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Heavy tail phenomenon and convergence to stable laws for iterated Lipschitz maps

Mariusz Mirek

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Abstract We consider the Markov chain $\{X_n^x\}_{n=0}^{\infty}$ on $\mathbb{R}^d$ defined by the stochastic recursion $X_n^x = \psi_{\theta_n}(X_{n-1}^x)$, starting at $x \in \mathbb{R}^d$, where $\theta_1, \theta_2, \ldots$ are i.i.d. random variables taking their values in a metric space $(\Theta, \mathfrak{r})$, and $\psi_{\theta_n} : \mathbb{R}^d \mapsto \mathbb{R}^d$ are Lipschitz maps. Assume that the Markov chain has a unique stationary measure ν. Under appropriate assumptions on ψ_{θ_n}, we will show that the measure ν has a heavy tail with the exponent $\alpha > 0$ i.e. $\nu(\{x \in \mathbb{R}^d : |x| > t\}) \asymp t^{-\alpha}$. Using this result we show that properly normalized Birkhoff sums $S_n^x = \sum_{k=1}^n X_k^x$, converge in law to an α-stable law for $\alpha \in (0, 2]$.

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1 Introduction and statement of results

We consider the Euclidean space $\mathbb{R}^d$ endowed with the scalar product $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$, the norm $|x| = \sqrt{\langle x, x\rangle}$, and its Borel σ-field $\mathcal{B}or(\mathbb{R}^d)$. An iterated random function is a sequence of the form

$$X_n^x = \psi\left(X_{n-1}^x, \theta_n\right), \tag{1.1}$$

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M. Mirek (✉)
Institute of Mathematics, University of Wroclaw, Plac Grunwaldzki 2/4, 50-384 Wroclaw, Poland
e-mail: mirek@math.uni.wroc.pl

where $n \in \mathbb{N}$, $X_0^x = x$ and $\theta_1, \theta_2, \ldots \in \Theta$ are independent and identically distributed according to the measure μ on a metric space $\Theta = (\Theta, \mathfrak{r})$. We assume that $\psi : \mathbb{R}^d \times \Theta \mapsto \mathbb{R}^d$ is jointly measurable and we write $\psi_\theta(x) = \psi(x, \theta)$. Then the sequence $(X_n^x)_{n\geq 0}$ is a Markov chain with the state space $\mathbb{R}^d$, the initial Dirac distribution δ_x, and the transition probability P defined by $P(x, B) = \int_\Theta \mathbf{1}_B(\psi_\theta(x))\mu(d\theta)$ for all $x \in \mathbb{R}^d$ and $B \in \mathcal{B}or(\mathbb{R}^d)$. Unless otherwise stated we assume throughout this paper that for every $\theta \in \Theta$, $\psi_\theta : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a Lipschitz map with the Lipschitz constant

$$L_\theta = \sup_{x \neq y} \frac{|\psi_\theta(x) - \psi_\theta(y)|}{|x - y|} < \infty.$$

Matrix recursions

$$X_n^x = \psi_{\theta_n}(X_{n-1}^x) = M_n X_{n-1}^x + Q_n \in \mathbb{R}^d, \tag{1.2}$$

where $\theta_n = (M_n, Q_n) \in Gl(\mathbb{R}^d) \times \mathbb{R}^d = \Theta$ and $X_0^x = x \in \mathbb{R}^d$ ($Gl(\mathbb{R}^d)$ is the group of $d \times d$ invertible matrices) are probably the best known examples of the situation we have in mind [5,6,14,24,25].

If the Lipschitz constant L_θ is contracting in average i.e. $\int_\Theta \log(L_\theta)\mu(d\theta) < 0$ and $\int_\Theta |\log(L_\theta)| + \log^+(|\psi_\theta(x_0)|)\mu(d\theta) < \infty$ for some $x_0 \in \mathbb{R}^d$, then (1.1) has a unique (in law) stationary solution S with law ν. In fact, $S = \lim_{n\to\infty} \psi_{\theta_1} \circ \psi_{\theta_2} \circ \ldots \circ \psi_{\theta_n}(x)$ a.s. and does not depend on the starting point $x \in \mathbb{R}^d$ (see [6,27] for more details). Throughout this paper we assume that our Lipschitz maps ψ_θ's satisfy above conditions and recursion (1.1) has the stationary solution S with law ν.

We are going to describe the asymptotic behavior of Birkhoff sums $S_n^x = \sum_{k=1}^n X_k^x$ of (non independent) random variables X_k^x. We prove that the sequence S_n^x normalized appropriately converges to a stable law (see Theorem 1.15).

The problem has been recently studied in [4] for the recursion (1.2) with $M \in \mathbb{R}_+^* \times O(\mathbb{R}^d)$ ($O(\mathbb{R}^d)$ is the orthogonal group) and a central limit theorem has been proved. Depending on the growth of M and Q, a stable law or a Gaussian law appear as the limit. In the first case the heavy tail behavior of the stationary solution of (1.2) at infinity is vital for the proof. (See [5]).

On the other hand being linear is not that crucial for ψ_θ and so, it is tempting to generalize the result of [4] for a larger class of possible ψ_θ. Lipschitz transformations fit perfectly into the scheme—see examples in Sect. 2 due to Goldie [7] and Borkovec and Klüppelberg [3].

To give an idea of our result let us formulate it in the special case of the recursion

$$X_n^x = \max\left(M_n X_{n-1}^x, Q_n\right), \tag{1.3}$$

where $(M_n, Q_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+ \times \mathbb{R}$ and $X_0^x = x \in \mathbb{R}$. For the stationary solution S of (1.3) with law ν, $\lim_{t\to\infty} t^\alpha \mathbb{P}(\{S > t\})$ exists, and under appropriate assumption, it is positive [7]. Then the limit Theorem 1.15 is:

Theorem 1.4 *Assume that* $(M_n, Q_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_+ \times \mathbb{R}$ *is the sequence of i.i.d. pairs with the law* μ *such that* $\mathbb{E}(M^\alpha) = 1$, *and* $\mathbb{E}(M^\alpha|\log M|) < \infty$, *for some* $\alpha \in (0, 2]$, *the conditional law of* $\log|M|$, *given* $M \neq 0$ *is non arithmetic,* $\mathbb{P}(Q > 0) > 0$, *and*

$\mathbb{E}(|Q|^\alpha) < \infty$. Let $S_n^x = \sum_{k=1}^n X_k^x$ for $n \in \mathbb{N}$. Then given $0 < \alpha < 2$ (for simplicity we assume here $\alpha \neq 1$), there is a sequence $d_n = d_n(\alpha)$ and a constant $C_\alpha \in \mathbb{C}$ such that the random variables $n^{-\frac{1}{\alpha}}(S_n^x - d_n)$ converge in law to the α-stable random variable with the characteristic function

$$\Upsilon_\alpha(t) = \exp(C_\alpha t^\alpha), \quad \textit{for } t > 0,$$

If $\alpha = 2$, there is a sequence $d_n = d_n(2)$ and a constant $C_2 \in \mathbb{R}$ such that the random variables $(n \log n)^{-\frac{1}{2}}(S_n^x - d_n)$ converge in law to the random variable with characteristic function

$$\Upsilon_2(t) = exp(C_2 t^2), \quad \textit{for } t > 0.$$

If $\alpha \in (0, 1)$, then $d_n = 0$, and if $\alpha \in (1, 2]$, then $d_n = nm$, where $m = \int_{\mathbb{R}^d} x\nu(dx)$. Furthermore, $\Re C_\alpha < 0$ for every $\alpha \in (0, 2]$.

The paper is divided into three parts. In the first one (Sect. 3) we describe the support of the stationary law ν of (1.1) in the terms of the fixed points for maps ψ_θ—see Theorem 3.1. Secondly, in section 4 we take care of the tail of ν. (See Theorem 1.8 saying that $\nu(\{x \in \mathbb{R}^d : |x| > t\}) \asymp t^{-\alpha}$). Finally, Sects. 5 and 6 are devoted to the proof of the limit Theorem 1.15. The limit law is a stable law with exponent $\alpha \in (0, 2]$. Thus we generalize the results for "$ax + b$" model stated in [4,15] for one dimensional and multidimensional situation respectively. The case where $\alpha > 2$ has been widely investigated in the general context of complete separable metric spaces by [2,18–20,29,33]. Recently, in [22] the authors proved α-stable theorem for $\alpha \in (0, 2)$ for additive functionals on metric spaces using martingale approximation method, but our situation does not fit into their framework. Convergence to stable laws were also studied by [1,9].

Now we are ready to formulate assumptions and to state theorems.

1.1 Heavy tail phenomena

In this section we state conditions that guarantee a heavy tail of ν. Contrary to the affine recursion

$$X_n^x = \psi_{\theta_n}(X_{n-1}^x) = M_n X_{n-1}^x + Q_n \in \mathbb{R}, \tag{1.5}$$

where $\theta_n = (M_n, Q_n) \in \mathbb{R} \times \mathbb{R} = \Theta$, we need more than just the behavior of the Lipschitz constant L_θ.

Assumption 1.6 *(Shape of the mappings ψ)* For every $t > 0$, let $\psi_{\theta,t} : \mathbb{R}^d \mapsto \mathbb{R}^d$ be defined by $\psi_{\theta,t}(x) = t\psi_\theta(t^{-1}x)$, where $x \in \mathbb{R}^d$ and $\theta \in \Theta$. $\psi_{\theta,t}$ are called dilatations of ψ_θ.

(H1) For every $\theta \in \Theta$, there exists a map $\overline{\psi}_\theta : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that $\lim_{t\to 0} \psi_{\theta,t}(x) = \overline{\psi}_\theta(x)$ for every $x \in \mathbb{R}^d$, and $\overline{\psi}_\theta(x) = M_\theta x$ for every $x \in \operatorname{supp}\nu$. The random variable M_θ takes its values in the group $G = \mathbb{R}_+^* \times K$, where K is a closed subgroup of orthogonal group $O(\mathbb{R}^d)$.

(H2) For every $\theta \in \Theta$, there is a random variable N_θ such that ψ_θ satisfies a cancellation condition i.e. $|\psi_\theta(x) - M_\theta x| \leq |N_\theta|$, for every $x \in \text{supp}\nu$.

To get the idea what is the meaning of (H1)–(H2) the reader may think of the affine recursion (1.5) with $\theta = (M, Q) \in G \times \mathbb{R}^d = \Theta$ or the recursion $\psi_\theta(x) = \max\{Mx, Q\}$, where $\theta = (M, Q) \in \mathbb{R}_+^* \times \mathbb{R} = \Theta$ (see Sect. 2). Then $\overline{\psi}_\theta(x) = Mx$ or $\overline{\psi}_\theta(x) = \max\{Mx, 0\}$ respectively. It is recommended to have in mind $\psi_\theta(x) = \max\{Mx, Q\}$ to get the first approximation of what the hypotheses mean. Notice that for the max recursion (H2) is not satisfied on $\mathbb{R}$, but only on $[0, \infty) \supseteq \text{supp}\nu$.

In one dimensional case condition (H2) has a very natural geometrical interpretation, namely it can be written in an equivalent form $M_\theta x - |N_\theta| \leq \psi_\theta(x) \leq M_\theta x + |N_\theta|$. It means that the graph of $\psi_\theta(x)$'s lies between the graphs of $M_\theta x - |N_\theta|$ and $M_\theta x + |N_\theta|$ for every $x \in \text{supp}\nu$. This allows us to think that the recursion is, in a sense, close to the affine recursion.

For simplicity we write X instead of X_θ.

Assumption 1.7 *(Moments condition for the heavy tail)* Let $\kappa(s) = \mathbb{E}|M|^s$ for $s \in [0, s_\infty)$, where $s_\infty = \sup\{s \in \mathbb{R}_+ : \kappa(s) < \infty\}$. Let $\bar{\mu}$ be the law of M.

(H3) G is the smallest closed semigroup generated by the support of $\bar{\mu}$ i.e. $G = \overline{\langle \text{supp}\bar{\mu} \rangle}$.

(H4) The conditional law of $\log |M|$, given $M \neq 0$ is non arithmetic.

(H5) M satisfies Cramér condition with exponent $\alpha > 0$, i.e. there exists $\alpha \in (0, s_\infty)$ such that $\kappa(\alpha) = \mathbb{E}(|M|^\alpha) = 1$.

(H6) Moreover, $\mathbb{E}(|M|^\alpha |\log |M||) < \infty$.

(H7) For the random variable N defined in (H2) we have $\mathbb{E}(|N|^\alpha) < \infty$.

Conditions (H4)–(H7) are natural in this context, see [3–5,7,10–12,15,16,24] and [32]. Now we are ready to formulate the main result.

Theorem 1.8 *Assume that ψ_θ satisfies Assumptions 1.6 and 1.7 for every $\theta \in \Theta$. Then there is a unique stationary solution S of (1.1) with law ν, and there is a unique Radon measure Λ on $\mathbb{R}^d \setminus \{0\}$ such that*

$$\lim_{g \in G, |g| \to 0} |g|^{-\alpha} \mathbb{E} f(gS) = \Lambda(f). \tag{1.9}$$

The convergence is valid for every bounded continuous function f that vanishes in a neighbourhood of zero. Furthermore the recursion defined in (1.1) has a heavy tail

$$\lim_{t \to \infty} t^\alpha \mathbb{P}(\{|S| > t\}) = \frac{1}{\alpha m_\alpha} \mathbb{E}\left(|\psi(S)|^\alpha - |MS|^\alpha\right), \tag{1.10}$$

where $m_\alpha = \mathbb{E}(|M|^\alpha \log |M|) > 0$. *If additionally the support of* ν *is unbounded, and one of the following condition is satisfied*

$$s_\infty < \infty \quad and \quad \lim_{s \to s_\infty} \frac{\mathbb{E}(|N|^s)}{\kappa(s)} = 0, \tag{1.11}$$

$$s_\infty = \infty \quad and \quad \lim_{s \to \infty} \left(\frac{\mathbb{E}(|N|^s)}{\kappa(s)}\right)^{\frac{1}{s}} < \infty, \tag{1.12}$$

then the measure Λ *is nonzero.*

Remark 1.13 Contrary to Theorems 1.15 and 3.1 the assumption that ψ_θ's are Lipschitz is not necessary for Theorem 1.8. The same conclusion holds if $\psi_\theta : \mathbb{R}^d \mapsto \mathbb{R}^d$ is continuous for every $\theta \in \Theta$, and the map $\Theta \ni \theta \mapsto \psi_\theta(x) \in \mathbb{R}^d$ is continuous for every $x \in \mathbb{R}^d$, 1.6 and 1.7 are satisfied and $S = \lim_{n\to\infty} \psi_{\theta_1} \circ \psi_{\theta_2} \circ \ldots \circ \psi_{\theta_n}(x)$ exists a.s. and does not depend on $x \in \mathbb{R}^d$.

In view of Letac's principle [27] the random variable S with law ν is a unique stationary solution of the recursion (1.1).

Theorem 1.8 on one hand generalizes Theorem 1.6 of [5] for multidimensional affine recursions and on the other, the results of Goldie [7] for a family of one-dimensional recursions modeled on $ax + b$. (H4)–(H6) were already assumed by Goldie. (H3) was introduced in [5] and the whole proof is based on it. (H1)–(H2) say that asymptotically (1.1) looks like an affine recursion and it allows us to use the methods of [5].

1.2 Limit theorem for Birhhoff sums

Now we introduce conditions necessary to obtain convergence in law of appropriately normalized sums $S_n^x = \sum_{k=1}^n X_k^x$ to an α-stable distribution.

Assumption 1.14 (*For the limit theorem*)

- (L1) For every $\theta \in \Theta$, $L_\theta \le |M_\theta|$.
- (L2) For every $\theta \in \Theta$, there is a random variable Q_θ, such that $\overline{\psi}_\theta$ satisfies a smoothness condition with respect to $t > 0$, i.e. $|\psi_{\theta,t}(x) - \overline{\psi}_\theta(x)| \le t\,|Q_\theta|$, for every $x \in \mathbb{R}^d$.
- (L3) For the random variable Q we have $\mathbb{E}(|Q|^\alpha) < \infty$.

Clearly, if $\overline{\psi}_\theta(x) = M_\theta x$ for every $x \in \text{supp}\nu$, then (L2) implies (H1), and (L2) together with (L3) imply (H2) and (H7). Now we are ready to formulate the limit theorem.

Theorem 1.15 *Assume that* ψ_θ *satisfies assumptions* (1.6), (1.7) *and* (1.14) *for every* $\theta \in \Theta$. *We define* $S_n^x = \sum_{k=1}^n X_k^x$ *for* $n \in \mathbb{N}$. *Let* $h_v(x) = \mathbb{E}(e^{i\langle v, \sum_{k=1}^\infty \overline{\psi}_{\theta_k} \circ \ldots \circ \overline{\psi}_{\theta_1}(x)\rangle})$ *for* $x \in \mathbb{R}^d$, *where* $\overline{\psi}_{\theta_k}$*'s were defined in (H1) of Assumption* 1.6, *and let* ν *be the stationary measure for the recursion* (1.1).

- *If* $\alpha \in (0,1) \cup (1,2)$*, then there is a sequence* $d_n = d_n(\alpha)$ *and a function* $C_\alpha : \mathbb{S}^{d-1} \mapsto \mathbb{C}$ *such that the random variables* $n^{-\frac{1}{\alpha}}\left(S_n^x - d_n\right)$ *converge in law to the* α*-stable random variable with characteristic function*

$$\Upsilon_\alpha(tv) = \exp(t^\alpha C_\alpha(v)), \quad \textit{for } t > 0 \textit{ and } v \in \mathbb{S}^{d-1}.$$

- *If* $\alpha = 1$*, then there are functions* $\xi, \tau : (0,\infty) \mapsto \mathbb{R}$ *and* $C_1 : \mathbb{S}^{d-1} \mapsto \mathbb{C}$ *such that the random variables* $n^{-1}S_n^x - n\xi(n^{-1})$ *converge in law to the random variable with characteristic function*

$$\Upsilon_1(tv) = \exp(tC_1(v) + it\langle v, \tau(t)\rangle), \quad \textit{for } t > 0 \textit{ and } v \in \mathbb{S}^{d-1}.$$

- *If* $\alpha = 2$*, then there is a sequence* $d_n = d_n(2)$ *and a function* $C_2 : \mathbb{S}^{d-1} \mapsto \mathbb{R}$ *such that the random variables* $(n\log n)^{-\frac{1}{2}}(S_n^x - d_n)$ *converge in law to the random variable with characteristic function*

$$\Upsilon_2(tv) = \exp(t^2 C_2(v)), \quad \textit{for } t > 0 \textit{ and } v \in \mathbb{S}^{d-1}.$$

If $\alpha \in (0,1)$*, then* $d_n = 0$*, and if* $\alpha \in (1,2]$*, then* $d_n = nm$*, where* $m = \int_{\mathbb{R}^d} x\nu(dx)$*. In all the above cases the function* C_α *depends on the function* h_v *and the measure* Λ *defined in Theorem 1.8. Moreover,* $C_\alpha(tv) = t^\alpha C_\alpha(v)$ *for every* $t > 0$*,* $v \in \mathbb{S}^{d-1}$ *and* $\alpha \in (0,1) \cup (1,2]$*. If*
suppΛ *spans* $\mathbb{R}^d$ *as a linear space, then* $\Re C_\alpha(v) < 0$ *for every* $v \in \mathbb{S}^{d-1}$*.*

The proof of the above theorem will be based on the spectral method that was initiated by Nagaev in [30] and then used and improved by many authors (see [1,4,9,15,18–20] and the references given there). The method is based on quasi-compactness of transition operators $Pf(x) = \mathbb{E}(f(\psi(x))) = \int_\Theta f(\psi_\theta(x))\mu(d\theta)$ on appropriate function spaces (see [4,15,18–20]). They are perturbed by adding Fourier characters.

The standard use of the perturbation theory requires exponential moments of μ, but there is some development towards μ's with polynomial moments [18] and their improvements [20], or even fractional moments [4,15]. They are based on a theorem of Keller and Liverani [23] (we refer also to [28] for an improvement of [23]). It says that the spectral properties of the operator P can be approximated by those of its Fourier perturbations

$$P_{t,v}f(x) = \mathbb{E}\left(e^{i\langle tv,\psi(x)\rangle}f(\psi(x))\right) = \int_\Theta e^{i\langle tv,\psi_\theta(x)\rangle}f(\psi_\theta(x))\mu(d\theta), \quad (1.16)$$

(with convention that $P_{0,v} = P$). Indeed,

$$P_{t,v} = k_v(t)\Pi_{P,t} + Q_{P,t}, \quad (1.17)$$

where $\lim_{t\to 0} k_v(t) = 1$, $\Pi_{P,t}$ is a projection on a one dimensional subspace and the spectral radii of $Q_{P,t}$ are smaller than $\varrho < 1$, when $t \le t_0$. To obtain Theorem 1.15 we need to expand the dominant eigenvalue $k_v(t)$ at 0.

When $\alpha \in (0, 2]$, $k_v(t)$ is neither analytic nor differentiable, hence their asymptotics at zero is much harder to obtain. The method used in [4] does not work here and so we propose another approach which is applicable to general Lipschitz models (see Sect. 6).

2 Examples

The following examples will help the reader to understand the meaning of the assumptions formulated in the introduction as well as to feel the breadth of the method.

2.1 An affine recursion

Let $G = \mathbb{R}_+^* \times O(\mathbb{R}^d)$ and take the sequence of i.i.d. random pairs $(A_n, B_n)_{n\in\mathbb{N}} \subseteq \Theta = G \times \mathbb{R}^d$ with the same law μ on Θ and define the affine map $\psi_n(x) = A_n x + B_n$, where $x \in \mathbb{R}^d$. This example was also widely considered in the context of discrete subgroups of $\mathbb{R}_+^*$ see [4,5].

2.2 An extremal recursion

Let $G = \mathbb{R}_+^*$ and $\Theta = G \times \mathbb{R}$. We consider the sequence of i.i.d. pairs $(A_n, B_n)_{n\in\mathbb{N}} \subseteq \Theta$ with the same law μ on Θ. Let $\psi_n(x) = \max\{A_n x, B_n\}$, where $x \in \mathbb{R}$. Assume that $\mathbb{P}(B > 0) > 0$. Then

- $\lim_{t\to 0} \psi_{n,t}(x) = \overline{\psi}_n(x)$, where $\overline{\psi}_n(x) = \max\{A_n x, 0\}$ and $M_n = A_n$.
- The stationary solution S with law ν is given by the explicit formula,

$$S = \max_{1\leq k<\infty} \{A_1 A_2 \cdot \ldots \cdot A_{k-1} B_k\},$$

  where $A_0 = 1$ a.s. [7].
- $\mathbb{P}(B > 0) > 0$ implies that $\operatorname{supp}\nu \subseteq [0, \infty)$ and $\operatorname{supp}\nu$ is unbounded.
- In order to check cancellation condition (H2) notice that $S \geq 0$ a.s and for $x > 0$

$$\begin{aligned} |\psi_{n,t}(x) - A_n x| &= |\max\{A_n x, B_n\} - A_n x| \mathbf{1}_{\{A_n x < B_n\}} \\ &\leq (|B_n| + |A_n x|)\, \mathbf{1}_{\{A_n x < B_n\}} \leq 2|B_n|, \end{aligned}$$

  so (H2) is fulfilled with $|N_n| = 2|B_n|$. Moreover, we assume (H4)–(H7) for $M_n = A_n$ and $N_n = 2B_n$
- Notice, that $|\psi_{n,t}(x) - \overline{\psi}_n(x)| = |\max\{A_n x, tB_n\} - \max\{A_n x, 0\}| \leq |t||B_n|$,

so (L2) is satisfied with $|Q_n| = |B_n|$.

We deal similarly with next example.

2.3 A model due to Letac

Let G be as above and take the sequence of i.i.d. random triples $(A_n, B_n, C_n)_{n\in\mathbb{N}} \subseteq \Theta = G \times \mathbb{R}_+ \times \mathbb{R}_+$ with the same law μ on Θ. Consider the map $\psi_n(x) = A_n \max\{x, B_n\} + C_n$, where $x \in \mathbb{R}$. If $C \geq 0$ a.s. and $\mathbb{P}(B > 0) + \mathbb{P}(C > 0) > 0$, then the support of the stationary measure ν is unbounded [7]. The others assumptions are also satisfied.

2.4 Another example

Take the sequence of i.i.d. random triples $(A_n, B_n, C_n)_{n\in\mathbb{N}} \subseteq \Theta = \mathbb{R}_+^* \times \mathbb{R}_+ \times \mathbb{R}_+$ with the same law μ on Θ, such that $B_n^2 - 4A_nC_n < 0$. Consider the map $\psi_n(x) = \sqrt{A_nx^2 + B_nx + C_n}$, where $x \in \mathbb{R}$. If $\mathbb{P}(B > 0) + \mathbb{P}(C > 0) > 0$, then the support of the stationary measure ν is unbounded [7]. Notice that $\psi_n(x)$ can be written in the equivalent form $\psi_n(x) = \sqrt{A_n(x + U_n)^2 + V_n}$, where $U_n = \frac{B_n}{2A_n}$ and $V_n = C_n - \frac{B_n^2}{4A_n} > 0$. Now we can easily verify that $\psi_n(x)$ is Lipschitz and (L2) is satisfied. Indeed,

$$\frac{|\psi_n(x) - \psi_n(y)|}{|x - y|} = \frac{A_n|(x + U_n)^2 - (y + U_n)^2|}{|x - y|\left(\sqrt{A_n(x + U_n)^2 + V_n} + \sqrt{A_n(y + U_n)^2 + V_n}\right)} \leq \sqrt{A_n}.$$

Next observe that $\overline{\psi}_n(x) = \sqrt{A_n}|x|$, and

$$\begin{aligned}
|\psi_{n,t}(x) - \overline{\psi}_n(x)| &= \frac{\left|A_n(x + tU_n)^2 + t^2V_n - A_nx^2\right|}{\sqrt{A_n(x + tU_n)^2 + t^2V_n} + \sqrt{A_nx^2}} \\
&\leq \frac{2tA_nU_n|x|}{\sqrt{A_n}|x|} + \frac{t^2A_nU_n^2 + t^2V_n}{t\sqrt{V_n}} \\
&\leq t\left(\frac{B_n}{\sqrt{A_n}} + \frac{C_n}{\sqrt{V_n}}\right),
\end{aligned}$$

this shows that (L2) is fulfilled.

For the above examples statements 1.8, 1.15 and 3.1 apply straightforwardly.

2.5 An autoregressive process with ARCH(1) errors

Now we consider an example described by Borkovec and Klüppelberg in [3]. For $x \in \mathbb{R}$, let $\psi(x) = \left|\gamma|x| + \sqrt{\beta + \lambda x^2}A\right|$, where $\gamma \geq 0, \beta > 0, \lambda > 0$ are constants and A is a symmetric random variable with continuous Lebesgue density p, finite second moment and with the support equal the whole of $\mathbb{R}$. (see section 2 in [3] for more details). Now consider the sequence $(\psi_n(x))_{n\in\mathbb{N}}$ of i.i.d. copies of $\psi(x)$ and observe that

- $\lim_{t\to 0} \psi_{n,t}(x) = \overline{\psi}_n(x)$, where $\overline{\psi}_n(x) = M_n|x|$ and $M_n = \left|\gamma + \sqrt{\lambda} A_n\right|$.
- $|\psi_{n,t}(x) - M_n|x|| = \left|\left|\gamma|x| + \sqrt{\beta t^2 + \lambda x^2} A_n\right| - \left|\gamma + \sqrt{\lambda} A_n\right| |x|\right|$
  $\leq |t|\sqrt{\beta}|A_n|$,

so (L2) holds with $|Q_n| = \sqrt{\beta}|A_n|$. Notice that (H2) holds for every $x \in [0, \infty)$ with $|N_n| = \sqrt{\beta}|A_n|$. In [3] the authors showed that it is possible to choose parameters $\gamma \geq 0, \beta > 0, \lambda > 0$ such that $\mathbb{E}(\log M_n) < 0$ and $\mathbb{E}\left(M_n^\alpha\right) = 1$ for some $0 < \alpha \leq 2$. Observe that $\mathbb{P}\left(\{M_n \in \mathbb{R}_+^*\}\right) = 1$. We are not able to verify conditions (1.11) and (1.12) to conclud that Λ is not zero, but this property follows from [3] and so Theorem 1.15 applies.

3 Stationary measure

3.1 Support of the stationary measure

Let $\mathcal{C}(\mathbb{R}^d)$ be the set of continuous functions on $\mathbb{R}^d$ and $\mathcal{C}_b(\mathbb{R}^d)$ be the set of bounded and continuous functions on $\mathbb{R}^d$. Recall that unless otherwise stated we assume (as in Introduction) that for every $\theta \in \Theta$, $\psi_\theta : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a Lipschitz map with the Lipschitz constant $L_\theta < \infty$.

Let $\mathcal{L}_\Theta^\mu = \overline{\{\psi_{\theta_1} \circ \ldots \circ \psi_{\theta_n}(\,\cdot\,) : \forall_{n\in\mathbb{N}} \forall_{1\leq i\leq n}\ \theta_i \in \mathrm{supp}\mu\}}$ i.e. $\mathcal{L}_\Theta^\mu$ is the closed semigroup generated by the maps ψ_θ, where $\theta \in \mathrm{supp}\mu$. Given ψ_θ with $L_\theta < 1$, let $\psi_\theta^\bullet$ be the unique fixed point of ψ_θ. Then we can formulate the main Theorem of this section:

Theorem 3.1 *Assume that $\int_\Theta \log(L_\theta)\mu(d\theta) < 0$, $\int_\Theta |\log(L_\theta)| + \log^+(|\psi_\theta(x_0)|)\mu(d\theta) < \infty$ for some $x_0 \in \mathbb{R}^d$ and $\Theta \ni \theta \mapsto \psi_\theta(x) \in \mathbb{R}^d$ is continuous for every $x \in \mathbb{R}^d$. If $\mathcal{S} = \{\psi_\theta^\bullet \in \mathbb{R}^d : \psi_\theta(\psi_\theta^\bullet) = \psi_\theta^\bullet$, where $\psi_\theta \in \mathcal{L}_\Theta^\mu$ and $L_\theta < 1\} \subseteq \mathbb{R}^d$, then $\mathrm{supp}\nu = \overline{\mathcal{S}}$, where ν is the law of the stationary solution S for the recursion (1.1).*

Theorem 3.1 generalizes similar theorems for affine random walks, see [5,14] for more details. Notice that conditions $\int_\Theta \log(L_\theta)\mu(d\theta) < 0$ and $\int_\Theta |\log(L_\theta)| + \log^+(|\psi_\theta(x_0)|)\mu(d\theta) < \infty$ for some $x_0 \in \mathbb{R}^d$, in view of [6] (see also [27]), give the existence of the stationary measure ν for the recursion (1.1).

Before proving Theorem 3.1, we need two lemmas. Given, ψ_θ with $L_\theta < 1$, the Banach fixed point theorem implies the existence of a unique fixed point $\psi_\theta^\bullet \in \mathbb{R}^d$ of the map ψ_θ. Moreover, for every $x \in \mathbb{R}^d$

$$\lim_{n\to\infty} \psi_\theta^n(x) = \psi_\theta^\bullet. \tag{3.2}$$

Lemma 3.3 *Assume that for the map ψ_θ we have $L_\theta < 1$ and $\psi \in \mathcal{L}_\Theta^\mu$. Then there exists*

$$\lim_{n\to\infty} (\psi \circ \psi_\theta^n)^\bullet = \psi(\psi_\theta^\bullet), \tag{3.4}$$

where $(\psi \circ \psi_\theta^n)^\bullet \in \mathbb{R}^d$ is the fixed point of the map $\psi \circ \psi_\theta^n$, for $n \in \mathbb{N}$.

Proof Notice that for n sufficiently large $\psi\psi_\theta^n = \psi \circ \psi_\theta^n$ is contracting. Fix $\varepsilon > 0$, then there exist $N_\varepsilon \in \mathbb{N}$ such that $\frac{L_\psi L_\theta^n}{1-L_\psi L_\theta^n} < \varepsilon$ for all $n \geq N_\varepsilon$, where L_ψ is the Lipschitz constant associated to ψ. For every $m \in \mathbb{N}$ we have

$$\left|(\psi\psi_\theta^n)^m(\psi_\theta^\bullet)-\psi(\psi_\theta^\bullet)\right| \leq \left(\sum_{k=1}^{\infty}(L_\psi L_\theta^n)^k\right)\cdot\left|\psi(\psi_\theta^\bullet)-\psi_\theta^\bullet\right| = \frac{L_\psi L_\theta^n \cdot \left|\psi(\psi_\theta^\bullet)-\psi_\theta^\bullet\right|}{1-L_\psi L_\theta^n}.$$

By (3.2) we can find $m \in \mathbb{N}$ such that $\left|(\psi\psi_\theta^n)^\bullet - (\psi\psi_\theta^n)^m(\psi_\theta^\bullet)\right| < \varepsilon$. Then

$$\begin{aligned}\left|(\psi\psi_\theta^n)^\bullet - \psi(\psi_\theta^\bullet)\right| &\leq \left|(\psi\psi_\theta^n)^\bullet - (\psi\psi_\theta^n)^m(\psi_\theta^\bullet)\right| + \left|(\psi\psi_\theta^n)^m(\psi_\theta^\bullet) - \psi(\psi_\theta^\bullet)\right| \\ &\leq \varepsilon + \left|\psi(\psi_\theta^\bullet) - \psi_\theta^\bullet\right| \cdot \frac{L_\psi L_\theta^n}{1-L_\psi L_\theta^n} \leq \varepsilon\left(1 + \left|\psi(\psi_\theta^\bullet) - \psi_\theta^\bullet\right|\right),\end{aligned}$$

for all $n \geq N_\varepsilon$. Since ε is arbitrary, (3.4) is established. □

Lemma 3.5 *If $\psi_\theta : \mathbb{R}^d \mapsto \mathbb{R}^d$ is continuous for every $\theta \in \Theta$ (not necessarily Lipschitz) and $\Theta \ni \theta \mapsto \psi_\theta(x) \in \mathbb{R}^d$ is continuous for every $x \in \mathbb{R}^d$, then for every $\theta \in supp\mu$*

$$\psi_\theta[supp\nu] \subseteq supp\nu,$$

where the measure ν is μ stationary i.e. $\int_{\mathbb{R}^d}\int_\Theta f(\psi_\theta(x))\mu(d\theta)\nu(dx) = \int_{\mathbb{R}^d} f(x)\nu(dx)$ for any $f \in \mathcal{C}_b(\mathbb{R}^d)$.

Proof Suppose for contradiction that $\psi_\theta(\text{supp}\nu) \subsetneq \text{supp}\nu$. Then for some $\theta_0 \in \text{supp}\mu$ and $x_0 \in \text{supp}\nu$, there exists an open neighborhood U of $\psi_{\theta_0}(x_0)$ such that $U \cap \text{supp}\nu = \emptyset$. Notice, that $\mu(\{\theta \in \Theta : \int_{\mathbb{R}^d} \mathbf{1}_U(\psi_\theta(x))\nu(dx) > 0\}) = 0$, since the measure ν is μ stationary. By the assumptions $\{\theta \in \Theta : \int_{\mathbb{R}^d} \mathbf{1}_U(\psi_\theta(x))\nu(dx) > 0\}$ is an open subset of Θ. $\psi_{\theta_0}^{-1}[U]$ is an open neighborhood of $x_0 \in \text{supp}\nu$, so $\theta_0 \in \{\theta \in \Theta : \int_{\mathbb{R}^d} \mathbf{1}_U(\psi_\theta(x))\nu(dx) > 0\}$, but this contradicts $\theta_0 \in \text{supp}\mu$. □

Proof of Theorem 3.1 It is a consequence of Lemmas 3.3 and 3.5. Compare also with Lemma 2.7 in [5]. □

3.2 Simple properties of recursions and their stationary measures

Lemma 3.6 *Assume that $Y_{n,t}^x = \psi_{\theta_1,t} \circ \psi_{\theta_2,t} \circ \ldots \circ \psi_{\theta_n,t}(x)$ for any $n \in \mathbb{N}$ and $t > 0$. Then,*

$$\left|Y_{n,t}^x - Y_{n,t}^y\right| \leq \prod_{i=1}^n L_{\theta_i}|x-y|, \tag{3.7}$$

$$\left|Y_{n,t}^x - Y_{n+m,t}^x\right| \leq \prod_{i=1}^n L_{\theta_i}|x - \psi_{\theta_{n+1},t} \circ \ldots \circ \psi_{\theta_{n+m},t}(x)|, \tag{3.8}$$

$$|x - \psi_{\theta_{n+1},t} \circ \ldots \circ \psi_{\theta_{n+m},t}(x)| \leq \sum_{k=1}^{m} \left(\prod_{i=n+1}^{n+k-1} L_{\theta_i} \right) |x - \psi_{\theta_{n+k},t}(x)|, \tag{3.9}$$

for any $x, y \in \mathbb{R}^d$ *and* $m, n \in \mathbb{N}$.

Proof It is easy to see that $|\psi_{\theta,t}(x) - \psi_{\theta,t}(y)| = |t\psi_\theta(t^{-1}x) - t\psi_\theta(t^{-1}y)| \leq L_\theta|x - y|$ for any $x, y \in \mathbb{R}^d$, so (3.7), (3.8) and (3.9) follow by induction. □

The next Lemma is obvious in view of what has just been established.

Lemma 3.10 *Under the assumptions of the previous Lemma, if (H2), (H5), (H7) and (L1) are satisfied, then for every* $\beta \in (0, \alpha)$, $x \in \mathbb{R}^d$,

$$\sup_{n \in \mathbb{N}} \left(\mathbb{E}|X_{n,t}^x|^\beta\right)^{\frac{1}{\beta}} = \sup_{n \in \mathbb{N}} \left(\mathbb{E}|Y_{n,t}^x|^\beta\right)^{\frac{1}{\beta}} < \infty,$$

where $X_{n,t}^x = \psi_{\theta_n,t} \circ \psi_{\theta_{n-1},t} \circ \ldots \circ \psi_{\theta_1,t}(x)$ *for any* $n \in \mathbb{N}$ *and* $t > 0$. *In particular, we obtain* $\left(\mathbb{E}|S|^\beta\right)^{\frac{1}{\beta}} < \infty$ *for every* $\beta \in (0, \alpha)$, *where* S *is the stationary solution of* (1.1).

4 The tail measure

This section deals with a heavy tail phenomenon for Lipschitz recursions satisfying Assumptions 1.7 modeled on analogous hypotheses for matrix recursions (1.5). (H1) and (H2) say that recursion (1.1) is in a sense close to the affine recursion with the linear part $M \in \mathbb{R}_+^* \times K$. This allows us to use techniques of [5], in particular a generalized renewal theorem.

Conditions 1.7 are typical for considerations of this type and they decide of asymptotic behavior of stationary measure; especially condition (H5) is crucial. Goldie and Grübel [8] showed that $\mathbb{P}(\{S > t\})$ can decay exponentially fast to zero if (H5) is not satisfied.

A closed subgroup of $\mathbb{R}_+^* \times O(\mathbb{R}^d)$ containing $\mathbb{R}_+^*$ is necessarily $G = \mathbb{R}_+^* \times K$, where K is a closed subgroup of the orthogonal group $O(\mathbb{R}^d)$, see e.g. Appendix C in [5] and Appendix A in [4]. Let $\frac{dr}{r}$ be the Haar measure of $\mathbb{R}_+^*$ and let ρ be the Haar measure of K such that $\rho(K) = 1$. Any element $g \in \mathbb{R}_+^* \times K$ can be uniquely written as $g = rk$, where $r \in \mathbb{R}_+^*$ and $k \in K$, and so the Haar measure λ on $\mathbb{R}_+^* \times K$ is $\int_G f(g)\lambda(dg) = \int_{\mathbb{R}_+^*} \int_K f(rk)\rho(dk)\frac{dr}{r}$. Clearly, G is unimodular.

Define convolution of a function f with a measure μ on the group G as

$$f * \mu(g) = \int_G f(gh)\mu(dh).$$

Given $f \in \mathcal{C}_b(\mathbb{R}^d)$, let

$$\bar{f}(g) = \mathbb{E}(f(gS)), \quad \text{and} \quad \chi_f(g) = \bar{f}(g) - \bar{f} * \bar{\mu}(g).$$

The functions $\bar{\mu}$ and χ_f are bounded and continuous. We are going to express the function $\bar{f}$ in the terms of the potential $U = \sum_{k=0}^{\infty} \bar{\mu}^{*k}$. Notice that for any $n \in \mathbb{N} \cup \{0\}$

$$\mathbb{E}(f(gM_{\theta_1}M_{\theta_2} \cdot \ldots \cdot M_{\theta_n}S)) = \int_G \mathbb{E}(f(ghS))\bar{\mu}^{*n}(dh) = \bar{f} * \bar{\mu}^{*n}(g).$$

Remark 4.1 Conditions (H5), (H6) imply that the function $\kappa(s) = \mathbb{E}(|M|^s)$ is well defined on $[0, \alpha]$ and $\kappa(0) = \kappa(\alpha) = 1$. Since κ is convex, we have

$$\mathbb{E}\left(\log(|M|)\right) < 0, \quad \text{and} \quad m_\alpha = \mathbb{E}(|M|^\alpha \log|M|) > 0.$$

For more details we refer to [7].

Let $\bar{\mu}_\alpha(dg) = |g|^\alpha \bar{\mu}(dg)$. In view of Remark 4.1 $\bar{\mu}_\alpha$ is a probability measure with positive mean and $\bar{\mu}_\alpha^{*n}(dg) = |g|^\alpha \bar{\mu}^{*n}(dg)$ for all $n \in \mathbb{N}$. Let $U_\alpha = \sum_{k=0}^{\infty} \bar{\mu}_\alpha^{*k}$ be the potential kernel built out of the measure $\bar{\mu}_\alpha$.

The aim of this section is to prove Theorem 4.3 which implies Theorem 1.8. Given any Radon measure Λ on $\mathbb{R}^d \backslash \{0\}$, let define

$$\begin{aligned}\mathcal{F}_\Lambda = \{f : \mathbb{R}^d \mapsto \mathbb{R} : f \text{ is measurable function such that } \Lambda(\mathrm{Dis}(f)) = 0, \text{ and} \\ \sup_{x \in \mathbb{R}^d} |x|^{-\alpha} |\log|x||^{1+\varepsilon} |f(x)| < \infty \text{ for some } \varepsilon > 0\},\end{aligned} \tag{4.2}$$

where $\mathrm{Dis}(f)$ is the set of all discontinuities of function f.

Theorem 4.3 *Suppose that 1.6 and 1.7 are satisfied. Then there is a unique stationary solution S of (1.1) with the law ν, and there is a unique Radon measure Λ on $\mathbb{R}^d \backslash \{0\}$ such that*

$$\lim_{|g| \to 0} |g|^{-\alpha} \mathbb{E} f(gS) = \lim_{|g| \to 0} |g|^{-\alpha} \int_{\mathbb{R}^d} f(gx)\nu(dx) = \int_{\mathbb{R}^d \backslash \{0\}} f(x)\Lambda(dx), \tag{4.4}$$

for every function $f \in \mathcal{F}_\Lambda$. The measure Λ is homogeneous with degree α i.e. $\int_{\mathbb{R}^d} f(gx)\Lambda(dx) = |g|^\alpha \Lambda(f)$ for every $g \in G$. There exists a measure σ_Λ on $\mathbb{S}^{d-1}$ such that Λ has the polar decomposition

$$\int_{\mathbb{R}^d \backslash \{0\}} f(x)\Lambda(dx) = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(rx)\sigma_\Lambda(dx)\frac{dr}{r^{\alpha+1}}, \tag{4.5}$$

where $\sigma_\Lambda(\mathbb{S}^{d-1}) = \frac{1}{m_\alpha}\mathbb{E}\left(|\psi(S)|^\alpha - |MS|^\alpha\right)$ and $m_\alpha = \mathbb{E}(|M|^\alpha \log|M|) \in (0, \infty)$. Furthermore, recursion defined in (1.1) has a heavy tail

$$\lim_{t \to \infty} t^\alpha \mathbb{P}(\{|S| > t\}) = \frac{1}{\alpha m_\alpha}\mathbb{E}\left(|\psi(S)|^\alpha - |MS|^\alpha\right). \tag{4.6}$$

If additionally the support of ν is unbounded, and one of the following condition is satisfied

$$s_\infty < \infty \quad and \quad \lim_{s\to s_\infty} \frac{\mathbb{E}(|N|^s)}{\kappa(s)} = 0, \tag{4.7}$$

$$s_\infty = \infty \quad and \quad \lim_{s\to\infty} \left(\frac{\mathbb{E}(|N|^s)}{\kappa(s)}\right)^{\frac{1}{s}} < \infty, \tag{4.8}$$

then the measures Λ and σ_Λ are nonzero.

We divide the proof into three steps. Step 1. (Existence of the tail measure Λ) and Step 3. (Nontriviality of the tail measure Λ) go along the same lines as the Main Theorem 1.6. in [5] so we give only outlines of proofs. The proof of the existence of a polar decomposition for the measure Λ is shorter here and it is given in the Step 2..

Proof Step 1. Existence of the tail measure Λ. Assumptions (1.7) imply the existence of the stationary solution S for the recursion (1.1) with the law ν (see [6,27] for more details). Now, for an $\varepsilon \in (0, 1]$, we define the set of Hölder functions by

$$\mathcal{H}_\varepsilon = \{f \in \mathcal{C}_b(\mathbb{R}^d) : \forall_{x,y\in\mathbb{R}^d}\ |f(x) - f(y)| \le C_f |x-y|^\varepsilon, \\ \text{and } f \text{ vanishes in a neighbourhood of } 0\}.$$

Given $f \in \mathcal{H}_\varepsilon$ for some $\varepsilon \in (0, 1]$ and $\varepsilon < \alpha$, we write $\chi_{f,\alpha}(g) = |g|^{-\alpha}\chi_f(g)$. Using cancellation condition (H2) and arguing in a similar way as in [5] (see Lemma 2.19) we obtain that the function $\chi_{f,\alpha}(g) = |g|^{-\alpha}\chi_f(g)$ is $d\mathcal{R}i$ (*direct Riemann integrable* on G, definition of $d\mathcal{R}i$ functions can be found in [5]). Now we can use a renewal theorem for closed subgroups of $\mathbb{R}_+^* \times K$, where K is a metrizable group not necessarily Abelian, (see Appendix A of [5], also [13,31]). It is applied to the function $\chi_{f,\alpha}(g)$, to obtain

$$\lim_{|g|\to 0} |g|^{-\alpha}\bar{f}(g) = \lim_{|g|\to 0} U_\alpha(\chi_{f,\alpha})(g) = \frac{1}{m_\alpha}\int_G \chi_{f,\alpha}(g)\lambda(dg). \tag{4.9}$$

The formula

$$\Lambda(f) = \frac{1}{m_\alpha}\int_G \chi_{f,\alpha}(g)\lambda(dg) = \frac{1}{m_\alpha}\int_G |g|^{-\alpha}(\mathbb{E}(f(gS) - f(gMS)))\lambda(dg),$$

defines a nonnegative Radon measure on $\mathbb{R}^d\backslash\{0\}$, which is α homogeneous. Convergence in (4.9) holds also for $f \in \mathcal{F}_\Lambda$ (compare with the proof of Theorem 2.8 in [5]).

Step 2. Polar decomposition for the measure Λ. Being homogeneous Λ can be nicely expressed in polar coordinates. Let $\Phi : \mathbb{R}^d\backslash\{0\} \mapsto (0,\infty)\times\mathbb{S}^{d-1}$ be defined as follows $\Phi(x) = \left(|x|, \frac{x}{|x|}\right)$ and its inverse $\Phi^{-1} : (0,\infty)\times\mathbb{S}^{d-1} \mapsto \mathbb{R}^d\backslash\{0\}$ by $\Phi^{-1}(r,z) = rz$.

Next we define the measures σ^s on $\mathbb{S}^{d-1}$

$$\sigma^s(F) = s\Lambda^s\big(\Phi^{-1}\,[[1,\infty)\times F]\big),$$

where $\Lambda^s(f) = \frac{1}{m_\alpha}\int_G |g|^{-s}(\mathbb{E}(f(gS) - f(gMS)))\lambda(dg)$ for $s < \alpha$ and $F \in \mathcal{B}or(\mathbb{S}^{d-1})$. $\mathcal{B}or(X)$ means the Borel σ-field of X. Fix $0 < \beta < \gamma$ and notice that for any $[\beta,\gamma)\times F \in \mathcal{B}or((0,\infty))\otimes\mathcal{B}or(\mathbb{S}^{d-1})$,

$$\big(\Lambda^s\circ\Phi^{-1}\big)([\beta,\gamma)\times F) = \sigma^s(F)\int_\beta^\gamma \frac{dr}{r^{s+1}}.$$

The above proves (4.5) with the measure Λ^s instead of Λ. Now notice that

$$\sigma^s(\mathbb{S}^{d-1}) = s\Lambda^s\left(\Phi^{-1}\left[[1,\infty)\times\mathbb{S}^{d-1}\right]\right) = \frac{1}{m_\alpha}\mathbb{E}\left(|\psi(S)|^s - |MS|^s\right).$$

Hence (4.5) holds. Furthermore,

$$\lim_{t\to\infty} t^\alpha\mathbb{P}(\{|S|>t\}) = \lim_{t\to\infty} t^\alpha\int_t^\infty \frac{dr}{r^{\alpha+1}}\sigma_\Lambda(\mathbb{S}^{d-1}) = \frac{1}{\alpha m_\alpha}\mathbb{E}\left(|\psi(S)|^\alpha - |MS|^\alpha\right),$$

and (4.6) also holds.

Step 3. Nontriviality of the tail measure Λ. In order to prove that measure Λ is nontrivial in view of (4.5) we have to show that $\sigma_\Lambda \neq 0$. Suppose for a contradiction that $\sigma_\Lambda(\mathbb{S}^{d-1}) = 0$. Applying the method from section 3 from [5] together with condition (H2) we obtain that the stationary solution S is bounded, but it contradicts with the fact that the support of the measure ν is unbounded and this finishes the proof of Theorem 4.3. □

The example below shows that for (4.7) and (4.8) the hypothesis that the support of the measure ν is unbounded is crucial. Consider $\psi_n(x) = A_n\max\{x, B_n\}+C_n$ and assume that $\mathbb{P}(\{A_n = \frac{1}{3}\}) = \frac{3}{4}$, $\mathbb{P}(\{A_n = 2\}) = \frac{1}{4}$ and $\mathbb{P}(\{B_n = \frac{1}{2}\}) = \mathbb{P}(\{C_n = -1\}) = 1$. Then $\mathbb{E}(\log A_n) < 0$ and $\mathbb{E}(A_n^\alpha) = 1$, where $\alpha \approx 1{,}851$. It is easy to see that the stationary measure ν is supported by the set $\{-\frac{5}{6}, 0\}$ though the function $\psi_n(x)$ is unbounded.

5 Fourier operators and their properties

As it is mentioned in Sect. 1.2, for the limit Theorem 1.15, we study the Markov operator P associated to the recursion (1.1) as well as the perturbations $P_{t,v}$ of P defined in (1.16). To expand the dominant eigenvalue $k_v(t)$ defined in (1.17) we need some information about the eigenfunctions $\Pi_{P,t}1$, $t \geq 0$. While t varies, the normalization of $\Pi_{P,t}1$ counts. This means that although the corresponding eigenspaces are one dimansional, the choice of a multiple of $\Pi_{P,t}1$ is delicate. Properties (L1)–(L3) allow us to proceed similarly as in [4], but not exactly. The major difference is related to auxiliary operators $T_{t,v}$. They are used in [4] to obtain an explicit expression for $\Pi_{P,t}1$, but they are written there by the formula that does not work beyond the affine recursion. However, a careful analysis of operators $T_{t,v}$ suggests to write them abstractly as

$$T_{t,v} = \Delta_t^{-1} \circ P_{t,v} \circ \Delta_t, \tag{5.1}$$

where Δ_t is the dilatation $\Delta_t f(x) = f(tx)$. The "abstract" $T_{t,v}$'s do the same job making the method applicable to a much more general context.

We start by introducing two Banach spaces $\mathcal{C}_\rho(\mathbb{R}^d)$ and $\mathcal{B}_{\rho,\epsilon,\lambda}(\mathbb{R}^d)$ of continuous functions [26] (see also [4,15,18–20]).

$$\mathcal{C}_\rho = \mathcal{C}_\rho(\mathbb{R}^d) = \left\{ f \in \mathcal{C}(\mathbb{R}^d) : |f|_\rho = \sup_{x\in\mathbb{R}^d} \frac{|f(x)|}{(1+|x|)^\rho} < \infty \right\},$$
$$\mathcal{B}_{\rho,\epsilon,\lambda} = \mathcal{B}_{\rho,\epsilon,\lambda}(\mathbb{R}^d) = \{ f \in \mathcal{C}(\mathbb{R}^d) : \|f\|_{\rho,\epsilon,\lambda} = |f|_\rho + [f]_{\epsilon,\lambda} < \infty \},$$

where

$$[f]_{\epsilon,\lambda} = \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\epsilon (1+|x|)^\lambda (1+|y|)^\lambda}.$$

Remark 5.2 If $\epsilon + \lambda < \rho$, then $[f]_{\epsilon,\lambda} < \infty$ implies $|f|_\rho < \infty$. As a simple application of Arzelà–Ascoli theorem we obtain that the injection operator $\mathcal{B}_{\rho,\epsilon,\lambda} \hookrightarrow \mathcal{C}_\rho$ is compact.

From now on we assume that ψ_θ satisfies 1.6, 1.7 and 1.14 for every $\theta \in \Theta$. For the sake of simplicity we write ψ instead of ψ_θ. On $\mathcal{C}_\rho$ and $\mathcal{B}_{\rho,\epsilon,\lambda}$ we consider the transition operator $Pf(x) = \mathbb{E}(f(\psi(x)))$ and its perturbations

$$P_{t,v} f(x) = \mathbb{E}\left(e^{i\langle tv, \psi(x)\rangle} f(\psi(x)) \right),$$

defined in (1.16), where $x \in \mathbb{R}^d$, $v \in \mathbb{S}^{d-1}$ and $t \in [0,1]$. Observe that $P_{0,v} = P$. For convenience we write $\psi_t = \overline{\psi}$ for $t = 0$, ($\overline{\psi}$ was defined in (H1) of assumption (1.6)). We will also use the family of Fourier operators $T_{t,v}$ defined in (5.1). Notice that

$$T_{t,v} f(x) = \left(\Delta_t^{-1} \circ P_{t,v} \circ \Delta_t \right) f(x) = \mathbb{E}\left(e^{i\langle v, \psi_t(x)\rangle} f(\psi_t(x)) \right),$$

for $x \in \mathbb{R}^d$, where $t \in [0,1]$ and $v \in \mathbb{S}^{d-1}$. For simplicity we will write $T_v = T_{0,v}$. $T_{t,v}$ are perturbations of T_v.

Clearly, for $t > 0$, $T_{t,v}$ and $P_{t,v}$, as being dilations of each other, have the same peripherical eigenvalues $k_v(t)$, but for $t = 0$, the relation between T_v and P is not that close. Therefore, by considering T_v and $T_{t,v}$ we obtain some extra information when

$t \to 0$. In particular, the eigenfunction h_v of T_v with the eigenvalue 1 plays a vital role in approximating peripherical eigenvectors of $P_{t,v}$.

To treat both $P_{t,v}$ and $T_{t,v}$ in a unified way we write

$$\mathcal{F}_{s,t,v} f(x) = \mathbb{E}\left(e^{i\langle sv, \psi_t(x)\rangle} f(\psi_t(x))\right) = \int\limits_{\Theta} e^{i\langle sv, \psi_{\theta,t}(x)\rangle} f(\psi_{\theta,t}(x))\mu(d\theta).$$

Notice that $\mathcal{F}_{s,0,v} f(x) = \mathbb{E}(e^{i\langle sv, \overline{\psi}(x)\rangle} f(\overline{\psi}(x))) = \int_{\Theta} e^{i\langle sv, \overline{\psi}_\theta(x)\rangle} f(\overline{\psi}_\theta(x))\mu(d\theta)$, and $\mathcal{F}_{0,t,v} f(x) = \mathbb{E}(f(\psi_t(x))) = \int_{\Theta} f(\psi_{\theta,t}(x))\mu(d\theta)$, for $x \in \mathbb{R}^d$, where $s, t \in [0, 1]$ and $v \in \mathbb{S}^{d-1}$. Observe that, $\mathcal{F}_{s,1,v} = P_{s,v}$ and $\mathcal{F}_{1,t,v} = T_{t,v}$.

Now by the definition (5.1) it is easy to see, that for every $n \in \mathbb{N}$ and $t \in [0, 1]$,

$$P^n_{t,v} \circ \Delta_t = \Delta_t \circ T^n_{t,v}. \tag{5.3}$$

Moreover, if $f \in \mathcal{C}_\rho$ is eigenfunction of operator $T_{t,v}$ with eigenvalue $k_v(t)$, then $\Delta_t f$ is an eigenfunction of the operator $P_{t,v}$ with the same eigenvalue. The main result of this section is the following

Proposition 5.4 *Assume that* $0 < \epsilon < 1$, $\lambda > 0$, $\lambda + 2\epsilon < \rho = 2\lambda$ *and* $2\lambda + \epsilon < \alpha$, *then there exist* $0 < \varrho < 1$, $\delta > 0$ *and* $t_0 > 0$ *such that* $\varrho < 1 - \delta$, *and for every* $t \in [0, t_0]$ *and every* $v \in \mathbb{S}^{d-1}$

- $\sigma(P_{t,v})$ *and* $\sigma(T_{t,v})$ *are contained in* $\mathcal{D} = \{z \in \mathbb{C} : |z| \le \varrho\} \cup \{z \in \mathbb{C} : |z-1| \le \delta\}$.
- *The sets* $\sigma(P_{t,v}) \cap \{z \in \mathbb{C} : |z-1| \le \delta\}$ *and* $\sigma(T_{t,v}) \cap \{z \in \mathbb{C} : |z-1| \le \delta\}$ *consist of exactly one eigenvalue* $k_v(t)$, *where* $\lim_{t\to 0} k_v(t) = 1$, *and the corresponding eigenspace is one dimensional.*
- *For any* $z \in \mathcal{D}^c$ *and every* $f \in \mathcal{B}_{\rho,\epsilon,\lambda}$

$$\left\|(z - P_{t,v})^{-1} f\right\|_{\rho,\epsilon,\lambda} \le D\|f\|_{\rho,\epsilon,\lambda}, \quad \text{and} \quad \left\|(z - T_{t,v})^{-1} f\right\|_{\rho,\epsilon,\lambda} \le D\|f\|_{\rho,\epsilon,\lambda},$$

  where $D > 0$ *is universal constant which does not depend on* $t \in [0, t_0]$.
- *Moreover, we can express operators* $P_{t,v}$ *and* $T_{t,v}$ *in the following form*

$$P^n_{t,v} = k_v(t)^n \Pi_{P,t} + Q^n_{P,t}, \quad \text{and} \quad T^n_{t,v} = k_v(t)^n \Pi_{T,t} + Q^n_{T,t},$$

  for every $n \in \mathbb{N}$, *where* $\Pi_{P,t}$ *and* $\Pi_{T,t}$ *are projections onto the one dimensional eigenspaces mentioned above.* $Q_{P,t}$ *and* $Q_{T,t}$ *are the complementary operators to projections* $\Pi_{P,t}$ *and* $\Pi_{T,t}$ *respectively, such that* $\Pi_{P,t}Q_{P,t} = Q_{P,t}\Pi_{P,t} = 0$ *and* $\Pi_{T,t}Q_{T,t} = Q_{T,t}\Pi_{T,t} = 0$. *Furthermore,* $\|Q^n_{P,t}\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} = O(\varrho^n)$ *and* $\|Q^n_{T,t}\|_{\mathcal{B}_{\rho,\epsilon,\lambda}} = O(\varrho^n)$ *for every* $n \in \mathbb{N}$. *The operators* $\Pi_{P,t}$, $\Pi_{T,t}$, $Q_{P,t}$ *and* $Q_{T,t}$ *depend on* $v \in \mathbb{S}^{d-1}$, *but this is omitted for simplicity.*

- *The above operators can be expressed in the terms of the resolvents of* $P_{t,v}$ *and* $T_{t,v}$*. Indeed, for appropriately chosen* $\xi_1 > 0$ *and* $\xi_2 > 0$ *we have*

$$k_v(t)\Pi_{F,t} = \frac{1}{2\pi i}\int\limits_{|z-1|=\xi_1} z(z - F_{t,v})^{-1}\,dz,$$

$$\Pi_{F,t} = \frac{1}{2\pi i}\int\limits_{|z-1|=\xi_1} (z - F_{t,v})^{-1}\,dz,$$

$$Q_{F,t} = \frac{1}{2\pi i}\int\limits_{|z|=\xi_2} z(z - F_{t,v})^{-1}\,dz,$$

where $F_{t,v} = P_{t,v}$ *or* $F_{t,v} = T_{t,v}$*.*

Proposition 5.4 is a consequence of the perturbation theorem of Keller and Liverani [23] see also [28]. Before we apply their theorem we will check in a number of Lemmas that its assumptions are satisfied.

Lemma 5.5 *For every* $n \in \mathbb{N}$

$$\mathcal{F}^n_{s,t,v} f(x) = \mathbb{E}\left(e^{i\langle sv, S^x_{n,t}\rangle} f\left(X^x_{n,t}\right)\right), \tag{5.6}$$

where $X^x_{n,t}$ *is defined as in Lemma 3.10 and* $S^x_{n,t} = \sum_{k=1}^n X^x_{k,t}$*.*

Proof Formula (5.6) is obvious, since $\{X^x_{n,t}\}_{n\geq 1}$ is a Markov chain. □

Remark 5.7 The formula (5.6) implies that for every $0 < \rho < \alpha$, there exists a constant $C_1 > 0$ independent of $s, t \in [0, 1]$ and $v \in \mathbb{S}^{d-1}$ such that for every $n \in \mathbb{N}$

$$\left|\mathcal{F}^n_{s,t,v} f\right|_\rho \leq C_1 |f|_\rho. \tag{5.8}$$

Let denote $\Pi_n = L_{\theta_1} \cdot \ldots \cdot L_{\theta_n}$ for $n \in \mathbb{N}$ and $\Pi_0 = 1$. The inequality $|e^{ix} - 1| \leq 2|x|^\varepsilon$ for $0 < \varepsilon \leq 1$ and $x \in \mathbb{R}$ will be used repeatedly.

Lemma 5.9 *Assume that* $0 < \epsilon < 1$, $\lambda > 0$, $2\lambda + \epsilon < \alpha$, *and* $\rho = 2\lambda$*. Then there exist constants* $C_2 > 0$, $C_3 > 0$ *and* $0 < \varrho < 1$ *independent of* $s, t \in [0, 1]$ *and* $v \in \mathbb{S}^{d-1}$ *such that for every* $f \in \mathcal{B}_{\rho,\epsilon,\lambda}$ *and* $n \in \mathbb{N}$

$$\left[\mathcal{F}^n_{s,t,v} f\right]_{\epsilon,\lambda} \leq C_2 \varrho^n [f]_{\epsilon,\lambda} + C_3 |f|_\rho. \tag{5.10}$$

Proof By the definition of the seminorm $[\,\cdot\,]_{\epsilon,\lambda}$ we have

$$\mathcal{F}^n_{s,t,v} f(x) - \mathcal{F}^n_{s,t,v} f(y) = \mathbb{E}\left(e^{i\langle sv, S^x_{n,t}\rangle}\left(f\left(X^x_{n,t}\right) - f\left(X^y_{n,t}\right)\right)\right) \tag{5.11}$$

$$+\mathbb{E}\left(\left(e^{i\langle sv, S^x_{n,t}\rangle} - e^{i\langle sv, S^y_{n,t}\rangle}\right) f\left(X^y_{n,t}\right)\right). \tag{5.12}$$

To obtain (5.10) we have to estimate (5.11) and (5.12) separately. Indeed,

$$
\begin{aligned}
&\frac{\left|\mathbb{E}\left(e^{i\langle sv, S^x_{n,t}\rangle}\left(f\left(X^x_{n,t}\right)-f\left(X^y_{n,t}\right)\right)\right)\right|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\\
&\quad\le [f]_{\epsilon,\lambda}\cdot\mathbb{E}\left(\frac{\left|X^x_{n,t}-X^y_{n,t}\right|^{\epsilon}\left(1+\left|X^x_{n,t}\right|\right)^{\lambda}\left(1+\left|X^y_{n,t}\right|\right)^{\lambda}}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\right)\\
&\quad\le [f]_{\epsilon,\lambda}\cdot\mathbb{E}\left(\frac{\Pi_n^{\epsilon}\left(1+\left|X^0_{n,t}\right|+\Pi_n|x|\right)^{\lambda}\left(1+\left|X^0_{n,t}\right|+\Pi_n|y|\right)^{\lambda}}{(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\right)\\
&\quad\le [f]_{\epsilon,\lambda}\cdot\mathbb{E}\left(\Pi_n^{\epsilon}\left(\Pi_n+\left|X^0_{n,t}\right|+1\right)^{2\lambda}\right)\\
&\quad\le 3^{2\lambda}[f]_{\epsilon,\lambda}\cdot\left(\mathbb{E}\left(\Pi_n^{2\lambda+\epsilon}\right)+\mathbb{E}\left(\Pi_n^{\epsilon}\left|X^0_{n,t}\right|^{2\lambda}\right)+\mathbb{E}\left(\Pi_n^{\epsilon}\right)\right). \qquad (5.13)
\end{aligned}
$$

Now let $\varrho=\max\{\kappa(\epsilon),\kappa(2\lambda+\epsilon),\kappa^{\frac{\epsilon}{2\lambda+\epsilon}}(2\lambda+\epsilon)\}<1$. Applying the Hölder inequality to the last expression, we obtain

$$
\begin{aligned}
&3^{2\lambda}[f]_{\epsilon,\lambda}\cdot\left(\mathbb{E}\left(\Pi_n^{2\lambda+\epsilon}\right)+\mathbb{E}\left(\Pi_n^{\epsilon}\left|X^0_{n,t}\right|^{2\lambda}\right)+\mathbb{E}\left(\Pi_n^{\epsilon}\right)\right)\\
&\quad\le 3^{2\lambda}[f]_{\epsilon,\lambda}\cdot\left(\kappa(2\lambda+\epsilon)^n+t^{2\lambda}\left(\kappa^{\frac{\epsilon}{2\lambda+\epsilon}}(2\lambda+\epsilon)\right)^n\mathbb{E}\left(\left|X^0_n\right|^{2\lambda+\epsilon}\right)^{\frac{2\lambda}{2\lambda+\epsilon}}+\kappa(\epsilon)^n\right)\\
&\quad\le 3^{2\lambda}\varrho^n[f]_{\epsilon,\lambda}\cdot\left(2+t^{2\lambda}\mathbb{E}\left(\left|X^0_n\right|^{2\lambda+\epsilon}\right)^{\frac{2\lambda}{2\lambda+\epsilon}}\right)\le C_2\varrho^n[f]_{\epsilon,\lambda}, \qquad (5.14)
\end{aligned}
$$

where by Lemma 3.10 the constant $C_2=3^{2\lambda}\sup_{n\in\mathbb{N}}(2+\mathbb{E}(|X^0_n|^{2\lambda+\epsilon})^{\frac{2\lambda}{2\lambda+\epsilon}})$ is finite.

In order to estimate (5.12) notice that we have

$$
\left|S^x_{n,t}-S^y_{n,t}\right|\le\sum_{k=1}^{n}\left|X^x_{k,t}-X^y_{k,t}\right|\le\sum_{k=1}^{n}\Pi_k|x-y|\le B_n|x-y|,
$$

where $B_n=\sum_{k=0}^{n}\Pi_k$. Assume that $|y|\le|x|$, then

$$
\begin{aligned}
&\frac{\left|\mathbb{E}\left(\left(e^{i\langle sv,S^x_{n,t}\rangle}-e^{i\langle sv,S^y_{n,t}\rangle}\right)f\left(X^y_{n,t}\right)\right)\right|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\\
&\quad\le |f|_{\rho}\cdot\mathbb{E}\left(\frac{\left|e^{i\langle sv,S^x_{n,t}-S^y_{n,t}\rangle}-1\right|\left(1+\left|X^y_{n,t}\right|\right)^{\rho}}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\right)
\end{aligned}
$$

$$\leq 2s^{\epsilon}|f|_{\rho} \cdot \mathbb{E}\left(\frac{B_n^{\epsilon}\left(1+\left|X_{n,t}^0\right|+\Pi_n\right)^{\rho}(1+|y|)^{\rho}}{(1+|y|)^{2\lambda}}\right)$$

$$\leq 2 \cdot 3^{\rho} s^{\epsilon}|f|_{\rho} \cdot \mathbb{E}\left(B_n^{\epsilon}+t^{\rho} B_n^{\epsilon}\left|X_n^0\right|^{\rho}+B_n^{\epsilon} \Pi_n^{\rho}\right) \leq C_3|f|_{\rho}, \quad (5.15)$$

where the constant $C_3 = \sup_{n\in\mathbb{N}} 2 \cdot 3^{\rho} \cdot \mathbb{E}(B_n^{\epsilon} + B_n^{\epsilon}|X_n^0|^{\rho} + B_n^{\epsilon}\Pi_n^{\rho})$ is finite by the similar argument as in the previous case. This completes the proof of the Lemma.

□

Lemma 5.16 *Assume that* $0 < \epsilon < 1$, $\lambda > 0$, $2\lambda + \epsilon < \alpha$, $\rho = 2\lambda$ *and* $\lambda + 2\epsilon < \rho$. *Then there exist finite constants* $C_4 > 0$ *and* $C_5 > 0$ *independent of* $s, t \in [0, 1]$ *and of* $v \in \mathbb{S}^{d-1}$ *such that for every* $f \in \mathcal{B}_{\rho,\epsilon,\lambda}$

$$\left|\left(\mathcal{F}_{s,t,v} - \mathcal{F}_{s,0,v}\right) f\right|_{\rho} \leq C_4 t^{\epsilon} \|f\|_{\rho,\epsilon,\lambda}, \quad (5.17)$$

$$\left|\left(\mathcal{F}_{s,t,v} - \mathcal{F}_{0,t,v}\right) f\right|_{\rho} \leq C_5 s^{\epsilon} \|f\|_{\rho,\epsilon,\lambda}. \quad (5.18)$$

Notice that this Lemma also applies to the special case when $\mathcal{F}_{1,0,v} = T_v$.

Proof In order to prove (5.17) we write

$$\left(\mathcal{F}_{s,t,v} - \mathcal{F}_{s,0,v}\right) f(x) = \mathbb{E}\left(e^{i\langle sv,\psi_t(x)\rangle}\left(f(\psi_t(x)) - f\left(\overline{\psi}(x)\right)\right)\right) \quad (5.19)$$

$$+\mathbb{E}\left(\left(e^{i\langle sv,\psi_t(x)\rangle} - e^{i\langle sv,\overline{\psi}(x)\rangle}\right) f\left(\overline{\psi}(x)\right)\right). \quad (5.20)$$

Now we estimate (5.19) and (5.20) separately. By the definition of map $\overline{\psi}$ we know that $\overline{\psi}(0) = 0$, so $\left|\overline{\psi}(x)\right| \leq |M||x|$. Then condition (L2) implies that $|\psi_t(x)| \leq t|Q| + |M||x|$ and so

$$\frac{\left|\mathbb{E}\left(e^{i\langle sv,\psi_t(x)\rangle}\left(f(\psi_t(x)) - f\left(\overline{\psi}(x)\right)\right)\right)\right|}{(1+|x|)^{\rho}} \leq \mathbb{E}\left(\frac{\left|f(\psi_t(x)) - f\left(\overline{\psi}(x)\right)\right|}{(1+|x|)^{\rho}}\right)$$

$$\leq [f]_{\epsilon,\lambda} \cdot \mathbb{E}\left(\frac{\left|\psi_t(x) - \overline{\psi}(x)\right|^{\epsilon}(1+|\psi_t(x)|)^{\lambda}\left(1+\left|\overline{\psi}(x)\right|\right)^{\lambda}}{(1+|x|)^{\rho}}\right)$$

$$\leq t^{\epsilon}[f]_{\epsilon,\lambda} \cdot \mathbb{E}\left(\frac{|Q|^{\epsilon}(1+t|Q|+|M||x|)^{2\lambda}}{(1+|x|)^{\rho}}\right)$$

$$\leq 3^{2\lambda} t^{\epsilon}[f]_{\epsilon,\lambda} \cdot \mathbb{E}\left(|Q|^{\epsilon} + t^{2\lambda}|Q|^{2\lambda+\epsilon} + |Q|^{\epsilon}|M|^{2\lambda}\right) \leq D_1 t^{\epsilon}[f]_{\epsilon,\lambda}. \quad (5.21)$$

The quantity $D_1 = 3^{2\lambda} \cdot \mathbb{E}(|Q|^{\epsilon} + |Q|^{2\lambda+\epsilon} + |Q|^{\epsilon}|M|^{2\lambda})$ is finite according to Hölder's inequality, (H5) and (L3).

For (5.20), we have

$$\begin{aligned}
&\frac{\left|\mathbb{E}\left(\left(e^{i\langle sv,\psi_t(x)\rangle}-e^{i\langle sv,\overline{\psi}(x)\rangle}\right)f\left(\overline{\psi}(x)\right)\right)\right|}{(1+|x|)^{\rho}}\\
&\le |f|_{\rho}\cdot\mathbb{E}\left(\frac{\left|e^{i\langle sv,\psi_t(x)-\overline{\psi}(x)\rangle}-1\right|(1+|M||x|)^{\rho}}{(1+|x|)^{\rho}}\right)\\
&\le 2s^{\epsilon}t^{\epsilon}|f|_{\rho}\cdot\mathbb{E}\left(|Q|^{\epsilon}(1+|M|)^{\rho}\right)\\
&\le 2^{\rho+1}s^{\epsilon}t^{\epsilon}|f|_{\rho}\cdot\mathbb{E}\left(|Q|^{\epsilon}+|Q|^{\epsilon}|M|^{\rho}\right)\le D_2t^{\epsilon}|f|_{\rho}, \qquad (5.22)
\end{aligned}$$

where the constant $D_2 = 2^{\rho+1}\cdot\mathbb{E}(|Q|^{\epsilon}+|Q|^{\epsilon}|M|^{\rho})$ is also finite by the Hölder inequality, (H5) and (L3). Combining (5.21) with (5.22) we obtain (5.17) with $C_4 = \max\{D_1, D_2\}$.

In order to prove (5.18) notice that

$$\begin{aligned}
&\frac{\left|\left(\mathcal{F}_{s,t,v}-\mathcal{F}_{0,t,v}\right)f(x)\right|}{(1+|x|)^{\rho}}\le\mathbb{E}\left(\frac{\left|e^{i\langle sv,\psi_t(x)\rangle}f(\psi_t(x))-f(\psi_t(x))\right|}{(1+|x|)^{\rho}}\right)\\
&\le\mathbb{E}\left(\frac{|e^{i\langle sv,\psi_t(x)\rangle}-1||f(\psi_t(x))-f(0)|}{(1+|x|)^{\rho}}\right)+\mathbb{E}\left(\frac{|e^{i\langle sv,\psi_t(x)\rangle}-1||f(0)|}{(1+|x|)^{\rho}}\right)\\
&\le 2s^{\epsilon}\left([f]_{\epsilon,\lambda}\cdot\mathbb{E}\left(\frac{|\psi_t(x)|^{2\epsilon}(1+|\psi_t(x)|)^{\lambda}}{(1+|x|)^{\rho}}\right)+|f|_{\rho}\cdot\mathbb{E}\left(\frac{|\psi_t(x)|^{\epsilon}}{(1+|x|)^{\rho}}\right)\right)\\
&\le 2^{\lambda+1}s^{\epsilon}\|f\|_{\rho,\epsilon,\lambda}\cdot\mathbb{E}\left(\frac{|\psi_t(x)|^{2\epsilon}+|\psi_t(x)|^{\lambda+2\epsilon}+|\psi_t(x)|^{\epsilon}}{(1+|x|)^{\rho}}\right)\\
&\le C_5s^{\epsilon}\|f\|_{\rho,\epsilon,\lambda}, \qquad (5.23)
\end{aligned}$$

where $C_5 = 2^{\lambda+1}\cdot\mathbb{E}\left((1+|M|+|Q|)^{2\epsilon}+(1+|M|+|Q|)^{\lambda+2\epsilon}+(1+|M|+|Q|)^{\epsilon}\right)$ is finite by (H5) and (L3). Hence (5.23) proves (5.18) and finally it completes the proof of the Lemma. □

Lemma 5.24 *The unique eigenvalue of modulus* 1 *for operator* P *acting on* $\mathcal{C}_{\rho}$ *is* 1 *and the eigenspace is one dimensional. The corresponding projection on* $\mathbb{C}\cdot 1$ *is given by the map* $f\mapsto\nu(f)$.

Proof The proof can be found in section 3 of [4]. □

Recall, that for every $n\in\mathbb{N}$,

$$T_v^n f(x)=\mathbb{E}\left(e^{i\langle v,\sum_{k=1}^{n}\overline{\psi}_k\circ\ldots\circ\overline{\psi}_1(x)\rangle}f\left(\overline{\psi}_n\circ\ldots\circ\overline{\psi}_1(x)\right)\right),$$

where $\overline{\psi}_k(x)=\overline{\psi}_{\theta_k}(x)$, and $\overline{\psi}_{\theta_k}$'s were defined in (H1) of Assumption 1.6. Let

$$h_v(x)=\mathbb{E}\left(e^{i\langle v,\sum_{k=1}^{\infty}\overline{\psi}_k\circ\ldots\circ\overline{\psi}_1(x)\rangle}\right). \qquad (5.25)$$

Lemma 5.26 *The function h_v defined in (5.25) belongs to $\mathcal{B}_{\rho,\epsilon,\lambda}$, and $h_v(tx) = h_{tv}(x)$ for every $x \in \mathbb{R}^d$ and $t > 0$. Moreover,*

$$|h_v(x)| \leq 1, \tag{5.27}$$

$$|h_v(x) - h_v(y)| \leq \frac{2}{1-\kappa(\delta)}|x-y|^\delta, \tag{5.28}$$

for every $x, y \in \mathbb{R}^d$ and every $0 < \delta \leq 1$ such that $0 < \delta < \alpha$.

Proof Inequality, (5.27) is obvious, (5.28) follows from the definition of function h_v and inequality $|\overline{\psi}_k(x) - \overline{\psi}_k(y)| \leq |M||x-y|$ for $k \in \mathbb{N}$, where $\overline{\psi}_k$ was defined in (H1) of Assumption 1.6. In order to prove that $h_v(tx) = h_{tv}(x)$ it is enough to show that for a fixed $s > 0$ and every $x \in \mathbb{R}^d$, $\overline{\psi}(sx) = s\overline{\psi}(x)$. Indeed, for every $\varepsilon > 0$ there exists $\eta > 0$ such that $|t\psi(t^{-1}sx) - \overline{\psi}(sx)| < \varepsilon$ for every $0 < t < s\eta$. Hence if $t = rs$ and $0 < r < \eta$ then $|sr\psi(r^{-1}x) - \overline{\psi}(sx)| < \varepsilon$. Letting r tend to 0 we obtain $s\overline{\psi}(x) = \overline{\psi}(sx)$. □

Lemma 5.29 *The unique eigenvalue of modulus 1 for operator T_v acting on $\mathcal{C}_\rho$ is 1 with the eigenspace $\mathbb{C} \cdot h_v(x)$, where function h_v was defined in (5.25).*

Proof Notice that $\lim_{n\to\infty} \overline{\psi}_n \circ \ldots \circ \overline{\psi}_1(x) = 0$ a.e., ($\overline{\psi}_k$'s were defined in (H1) of Assumption 1.6). Take $f \in \mathcal{C}_\rho$, then by the Lebesgue dominated convergence theorem we have

$$\begin{aligned} T_v^n f(x) &= \mathbb{E}\left(e^{i\langle v, \sum_{k=1}^n \overline{\psi}_k \circ \ldots \circ \overline{\psi}_1(x)\rangle}\left(f\left(\overline{\psi}_n \circ \ldots \circ \overline{\psi}_1(x)\right) - f(0)\right)\right) \\ &\quad + \mathbb{E}\left(e^{i\langle v, \sum_{k=1}^n \overline{\psi}_k \circ \ldots \circ \overline{\psi}_1(x)\rangle} f(0)\right) \underset{n\to\infty}{\longrightarrow} f(0)h_v(x), \end{aligned}$$

for every $x \in \mathbb{R}^d$. Since $h_v(0) = 1$ the above convergence shows that 1 is a simple eigenvalue for the action of T_v on $\mathcal{B}_{\rho,\epsilon,\lambda}$ with h_v as the unique associated eigenfunction (up to multiplicative constant). It also proves that 1 is the unique peripheral eigenvalue. □

Proof of Proposition 5.4 Remark 5.7 implies that $\ker(T_v - I) = \ker(T_v - I)^2$. By Lemmas 5.9, 5.24 and 5.29, we have thanks to [17,21], that for every $n \in \mathbb{N}$

$$\forall f \in \mathcal{B}_{\rho,\epsilon,\lambda} \ P^n f = P^n_{0,v} f = \nu(f) \cdot 1 + Q^n_{P,0} f, \quad \text{and} \quad T^n_v f = T^n_{0,v} f = f(0) \cdot h_v + Q^n_{T,0} f,$$

where $Q_{P,0}$ and $Q_{T,0}$ are the complementary operators to the projections $\Pi_{P,0}$ ($\Pi_{P,0} f = \nu(f) \cdot 1$) and $\Pi_{T,0}$ ($\Pi_{T,0} f = f(0) \cdot h_v$) respectively.

Besides, in view of Lemmas 5.9 and 5.16, (in particular (5.17) and (5.18) imply

$$\forall f \in \mathcal{B}_{\rho,\epsilon,\lambda} \ \left|\left(T_{t,v} - T_v\right) f\right|_\rho \leq C_4 t^\epsilon \|f\|_{\rho,\epsilon,\lambda}, \quad \text{and} \quad \left|\left(P_{t,v} - P\right) f\right|_\rho \leq C_5 t^\epsilon \|f\|_{\rho,\epsilon,\lambda},$$

respectively for every $t \in [0, 1]$) we may use the perturbation theorem of Keller and Liverani [23] (we refer also to [28] for an improvement of [23]) for the operators $P_{t,v}$ and $T_{t,v}$ to get Proposition 5.4. □

Remark 5.30 If $z \in \sigma(P_{t,v})$ or $z \in \sigma(T_{t,v})$ and $|z| > \varrho$, where $0 < \varrho < 1$ is defined as in Lemma 5.9, then z does not belong to the residual spectrum of the operator $P_{t,v}$ or $T_{t,v}$ (see [17,21]). But thanks to the improvement of [23] given in [28] (see also [20]) condition on the essential spectral radius of $P_{t,v}$ (and $T_{t,v}$) is not required for $t \neq 0$.

6 Rate of convergence and fractional expansions

6.1 Rate of convergence of projections

As it has been already mentioned, to write down an expansion of $k_v(t)$ sufficiently good for the limit Theorem 1.15, we study the peripherical eigenfunctions of $P_{t,v}$ and, when t varies, the normalization is important. We have two natural candidates $\Pi_{P,t}1$ and $\Delta_t \Pi_{T,t} h_v$ one being a multiple of the other

$$\Delta_t \Pi_{T,t} h_v(x) = c_t \Pi_{P,t}(x).$$

Notice that, if $P_{t,v} f_t = k_v(t) f_t$, then

$$\begin{aligned}(k_v(t) - 1) \cdot \nu(f_t) &= \nu(P_{t,v} - 1) f_t) \\ &= \int (e^{it\langle v, \psi_\theta(x)\rangle} - 1) f_t(\psi_\theta(x))\, \mu(d\theta)\nu(dx) \\ &= \int (e^{it\langle v, x\rangle} - 1) f_t(x)\, d\nu(dx).\end{aligned}$$

Therefore, for both $\Pi_{P,t}1$ and $\Delta_t \Pi_{T,t} h_v$ we have

$$(k_v(t) - 1) \cdot \nu(\Pi_{P,t}1) = \nu((e^{it\langle v, \cdot\rangle} - 1) \cdot (\Pi_{P,t}1)), \tag{6.1}$$

and

$$(k_v(t) - 1) \cdot \nu(\Delta_t \Pi_{T,t} f) = \nu((e^{it\langle v, \cdot\rangle} - 1) \cdot (\Delta_t \Pi_{T,t} f)). \tag{6.2}$$

We may use either $\Pi_{P,t}1$ and approximate it by 1, or $\Delta_t \Pi_{T,t} h_v$ and approximate it by h_v. We choose the second possibility and we prove the following

Theorem 6.3 *Let h_v be the eigenfunction for operator T_v defined in (5.25). Then for any $0 < \delta \leq 1$ such that $\epsilon < \delta < \alpha$, there exist $C > 0$ and $D > 0$ such that*

$$\left\| \Delta_t (\Pi_{T,t} - \Pi_{T,0}) h_v \right\|_{\rho,\epsilon,\lambda} \leq C t^\delta, \tag{6.4}$$

and

$$\nu(\Delta_t \Pi_{T,t} h_v - 1) \leq D t^\delta, \tag{6.5}$$

for every $0 < t \le t_0$. *Moreover, for every* $x \in \mathbb{R}^d$ *and every* $0 < t \le t_0$

$$\left|\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)\right| \le Ct^{\delta}(1+|x|)^{\rho}. \tag{6.6}$$

Remark 6.7 The use of the family $\{T_{t,v}\}_{t>0}$ above is much more efficient than that of $\{P_{t,v}\}_{t>0}$. Indeed, the difference $|P_{t,v}f(x) - Pf(x)|$ involves the term $|e^{i\langle tv,\psi(x)\rangle} - 1|$ which depends on x, while $|T_{t,v}f(x) - T_v f(x)|$, leads to $|e^{i\langle v,\psi_t(x)\rangle} - e^{i\langle v,\overline{\psi}(x)\rangle}| \le |t||Q|$ independently of x by assumption (L2). This is the main new idea in comparison to [4] and it allows to prove (6.6). In (6.6) $0 < \delta \le 1$ satisfies $\epsilon < \delta < \alpha$ while ρ can be chosen small. This cannot be proved for $\Pi_{P,t}$ directly. Moreover, in Subsection 6.2, we shall deduce from (6.6) the following important property

$$\int_{\mathbb{R}^d} (e^{it\langle v,x\rangle} - 1)(\Pi_{T,t}(h_v)(tx) - h_v(tx))\nu(dx) = o(t^{\alpha}) \quad \text{as } t \to 0, \tag{6.8}$$

or $o(t^2 \log t)$ when $\alpha = 2$. Therefore, for $0 < \alpha < 2$

$$\frac{k_v(t) - 1}{t^{\alpha}} \approx \frac{1}{t^{\alpha}} \int_{\mathbb{R}^d} (e^{it\langle v,x\rangle} - 1)h_v(tx)\nu(dx), \tag{6.9}$$

when $t \to 0$ (if $\alpha = 2$ we have $t^2|\log t|$ instead of t^{α} in the above denominators) and the right hand side of (6.9) is further studied in the next subsection.

Before we prove Theorem 6.3 we need two lemmas.

Lemma 6.10 *Assume that the function* f *satisfies* $|f(x)| \le C$ *for any* $x \in \mathbb{R}^d$, *and* $|f(x) - f(y)| \le C|x-y|^{\delta}$ *for any* $0 < \delta \le 1$ *and* $x, y \in \mathbb{R}^d$, *where constant* $C > 0$ *depends on* δ. *Then for every* $\delta \in (\epsilon, \alpha)$

$$[(T_{t,v} - T_v)f]_{\epsilon,\lambda} \le C_1 t^{\delta-\epsilon}, \tag{6.11}$$

$$|(T_{t,v} - T_v)f|_{\rho} \le C_2 t^{\delta}, \tag{6.12}$$

where $C_1 > 0$ *and* $C_2 > 0$ *do not depend on* $0 < t \le t_0$.

Proof In order to show (6.11) we have to estimate the seminorm $[(T_{t,v} - T_v)f]_{\epsilon,\lambda}$. Notice, that

$$\begin{aligned}[(T_{t,v} - T_v)f]_{\epsilon,\lambda} \le &\sup_{x\neq y, |x-y|\le t} \frac{|(T_{t,v} - T_v)f(x) - (T_{t,v} - T_v)f(y)|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}} \\ &+ \sup_{x\neq y, |x-y|> t} \frac{|(T_{t,v} - T_v)f(x) - (T_{t,v} - T_v)f(y)|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}. \end{aligned} \tag{6.13}$$

For the first term in (6.13) ($|x-y|\le t$) we observe that

$$
\begin{aligned}
&(T_{t,v}-T_v)f(x)-(T_{t,v}-T_v)f(y)\\
&=\mathbb{E}((e^{i\langle v,\psi_t(x)\rangle}-e^{i\langle v,\psi_t(y)\rangle})f(\psi_t(x))) \quad (6.14)\\
&\quad+\mathbb{E}(e^{i\langle v,\psi_t(y)\rangle}(f(\psi_t(x))-f(\psi_t(y)))) \quad (6.15)\\
&\quad-\mathbb{E}((e^{i\langle v,\overline{\psi}(x)\rangle}-e^{i\langle v,\overline{\psi}(y)\rangle})f(\overline{\psi}(x))) \quad (6.16)\\
&\quad-\mathbb{E}(e^{i\langle v,\overline{\psi}(y)\rangle}(f(\overline{\psi}(x))-f(\overline{\psi}(y)))). \quad (6.17)
\end{aligned}
$$

We will estimate (6.14), (6.15), (6.16) and (6.17) separately. By assumptions on the function f observe, that for every $0<\delta\le 1$ such that $\epsilon<\delta<\alpha$ we have

$$
\begin{aligned}
&\mathbb{E}\left(\frac{\left|e^{i\langle v,\psi_t(x)\rangle}-e^{i\langle v,\psi_t(y)\rangle}\right||f(\psi_t(x))|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\right)\le 2C\mathbb{E}\left(\frac{|\psi_t(x)-\psi_t(y)|^{\delta}}{|x-y|^{\epsilon}}\right)\\
&\le 2C\mathbb{E}\left(|M|^{\delta}\right)|x-y|^{\delta-\epsilon}\le 2C\mathbb{E}\left(|M|^{\delta}\right)t^{\delta-\epsilon}. \quad (6.18)
\end{aligned}
$$

Similarly, we obtain the estimate of the second term. Indeed,

$$
\begin{aligned}
&\mathbb{E}\left(\frac{\left|e^{i\langle v,\psi_t(y)\rangle}(f(\psi_t(x))-f(\psi_t(y)))\right|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\right)\le \mathbb{E}\left(\frac{|f(\psi_t(x))-f(\psi_t(y))|}{|x-y|^{\epsilon}}\right)\\
&\le 2C\mathbb{E}\left(|M|^{\delta}\right)|x-y|^{\delta-\epsilon}\le 2C\mathbb{E}\left(|M|^{\delta}\right)t^{\delta-\epsilon}. \quad (6.19)
\end{aligned}
$$

Remaining (6.16) and (6.17) are estimated in the similar way, since $\left|\overline{\psi}(x)-\overline{\psi}(y)\right|\le |M||x-y|$ by definition of $\overline{\psi}$. Now consider the second term of (6.13) ($|x-y|>t$) and notice, that

$$
\begin{aligned}
&(T_{t,v}-T_v)f(x)-(T_{t,v}-T_v)f(y)\\
&=\mathbb{E}((e^{i\langle v,\psi_t(x)\rangle}-e^{i\langle v,\overline{\psi}(x)\rangle})f(\psi_t(x))) \quad (6.20)\\
&\quad+\mathbb{E}(e^{i\langle v,\overline{\psi}(x)\rangle}(f(\psi_t(x))-f(\overline{\psi}(x)))) \quad (6.21)\\
&\quad-\mathbb{E}((e^{i\langle v,\psi_t(y)\rangle}-e^{i\langle v,\overline{\psi}(y)\rangle})f(\psi_t(y))) \quad (6.22)\\
&\quad-\mathbb{E}(e^{i\langle v,\overline{\psi}(y)\rangle}(f(\psi_t(y))-f(\overline{\psi}(y)))). \quad (6.23)
\end{aligned}
$$

As before, we will estimate (6.20), (6.21), (6.22) and (6.23) separately using (L2) and (L3). Indeed, for every $0<\delta\le 1$ such that $\epsilon<\delta<\alpha$ we have

$$
\begin{aligned}
&\mathbb{E}\left(\frac{\left|e^{i\langle v,\psi_t(x)\rangle}-e^{i\left\langle v,\overline{\psi}(x)\right\rangle}\right||f(\psi_t(x))|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\right)\le 2C\mathbb{E}\left(\frac{\left|\psi_t(x)-\overline{\psi}(x)\right|^{\delta}}{|x-y|^{\epsilon}}\right)\\
&\le 2C\mathbb{E}\left(\frac{t^{\delta}|Q|^{\delta}}{|x-y|^{\epsilon}}\right)\le 2C\mathbb{E}\left(|Q|^{\delta}\right)t^{\delta-\epsilon}. \quad (6.24)
\end{aligned}
$$

Similarly, we obtain the estimate for the second term. Indeed,

$$\mathbb{E}\left(\frac{\left|e^{i\langle v,\overline{\psi}(x)\rangle}\left(f(\psi_t(x))-f\left(\overline{\psi}(x)\right)\right)\right|}{|x-y|^{\epsilon}(1+|x|)^{\lambda}(1+|y|)^{\lambda}}\right)\le\mathbb{E}\left(\frac{\left|f(\psi_t(x))-f\left(\overline{\psi}(x)\right)\right|}{|x-y|^{\epsilon}}\right)$$
$$\le 2C\mathbb{E}\left(\frac{\left|\psi_t(x)-\overline{\psi}(x)\right|^{\delta}}{|x-y|^{\epsilon}}\right)\le 2C\mathbb{E}\left(\frac{t^{\delta}|Q|^{\delta}}{|x-y|^{\epsilon}}\right)\le 2C\mathbb{E}\left(|Q|^{\delta}\right)t^{\delta-\epsilon}. \tag{6.25}$$

Also remaining (6.22) and (6.23) can be estimated analogously. Hence, in view of (6.18), (6.19), (6.24) and (6.25), we obtain (6.11). For (6.12) notice that

$$(T_{t,v}-T_v)f(x)=\mathbb{E}((e^{i\langle v,\psi_t(x)\rangle}-e^{i\langle v,\overline{\psi}(x)\rangle})f(\psi_t(x)))$$
$$+\mathbb{E}(e^{i\langle v,\overline{\psi}(x)\rangle}(f(\psi_t(x))-f(\overline{\psi}(x)))). \tag{6.26}$$

We have,

$$\mathbb{E}\left(\frac{|e^{i\langle v,\psi_t(x)\rangle}-e^{i\langle v,\overline{\psi}(x)\rangle}||f(\psi_t(x))|}{(1+|x|)^{\rho}}\right)\le 2C\mathbb{E}(|\psi_t(x)-\overline{\psi}(x)|^{\delta})$$
$$\le 2C\mathbb{E}(t^{\delta}|Q|^{\delta})\le 2C\mathbb{E}\left(|Q|^{\delta}\right)t^{\delta}, \tag{6.27}$$

and

$$\mathbb{E}\left(\frac{|e^{i\langle v,\overline{\psi}(x)\rangle(f(\psi_t(x))-f(\overline{\psi}(x)))|}}{(1+|x|)^{\rho}}\right)\le\mathbb{E}(|f(\psi_t(x))-f(\overline{\psi}(x))|)$$
$$\le 2C\mathbb{E}(|\psi_t(x)-\overline{\psi}(x)|^{\delta})\le 2C\mathbb{E}(t^{\delta}|Q|^{\delta})\le 2C\mathbb{E}(|Q|^{\delta})t^{\delta}. \tag{6.28}$$

Combining (6.27) with (6.28) we obtain (6.12) which completes the proof of the Lemma. □

Lemma 6.29 *Let h_v be the eigenfunction for operator T_v defined in (5.25), then*

$$\Delta_t(\Pi_{T,t}-\Pi_{T,0})h_v=\frac{1}{2\pi i}\int_{|z-1|=\xi_1}\frac{1}{z-1}(z-P_{t,v})^{-1}\Delta_t(T_{t,v}-T_v)h_v\,dz, \tag{6.30}$$

where $\xi_1>0$ was defined in Proposition 5.4.

Proof Notice, that $(z-T_v)^{-1}h_v=\frac{1}{z-1}h_v$,

$$(z-T_{t,v})^{-1}-(z-T_v)^{-1}=(z-T_{t,v})^{-1}(T_{t,v}-T_v)(z-T_v)^{-1},$$

and by the definition (5.1)

$$(z-P_{t,v})^{-1}\Delta_t=\Delta_t(z-T_{t,v})^{-1},$$

Then

$$\begin{aligned}\Delta_t(\Pi_{T,t}-\Pi_{T,0})h_v &= \frac{1}{2\pi i}\int\limits_{|z-1|=\xi_1}\Delta_t(z-T_{t,v})^{-1}(T_{t,v}-T_v)(z-T_v)^{-1}h_v\,dz\\ &= \frac{1}{2\pi i}\int\limits_{|z-1|=\xi_1}\frac{1}{z-1}\Delta_t(z-T_{t,v})^{-1}(T_{t,v}-T_v)h_v\,dz\\ &= \frac{1}{2\pi i}\int\limits_{|z-1|=\xi_1}\frac{1}{z-1}(z-P_{t,v})^{-1}\Delta_t(T_{t,v}-T_v)h_v\,dz,\end{aligned}$$

which completes the proof of (6.30). □

Proof of Theorem 6.3 For every $f \in \mathcal{B}_{\rho,\epsilon,\lambda}$ and $|t| \le 1$ we have

$$\|\Delta_t f\|_{\rho,\epsilon,\lambda} \le |f|_\rho + |t|^\epsilon [f]_{\epsilon,\lambda}, \tag{6.31}$$

In view of (6.30), Proposition 5.4, inequalities (6.31), (6.11) and (6.12) with the function h_v we have

$$\begin{aligned}&\|\Delta_t(\Pi_{T,t}-\Pi_{T,0})h_v\|_{\rho,\epsilon,\lambda}\\ &\quad\le \frac{1}{2\pi\xi_1}\int\limits_{|z-1|=\xi_1}\left\|(z-P_{t,v})^{-1}\Delta_t(T_{t,v}-T_v)h_v\right\|_{\rho,\epsilon,\lambda}dz\\ &\quad\le D(|\Delta_t(T_{t,v}-T_v)h_v|_\rho + [\Delta_t(T_{t,v}-T_v)h_v]_{\epsilon,\lambda})\\ &\quad\le D(|(T_{t,v}-T_v)h_v|_\rho + t^\epsilon[(T_{t,v}-T_v)h_v]_{\epsilon,\lambda})\\ &\quad\le D(C_2t^\delta + t^\epsilon C_1 t^{\delta-\epsilon}) \le Ct^\delta,\end{aligned}$$

for every $0 \le t \le t_0$ and it completes the proof of (6.4). Now (6.6) follows. In order to prove (6.5) apply inequality (6.6) and (5.28) to obtain

$$\begin{aligned}\left|(\Delta_t\Pi_{T,t}h_v)(x)-1\right| &\le \left|\Pi_{T,t}(h_v)(tx)-\Pi_{T,0}(h_v)(tx)\right| + |h_v(tx)-1|\\ &\le t^\delta\left(C(1+|x|)^\rho + \frac{2}{1-\kappa(\delta)}|x|^\delta\right).\end{aligned}$$

Above inequality implies (6.5) and the proof is finished. □

6.2 Rate of convergence of eigenvalues and fractional expansions

In this section we study the fractional expansions (6.9) when $t \to 0$. Their behavior is strongly related to the asymptotics of the stationary measure ν at infinity. While Theorem 6.37 is proved then the limit Theorem 1.15 follows as in [4]. First we establish (6.8).

The proof goes along the same lines as in [4] with the function h_v playing the role of $\hat{\eta}_v$ there. Therefore, the details have been omitted. We shall use radial coordinates in $\mathbb{R}^d$ i.e. every point is expressed as tv where $t > 0$ and $v \in \mathbb{S}^{d-1}$.

Condition 6.32 *Assume that* $0 < \epsilon < 1$, $\lambda > 0$, $\lambda + 2\epsilon < \rho = 2\lambda$ *and* $2\lambda + \epsilon < \alpha$ *as in Proposition* 5.4 *and additionally*

- *If* $0 < \alpha \le 1$, *take any* $0 < \beta < \frac{1}{2}$ *such that* $\rho + 2\beta < \alpha$.
- *If* $1 < \alpha \le 2$, *take any* $\lambda > 0$ *such that* $\rho = 2\lambda < 1$ *and* $\rho + 1 < \alpha$.

Proposition 6.33 *Let* h_v *be the eigenfunction of operator* T_v *defined in* (5.25). *If* $0 < \alpha < 2$, *then*

$$\lim_{t\to 0} \frac{1}{t^\alpha} \int_{\mathbb{R}^d} (e^{it\langle v,x\rangle} - 1)(\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx))\nu(dx) = 0. \tag{6.34}$$

If $\alpha = 2$, *then*

$$\lim_{t\to 0} \frac{1}{t^2|\log t|} \int_{\mathbb{R}^d} (e^{it\langle v,x\rangle} - 1)(\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx))\nu(dx) = 0. \tag{6.35}$$

Proof In estimations below in view of Condition 6.32 we have to use appropriate parameters ϵ, λ, ρ, δ and η which are determined by α.

- If $0 < \alpha \le 1$, we take $\delta = \alpha - \beta > \rho + \beta > \epsilon$ and $\eta = 2\beta$.
- If $1 < \alpha \le 2$, we take $\delta = 1 > \epsilon$ and $\eta = 1$.

In view of (6.6), we have

$$\begin{aligned} &\left| \frac{1}{t^\alpha} \int_{\mathbb{R}^d} \left(e^{it\langle v,x\rangle} - 1\right) \left(\Pi_{T,t}(h_v)(tx) - \Pi_{T,0}(h_v)(tx)\right) \nu(dx) \right| \\ &\quad \le C_1 t^{\eta+\delta-\alpha} \int_{\mathbb{R}^d} |x|^\eta (1+|x|)^\rho \nu(dx) \le C_2 t^{\eta+\delta-\alpha}. \end{aligned} \tag{6.36}$$

for every $0 < t \le t_0 \le 1$. Notice that, if

- $0 < \alpha \le 1$, then $\eta + \delta - \alpha = 2\beta + \alpha - \beta - \alpha = \beta > 0$ and $\rho + \eta = \rho + 2\beta < \alpha$.
- $1 < \alpha \le 2$, then $\eta + \delta - \alpha = 1 + 1 - \alpha = 2 - \alpha \ge 0$ and $\rho + \eta = \rho + 1 < \alpha$.

This justifies inequalities in (6.36) and completes the proof of (6.34) and (6.35). □

Theorem 6.37 *Assume that* ψ_θ *satisfies Assumptions* 1.6, 1.7 *and* 1.14 *for* $\theta \in \Theta$. *We define* $S_n^x = \sum_{k=1}^n X_k^x$ *for* $n \in \mathbb{N}$. *Let* $h_v(x) = \mathbb{E}(e^{i\langle v, \sum_{k=1}^\infty \overline{\psi}_k \circ \ldots \circ \overline{\psi}_1(x)\rangle})$ *for* $x \in \mathbb{R}^d$, *where* $\overline{\psi}_k(x) = \overline{\psi}_{\theta_k}(x)$, *and* $\overline{\psi}_{\theta_k}$*'s were defined in (H1) of Assumption* 1.6. *Measure* ν *is the stationary measure for the recursion* (1.1), *and* Λ *and* σ_Λ *are the measures defined in Theorem* 4.3.

Case $0 < \alpha < 1$. *Let* Ξ_α^n *be the characteristic function of the random variable* $n^{-\frac{1}{\alpha}} S_n^x$. *Then for every* $t > 0$ *and* $v \in \mathbb{S}^{d-1}$

$$\lim_{n\to\infty} \Xi_\alpha^n(tv) = \Upsilon_\alpha(tv) = \exp(t^\alpha C_\alpha(v)), \tag{6.38}$$

where

$$C_\alpha(v) = \lim_{t\to 0} \frac{k_v(t) - 1}{t^\alpha} = \int_{\mathbb{R}^d} \left(e^{i\langle v,x\rangle} - 1\right) h_v(x)\Lambda(dx). \tag{6.39}$$

Case $\alpha = 1$. *Let* Ξ_1^n *be the characteristic function of the random variable* $n^{-1} S_n^x - n\xi(n^{-1})$. *Then for every* $t > 0$ *and* $v \in \mathbb{S}^{d-1}$

$$\lim_{n\to\infty} \Xi_1^n(tv) = \Upsilon_1(tv) = \exp(tC_1(v) + it\langle v, \tau(t)\rangle), \tag{6.40}$$

where

$$\begin{aligned} C_1(v) &= \lim_{t\to 0} \frac{k_v(t) - 1 - i\langle v, \xi(t)\rangle}{t} \\ &= \int_{\mathbb{R}^d} \left(\left(e^{i\langle v,x\rangle} - 1\right) h_v(x) - \frac{i\langle v, x\rangle}{1 + |x|^2}\right) \Lambda(dx), \end{aligned} \tag{6.41}$$

$\xi(t) = \int_{\mathbb{R}^d} \frac{tx}{1+|tx|^2} \nu(dx)$ *and* $\tau(t) = \int_{\mathbb{R}^d} \left(\frac{x}{1+|tx|^2} - \frac{x}{1+|x|^2}\right) \Lambda(dx)$.
Case $1 < \alpha < 2$. *Assume that* $m = \int_{\mathbb{R}^d} x\nu(dx)$. *Let* Ξ_α^n *be the characteristic function of the random variable* $n^{-\frac{1}{\alpha}}(S_n^x - nm)$. *Then for every* $t > 0$ *and* $v \in \mathbb{S}^{d-1}$

$$\lim_{n\to\infty} \Xi_\alpha^n(tv) = \Upsilon_\alpha(tv) = \exp(t^\alpha C_\alpha(v)), \tag{6.42}$$

where

$$\begin{aligned} C_\alpha(v) &= \lim_{t\to 0} \frac{k_v(t) - 1 - i\langle v, tm\rangle}{t^\alpha} \\ &= \int_{\mathbb{R}^d} ((e^{i\langle v,x\rangle} - 1)h_v(x) - i\langle v, x\rangle)\Lambda(dx). \end{aligned} \tag{6.43}$$

Case $\alpha = 2$. *Assume that* $m = \int_{\mathbb{R}^d} x\nu(dx)$. *Let* Ξ_2^n *be the characteristic function of the random variable* $(n\log n)^{-\frac{1}{2}} \left(S_n^x - nm\right)$. *Then for every* $t > 0$ *and* $v \in \mathbb{S}^{d-1}$

$$\lim_{n\to\infty} \Xi_2^n(tv) = \Upsilon_2(tv) = \exp(t^2 C_2(v)), \tag{6.44}$$

where

$$C_2(v) = \lim_{t\to 0} \frac{k_v(t) - 1 - i\langle v, tm\rangle}{2t^2|\log t|}$$
$$= -\frac{1}{4} \int_{\mathbb{S}^{d-1}} (\langle v, w\rangle^2 + 2\langle v, w\rangle\langle v, \mathbb{E}(\varphi(w))\rangle)\sigma_\Lambda(dw), \qquad (6.45)$$

and $\varphi(x) = \sum_{k=1}^{\infty} \overline{\psi}_k \circ \ldots \circ \overline{\psi}_1(x) = \sum_{k=1}^{\infty} M_k \cdot \ldots \cdot M_1 x$*, where* M_k*'s were defined in (H1) of Assumption 1.6.*

Moreover, $C_\alpha(tv) = t^\alpha C_\alpha(v)$ *for every* $t > 0$, $v \in \mathbb{S}^{d-1}$ *and* $\alpha \in (0, 1) \cup (1, 2]$*. If* $supp\sigma_\Lambda$ *spans* $\mathbb{R}^d$ *as a linear space, then* $\Re C_\alpha(v) < 0$ *for every* $v \in \mathbb{S}^{d-1}$ *and* $\alpha \in (0, 2]$.

Proof In order to obtain fractional expansions (6.39), (6.41), (6.43) and (6.45), we have to proceed as in Theorem 5.1 from [4] (see also [15]) using formula (6.2), Proposition 6.33, inequality (6.5) and convergence (4.4) which holds for every function from family $\mathcal{F}_\Lambda$. Proof of (6.38), (6.40), (6.42) and (6.44) base on section 6 from [4] (see also [15]). Using formula $h_v(tx) = h_{tv}(x)$ from Lemma 5.26 we obtain that $C_\alpha(tv) = t^\alpha C_\alpha(v)$ is valid for every $t > 0$, $v \in \mathbb{S}^{d-1}$ and $\alpha \in (0, 1) \cup (1, 2]$. Method developed in section 5.6 in [4] allows us to show that $\Re C_\alpha(v) < 0$ for every $v \in \mathbb{S}^{d-1}$ and $\alpha \in (0, 2]$. □

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References

1. Bálint, P., Gouëzel, S.: Limit theorems in the stadium billard. Commuun. Math. Phys. **263**, 451–512 (2006)
2. Benda, M.: A central limit theorem for contractive stochastic dynamical systems. J. Appl. Probab. **35**, 200–205 (1998)
3. Borkovec, M., Klüppelberg, C.: The tail of the stationary distribution of an autoregressive process with ARCH(1) errors. Ann. Appl. Probab. **11**(4), 1220–1241 (2001)
4. Buraczewski, D., Damek, E., Guivarc'h, Y.: Convergence to stable laws for a class multidimensional stochastic recursions. Probab. Theory Relat. Fields (2009, accepted)
5. Buraczewski, D., Damek, E., Guivarc'h, Y., Hulanicki, A., Urban, R.: Tail—homogeneity of stationary measures for some multidimensional stochastic recursions. Probab. Theory Relat. Fields **145**, 385–420 (2009)
6. Diaconis, P., Freedman, D.: Iterated random functions. SIAM Rev. **41**(1), 45–76 (1999)
7. Goldie, Ch.M.: Implicit renewal theory and tails of solutions of random equations. Ann. Appl. Probab. **1**(1), 126–166 (1991)
8. Goldie, Ch.M., Grübel, R.: Perpetuites with thin tails. Adv. Appl. Probab. **28**, 463–480 (1996)
9. Gouëzel, S.: Characterization of weak convergence of Birkhoff sums for Gibbs–Markov maps. Preprint (2008)
10. Grey, D.R.: Regular variation in the tail behaviour of solutions of random difference equations. Ann. Appl. Probab. **4**(1), 169–183 (1994)
11. Grincevičius, A.K.: On limit distribution for a random walk on the line. Lithuanian Math. J. **15**, 580–589 (1975)

12. Grincevičius, A.K.: Products of random affine transformations. Lithuanian Math. J. **20**, 279–282 (1980)
13. Guivarc'h, Y.: Extension d'un théorèm de Choquet-Deny à une class de group non abéliens. Astérisque **4** Soc. Math. France, pp. 41–60 (1973)
14. Guivarc'h, Y.: Heavy tail properties of multidimensional stochastic recursions. IMS Lect. Notes Monogr. Ser. Dyn. Stoch. **48**, 85–99 (2006)
15. Guivarc'h, Y., Le Page, É.: On spectral properties of a family of transfer operators and convergence to stable laws for affine random walks. Ergod. Theory Dyn. Syst. **28**, 423–446 (2008)
16. Guivarc'h, Y., Le Page, É.: On the tails of the stationary measure of affine random walks on the line (2010, preprint)
17. Hennion, H.: Sur un théorème spectral et son application aux noyaux lipchitziens. Proc. Am. Math. Soc. **118**, 627–634 (1993)
18. Hennion, H., Hervé, L.: Central limit theorems for iterated random Lipschitz mappings. Ann. Probab. **32**(3A), 1934–1984 (2004)
19. Hennion, H., Hervé, L.: Limit theorems for Markov chains and stochastic properties of dynamical systems by quasi-compactness. Lecture Notes in Mathematics, vol. 1766. Springer, Berlin (2001)
20. Hervé, L., Pène, F.: The Nagaev method via the Keller–Liverani theorem. Bull. Soc. Math. France (2010, accepted)
21. Ionescu-Tulcea, C.T., Marinescu, G.: Theoreme ergodique pour des classes doperations non complet-ement continues. Ann. Math. **52**(1), 140–147 (1950)
22. Jara, M., Komorowski, T., Olla, S.: Limit theorems for additive functionals of a Markov chain. Ann. Appl. Probab. **19**(6), 2270–2300 (2009)
23. Keller, G., Liverani, C.: Stability of the spectrum for transfer operators. Ann. Scuola Norm. Sup. Pisa. CI. Sci. **28**(4), 141–152 (1999)
24. Kesten, H.: Random difference equations and renewal theory for products of random matrices. Acta Math. **131**, 207–248 (1973)
25. Kesten, H.: Renewal theory for functionals of a Markov chain with general state space. Ann. Probab. **2**, 355–386 (1974)
26. Le Page, É.: Théorèmes de renouvellement pour les produits de matrices aléatoires. Équations aux différences aléatoires. Séminaires de probabilités Rennes 1983. Publ. Sém. Math., vol. 1. Univ. Rennes (1983)
27. Letac, G.: A contraction principle for certain Markov chains and its applications. Random matrices matrices and their applications (Brunswick, Maine, 1984). Contemporary Mathematic, vol. 50. Amer. Math. Soc., Providence, RI, pp. 263–273 (1986)
28. Liverani, C.: Invariant measures and their properties. A functional analytic point of view. In: Dynamical Systems. Part II: topological Geometrical and Ergodic Properties of Dynamics. Pubblicazioni della Classe di Scienze, Scuola Normale Superiore, Pisa. Centro di Ricerca Matematica "Ennio De Giorgi": Proceedings. Scuola Normale Superiore, Pisa (2004)
29. Maxwell, M., Woodroofe, M.: A central limit theorem for additive functions of a Markov chain. Ann. Probab. **28**, 713–724 (2000)
30. Nagaev, S.V.: Some limit theorems for stationary Markov chains. Theory Probab. Appl. **11**, 378–406 (1957)
31. Raugi, A.: A general Choquet–Deny theorem for nilpotent groups. Ann. Inst. H. Poincaré Probab. Stat. **40**, 677–683 (2004)
32. Vervaat, W.: On a stochastic difference equation and a representation of non-negativeve infinitely divisible random variables. Adv. Appl. Prob. **11**, 750–783 (1979)
33. Woodroofe, M., Wu, W.B.: A central limit theorem for iterated random functions. J. Appl. Probab. **37**, 748–755 (2000)