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Regularization properties of the 2D homogeneous Boltzmann equation without cutoff

Vlad Bally · Nicolas Fournier

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Abstract We consider the 2-dimensional spatially homogeneous Boltzmann equation for hard potentials. We assume that the initial condition is a probability measure that has some exponential moments and is not a Dirac mass. We prove some regularization properties: for a class of very hard potentials, the solution instantaneously belongs to H^r, for some $r \in (-1, 2)$ depending on the parameters of the equation. Our proof relies on the use of a well-suited Malliavin calculus for jump processes.

Keywords Kinetic equations · Hard potentials without cutoff · Malliavin calculus · Jump processes

Mathematics Subject Classification (2000) 60H07 · 82C40

1 Introduction

The Boltzmann equation

We consider a spatially homogeneous gas in dimension 2 modeled by the Boltzmann equation. The density $f_t(v)$ of particles with velocity $v \in \mathbb{R}^2$ at time $t \geq 0$ solves

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V. Bally (✉)
LAMA UMR 8050, Université Paris Est, Cité Descartes, 5 boulevard Descartes,
Champs sur Marne, 77454 Marne la Vallée Cedex, France
e-mail: vlad.bally@univ-mlv.fr

N. Fournier
LAMA UMR 8050, Faculté de Sciences et Technologies,
Université Paris Est, 61, avenue du Général de Gaulle, 94010 Créteil Cedex, France
e-mail: nicolas.fournier@univ-paris12.fr

$$\partial_t f_t(v) = \int_{\mathbb{R}^2} dv_* \int_{-\pi/2}^{\pi/2} d\theta\, B(|v-v_*|,\theta)\big[f_t(v')f_t(v'_*) - f_t(v)f_t(v_*)\big], \qquad (1.1)$$

where

$$v' = \frac{v+v_*}{2} + R_\theta\left(\frac{v-v_*}{2}\right), \quad v'_* = \frac{v+v_*}{2} - R_\theta\left(\frac{v-v_*}{2}\right)$$

and where R_θ is the rotation of angle θ. One usually integrates θ on $(-\pi,\pi)$, but a famous trick allows one to restrict the integration to $[-\pi/2,\pi/2]$ without loss of generality, see e.g. the argument in the introduction of [2]. The cross section $B(|v-v_*|,\theta) \geq 0$ is given by physics and depends on the type of interaction between particles. We refer to the book of Cercignani [9] for a good physical reference on the Boltzmann equation and to the review papers of Villani [24] and Alexandre [1] for many details on what is known from the mathematical point of view.

Conservation of mass, momentum and kinetic energy hold for reasonable solutions to (1.1):

$$\forall t \geq 0, \quad \int_{\mathbb{R}^2} f_t(v)\psi(v)\,dv = \int_{\mathbb{R}^2} f_0(v)\psi(v)\,dv, \qquad \psi = 1, v, |v|^2$$

and we classically may assume without loss of generality that $\int_{\mathbb{R}^2} f_0(v)\,dv = 1$ and $\int_{\mathbb{R}^2} v f_0(dv) = 0$.

Assumptions

We shall assume here that for some $\gamma \in (0,1)$, $\nu \in (0,1/2)$, some even function $b : [-\pi/2,\pi/2]\backslash\{0\} \mapsto \mathbb{R}_+$,

$$\begin{cases} B(|v-v_*|,\theta) = |v-v_*|^\gamma b(\theta), \\ \exists\, 0 < c < C, \quad \forall\, \theta \in (0,\pi/2], \quad c\theta^{-1-\nu} \leq b(\theta) \leq C\theta^{-1-\nu}, \\ \forall\, k \geq 1, \quad \exists\, C_k, \quad \forall\, \theta \in (0,\pi/2], \quad |b^{(k)}(\theta)| \leq C_k \theta^{-1-\nu-k}. \end{cases} \qquad (\mathbf{A}(\gamma,\nu))$$

This assumption is made by analogy to the case where particles collide by pairs due to a repulsive force proportional to $1/r^s$ for some $s > 2$ in dimension 3, for which $\gamma = (s-5)/(s-1)$ and $b(\theta) \simeq |\theta|^{-1-\nu}$, with $\nu = 2/(s-1)$. We aim to study here hard potentials ($s > 5$), for which $\gamma \in (0,1)$ and $\nu \in (0,1/2)$.

Weak solutions

For $\theta \in (-\pi/2,\pi/2)$, we introduce

$$A(\theta) = \frac{1}{2}(R_\theta - I) = \frac{1}{2}\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix}.$$

Note that $v' = v + A(\theta)(v - v_*)$ and that for $X \in \mathbb{R}^2$,

$$|A(\theta)X|^2 = \frac{1}{2}(1 - \cos\theta)|X|^2 \leq \frac{\theta^2}{4}|X|^2. \tag{1.2}$$

Definition 1.1 Assume ($\mathbf{A}(\gamma, \nu)$) for some $\nu \in (0, 1)$ and $\gamma \in (0, 1]$. A family $(f_t)_{t\in[0,T]}$ of probability measures on $\mathbb{R}^2$ is said to be a weak solution of (1.1) if for all $t \in [0, T]$,

$$\int_{\mathbb{R}^2} v f_t(dv) = \int_{\mathbb{R}^2} v f_0(dv) \quad \text{and} \quad \int_{\mathbb{R}^2} |v|^2 f_t(dv) = \int_{\mathbb{R}^2} |v|^2 f_0(dv) < \infty \tag{1.3}$$

and if for any $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ globally Lipschitz continuous and any $t \in [0, T]$,

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi(v) f_t(dv) = \int_{\mathbb{R}^2} f_t(dv) \int_{\mathbb{R}^2} f_t(dv_*) \int_{-\pi/2}^{\pi/2} b(\theta)d\theta |v - v_*|^\gamma \\ [\psi(v + A(\theta)(v - v_*)) - \psi(v)]. \tag{1.4}$$

The right hand side of (1.4) is well-defined due to (1.3), (1.2) and because $\int_{-\pi/2}^{\pi/2} |\theta| b(\theta)d\theta < \infty$ thanks to ($\mathbf{A}(\gamma, \nu)$) with $\nu \in (0, 1)$. As shown in [17, Corollary 2.3 and Lemma 4.1], we have the following result.

Theorem 1.2 *Assume ($\mathbf{A}(\gamma, \nu)$) for some $\nu \in (0, 1)$ and $\gamma \in (0, 1]$. Assume also that $b(\theta) = \tilde{b}(\cos\theta)$, for some nondecreasing convex C^1 function $\tilde{b}$ on $[0, 1)$. Let f_0 be a probability measure on $\mathbb{R}^2$ such that for some $\delta \in (\gamma, 2)$, $\int_{\mathbb{R}^2} e^{|v|^\delta} f_0(dv) < \infty$. There exists a unique weak solution $(f_t)_{t\geq 0}$ to (1.1) starting from f_0. Furthermore, for all $\kappa \in (0, \delta)$, $\sup_{t\geq 0} \int_{\mathbb{R}^2} e^{|v|^\kappa} f_t(dv) < \infty$.*

The additional condition that $\tilde{b}$ is nondecreasing and convex is made for convenience and typically holds if $b(\theta) \simeq |\theta|^{-1-\nu}$.

Sobolev spaces

For f a probability measure on $\mathbb{R}^2$, we set, for $\xi \in \mathbb{R}^2$, $\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{i\langle \xi, x\rangle} f(dx)$. Recall that for $r \in \mathbb{R}$,

$$H^r(\mathbb{R}^2) = \left\{f, ||f||_{H^r(\mathbb{R}^2)} < \infty\right\}, \quad \text{where} \quad ||f||^2_{H^r(\mathbb{R}^2)} = \int_{\mathbb{R}^2} (1 + |\xi|^2)^r |\widehat{f}(\xi)|^2 d\xi.$$

Let us recall the following classical results. For f a probability measure on $\mathbb{R}^2$,

- $f \in H^r(\mathbb{R}^2)$ for every $r < -1$;
- if $f \in H^r(\mathbb{R}^2)$ for some $r \geq 0$, then f has a density that belongs to $L^2(\mathbb{R}^2)$;
- if $f \in H^r(\mathbb{R}^2)$ for some $r > 1$, then f has a bounded and continuous density.

Main result

We need to introduce, for $\nu \in (0, 1/2)$ and $\gamma \in (0, 1)$ satisfying $\gamma > \nu^2/(1-2\nu)$,

$$a_{\gamma,\nu} = \frac{1}{2}\left[\sqrt{(\gamma+\nu+1)^2 + 4\left(\frac{\gamma(1-2\nu)}{\nu} - \nu\right)} - (\gamma+\nu+1)\right] > 0, \quad (1.5)$$

$$q_{\gamma,\nu} = \begin{cases} a_{\gamma,\nu} & \text{if } a_{\gamma,\nu} \le 2, \\ \dfrac{(2+\gamma)(1-2\nu)-\nu^2}{(1+\gamma+\nu)\nu+1} & \text{if } a_{\gamma,\nu} > 2. \end{cases} \quad (1.6)$$

As we will see in Lemma 5.3, $q_{\gamma,\nu} > 2$ in the latter case. We would like to comment on these values. However, we believe that they have no physical or mathematical meaning: we found these values for $q_{\gamma,\nu}$ after some very technical computations, which are probably not optimal in many places.

Theorem 1.3 *Assume* $(\boldsymbol{A}(\gamma,\nu))$*, for some* $\gamma \in (0,1)$, $\nu \in (0, 1/2)$*, such that* $\gamma > \nu^2/(1-2\nu)$*. Consider a weak solution* $(f_t)_{t\in[0,T]}$ *to* (1.1) *such that* f_0 *is not a Dirac mass and, for some* $\delta \in (\gamma \vee \nu, 1)$,

$$\sup_{t\in[0,T]} \int_{\mathbb{R}^2} e^{|v|^\delta} f_t(dv) < \infty. \quad (1.7)$$

(i) *For all* $t_0 \in (0, T]$,

$$\forall q \in (0, q_{\gamma,\nu}), \quad \forall \xi \in \mathbb{R}^2, \quad \sup_{[t_0,T]} |\widehat{f_t}(\xi)| \le C_{t_0,T,q}(1+|\xi|)^{-q},$$
$$\forall r < q_{\gamma,\nu} - 1, \quad \sup_{[t_0,T]} ||f_t||_{H^r(\mathbb{R}^2)} < \infty,$$
$$\forall q \in (0, q_{\gamma,\nu}), \quad \forall v_0 \in \mathbb{R}^2, \quad \forall \epsilon > 0, \quad \sup_{[t_0,T]} f_t(Ball(v_0,\epsilon)) \le C_{t_0,T,q}\epsilon^q.$$

(ii) *If* $\nu \in (0, 1/3)$ *and* $\gamma > (2\nu + 2\nu^2)/(1-3\nu)$*, then* $q_{\gamma,\nu} > 1$*. Thus* f_t *has a density belonging to* $L^2(\mathbb{R}^2)$ *for all* $t \in (0, T]$.
(iii) *If finally* $\nu \in (0, 1/4)$ *and* $\gamma > (6\nu + 3\nu^2)/(1-4\nu)$*, then* $q_{\gamma,\nu} > 2$*. Thus* f_t *has a continuous and bounded density for all* $t \in (0, T]$.

Discussion about the result

In the realistic case where $\gamma = (s-5)/(s-1)$ and $\nu = 2/(s-1)$, point (i) applies if $s > 7$, point (ii) applies if $s > 8+\sqrt{33} \simeq 13.75$, point (iii) applies if $s > 13+2\sqrt{31} \simeq 24.14$.

When at least point (ii) applies, this shows in particular that for all $t > 0$, $H(f_t) < \infty$, where the entropy is defined as $H(f) := \int_{\mathbb{R}^2} f(v)\log f(v)dv$. This allows us to apply many results concerning regularization (see e.g. Villani [23] or Alexandre-Desvillettes-Villani-Wennberg [2]) or large time behavior (see e.g. Villani [24]) where the finiteness of entropy is required.

Until the middle of the 90's, almost all the works on the Boltzmann equation were assuming Grad's angular cutoff, where the cross section B, which physically satisfies $\int_0^{\pi/2} B(|v-v_*|,\theta)d\theta = \infty$ was replaced by an integrable cross section. A fully general existence result was obtained by Villani [22] for true physical cross sections without cutoff. As shown in Mouhot-Villani [20], no regularization may occur under Grad's angular cutoff. Intuitively, this comes from the fact that each particle is subjected to finitely (resp. infinitely) many collisions on each time interval in the case with (resp. without) cutoff. See however [15] where it is shown on a simplified model that some regularization might occur under Grad's angular cutoff, but for some very soft potentials (i.e. with $\gamma < -1$).

Here we deal with *true* hard potentials and we thus have to overcome the three following difficulties: $|w|^\gamma$ vanishes at 0, explodes at infinity and is not smooth at 0. This lack of regularity is the basis of many technical complications.

The first papers on regularization for the homogeneous Boltzmann equation seem to be those of Desvillettes [10,11], that concern Maxwell molecules, that is $\gamma = 0$. This is the most simple case, since then $|v-v_*|^\gamma$ is constant. For the one-dimensional Kac equation, he proves that $f_t \in C^\infty$ for all $t > 0$, while for the 2D Boltzmann equation, he shows that f_t almost lies in H^1 for all $t > 0$. Still in the case of Maxwell molecules, Alexandre-El Safadi [3] have shown, in the realistic 3D case, that $f_t \in C^\infty$ as soon as $t > 0$. In all these works, $H(f_0)$ is supposed to be finite. Using a probabilistic approach, Graham-Méléard [18] (for the Kac equation) and [14] (for the 2D case) proved that if f_0 is a measure with some moments of all orders and is not a Dirac mass, then $f_t \in C^\infty$ for all $t > 0$. The main advantage of these works is that the finiteness of entropy is not required, but they still have not been extended to the 3D case.

Many papers deal with the case of regularized hard potentials, where $|v-v_*|^\gamma$ is replaced by something like $(\epsilon^2 + |v-v_*|^2)^{\gamma/2}$. In this situation, Desvillettes-Wennberg [13], Alexandre-El Safadi [4], Huo-Morimoto-Ukai-Yang [19] have shown that if $H(f_0) < \infty$, then $f_t \in C^\infty$ for all $t > 0$ for any $\gamma \in (0,1)$, any $\nu \in (0,2)$, in any dimension. See Alexandre [1] for a review.

In the case of the Landau equation, which is a diffusion approximation of the Boltzmann equation, Desvillettes-Villani [12] have obtained a very complete regularization result, for *true* hard potentials and initial conditions with a finite entropy. The point is that for the Landau equation, the computations are much less intricate.

But to our knowledge, the only regularization result that concerns the homogeneous Boltzmann equation for *true* hard potentials is that of Alexandre-Desvillettes-Villani-Wennberg [2]: in any dimension $d \geq 2$, if f_0 is a function such that $H(f_0) < \infty$, then any weak solution satisfies $\sqrt{f_t} \in H^{\nu/2}_{loc}(\mathbb{R}^2)$ for all $t > 0$, for any value of $\gamma \in (-d, 1)$ and any value of $\nu \in (0,2)$. The main idea of this paper is very simple. Since the entropy $H(f_t)$ is bounded below and nonincreasing as a function of time, its derivative, called *entropy dissipation*, is finite. In [2], a lowerbound of this *entropy dissipation* involving the regularity of $\sqrt{f_t}$ is proved. Let us insist on the fact that regularization is only one of the many applications of [2].

The main limitation of our study is that we work in dimension 2. Furthermore we have to assume at least $s > 7$. The two main positive points of the present paper are that (i) we deal with *true* hard potentials and (ii) we assume no regularity at all on the initial condition (in [2–4,10,11,13,19], f_0 is already a function): we only suppose that f_0 is not a Dirac mass. This is a necessary condition for regularization, since Dirac masses are stationary solutions to (1.1).

If $\nu > 0$ is small and $\gamma \in (0,1)$ is large, our result seems really competitive. For example if $\gamma = (s-5)/(s-1)$ and $\nu = 2/(s-1)$, then (denoting by $H^{r-} = \cap_{s\in(0,r)} H^s$),

- with $s = 15$ we obtain $f_t \in H^{(1/7)-}(\mathbb{R}^2)$,
- with $s = 25$ we obtain get $f_t \in H^{(172/167)-}(\mathbb{R}^2)$,
- with $s = 101$, we obtain $f_t \in H^{(4504/2599)-}(\mathbb{R}^2)$;

Let us finally mention that for any values of $\gamma \in (0,1)$ and $\nu \in (0,1/2)$, our result will never provide a better estimate than $f_t \in H^{2-}(\mathbb{R}^2)$. Here again, we believe that this is a technical limitation, but our tedious proof leads to such a maximal regularity.

It might seem surprising that we assume some smoothness on the angular cross section b. In most works, the third line of $(A(\gamma,\nu))$ is not required, see [2–4,13,14,18]. For example, the lowerbound $b(\theta) \geq c\theta^{-1-\nu}$ is clearly sufficient if one uses a lowerbound of the entropy dissipation: by monotonicity, one then can assume that $b(\theta) = c\theta^{-1-\nu}$. Intuitively, the main idea is the following. Write $b = b_0 + b_1$, with b_1 possibly non smooth and $b_0(\theta) = c\theta^{-1-\nu}$. The collisions due to b_0 produce smooth collisions, from which the regularizating effect is deduced, while b_1 produce some (possibly) non smooth collisions, which can only have a (non-quantified) regularizing effect. We have not been able to make rigorous such considerations while using the present method. However, the third line of $(A(\gamma,\nu))$ does not seem so restrictive from a physical point of view.

We conclude this subsection with a remark on regularized hard potentials: if $\nu \in (0,1/3)$, our method allows us to extend the result of Desvillettes-Wennberg [13] to initial conditions with infinite entropy.

Remark 1.4 Assume that $B(|v-v_*|,\theta) = (\epsilon^2 + |v-v_*|^2)^{\gamma/2} b(\theta)$, for some $\epsilon > 0$, some $\gamma \in (0,1)$ and some b satisfying the same conditions as in $(\mathbf{A}(\gamma,\nu))$ for some $\nu \in (0,1/2)$. With our method, it is possible to prove that for $(f_t)_{t\in[0,T]}$ a weak solution to (1.1) satisfying (1.7) and such that f_0 is not a Dirac mass, for $0 < t_0 < T$, $\sup_{[t_0,T]} |\widehat{f_t}(\xi)| \leq C_{t_0,T,r}(1+|\xi|)^{-r}$ for all $r \in (0, 1/\nu - 2)$. In particular if $\nu \in (0,1/3)$, we deduce that $f_t \in L^2(\mathbb{R}^2)$ so that $H(f_t) < \infty$ for any $t > 0$. Thus we can apply the result of [13] and deduce that $f_t \in C^\infty(\mathbb{R}^2)$ for all $t > 0$.

Discussion about the method

Following the seminal work of Tanaka [21], we will build a stochastic process $(V_t)_{t\in[0,T]}$ such that for each $t \in [0,T]$, $\mathcal{L}(V_t) = f_t$. This process will solve a jumping stochastic differential equation. Then we will use some Malliavin calculus to study the smoothness of f_t, in the spirit of Graham-Méléard [18]. When using the classical Malliavin calculus for jumps processes of Bichteler-Gravereaux-Jacod [7], one can only treat the case of a constant rate of jump, which corresponds here to the case where $\gamma = 0$. This was done in [14,18]. We thus have to build a suitable Malliavin calculus.

Recently Bally-Clément [5] introduced a new method, still inspired by [7] which allows one to deal with equations with a non-constant rate of jump. They discuss equations with a similar structure as (1.1), but with much more regular coefficients. Here we use the same method, but we have to overcome some nontrivial difficulties related to the singularity and unboundedness of the coefficients. The nondegeneracy property is also quite complicated to establish, in particular because $|v - v_*|^\gamma$ vanishes on the diagonal and because (1.1) is nonlinear.

Plan of the paper

In the next section, we give the probabilistic interpretation of (1.1) in terms of a jumping S.D.E. We also build some approximations of the process and study their rate of convergence. Another representation of the approximating processes is given in Sect. 3. In Sect. 4, we prove an integration by parts formula for the approximating process, using the Malliavin calculus introduced in [5]. We conclude the proof in Sect. 5. An appendix containing technical results lies at the end of the paper.

Notation

In the whole paper, we assume without loss of generality that

$$\int_{\mathbb{R}^2} v f_0(dv) = 0 \quad \text{and} \quad e_0 = \int_{\mathbb{R}^2} |v|^2 f_0(dv) > 0. \tag{1.8}$$

Observe that $e_0 > 0$, because else, f_0 would be the Dirac mass at 0. We always assume at least that $(\mathbf{A}(\gamma, \nu))$ hold for some $\gamma \in (0, 1)$, some $\nu \in (0, 1)$. We denote by $(f_t)_{t \geq 0}$ a weak solution to (1.1) satisfying (1.7) for some $\delta > \gamma$. We consider η_0 such that

$$\eta_0 \in (1/\delta, 1/(\gamma \vee \nu)). \tag{1.9}$$

For $v_0 \in \mathbb{R}^2$ and $r > 0$, we denote by

$$Ball(v_0, r) = \{v \in \mathbb{R}^2, |v - v_0| < r\}$$

the open ball centered at v_0 with radius r. We will always write C for a finite (large) constant and c for a positive (small) constant, of which the values may change from line to line and which depend only on $b, \nu, \gamma, \delta, \eta_0, T, f_0$. When a constant depends on another quantity, we will always indicate it. For example, C_{t_0} or c_{t_0} stand for constants depending on $b, \nu, \gamma, \delta, \eta_0, T, f_0$ and t_0.

2 Probabilistic interpretation and approximation

Recall that we are given a fixed weak solution $(f_t)_{t\in[0,T]}$ to (1.1). We wish to build a Markov process $(V_t)_{t\in[0,T]}$, solution to a jumping stochastic differential equation, whose time marginals are equal to $(f_t)_{t\in[0,T]}$.

We consider a Poisson measure $N(ds, d\theta, dv, du)$ on $[0, T] \times [-\pi/2, \pi/2] \times \mathbb{R}^2 \times [0, \infty)$ with intensity measure $ds b(\theta) d\theta f_s(dv) du$. Then for a $\mathbb{R}^2$-valued f_0-distributed random variable V_0 independent of N, we consider the $\mathbb{R}^2$-valued stochastic differential equation, setting $E = [-\pi/2, \pi/2] \times \mathbb{R}^2 \times [0, \infty)$,

$$V_t = V_0 + \int_0^t \int_E A(\theta)(V_{s-} - v) \mathbb{1}_{\{u \le |V_{s-} - v|^\gamma\}} N(ds, d\theta, dv, du). \qquad (2.1)$$

We will prove that this equation has a unique solution, which furthermore satisfies $\mathcal{L}(V_t) = f_t$ for all $t \in [0, T]$.

We also introduce some approximations of the process $(V_t)_{t\in[0,T]}$. We consider a C^∞ even nonnegative function χ supported by $(-1, 1)$ satisfying $\int_{\mathbb{R}} \chi(x)dx = 1$. Then we introduce, for $x \in \mathbb{R}$ and $\epsilon \in (0, 1)$, (recall (1.9))

$$\Gamma_\epsilon = [\log(1/\epsilon)]^{\eta_0}, \quad \phi_\epsilon(x) = \int_{\mathbb{R}} ((y \vee 2\epsilon) \wedge \Gamma_\epsilon) \frac{\chi((x-y)/\epsilon)}{\epsilon} dy. \qquad (2.2)$$

Observe that we have $2\epsilon \le \phi_\epsilon(x) \le \Gamma_\epsilon$ for all $x \ge 0$, $\phi_\epsilon(x) = x$ for $x \in [3\epsilon, \Gamma_\epsilon - 1]$, $\phi_\epsilon(x) = 2\epsilon$ for $x \in [0, \epsilon]$ and $\phi_\epsilon(x) = \Gamma_\epsilon$ for $x \ge \Gamma_\epsilon + 1$. We find $\epsilon_0 > 0$ small enough, in such a way that for $\epsilon \in (0, \epsilon_0)$, $3\epsilon < 1 < \Gamma_\epsilon - 1$ and consider, for $\epsilon \in (0, \epsilon_0)$, the equation

$$V_t^\epsilon = V_0 + \int_0^t \int_E A(\theta)(V_{s-}^\epsilon - v) \mathbb{1}_{\{u \le \phi_\epsilon^\gamma(|V_{s-}^\epsilon - v|)\}} N(ds, d\theta, dv, du), \qquad (2.3)$$

Next we introduce, for $\zeta \in (0, 1)$, a function $I_\zeta : \mathbb{R}_+ \mapsto [0, 1]$ such that $I_\zeta(x) = 1$ for $x \ge \zeta$ and vanishing on a neighborhood of 0. We will choose I_ζ in the next section as a smooth version of $\mathbb{1}_{\{x \ge \zeta\}}$. We consider the equation

$$V_t^{\epsilon,\zeta} = V_0 + \int_0^t \int_E A(\theta)(V_{s-}^{\epsilon,\zeta} - v) \mathbb{1}_{\{u \le \phi_\epsilon^\gamma(|V_{s-}^{\epsilon,\zeta} - v|)\}} I_\zeta(|\theta|) N(ds, d\theta, dv, du). \quad (2.4)$$

The goal of this section is to check the following results.

Proposition 2.1 (i) *There exists a unique càdlàg adapted solution* $(V_t)_{t\in[0,T]}$ *to* (2.1). *For each* $\epsilon \in (0, \epsilon_0)$ *and each* $\zeta \in (0, 1)$, *there exist some unique càdlàg adapted solutions* $(V_t^\epsilon)_{t\in[0,T]}$ *and* $(V_t^{\epsilon,\zeta})_{t\in[0,T]}$ *to* (2.3) *and* (2.4).

(ii) *For all $t \in [0, T]$, V_t is f_t-distributed.*
(iii) *For any $\kappa \in (\nu, \delta)$, any $\epsilon \in (0, \epsilon_0)$, any $\zeta \in (0, 1)$,*

$$\mathbb{E}\left[\sup_{[0,T]}\left(e^{|V_t|^\kappa} + e^{|V_t^\epsilon|^\kappa} + e^{|V_t^{\epsilon,\zeta}|^\kappa}\right)\right] \le C_\kappa.$$

(iv) *For any $\beta \in (\nu, 1]$, any $\epsilon \in (0, \epsilon_0)$, any $\zeta \in (0, 1)$,*

$$\sup_{[0,T]} \mathbb{E}\left[|V_t^\epsilon - V_t^{\epsilon,\zeta}|^\beta\right] \le C_\beta e^{C_\beta \Gamma_\epsilon^\gamma} \zeta^{\beta-\nu}.$$

(v) *Assume furthermore that for some $\alpha \ge 0$, some K, for all $v_0 \in \mathbb{R}^2$, for all $\epsilon \in (0, 1]$,*

$$\sup_{[0,T]} f_t(Ball(v_0, \epsilon)) \le K\epsilon^\alpha.$$

*This **always** holds with $K = 1$, $\alpha = 0$. Then for any $\beta \in (\nu, 1]$, any $\epsilon \in (0, \epsilon_0)$, any $\zeta \in (0, 1)$,*

$$\sup_{[0,T]} \mathbb{E}\left[|V_t - V_t^\epsilon|^\beta\right] \le C_{\beta,K} e^{C_\beta \Gamma_\epsilon^\gamma} \epsilon^{\beta+\gamma+\alpha}.$$

Observe that $e^{C\Gamma_\epsilon^\gamma}$ is not very large: since $\Gamma_\epsilon^\gamma = [\log(1/\epsilon)]^{\gamma\eta_0}$ with $\gamma\eta_0 < 1$ (recall (1.9)), we have $e^{C\Gamma_\epsilon^\gamma} \le C_\eta \epsilon^{-\eta}$, for any $\eta > 0$.

To check this proposition, we need the two following Lemmas, of which the proofs can be found in the appendix. First, we state some estimates concerning the exponential moments for the linearized Boltzmann equation. The study of exponential moments for the nonlinear Boltzmann equation was initiated by Bobylev [8], see also [17] and the references therein. These results really use the nonlinear structure of the Boltzmann equation and we can unfortunately not use them.

Lemma 2.2 *For any $\kappa \in (\nu, 1)$, any $v, V \in \mathbb{R}^2$, for some constants $C > 0$, $c_\kappa > 0$, $C_\kappa > 0$,*

$$\int_{-\pi/2}^{\pi/2} \left(e^{|V+A(\theta)(V-v)|^\kappa} - e^{|V|^\kappa}\right) b(\theta)d\theta \le e^{|V|^\kappa}$$
$$\times\left[-c_\kappa \mathbb{1}_{\{|V|\ge 1, |V|\ge C|v|\}} + C_\kappa(|V| \vee 1)^{\kappa+\nu-2} e^{C_\kappa |v|^\kappa}\right],$$
$$\int_{-\pi/2}^{\pi/2} \left|e^{|V+A(\theta)(V-v)|^\kappa} - e^{|V|^\kappa}\right| b(\theta)d\theta \le C_\kappa e^{C_\kappa|v|^\kappa} e^{C_\kappa|V|^\kappa}.$$

Next, we state some regularity estimates for the cutoff function ϕ_ϵ.

Lemma 2.3 *Consider the function ϕ_ϵ introduced in (2.2).*

(i) *For $\beta \in (0, 1]$, for all $x, y \geq 0$, all $\epsilon \in (0, \epsilon_0)$,*

$$x^\beta |\phi^\gamma_\epsilon(x) - \phi^\gamma_\epsilon(y)| \leq C_\beta \Gamma^\gamma_\epsilon |x - y|^\beta.$$

(ii) *For every $l \geq 1$, for every multi-index $q = (q_1, \dots, q_l) \in \{1, 2\}^l$,*

$$\left|\partial_{v_{q_l}} \dots \partial_{v_{q_1}} [\log \phi_\epsilon(|v|)]\right| \leq C_l \left(\mathbb{1}_{\{|v| \in (\epsilon, \Gamma_\epsilon - 1]\}} |v|^{-l} + \mathbb{1}_{|v| \in (\Gamma_\epsilon - 1, \Gamma_\epsilon + 1)} \Gamma^{-1}_\epsilon\right),$$
$$\left|\partial_{v_{q_l}} \dots \partial_{v_{q_1}} [\phi^\gamma_\epsilon(|v|)]\right| \leq C_l \left(\mathbb{1}_{\{|v| \in (\epsilon, \Gamma_\epsilon - 1]\}} |v|^{\gamma - l} + \mathbb{1}_{|v| \in (\Gamma_\epsilon - 1, \Gamma_\epsilon + 1)} \Gamma^{\gamma - 1}_\epsilon\right).$$

Proof of Proposition 2.1. We handle the proof in several steps. In Steps 1–5, we assume that $(V_t)_{t \in [0,T]}$, $(V^\epsilon_t)_{t \in [0,T]}$ and $(V^{\epsilon,\zeta}_t)_{t \in [0,T]}$ exist and prove points (iii)–(v). Points (i) and (ii) are then checked in Steps 6–7.

Step 1 We first check that for $\kappa \in (\nu, \delta)$,

$$\sup_{[0,T]} \mathbb{E}\left[e^{|V_t|^\kappa} + e^{|V^\epsilon_t|^\kappa} + e^{|V^{\epsilon,\zeta}_t|^\kappa}\right] \leq C_\kappa.$$

Let us for example treat the case of $(V^\epsilon_t)_{t \in [0,T]}$. We have

$$e^{|V^\epsilon_t|^\kappa} = e^{|V_0|^\kappa} + \int_0^t \int_E \left[e^{\left|V^\epsilon_{s-} + A(\theta)(V^\epsilon_{s-} - v)\right|^\kappa} - e^{|V^\epsilon_{s-}|^\kappa}\right] \times \mathbb{1}_{\{u \leq \phi^\gamma_\epsilon(|V^\epsilon_{s-} - v|)\}} N(ds, d\theta, dv, du). \tag{2.5}$$

Taking expectations and using Lemma 2.2,

$$\begin{aligned}\mathbb{E}\left[e^{|V^\epsilon_t|^\kappa}\right] &= \mathbb{E}\left[e^{|V_0|^\kappa}\right] + \int_0^t ds \int_{-\pi/2}^{\pi/2} b(\theta) d\theta \int_{\mathbb{R}^2} f_s(dv) \\ &\quad \times \mathbb{E}\left[\left(e^{\left|V^\epsilon_s + A(\theta)(V^\epsilon_s - v)\right|^\kappa} - e^{|V^\epsilon_s|^\kappa}\right) \phi^\gamma_\epsilon(|V^\epsilon_s - v|)\right] \\ &\leq \mathbb{E}\left[e^{|V_0|^\kappa}\right] + \int_0^t ds \int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\Big[\phi^\gamma_\epsilon(|V^\epsilon_s - v|) e^{|V^\epsilon_s|^\kappa} \\ &\quad \times \left(-c_\kappa \mathbb{1}_{\{|V^\epsilon_s| \geq 1, |V^\epsilon_s| \geq C|v|\}} + C_\kappa (|V^\epsilon_s| \vee 1)^{\kappa + \nu - 2} e^{C_\kappa |v|^\kappa}\right)\Big].\end{aligned}$$

But $\kappa + \nu - 2 < 0$, so that for $|V| \geq M_\kappa(v) := \max\{1, C|v|, [C_\kappa e^{C_\kappa |v|^\kappa} / c_\kappa]^{1/(2-\nu-\kappa)}\}$, we have

$$-c_\kappa \mathbb{1}_{\{|V| \geq 1, |V| \geq C|v|\}} + C_\kappa (|V| \vee 1)^{\kappa + \nu - 2} e^{C_\kappa |v|^\kappa} \leq 0.$$

Changing the values of the constants, $M_\kappa(v) \le C_\kappa e^{C_\kappa |v|^\kappa}$. Thus

$$\mathbb{E}\left[e^{|V_t^\epsilon|^\kappa}\right] \le \mathbb{E}\left[e^{|V_0|^\kappa}\right] + C_\kappa \int_0^t ds \int_{\mathbb{R}^2} f_s(dv)$$
$$\mathbb{E}\left[\phi_\epsilon^\gamma(|V_s^\epsilon - v|) e^{|V_s^\epsilon|^\kappa} \mathbb{1}_{\{|V_s^\epsilon| \le C_\kappa e^{C_\kappa |v|^\kappa}\}} e^{C_\kappa |v|^\kappa}\right].$$

Since now $\phi_\epsilon^\gamma(|V - v|) \le (1 + |V| + |v|)^\gamma$, we deduce that $\phi_\epsilon^\gamma(|V - v|) \mathbb{1}_{\{|V| \le C_\kappa e^{C_\kappa |v|^\kappa}\}} e^{C_\kappa |v|^\kappa} \le C_\kappa e^{C_\kappa |v|^\kappa}$, whence

$$\mathbb{E}\left[e^{|V_t^\epsilon|^\kappa}\right] \le \mathbb{E}\left[e^{|V_0|^\kappa}\right] + C_\kappa \int_0^t ds \int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\left[e^{|V_s^\epsilon|^\kappa}\right] e^{C_\kappa |v|^\kappa}$$
$$\le C_\kappa + C_\kappa \int_0^t ds \mathbb{E}\left[e^{|V_s^\epsilon|^\kappa}\right].$$

We finally used (1.7), that $\kappa < \delta$ and that $V_0 \sim f_0$. The Gronwall Lemma allows us to conclude.

Step 2 We now prove (iii), for example with $(V_t^\epsilon)_{t\in[0,T]}$. Using (2.5) and Lemma 2.2, we obtain

$$\mathbb{E}\left[\sup_{[0,T]} e^{|V_t^\epsilon|^\kappa}\right] \le \mathbb{E}\left[e^{|V_0|^\kappa}\right] + \int_0^T ds \int_{-\pi/2}^{\pi/2} b(\theta) d\theta \int_{\mathbb{R}^2} f_s(dv)$$
$$\times \mathbb{E}\left[\left| e^{\left|V_s^\epsilon + A(\theta)(V_s^\epsilon - v)\right|^\kappa} - e^{|V_s^\epsilon|^\kappa} \right| \phi_\epsilon^\gamma(|V_s^\epsilon - v|)\right]$$
$$\le C_\kappa + C_\kappa \int_0^T ds \int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\left[\phi_\epsilon^\gamma(|V_s^\epsilon - v|) e^{C_\kappa |v|^\kappa} e^{C_\kappa |V_s^\epsilon|^\kappa}\right]$$
$$\le C_\kappa + C_\kappa \int_0^T ds \int_{\mathbb{R}^2} f_s(dv) e^{C_\kappa |v|^\kappa} \mathbb{E}\left[e^{C_\kappa |V_s^\epsilon|^\kappa}\right].$$

We used here that $\phi_\epsilon^\gamma(|V - v|) e^{C_\kappa |V|^\kappa} e^{C_\kappa |v|^\kappa} \le (1 + |V| + |v|)^\gamma e^{C_\kappa |V|^\kappa} e^{C_\kappa |v|^\kappa} \le e^{C_\kappa |V|^\kappa} e^{C_\kappa |v|^\kappa}$. Step 1 and (1.7) allow us to conclude, for $\kappa \in (\nu, \delta)$.

Step 3 We set

$$h(u, v, \theta, w) = A(\theta)(w - v) \mathbb{1}_{\{u \le |w - v|^\gamma\}} \text{ and } h_\epsilon(u, v, \theta, w)$$
$$= A(\theta)(w - v) \mathbb{1}_{\{u \le \phi_\epsilon^\gamma(|w - v|)\}}$$

and we prove that for $\beta \in (0,1]$,

$$\int_0^\infty |(h-h_\epsilon)(u,v,\theta,w)|^\beta du \le C|\theta|^\beta |w-v|^\beta (\epsilon^\gamma \mathbb{1}_{\{|w-v|\le 3\epsilon\}} + |w-v|^\gamma \mathbb{1}_{\{|w-v|\ge \Gamma_\epsilon - 1\}}), \tag{2.6}$$

$$\int_0^\infty |h_\epsilon(u,v,\theta,w)-h_\epsilon(u,v,\theta,\tilde{w})|^\beta du \le C_\beta |\theta|^\beta \Gamma_\epsilon^\gamma |w-\tilde{w}|^\beta. \tag{2.7}$$

We note that $|A(\theta)| \le |\theta|$ (see (1.2)) and recall that $\phi_\epsilon(x) = x$ for $x \in [3\epsilon, \Gamma_\epsilon - 1]$, that $\phi_\epsilon(x) \le 3\epsilon$ for $x \in [0, 3\epsilon]$ and that $\phi_\epsilon(x) \le x$ for $x \ge \Gamma_\epsilon - 1$. The left hand side of (2.6) is bounded by

$$\begin{aligned}
&|\theta|^\beta |w-v|^\beta \int_0^\infty \left| \mathbb{1}_{\{u\le |v-w|^\gamma\}} - \mathbb{1}_{\{u \le \phi_\epsilon^\gamma(|v-w|)\}} \right| du \\
&\le |\theta|^\beta |w-v|^\beta \left| |v-w|^\gamma - \phi_\epsilon^\gamma(|v-w|) \right| \\
&\le |\theta|^\beta |w-v|^\beta (\mathbb{1}_{\{|w-v|\le 3\epsilon\}} + \mathbb{1}_{\{|w-v|\ge \Gamma_\epsilon - 1\}}) \left| |w-v|^\gamma - \phi_\epsilon^\gamma(|w-v|) \right| \\
&\le |\theta|^\beta |w-v|^\beta \left(\mathbb{1}_{\{|w-v|\le 3\epsilon\}} (3\epsilon)^\gamma + \mathbb{1}_{\{|w-v|\ge \Gamma_\epsilon - 1\}} |w-v|^\gamma \right).
\end{aligned}$$

Similarly, using Lemma 2.3-(i) and that $\phi_\epsilon \le \Gamma_\epsilon$, the left hand side of (2.7) is bounded by

$$\begin{aligned}
&|\theta|^\beta \left|(w-v)-(\tilde{w}-v)\right|^\beta \phi_\epsilon^\gamma(|w-v|) + |\theta|^\beta |\tilde{w}-v|^\beta \left|\phi_\epsilon^\gamma(|w-v|) - \phi_\epsilon^\gamma(|\tilde{w}-v|)\right| \\
&\le |\theta|^\beta |w-\tilde{w}|^\beta \Gamma_\epsilon^\gamma + C_\beta |\theta|^\beta \Gamma_\epsilon^\gamma \left||w-v| - |\tilde{w}-v|\right|^\beta \le C_\beta |\theta|^\beta |w-\tilde{w}|^\beta \Gamma_\epsilon^\gamma.
\end{aligned}$$

Step 4 We now prove (iv). Let thus $\beta \in (\nu, 1]$. Since $x \mapsto x^\beta$ is sub-additive, we can write

$$\begin{aligned}
\mathbb{E}\left[|V_t^\epsilon - V_t^{\epsilon,\zeta}|^\beta \right] \le \int_0^t ds \int_{-\pi/2}^{\pi/2} b(\theta) d\theta \int_{\mathbb{R}^2} f_s(dv) \\
\times \int_0^\infty du \mathbb{E}\left[|h_\epsilon(u,v,\theta,V_s^{\epsilon,\zeta}) - h_\epsilon(u,v,\theta,V_s^\epsilon)|^\beta \right]
\end{aligned}$$

$$+\int_0^t ds \int_{-\pi/2}^{\pi/2} (1 - I_\zeta(|\theta|))^\beta b(\theta) d\theta \int_{\mathbb{R}^2} f_s(dv)$$

$$\times \int_0^\infty du \mathbb{E}\left[|h_\epsilon(u, v, \theta, V_s^{\epsilon,\zeta})|^\beta\right].$$

Using (2.7) and that $0 \le 1 - I_\zeta(|\theta|) \le \mathbb{1}_{\{|\theta| \le \zeta\}}$, we get

$$\mathbb{E}\left[|V_t^\epsilon - V_t^{\epsilon,\zeta}|^\beta\right] \le C_\beta \Gamma_\epsilon^\gamma \int_0^t ds \int_{-\pi/2}^{\pi/2} b(\theta) d\theta |\theta|^\beta \mathbb{E}\left[|V_s^\epsilon - V_s^{\epsilon,\zeta}|^\beta\right]$$
$$+ C_\beta \int_0^t ds \int_{-\zeta}^{\zeta} b(\theta) d\theta |\theta|^\beta \int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\left[\phi_\epsilon^\gamma(|V_s^{\epsilon,\zeta} - v|)|V_s^{\epsilon,\zeta} - v|^\beta\right].$$

Using ($\mathbf{A}(\gamma, \nu)$), since $\beta > \nu$ and since $\phi_\epsilon^\gamma(|V - v|)|V - v|^\beta \le C(1 + |v|^2 + |V|^2)$, this yields

$$\mathbb{E}\left[|V_t^\epsilon - V_t^{\epsilon,\zeta}|^\beta\right] \le C_\beta \Gamma_\epsilon^\gamma \int_0^t \mathbb{E}\left[|V_s^\epsilon - V_s^{\epsilon,\zeta}|^\beta\right] ds$$
$$+ C_\beta \zeta^{\beta-\nu} \int_0^t ds \int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\left[1 + |V_s^{\epsilon,\zeta}|^2 + |v|^2\right]$$
$$\le C_\beta \Gamma_\epsilon^\gamma \int_0^t \mathbb{E}\left[|V_s^\epsilon - V_s^{\epsilon,\zeta}|^\beta\right] ds + C_\beta \zeta^{\beta-\nu},$$

where we used (1.7) and point (iii). The Gronwall Lemma allows us to conclude.

Step 5 Let us check (v), for some $\beta \in (\nu, 1]$ fixed. Using again the sub-additivity of $x \mapsto x^\beta$, (2.6–2.7), ($\mathbf{A}(\gamma, \nu)$) and that $\beta > \nu$, we obtain

$$\mathbb{E}\left[|V_t - V_t^\epsilon|^\beta\right] \le \int_0^t ds \int_{-\pi/2}^{\pi/2} b(\theta) d\theta \int_{\mathbb{R}^2} f_s(dv) \int_0^\infty du \mathbb{E}$$
$$\times \left[|h(u, v, \theta, V_s) - h_\epsilon(u, v, \theta, V_s^\epsilon)|^\beta\right].$$

We infer from (2.6–2.7), ($\mathbf{A}(\gamma, \nu)$) and the fact that $\beta > \nu$ that

$$\mathbb{E}\left[|V_t - V_t^\epsilon|^\beta\right] \le C_\beta \int_0^t ds \int_{-\pi/2}^{\pi/2} b(\theta) d\theta |\theta|^\beta \int_{\mathbb{R}^2} f_s(dv)$$
$$\times \mathbb{E}\left(|V_s - v|^\beta (\epsilon^\gamma \mathbb{1}_{\{|V_s - v| \le 3\epsilon\}} + |V_s - v|^\gamma \mathbb{1}_{\{|V_s - v| \ge \Gamma_\epsilon - 1\}}) + \Gamma_\epsilon^\gamma |V_s - V_s^\epsilon|^\beta\right)$$

$$\leq C_\beta \epsilon^{\beta+\gamma} \int_0^t ds \mathbb{E}\left[f_s(Ball(V_s, 3\epsilon))\right] + C_\beta \Gamma_\epsilon^\gamma \int_0^t ds \mathbb{E}\left[|V_s - V_s^\epsilon|^\beta\right]$$

$$+C_\beta \int_0^t ds \int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\left[|V_s - v|^{\beta+\gamma} \mathbb{1}_{\{|V_s - v| \geq \Gamma_\epsilon - 1\}})\right].$$

By assumption, we have

$$\sup_{[0,T]} \mathbb{E}\left[f_s(Ball(V_s, 3\epsilon))\right] \leq 3^\alpha K \epsilon^\alpha.$$

Next (1.7) and point (iii) yield, for $\kappa \in (1/\eta_0, \delta)$,

$$\int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\left[|V_s - v|^{\beta+\gamma} \mathbb{1}_{\{|V_s - v| \geq \Gamma_\epsilon - 1\}})\right]$$

$$\leq \int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\left[(|V_s| + |v|)^{\beta+\gamma} \mathbb{1}_{\{|V_s| + |v| \geq \Gamma_\epsilon - 1\}})\right]$$

$$\leq e^{-(\Gamma_\epsilon - 1)^\kappa} \int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\left[(|V_s| + |v|)^{\beta+\gamma} e^{(|V_s| + |v|)^\kappa}\right]$$

$$\leq C_\kappa e^{-\Gamma_\epsilon^\kappa} \int_{\mathbb{R}^2} f_s(dv) \mathbb{E}\left[e^{C_\kappa (|V_s| + |v|)^\kappa}\right] \leq C_\kappa e^{-\Gamma_\epsilon^\kappa}.$$

Thus we have

$$\mathbb{E}\left[|V_t - V_t^\epsilon|^\beta\right] \leq C_{\beta,\kappa,K}(\epsilon^{\beta+\gamma+\alpha} + e^{-\Gamma_\epsilon^\kappa}) + C_\beta \Gamma_\epsilon^\gamma \int_0^t ds \mathbb{E}\left[|V_s - V_s^\epsilon|^\beta\right],$$

whence $\mathbb{E}\left[|V_t - V_t^\epsilon|^\beta\right] \leq C_{\beta,\kappa,K}(\epsilon^{\beta+\gamma+\alpha} + e^{-\Gamma_\epsilon^\kappa}) e^{C_\beta \Gamma_\epsilon^\gamma T}$ by the Gronwall Lemma. We easily conclude, since $\kappa > \gamma$ and since $\Gamma_\epsilon^\kappa = [\log(1/\epsilon)]^{\kappa \eta_0}$, with $\kappa \eta_0 > 1$.

Step 6 We now prove point (i). First, the strong existence and uniqueness of a solution $(V_t^{\epsilon,\zeta})_{t\in[0,T]}$ to (2.4) is obvious, since the Poisson measure used in (2.4) is a.s. finite because since I_ζ vanishes on a neighborhood of 0,

$$\int_0^T \int_E \mathbb{1}_{\{I_\zeta(|\theta|) \neq 0, u \leq \Gamma_\epsilon^\gamma\}} ds b(\theta) d\theta f_s(dv) du < \infty.$$

Similar arguments as in point (iv) allow us to pass to the limit as $\zeta \to 0$ (recall that $I_\zeta(|\theta|) \to \mathbb{1}_{\{\theta \neq 0\}}$) and to deduce that there exists a unique solution to $(V_t^\epsilon)_{t\in[0,T]}$

to (2.3). Finally, we use similar arguments as in point (v) to prove the existence and uniqueness of a solution $(V_t)_{t\in[0,T]}$ to (2.1), by taking the limit $\epsilon \to 0$.

Step 7 It remains to show that $V_t \sim f_t$ for all $t \in [0, T]$. To this end, we denote by g_t the law of V_t. Then $g_0 = f_0$ by assumption. Using the Itô formula for jump processes and taking expectations, we see that $(g_t)_{t\in[0,T]}$ solves the following linear Boltzmann equation: for all $\psi : \mathbb{R}^2 \mapsto \mathbb{R}$ globally Lipschitz continuous,

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi(v)\, g_t(dv) = \int_{\mathbb{R}^2} g_t(dv) \int_{\mathbb{R}^2} f_t(dv_*)$$
$$\times \int_{-\pi/2}^{\pi/2} b(\theta)d\theta |v - v_*|^\gamma \left[\psi(v + A(\theta)(v - v_*)) - \psi(v)\right].$$

Of course, $(f_t)_{t\in[0,T]}$ also solves this linear equation. Thus $(g_t)_{t\in[0,T]} = (f_t)_{t\in[0,T]}$ by a uniqueness argument. The uniqueness for this linear equation can be derived from the uniqueness of the solution to (2.1), by using the results of Bhatt-Karandikar [6, Theorem 5.2], see [16, Lemma 4.6] for very similar considerations in a very close situation.

3 Some substitutions

We will use some Malliavin calculus for the process $(V_t^{\epsilon,\zeta})_{t\in[0,T]}$, solution to (2.4). Since $\phi_\epsilon^\gamma \le \Gamma_\epsilon^\gamma \le 2\Gamma_\epsilon^\gamma$, we can write

$$V_t^{\epsilon,\zeta} = V_0 + \int_0^t \int_{-\pi/2}^{\pi/2} \int_{\mathbb{R}^2} \int_0^{2\Gamma_\epsilon^\gamma} A(\theta)(V_{s-}^{\epsilon,\zeta} - v) I_\zeta(|\theta|) \mathbb{1}_{\{u \le \phi_\epsilon^\gamma(|V_{s-}^{\epsilon,\zeta} - v|)\}} N(ds, d\theta, dv, du).$$

Recall that the intensity measure of N is given by $dsb(\theta)d\theta f_s(dv)du$. Our goal in this section is to modify this formula in order to get an expression in adequacy with [5]. First of all, we use the Skorokhod representation Theorem to find a measurable application $v_t : [0, 1] \mapsto \mathbb{R}^2$ such that for all $\psi : \mathbb{R}^2 \mapsto \mathbb{R}_+$,

$$\int_0^1 \psi(v_t(\rho))d\rho = \int_{\mathbb{R}^2} \psi(v) f_t(dv). \tag{3.1}$$

Next, we consider the following function $G : x \in (0, \pi/2) \mapsto (0, \infty)$

$$G(x) = \int_x^{\pi/2} b(\theta)d\theta$$

and its inverse $\vartheta : (0,\infty) \mapsto (0,\pi/2)$ (i.e. $G(\vartheta(z)) = z$) and we set $\vartheta(z) = -\vartheta(-z)$ if $z < 0$. Then for all $\psi : [-\pi/2, \pi/2]\backslash\{0\} \mapsto \mathbb{R}_+$,

$$\int_{-\pi/2}^{\pi/2} \psi(\theta)b(\theta)d\theta = \int_{\mathbb{R}_*} \psi(\vartheta(z))dz. \tag{3.2}$$

Note that ϑ is smooth on $(-\infty,0)\cup(0,\infty)$. Since $b(\theta) \simeq |\theta|^{-1-\nu}$ by assumption, we have $G(x) \simeq \nu^{-1}(x^{-\nu} - (\pi/2)^{-\nu})$ and thus $\vartheta(z) \simeq (\nu z + (2/\pi)^{\nu})^{-1/\nu} \simeq (1+z)^{-1/\nu}$. More precisely, the following estimates will be checked in the appendix.

Lemma 3.1 *The function* ϑ *is* C^∞ *on* $(0,\infty)$. *For all* $z > 0$,

$$\begin{aligned}
&\text{(i)} \quad c(1+z)^{-1/\nu} \le \vartheta(z) \le C(1+z)^{-1/\nu},\\
&\text{(ii)} \quad c(1+z)^{-1/\nu-1} \le |\vartheta'(z)| \le C(1+z)^{-1/\nu-1},\\
&\text{(iii)} \quad |\vartheta^{(k)}(z)| \le C_k(1+z)^{-1/\nu-1}, \quad k \ge 1,\\
&\text{(iv)} \quad |(A(\vartheta(z)))^{(k)}| \le C_k(1+z)^{-1/\nu-1}, \quad k \ge 1.
\end{aligned}$$

Observe now that for all $z \in \mathbb{R}_*$,

$$|\vartheta(z)| > \zeta \iff |z| < G(\zeta). \tag{3.3}$$

We choose I_ζ in such a way that for $\mathbf{I}_\zeta(z) = I_\zeta(\vartheta(|z|))$, $\mathbf{I}_\zeta : \mathbb{R} \mapsto [0,1]$ is smooth (with all its derivatives bounded uniformly in ζ) and verifies $\mathbf{I}_\zeta(z) = 1$ for $|z| \le G(\zeta)$ and $\mathbf{I}_\zeta(z) = 0$ for $|z| \ge G(\zeta)+1$.

We can write, using the substitutions $\theta = \vartheta(z)$ and $v = v_s(\rho)$,

$$\begin{aligned}
V_t^{\epsilon,\zeta} = V_0 + \int_0^t \int_0^1 \int_{-G(\zeta)-1}^{G(\zeta)+1} \int_0^{2\Gamma_\epsilon^\gamma} A(\vartheta(z))(V_{s-}^{\epsilon,\zeta} - v_s(\rho))\mathbf{I}_\zeta(z)\\
\times \mathbb{1}_{\{u \le \phi_\epsilon^\gamma(|V_{s-}^{\epsilon,\zeta} - v_s(\rho)|)\}} M(ds, d\rho, dz, du),
\end{aligned}$$

where M is a Poisson measure on $[0,T]\times[0,1]\times\mathbb{R}_*\times[0,\infty)$ with intensity measure $dsd\rho dzdu$. These substitutions are used for technical convenience: for example, it would have been technically complicated to use a smooth version of $\mathbb{1}_{\{|\theta|\ge\zeta\}}$ (with ζ small), while it is easy to build a smooth version of $\mathbb{1}_{\{|z|\ge G(\zeta)\}}$ (with $G(\zeta)$ large), see also Remark 4.2 below.

Consequently, there exists a standard Poisson process $J_t^{\epsilon,\zeta} = \sum_{k\ge1} \mathbb{1}_{\{T_k^{\epsilon,\zeta} \le t\}}$ with rate

$$\lambda_{\epsilon,\zeta} = \int_0^1 d\rho \int_{-G(\zeta)-1}^{G(\zeta)+1} dz \int_0^{2\Gamma_\epsilon^\gamma} du = 4(G(\zeta)+1)\Gamma_\epsilon^\gamma$$

and a family $(\bar{R}_k^{\epsilon,\zeta}, \bar{Z}_k^{\epsilon,\zeta}, \bar{U}_k^{\epsilon,\zeta})_{k\geq 1}$ of i.i.d. $[0,1]\times[-G(\zeta)-1, G(\zeta)+1]\times[0, 2\Gamma_\epsilon^\gamma]$-valued random variables with law $\lambda_{\epsilon,\zeta}^{-1} d\rho dz du$ such that, with the conventions $\sum_1^0 = 0$ and $T_0^{\epsilon,\zeta} = 0$,

$$V_t^{\epsilon,\zeta} = V_0 + \sum_{k=1}^{J_t^{\epsilon,\zeta}} A(\vartheta(\bar{Z}_k^{\epsilon,\zeta}))\left(V_{T_{k-1}^{\epsilon,\zeta}}^{\epsilon,\zeta} - v_{T_k^{\epsilon,\zeta}}(\bar{R}_k^{\epsilon,\zeta})\right)\mathbf{I}_\zeta(Z_k^{\epsilon,\zeta})$$
$$\times \mathbb{1}_{\left\{\bar{U}_k^{\epsilon,\zeta} \leq \phi_\epsilon^\gamma\left(\left|V_{T_{k-1}^{\epsilon,\zeta}}^{\epsilon,\zeta} - v_{T_k^{\epsilon,\zeta}}(\bar{R}_k^{\epsilon,\zeta})\right|\right)\right\}}.$$

For $t \in [0,T]$, $w \in \mathbb{R}^2$, (recall that $\phi_\epsilon \leq \Gamma_\epsilon$), define

$$g_{\epsilon,\zeta}(t,w) = 1 - \frac{1}{\lambda_{\epsilon,\zeta}} \int_0^1 d\rho \int_{-G(\zeta)-1}^{G(\zeta)+1} dz \phi_\epsilon^\gamma(|w - v_t(\rho)|)$$
$$= 1 - \frac{1}{2\Gamma_\epsilon^\gamma} \int_0^1 d\rho \phi_\epsilon^\gamma(|w - v_t(\rho)|) \in [1/2, 1].$$

Consider a C^∞ function $\chi : \mathbb{R} \mapsto [0,1]$ supported by $(-1,1)$ such that $\int_{-1}^1 \chi(x)dx = 1$. Setting

$$q_{\epsilon,\zeta}(t,w,\rho,z) = g_{\epsilon,\zeta}(t,w)\chi(z - G(\zeta) - 3) + \frac{\phi_\epsilon^\gamma(|w - v_t(\rho)|)}{\lambda_{\epsilon,\zeta}} \mathbb{1}_{\{|z|\leq G(\zeta)+1\}}$$

we see that for each $t \in [0,T]$, $w \in \mathbb{R}^2$, $q_{\epsilon,\zeta}(t,w,\rho,z)d\rho dz$ is a probability measure on $[0,1]\times\mathbb{R}_*$. Since $\chi(z-G(\zeta)-3) = 0$ for $|z| \leq G(\zeta)+1$ and $\chi(z-G(\zeta)-3) > 0$ implies $|z| > G(\zeta)+1$ and thus $\mathbf{I}_\zeta(z) = 0$, we see that for all $k \geq 0$, all $\psi : \mathbb{R}^2 \mapsto \mathbb{R}_+$,

$$\mathbb{E}\left[\psi\left(V_{T_{k+1}^{\epsilon,\zeta}}^{\epsilon,\zeta}\right)\middle| V_{T_k^{\epsilon,\zeta}}^{\epsilon,\zeta}, T_k^{\epsilon,\zeta}, T_{k+1}^{\epsilon,\zeta}\right]$$
$$= \int_0^1 \int_{\mathbb{R}_*} \psi\left(V_{T_k^{\epsilon,\zeta}}^{\epsilon,\zeta} + A(\vartheta(z))(V_{T_k}^{\epsilon,\zeta} - v_{T_{k+1}^{\epsilon,\zeta}}(\rho))\mathbf{I}_\zeta(z)\right)\phi_\epsilon^\gamma\left(\left|V_{T_k^{\epsilon,\zeta}}^{\epsilon,\zeta} - v_{T_{k+1}^{\epsilon,\zeta}}(\rho)\right|\right)\frac{d\rho dz}{\lambda_{\epsilon,\zeta}}$$
$$= \int_0^1 \int_{\mathbb{R}_*} \psi\left(V_{T_k^{\epsilon,\zeta}}^{\epsilon,\zeta} + A(\vartheta(z))(V_{T_k}^{\epsilon,\zeta} - v_{T_{k+1}^{\epsilon,\zeta}}(\rho))\mathbf{I}_\zeta(z)\right) q_{\epsilon,\zeta}(T_{k+1}^{\epsilon,\zeta}, V_{T_k^{\epsilon,\zeta}}^{\epsilon,\zeta}, \rho, z) d\rho dz.$$

Consequently, we can build, on a possibly enlarged probability space, a sequence $(R_k^{\epsilon,\zeta}, Z_k^{\epsilon,\zeta})_{k\geq 1}$ of random variables such that $V_0^{\epsilon,\zeta} = V_0$ and for all $k \in \{0,\ldots, J_T^{\epsilon,\zeta} - 1\}$,

$$V_t^{\epsilon,\zeta} = V_{T_k^{\epsilon,\zeta}}^{\epsilon,\zeta} \quad \text{for all } t \in [T_k^{\epsilon,\zeta}, T_{k+1}^{\epsilon,\zeta}),$$

$$V_{T_{k+1}^{\epsilon,\zeta}}^{\epsilon,\zeta} = \sum_{k=1}^{J_t^{\epsilon,\zeta}} A(\vartheta(Z_{k+1}^{\epsilon,\zeta}))(V_{T_k}^{\epsilon,\zeta} - v_{T_{k+1}^{\epsilon,\zeta}}(R_{k+1}^{\epsilon,\zeta}))\mathbf{I}_\zeta(Z_{k+1}^{\epsilon,\zeta}),$$

$$\mathcal{L}\left((R_{k+1}^{\epsilon,\zeta}, Z_{k+1}^{\epsilon,\zeta})|V_{T_k^{\epsilon,\zeta}}^{\epsilon,\zeta}, T_k^{\epsilon,\zeta}, T_{k+1}^{\epsilon,\zeta}\right) = q_{\epsilon,\zeta}(T_{k+1}^{\epsilon,\zeta}, V_{T_k^{\epsilon,\zeta}}^{\epsilon,\zeta}, \rho, z)d\rho dz.$$

Observe that by construction, we have

$$V_t^{\epsilon,\zeta} = V_0 + \sum_{k=1}^{J_t^{\epsilon,\zeta}} A(\vartheta(Z_k^{\epsilon,\zeta}))(V_{T_{k-1}^{\epsilon,\zeta}}^{\epsilon,\zeta} - v_{T_k^{\epsilon,\zeta}}(R_k^{\epsilon,\zeta}))\mathbf{I}_\zeta(Z_k^{\epsilon,\zeta})$$

for all $t \in [0, T]$. The following observation will allow us to handle several computations.

Remark 3.2 Recall that $M(ds, d\rho, dz, du)$ is a Poisson measure on $[0, T] \times [0, 1] \times \mathbb{R}_* \times [0, \infty)$ with intensity measure $dsd\rho dzdu$. For any $\psi : [0, T] \times \mathbb{R}^2 \times [0, 1] \times \mathbb{R}_* \mapsto \mathbb{R}_+$, any $t \in [0, T]$,

$$\sum_{k=1}^{J_t^{\epsilon,\zeta}} \psi(T_k^{\epsilon,\zeta}, V_{T_{k-1}^{\epsilon,\zeta}}^{\epsilon,\zeta}, R_k^{\epsilon,\zeta}, Z_k^{\epsilon,\zeta})\mathbf{I}_\zeta(Z_k^{\epsilon,\zeta})$$
$$= \int_0^t \int_0^1 \int_{\mathbb{R}_*} \int_0^\infty \psi(s, V_{s-}^{\epsilon,\zeta}, \rho, z)\mathbf{I}_\zeta(z)\mathbb{1}_{\{u \le \phi_\epsilon^\gamma(|V_{s-}^{\epsilon,\zeta} - v_s(\rho)|)\}} M(ds, d\rho, dz, du).$$

Remark 3.3 We finally compute the law of $((R_1^{\epsilon,\zeta}, Z_1^{\epsilon,\zeta}), \dots, (R_l^{\epsilon,\zeta}, Z_l^{\epsilon,\zeta}))$. We can write, for each $k \ge 0$,

$$V_{T_k}^{\epsilon,\zeta} = \mathcal{H}_k(V_0, (T_1^{\epsilon,\zeta}, R_1^{\epsilon,\zeta}, Z_1^{\epsilon,\zeta}), \dots, (T_k^{\epsilon,\zeta}, R_k^{\epsilon,\zeta}, Z_k^{\epsilon,\zeta})),$$

for some function $\mathcal{H}_k : \mathbb{R}^2 \times (\mathbb{R}_+ \times [0, 1] \times \mathbb{R}_*)^k \mapsto \mathbb{R}^2$. Indeed, set $\mathcal{H}_0(v) = v$ and

$$\mathcal{H}_{k+1}(v, (t_1, \rho_1, z_1), \dots, (t_{k+1}, \rho_{k+1}, z_{k+1})) = \mathcal{H}_k(v, (t_1, \rho_1, z_1), \dots, (t_k, \rho_k, z_k)))$$
$$+ A(\vartheta(z_{k+1}))\left(\mathcal{H}_k(v, (t_1, \rho_1, z_1), \dots, (t_k, \rho_k, z_k))) - v_{t_{k+1}}(\rho_{k+1})\right)\mathbf{I}_\zeta(z_{k+1}).$$

Conditionally on $\sigma(V_0, J_t^{\epsilon,\zeta}, t \ge 0)$, the law of $((R_1^{\epsilon,\zeta}, Z_1^{\epsilon,\zeta}), \dots, (R_l^{\epsilon,\zeta}, Z_l^{\epsilon,\zeta}))$ has the density

$$\prod_{k=1}^{l} q_{\epsilon,\zeta}(T_k^{\epsilon,\zeta}, \mathcal{H}_{k-1}(V_0, (T_1^{\epsilon,\zeta}, \rho_1, z_1), \dots, (T_{k-1}^{\epsilon,\zeta}, \rho_{k-1}, z_{k-1})), \rho_k, z_k),$$

with respect to the Lebesgue measure on $([0, 1] \times \mathbb{R}_*)^k$.

4 An integration by parts formula

The aim of this section is to prove an integration by parts formula for $V_t^{\epsilon,\zeta}$. Clearly, on the event $\{T_1^{\epsilon,\zeta} > t\}$, $V_t^{\epsilon,\zeta} = V_0$, so that no regularization may occur. To avoid this degeneracy, we consider (Z_{-1}, Z_0) with law $\mathcal{N}(0, I_2)$ independent of everything else. We also introduce a C^∞ non-decreasing function $\Phi_\epsilon : \mathbb{R} \mapsto [0, 1]$ such that $\Phi_\epsilon(x) = 0$ for $x \leq \Gamma_\epsilon - 1$ and $\Phi_\epsilon(x) = 1$ for $x \geq \Gamma_\epsilon$. We may assume that the derivatives of all orders of Φ_ϵ are bounded uniformly with respect to $\epsilon \in (0, \epsilon_0)$. Finally, we consider a C^∞ function $\Psi : \mathbb{R} \mapsto [0, 1]$ such that $\Psi(x) = 1$ for $x \leq 1/4$ and $\Psi(x) = 0$ for $x \geq 3/4$. We set

$$\Sigma_t^{\epsilon,\zeta} = \Phi_\epsilon(|V_0|) + \sum_{k=1}^{J_t^{\epsilon,\zeta}} \Phi_\epsilon(|V_{T_k^{\epsilon,\zeta}}^{\epsilon,\zeta}|) \quad \text{and} \quad G_t^{\epsilon,\zeta} = \Psi(\Sigma_t^{\epsilon,\zeta}). \tag{4.1}$$

Observe that since $\sup_{[0,t]} |V_s^{\epsilon,\zeta}| = \max\{|V_0|, |V_{T_1^{\epsilon,\zeta}}^{\epsilon,\zeta}|, \ldots, |V_{T_{J_t}^{\epsilon,\zeta}}^{\epsilon,\zeta}|\}$, we have

$$\mathbb{1}_{\{\sup_{[0,t]} |V_s^{\epsilon,\zeta}| \leq \Gamma_\epsilon - 1\}} \leq G_t^{\epsilon,\zeta} \leq \mathbb{1}_{\{\sup_{[0,t]} |V_s^{\epsilon,\zeta}| \leq \Gamma_\epsilon\}}. \tag{4.2}$$

Theorem 4.1 *We set $u_\zeta(t) := t\zeta^{4+\nu}$. For any $\psi \in C_b^\infty(\mathbb{R}^2, \mathbb{R})$, any $0 < t_0 \leq t \leq T$, any $\kappa \in (1/\eta_0, \delta)$, any $q \geq 1$, any multi-index $\beta \in \{1, 2\}^q$,*

$$\left| \mathbb{E}\left[\partial_\beta^q \psi \left(\sqrt{u_\zeta(t)} \begin{pmatrix} Z_{-1} \\ Z_0 \end{pmatrix} + V_t^{\epsilon,\zeta} \right) G_t^{\epsilon,\zeta} \right] \right| \\ \leq C_{q,t_0,\kappa} e^{C_{q,\kappa}\Gamma_\epsilon^\gamma} ||\psi||_\infty \left[\epsilon^{-q} \zeta^{-\nu q} + e^{-\Gamma_\epsilon^\kappa} \zeta^{-2\nu q} \right].$$

In the whole section, $\zeta \in (0, 1)$ and $\epsilon \in (0, \epsilon_0)$ are fixed. We set for simplicity $\lambda = \lambda_{\epsilon,\zeta}$, $T_k = T_k^{\epsilon,\zeta}$, $R_k = R_k^{\epsilon,\zeta}$, $Z_k = Z_k^{\epsilon,\zeta}$, but we track the dependance of all the constants with respect to ϵ and ζ.

4.1 The Malliavin calculus

We recall here the Malliavin calculus defined in [5]. This calculus is based on the variables $(Z_k)_{k\geq 1}$ (they correspond to the variables $(V_k)_{k\geq 1}$ in [5]). The σ-field with respect to which we will take conditional expectations is

$$\mathcal{G} = \sigma(V_0, T_k, R_k, k \geq 1).$$

The calculus presented below is slightly different from the one used in [5]: there one employs as basic random variables $(R_k, Z_k)_{k\geq 1}$, while here we use only $(Z_k)_{k\geq 1}$. This is because we have no information about the derivability of the coefficients of the equation with respect to ρ. We also note that our coefficients depend on time, but since the bounds of the coefficients and of their derivatives are uniform with respect to time, the estimates from [5] hold in our framework.

Recall that (Z_{-1}, Z_0) is independent of everything else and $\mathcal{N}(0, I_2)$-distributed. We set

$$\mathbf{Z}_t = (Z_{-1}, Z_0, Z_1, \dots, Z_{J_t}).$$

We now use Remark 3.3. Conditionally on $\mathcal{G}$, the law of $\mathbf{Z}_t$ has the following density with respect to the Lebesgue measure on $\mathbb{R}^2 \times (\mathbb{R}_*)^{J_t}$: setting $z = (z_{-1}, \dots, z_{J_t})$,

$$p_{\epsilon,\zeta}(z) = \mathcal{W}_t e^{-\frac{|z_{-1}|^2+|z_0|^2}{2}} \prod_{k=1}^{J_t} q_{\epsilon,\zeta}(T_k, \mathcal{H}_{k-1}(V_0, (T_1, R_1, z_1), \dots, (T_{k-1}, R_{k-1}, z_{k-1})), R_k, z_k),$$

the normalization constant

$$\mathcal{W}_t = \left(2\pi \int_{[0,1]^{J_t}} \left[\prod_{k=1}^{J_t} q_{\epsilon,\zeta}(T_k, \mathcal{H}_{k-1}(V_0, (T_1, R_1, z_1), \dots, \times (T_{k-1}, R_{k-1}, z_{k-1})), R_k, z_k) \right] dz_1 \dots dz_{J_t} \right)^{-1}$$

being $\mathcal{G}$-measurable.

We denote by $U_\zeta : \mathbb{R}_* \mapsto [0, 1]$ a C^∞ function such that $U_\zeta(z) = 1$ for $|z| \in (1, G(\zeta) - 1)$ and $U_\zeta(z) = 0$ for $|z| \le 1/2$ and $|z| \ge G(\zeta) - 1/2$. We may of course choose U_ζ in such a way that its derivatives of all orders are uniformly bounded (with respect to ζ). Then we define

$$\pi_{-1} = \pi_0 = 1, \quad \pi_k = U_\zeta(Z_k), \quad k \ge 1.$$

Remark 4.2 Note that π_k is smooth with respect to Z_k and that all its derivatives are bounded uniformly with respect to ζ. This is the reason why we used the substitution $\theta = \vartheta(z)$ in the previous section.

A random variable F is said to be a *simple functional* if it is of the form

$$F = h(\omega, (Z_{-1}, \dots, Z_{J_t})) = h(\omega, \mathbf{Z}_t)$$

for some $t \ge 0$, some $\mathcal{G}$-measurable $h : \{(\omega, z), \omega \in \Omega, z \in \mathbb{R}^2 \times (\mathbb{R}_*)^{J_t(\omega)}\} \mapsto \mathbb{R}$, such that for almost all $\omega \in \Omega$, for all $k \in \{-1, \dots, J_t(\omega)\}$, $z \mapsto f(\omega, z)$ is smooth with respect to z_k on the set $\pi_k > 0$. For such a functional we define the Malliavin derivatives: for $k \ge -1$,

$$D_k F = \pi_k \partial_{z_k} h(\omega, \mathbf{Z}_t).$$

Remark 4.3 We note that Remark 3.3 ensures us that $V_t^{\epsilon,\zeta}$ is a simple functional for each $t \in [0, T]$. Indeed, $\mathcal{H}_k$ is smooth with respect to z_l for $l \in \{1, \dots, k\}$ on $\{z_l \in (-G(\zeta), 0) \cup (0, G(\zeta))$, which contains $\{\pi_l > 0\}$. This explains our choice for π_l.

Observe that if F is a simple functional, $D_k F$ is also a simple functional (in particular because the weights π_k are smooth functions of Z). Thus for a multi-index $\beta = (k_1, \dots, k_m)$ with length $|\beta| = m$, we may define

$$D^\beta F = D_{k_m} \dots D_{k_1} F.$$

For $m \geq 1$, we will use the norm

$$|F|_m = |F| + \sum_{1 \leq |\beta| \leq m} |D^\beta F|.$$

Given a d-dimensional simple functional $F = (F_1, \dots, F_d)$, we set $|F|_m = \sum_{i=1}^d |F_i|_m$. The Malliavin covariance matrix of F is defined by

$$\sigma^{i,j}(F) = \sum_{k=-1}^{J_t} D_k F_i \times D_k F_j, \quad 1 \leq i, j \leq d.$$

Finally, we introduce the divergence operator L: for a simple functional F,

$$LF = -\sum_{k=-1}^{J_t} \left[\frac{1}{\pi_k} D_k(\pi_k D_k F) + D_k F \times D_k \log p_{\varepsilon,\zeta}(\mathbf{Z}_t) \right].$$

We now are able to state the integration by parts formula obtained in [5], of which the assumptions are satisfied.

Theorem 4.4 ([5, Theorems 1 and 3]) *Let G and $F = (F_1, \dots, F_d)$ be simple functionals. We suppose that* $\det \sigma(F) \neq 0$ *almost surely. Then for every $\psi \in C_b^\infty(\mathbb{R}^d, \mathbb{R})$ and every multi-index $\beta = (\beta_1, \dots, \beta_q) \in \{1, \dots, d\}^q$, we have*

$$\mathbb{E}\left(\partial_\beta^q \psi(F) G\right) = \mathbb{E}\left(\psi(F) K_{\beta,q}(F, G)\right),$$

with the following estimate:

$$\left|K_{\beta,q}(F, G)\right| \leq C_{q,d} \frac{|G|_q \, (1 + |F|_{q+1})^{q(6d+1)}}{|\det \sigma(F)|^{3q-1}} \left(1 + \sum_{j=1}^q \sum_{k_1+\dots+k_j \leq q-j} \prod_{i=1}^j |LF|_{k_i}\right).$$

4.2 Lower-bound of the covariance matrix

The aim of this subsection is to show the following proposition. We denote by I the identity matrix of $M_{2\times 2}(\mathbb{R})$. As we will see below (see Subsect. 4.4), the Malliavin covariance matrix of $\sqrt{u_\zeta(t)}\begin{pmatrix} Z_{-1} \\ Z_0 \end{pmatrix} + V_t^{\epsilon,\zeta}$ is nothing but $u_\zeta(t)I + \sigma(V_t^{\epsilon,\zeta})$.

Proposition 4.5 *Recall that* $u_\zeta(t) := t\zeta^{4+\nu}$. *For all* $p \geq 1$, *all* $0 < t_0 < t < T$,

$$\mathbb{E}\left[\left(\det\left[u_\zeta(t)I + \sigma(V_t^{\epsilon,\zeta})\right]\right)^{-p}\right] \leq C_{t_0,p}e^{C_p\Gamma_\epsilon^\gamma}.$$

First, we compute the derivatives of $V_t^{\epsilon,\zeta}$ for $t \in [0,T]$. If we have a family $(M_k)_{k\in\{1,\dots,j\}}$ in $M_{2\times 2}(\mathbb{R})$, we write $\prod_{k=1}^j M_k = M_j \dots M_1$.

Lemma 4.6 *Let* $(Y_t)_{t\in[0,T]}$ *be the* $M_{2\times 2}(\mathbb{R})$ *-valued process defined by*

$$Y_t = \prod_{k=1}^{J_t}\left[I + A(\vartheta(Z_k))\mathbf{I}_\zeta(Z_k)\right] \quad (\textit{with } Y_t = I \textit{ if } J_t = 0).$$

This process solves

$$Y_t = I + \sum_{k=1}^{J_t} A(\vartheta(Z_k))\mathbf{I}_\zeta(Z_k)Y_{T_{k-1}}$$

and Y_t *is invertible for all* $t \in [0,T]$, *because* $I + A(\theta)$ *is invertible for* $|\theta| \leq \pi/2$. *Set, for* $k \geq 1$,

$$H_k = \vartheta'(Z_k)A'(\vartheta(Z_k))(V_{T_{k-1}}^{\epsilon,\zeta} - v_{T_k}(R_k)).$$

Then for $k \geq 1$, *for* $t \in [0,T]$,

$$D_k V_t^{\epsilon,\zeta} = \pi_k Y_t Y_{T_k}^{-1} H_k \mathbb{1}_{t\geq T_k}.$$

Proof Since $V_t^{\epsilon,\zeta}$ and Y_t are constant on $[T_j, T_{j+1})$, it suffices to check the result for $V_{T_j}^{\epsilon,\zeta}$, for all $j \geq 0$, that is, on the set $\pi_k > 0$ (i.e. $|Z_k| \in [1/2, G(\zeta) - 1/2]$),

$$\partial_{z_k} V_{T_j}^{\epsilon,\zeta} = Y_{T_j} Y_{T_k}^{-1} H_k \mathbb{1}_{j\geq k}.$$

Since $V_{T_j}^{\epsilon,\zeta}$ does not depend on Z_k if $j < k$, the result is obvious for $j < k$. We now work by induction on $j \geq k$. First, $V_{T_k}^{\epsilon,\zeta} = V_{T_{k-1}}^{\epsilon,\zeta} + A(\vartheta(Z_k))(V_{T_{k-1}}^{\epsilon,\zeta} -$

$v_{T_k}(R_k,))\mathbf{I}_\zeta(Z_k)$. Derivating this formula with respect to z_k yields (recall that $|Z_k| \in [1/2, G(\zeta - 1/2)]$ and thus $\mathbf{I}_\zeta(Z_k) = 1$),

$$\partial_{z_k} V_{T_k}^{\epsilon,\zeta} = \vartheta'(Z_k)A'(\vartheta(Z_k))(V_{T_{k-1}}^{\epsilon,\zeta} - v_{T_k}(R_k)) = Y_{T_k} Y_{T_k}^{-1} H_k.$$

We now assume that the result holds for some $j \geq k$ and we recall that due to Sect. 3, $V_{T_{j+1}}^{\epsilon,\zeta} = V_{T_j}^{\epsilon,\zeta} + A(\vartheta(Z_{j+1}))(V_{T_j}^{\epsilon,\zeta} - v_{T_{j+1}}(R_{j+1}))\mathbf{I}_\zeta(Z_{j+1})$. Hence

$$\begin{aligned}\partial_{z_k} V_{T_{j+1}}^{\epsilon,\zeta} &= \big(I + A(\vartheta(Z_{j+1}))\mathbf{I}_\zeta(Z_{j+1})\big)\, \partial_{z_k} V_{T_j}^{\epsilon,\zeta} \\ &= \big(I + A(\vartheta(Z_{j+1}))\mathbf{I}_\zeta(Z_{j+1})\big)\, Y_{T_j} Y_{T_k}^{-1} H_k = Y_{T_{j+1}} Y_{T_k}^{-1} H_k\end{aligned}$$

as desired. □

We deduce the following expression.

Lemma 4.7 *For all $t \in [0, T]$, $\sigma(V_t^{\epsilon,\zeta}) = Y_t S_t Y_t^*$, where*

$$S_t := \sum_{k=1}^{J_t} \pi_k^2 Y_{T_k}^{-1} H_k H_k^* (Y_{T_k}^{-1})^*.$$

Proof Due to Lemma 4.6, we have

$$\sigma(V_t^{\epsilon,\zeta}) = \sum_{k=1}^{J_t} \pi_k^2 \left[Y_t Y_{T_k}^{-1} H_k\right]\left[Y_t Y_{T_k}^{-1} H_k\right]^* = Y_t \left(\sum_{k=1}^{J_t} \pi_k^2 Y_{T_k}^{-1} H_k H_k^* (Y_{T_k}^{-1})^*\right) Y_t^*,$$

whence the result. □

Next, we prove some estimates concerning $(Y_t)_{t\in[0,T]}$.

Lemma 4.8 *Almost surely, for all $t \geq 0$, $|Y_t| \leq 1$. Furthermore, for all $p \geq 1$,*

$$\mathbb{E}\left[\sup_{[0,T]} |Y_t^{-1}|^p\right] \leq \exp(C_p \Gamma_\epsilon^\gamma).$$

Proof First, an immediate computation shows that

$$|I + A(\theta)|^2 = \sup_{|\xi|=1} |(I + A(\theta))\xi|^2 = \frac{1 + \cos\theta}{2} \leq 1,$$

so that $|Y_t| \leq 1$. Next, one can check that for $\theta \in (-\pi/2, \pi/2)$,

$$|(I + A(\theta))^{-1}|^2 = \frac{2}{1 + \cos\theta} \leq 1 + \theta^2 \leq \exp(\theta^2).$$

Thus for $0 \leq t \leq T$,

$$|Y_t^{-1}|^2 \leq \prod_{k=1}^{J_t} |(I + A(\vartheta(Z_k))\mathbf{I}_\zeta(Z_k))^{-1}|^2 \leq \exp\left(\sum_{k=1}^{J_T} \vartheta^2(Z_k)\mathbf{I}_\zeta(Z_k)\right) =: \exp(L_T).$$

We infer from Remark 3.2 that for some Poisson measure M with intensity measure $dsd\rho dzdu$,

$$\begin{aligned} L_T &= \int_0^T \int_0^1 \int_{\mathbb{R}_*} \int_0^\infty |\vartheta(z)|^2 \mathbf{I}_\zeta(z) \mathbb{1}_{\{u \leq \phi_\epsilon^\gamma(|V_{s-}^{\epsilon,\zeta} - v_s(\rho)|)\}} M(ds, d\rho, dz, du) \\ &\leq \int_0^T \int_0^1 \int_{\mathbb{R}_*} \int_0^\infty |\vartheta(z)|^2 \mathbb{1}_{\{u \leq \Gamma_\epsilon^\gamma\}} M(ds, d\rho, dz, du). \end{aligned}$$

Hence for any $p > 0$,

$$\mathbb{E}\left[\exp(pL_T)\right] \leq \exp\left(\Gamma_\epsilon^\gamma T \int_{\mathbb{R}_*} (e^{p\vartheta^2(z)} - 1)dz\right) \leq \exp\left(C_p T \Gamma_\epsilon^\gamma\right),$$

since $\vartheta^2(z) \leq (\pi/2)^2$ and since $\int_{\mathbb{R}_*} \vartheta^2(z)dz = \int_{-\pi/2}^{\pi/2} \theta^2 b(\theta)d\theta < \infty$ by (3.2) and $(\mathbf{A}(\gamma, \nu))$. □

To bound S_t from below, we need a lower-bound of f_t. The following estimate is probably standard and will be verified in the appendix. Recall (3.1).

Lemma 4.9 *One may find $r_0 > 0$ and $q_0 > 0$ such that for any $w \in \mathbb{R}^2$, any $t \in [0, T]$,*

$$f_t(\{v, |v - w| \geq r_0\}) = \int_0^1 \mathbb{1}_{\{|v_t(\rho) - w| \geq r_0\}} d\rho \geq q_0.$$

We now prove some basic but fundamental estimates.

Lemma 4.10 *For $\xi \in \mathbb{R}^2$, $X \in \mathbb{R}^2$, consider*

$$I(\xi, X) = \left\{\theta \in [-\pi/2, \pi/2], \left\langle \xi, (I + A(\theta))^{-1} A'(\theta) X \right\rangle^2 \geq \theta^2 |X|^2 |\xi|^2 / 128 \right\}.$$

For any $\xi, X \in \mathbb{R}^2$, we always have either $(0, \pi/2] \subset I(\xi, X)$ or $[-\pi/2, 0) \subset I(\xi, X)$.

Proof We may assume, by homogeneity, that $|X| = |\xi| = 1$. We have

$$(I + A(\theta))^{-1}A'(\theta) = \frac{1}{2}\begin{pmatrix} \frac{-\sin\theta}{1+\cos\theta} & -1 \\ 1 & \frac{-\sin\theta}{1+\cos\theta} \end{pmatrix} =: \frac{1}{2}\left[\frac{-\sin\theta}{1+\cos\theta}I + P\right],$$

$$\left\langle \xi, (I + A(\theta))^{-1}A'(\theta)X\right\rangle^2 = \frac{1}{4}\left[\frac{\sin^2\theta}{(1+\cos\theta)^2}\langle \xi, X\rangle^2 + \langle \xi, PX\rangle^2 \right.$$
$$\left. -2\frac{\sin\theta}{1+\cos\theta}\langle \xi, X\rangle \langle \xi, PX\rangle\right].$$

Since $\langle X, PX\rangle = 0$ and $|X| = |\xi| = 1$, we always have either $\langle \xi, X\rangle^2 \geq 1/2$ or $\langle \xi, PX\rangle^2 \geq 1/2$. Thus for all θ such that $\langle \xi, X\rangle \langle \xi, PX\rangle \sin\theta \leq 0$ (this holds either on $[0, \pi/2]$ or on $[-\pi/2, 0]$),

$$\left\langle \xi, (I + A(\theta))^{-1}A'(\theta)X\right\rangle^2 \geq \frac{1}{8}\min\left[\frac{\sin^2\theta}{(1+\cos\theta)^2}, 1\right] \geq \frac{\sin^2\theta}{32}.$$

We easily conclude, since $|\sin\theta| \geq |\theta|/2$ on $[-\pi/2, \pi/2]$. □

We deduce the following estimate.

Lemma 4.11 *There are some constants $c > 0, C > 0$ such that for all $\xi \in \mathbb{R}^2$, all $t \in [0, T]$,*

$$\mathbb{E}[\exp(-\xi^* S_t \xi)] \leq C\exp\left(-ct[|\xi|^{\nu/(2+\nu)} \wedge \zeta^{-\nu}]\right).$$

Proof Recalling Lemmas 4.6, 4.7, the definition of π_k and using that $Y_{T_k} = (I + A(\vartheta(Z_k)))Y_{T_{k-1}}$ on $\pi_k > 0$ (because $\pi_k > 0$ implies $\mathbf{I}_\zeta(Z_k) = 1$), we see that

$$\xi^* S_t \xi = \sum_{k=1}^{J_t}\pi_k^2\left\langle Y_{T_k}^{-1}H_k, \xi\right\rangle^2 = \sum_{k=1}^{J_t}\pi_k^2\left\langle (I + A(\vartheta(Z_k)))^{-1}H_k, (Y_{T_{k-1}}^{-1})^*\xi\right\rangle^2$$
$$\geq \sum_{k=1}^{J_t}\mathbb{1}_{\{|Z_k|\in[1/2, G(\zeta)-1/2]\}}(\vartheta'(Z_k))^2$$
$$\times\left\langle (I + A(\vartheta(Z_k)))^{-1}A'(\vartheta(Z_k))(V_{T_{k-1}}^{\epsilon,\zeta} - v_{T_k}(R_k)), \xi_{T_{k-1}}\right\rangle^2,$$

where $\xi_t := (Y_t^{-1})^*\xi$. We observe that a.s., $|\xi_t| \geq |\xi|$ because $|Y_t| \leq 1$ by Lemma 4.8. We splitted $Y_{T_k} = (I + A(\vartheta(Z_k)))Y_{T_{k-1}}$ in order to make rigorous the stochastic calculus below ($\xi_{T_{k-1}}$ will be predictable). We recall that r_0 and q_0 were defined in

Lemma 4.9. Thus, due to Lemma 4.10,

$$\begin{aligned}
\xi^* S_t \xi &\ge \sum_{k=1}^{J_t} \mathbb{1}_{\{|Z_k|\in[1/2,G(\zeta)-1/2]\}} \mathbb{1}_{\{\vartheta(Z_k)\in I(\xi_{T_{k-1}}, V^{\epsilon,\zeta}_{T_{k-1}} - v_{T_k}(R_k))\}} \mathbb{1}_{\{|V^{\epsilon,\zeta}_{T_{k-1}} - v_{T_k}(R_k))|\ge r_0\}} \\
&\quad \times \frac{(\vartheta'(Z_k))^2 \vartheta^2(Z_k) r_0^2 |\xi_{T_{k-1}}|^2}{128} \\
&\ge \frac{|\xi|^2 r_0^2}{128} \sum_{k=1}^{J_t} \mathbb{1}_{\{|Z_k|\in[1/2,G(\zeta)-1/2]\}} \mathbb{1}_{\{\vartheta(Z_k)\in I(\xi_{T_{k-1}}, V^{\epsilon,\zeta}_{T_{k-1}} - v_{T_k}(R_k))\}} \\
&\quad \times \mathbb{1}_{\{|V^{\epsilon,\zeta}_{T_{k-1}} - v_{T_k}(R_k))|\ge r_0\}} (\vartheta'(Z_k))^2 \vartheta^2(Z_k) \\
&= \frac{|\xi|^2 r_0^2}{128} \int_0^t \int_0^1 \int_{\mathbb{R}_*} \int_0^\infty \vartheta^2(z)(\vartheta'(z))^2 \mathbb{1}_{\{|z|\in[1/2,G(\zeta)-1/2]\}} \mathbb{1}_{\{\vartheta(z)\in I(\xi_{s-}, V^{\epsilon,\zeta}_{s-} - v_s(\rho))\}} \\
&\quad \times \mathbb{1}_{\{|V^{\epsilon,\zeta}_{s-} - v_s(\rho)|\ge r_0\}} \mathbb{1}_{\{u\le \phi^\gamma_\epsilon(|V^{\epsilon,\zeta}_{s-} - v_s(\rho)|)\}} M(ds,d\rho,dz,du),
\end{aligned}$$

where M is a Poisson measure on $[0,T]\times[0,1]\times\mathbb{R}_*\times[0,\infty)$ with intensity measure $dsd\rho dzdu$. We used Remark 3.2. Since $\phi^\gamma_\epsilon(x) \ge r_0^\gamma$ for $x > r_0$ we get $\xi^* S_t \xi \ge \frac{|\xi|^2 r_0^2}{128} L_t$, where

$$\begin{aligned}
L_t := \int_0^t \int_0^1 \int_{\mathbb{R}_*} \int_0^\infty & \vartheta^2(z)(\vartheta'(z))^2 \mathbb{1}_{\{|z|\in[1/2,G(\zeta)-1/2]\}} \mathbb{1}_{\{\vartheta(z)\in I(\xi_{s-}, V^{\epsilon,\zeta}_{s-} - v_s(\rho))\}} \\
& \times \mathbb{1}_{\{|V^{\epsilon,\zeta}_{s-} - v_s(\rho)|\ge r_0\}} \mathbb{1}_{\{u\le r_0^\gamma\}} M(ds,d\rho,dz,du).
\end{aligned}$$

Using the Itô formula for jump processes, taking expectations and differentiating with respect to time, we get, for $x > 0$,

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}\left[e^{-xL_t}\right] = - \int_0^1 \int_{\mathbb{R}_*} \int_0^\infty \mathbb{E}\Big[& e^{-xL_t}\left(1 - e^{-x\vartheta^2(z)(\vartheta'(z))^2}\right) \mathbb{1}_{\{|z|\in[1/2,G(\zeta)-1/2]\}} \\
& \times \mathbb{1}_{\{\vartheta(z)\in I(\xi_t, V^{\epsilon,\zeta}_t - v_t(\rho))\}} \mathbb{1}_{\{|V^{\epsilon,\zeta}_t - v_t(\rho)|\ge r_0\}} \mathbb{1}_{\{u\le r_0^\gamma\}}\Big] dudzd\rho.
\end{aligned}$$

The integration with respect to u is explicit. Using Lemma 4.10, we see that the set $\{\vartheta(z) \in I(\xi_t, V^{\epsilon,\zeta}_t - v_t(\rho))\}$ a.s. contains $\{\vartheta(z)\in(0,\pi/2)\} = \{z\in(0,\infty)\}$ or $\{\vartheta(z)\in(-\pi/2,0)\} = \{z\in(-\infty,0)\}$. Since $(\vartheta\vartheta')^2$ is even, this yields

$$\frac{d}{dt}\mathbb{E}\left[e^{-xL_t}\right] \le -r_0^\gamma \int_0^1 \int_{1/2}^{G(\zeta)-1/2} \mathbb{E}\Big[e^{-xL_t}\left(1 - e^{-x\vartheta^2(z)(\vartheta'(z))^2}\right) \mathbb{1}_{\{|V^{\epsilon,\zeta}_t - v_t(\rho)|\ge r_0\}}\Big] dzd\rho.$$

Finally we use Lemma 4.9 to deduce

$$\frac{d}{dt}\mathbb{E}\left[e^{-xL_t}\right] \le -\left(r_0^\gamma q_0 \int_{1/2}^{G(\zeta)-1/2} \left(1-e^{-x\vartheta^2(z)(\vartheta'(z))^2}\right) dz\right)\mathbb{E}\left[e^{-xL_t}\right].$$

Since $L_0 = 0$, this implies

$$\mathbb{E}\left[e^{-xL_t}\right] \le \exp\left(-tr_0^\gamma q_0 \int_{1/2}^{G(\zeta)-1/2} \left(1-e^{-x\vartheta^2(z)(\vartheta'(z))^2}\right) dz\right).$$

Recalling that $\xi^* S_t \xi \ge \frac{|\xi|^2 r_0^2}{128} L_t$, we get

$$\mathbb{E}[\exp(-\xi^* S_t\xi)] \le \exp\left(-tr_0^\gamma q_0 \int_{1/2}^{G(\zeta)-1/2} \left(1-e^{-|\xi|^2 r_0^2\vartheta^2(z)(\vartheta'(z))^2/128}\right) dz\right).$$

We observe that due to $(\mathbf{A}(\gamma,\nu))$,

$$G(\zeta) - 1/2 \ge c(\zeta^{-\nu} - (\pi/2)^{-\nu}) - 1/2 \ge c\zeta^{-\nu}$$

for $\zeta > 0$ small enough. By Lemma 3.1, we have $\vartheta^2(z)(\vartheta'(z))^2 \ge c(1+z)^{-4/\nu-2} \ge cz^{-4/\nu-2}$ for $z \ge 1/2$. We thus have

$$\mathbb{E}[\exp(-\xi^* S_t\xi)] \le \exp\left(-tr_0^\gamma q_0 \int_{1/2}^{c\zeta^{-\nu}} \left(1-e^{-c|\xi|^2 z^{-4/\nu-2}}\right) dz\right).$$

But for $z < |\xi|^{\nu/(2+\nu)}$, we have $|\xi|^2 z^{-4/\nu-2} \ge 1$, whence $1-e^{-c|\xi|^2 z^{-4/\nu-2}} \ge 1-e^{-c}$. Consequently,

$$\mathbb{E}[\exp(-\xi^* S_t\xi)] \le \exp\left(-ct\left((c\zeta^{-\nu}) \wedge |\xi|^{\nu/(2+\nu)} - 1/2\right)\right).$$

The conclusion follows. □

We are finally able to conclude this subsection.

Proof of Proposition 4.5. We recall that due to [7, p 92], for all $p \ge 1$, there is a constant C_p such that for all nonnegative symmetric $A \in M_{2\times 2}(\mathbb{R})$,

$$|\det A|^{-p} \le C_p \int_{\xi\in\mathbb{R}^2} |\xi|^{4p-2} e^{-\xi^* A\xi} d\xi.$$

We set $d_t = \det(u_\zeta(t) I + \sigma(V_t^{\epsilon,\zeta}))$. Using Lemma 4.7, we have $\sigma(V_t^{\epsilon,\zeta}) = Y_t S_t Y_t^*$, whence $d_t = \det^2(Y_t)\det(u_\zeta(t)(Y_t^* Y_t)^{-1} + S_t)$. Lemma 4.8 and the Cauchy-Schwarz inequality yield

$$\begin{aligned}\mathbb{E}[d_t^{-p}] &\le \mathbb{E}\left[\det(Y_t)^{-2p}\det\left(u_\zeta(t)(Y_t^*Y_t)^{-1} + S_t\right)^{-p}\right]\\ &\le e^{C_p\Gamma_\epsilon^\gamma}\,\mathbb{E}\left[\det\left(u_\zeta(t)(Y_t^*Y_t)^{-1} + S_t\right)^{-2p}\right]^{1/2}.\end{aligned}$$

Thus due to (4.3) and Lemma 4.11, since $\xi^*(Y_t^*Y_t)^{-1}\xi = |(Y_t^{-1})^*\xi|^2 \ge |\xi|^2$ by Lemma 4.8 ,

$$\begin{aligned}\mathbb{E}[d_t^{-p}] &\le C_p e^{C_p\Gamma_\epsilon^\gamma}\left(\int_{|\xi|\in\mathbb{R}^2} |\xi|^{8p-2} e^{-u_\zeta(t)|\xi|^2}\,\mathbb{E}\left[e^{-\xi^* S_t \xi}\right] d\xi\right)^{1/2}\\ &\le C_p e^{C_p\Gamma_\epsilon^\gamma}\left(\int_{|\xi|\in\mathbb{R}^2} |\xi|^{8p-2}\exp\left(-u_\zeta(t)|\xi|^2 - ct[|\xi|^{\nu/(2+\nu)}\wedge\zeta^{-\nu}]\right) d\xi\right)^{1/2}\\ &\le C_p e^{C_p\Gamma_\epsilon^\gamma}\left(\int_{|\xi|\in\mathbb{R}^2} |\xi|^{8p-2}\exp\left(-ct|\xi|^{\nu/(2+\nu)}\right) d\xi\right)^{1/2}.\end{aligned}$$

To get the last inequality, observe that if $|\xi|^{\nu/(2+\nu)} \ge \zeta^{-\nu}$, then $|\xi|^{2-\nu/(2+\nu)} \ge \zeta^{-4-\nu}$, so that

$$u_\zeta(t)|\xi|^2 = t\zeta^{4+\nu}|\xi|^2 = t\zeta^{4+\nu}|\xi|^{\nu/(2+\nu)}|\xi|^{2-\nu/(2+\nu)} \ge t|\xi|^{\nu/(2+\nu)}.$$

Thus for $0 < t_0 < t < T$, we have

$$\mathbb{E}[d_t^{-p}] \le C_{t_0,p} e^{C_p\Gamma_\epsilon^\gamma}$$

as desired. □

4.3 Upper-bounds of the derivatives

This subsection is devoted to the following estimates.

Proposition 4.12 *For all $l \geq 1$, all $p \geq 1$,*

$$\mathbb{E}\left(\mathbb{1}_{\{\sup_{[0,T]}|V_s^{\epsilon,\zeta}|\leq\Gamma_\epsilon\}} \sup_{[0,T]} |V_s^{\epsilon,\zeta}|_l^p\right) \leq C_{l,p} e^{C_{l,p}\Gamma_\epsilon^\gamma},$$
$$\mathbb{E}\left(\mathbb{1}_{\{\sup_{[0,T]}|V_s^{\epsilon,\zeta}|\leq\Gamma_\epsilon\}} \sup_{[0,T]} |LV_s^{\epsilon,\zeta}|_l^p\right) \leq C_{l,p} \frac{e^{C_{l,p}\Gamma_\epsilon^\gamma}}{\epsilon^{p(l+1)}\zeta^{\nu p}}.$$

Proof We will use the estimates from [5, Sect. 4]. In [5], the coefficients are bounded. But, as long as we are on the set $\{\sup_{[0,T]}\left|V_s^{\epsilon,\zeta}\right| \leq \Gamma_\epsilon\}$, we do not need to take a supremum over all $w \in \mathbb{R}^2$. For a function $\psi = [0,\infty) \times \mathbb{R}^2 \times [0,1] \times \mathbb{R}_* \mapsto \mathbb{R}$ (or $\mapsto \mathbb{R}^2$) which is infinitely differentiable with respect to $z \in \mathbb{R}_*$ and to $w \in \mathbb{R}^2$, we set, for $\epsilon \in (0,\epsilon_0)$, $l \geq 1$,

$$\bar{\psi}_\epsilon^l(t,\rho,z) := \sup_{\{|w|\leq\Gamma_\epsilon\}} \sum_{0\leq|\beta|+k\leq l} |\partial_w^\beta \partial_z^k \psi(t,w,\rho,z)|.$$

Let $c(t,w,\rho,z) = A(\vartheta(z))(w - v_t(\rho))\mathbf{I}_\zeta(z)$, for which $\sup_{w\in\mathbb{R}^2} |\nabla_w c(t,w,\rho,z)| = |A(\vartheta(z))|\mathbf{I}_\zeta(z)$. Due to [5, Lemma 7], we know that

$$\begin{aligned} Y_l(t) &:= \mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\leq\Gamma_\epsilon\}} \sup_{[0,t]} |V_s^{\epsilon,\zeta}|_l \\ &\leq \mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\leq\Gamma_\epsilon\}} \sup_{[0,t]} |V_s^{\epsilon,\zeta}| + C_l\left(1+\sum_{k=1}^{J_t} \bar{c}_\epsilon^l(T_k,R_k,Z_k)\right)^{l\times l!} \sup_{[0,t]} (\mathcal{E}_s)^{l\times l!}, \end{aligned}$$

where

$$\mathcal{E}_t = 1 + C_l \sum_{k=1}^{J_t} |A(\vartheta(Z_k))|\mathbf{I}_\zeta(Z_k)\mathcal{E}_{T_k-} = \prod_{k=1}^{J_t}(1 + C_l|A(\vartheta(Z_k))|\mathbf{I}_\zeta(Z_k)).$$

First, we prove exactly as in Lemma 4.8 that for all $p \geq 1$, $0 \leq t \leq T$,

$$\mathbb{E}\left[\sup_{[0,t]} \mathcal{E}_s^p\right] \leq e^{C_{p,l}\Gamma_\epsilon^\gamma}.$$

Due to Lemma 3.1, since $|A(\theta)| \leq |\theta|$ and since the derivatives of $\mathbf{I}_\zeta$ are bounded uniformly with respect to ζ, we have $\bar{c}_\epsilon^l(t,\rho,z) \leq C_l(1+|z|)^{-1/\nu}(\Gamma_\epsilon + |v_t(\rho)|) \leq C_l\Gamma_\epsilon(1+|z|)^{-1/\nu}(1+|v_t(\rho)|)$. We thus have, using the Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbb{E}\left[Y_l(t)^p\right] &\leq C_p\Gamma_\epsilon^p + C_{p,l}e^{C_{p,l}\Gamma_\epsilon^\gamma}\Gamma_\epsilon^{pl\times l!}\mathbb{E}\left[1+\left(\sum_{k=1}^{J_t}(1+|Z_k|)^{-1/\nu}(1+|v_{T_k}(R_k)|)\right)^{2pl\times l!}\right]^{1/2} \\ &\leq C_{p,l}e^{C_{p,l}\Gamma_\epsilon^\gamma}\mathbb{E}\left[1+X_t^{2pl\times l!}\right]^{1/2}, \end{aligned}$$

where $X_t := \sum_{k=1}^{J_t}(1+|Z_k|)^{-1/\nu}(1+|v_{T_k}(R_k)|)$. We now prove that for any $p \geq 1$, $\mathbb{E}[X_t^p] \leq C_p e^{C_p \Gamma_\epsilon^\gamma}$, which will end the proof of the first inequality. Using Remark 3.2, one may find a Poisson measure M on $[0,T]\times[0,1]\times\mathbb{R}_*\times[0,\infty)$ with intensity measure $ds d\rho dz du$ such that

$$X_t = \int_0^t \int_0^1 \int_{\mathbb{R}_*} \int_0^\infty (1+|z|)^{-1/\nu}(1+|v_s(\rho)|)\mathbb{1}_{\{u\leq \phi_\epsilon^\gamma(|V_{s-}^{\epsilon,\zeta}-v_t(\rho)|)\}}\mathbf{I}_\zeta(z) M(ds,d\rho,dz,du)$$
$$\leq \int_0^t \int_0^1 \int_{\mathbb{R}_*} \int_0^\infty (1+|z|)^{-1/\nu}(1+|v_s(\rho)|)\mathbb{1}_{\{u\leq \Gamma_\epsilon^\gamma\}} M(ds,d\rho,dz,du) =: \tilde{X}_t.$$

A simple computation shows that

$$\mathbb{E}[\tilde{X}_t^p] \leq \Gamma_\epsilon^\gamma \int_0^t ds \int_0^1 d\rho \int_{\mathbb{R}_*} dz \mathbb{E}\Big[(\tilde{X}_s + (1+|z|)^{-1/\nu}(1+|v_s(\rho)|))^p - \tilde{X}_s^p\Big]$$
$$\leq C_p\Gamma_\epsilon^\gamma \int_0^t ds \int_0^1 d\rho \int_{\mathbb{R}_*} dz (1+|z|)^{-1/\nu}(1+|v_s(\rho)|)\mathbb{E}\Big[1+\tilde{X}_s^p + |v_s(\rho)|^p\Big].$$

Since $\int_{\mathbb{R}_*}(1+|z|)^{-1/\nu}dz < \infty$ and since $\int_0^1 |v_t(\rho)|^q d\rho = \int_{\mathbb{R}^2}|v|^q f_t(dv) \leq C_q$ for all $q \geq 1$ due to (1.7), we conclude that $\mathbb{E}[\tilde{X}_t^p] \leq C_p\Gamma_\epsilon^\gamma \int_0^t \mathbb{E}[\tilde{X}_s^p]ds + C_p\Gamma_\epsilon^\gamma$, whence $\mathbb{E}[\tilde{X}_t^p] \leq C_p\Gamma_\epsilon^\gamma e^{C_p\Gamma_\epsilon^\gamma} \leq C_p e^{C_p\Gamma_\epsilon^\gamma}$ by the Gronwall Lemma. This ends the proof of the first inequality.

We now prove the second inequality. We use [5, Lemmas 11 and 12]. We introduce the functions

$$g(t,w) = 1 - \frac{1}{\lambda_{\epsilon,\zeta}} \int_0^1 d\rho \int_{\mathbb{R}_*} dz \mathbb{1}_{\{|z|<G(\zeta)+1\}} \phi_\epsilon^\gamma(|w - v_t(\rho)|)$$
$$= 1 - \frac{1}{2\Gamma_\epsilon^\gamma} \int_0^1 d\rho \phi_\epsilon^\gamma(|w-v_t(\rho)|),$$
$$h(t,w,\rho) = \phi_\epsilon^\gamma(|w-v_t(\rho)|).$$

Then by [5, Lemma 11], for $k = 1,\ldots,J_t$,

$$|LZ_k|_l \leq C_l\Big(\overline{(\log h)}_\epsilon^{l+1}(T_k,R_k)$$
$$+(1+\sup_{[0,t]}|V_s^{\epsilon,\zeta}|_{l+1})^{l+1} \sum_{j=k+1}^{J_t}[\overline{(\log g)}_\epsilon^{l+1}(T_j) + \overline{(\log h)}_\epsilon^{l+1}(T_j,R_j)]\Big).$$

Making use of Lemma 2.3-(ii), one easily checks that $\overline{(\log h)}^{l}_{\epsilon}(t,\rho) \le C_l \epsilon^{-l}$ and that for any multi-index $q = (q_1, \dots, q_l) \in \{1,2\}^l$, $|\partial^l_q g_\epsilon(t,w)| \le C_l \Gamma_\epsilon^{-1} \epsilon^{\gamma - l}$. Hence, using the Faa di Bruno formula (5.5) and the fact that $g_\epsilon(t,w) \ge 1/2$,

$$\overline{(\log g)}^{l}_{\epsilon}(t) \le C_l \epsilon^{\gamma - l}.$$

Thus for $k = 1, \dots, J_t$,

$$|LZ_k|_l \le C_l \epsilon^{-l-1} \left(1 + \sup_{[0,t]} |V_s^{\epsilon,\zeta}|_{l+1}\right)^{l+1} (1 + J_t).$$

We now infer from [5, Lemma 12] that

$$\sup_{[0,t]} |LV_s^{\epsilon,\zeta}|_l \le C_l \left(1 + \sup_{k=1,\dots,J_t} |LZ_k|_l\right)\left(1 + \sum_{k=1}^{J_t} \bar{c}^l_\epsilon(T_k, R_k, Z_k)\right)^{l+1}$$
$$\times \left(1 + \sup_{[0,t]} |V_s^{\epsilon,\zeta}|^{l+2}_{l+1}\right)^{l+1} \sup_{[0,t]} \mathcal{E}_s^{l+1}$$

Using the above estimates, we can upperbound $\sup_{[0,t]} |LV_s^{\epsilon,\zeta}|_l$ with

$$C_l \epsilon^{-l-1}(1 + J_t)\left(1 + \sup_{[0,t]} |V_s^{\epsilon,\zeta}|^{(l+1)(l+3)}_{l+1}\right)$$
$$\times \left(1 + \Gamma_\epsilon \sum_{k=1}^{J_t} |\vartheta(Z_k)|(1 + |v_{T_k}(R_k)|)\right)^{l+1} \sup_{[0,t]} \mathcal{E}_s^{l+1}.$$

Thus using the Cauchy-Schwarz inequality and similar arguments as in the proof of the first inequality, we get

$$\mathbb{E}\left[\sup_{[0,t]} |LV_s^{\epsilon,\zeta}|^p_l\right] \le C_{l,p} \epsilon^{-p(l+1)} e^{C_{l,p}\Gamma_\epsilon^\gamma} \mathbb{E}\left[(1 + J_t)^{2p}\right]^{1/2}.$$

Recall now that J_t is a Poisson process with rate $\lambda = \lambda_{\epsilon,\zeta} = 4(G(\zeta) + 1)\Gamma_\epsilon^\gamma \le C\Gamma_\epsilon^\gamma \zeta^{-\nu}$ by $(\mathbf{A}(\gamma,\nu))$. Hence $\mathbb{E}[J_t^p] \le C_p(\lambda_{\epsilon,\zeta} T + (\lambda_{\epsilon,\zeta} T)^p) \le C_p \Gamma_\epsilon^{\gamma p} \zeta^{-\nu p}$. The second inequality follows. □

4.4 Proof of the formula

We prove a final lemma to compute the norm of $G_t^{\epsilon,\zeta}$.

Lemma 4.13 *Recall* (4.1). *For all* $l \geq 1$, *all* $t \in [0,T]$,

$$|G_t^{\epsilon,\zeta}|_l \leq C_l \mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\leq \Gamma_\epsilon\}} \left[1 + \mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\geq \Gamma_\epsilon - 1\}}(1+J_t)^l (\sup_{[0,t]}|V_s^{\epsilon,\zeta}|_l^l)^l\right].$$

Proof Using [5, Lemma 8], we have

$$|G_t^{\epsilon,\zeta}|_l \leq |G_t^{\epsilon,\zeta}| + C_l \left(\sup_{\{k=1,\dots,l\}} |\Psi^{(k)}(\Sigma_t^{\epsilon,\zeta})|\right) |\Sigma_t^{\epsilon,\zeta}|_l^l.$$

By definition of Ψ, we see that $\sup_{\{k=1,\dots,l\}} |\Psi^{(k)}(x)| \leq C_l \mathbb{1}_{\{1/4\leq x\leq 3/4\}}$. Next we observe that by definition, $\Sigma_t^{\epsilon,\zeta} \in [1/4, 3/4]$ implies $\sup_{[0,t]}|V_s^{\epsilon,\zeta}| \in [\Gamma_\epsilon - 1, \Gamma_\epsilon]$. Recalling (4.2), we only have to prove that $|\Sigma_t^{\epsilon,\zeta}|_l \leq C_l(1+J_t)(\sup_{[0,t]}|V_s^{\epsilon,\zeta}|_l^l)$. But of course, $|\Sigma_t^{\epsilon,\zeta}|_l \leq |\Phi_\epsilon(|V_0|)|_l + \sum_1^{J_t} |\Phi_\epsilon(|V_{T_k}^{\epsilon,\zeta}|)|_l \leq (1+J_t)\sup_{[0,t]}|\Phi_\epsilon(|V_s^{\epsilon,\zeta}|)|_l$. It only remains to check that for all $s \in [0,T]$, $|\Phi_\epsilon(|V_s^{\epsilon,\zeta}|)|_l \leq C_l |V_s^{\epsilon,\zeta}|_l^l$. But this is an immediate consequence of the chain rule (see [5, Lemma 8]) and the fact that $v \mapsto \Phi_\epsilon(|v|)$ has bounded derivative of all orders, uniformly in ϵ. □

Finally, we have all the weapons in hand to give the

Proof of Theorem 4.1. We apply Theorem 4.4 with

$$F = V_t^{\epsilon,\zeta} + \sqrt{u_\zeta(t)} \begin{pmatrix} Z_{-1} \\ Z_0 \end{pmatrix}, \quad G = G_t^{\epsilon,\zeta}.$$

We first note that for $k \geq 1$, $D_k F = D_k V_t^{\epsilon,\zeta}$, that $D_{-1}F = \sqrt{u_\zeta(t)} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $D_0 F = \sqrt{u_\zeta(t)} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. We also have $LF = LV_t^{\epsilon,\zeta} + \sqrt{u_\zeta(t)} \begin{pmatrix} LZ_{-1} \\ LZ_0 \end{pmatrix}$. A simple computation shows that $LZ_0 = Z_0$, so that $D_k(LZ_0) = \mathbb{1}_{k=0}$ and thus so that $D_l D_k(LZ_0) = 0$. This yields $|LZ_0|_l = 1 + |Z_0|$. By the same way, $|LZ_{-1}|_l = 1 + |Z_{-1}|$. Since $u_\zeta(t) \leq 1$,

$$|F|_l \leq C_l(1 + |V_t^{\epsilon,\zeta}|_l), \quad |LF|_l \leq 2 + |Z_{-1}| + |Z_0| + |LV_t^{\epsilon,\zeta}|_l$$
$$\text{and} \quad \sigma(F) = u_\zeta(t) I + \sigma(V_t^{\epsilon,\zeta}).$$

Using Theorem 4.4, we deduce that for β a multi-index with length q,

$$\left|\mathbb{E}\left[\partial_\beta^q \psi(F) G_t^{\epsilon,\zeta}\right]\right| \leq C_q \mathbb{E}[K_{\beta,q}] ||\psi||_\infty,$$

where

$$K_{\beta,q} = \frac{|G_t^{\epsilon,\zeta}|_q (1+\sup_{[0,t]}|V_t^{\epsilon,\zeta}|_{q+1})^{13q}}{(\det(u_\zeta(t)I+\sigma(V_t^{\epsilon,\zeta})))^{3q-1}}$$
$$\times\left[1+\sum_{j=1}^{q}\sum_{k_1+\cdots+k_j\le q-j}\prod_{i=1}^{j}(2+|Z_{-1}|+|Z_0|+|LV_t^{\epsilon,\zeta}|_{k_i})\right]$$
$$\le C_q \mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\le\Gamma_\epsilon\}}\frac{(1+\sup_{[0,t]}|V_t^{\epsilon,\zeta}|_{q+1})^{13q+q^2}}{(\det(u_\zeta(t)I+\sigma(V_t^{\epsilon,\zeta})))^{3q-1}}\left(1+J_t^q\mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\ge\Gamma_\epsilon-1\}}\right)$$
$$\times\left[1+\sum_{j=1}^{q}\sum_{k_1+\cdots+k_j\le q-j}\prod_{i=1}^{j}(2+|Z_{-1}|+|Z_0|+|LV_t^{\epsilon,\zeta}|_{k_i})\right]$$

due to Lemma 4.13. Using the Cauchy the Cauchy-Schwarz inequality, we obtain

$$\mathbb{E}[K_{\beta,q}]\le C_q I_1 I_2 I_3 I_4,$$

where

$$I_1=\mathbb{E}\left[\mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\le\Gamma_\epsilon\}}(1+\sup_{[0,t]}|V_t^{\epsilon,\zeta}|_{q+1})^{4(13q+q^2)}\right]^{1/4},$$
$$I_2=\mathbb{E}\left[(\det(u_\zeta(t)I+\sigma(V_t^{\epsilon,\zeta})))^{-4(3q-1)}\right]^{1/4},$$
$$I_3=\mathbb{E}\left[1+J_t^{4q}\mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\ge\Gamma_\epsilon-1\}}\right]^{1/4},$$
$$I_4=\mathbb{E}\left[1+\sum_{j=1}^{q}\sum_{k_1+\cdots+k_j\le q-j}\prod_{i=1}^{j}(2+|Z_{-1}|+|Z_0|+|LV_t^{\epsilon,\zeta}|_{k_i})^4\mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\le\Gamma_\epsilon\}}\right]^{1/4}.$$

Making use of Lemmas 4.5 and 4.12, we immediately get, for $0\le t_0\le t\le T$,

$$I_1\le C_q e^{C_q\Gamma_\epsilon^\gamma}\quad\text{and}\quad I_2\le C_{t_0,q}e^{C_q\Gamma_\epsilon^\gamma}.$$

Recall now that J_t is a Poisson process with rate $4\Gamma_\epsilon^\gamma(G(\zeta)+1)\le C\Gamma_\epsilon^\gamma\zeta^{-\nu}$, so that $\mathbb{E}[J_t^p]\le C_p\Gamma_\epsilon^{\gamma p}\zeta^{-\nu p}$ for all $p\ge 1$. Using Proposition 2.1-(iii) with some $1/\eta_0<\kappa<\delta$, and the Cauchy-Schwarz inequality, we obtain

$$I_3\le C_q+C_q\mathbb{E}\left[J_t^{8q}\right]^{1/8}\mathbb{P}\left[\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\ge\Gamma_\epsilon-1\right]^{1/8}$$
$$\le C_q+C_q\Gamma_\epsilon^{\gamma q}\zeta^{-\nu q}e^{-4(\Gamma_\epsilon-1)^\kappa})\mathbb{E}\left[\sup_{[0,t]}e^{32|V_s^{\epsilon,\zeta}|^\kappa}\right]^{1/8}\le C_{q,\kappa}(1+\zeta^{-\nu q}e^{-2\Gamma_\epsilon^\kappa}).$$

Finally, using Lemma 4.12, we see that for $j = 1, \dots, q$ and $k_1 + \cdots + k_j \le q - j$,

$$
\begin{aligned}
&\mathbb{E}\left[\prod_{i=1}^{j}(2+|Z_{-1}|+|Z_0|+|LV_t^{\epsilon,\zeta}|_{k_i})^4 \mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\le\Gamma_\epsilon\}}\right]^{1/4}\\
&\le \prod_{i=1}^{j}\mathbb{E}\left[(2+|Z_{-1}|+|Z_0|+|LV_t^{\epsilon,\zeta}|_{k_i})^{4j}\mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\le\Gamma_\epsilon\}}\right]^{1/(4j)}\\
&\le C_q\prod_{i=1}^{j}\mathbb{E}\left[1+|LV_t^{\epsilon,\zeta}|_{k_i}^{4j}\mathbb{1}_{\{\sup_{[0,t]}|V_s^{\epsilon,\zeta}|\le\Gamma_\epsilon\}}\right]^{1/(4j)}\\
&\le C_q e^{C_q\Gamma_\epsilon^\gamma}\left[\prod_{i=1}^{j}(1+\zeta^{-4j\nu}\epsilon^{-4j(k_i+1)})\right]^{1/(4j)}\\
&\le C_q e^{C_q\Gamma_\epsilon^\gamma}\left[\prod_{i=1}^{j}\zeta^{-4j\nu}\epsilon^{-4j(k_i+1)}\right]^{1/(4j)} \le C_q e^{C_q\Gamma_\epsilon^\gamma}\zeta^{-j\nu}\epsilon^{-q}\\
&\le C_q e^{C_q\Gamma_\epsilon^\gamma}\zeta^{-q\nu}\epsilon^{-q},
\end{aligned}
$$

whence $I_4 \le C_q e^{C_q\Gamma_\epsilon^\gamma}\zeta^{-q\nu}\epsilon^{-q}$. All this yields

$$
\begin{aligned}
E[K_{\beta,q}] &\le C_{t_0,q,\kappa}e^{C_q\Gamma_\epsilon^\gamma}\zeta^{-q\nu}\epsilon^{-q}(1+\zeta^{-\nu q}e^{-2\Gamma_\epsilon^\kappa})\\
&\le C_{t_0,q,\kappa}e^{C_q\Gamma_\epsilon^\gamma}\left(\zeta^{-q\nu}\epsilon^{-q}+\zeta^{-2\nu q}e^{-\Gamma_\epsilon^\kappa}\right).
\end{aligned}
$$

For the last inequality, we used that $\Gamma_\epsilon = [\log(1/\epsilon)]^{\eta_0}$ and that $\gamma\eta_0 < 1 < \kappa\eta_0$. Theorem 4.1 is checked. □

5 Conclusion

We now wish to end the proof of our main result.

Lemma 5.1 *Assume that for some $\alpha \in [0, 2)$, some $K > 0$, for all $\epsilon \in (0, 1)$,*

$$\sup_{[0,T]}\sup_{v_0\in\mathbb{R}^2} f_s(Ball(v_0,\epsilon)) \le K\epsilon^\alpha.$$

Then for $\eta \in (0, 1-\nu)$ and $p \ge 1$, for $0 < t_0 \le t \le T$, for $\epsilon \in (0, \epsilon_0)$ and $\zeta \in (0, 1)$, for $q \ge 1$, for all $\xi \in \mathbb{R}^2$ with $|\xi| \ge 1$,

$$
\begin{aligned}
|\widehat{f_t}(\xi)| = \left|\mathbb{E}\left[e^{i\langle\xi,V_t\rangle}\right]\right| \le C_{q,t_0,\eta,p}\Big[&|\xi|^{-q}(\epsilon^{-q-\eta}\zeta^{-\nu q}+\epsilon^p\zeta^{-2\nu q})\\
&+|\xi|^{\nu+\eta}\epsilon^{\nu+\gamma+\alpha}+|\xi|\epsilon^{-\eta}\zeta^{1-\nu}\Big].
\end{aligned}
$$

Proof We have $|\widehat{f}_t(\xi)| = \left|\mathbb{E}\left[e^{i\langle\xi,V_t\rangle}\right]\right|$ by Proposition (2.1)-(ii). We set $X_t^\zeta := \sqrt{u_\zeta(t)}(Z_{-1}, Z_0)$ for simplicity and write

$$\begin{aligned}
|\widehat{f}_t(\xi)| \le\ & \left|\mathbb{E}\left[e^{i\langle\xi,V_t\rangle} - e^{i\langle\xi,V_t^\epsilon\rangle}\right]\right| + \left|\mathbb{E}\left[e^{i\langle\xi,V_t^\epsilon\rangle} - e^{i\left\langle\xi,V_t^{\epsilon,\zeta}\right\rangle}\right]\right| \\
& + \left|\mathbb{E}\left[e^{i\left\langle\xi,V_t^{\epsilon,\zeta}\right\rangle} - e^{i\left\langle\xi,V_t^{\epsilon,\zeta}+X_t^\zeta\right\rangle}\right]\right| \\
& + \left|\mathbb{E}\left[e^{i\left\langle\xi,V_t^{\epsilon,\zeta}+X_t^\zeta\right\rangle}(1-G_t^{\epsilon,\zeta})\right]\right| + \left|\mathbb{E}\left[e^{i\left\langle\xi,V_t^{\epsilon,\zeta}+X_t^\zeta\right\rangle}G_t^{\epsilon,\zeta}\right]\right| \\
=:\ & A_1 + \cdots + A_5.
\end{aligned}$$

First, we apply Theorem 4.1 with $\psi(v) = e^{i\langle\xi,v\rangle}$ and the multi-indexes $\beta_1 = (1,\ldots,1)$ and $\beta_2 = (2,\ldots,2)$ with length q, for which $\partial_{\beta_1}^q\psi(v) = (i\xi_1)^q e^{i\langle\xi,v\rangle}$ and $\partial_{\beta_2}^q\psi(v) = (i\xi_2)^q e^{i\langle\xi,v\rangle}$. For any $\kappa \in (1/\eta_0, \delta)$,

$$\begin{aligned}
A_5 &\le C_{q,t_0,\kappa}|\xi|^{-q}e^{C_q\Gamma_\epsilon^\gamma}(\zeta^{-\nu q}\epsilon^{-q} + \zeta^{-2\nu q}e^{-\Gamma_\epsilon^\kappa}) \\
&\le C_{q,t_0,\eta,p}|\xi|^{-q}(\zeta^{-\nu q}\epsilon^{-q-\eta} + \zeta^{-2\nu q}\epsilon^p),
\end{aligned}$$

because $\Gamma_\epsilon = \log(1/\epsilon)^{\eta_0}$ and $\gamma\eta_0 < 1 < \kappa\eta_0$. Next, by (4.2) and Proposition 2.1-(iii),

$$A_4 \le \mathbb{P}\left[\sup_{[0,T]}|V_t^{\epsilon,\zeta}| \ge \Gamma_\epsilon - 1\right] \le C_\kappa e^{-(\Gamma_\epsilon-1)^\kappa} \le C\epsilon^{\nu+\alpha+\gamma}.$$

We could have chosen any other positive power of ϵ. We also have, since $|e^{i\langle\xi,x\rangle} - e^{i\langle\xi,y\rangle}| \le |\xi||x-y|$,

$$A_3 \le |\xi|\mathbb{E}\left[|X_t^\zeta|\right] \le C|\xi|\sqrt{u_\zeta(t)} \le C|\xi|\zeta^{2+\nu/2}.$$

Proposition 2.1-(iv) (with $\beta = 1$) implies

$$A_2 \le |\xi|\mathbb{E}\left[|V_t^{\epsilon,\zeta} - V_t^\epsilon|\right] \le C|\xi|e^{C\Gamma_\epsilon^\gamma}\zeta^{1-\nu} \le C_\eta|\xi|\epsilon^{-\eta}\zeta^{1-\nu}.$$

Finally, we note that for $\beta \in (0, 1]$,

$$|e^{i\langle\xi,x\rangle} - e^{i\langle\xi,y\rangle}| \le \min(|\xi||x-y|, 2) \le 2^{1-\beta}|\xi|^\beta|x-y|^\beta.$$

Hence using Proposition 2.1-(v) with $\beta = \nu + \eta$ (which is smaller than 1),

$$A_1 \le 2^{1-\beta}\mathbb{E}\left[|\xi|^{\nu+\eta}|V_t^\epsilon - V_t|^{\nu+\eta}\right] \le C_\eta|\xi|^{\nu+\eta}\epsilon^{\nu+\eta+\gamma+\alpha}e^{C_\eta\Gamma_\epsilon^\gamma},$$

which we can bound by $C_\eta|\xi|^{\nu+\eta}\epsilon^{\nu+\gamma+\alpha}$ as usual. To conclude the proof, it suffices to note that we obviously have $\epsilon^{\nu+\alpha+\gamma} \le |\xi|^{\nu+\eta}\epsilon^{\nu+\alpha+\gamma}$ and $|\xi|\zeta^{2+\nu/2} \le |\xi|\epsilon^{-\eta}\zeta^{1-\nu}$. □

Next, we optimize the previous formula.

Lemma 5.2 *Assume that for some* $\alpha \in [0, 2)$*, some* $K > 0$*, for all* $\epsilon \in (0, 1)$*,*

$$\sup_{[0,T]} \sup_{v_0 \in \mathbb{R}^2} f_s(Ball(v_0, \epsilon)) \le K\epsilon^\alpha.$$

Assume that $\nu \in (0, 1/2)$ *and that* $\gamma > \nu^2/(1-2\nu)$*. Define*

$$p(\alpha) = \frac{(\alpha+\gamma)(1-2\nu) - \nu^2}{(\alpha+\gamma+\nu-1)\nu + 1} > 0.$$

Then for all $r \in (0, p(\alpha))$*, all* $0 < t_0 \le t \le T$ *and all* $\xi \in \mathbb{R}^2$*,*

$$|\widehat{f_t}(\xi)| \le C_{r,t_0}|\xi|^{-r}.$$

Proof We can assume that $|\xi| \ge 1$, because f_t is a probability measure, so that $||\widehat{f_t}||_\infty = 1$. We use Lemma 5.1 with $\epsilon = |\xi|^{-a}$ and $\zeta = |\xi|^{-b}$, for some $a > 0$, $b > 0$ such that $a + \nu b = 1 - \eta_1$, for some small $\eta_1 \in (0, 1)$ to be chosen later. We thus get, for some small $\eta \in (0, 1-\nu)$ and some large $p \ge 1$, $q \ge 1$ to be chosen later, for all $|\xi| \ge 1$,

$$\begin{aligned}
|\widehat{f_t}(\xi)| &\le C_{q,t_0,\eta,p}\Big(|\xi|^{-q+a\eta+(a+\nu b)q} + |\xi|^{-q-ap+2\nu qb} + |\xi|^{\nu+\eta-a(\nu+\gamma+\alpha)} \\
&\qquad + |\xi|^{1+a\eta-b(1-\nu)}\Big) \\
&= C_{q,t_0,\eta,p}\Big(|\xi|^{-\eta_1 q+a\eta} + |\xi|^{-q-ap+2q(1-\eta_1-a)} + |\xi|^{\nu+\eta-a(\nu+\gamma+\alpha)} \\
&\qquad + |\xi|^{1+a\eta-(1-\eta_1-a)(1/\nu-1)}\Big) \\
&\le C_{q,t_0,\eta,p}\Big(|\xi|^{-\eta_1 q+1} + |\xi|^{q-ap} + |\xi|^{\nu+\eta-a(\nu+\gamma+\alpha)} \\
&\qquad + |\xi|^{1+a(\eta+1/\nu-1)-(1-\eta_1)(1/\nu-1)}\Big).
\end{aligned}$$

We used here that $0 < a\eta \le 1$ and $1-\eta_1-a \le 1$. Let now $r \in (0, p(\alpha))$. It remains to show that one may find $q \ge 1$, $p \ge 1$, $\eta_1 \in (0, 1)$, $\eta \in (0, 1-\nu)$ and $a \in (0, 1-\eta_1)$ in such a way that

$$\eta_1 q - 1 \ge r, \tag{5.1}$$
$$ap - q \ge r, \tag{5.2}$$
$$a(\nu+\gamma+\alpha) - \nu - \eta \ge r, \tag{5.3}$$
$$(1-\eta_1)(1/\nu-1) - 1 - a(\eta + 1/\nu - 1) \ge r. \tag{5.4}$$

It suffices to show that (5.3) and (5.4) hold for some $\eta \in (0, 1-\nu)$, some $\eta_1 \in (0, 1)$ and some $a \in (0, 1-\eta_1)$ small enough. Indeed, it will then suffice to choose q large enough to get (5.1) and then p large enough to obtain (5.2). Hence it suffices to check that there is $a \in (0, 1)$ such that

$$a(\nu + \gamma + \alpha) - \nu > r \quad \text{and} \quad 1/\nu - 2 - a(1/\nu - 1) > r.$$

But setting $a = (1 - 2\nu + \nu^2)/[1 + \nu(\nu + \gamma + \alpha - 1)]$, we get

$$a(\nu + \gamma + \alpha) - \nu = 1/\nu - 2 - a(1/\nu - 1) = p(\alpha) > r.$$

To conclude the proof, it only remains to check that $a \in (0, 1)$. Clearly, $a > 0$. To check that $a < 1$, it suffices to prove that $1 - 2\nu + \nu^2 < 1 + \nu(\nu - 1)$, which always holds for $\nu > 0$. □

The last preliminary consists of studying the function $\alpha \mapsto p(\alpha)$.

Lemma 5.3 *Assume that* $\nu \in (0, 1/2)$ *and that* $\gamma > \nu^2/(1 - 2\nu)$.

(i) *The map* $\alpha \mapsto p(\alpha)$ *is increasing on* $[0, \infty)$. *The function* $\alpha \mapsto p(\alpha)/\alpha$ *is decreasing on* $(0, \infty)$ *and* $p(a_{\gamma,\nu})/a_{\gamma,\nu} = 1$, *where* $a_{\gamma,\nu}$ *was defined by* (1.5).

(ii) *Furthermore, we have, recalling* (1.6)

$$q_{\gamma,\nu} > 1 \Longleftrightarrow a_{\gamma,\nu} > 1 \quad \Longleftrightarrow \quad \nu < 1/3 \quad \textit{and} \quad \gamma > (2\nu + 2\nu^2)/(1 - 3\nu),$$
$$q_{\gamma,\nu} > 2 \Longleftrightarrow a_{\gamma,\nu} > 2 \quad \Longleftrightarrow \quad \nu < 1/4 \quad \textit{and} \quad \gamma > (6\nu + 3\nu^2)/(1 - 4\nu).$$

Observe that $q_{\gamma,\nu} = p(2 \wedge a_{\gamma,\nu})$.

(iii) *For* $q \in (0, q_{\gamma,\nu})$, *one may find* $n_0 \geq 1$ *and* $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n_0}$ *such that for all* $k \in \{0, \ldots, n_0 - 1\}$, $\alpha_k \in [0, 2)$ *and* $\alpha_{k+1} < p(\alpha_k)$, *with furthermore* $\alpha_{n_0} \geq q$, *all these quantities depending only on* q, γ, ν.

Proof We start with point (i). To show that p is increasing, it suffices to note that its derivative is positive if and only if $(1 - 2\nu)[(\gamma + \nu - 1)\nu + 1] > \nu[\gamma(1 - 2\nu) - \nu^2]$, i.e. $1 - 3\nu + 3\nu^2 - \nu^3 > 0$, which always holds for $\nu \in (0, 1)$. We also have

$$\frac{p(\alpha)}{\alpha} = \frac{1 - 2\nu}{\alpha\nu + [(\gamma + \nu - 1)\nu + 1]} + \frac{\gamma(1 - 2\nu) - \nu^2}{\alpha^2\nu + \alpha[(\gamma + \nu - 1)\nu + 1]},$$

which is obviously decreasing, because under our assumptions, $1 - 2\nu > 0$, $\gamma(1 - 2\nu) - \nu^2 > 0$ and $(\gamma + \nu - 1)\nu + 1 > 0$. Next, $a_{\gamma,\nu} > 0$ is designed to solve $\nu a_{\gamma,\nu}^2 + \nu(\gamma + \nu + 1)a_{\gamma,\nu} = \gamma(1 - 2\nu) - \nu^2$, whence

$$\frac{p(a_{\gamma,\nu})}{a_{\gamma,\nu}} = \frac{a_{\gamma,\nu}(1 - 2\nu) + \gamma(1 - 2\nu) - \nu^2}{\nu a_{\gamma,\nu}^2 + \nu(\gamma + \nu + 1)a_{\gamma,\nu} + (1 - 2\nu)a_{\gamma,\nu}} = 1.$$

We now prove (ii). Due to (i), we clearly have $a_{\gamma,\nu} > 1$ if and only if $p(1)/1 > 1$, i.e. $[(1 + \gamma)(1 - 2\nu) - \nu^2]/[(\gamma + \nu)\nu + 1] > 1$, which is equivalent to $\nu > 1/3$ and $\gamma > (2\nu + 2\nu^2)/(1 - 3\nu)$. By the same way, $a_{\gamma,\nu} > 2$ if and only if $p(2)/2 > 1$, i.e.

$[(2+\gamma)(1-2\nu)-\nu^2]/[(1+\gamma+\nu)\nu+1] > 2$, which is equivalent to $\nu > 1/4$ and $\gamma > (6\nu+3\nu^2)/(1-4\nu)$. Next we note that we always have $q_{\gamma,\nu} = p(a_{\gamma,\nu} \wedge 2)$. Thus we have $a_{\gamma,\nu} > 2$ if and only if $p(2)/2 > 1$ if and only if $q_{\gamma,\nu} > 2$. Similarly, $a_{\gamma,\nu} > 1$ if and only if $p(1)/1 > 1$ if and only if $q_{\gamma,\nu} > 1$.

Let us now check point (iii). We fix $q \in (0, q_{\gamma,\nu})$.

We first assume that $a_{\gamma,\nu} \le 2$, whence $q_{\gamma,\nu} = a_{\gamma,\nu}$. We fix $q' \in (q, q_{\gamma,\nu})$, we observe that due to (i), $p(q')/q' > 1$ and we consider $\eta > 0$ such that $(1-\eta)p(q')/q' = 1$. Then by (i), we deduce that the sequence $\alpha_0 = 0, \alpha_{k+1} = (1-\eta)p(\alpha_k)$ takes its values in $[0, q'] \subset [0, 2)$ and increases to q'. Thus for some $n_0, \alpha_{n_0} \ge q$. Of course, we have $\alpha_{k+1} < p(\alpha_k)$ for all $k \in \{0, \dots, n_0 - 1\}$, so that $(\alpha_0, \dots, \alpha_{n_0})$ solves our problem.

Next we assume that $a_{\gamma,\nu} > 2$, whence $q_{\gamma,\nu} = p(2) > 2$. We may assume that $q \in (2, p(2))$. We consider $\eta > 0$ such that $(1-\eta)p(2)/2 = 1$, whence $(1-\eta)p(\alpha)/\alpha > 1$ for all $\alpha \in [0, 2)$. Then by (i), the sequence $\alpha_0 = 0$, $\alpha_{k+1} = (1-\eta)p(\alpha_k)$ takes its values in $[0, 2)$ and increases to 2. Consider now $x \in (0, 2)$ such that $p(x) = q$ (recall that $q \in (2, p(2))$ is fixed). Then for n_0 sufficiently large, we have $\alpha_{n_0-1} > x$ and thus $\alpha_{n_0-1} < q < p(\alpha_{n_0-1})$. Hence $(\alpha_0, \dots, \alpha_{n_0-1}, q)$ solves our problem. □

The last preliminary consists of an easy result on Fourier transforms. Recall that for f a probability measure on $\mathbb{R}^2$ and $\xi \in \mathbb{R}^2$, we denote by $\widehat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^2} e^{i\langle \xi, v\rangle} f(dv)$.

Lemma 5.4 *Let f be a probability measure on $\mathbb{R}^2$ such that $|\widehat{f}(\xi)| \le K|\xi|^{-\alpha}$, for some $\alpha \in (0, 2)$. Then for all $v_0 \in \mathbb{R}^2$, all $\epsilon \in (0, 1)$, one has $f(Ball(v_0, \epsilon)) \le C_{K,\alpha}\epsilon^\alpha$.*

This Lemma will be checked in the appendix. We can now give the

Proof of Theorem 1.3 Points (ii) and (iii) follow from (i) and Lemma 5.3. We fix $0 < t_0 < T$ and $q \in (0, q_{\gamma,\nu})$. The only thing we have to check is that for all $\xi \in \mathbb{R}^2$, all $t \in [t_0, T]$, $|\widehat{f}_t(\xi)| \le C_{t_0,q}(1+|\xi|)^{-q}$. Then the Sobolev norm estimate and the ball estimate will follow (see Lemma 5.4). By Lemma 5.3, we may consider $n_0 \ge 1$ and $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{n_0}$ such that for all $k \in \{0, \dots, n_0 - 1\}$, $\alpha_k \in [0, 2)$ and $\alpha_{k+1} < p(\alpha_k)$, with $\alpha_{n_0} \ge q$.

Step 1 First, we apply Lemma 5.2 with $\alpha = \alpha_0 = 0$. Since $\alpha_1 < p(\alpha_0)$, we deduce that

$$\sup_{t\in[t_0/n_0,T]} |\widehat{f}_t(\xi)| \le C|\xi|^{-\alpha_1}.$$

By Lemma 5.4, we deduce that $\sup_{[t_0/n_0,T]} \sup_{v_0\in\mathbb{R}^2} f_t(Ball(v_0,\epsilon)) \le C_{t_0,q}\epsilon^{\alpha_1}$.

Step 2 Define now $(f_t^1)_{t\in[0,T-t_0/n_0]}$ by $f_t^1 = f_{t+t_0/n_0}$. This is also a weak solution of (1.1). It satisfies the same properties as $(f_t)_{t\in[0,T]}$, and the additional property that

$$\sup_{[0,T-t_0/n_0]} \sup_{v_0\in\mathbb{R}^2} f_t^1(Ball(v_0,\epsilon)) \le C_{t_0,q}\epsilon^{\alpha_1}.$$

We thus can apply Lemma 5.2 with $\alpha = \alpha_1$ and $r = \alpha_2 < p(\alpha_1)$, to get

$$\sup_{t\in[2t_0/n_0,T]} |\widehat{f_t}(\xi)| = \sup_{t\in[t_0/n_0,T-t_0/n_0]} |\widehat{f_t^1}(\xi)| \le C|\xi|^{-\alpha_2},$$

whence $\sup_{[2t_0/n_0,T]} \sup_{v_0\in\mathbb{R}^2} f_t(Ball(v_0,\epsilon)) \le C_{t_0,q}\epsilon^{\alpha_2}$ by Lemma 5.4.
Step 3 Iterating this procedure (n_0 times), we deduce that

$$\sup_{t\in[t_0,T]} |\widehat{f_t}(\xi)| \le C_{t_0,r}|\xi|^{-\alpha_{n_0}}.$$

But f_t is a probability measure, so that $|\widehat{f_t}(\xi)| \le 1$. Thus

$$\sup_{t\in[t_0,T]} |\widehat{f_t}(\xi)| \le C_{t_0,r}(1+|\xi|)^{-\alpha_{n_0}},$$

which ends the proof since $\alpha_{n_0} \ge q$. □

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Appendix

Fourier transforms

We first check Lemma 5.4. This result is probably standard, but we found no reference and the proof is short and easy.

Proof of Lemma 5.4 We use the Plancherel identity. Recall that

$$\mathcal{F}(\mathbb{1}_{[x_0-\epsilon,x_0+\epsilon]\times[y_0-\epsilon,y_0+\epsilon]})(\xi_1,\xi_2) = 4e^{i\xi_1 x_0+i\xi_2 y_0}\sin(\xi_1\epsilon)\sin(\xi_2\epsilon)/(\xi_1\xi_2).$$

Setting $v_0 = (x_0, y_0)$,

$$\begin{aligned} f(Ball(v_0,\epsilon)) &\le \int_{\mathbb{R}^2} f(dv)\mathbb{1}_{[x_0-\epsilon,x_0+\epsilon]\times[y_0-\epsilon,y_0+\epsilon]}(v) \\ &\le C\int_{\mathbb{R}^2} \left|\widehat{f}(\xi)\frac{\sin(\xi_1\epsilon)\sin(\xi_2\epsilon)}{\xi_1\xi_2}\right| d\xi \\ &\le C_K\int_{\mathbb{R}^2} |\xi|^{-\alpha}\frac{|\sin(\xi_1\epsilon)\sin(\xi_2\epsilon)|}{|\xi_1\xi_2|}d\xi \le C_K\int_{\mathbb{R}^2} \frac{|\sin(\xi_1\epsilon)\sin(\xi_2\epsilon)|}{|\xi_1\xi_2|^{1+\alpha/2}}d\xi, \end{aligned}$$

because $|\xi| \geq \sqrt{2|\xi_1\xi_2|}$. We handle the substitution $\xi = x/\epsilon$ and get

$$f(Ball(v_0,\epsilon)) \leq C_K\epsilon^\alpha \int_{\mathbb{R}^2} \frac{|\sin(x_1)|}{|x_1|^{1+\alpha/2}}\frac{|\sin(x_2)|}{|x_2|^{1+\alpha/2}}dx \leq C_K\epsilon^\alpha\left(\int_{\mathbb{R}} \frac{|\sin(x_1)|}{|x_1|^{1+\alpha/2}}dx_1\right)^2.$$

We easily conclude, since $\alpha \in (0,2)$. □

Lowerbound

We now handle the

Proof of Lemma 4.9 Recall that by (1.8), we have $\int_{\mathbb{R}^2}|v|^2 f_t(dv) = e_0 > 0$ and $\int_{\mathbb{R}^2} v f_t(dv) = 0$. First, we observe that for all w such that $|w| \geq \sqrt{2e_0}+1 =: a$, we have

$$f_t(\{v, |v-w|\geq 1\}) \geq f_t(\{v,|v|\leq |w|-1\}) = 1 - f_t(\{v,|v|>|w|-1\}) \\ \geq 1 - e_0/(|w|-1)^2 \geq 1/2.$$

Thus it suffices to prove the result for $(t,w) \in [0,T]\times\overline{Ball(0,a)}$. We note that for each $t\geq 0$, f_t is not a Dirac mass. Indeed, since $\int_{\mathbb{R}^2} v f_t(dv) = 0$, the only possible Dirac mass is δ_0, but this would imply $\int_{\mathbb{R}^2}|v|^2 f_t(dv) = 0$.

As a consequence, we can find, for each $(t,w)\in[0,T]\times\overline{Ball(0,a)}$, some numbers $r_{t,w} > 0$ and $q_{t,w} > 0$ such that $f_t(\{v,|v-w|\geq r_{t,w}\}) \geq q_{t,w}$.

Now we prove that for each $(t,w)\in[0,T]\times\overline{Ball(0,a)}$, we can find a neighborhood $\mathcal{V}_{t,w}$ of (t,w) such that for all $(t',w')\in\mathcal{V}_{t,w}$, $f_{t'}(\{v,|v-w'|\geq r_{t,w}/2\}) \geq q_{t,w}/2$. To do so, we first observe that it is clear from Definition 1.1 that $t\mapsto f_t$ is weakly continuous. Hence for all continuous-bounded function $\varphi:\mathbb{R}\mapsto\mathbb{R}_+$, $(t',w')\mapsto \int_{\mathbb{R}^2}\varphi(|w'-v|)f_{t'}(dv)$ is continuous. Consider now a continuous-bounded nonnegative function $\varphi:\mathbb{R}_+\mapsto\mathbb{R}_+$ such that $\mathbb{1}_{\{x\geq r_{t,w}\}} \leq \varphi \leq \mathbb{1}_{\{x\geq r_{t,w}/2\}}$. By continuity, there is a neighborhood $\mathcal{V}_{t,w}$ of (t,w) such that for all $(t',w')\in\mathcal{V}_{t,w}$, there holds $\int_{\mathbb{R}^2}\varphi(|w'-v|)f_{t'}(dv) \geq \frac{1}{2}\int_{\mathbb{R}^2}\varphi(|w-v|)f_t(dv)$, which implies

$$f_{t'}(\{v,|v-w'|\geq r_{t,w}/2\}) \geq \tfrac{1}{2}f_t(\{v,|v-w|\geq r_{t,w}\}) \geq q_{t,w}/2.$$

Since $[0,T]\times\overline{Ball(0,a)}$ is compact, we can find a finite covering $[0,T]\times\overline{Ball(0,a)} \subset \cup_{i=1}^n \mathcal{V}_{t_i,w_i}$. We conclude choosing $r_0 = \min(r_{t_i,w_i}/2)\wedge 1$ and $q_0 = \min(q_{t_i,w_i}/2)\wedge(1/2)$. □

Derivatives

We recall here the Faa di Bruno formula. Let $l\geq 1$ be fixed. The exist some coefficients $a^{l,r}_{i_1,\dots,i_r} > 0$ such that for $\phi:\mathbb{R}\mapsto\mathbb{R}$ and $\tau:\mathbb{R}\mapsto\mathbb{R}$ of class $C^l(\mathbb{R})$,

$$[\phi(\tau)]^{(l)} = [\tau']^l \phi^{(l)}(\tau) + \sum_{r=1}^{l-1} \left(\sum_{i_1+\cdots+i_r=l} a^{l,r}_{i_1,\ldots,i_r} \prod_{j=1}^{r} \tau^{(i_j)} \right) \phi^{(r)}(\tau), \tag{5.5}$$

where the sum is taken over $i_1 \geq 1, \ldots, i_r \geq 1$ with $i_1 + \cdots + i_r = l$.

We also need another formula. For $l \geq 2$ fixed, there exist some coefficients $c^{l,r}_{i_1,..,i_q} \in \mathbb{R}$ such that for $\phi : \mathbb{R} \mapsto \mathbb{R}$ a C^l-diffeomorphism and for τ its inverse function,

$$\tau^{(l)} = \sum_{r=l+1}^{2l-1} \frac{1}{(\phi'(\tau))^r} \sum_{i_1+\cdots+i_q=r-1} c^{l,r}_{i_1,..,i_q} \prod_{j=1}^{q} \phi^{(i_j)}(\tau), \tag{5.6}$$

where the sum is taken over $q \in \mathbb{N}$, over $i_1, \ldots, i_q \in \{2, \ldots, l\}$ with $i_1 + \cdots + i_q = r - 1$. This formula can be checked by induction on $l \geq 2$.

Regularity of the modified cross section

We still have to give the

Proof of Lemma 3.1 Due to $(\mathbf{A}(\gamma, \nu))$, we have $c(x^{-\nu} - (\pi/2)^{-\nu}) \leq G(x) \leq C(x^{-\nu} - (\pi/2)^{-\nu})$, for all $x \in (0, \pi/2]$. Since ϑ is nonincreasing, we easily deduce that for all $z \in [0, \infty)$, $(z/c + (\pi/2)^{-\nu})^{-1/\nu} \leq \vartheta(z) \leq (z/C + (\pi/2)^{-\nu})^{-1/\nu}$ and (i) follows. Next, we have $|\vartheta'(z)| = 1/|b(\vartheta(z))|$. But $b(x) \in [cx^{-1-\nu}, Cx^{-1-\nu}]$, so that $|\vartheta'(z)| \in [\vartheta^{1+\nu}(z)/C, \vartheta^{1+\nu}(z)/c]$. Using (i), we deduce (ii). Next, (iii) is obtained from (5.6): using that for any $k \geq 2$, $|G^{(k)}(x)| = |b^{(k-1)}(x)| \leq C_k |x|^{-\nu-k}$, we get

$$|\vartheta^{(k)}(z)| \leq C_k \sum_{r=k+1}^{2k-1} |\vartheta(z)|^{r(\nu+1)} \sum_{i_1+\cdots+i_q=r-1} |\vartheta(z)|^{-\nu q - r + 1}.$$

Since we have $i_1, \ldots, i_q \in \{2, \ldots, k\}$ such that $i_1 + \cdots + i_q = r - 1$, we see that $q \leq (r-1)/2$. Consequently, for $k \geq 2$,

$$\begin{aligned} |\vartheta^{(k)}(z)| &\leq C_k \sum_{r=k+1}^{2k-1} |\vartheta(z)|^{r(\nu+1)} |\vartheta(z)|^{-\nu(r-1)/2 - r + 1} \\ &= C_k \sum_{r=k+1}^{2k-1} |\vartheta(z)|^{(r+1)\nu/2+1} \leq C_k |\vartheta(z)|^{(k+2)\nu/2+1} \leq C_k (1+|z|)^{-1/\nu - 1}, \end{aligned}$$

where we finally used (i). Since $|A^{(l)}(\theta)| \leq C_l$ for all $l \geq 1$, (iv) follows from (5.5) and (iii). □

Regularity of the cutoff function

We now prove the regularity properties of our cutoff function ϕ_ϵ stated in Lemma 2.3.

Proof of Lemma 2.3 We first prove (i). We recall that for any $a, b > 0$, there are some constants $0 < c_{a,b} < C_{a,b}$ such that for any $x, y \geq 0$, $c_{a,b}|x^{a+b} - y^{a+b}| \leq (x^a + y^a)|x^b - y^b| \leq C_{a,b}|x^{a+b} - y^{a+b}|$. We also recall that ϕ_ϵ is globally Lipschitz continuous with constant 1, that $\phi_\epsilon(x) = \Gamma_\epsilon$ for $x \geq \Gamma_\epsilon + 1$ and that $\phi_\epsilon(x) \geq x/2$ for $x \in [0, \Gamma_\epsilon + 1]$, since $\phi_\epsilon(x) \geq x$ for $x \in [0, \Gamma_\epsilon - 1]$ and since ϕ_ϵ is non-decreasing. We set $\Delta_\epsilon(x, y) = x^\beta|\phi_\epsilon^\gamma(x) - \phi_\epsilon^\gamma(y)|$. If $x, y \geq \Gamma_\epsilon + 1$, then $\Delta_\epsilon(x, y) = 0$. If now $x \leq \Gamma_\epsilon + 1$, then

$$\begin{aligned}
\Delta_\epsilon(x, y) &\leq 2^\beta \phi_\epsilon^\beta(x)|\phi_\epsilon^\gamma(x) - \phi_\epsilon^\gamma(y)| \\
&\leq 2^\beta(\phi_\epsilon^\beta(x) + \phi_\epsilon^\beta(y))|\phi_\epsilon^\gamma(x) - \phi_\epsilon^\gamma(y)| \\
&\leq 2^\beta C_{\beta,\gamma}|\phi_\epsilon^{\beta+\gamma}(x) - \phi_\epsilon^{\beta+\gamma}(y)| \\
&\leq 2^\beta \frac{C_{\beta,\gamma}}{c_{\gamma,\beta}}(\phi_\epsilon^\gamma(x) + \phi_\epsilon^\gamma(y))|\phi_\epsilon^\beta(x) - \phi_\epsilon^\beta(y)| \\
&\leq 2^{\beta+\gamma} \frac{C_{\beta,\gamma}}{c_{\gamma,\beta}}\Gamma_\epsilon^\gamma|\phi_\epsilon(x) - \phi_\epsilon(y)|^\beta \\
&\leq 2^{\beta+\gamma} \frac{C_{\beta,\gamma}}{c_{\gamma,\beta}}\Gamma_\epsilon^\gamma|x - y|^\beta.
\end{aligned}$$

We used here that $\beta < 1$. Finally, if $x \geq \Gamma_\epsilon + 1$ and $y \leq \Gamma_\epsilon + 1$,

$$\begin{aligned}
\Delta_\epsilon(x, y) &= x^\beta|\Gamma_\epsilon^\gamma - \phi_\epsilon^\gamma(y)| \\
&\leq (|x - y|^\beta + |y|^\beta)(\Gamma_\epsilon^\gamma - \phi_\epsilon^\gamma(y)) \\
&\leq |x - y|^\beta\Gamma_\epsilon^\gamma + |y|^\beta|\phi_\epsilon^\gamma(x) - \phi_\epsilon^\gamma(y)| \\
&\leq |x - y|^\beta\Gamma_\epsilon^\gamma + 2^{\beta+\gamma}\frac{C_{\beta,\gamma}}{c_{\gamma,\beta}}\Gamma_\epsilon^\gamma|x - y|^\beta,
\end{aligned}$$

the last inequality being obtained as previously, since $y \leq \Gamma_\epsilon + 1$.

To prove (ii), we first observe that for $k \geq 1$,

$$|\phi_\epsilon^{(k)}(x)| \leq C_k \left(\epsilon^{1-k}\mathbb{1}_{\{x\in(\epsilon,3\epsilon)\}} + \mathbb{1}_{\{k=1\}}\mathbb{1}_{\{x\in[3\epsilon,\Gamma_\epsilon-1]\}} + \mathbb{1}_{\{x\in(\Gamma_\epsilon-1,\Gamma_\epsilon+1)\}}\right).$$

Using the Faa di Bruno formula (5.5), one easily deduces that for $l \geq 1$,

$$|[\log\phi_\epsilon(x)]^{(l)}| \leq C_l\left(\mathbb{1}_{\{x\in(\epsilon,\Gamma_\epsilon]\}}x^{-l} + \mathbb{1}_{\{x\in(\Gamma_\epsilon-1,\Gamma_\epsilon+1)\}}\Gamma_\epsilon^{-1}\right)$$

and

$$|[\phi_\epsilon^\gamma(x)]^{(l)}| \leq C_l\left(\mathbb{1}_{\{x\in(\epsilon,\Gamma_\epsilon]\}}x^{\gamma-l} + \mathbb{1}_{\{x\in(\Gamma_\epsilon-1,\Gamma_\epsilon+1)\}}\Gamma_\epsilon^{\gamma-1}\right).$$

Using again (5.5) and that any derivative of order $k \geq 1$ of $v \mapsto |v|$ is smaller than $C_k |v|^{1-k}$, one easily concludes. □

Exponential estimates

Finally, we conclude with the

Proof of Lemma 2.2 We start with the first inequality. Recall that by (1.2), $|A(\theta)V|^2 = \frac{1+\cos\theta}{2}|V|^2$. We also have $\langle V, A(\theta)V\rangle = -\frac{1-\cos\theta}{2}|V|^2$, $|A(\theta)| \leq |\theta|$ and $\theta^2/4 \leq 1-\cos\theta \leq \theta^2$ for $\theta \in [-\pi/2, \pi/2]$. Thus

$$
\begin{aligned}
|V + A(\theta)(V-v)|^2 &= |V|^2 + \frac{1-\cos\theta}{2}(|V|^2 + |v|^2 - 2\langle V, v\rangle) \\
&\quad +2\langle V, A(\theta)V\rangle - 2\langle V, A(\theta)v\rangle \\
&= \frac{1+\cos\theta}{2}|V|^2 + \frac{1-\cos\theta}{2}(|v|^2 - 2\langle V, v\rangle) - 2\langle V, A(\theta)v\rangle \\
&\leq |V|^2(1-\theta^2/8) + \theta^2|v|^2 + 4|\theta||V||v|.
\end{aligned}
$$

An simple computation shows that

$$
|V + A(\theta)(V-v)|^2 \leq \left\{ \begin{array}{ll} |V|^2(1-\theta^2/16) & \text{if } |V| \geq 130|v|/|\theta| \\ |V|^2 + \theta^2|v|^2 + 4|\theta||V||v| & \text{if } |V| \leq 130|v|/|\theta| \end{array} \right\}.
$$

In the case where $|V| \leq 1$, we observe that, since $\kappa \in (0,1)$,

$$
|V + A(\theta)(V-v)|^\kappa \leq (|V| + |\theta|(|V|+|v|))^\kappa \leq |V|^\kappa + |\theta|^\kappa(1+|v|^\kappa).
$$

We thus may write

$$
\begin{aligned}
\Delta(V, v) &:= \int_{-\pi/2}^{\pi/2} \left(e^{|V+A(\theta)(V-v)|^\kappa} - e^{|V|^\kappa}\right) b(\theta)d\theta \\
&\leq -\int_{-\pi/2}^{\pi/2} \left(e^{|V|^\kappa} - e^{|V|^\kappa(1-\theta^2/16)^{\kappa/2}}\right) \mathbb{1}_{\{|\theta|\geq 130|v|/|V|\}} b(\theta)d\theta \\
&\quad + \mathbb{1}_{\{|V|\geq 1\}} \int_{-\pi/2}^{\pi/2} \left(e^{(|V|^2+\theta^2|v|^2+4|\theta||V||v|)^{\kappa/2}} - e^{|V|^\kappa}\right) \mathbb{1}_{\{|\theta|\leq 130|v|/|V|\}} b(\theta)d\theta \\
&\quad + \mathbb{1}_{\{|V|\leq 1\}} \int_{-\pi/2}^{\pi/2} \left(e^{|V|^\kappa + C_\kappa|\theta|(1+|v|^\kappa)} - e^{|V|^\kappa}\right) b(\theta)d\theta \\
&=: -\Delta_1(V, v) + \Delta_2(V, v) + \Delta_3(V, v).
\end{aligned}
$$

We now compute carefully. First, we have

$$\Delta_1(V,v) \geq \mathbb{1}_{\{|V|\geq 1, |V|\geq 130|v|\}} \int_{-\pi/2}^{\pi/2} \left(e^{|V|^\kappa} - e^{|V|^\kappa(1-\theta^2/16)^{\kappa/2}}\right) \mathbb{1}_{\{|\theta|\geq 1\}} b(\theta)d\theta.$$

But for $|\theta| \geq 1$ and $|V| \geq 1$,

$$\begin{aligned} e^{|V|^\kappa} - e^{|V|^\kappa(1-\theta^2/16)^{\kappa/2}} &\geq e^{|V|^\kappa} - e^{|V|^\kappa(1-1/16)^{\kappa/2}} \\ &\geq e^{|V|^\kappa}(1 - e^{-|V|^\kappa(1-(1-1/16)^{\kappa/2})}) \geq c_\kappa e^{|V|^\kappa}, \end{aligned}$$

whence, since $b([1,\pi/2]) > 0$ by assumption,

$$\Delta_1(V,v) \geq c_\kappa \mathbb{1}_{\{|V|\geq 1, |V|\geq 130|v|\}} e^{|V|^\kappa}.$$

Next we observe that for $x, y \geq 0$, since $\kappa/2 \in (0,1)$, $e^{(x+y)^{\kappa/2}} - e^{x^{\kappa/2}} \leq (\kappa/2) y x^{\kappa/2-1} e^{x^{\kappa/2}} e^{y^{\kappa/2}}$. As a consequence in Δ_2, since $|\theta||V| \leq 130|v|$,

$$\begin{aligned} e^{(|V|^2+\theta^2|v|^2+4|\theta||V||v|)^{\kappa/2}} - e^{|V|^\kappa} &\leq C_\kappa(\theta^2|v|^2 + |\theta||V||v|)|V|^{\kappa-2} e^{|V|^\kappa} e^{C_\kappa(\theta^2|v|^2+|\theta||V||v|)^{\kappa/2}} \\ &\leq C_\kappa(\theta^2|v|^2 + |\theta||V||v|)|V|^{\kappa-2} e^{|V|^\kappa} e^{C_\kappa|v|^\kappa}. \end{aligned}$$

Integrating this formula against $b(\theta)d\theta$ (on $|\theta| \in [0, \min(\pi/2, 130|v|/|V|)]$) and using $(\mathbf{A}(\gamma,\nu))$ yields

$$\begin{aligned} \Delta_2(V,v) &\leq C_\kappa \mathbb{1}_{\{|V|\geq 1\}} |V|^{\kappa-2} e^{|V|^\kappa} e^{C_\kappa|v|^\kappa} \Big[|v|^2 \min(1, (|v|/|V|)^{2-\nu}) \\ &\qquad + |V||v| \min(1, (|v|/|V|)^{1-\nu})\Big] \\ &\leq C_\kappa \mathbb{1}_{\{|v|\geq|V|\geq 1\}} e^{|V|^\kappa} e^{C_\kappa|v|^\kappa} |V|^{\kappa-2}|v|^2 \\ &\quad + C_\kappa \mathbb{1}_{\{|V|\geq 1, |V|\geq|v|\}} e^{|V|^\kappa} e^{C_\kappa|v|^\kappa} (|v|^{4-\nu}|V|^{\kappa+\nu-4} + |v|^{2-\nu}|V|^{\kappa+\nu-2}) \\ &\leq C_\kappa \mathbb{1}_{\{|V|\geq 1\}} |V|^{\kappa+\nu-2} e^{|V|^\kappa} e^{C_\kappa|v|^\kappa}. \end{aligned}$$

We finally used that $\kappa + \nu - 4 \leq \kappa - 2 \leq \kappa + \nu - 2 < 0$. Recall now that for $x \geq 0$, $e^x - 1 \leq x e^x$, so that in Δ_3, since $|V| \leq 1$,

$$e^{|V|^\kappa + |\theta|^\kappa(1+|v|^\kappa)} - e^{|V|^\kappa} = e^{|V|^\kappa}(e^{|\theta|^\kappa(1+|v|^\kappa)} - 1) \leq C_\kappa |\theta|^\kappa e^{C_\kappa|v|^\kappa}.$$

Thus, using $(\mathbf{A}(\gamma,\nu))$ and that $\kappa > \nu$,

$$\Delta_3(V,v) \leq C_\kappa \mathbb{1}_{\{|V|\leq 1\}} e^{C_\kappa|v|^\kappa} \int_{-\pi/2}^{\pi/2} |\theta|^\kappa b(\theta)d\theta \leq C_\kappa \mathbb{1}_{\{|V|\leq 1\}} e^{C_\kappa|v|^\kappa}.$$

We have proved that

$$\Delta(V, v) \leq -c_\kappa e^{|V|^\kappa} \mathbb{1}_{\{|V|\geq 1, |V|\geq 130|v|\}} + C_\kappa \mathbb{1}_{\{|V|\geq 1\}} |V|^{\kappa+\nu-2} e^{|V|^\kappa} e^{C_\kappa |v|^\kappa} + C_\kappa \mathbb{1}_{\{|V|\leq 1\}} e^{C_\kappa |v|^\kappa},$$

which ends the proof of the first inequality.

The second inequality is much easier. Since $\kappa \in (0, 1)$, we have for all $x, y \geq 0$,

$$|e^{x^\kappa} - e^{y^\kappa}| \leq \kappa |x^\kappa - y^\kappa| e^{(x\vee y)^\kappa} \leq |x - y|^\kappa e^{(x\vee y)^\kappa}.$$

Thus, since $|A(\theta)| \leq |\theta| \leq \pi/2$,

$$\left| e^{|V+A(\theta)(V-v)|^\kappa} - e^{|V|^\kappa} \right| \leq |\theta|^\kappa (|V| + |v|)^\kappa e^{(|V|+2|\theta|(|V|+|v|))^\kappa} \leq C_\kappa |\theta|^\kappa e^{C_\kappa |V|^\kappa} e^{C_\kappa |v|^\kappa}.$$

Since $\int_{-\pi/2}^{\pi/2} |\theta|^\kappa b(\theta) d\theta < \infty$ by $(\mathbf{A}(\gamma, \nu))$, the second inequality holds true. □

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