) \le \overline{\lambda}(2) \le \lambda^2/2$.

We know from Theorem 1.1 that the positions of the farthest peaks grow linearly with time. Theorem 1.3 describes an explicit interval in which the farthest high peaks necessarily fall. Moreover, this interval does not depend on the value of the diffusion coefficient κ. In intuitive terms, these remarks can be summed up as follows: "Any amount of noise leads to totally intermittent behavior." This observation was made, much earlier, in various physical contexts; see, for example, Zeldovich et al. [30, pp. 35–37].

We mention that the main ideas in the proofs of Theorems 1.1 and 1.3 apply also in other settings. For example, in Sect. 5 below we study a hyperbolic SPDE, and prove that $\underline{\lambda}(2) = \underline{\lambda}(\nu) = \overline{\lambda}(\nu) = \overline{\lambda}(2)$ for $\nu \geq 2$, under some regularity hypotheses. This implies the existence of a sharp phase transition between exponential growth and exponential decay of those hyperbolic SPDEs. Moreover, we will see that the intermittent behavior of the stochastic wave equation differs from (1.1) in two fundamental ways: (a) The variance of the noise affects the strength of intermittency; and (b) the rate of growth of σ does not.

We conclude the introduction with two questions that have eluded us.

Open problems 1. Is there a unique phase transition in the exponential growth of (1.1). In other words, we ask:

$$\text{Is } \underline{\lambda}(\nu) = \overline{\lambda}(\nu)?$$

Although we have no conjectures about this in the present setting of parabolic equations, Theorem 5.1 below answers this question affirmatively for some hyperbolic SPDEs.

2. Suppose $u_0 \in \mathscr{D}_{exp}$ and $\mathscr{L} = -(-\Delta)^{\alpha/2}$ denote the fractional Laplacian for some exponent $\alpha \in (1\,,2)$. Does $\sup_{x\in\mathbf{R}} \mathrm{E}(|u_t(x)|^2)$ grow exponentially with t? We mention the following related fact: It is possible to adapt the proof of [16, Theorem 2.1] to show that if $u_0 \in L^2(\mathbf{R})$, then $\int_{-\infty}^{\infty} \mathrm{E}(|u_t(x)|^2)\,\mathrm{d}x$ grows exponentially with t. The remaining difficulty is to establish "localization." The results of the present paper accomplish all this if the fractional Laplacian—which is the generator of a symmetric stable process—were replaced by the generator of a truncated symmetric stable process; see Remark 1.2.

Before proceeding to the proofs of Theorems 1.1 and 1.3, we introduce some notation. We write $\|\cdot\|_\nu$ the standard norm on $L^\nu(\mathrm{P})$. That is,

$$\|Y\|_\nu := \{\mathrm{E}(|Y|^\nu)\}^{1/\nu}, \qquad \text{for all} \nu \in [1\,,\infty) \text{and} Y \in L^\nu(\mathrm{P}).$$

We now recall the following form of Burkholder's inequality that will be used here and throughout.

Theorem 1.4 (The Burkholder–Davis–Gundy inequality [2–4]) *Let $\{M_t\}_{t\geq 0}$ be a continuous martingale. Then, for all $k \geq 1$ and for all $t > 0$ there exists a constant z_k such that*

$$\|M_t\|_k \leq z_k \|\langle M\rangle_t\|_{k/2}^{1/2}, \tag{1.13}$$

where $\langle M\rangle$ denotes the quadratic variation of M.

Throughout this paper, we always choose the constant z_k of Burkholder's inequality to denote the optimal constant in Burkholder's $L^k(\mathrm{P})$-inequality for continuous square-integrable martingales. The precise value of z_k involves the zeros of Hermite polynomials; see Davis [14].

By the Itô isometry, $z_2 = 1$. Carlen and Kree [5, Appendix] have shown that $z_k \leq 2\sqrt{k}$ for all $k \geq 2$, and moreover $z_k = (2 + o(1))\sqrt{k}$ as $k \to \infty$.

2 Proof of Theorem 1.1: upper bound

In this section we prove that $\overline{\lambda}(\nu) < \infty$ for all $\nu \in [2\,,\infty)$.

If $\nu_1 \le \nu_2$ are both in $[1\,,\infty)$, then by Jensen's inequality,

$$\limsup_{t\to\infty}\frac{1}{t}\sup_{|x|\ge\alpha t}\ln \mathrm{E}\left(|u_t(x)|^{\nu_1}\right) \le \frac{\nu_1}{\nu_2}\limsup_{t\to\infty}\frac{1}{t}\sup_{|x|\ge\alpha t}\ln \mathrm{E}\left(|u_t(x)|^{\nu_2}\right). \tag{2.1}$$

This leads to the inclusion

$$\begin{aligned}&\left\{\alpha>0:\ \limsup_{t\to\infty}\frac{1}{t}\sup_{|x|\ge\alpha t}\ln \mathrm{E}\left(|u_t(x)|^{\nu_2}\right)<0\right\}\\ &\quad\subseteq\left\{\alpha>0:\ \limsup_{t\to\infty}\frac{1}{t}\sup_{|x|\ge\alpha t}\ln \mathrm{E}\left(|u_t(x)|^{\nu_1}\right)<0\right\},\end{aligned} \tag{2.2}$$

and hence the inequality $\overline{\lambda}(\nu_1) \le \overline{\lambda}(\nu_2)$. Therefore, it suffices to prove the result in the case that ν is an even integer ≥ 2. Our method is motivated strongly by ideas of Lunardi [23] on optimal regularity of analytic semigroups.

Dalang's condition (1.2) implies that the Lévy process X has transition functions $p_t(x)$ [18, Lemma 8.1]; that is, for all measurable $f:\mathbf{R}\to\mathbf{R}_+$,

$$(P_t f)(x) := \mathrm{E}f(X_t) = \int_{-\infty}^{\infty} p_t(z)f(z)\,\mathrm{d}z \qquad \text{for all } t>0. \tag{2.3}$$

And Dalang's theory implies that the solution can be written in mild form, in the sense of Walsh [29], as

$$u_t(x) = (P_t u_0)(x) + \int_{[0,t]\times\mathbf{R}} p_{t-s}(y-x)\sigma(u_s(y))\,W(\mathrm{d}s\,\mathrm{d}y), \tag{2.4}$$

where $\{P_t\}_{t\ge0}$ denotes the semigroup associated to the process X. Henceforth, we will be concerned solely with the mild formulation of the solution, as given to us by (2.4).

The following implies part 1 of Theorem 1.1 immediately.

Proposition 2.1 *If* $\sup_{x\in\mathbf{R}}|e^{cx/2}u_0(x)|$ *and* $\mathrm{E}\exp(cX_1)$ *are both finite for some* $c\in\mathbf{R}$, *then for every even integer* $\nu\ge 2$ *and for all*

$$\beta > \ln \mathrm{E}e^{cX_1} + \frac{1}{2}\Upsilon^{-1}\left(\left(2z_\nu \mathrm{Lip}_\sigma\right)^{-2}\right), \tag{2.5}$$

there exists a finite constant $A_{\beta,\nu}$ *such that* $\mathrm{E}(|u_t(x)|^\nu) \le A_{\beta,\nu}\exp(\beta t - cx)$, *uniformly for all* $t\ge0$ *and* $x\in\mathbf{R}$.

Proposition 2.1 will be proved in Sect. 2.2.

Remark 2.2 The proof shows that we require only that $\sigma(0) = 0$; the positivity of L_σ—see (1.5)—is not required for this portion.

Remark 2.3 Proposition 2.1 can frequently be used to give an explicit bound on $\overline{\lambda}(\nu)$. For example, if $\mathrm{E}e^{c|X_1|} < \infty$ for all $c \in \mathbf{R}$ and u_0 has compact support, then Proposition 2.1 implies that

$$\limsup_{t\to\infty} \frac{1}{t} \ln \sup_{|x|\geq\alpha t} \mathrm{E}\left(|u_t(x)|^\nu\right) \leq -\Lambda(\alpha) + \frac{1}{2}\Upsilon^{-1}\left(\left(2z_\nu \mathrm{Lip}_\sigma\right)^{-2}\right), \tag{2.6}$$

where $\Lambda(\alpha) := \sup_{c\in\mathbf{R}}(\alpha c - \ln \mathrm{E}e^{cX_1})$ is the Legendre transformation of the logarithmic moment-generating function of X_1; see, for example, Dembo and Zeitouni [15]. Thus, the left-hand side of (2.6) is negative as soon as $\Lambda(\alpha) > \frac{1}{2}\Upsilon^{-1}((2z_\nu \mathrm{Lip}_\sigma)^{-2})$, and hence

$$\overline{\lambda}(\nu) \leq \inf\left\{\alpha > 0 :\ \Lambda(\alpha) > \frac{1}{2}\Upsilon^{-1}\left(\left(2z_\nu \mathrm{Lip}_\sigma\right)^{-2}\right)\right\}. \tag{2.7}$$

We do not know how to obtain useful explicit lower bounds for $\underline{\lambda}(\nu)$ in general. However, when $\mathscr{L}f = \frac{\kappa}{2}f''$, Theorem 1.3 contains more precise bounds for both indices $\underline{\lambda}(2)$ and $\overline{\lambda}(2)$.

2.1 Stochastic weighted Young inequalities

Proposition 2.1 is based on general principles that might be of independent interest. These results will also be used in Sect. 5 to study a family of hyperbolic SPDEs. Throughout this subsection, $\Gamma_t(x)$ defines a nonrandom measurable function on $(0\,,\infty) \times \mathbf{R}$, and Z a predictable random field [29, p. 292].

Consider the stochastic convolution

$$(\Gamma * Z\dot{W})_t(x) := \int_{[0,t]\times\mathbf{R}} \Gamma_{t-s}(y-x) Z_s(y)\, W(\mathrm{d}s\,\mathrm{d}y), \tag{2.8}$$

provided that it is defined in the sense of Walsh [29, Theorem 2.5]. According to the theory of Walsh, when it is defined, $\Gamma * Z\dot{W}$ defines a predictable random field. We study its $L^\nu(\mathrm{P})$ norm next.

Lemma 2.4 *For all even integers $\nu \geq 2$, $t \geq 0$, and $x \in \mathbf{R}$,*

$$\left\|(\Gamma * Z\dot{W})_t(x)\right\|_\nu \leq z_\nu \left(\int_{[0,t]\times\mathbf{R}} \Gamma^2_{t-s}(y-x)\, \|Z_s(y)\|^2_\nu\, \mathrm{d}s\, \mathrm{d}y\right)^{1/2}, \tag{2.9}$$

where z_ν was defined in Theorem 1.4.

Proof For fixed $t > 0$ and $x \in \mathbf{R}$, we apply Burkholder's inequality (Theorem 1.4) to the martingale

$$r \mapsto \int_{[0,r]\times\mathbf{R}} \Gamma_{t-s}(y-x)Z_s(y)\, W(\mathrm{d}s\,\mathrm{d}y), \tag{2.10}$$

which has quadratic variation given by

$$r \mapsto \int_{[0,r]\times\mathbf{R}} \Gamma^2_{t-s}(y-x)Z_s(y)^2\,\mathrm{d}s\,\mathrm{d}y. \tag{2.11}$$

We let $r = t$ to obtain

$$\begin{aligned}\left\|(\Gamma * Z\dot{W})_t(x)\right\|_\nu^\nu &\le z_\nu^\nu \mathrm{E}\left(\left|\int_{[0,t]\times\mathbf{R}} \Gamma^2_{t-s}(y-x)Z_s(y)^2\,\mathrm{d}s\,\mathrm{d}y\right|^{\nu/2}\right)\\ &= z_\nu^\nu \mathrm{E}\left(\int_{([0,t]\times\mathbf{R})^{\nu/2}} \prod_{j=1}^{\nu/2} \Gamma^2_{t-s_j}(y_j-x)|Z_{s_j}(y_j)|^2\,\mathrm{d}\boldsymbol{s}\,\mathrm{d}\boldsymbol{y}\right).\end{aligned} \tag{2.12}$$

The generalized Hölder inequality implies that

$$\mathrm{E}\left(\prod_{j=1}^{\nu/2} |Z_{s_j}(y_j)|^2\right) \le \prod_{j=1}^{\nu/2} \|Z_{s_j}(y_j)\|_\nu^2, \tag{2.13}$$

and the result follows. □

We say that $\vartheta : \mathbf{R} \to \mathbf{R}_+$ is a *weight* when ϑ is measurable and

$$\vartheta(a+b) \le \vartheta(a)\vartheta(b) \qquad \text{for all } a, b \in \mathbf{R}. \tag{2.14}$$

As usual, the weighted L^2-space $L^2_\vartheta(\mathbf{R})$ denotes the collection of all measurable functions $h : \mathbf{R} \to \mathbf{R}$ such that $\|h\|_{L^2_\vartheta(\mathbf{R})} < \infty$, where

$$\|h\|^2_{L^2_\vartheta(\mathbf{R})} := \int_{-\infty}^{\infty} |h(x)|^2\, \vartheta(x)\,\mathrm{d}x. \tag{2.15}$$

Define, for all predictable processes v, $\nu \in [1\,,\infty)$, and $\beta > 0$,

$$\mathcal{N}_{\beta,\nu,\vartheta}(v) := \left[\sup_{t\ge 0}\sup_{x\in\mathbf{R}} \mathrm{e}^{-\beta t}\vartheta(x)\,\|v_t(x)\|_\nu^2\right]^{1/2}. \tag{2.16}$$

Proposition 2.5 (A stochastic Young inequality) *For all weights ϑ, all $\beta > 0$, and all even integers $\nu \geq 2$,*

$$\mathcal{N}_{\beta,\nu,\vartheta}(\Gamma * Z\dot{W}) \leq z_\nu \left(\int_0^\infty e^{-\beta t} \, \|\Gamma_t\|^2_{L^2_\vartheta(\mathbf{R})} \, \mathrm{d}t\right)^{1/2} \cdot \mathcal{N}_{\beta,\nu,\vartheta}(Z). \tag{2.17}$$

Proof We apply Lemma 2.4 together with (2.14) to find that

$$\begin{aligned} &\mathrm{e}^{-\beta t}\vartheta(x) \left\|(\Gamma * Z\dot{W})_t(x)\right\|_\nu^2 \\ &\leq z_\nu^2 \int_{[0,t]\times\mathbf{R}} \mathrm{e}^{-\beta(t-s)}\vartheta(y-x)\Gamma_{t-s}^2(y-x)\,\mathrm{e}^{-\beta s}\vartheta(y)\,\|Z_s(y)\|_\nu^2 \,\mathrm{d}s\,\mathrm{d}y \\ &\leq z_\nu^2 \left|\mathcal{N}_{\beta,\nu,\vartheta}(Z)\right|^2 \cdot \int_{[0,t]\times\mathbf{R}} \mathrm{e}^{-\beta r}\vartheta(z)\Gamma_r^2(z)\,\mathrm{d}r\,\mathrm{d}z. \end{aligned} \tag{2.18}$$

The proposition follows from optimizing this expression over all $t \geq 0$ and $x \in \mathbf{R}$. □

Proposition 2.6 *If* $\mathrm{E}\exp(cX_1) < \infty$ *for some* $c \in \mathbf{R}$, *then for all predictable random fields* Z, *all* $\beta > \ln \mathrm{E}e^{cX_1}$, *and all even integers* $\nu \geq 2$,

$$\mathcal{N}_{\beta,\nu,\vartheta_c}(p * Z\dot{W}) \leq z_\nu \left(2\Upsilon\left(2\beta - 2\ln \mathrm{E}e^{cX_1}\right)\right)^{1/2} \cdot \mathcal{N}_{\beta,\nu,\vartheta_c}(Z), \tag{2.19}$$

where $\vartheta_c(x) := \exp(cx)$.

Proof If ϑ is an arbitrary weight, then $\|p_t\|^2_{L^2_\vartheta(\mathbf{R})} \leq \sup_{z\in\mathbf{R}} p_t(z)\cdot \mathrm{E}\,\vartheta(X_t)$. According to the inversion formula,

$$\sup_{z\in\mathbf{R}} p_t(z) \leq \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-t\mathrm{Re}\Psi(\xi)}\,\mathrm{d}\xi, \tag{2.20}$$

whence

$$\int_0^\infty \mathrm{e}^{-\beta t}\|p_t\|^2_{L^2_\vartheta(\mathbf{R})}\,\mathrm{d}t \leq \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\int_0^\infty \mathrm{d}t\;\mathrm{e}^{-t(\beta+\mathrm{Re}\Psi(\xi))}\mathrm{E}\,\vartheta(X_t). \tag{2.21}$$

The preceding is valid for all weights ϑ. Now consider the following special case of $\vartheta := \vartheta_c$. Clearly, this is a weight and, in addition, by standard facts about Lévy processes,

$$\mathrm{E}\,\vartheta_c(X_t) = \left(\mathrm{E}e^{cX_1}\right)^t. \tag{2.22}$$

Consequently, for all $\beta > M(c) := \ln \mathrm{E}\mathrm{e}^{cX_1}$,

$$\int_0^\infty \mathrm{e}^{-\beta t}\|p_t\|^2_{L^2_{\vartheta_c}(\mathbf{R})}\,\mathrm{d}t \le \frac{1}{2\pi}\int_{-\infty}^\infty \mathrm{d}\xi \int_0^\infty \mathrm{d}t\ \mathrm{e}^{-t(\beta+\mathrm{Re}\Psi(\xi)-M(c))} = 2\Upsilon(2\beta-2M(c)). \tag{2.23}$$

Proposition 2.5 completes the proof. □

Lemma 2.7 *For all weights ϑ, all $\beta > 0$, and all even integers $\nu \ge 2$,*

$$\mathcal{N}_{\beta,\nu,\vartheta}(P_\bullet u_0) \le \mathcal{N}_{\beta,\nu,\vartheta}(u_0)\cdot \sup_{t\ge 0}\left(e^{-\beta t}\mathrm{E}\,\vartheta(X_t)\right)^{1/2}, \tag{2.24}$$

where $P_\bullet u_0$ stands for the function $t \mapsto (P_t u_0)(x)$. In particular, if $\mathrm{E}e^{cX_1} < \infty$ for some $c \in \mathbf{R}$, then for all $\beta > \ln \mathrm{E}e^{cX_1}$,

$$\mathcal{N}_{\beta,\nu,\vartheta_c}(P_\bullet u_0) \le \mathcal{N}_{\beta,\nu,\vartheta_c}(u_0). \tag{2.25}$$

Proof Thanks to (2.14),

$$|\vartheta(x)|^{1/2}(P_t u_0)(x) \le \int_{-\infty}^\infty |\vartheta(y-x)|^{1/2}p_t(y-x)|\vartheta(y)|^{1/2}u_0(y)\,\mathrm{d}y \le \sup_{y\in\mathbf{R}}\left[|\vartheta(y)|^{1/2}u_0(y)\right]\cdot \mathrm{E}\left(|\vartheta(X_t)|^{1/2}\right). \tag{2.26}$$

This and the Cauchy–Schwarz inequality together imply (2.24), and the remainder of the lemma follows from (2.22). □

2.2 Proof of Proposition 2.1

We begin by studying the Picard-scheme approximation to the solution u. Namely, let $u_t^{(0)}(x) := u_0(x)$, and then define iteratively

$$u_t^{(n+1)}(x) := (P_t u_0)(x) + \left(p * \left(\sigma \circ u^{(n)}\right)\dot{W}\right)_t(x), \tag{2.27}$$

for $t > 0$, $x \in \mathbf{R}$, and $n \ge 0$, where the stochastic convolution is defined in (2.8). Clearly,

$$\left\|u_t^{(n+1)}(x)\right\|_\nu \le |(P_t u_0)(x)| + \left\|\left(p * \left(\sigma \circ u^{(n)}\right)\dot{W}\right)_t(x)\right\|_\nu, \tag{2.28}$$

whence for all $\beta > \ln \mathrm{E}e^{cX_1}$,

$$\mathcal{N}_{\beta,\nu,\vartheta_c}\left(u^{(n+1)}\right) \le \mathcal{N}_{\beta,\nu,\vartheta_c}(u_0) + z_\nu \mathrm{Lip}_\sigma \mathscr{T}^{1/2}\cdot \mathcal{N}_{\beta,\nu,\vartheta_c}\left(u^{(n)}\right), \tag{2.29}$$

where $\mathcal{T} := 2\Upsilon(2\beta - 2\ln \mathrm{E}e^{cX_1})$; see Proposition 2.6 and Lemma 2.7. Condition (2.5) is equivalent to the inequality $z_\nu^2 \mathrm{Lip}_\sigma^2 \mathcal{T} < 1$. Therefore, it follows from iteration that the quantity $\mathcal{N}_{\beta,\nu,\vartheta_c}(u^{(n+1)})$ is bounded uniformly in n, for this choice of β. Dalang's theory [10, Theorem 13 and its proof] tells us that $\lim_{n\to\infty} u_t^{(n)}(x) = u_t(x)$ in probability for all $t \geq 0$ and $x \in \mathbf{R}$. Therefore, Fatou's lemma implies that $\mathcal{N}_{\beta,\nu,\vartheta_c}(u) < \infty$ when $\beta > \ln \mathrm{E}e^{cX_1}$. This completes the proof of Proposition 2.1 [and hence part 1 of Theorem 1.1]. □

3 Proof of Theorem 1.1: lower bound

Our present, and final, goal is to prove that for all $\nu \in [2\,,\infty)$, whenever $0 < \alpha$ is sufficiently small, $\limsup_{t\to\infty} t^{-1} \sup_{|x|>\alpha t} \ln \|u_t(x)\|_\nu > 0$. By Jensen's inequality, it suffices to prove this in the case that $\nu = 2$. We will borrow liberally several localization ideas from two related papers by Mueller [24] and Mueller and Perkins [25].

Define, for all predictable random fields v, and $\alpha, \beta > 0$,

$$\mathcal{M}_{\alpha,\beta}(v) := \left[\int_0^\infty e^{-\beta t}\,\mathrm{d}t \int_{\substack{x\in\mathbf{R}:\\ |x|\geq\alpha t}} \mathrm{d}x\ \|v_t(x)\|_2^2\right]^{1/2}. \tag{3.1}$$

Thus, $\{\mathcal{M}_{\alpha,\beta}\}_{\alpha,\beta>0}$ defines a family of norms on the family of predictable random fields.

Proposition 3.1 *If* $\mathrm{E}|X_1| < \infty$, *then* $\mathcal{M}_{\alpha,\beta}(u) = \infty$ *for all sufficiently small* $\alpha, \beta > 0$.

Proof Thanks to (2.4) and the Itô isometry for stochastic integrals,

$$\|u_t(x)\|_2^2 \geq |(P_t u_0)(x)|^2 + \mathrm{L}_\sigma^2 \cdot \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ |p_{t-s}(y-x)|^2\, \|u_s(y)\|_2^2. \tag{3.2}$$

Let us define

$$\mathcal{M}_{\alpha,\beta}^+(v) := \left[\int_0^\infty e^{-\beta t}\,\mathrm{d}t \int_{\substack{x\in\mathbf{R}:\\ x\geq\alpha t}} \mathrm{d}x\ \|v_t(x)\|_2^2\right]^{1/2}, \tag{3.3}$$

and

$$\mathcal{M}_{\alpha,\beta}^-(v) := \left[\int_0^\infty e^{-\beta t}\,\mathrm{d}t \int_{\substack{x\in\mathbf{R}:\\ x\leq-\alpha t}} \mathrm{d}x\ \|v_t(x)\|_2^2\right]^{1/2}. \tag{3.4}$$

If $x, y \in \mathbf{R}$ and $0 \le s \le t$, then the triangle inequality implies that

$$\mathbf{1}_{[\alpha t,\infty)}(x) \ge \mathbf{1}_{[\alpha(t-s),\infty)}(x-y) \cdot \mathbf{1}_{[\alpha s,\infty)}(y). \tag{3.5}$$

For all $r \ge 0$, let

$$T_\alpha^+(r) := \int_{\substack{z\in\mathbf{R}:\\ z\ge\alpha r}} |p_r(z)|^2\,\mathrm{d}z, \quad T_\alpha^-(r) := \int_{\substack{z\in\mathbf{R}:\\ z\le-\alpha r}} |p_r(z)|^2\,\mathrm{d}z, \tag{3.6}$$

and

$$S_\alpha^+(r) := \int_{\substack{y\in\mathbf{R}:\\ y\ge\alpha r}} \|u_r(y)\|_2^2\,\mathrm{d}y, \quad S_\alpha^-(r) := \int_{\substack{y\in\mathbf{R}:\\ y\le-\alpha r}} \|u_r(y)\|_2^2\,\mathrm{d}y. \tag{3.7}$$

According to (3.5),

$$\int_{x\ge\alpha t} \|u_t(x)\|_2^2\,\mathrm{d}x \ge \int_{x\ge\alpha t} |(P_t u_0)(x)|^2\,\mathrm{d}x + \mathrm{L}_\sigma^2 \cdot (T_\alpha^- * S_\alpha^+)(t), \tag{3.8}$$

where "$*$" denotes the usual convolution on $\mathbf{R}_+$.

We multiply both sides of (3.8) by $\exp(-\beta t)$ and integrate $[\mathrm{d}t]$ to find

$$\begin{aligned}\left|\mathscr{M}_{\alpha,\beta}^+(u)\right|^2 &\ge \left|\mathscr{M}_{\alpha,\beta}^+(P_\bullet u_0)\right|^2 + \mathrm{L}_\sigma^2 \cdot \widetilde{T}_\alpha^-(\beta)\widetilde{S}_\alpha^+(\beta)\\ &= \left|\mathscr{M}_{\alpha,\beta}^+(P_\bullet u_0)\right|^2 + \mathrm{L}_\sigma^2 \cdot \widetilde{T}_\alpha^-(\beta)\left|\mathscr{M}_{\alpha,\beta}^+(u)\right|^2,\end{aligned} \tag{3.9}$$

where $\widetilde{H}(\beta) := \int_0^\infty \exp(-\beta t)H(t)\,\mathrm{d}t$ defines the Laplace transform of H for every measurable function $H : \mathbf{R}_+ \to \mathbf{R}_+$. Also, we can apply a similar argument, run on the negative half of the real line, to deduce that

$$\left|\mathscr{M}_{\alpha,\beta}^-(u)\right|^2 \ge \left|\mathscr{M}_{\alpha,\beta}^-(P_\bullet u_0)\right|^2 + \mathrm{L}_\sigma^2 \cdot \widetilde{T}_\alpha^+(\beta)\left|\mathscr{M}_{\alpha,\beta}^-(u)\right|^2. \tag{3.10}$$

Next we add the inequalities (3.9) and (3.10): Because $\{X_t\}_{t\ge0}$ is symmetric, $\tilde{T}_\alpha^+(\beta) = \tilde{T}_\alpha^-(\beta)$; and it is easy to see that $\mathscr{M}_{\alpha,\beta}(u)^2 = \mathscr{M}_{\alpha,\beta}^+(u)^2 + \mathscr{M}_{\alpha,\beta}^-(u)^2$. Therefore, we can conclude that

$$\left|\mathscr{M}_{\alpha,\beta}(u)\right|^2 \ge \left|\mathscr{M}_{\alpha,\beta}(P_\bullet u_0)\right|^2 + \mathrm{L}_\sigma^2 \cdot \widetilde{T}_\alpha^+(\beta)\left|\mathscr{M}_{\alpha,\beta}(u)\right|^2. \tag{3.11}$$

Next we may observe that

$$|\mathscr{M}_{\alpha,\beta}(P_\bullet u_0)| > 0. \tag{3.12}$$

This holds because $u_0 \geq 0$, $u_0 > 0$ on a set of positive measure, and u_0 is lower semicontinuous. Indeed, if it were not so, then $\int_{|x|\geq \alpha t}(P_t u_0)(x)\,\mathrm{d}x = 0$ for almost all, hence all, $t > 0$. But then we would let $t \to 0$ to deduce from this and Fatou's lemma that $\int_{-\infty}^{\infty} u_0(x)\,\mathrm{d}x = 0$, which is a contradiction.

The preceding development implies the following:

$$\text{If } \mathscr{M}_{\alpha,\beta}(u) < \infty, \quad \text{then } \widetilde{T}_\alpha^+(\beta) < \mathrm{L}_\sigma^{-2}. \tag{3.13}$$

The symmetric Lévy process X is recurrent iff

$$\Upsilon(0^+) = \infty. \tag{3.14}$$

See, for example, Port and Stone [27, Sect. 16]. Therefore it remains to prove that the conditions of Theorem 1.1 imply (3.14).

The discrete-time process $\{X_n\}_{n=1}^{\infty}$ is a one-dimensional mean-zero [in fact symmetric] random walk, which is necessarily recurrent thanks to the Chung–Fuchs theorem [8]. Consequently, the Lévy process $\{X_t\}_{t\geq 0}$ is recurrent as well. Thanks to the preceding paragraph, (3.14) holds.

By the monotone convergence theorem,

$$\lim_{\alpha\downarrow 0} \widetilde{T}_\alpha(\beta) = \frac{1}{2}\int_0^\infty \mathrm{e}^{-\beta t}\|p_t\|_{L^2(\mathbf{R})}^2\,\mathrm{d}t = \frac{1}{2}\Upsilon(\beta) \qquad \text{for all } \beta > 0. \tag{3.15}$$

[The second identity follows from Plancherel's theorem.] Let $\beta \downarrow 0$ and appeal to (3.14) to conclude that $\widetilde{T}_\alpha^+(\beta) > \mathrm{L}_\sigma^{-2}$ for all sufficiently-small positive α and β. In light of (3.13), this completes our demonstration. □

Proof (of Part 2 of Theorem 1.1) Choose and fix α and β positive, but so small that $\mathscr{M}_{\alpha,\beta}(u) = \infty$ [Proposition 3.1]. According to Proposition 2.1, for all fixed $\alpha' > 0$,

$$\int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t \int_{|x|\geq\alpha' t} \mathrm{d}x\, \|u_t(x)\|_2^2 \leq A_{\alpha',2}\int_0^\infty \mathrm{e}^{(\beta'-\beta)t}\,\mathrm{d}t \int_{|x|\geq\alpha' t}\mathrm{d}x\,\mathrm{e}^{-c|x|},$$

provided that β' [in place of the variable β there] satisfies (2.5) with $\pm c$ [in place of the variable c there]. We choose and fix β' so large that the condition (2.5) is satisfied for β'. Then, choose and fix α' so large that the right-most integral in the preceding display is finite. Since $\mathscr{M}_{\alpha,\beta}(u) = \infty$, it follows from the preceding that

$$\int_0^\infty \mathrm{e}^{-\beta t}\,\mathrm{d}t \int_{\alpha t\leq|x|\leq\alpha' t}\mathrm{d}x\,\mathrm{E}\left(|u_t(x)|^2\right) = \infty. \tag{3.16}$$

Consequently,

$$\int_0^\infty t\mathrm{e}^{-\beta t} \sup_{|x|\geq\alpha t} \mathrm{E}\left(|u_t(x)|^2\right) \mathrm{d}t = \infty, \tag{3.17}$$

whence

$$\limsup_{t\to\infty} t^{-1} \sup_{|x|\geq\alpha t} \ln \mathrm{E}\left(|u_t(x)|^2\right) \geq \beta > 0 \tag{3.18}$$

for the present choice of α and β. This implies that $\underline{\lambda}(2) \geq \alpha > 0$. □

Remark 3.2 Theorem 1.1 requires less than the symmetry of the Lévy process $\{X_t\}_{t\geq 0}$. For instance, our proof continues to work provided that there exist finite and positive constants c_1 and c_2 such that

$$c_1 T_\alpha^-(r) \leq T_\alpha^+(r) \leq c_2 T_\alpha^-(r), \tag{3.19}$$

simultaneously for all $\alpha > 0$ and $r \geq 0$.

4 Proof of Theorem 1.3

Throughout the proof, we choose and fix some $\kappa > 0$. Thus, the operator $\mathscr{L} f = \frac{\kappa}{2} f''$ is the generator of a Lévy process given by $X_t = \sqrt{\kappa}\, B_t$, where $\{B_t\}_{t\geq 0}$ is a Brownian motion, and Theorem 1.1 obviously applies in this case. We now would like to prove the second claim of Theorem 1.3. We proceed as we did for Theorem 1.1, and divide the proof in two parts: One part is concerned with an upper bound for $\overline{\lambda}(2)$; and the other deals with a lower bound on $\underline{\lambda}(2)$.

4.1 Upper bound

In order to obtain an upper estimate for $\overline{\lambda}(2)$, we could follow the procedure outlined in Remark 2.3. But this turns out to be not optimal. In the case of Theorem 1.3, we know explicitly the transition functions $p_t^{(\kappa)}$:

$$p_t^{(\kappa)}(x) = \frac{1}{\sqrt{2\pi\kappa t}} \exp\left(-\frac{x^2}{2\kappa t}\right). \tag{4.1}$$

Therefore, we can use (4.1) directly and make exact computations in order to improve on the general bounds of Remark 2.3. We first prove the following; it sharpens Proposition 2.1 in the present setting.

Proposition 4.1 *If* $\mathscr{L} f = \frac{\kappa}{2} f''$ *and* $\sup_{x\in\mathbf{R}} |e^{cx/2}u_0(x)|$ *is finite for some* $c \in \mathbf{R}$, *then for every*

$$\beta > \frac{\kappa c^2}{4} + \frac{\mathrm{Lip}_\sigma^4}{4\kappa}, \tag{4.2}$$

there exists a finite constant A_β *such that* $\mathrm{E}(|u_t(x)|^2) \le A_\beta \exp(\beta t - cx)$, *uniformly for all* $t \ge 0$ *and* $x \in \mathbf{R}$.

Proof We follow the proof of Proposition 2.1, but use Proposition 2.5, instead of Proposition 2.6, in order to handle (2.28) better. Then, (2.29) is replaced by

$$\begin{aligned}&\mathscr{N}_{\beta,2,\vartheta_c}\left(u^{(n+1)}\right)\\ &\quad\le \mathscr{N}_{\beta,2,\vartheta_c}(u_0) + \mathrm{Lip}_\sigma \left(\int_0^\infty \mathrm{e}^{-\beta t} \left\| p_t^{(\kappa)} \right\|^2_{L^2_{\vartheta_c}(\mathbf{R})} \mathrm{d}t\right)^{1/2} \mathscr{N}_{\beta,2,\vartheta_c}\left(u^{(n)}\right).\end{aligned} \tag{4.3}$$

Next we complete the proof, in the same way we did for Proposition 2.1, and deduce that there exists a constant A_β such that $\mathrm{E}(|u_t(x)|^2) \le A_\beta \exp(\beta t - cx)$ uniformly for all $t \ge 0$ and $x \in \mathbf{R}$, provided that β is chosen to be large enough to satisfy

$$\mathrm{Lip}_\sigma^2 \cdot \int_0^\infty \mathrm{e}^{-\beta t} \left\| p_t^{(\kappa)} \right\|^2_{L^2_{\vartheta_c}(\mathbf{R})} \mathrm{d}t < 1. \tag{4.4}$$

Now we compute:

$$\begin{aligned}\left\| p_t^{(\kappa)} \right\|^2_{L^2_{\vartheta_c}(\mathbf{R})} &= \frac{1}{2\pi\kappa t} \int_{-\infty}^\infty \exp\left(-\frac{x^2}{\kappa t} + cx\right) \mathrm{d}x\\ &= \frac{1}{2\sqrt{\pi\kappa t}} \exp\left(\frac{\kappa c^2 t}{4}\right).\end{aligned} \tag{4.5}$$

Since $\int_0^\infty t^{-1/2} e^{-\beta t}\,\mathrm{d}t = \sqrt{\pi/\beta}$, we have the following for all $\beta > \kappa c^2/4$:

$$\mathrm{Lip}_\sigma^2 \cdot \int_0^\infty \mathrm{e}^{-\beta t} \left\| p_t^{(\kappa)} \right\|^2_{L^2_{\vartheta_c}(\mathbf{R})} \mathrm{d}t = \frac{1}{2}\mathrm{Lip}_\sigma^2 \left(\kappa\beta - \frac{\kappa^2 c^2}{4}\right)^{-1/2}. \tag{4.6}$$

And hence, (4.4) follows from (4.2). This proves Proposition 4.1. □

Proof (of the upper bound in Theorem 1.3) If u_0 has compact support, then the assumption of Proposition 4.1 is satisfied for all $c \in \mathbf{R}$. Consequently,

$$\limsup_{t\to\infty} \frac{1}{t} \ln \sup_{|x|\ge\alpha t} \mathrm{E}(|u_t(x)|^2) \le \beta - c\alpha, \tag{4.7}$$

and hence

$$\bar{\lambda}(2) \le \inf\{\alpha > 0 :\ \beta - c\alpha < 0\} = \frac{\beta}{c}. \tag{4.8}$$

This and (4.2) together imply that

$$\bar{\lambda}(2) \le \inf_{c \in \mathbf{R}} \left(\frac{\kappa c}{4} + \frac{\mathrm{Lip}_\sigma^4}{4\kappa c} \right) = \frac{\mathrm{Lip}_\sigma^2}{2}. \tag{4.9}$$

This concludes the proof of the upper bound. □

4.2 Lower bound

We first prove the following refinement of Proposition 3.1.

Proposition 4.2 *If $\mathscr{L} f = \frac{\kappa}{2} f''$ and α and β satisfy*

$$\left(\alpha - \frac{\mathrm{L}_\sigma^2}{4\pi} \right)^2 < \frac{\mathrm{L}_\sigma^4}{16\pi^2} - \kappa\beta, \tag{4.10}$$

then $\mathscr{M}_{\alpha,\beta}(u) = \infty$.

Proof In the case that we consider here, the Lévy process is a scaled Brownian motion. Hence, Proposition 3.1 applies, and in accord with (3.13), it suffices to prove the following:

$$\mathfrak{I} := \int_0^\infty \mathrm{e}^{-\beta t} \left(\int_{\substack{z \in \mathbf{R}: \\ z \ge \alpha t}} \left| p_t^{(\kappa)}(z) \right|^2 \mathrm{d}z \right) \mathrm{d}t > \mathrm{L}_\sigma^{-2}. \tag{4.11}$$

Let $\bar{\Phi}(z) := (2\pi)^{-1/2} \int_z^\infty \exp(-\tau^2/2)\, \mathrm{d}\tau$ for every $z \in \mathbf{R}$, then apply (4.1) and compute directly to find that

$$\begin{aligned} \mathfrak{I} &= \frac{1}{2\sqrt{\pi\kappa}} \int_0^\infty \frac{\mathrm{e}^{-\beta t}}{\sqrt{t}} \bar{\Phi}\left(\sqrt{\frac{2\alpha^2 t}{\kappa}} \right) \mathrm{d}t \\ &= \frac{\alpha}{4\pi\kappa} \int_0^\infty \frac{\mathrm{e}^{-\alpha^2 t/\kappa}}{\sqrt{t}} \left(\int_0^t \frac{\mathrm{e}^{-\beta s}}{\sqrt{s}}\, \mathrm{d}s \right) \mathrm{d}t, \end{aligned} \tag{4.12}$$

after we integrate by parts. Since $\mathrm{e}^{-\beta s} \ge \mathrm{e}^{-\beta t}$ for $s \le t$,

$$\mathfrak{I} \geq \frac{\alpha}{2\pi\kappa} \int_0^\infty \exp\left(-\left(\beta + \frac{\alpha^2}{\kappa}\right) t\right) \mathrm{d}t = \frac{\alpha}{2\pi(\beta\kappa + \alpha^2)}. \tag{4.13}$$

Hence, (4.10) implies (4.11), and hence the proposition. □

Remark 4.3 We notice that condition (4.10) is sufficient but not necessary. Indeed, as $\mathfrak{I}$ is decreasing in α, only the upper bound implied by (4.10) is relevant. Typically, (4.11) is satisfied for $\alpha = 0$.

Proof (of the lower bound in Theorem 1.3) The second part of the proof of Theorem 1.1 shows that $\underline{\lambda}(2) \geq \alpha$, provided that we choose α and β such that $\mathscr{M}_{\alpha,\beta}(u) = \infty$. In accordance with (4.10), and after maximizing over $\beta \leq \mathrm{L}_\sigma^4/(16\pi^2\kappa)$—that is, making β as small as possible—we obtain $\underline{\lambda}(2) \geq \alpha \geq \mathrm{L}_\sigma^2/2\pi$. This concludes the proof of Theorem 1.3. □

5 A nonlinear stochastic wave equation

In this section, we study the nonlinear stochastic wave equation

$$\frac{\partial^2}{\partial t^2} u_t(x) = \kappa^2 \left(\frac{\partial^2}{\partial x^2} u_t\right)(x) + \sigma(u_t(x)) \frac{\partial^2}{\partial t \partial x} W(t, x) \quad \text{for} \quad t > 0,\ x \in \mathbf{R}, \tag{5.1}$$

where: (i) $\sigma : \mathbf{R} \to \mathbf{R}$ is Lipschitz continuous with Lipschitz constant Lip_σ; (ii) W is two-parameter Brownian sheet, indexed by $(t, x) \in \mathbf{R}_+ \times \mathbf{R}$; (iii) the initial function $u_0 : \mathbf{R} \to \mathbf{R}_+$ and the initial derivative $v_0 : \mathbf{R} \to \mathbf{R}$ are both in $L^\infty(\mathbf{R})$; and (iv) $\kappa > 0$. In the present one-dimensional setting, the nonlinear equation (5.1) has been studied by Carmona and Nualart [7] and Walsh [29]. There are also results available in the more delicate setting where $x \in \mathbf{R}^d$ for $d > 1$; see Conus and Dalang [9], Dalang [10], Dalang and Frangos [11], and Dalang and Mueller [12].

It is well known that the fundamental solution for the wave equation in spatial dimension 1 is

$$\Gamma_t(x) := \frac{1}{2}\mathbf{1}_{[-\kappa t, \kappa t]}(x) \qquad \text{for } t > 0 \quad \text{and} \quad x \in \mathbf{R}. \tag{5.2}$$

According to the theory of Dalang [10], the stochastic wave Eq. (5.1) has an a.s.-unique mild solution. In the case that u_0 and v_0 are both constant functions, Dalang and Mueller [13] have shown that the solution to (5.1) is intermittent.

In this section we will use the stochastic weighted Young inequalities of Sect. 2.1 in order to deduce the weak intermittence of the solution to (5.1) for nonconstant functions u_0 and v_0. And more significantly, when u_0 and v_0 have compact support, we describe the precise rate at which the farthest peaks can move away from the origin.

Here and throughout, we assume that (1.5) holds, and define $\overline{\lambda}(\nu)$ and $\underline{\lambda}(\nu)$ as in (1.6) and (1.7).

Theorem 5.1 *If $u_0, v_0 \in \mathscr{D}_{exp}$, $u_0 > 0$ on a set of positive measure and $v_0 \geq 0$, then $0 < \underline{\lambda}(\nu) \leq \overline{\lambda}(\nu) < \infty$ for all $\nu \in [2, \infty)$. If, in addition, u_0 and v_0 have compact support, then $\underline{\lambda}(\nu) = \overline{\lambda}(\nu) = \kappa$ for all $\nu \in [2, \infty)$.*

Theorem 5.1 implies the weak intermittence of the solution to (5.1). And more significantly, it tells that when the initial data have compact support, we have a sharp phase transition $[\underline{\lambda}(\nu) = \overline{\lambda}(\nu) = \kappa]$: The solution has exponentially-large peaks inside $[-\kappa t + o(t), \kappa t + o(t)]$, and is exponentially small everywhere outside $[-\kappa t + o(t), \kappa t + o(t)]$. In particular, the farthest high peaks of the solution travel at sharp linear speed $\pm\kappa t + o(t)$. This speed corresponds to the speed of the traveling waves if we consider the deterministic equivalent of (5.1) [say, when $\sigma \equiv 0$]. We emphasize that, contrary to what happens in the stochastic heat equation (Theorem 1.3), the growth behavior of the solution to the stochastic wave Eq. (5.1) depends on the size of the noise (that is, the magnitude of κ), but not on the growth rate of the nonlinearity σ.

5.1 Proof of Theorem 5.1: upper bound

The proof of Theorem 5.1 follows closely those of Theorems 1.1 and 1.3.

We first show that $\overline{\lambda}(\nu) < \infty$. The solution to (5.1) can be written in mild form, as

$$u_t(x) = U_t^{(0)}(x) + V_t^{(0)}(x) + \int\limits_{[0,t]\times\mathbf{R}} \Gamma_{t-s}(y-x)\sigma(u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y), \tag{5.3}$$

where $U_t^{(0)}(x) = \frac{1}{2}(u_0(x+\kappa t) + u_0(x-\kappa t))$ and $V_t^{(0)}(x) = \frac{1}{2\kappa}\int_{x-\kappa t}^{x+\kappa t} v_0(y)\,\mathrm{d}y$.

The following Proposition implies immediately that $\overline{\lambda}(\nu) < \infty$ for $\nu \geq 2$.

Proposition 5.2 *Let $\nu \geq 2$ be an even integer, and assume that $\sup_{x\in\mathbf{R}} |e^{cx/2}u_0(x)|$ and $\sup_{x\in\mathbf{R}} |e^{cx/2}v_0(x)|$ are both finite for some $c \in \mathbf{R}$. Then for every*

$$\beta > \sqrt{\kappa^2 c^2 + \frac{z_\nu^2 \mathrm{Lip}_\sigma^2}{2}}, \tag{5.4}$$

there exists a finite constant A_β such that $\mathrm{E}(|u_t(x)|^\nu) \leq A_\beta \exp(\beta t - cx)$, uniformly for all $t \geq 0$ and $x \in \mathbf{R}$.

In order to prove Proposition 5.2, we will need the following Lemma. Let ϑ_c and $\mathscr{N}_{\beta,\nu,\vartheta}$ be defined as they were in Sect. 2.1.

Lemma 5.3 *For all $c \in \mathbf{R}$, $\beta > \kappa|c|/2$, and even integers $\nu \geq 2$,*

$$\mathscr{N}_{\beta,\nu,\vartheta_c}(U^{(0)}) \leq \mathscr{N}_{\beta,\nu,\vartheta_c}(u_0) \quad \text{and} \quad \mathscr{N}_{\beta,\nu,\vartheta_c}(V^{(0)}) \leq \frac{1}{\kappa c}\mathscr{N}_{\beta,\nu,\vartheta_c}(v_0). \tag{5.5}$$

Proof The first inequality of (5.5) follows from the definition of $U^{(0)}$. As regards the second, we have

$$\begin{aligned} e^{cx/2} V_t^{(0)}(x) &\leq \left(\sup_{y \in \mathbf{R}} e^{cy/2} v_0(y) \right) \frac{e^{cx/2}}{2\kappa} \int_{x-\kappa t}^{x+\kappa t} e^{-cy/2}\, dy \\ &\leq \frac{e^{c\kappa t/2}}{\kappa c} \left(\sup_{y \in \mathbf{R}} e^{cy/2} v_0(y) \right). \end{aligned} \tag{5.6}$$

Because $\beta > \kappa|c|/2$, this proves the lemma. □

Proof (of Proposition 5.2) As in the proof of Proposition 4.1, we apply a Picard-iteration scheme to approximate the solution u. Then, Lemma 5.3 and Proposition 2.5 yield

$$\begin{aligned} \mathcal{N}_{\beta,\nu,\vartheta_c}\left(u^{(n+1)}\right) &\leq \mathcal{N}_{\beta,\nu,\vartheta_c}(u_0) + \frac{1}{\kappa c} \mathcal{N}_{\beta,\nu,\vartheta_c}(v_0) \\ &\quad + z_\nu \mathrm{Lip}_\sigma \left(\int_0^\infty e^{-\beta t} \|\Gamma_t\|^2_{L^2_{\vartheta_c}(\mathbf{R})}\, dt \right)^{\frac{1}{2}} \cdot \mathcal{N}_{\beta,\nu,\vartheta_c}\left(u^{(n)}\right). \end{aligned} \tag{5.7}$$

A direct computation, using only (5.2), shows that

$$\int_0^\infty e^{-\beta t} \|\Gamma_t\|^2_{L^2_{\vartheta_c}(\mathbf{R})}\, dt < (z_\nu \mathrm{Lip}_\sigma)^{-2}. \tag{5.8}$$

And the same arguments that were used in the proof of Proposition 2.5 can be used to deduce from this bound that $\mathcal{N}_{\beta,\nu,\vartheta_c}(u)$ is finite. Now we use (5.2) in order to see that this condition is equivalent to (5.4). This concludes the proof of Proposition 5.2. □

Proof (of the upper bound in Theorem 5.1) Proposition 5.2 implies that $\overline{\lambda}(\nu) < \infty$. Now suppose u_0 and v_0 have compact support. In that case, c is an arbitrary real number. And similar arguments as in the proof of the upper bound of Theorem 1.3 imply that $\overline{\lambda}(\nu) \leq \beta/c$. Together with (4.2), this leads to the following estimate:

$$\overline{\lambda}(\nu) \leq \inf_{c \in \mathbf{R}} \sqrt{\kappa^2 + \frac{2 z_\nu^2 \mathrm{Lip}_\sigma^2}{c^2}} = \kappa. \tag{5.9}$$

This proves half of the theorem. □

5.2 Proof of Theorem 5.1: lower bound

The following proposition implies the requisite bound for the second half of the proof of Theorem 5.1; namely, that $\underline{\lambda}(\nu) > 0$ for $\nu \geq 2$. Let $\mathcal{M}_{\alpha,\beta}$ be defined as in (3.1).

Proposition 5.4 *$\mathscr{M}_{\alpha,\beta}(u) = \infty$ provided that*

$$0 < \alpha < \kappa - \frac{4\beta^2}{\mathrm{L}_\sigma^2}. \tag{5.10}$$

Proof Similar arguments as in the proof of Proposition 3.1 show that

$$|\mathscr{M}_{\alpha,\beta}(u)|^2 \geq |\mathscr{M}_{\alpha,\beta}(U^{(0)} + V^{(0)})|^2 + \mathrm{L}_\sigma^2 \cdot \tilde{T}_\alpha^+(\beta)|\mathscr{M}_{\alpha,\beta}(u)|^2, \tag{5.11}$$

where $\tilde{T}_\alpha^+(\beta)$ denotes the Laplace transform of $T_\alpha^+(r) := \int_{z \geq \alpha r} |\Gamma_r(z)|^2 \, \mathrm{d}z$. Since $u_0 > 0$ on a set of positive measure and $v_0 \geq 0$, we have $|\mathscr{M}_{\alpha,\beta}(U^{(0)} + V^{(0)})| > 0$. This shows that if $\mathrm{L}_\sigma^2 \cdot \tilde{T}_\alpha^+(\beta) > 1$, then $\mathscr{M}_{\alpha,\beta}(u) = \infty$. A direct computation reveals that

$$\tilde{T}_\alpha^+(\beta) = \begin{cases} (\kappa - \alpha)/(4\beta^2) & \text{if } \alpha \leq \kappa, \\ 0 & \text{otherwise.} \end{cases} \tag{5.12}$$

Hence, $\mathscr{M}_{\alpha,\beta}(u) = \infty$ if $\tilde{T}_\alpha^+(\beta) > \mathrm{Lip}_\sigma^{-2}$, and the latter condition is equivalent to (5.10). Since we also want $\alpha > 0$, Proposition 5.4 follows. □

Proof (of the lower bound in Theorem 5.1) For every α such that $\mathscr{M}_{\alpha,\beta}(u) = \infty$, we can apply the same arguments as in the proof of the lower bound of Theorem 1.1 in order to conclude that $\underline{\lambda}(2) \geq \alpha > 0$. Now, Proposition 5.4 shows that $\underline{\lambda}(2) \geq \kappa - 4\beta^2/\mathrm{L}_\sigma^2$ for all $\beta > 0$, whence $\underline{\lambda}(2) \geq \kappa$. Jensen's inequality then shows that $\underline{\lambda}(\nu) \geq \kappa$ as well. □

Remark 5.5 The condition $v_0 \geq 0$ is not necessary in Theorem 5.1. Indeed, the necessary condition is

$$\mathscr{M}_{\alpha,\beta}(U^{(0)} + V^{(0)}) > 0. \tag{5.13}$$

The easy-to-verify conditions on u_0 and v_0 [in Theorem 5.1] imply (5.13).

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