# A Bernstein type inequality and moderate deviations for weakly dependent sequences 

Florence Merlevède • Magda Peligrad • Emmanuel Rio

Received: 29 January 2009 / Revised: 7 May 2010 / Published online: 9 June 2010
© Springer-Verlag 2010


#### Abstract

In this paper we present a Bernstein-type tail inequality for the maximum of partial sums of a weakly dependent sequence of random variables that is not necessarily bounded. The class considered includes geometrically and subgeometrically strongly mixing sequences. The result is then used to derive asymptotic moderate deviation results. Applications are given for classes of Markov chains, iterated Lipschitz models and functions of linear processes with absolutely regular innovations.


Keywords Deviation inequality • Moderate deviations principle • Semiexponential tails • Weakly dependent sequences • Strong mixing • Absolute regularity • Linear processes

Mathematics Subject Classification (2000) 60E15 • 60F10

[^0]
## 1 Introduction and background results

In recent years there has been a great effort towards a better understanding of the structure and asymptotic behavior of stochastic processes. For processes with short memory one basic technique is approximation with independent random variables. In this approach, after a suitable blocking argument, the blocks are approximated by independent random vectors. For many examples including functionals of Gaussian processes, Harris recurrent Markov chains, time series, and for results including moment inequalities or central limit-type theorems with rate estimates, this method is very fruitful. Moreover, recently, the traditional measures of dependence that quantify the departure from independence have been fine tuned to include more examples than those covered by traditional mixing classes (see [14]).

However, at this point, this method is not developed enough to handle Bernsteintype exponential inequalities. For instance, by the traditional blocking methods most of the exponential inequalities for tails of sums of weakly dependent random variables are known to hold only in an interval close to the central limit theorem range. Moreover the results are often restricted to bounded random variables, as for example in Adamczak [1] (see the inequality (1.10)).

In this paper we will develop new methods to enlarge the interval on which the Bernstein inequality holds for some classes of weakly dependent random variables. The dependence coefficients used are weak enough to include sequences that are not necessarily strongly mixing in the traditional sense, so the results have a large applicability. Examples that can be treated this way include classes of Markov chains, iterated Lipschitz models and functions of linear processes with absolutely regular innovations.

Concerning the traditional large deviations principle, it is known from the paper by Bryc and Dembo [9] that it is not satisfied by many classes of weakly dependent random variables. This is the reason why it is convenient to look at moderate deviations principles, which are intermediate results between central limit theorem and large deviations. We shall use the new developed Bernstein-type inequalities to obtain sharp moderate deviations asymptotic results for some classes of dependent random variables. The study will be made in the nonstationary setting.

We recall now some known results concerning the Bernstein-type inequalities. Let us consider a sequence $X_{1}, X_{2}, \ldots$ of centered real valued random variables defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, and set $S_{n}=X_{1}+X_{2}+\cdots+X_{n}$. We first recall the Bernstein inequality for random variables satisfying Condition (1.1) below. Suppose that the random variables $X_{1}, X_{2}, \ldots$ are independent and satisfy

$$
\begin{equation*}
\log \mathbb{E} \exp \left(t X_{i}\right) \leq \frac{\sigma_{i}^{2} t^{2}}{2(1-t B)} \text { for positive constants } \sigma_{i} \text { and } B \tag{1.1}
\end{equation*}
$$

for any $t$ in $\left[0,1 / B\left[\right.\right.$. Set $V_{n}=\sigma_{1}^{2}+\sigma_{2}^{2}+\cdots+\sigma_{n}^{2}$. Then

$$
\mathbb{P}\left(S_{n} \geq \sqrt{2 V_{n} x}+B x\right) \leq \exp (-x)
$$

When the random variables $X_{1}, X_{2}, \ldots$ are centered and uniformly bounded by $M$ then (1.1) holds with $\sigma_{i}^{2}=\operatorname{Var} X_{i}, B=M$ and the above inequality implies the usual Bernstein inequality

$$
\begin{equation*}
\mathbb{P}\left(S_{n} \geq y\right) \leq \exp \left(-y^{2}\left(2 V_{n}+2 y M\right)^{-1}\right) \tag{1.2}
\end{equation*}
$$

It is well known that (1.1) also holds true when the variables are centered and satisfy: there exist positive constants $\sigma_{i}$ and $B$ such that $\mathbb{E}\left|X_{i}\right|^{k} \leq k!\sigma_{i}^{2} B^{k-2} / 2$ for all $k \geq 2$, and that this last condition is satisfied by variables having exponential moments.

Assume now that the random variables $X_{1}, X_{2}, \ldots$ are independent, centered and satisfy the following weaker tail condition: for some $\delta>0, \gamma \in(0,1)$ and $K>0$,

$$
\begin{equation*}
\sup _{i} \mathbb{E}\left(\exp \left(\delta\left|X_{i}\right|^{\gamma}\right)\right) \leq K \tag{1.3}
\end{equation*}
$$

By the proof of Corollary 5.1 in Borovkov [6] we infer that there exist two positive constants $c_{1}$ and $c_{2}$ depending only $\delta, \gamma$ and $K$ such that

$$
\begin{equation*}
\mathbb{P}\left(S_{n} \geq y\right) \leq \exp \left(-c_{1} y^{2} / n\right)+n \exp \left(-c_{2} y^{\gamma}\right) \tag{1.4}
\end{equation*}
$$

More precise results for large deviations of sums of independent random variables with semiexponential tails (i.e. (1.3) is satisfied for $\gamma \in(0,1)$ ) may be found in Borovkov [7].

Our interest is to extend the above inequalities to sequences of dependent random variables. Let us first assume that $X_{1}, X_{2}, \ldots$ is a strongly mixing sequence of realvalued and centered random variables (see (2.4) for the definition of the strong mixing coefficients $\alpha(n)$ ). Assume in addition that there exist two positive constants $\gamma_{1}$ and $c$ such that the strong mixing coefficients of the sequence satisfy

$$
\begin{equation*}
\alpha(n) \leq \exp \left(-c n^{\gamma_{1}}\right) \text { for any positive integer } n \tag{1.5}
\end{equation*}
$$

and there are constants $b \in] 0, \infty\left[\right.$ and $\gamma_{2}$ in $\left.] 0,+\infty\right]$ such that

$$
\begin{equation*}
\sup _{i>0} \mathbb{P}\left(\left|X_{i}\right|>t\right) \leq \exp \left(1-(t / b)^{\gamma_{2}}\right) \text { for any positive } t \tag{1.6}
\end{equation*}
$$

(when $\gamma_{2}=+\infty$ (1.6) means that $\left\|X_{i}\right\|_{\infty} \leq b$ for any positive $i$ ).
Obtaining exponential bounds for this case is a challenging problem. To understand the difficulty of the problem we shall mention a possible approach. One of the available tools in the literature is Theorem 6.2 in Rio [25] which is a Fuk-Nagaev type inequality (for a similar inequality, using the coupling coefficients $\tau$ recalled in Sect. 2 instead of the strong mixing ones, we refer to Theorem 2 in Dedecker and Prieur [14]). This tail inequality is sharp for sequences of random variables with polynomial strongly mixing rates and finite moments up to a certain order, and one could be tempted to apply it to sequences with exponential or subexponential mixing rates. We shall argue that for these cases the inequality does not provide optimal results. To explain the
situation, let $\gamma$ be defined by $1 / \gamma=\left(1 / \gamma_{1}\right)+\left(1 / \gamma_{2}\right)$. For any positive $\lambda$ and any $r \geq 1$, Theorem 6.2 in Rio [25] yields that there exists a positive constant $C$ such that

$$
\begin{equation*}
\mathbb{P}\left(\sup _{1 \leq k \leq n}\left|S_{k}\right| \geq 4 \lambda\right) \leq 4\left(1+\frac{\lambda^{2}}{r n s^{2}}\right)^{-r / 2}+4 \operatorname{Cn} \lambda^{-1} \exp \left(-c \frac{\lambda^{\gamma}}{b^{\gamma} r^{\gamma}}\right) \tag{1.7}
\end{equation*}
$$

where

$$
s^{2}=\sup _{i>0}\left(\mathbb{E}\left(X_{i}^{2}\right)+2 \sum_{j>i}\left|\mathbb{E}\left(X_{i} X_{j}\right)\right|\right)
$$

Selecting in (1.7) $r=\lambda^{\gamma /(\gamma+1)}$ leads to

$$
\begin{align*}
\mathbb{P}\left(\sup _{k \in[1, n]}\left|S_{k}\right| \geq 4 \lambda\right) \leq & 4 \exp \left(-\frac{\lambda^{\gamma /(\gamma+1)} \log 2}{2}\right) \\
& +4 C n \lambda^{-1} \exp \left(-c \frac{\lambda^{\gamma /(\gamma+1)}}{b^{\gamma}}\right) \tag{1.8}
\end{align*}
$$

for any $\lambda \geq 1 \vee\left(n s^{2}\right)^{(\gamma+1) /(\gamma+2)}$.
For stationary subgeometrically (absolutely regular) Markov chains $\left(Y_{i}\right)_{i \in \mathbb{Z}}$ having a petite set, and for bounded functions $f$ (here $\gamma=\gamma_{1}$ ), this gives the following exponential deviation inequality for $S_{n}(f)=f\left(Y_{1}\right)+f\left(Y_{2}\right)+\cdots+f\left(Y_{n}\right)$. Under the centering condition $\mathbb{E}\left(f\left(Y_{1}\right)\right)=0$, there exist positive constants $K$ and $L$ such that for any $\varepsilon>0$ and $n>n_{0}(\varepsilon)$,

$$
\begin{equation*}
\mathbb{P}\left(\left|S_{n}(f)\right| \geq n \varepsilon\right) \leq K \exp \left(-L(n \varepsilon)^{\gamma /(1+\gamma)}\right) \tag{1.9}
\end{equation*}
$$

which is, for $\gamma \in(0,1]$, also the inequality given in Theorem 10 of Douc et al. [21] in case where their drift condition implies subgeometrical ergodicity (and then (1.5), see Sect. 2.1.1 for details on this drift condition). When $\gamma=1$, this yields a power $\sqrt{n}$ in the exponential whereas in this case, Theorem 6 in Adamczak [1] provides the following inequality: for any positive $\lambda$,

$$
\begin{equation*}
\mathbb{P}\left(\left|S_{n}(f)\right| \geq \lambda\right) \leq C \exp \left(-\frac{1}{C} \min \left(\frac{\lambda^{2}}{n \sigma^{2}}, \frac{\lambda}{\log n}\right)\right), \tag{1.10}
\end{equation*}
$$

where $\sigma^{2}=\lim _{n} n^{-1} \operatorname{Var} S_{n}(f)$ and $C$ is a positive constant (here we take $m=1$ in his condition (14) on the petite set). In addition, when $\gamma \in(0,1)$, the hope is to achieve the power $n^{\gamma}$ in the exponential term of (1.9) instead of $n^{\gamma /(1+\gamma)}$, since the gap is filled asymptotically via the moderate deviations results (see Theorem 1 in Djellout and Guillin [19] or Theorem 7 in Douc et al. [21]).

In this paper, we extend the inequality (1.4) to dependent sequences allowing us to fill the gap above mentioned. To be more precise, we shall prove that, for $\alpha$-mixing
sequences satisfying (1.5) and (1.6) for $\gamma<1$, there exists a positive $\eta$ such that, for $n \geq 4$ and $\lambda \geq C(\log n)^{\eta}$

$$
\begin{equation*}
\mathbb{P}\left(\sup _{j \leq n}\left|S_{j}\right| \geq \lambda\right) \leq(n+1) \exp \left(-\lambda^{\gamma} / C_{1}\right)+\exp \left(-\lambda^{2} /\left(C_{2}+C_{2} n V\right)\right), \tag{1.11}
\end{equation*}
$$

where $C, C_{1}$ and $C_{2}$ are positive constants depending on $b, c, \gamma_{1}$ and $\gamma_{2}$ and $V$ is some constant, depending on the covariance properties of truncated random variables built from the initial sequence. In order to define precisely $V$ we need to introduce truncation functions $\varphi_{M}$.

Notation 1 For any positive $M$ let the function $\varphi_{M}$ be defined by $\varphi_{M}(x)=(x \wedge M) \vee$ $(-M)$.

With this notation, (1.11) holds with

$$
\begin{equation*}
V=\sup _{M>0} \sup _{i>0}\left(\operatorname{Var}\left(\varphi_{M}\left(X_{i}\right)\right)+2 \sum_{j>i}\left|\operatorname{Cov}\left(\varphi_{M}\left(X_{i}\right), \varphi_{M}\left(X_{j}\right)\right)\right|\right) . \tag{1.12}
\end{equation*}
$$

Let us mention that in the case $\gamma_{1}=1$ and $\gamma_{2}=\infty$, a Bernstein type inequality very close to (1.10) (up to a logarithmic term) has been obtained by Merlevède et al. [24]. The case $\gamma>1$ will certainly give a different bound than (1.11) for the deviation inequality as in the case for independent random variables (see for instance Theorems 3.1 and 3.2 in Liu and Watbled [23]). It is outside the scope of the present paper to consider this case.

The main tool to prove (1.11) is our Proposition 2 allowing to derive a sharp upper bound for the Laplace transform of the partial sums. The proof of this proposition can be described as follows: the variables (suitably truncated) are partitioned in blocks indexed by Cantor-type sets plus a remainder. The log-Laplace transform of each partial sum on the Cantor-type sets is then controlled with the help of our Proposition 1, and Lemma 3 of Appendix provides bounds for the log-Laplace transform of any sum of real-valued random variables. The proof of Proposition 1 is based on decorrelation arguments on the Cantor-type sets we consider together with adaptive truncations (at each decorrelation step, the variables get truncated at different levels, as in Bass [4], to compensate for the diminishing block size). The assumption $\gamma<1$ is crucial for the decorrelation steps. Proposition 2 alone does not lead directly to the inequality (1.11) and it has to be combined with coupling to reduce the number of variables considered in the partial sums. The strong mixing coefficients are thus not needed in their full generality: Theorem 1 gives (1.11) when (1.5) (up to a constant) is fulfilled by the dependence coefficients $\tau$, as introduced by Dedecker and Prieur [14], having exactly the coupling property in $\mathbb{L}^{1}$. In Sect. 2, we also prove the moderate deviations principle for the class of dependent sequences that we consider (see our Theorem 2). Section 2.1 is devoted to some applications and further comparisons with previous results are given (see Sect. 2.1.1). The proofs of Theorems 1 and 2 are postponed in Sect. 3. Technical results are given in Appendix.

## 2 Main results

We first define the dependence coefficients that we consider in this paper.
For any real random variable $X$ in $\mathbb{L}^{1}$ and any $\sigma$-algebra $\mathcal{M}$ of $\mathcal{A}$, let $\mathbb{P}_{X \mid \mathcal{M}}$ be a conditional distribution of $X$ given $\mathcal{M}$ and let $\mathbb{P}_{X}$ be the distribution of $X$. We consider the coefficient $\tau(\mathcal{M}, X)$ of weak dependence [14] which is defined by

$$
\begin{equation*}
\tau(\mathcal{M}, X)=\left\|\sup _{f \in \Lambda_{1}(\mathbb{R})}\left|\int f(x) \mathbb{P}_{X \mid \mathcal{M}}(d x)-\int f(x) \mathbb{P}_{X}(d x)\right|\right\|_{1} \tag{2.1}
\end{equation*}
$$

where $\Lambda_{1}(\mathbb{R})$ is the set of 1-Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$.
The coefficient $\tau$ has the following coupling property: If $\Omega$ is rich enough then the coefficient $\tau(\mathcal{M}, X)$ is the infimum of $\left\|X-X^{*}\right\|_{1}$ where $X^{*}$ is independent of $\mathcal{M}$ and distributed as $X$ (see Lemma 5 in [14]). This coupling property allows to relate the coefficient $\tau$ to the strong mixing coefficient. Rosenblatt [26] defined by

$$
\alpha(\mathcal{M}, \sigma(X))=\sup _{A \in \mathcal{M}, B \in \sigma(X)}|\mathbb{P}(A \cap B)-\mathbb{P}(A) \mathbb{P}(B)|,
$$

(see Lemma 6 in [14]).
If $Y$ is a random variable with values in $\mathbb{R}^{k}$ equipped with the norm $|\cdot|_{k}$ defined by $|x-y|_{k}=\sum_{i=1}^{k}\left|x_{i}-y_{i}\right|$, the coupling coefficient $\tau$ is defined as follows: If $Y \in \mathbb{L}^{1}\left(\mathbb{R}^{k}\right)$,

$$
\begin{equation*}
\tau(\mathcal{M}, Y)=\sup \left\{\tau(\mathcal{M}, f(Y)), f \in \Lambda_{1}\left(\mathbb{R}^{k}\right)\right\}, \tag{2.2}
\end{equation*}
$$

where $\Lambda_{1}\left(\mathbb{R}^{k}\right)$ is the set of 1-Lipschitz functions from $\mathbb{R}^{k}$ to $\mathbb{R}$.
The $\tau$-mixing coefficients $\tau_{X}(i)=\tau(i)$ of a sequence $\left(X_{i}\right)_{i \in \mathbb{Z}}$ of real-valued random variables are then defined by

$$
\begin{align*}
\tau_{k}(i)= & \max _{1 \leq \ell \leq k} \frac{1}{\ell} \sup \left\{\tau\left(\mathcal{M}_{p},\left(X_{j_{1}}, \ldots, X_{j_{\ell}}\right)\right), p+i \leq j_{1}<\cdots<j_{\ell}\right\} \\
& \quad \text { and } \tau(i)=\sup _{k \geq 0} \tau_{k}(i) \tag{2.3}
\end{align*}
$$

where $\mathcal{M}_{p}=\sigma\left(X_{j}, j \leq p\right)$ and the above supremum is taken over $p$ and $\left(j_{1}, \ldots j_{\ell}\right)$. Recall that the strong mixing coefficients $\alpha(i)$ are defined by:

$$
\begin{equation*}
\alpha(i)=\sup _{p \in \mathbb{Z}} \alpha\left(\mathcal{M}_{p}, \sigma\left(X_{j}, j \geq i+p\right)\right) . \tag{2.4}
\end{equation*}
$$

Define now the function $Q_{|Y|}$ by $Q_{|Y|}(u)=\inf \{t>0, \mathbb{P}(|Y|>t) \leq u\}$ for $u$ in $\left.] 0,1\right]$. To compare the $\tau$-mixing coefficients with the strong mixing ones, let us mention that, by Lemma 7 in Dedecker and Prieur [14],

$$
\begin{equation*}
\tau(i) \leq 2 \int_{0}^{2 \alpha(i)} Q(u) d u, \text { where } Q=\sup _{k \in \mathbb{Z}} Q_{\left|X_{k}\right|} \tag{2.5}
\end{equation*}
$$

Let $\left(X_{j}\right)_{j \in \mathbb{Z}}$ be a sequence of centered real valued random variables and let $\tau(i)$ be defined by (2.3). Let $\tau(x)=\tau([x])$ (square brackets denoting the integer part). Throughout, we assume that there exist positive constants $\gamma_{1}, a$ and $c$ such that

$$
\begin{equation*}
\tau(x) \leq a \exp \left(-c x^{\gamma_{1}}\right):=\tau^{*}(x) \text { for any } x \geq 1, \tag{2.6}
\end{equation*}
$$

and that, for some constants $\gamma_{2}$ in $\left.] 0,+\infty\right]$ and $b$ in $] 0,+\infty[$, the following tail condition is satisfied: for any positive $t$,

$$
\begin{equation*}
\sup _{k>0} \mathbb{P}\left(\left|X_{k}\right|>t\right) \leq \exp \left(1-(t / b)^{\gamma_{2}}\right):=H(t) \tag{2.7}
\end{equation*}
$$

Suppose furthermore that

$$
\begin{equation*}
\gamma<1 \text { where } \gamma \text { is defined by } 1 / \gamma=1 / \gamma_{1}+1 / \gamma_{2} . \tag{2.8}
\end{equation*}
$$

Theorem 1 Let $\left(X_{j}\right)_{j \in \mathbb{Z}}$ be a sequence of centered real valued random variables and let $V$ be defined by (1.12). Assume that (2.6), (2.7) and (2.8) are satisfied. Then $V$ is finite and, for any $n \geq 4$, there exist positive constants $C_{1}, C_{2}, C_{3}$ and $C_{4}$ depending only on $a, b, c, \gamma$ and $\gamma_{1}$ such that, for any positive $x$,

$$
\begin{aligned}
\mathbb{P}\left(\sup _{j \leq n}\left|S_{j}\right| \geq x\right) \leq & n \exp \left(-\frac{x^{\gamma}}{C_{1}}\right)+\exp \left(-\frac{x^{2}}{C_{2}(1+n V)}\right) \\
& +\exp \left(-\frac{x^{2}}{C_{3} n} \exp \left(\frac{x^{\gamma(1-\gamma)}}{C_{4}(\log x)^{\gamma}}\right)\right)
\end{aligned}
$$

Remark 1 Let us mention that if the sequence $\left(X_{j}\right)_{j \in \mathbb{Z}}$ satisfies (2.7) and is strongly mixing with strong mixing coefficients satisfying (1.5), then, from (2.5), (2.6) is satisfied (with an other constant), and Theorem 1 applies.

Remark 2 If there exists $\delta>0$ such that $\left.\mathbb{E} \exp \left(\delta\left|X_{i}\right|^{\gamma_{2}}\right)\right) \leq K$ for any positive $i$, then setting $C=1 \vee \log K$, Markov inequality yields that the process $\left(X_{i}\right)_{i \in \mathbb{Z}}$ satisfies (2.7) with $b=(C / \delta)^{1 / \gamma_{2}}$.

Remark 3 If $\left(X_{i}\right)_{i \in \mathbb{Z}}$ satisfies (2.6) and (2.7), then

$$
\begin{aligned}
V & \leq \sup _{i>0}\left(\mathbb{E}\left(X_{i}^{2}\right)+4 \sum_{k>0} \int_{0}^{\tau(k) / 2} Q_{\left|X_{i}\right|}(G(v)) d v\right) \\
& =\sup _{i>0}\left(\mathbb{E}\left(X_{i}^{2}\right)+4 \sum_{k>0} \int_{0}^{G(\tau(k) / 2)} Q_{\left|X_{i}\right|}(u) Q(u) d u\right),
\end{aligned}
$$

where $G$ is the inverse function of $x \mapsto \int_{0}^{x} Q(u) d u$ (see Sect. 3.3 for a proof). Here the random variables do not need to be centered. Note also that, in the strong mixing case, using (2.5), we have $G(\tau(k) / 2) \leq 2 \alpha(k)$.

This result is one of the main tools to derive the moderate deviations principle (MDP) given in Theorem 2 below. In our terminology the MDP stays for the following type of asymptotic behavior:

Definition 1 We say that the MDP holds for a sequence $\left(T_{n}\right)_{n}$ of random variables with speed $a_{n} \rightarrow 0$ and good rate function $I(\cdot)$, if the level sets $\{x, I(x) \leq \lambda\}$ are compact for all $\lambda<\infty$, and for each Borel set $A$,

$$
\begin{align*}
-\inf _{t \in A^{o}} I(t) & \leq \liminf _{n} a_{n} \log \mathbb{P}\left(\sqrt{a_{n}} T_{n} \in A\right) \\
& \leq \limsup _{n} a_{n} \log \mathbb{P}\left(\sqrt{a_{n}} T_{n} \in A\right) \leq-\inf _{t \in \bar{A}} I(t), \tag{2.9}
\end{align*}
$$

where $\bar{A}$ denotes the closure of $A$ and $A^{o}$ the interior of $A$.
Theorem 2 Let $\left(X_{i}\right)_{i \in \mathbb{Z}}$ be a sequence of random variables as in Theorem 1 and let $S_{n}=\sum_{i=1}^{n} X_{i}$ and $\sigma_{n}^{2}=\operatorname{Var} S_{n}$. Assume in addition that $\lim \inf _{n \rightarrow \infty} \sigma_{n}^{2} / n>0$. Then for all positive sequences $a_{n}$ with $a_{n} \rightarrow 0$ and $a_{n} n^{\gamma /(2-\gamma)} \rightarrow \infty,\left\{\sigma_{n}^{-1} S_{n}\right\}$ satisfies (2.9) with rate function $I(t)=t^{2} / 2$.

If we impose a stronger degree of stationarity we obtain the following corollary.
Corollary 1 Let $\left(X_{i}\right)_{i \in \mathbb{Z}}$ be a second order stationary sequence of centered real valued random variables. Assume that (2.6), (2.7) and (2.8) are satisfied. Let $S_{n}=$ $\sum_{i=1}^{n} X_{i}$ and $\sigma_{n}^{2}=\operatorname{Var} S_{n}$. Assume in addition that $\sigma_{n}^{2} \rightarrow \infty$. Then $\lim _{n \rightarrow \infty} \sigma_{n}^{2} / n=$ $\sigma^{2}>0$, and for all positive sequences $a_{n}$ with $a_{n} \rightarrow 0$ and $a_{n} n^{\gamma /(2-\gamma)} \rightarrow \infty$, $\left\{n^{-1 / 2} S_{n}\right\}$ satisfies (2.9) with rate function $I(t)=t^{2} /\left(2 \sigma^{2}\right)$.

The proof is direct by using the fact that (2.6) and (2.7) imply that $\sum_{k>0} k \mid \operatorname{Cov}\left(X_{0}\right.$, $\left.X_{k}\right) \mid<\infty$, and then $\lim _{n \rightarrow \infty} \sigma_{n}^{2} / n=\sigma^{2}>0$ since $\sigma_{n}^{2} \rightarrow \infty$ (see Lemma 1 in [8]).

### 2.1 Applications

### 2.1.1 Instantaneous functions of absolutely regular processes

Let $\left(Y_{j}\right)_{j \in \mathbb{Z}}$ be a strictly stationary sequence of random variables with values in a Polish space $E$, and let $f$ be a measurable function from $E$ to $\mathbb{R}$. Set $X_{j}=f\left(Y_{j}\right)$. Consider now the case where the sequence $\left(Y_{k}\right)_{k \in \mathbb{Z}}$ is absolutely regular (or $\beta$-mixing) in the sense of Rozanov and Volkonskii [27]. Setting $\mathcal{F}_{0}=\sigma\left(Y_{i}, i \leq 0\right)$ and $\mathcal{G}_{k}=$ $\sigma\left(Y_{i}, i \geq k\right)$, this means that

$$
\beta(k)=\beta\left(\mathcal{F}_{0}, \mathcal{G}_{k}\right) \rightarrow 0, \text { as } k \rightarrow \infty,
$$

with $\beta(\mathcal{A}, \mathcal{B})=\frac{1}{2} \sup \left\{\sum_{i \in I} \sum_{j \in J}\left|\mathbb{P}\left(A_{i} \cap B_{j}\right)-\mathbb{P}\left(A_{i}\right) \mathbb{P}\left(B_{j}\right)\right|\right\}$, the maximum being taken over all finite partitions $\left(A_{i}\right)_{i \in I}$ and $\left(B_{i}\right)_{i \in J}$ of $\Omega$ respectively with elements in $\mathcal{A}$ and $\mathcal{B}$. If we assume that

$$
\begin{equation*}
\beta(n) \leq 2 \exp \left(-c n^{\gamma_{1}}\right) \quad \text { for any positive } n, \tag{2.10}
\end{equation*}
$$

where $c>0$ and $\gamma_{1}>0$, and that the random variables $X_{j}$ are centered and satisfy (2.7) for some positive $\gamma_{2}$ such that $1 / \gamma=1 / \gamma_{1}+1 / \gamma_{2}>1$, then Theorem 1 and Corollary 1 apply to the sequence $\left(X_{j}\right)_{j \in \mathbb{Z}}$ (recall that $\left.\alpha(n) \leq \beta(n) / 2\right)$. Furthermore, as shown in Viennet [29], by Delyon's [16] covariance inequality,

$$
V \leq \mathbb{E}\left(f^{2}\left(Y_{0}\right)\right)+4 \sum_{k>0} \mathbb{E}\left(B_{k} f^{2}\left(Y_{0}\right)\right),
$$

for some sequence $\left(B_{k}\right)_{k>0}$ of random variables with values in [0,1] satisfying $\mathbb{E}\left(B_{k}\right) \leq \beta(k)$ (see [25, Section 1.6]) for more details).

We now give an example where $\left(Y_{j}\right)_{j \in \mathbb{Z}}$ satisfies (2.10). Let $\left(Y_{j}\right)_{j \geq 0}$ be an $E$-valued irreducible ergodic and stationary Markov chain with a transition probability $P$ having a unique invariant probability measure $\pi$ (by Kolmogorov extension Theorem one can complete $\left(Y_{j}\right)_{j \geq 0}$ to a sequence $\left.\left(Y_{j}\right)_{j \in \mathbb{Z}}\right)$. To simplify the exposition, we assume in the rest of the section that the chain has an atom, that is there exists $A \subset E$ with $\pi(A)>0$ and $\nu$ a probability measure such that $P(x, \cdot)=v(\cdot)$ for any $x$ in $A$. If

$$
\begin{equation*}
\text { there exists } \delta>0 \quad \text { and } \quad \gamma_{1}>0 \text { such that } \mathbb{E}_{v}\left(\exp \left(\delta \tau^{\gamma_{1}}\right)\right)<\infty \tag{2.11}
\end{equation*}
$$

where $\tau=\inf \left\{n \geq 0 ; Y_{n} \in A\right\}$, then the $\beta$-mixing coefficients of the sequence $\left(Y_{j}\right)_{j \geq 0}$ satisfy (2.10) with the same $\gamma_{1}$ (see Proposition 9.6 and Corollary 9.1 in [25] for more details). Consequently the following corollary holds:

Corollary 2 Suppose that $\pi(f)=0$ and that there exist $b \in] 0, \infty\left[\right.$ and $\gamma_{2} \in[0, \infty]$ such that

$$
\begin{equation*}
\pi(|f|>t) \leq \exp \left(1-(t / b)^{\gamma_{2}}\right) \text { for any positive } t . \tag{2.12}
\end{equation*}
$$

If (2.11) holds and $1 / \gamma_{1}+1 / \gamma_{2}>1$, then $\left(f\left(Y_{i}\right), i \in \mathbb{Z}\right\}$ satisfies both the conclusions of Theorem 1 and of Corollary 1 with rate function $I_{f}(t)=t^{2} /\left(2 \sigma_{f}^{2}\right)$ (here $\left.\sigma_{f}^{2}=\lim _{n \rightarrow \infty} n^{-1} \mathbb{E}\left(\sum_{i=1}^{n} f\left(Y_{i}\right)\right)^{2}\right)$.
Concerning the MDP, notice that Lemmas 5 and 7 in Djellout and Guillin [19] imply that under (2.11) with $\gamma_{1}<1$, the MDP holds for $\left\{n^{-1 / 2} \sum_{i=1}^{n} f\left(Y_{i}\right)\right\}$ with rate function $I_{f}(t)=t^{2} /\left(2 \sigma_{f}^{2}\right)$ and speed $a_{n}$ satisfying $a_{n} \searrow 0$, and $a_{n} n^{\gamma_{1} /\left(2-\gamma_{1}\right)} \rightarrow \infty$ as soon as

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} a_{n} \log \left(n \mathbb{P}\left(\sum_{k=0}^{\tau}\left|f\left(Y_{k}\right)\right| \geq \sqrt{\frac{n}{a_{n}}}\right)\right)=-\infty \tag{2.13}
\end{equation*}
$$

Their result extends, in the context of chains with an atom, works by de Acosta [11] and Chen and de Acosta [10] for bounded functionals of geometrically ergodic Markov chains.

Condition (2.13) originates from the use of the regeneration method constructed via the splitting technique on return times to the atom and links the rate of ergodicity of the chain with the growth of $f$. If the function $f$ is bounded then (2.11) implies (2.13). When $f$ is an unbounded functional, conditions ensuring that (2.13) holds are given in Douc et al. [21]. To describe them, introduce the following "subgeometric drift" condition due to Douc et al. [20]:
Assumption SGD. There exist a concave, non decreasing, differentiable function $\varphi:[1,+\infty) \rightarrow \mathbb{R}^{+}$, a measurable function $V: E \rightarrow[1, \infty)$ and a positive constant $r$ satisfying $\varphi(1)>0, \lim _{x \rightarrow \infty} \varphi(x)=\infty, \lim _{x \rightarrow \infty} \varphi^{\prime}(x)=\infty, \sup _{x \in A} V(x)<\infty$ and $P V \leq V-\varphi \circ V+r \mathbb{1}_{A}$.

By Proposition 2.2 in Douc et al. [20] (and their computations on page 1365), if Assumption SGD holds with $\varphi(x)=c(x+d)(\log (x+d))^{\left(1-\gamma_{1}\right) / \gamma_{1}}$ for $c>0$, $\gamma_{1} \in(0,1)$ and sufficiently large $d$, then (2.11) is satisfied. In addition if $\pi(V)<\infty$, according to Theorems 4 and 9 in Douc et al. [21], (2.13) is satisfied for all functions $f$ such that

$$
\begin{align*}
& \sup _{x \in E}|f(x)| / \psi \circ V(x)<\infty, \quad \text { where } \psi(x)=(\log (1+x))^{1 / \gamma_{2}} \\
& \quad \text { with } \gamma_{2} \text { a positive constant, } \tag{2.14}
\end{align*}
$$

as soon as $a_{n} \rightarrow 0$ and $a_{n} n^{\gamma /(2-\gamma)} \rightarrow \infty$, where $\gamma^{-1}=\gamma_{1}^{-1}+\gamma_{2}^{-1}$. Since $\pi(V)<\infty$, notice that (2.14) implies (2.12). It follows that for the MDP, our Corollary 2 gives alternative conditions to the ones imposed in Theorem 9 of Douc et al. [21]. In addition as in their paper, Corollary 2 can be extended to the one petite set case.

Let us now make some comments about the exponential deviation inequality obtained in Corollary 2. Assume that Assumption SGD holds with $V$ such that $\pi(V)<$ $\infty$ and $\varphi(x)=c(x+d)(\log (x+d))^{\left(1-\gamma_{1}\right) / \gamma_{1}}$ for $c>0, \gamma_{1} \in(0,1)$ and sufficiently large $d$. If $f$ is a bounded and centered function w.r.t. $\pi$, then Theorem 10 in Douc et al. [21] states that (1.9) holds with $\gamma=\gamma_{1}$, whereas Corollary 2 gives a better rate (the power is $\gamma_{1}$ instead of $\left.\gamma_{1} /\left(1+\gamma_{1}\right)\right)$. If we relax now the assumption that $f$ is bounded by assuming that (2.14) holds then Theorem 1 in Bertail and Clémençon [5] combined
with Theorem 4 in Douc et al. [21] implies (1.9) with $\gamma$ such that $\gamma^{-1}=\gamma_{1}^{-1}+\gamma_{2}^{-1}$ whereas Corollary 2 again gives a better rate.

### 2.1.2 Iterative Lipschitz models

In this section, we give an example of iterative Lipschitz model, which fails to be irreducible, to which our results apply. For the sake of simplicity, we do not take the iterative Lipschitz models in their full generality, as defined in Diaconis and Freedman [18] and Duflo [22].

Example: Autoregressive Lipschitz model. For $\delta$ in [0, $1[$ and $C$ in $] 0,1]$, let $\mathcal{L}(C, \delta)$ be the class of 1-Lipschitz functions $f$ which satisfy

$$
f(0)=0 \quad \text { and } \quad\left|f^{\prime}(t)\right| \leq 1-C(1+|t|)^{-\delta} \text { almost everywhere. }
$$

Let $\left(\varepsilon_{i}\right)_{i \in \mathbb{Z}}$ be a sequence of i.i.d. real-valued random variables. For $\left.\left.\eta \in\right] 0,1\right]$, let $A R L(C, \delta, \eta)$ be the class of Markov chains on $\mathbb{R}$ defined by

$$
\begin{equation*}
Y_{n}=f\left(Y_{n-1}\right)+\varepsilon_{n} \text { with } f \in \mathcal{L}(C, \delta) \text { and } \mathbb{E}\left(\exp \left(\lambda\left|\varepsilon_{0}\right|^{\eta}\right)\right)<\infty \text { for some } \lambda>0 \tag{2.15}
\end{equation*}
$$

For this model, there exists an unique invariant probability measure $\mu$ (see Proposition 2 of [15]) and the following result holds (see Sect. 4.2 for the proof):

Corollary 3 Assume that $\left(Y_{i}\right)_{i \in \mathbb{Z}}$ belongs to $\operatorname{ARL}(C, \delta, \eta)$. Let $g$ be a 1-Lipschitz function. Assume furthermore that, for some $\zeta$ in $[0,1]$ and some positive constant $c,|g(x)| \leq c\left(1+|x|^{\zeta}\right)$ for any real $x$. If $\delta+\zeta>0$, then $\left(g\left(Y_{i}\right)-\mathbb{E}\left(g\left(Y_{i}\right)\right)\right)_{i \in \mathbb{Z}}$ satisfies both the conclusions of Theorem 1 and of Corollary 1 with $\gamma_{2}=\eta(1-\delta) / \zeta$ and $\gamma_{1}=\eta(1-\delta)(\eta(1-\delta)+\delta)^{-1}$.

Note that $\gamma=\eta(1-\delta)(\eta(1-\delta)+\delta+\zeta)^{-1}$. An element of $\operatorname{ARL}(C, \delta, \eta)$ may fail to be irreducible and then strongly mixing in the general case. However, if the distribution of $\varepsilon_{0}$ has an absolutely continuous component which is bounded away from 0 in a neighborhood of the origin, then the chain is irreducible and fits in the example of Tuominen and Tweedie [28], Section 5.2. In this case, the rate of ergodicity can be derived from Theorem 2.1 in Tuominen and Tweedie [28] (cf. [2] for exact rates of ergodicity).

### 2.1.3 Functions of linear processes with absolutely regular innovations

Let $f$ be a 1-Lipschitz function. We consider here the case where

$$
X_{n}=f\left(\sum_{j \geq 0} a_{j} Y_{n-j}\right)-\mathbb{E} f\left(\sum_{j \geq 0} a_{j} Y_{n-j}\right)
$$

where $A=\sum_{j \geq 0}\left|a_{j}\right|<\infty$ and $\left(Y_{i}\right)_{i \in \mathbb{Z}}$ is a strictly stationary sequence of real-valued random variables which is absolutely regular in the sense of Rozanov and Volkonskii; namely, $\beta(k) \rightarrow 0$, as $k \rightarrow \infty$ (see Sect. 2.1.1 for the definitions).

According to Section 3.1 in Dedecker and Merlevède [13], if the innovations $\left(Y_{i}\right)_{i \in \mathbb{Z}}$ are in $\mathbb{L}^{2}$, the following bound holds for the $\tau$-mixing coefficient associated to the sequence $\left(X_{i}\right)_{i \in \mathbb{Z}}$ :

$$
\tau(i) \leq 2\left\|Y_{0}\right\|_{1} \sum_{j \geq i}\left|a_{j}\right|+4\left\|Y_{0}\right\|_{2} \sum_{j=0}^{i-1}\left|a_{j}\right| \beta^{1 / 2}(i-j)
$$

Assume that there exist $\gamma_{1}>0$ and $c^{\prime}>0$ such that, for any positive integer $k$,

$$
a_{k} \leq \exp \left(-c^{\prime} k^{\gamma_{1}}\right) \quad \text { and } \quad \beta(k) \leq \exp \left(-c^{\prime} k^{\gamma_{1}}\right)
$$

Then the $\tau$-mixing coefficients of $\left(X_{j}\right)_{j \in \mathbb{Z}}$ satisfy (2.6). Let us now focus on the tails of the random variables $X_{i}$. Assume that $\left(Y_{i}\right)_{i \in \mathbb{Z}}$ satisfies (2.7). Define the convex functions $\psi_{\eta}$ for $\eta>0$ in the following way: $\psi_{\eta}(-x)=\psi_{\eta}(x)$, and for any $x \geq 0$,

$$
\left.\left.\psi_{\eta}(x)=\exp \left(x^{\eta}\right)-1 \quad \text { for } \eta \geq 1 \quad \text { and } \quad \psi_{\eta}(x)=\int_{0}^{x} \exp \left(u^{\eta}\right) d u \text { for } \eta \in\right] 0,1\right]
$$

Let $\|\cdot\|_{\psi_{\eta}}$ be the usual corresponding Orlicz norm. Since the function $f$ is 1-Lipshitz, we get that $\left\|X_{0}\right\|_{\psi_{\gamma_{2}}} \leq 2 A\left\|Y_{0}\right\|_{\psi_{\gamma_{2}}}$. Next, if $\left(Y_{i}\right)_{i \in \mathbb{Z}}$ satisfies (2.7), then $\left\|Y_{0}\right\|_{\psi_{\gamma_{2}}}<\infty$. Moreover, it can easily be proven that, if $\|Z\|_{\psi_{\eta}} \leq 1$, then $\mathbb{P}(|Z|>t) \leq \exp \left(1-t^{\eta}\right)$ for any positive $t$. Hence, $\left(X_{i}\right)_{i \in \mathbb{Z}}$ satisfies (2.7) with the same parameter $\gamma_{2}$, and therefore the conclusions of Theorem 1 and Corollary 1 hold with $\gamma$ defined by $1 / \gamma=$ $1 / \gamma_{1}+1 / \gamma_{2}$, provided that $\gamma<1$.

## 3 Proofs

### 3.1 Some auxiliary results

For any positive real $M$, let $\varphi_{M}(x)=(x \wedge M) \vee(-M)$. The aim of this section is essentially to give suitable bounds of the Laplace transform of the truncated sums

$$
\begin{equation*}
\bar{S}_{M}(K)=\sum_{i \in K} \bar{X}_{M}(i) \quad \text { where } \bar{X}_{M}(i)=\varphi_{M}\left(X_{i}\right)-\mathbb{E}\left(\varphi_{M}\left(X_{i}\right)\right), \tag{3.1}
\end{equation*}
$$

and $K$ is a finite set of integers.
We first define some constants, depending only on $b, \gamma, \gamma_{1}$ that are needed in the following. Let

$$
\begin{align*}
& c_{0}=\left(2\left(2^{1 / \gamma}-1\right)\right)^{-1}\left(2^{(1-\gamma) / \gamma}-1\right), \quad c_{1}=\min \left(c^{1 / \gamma_{1}} c_{0} / 4,2^{-1 / \gamma}\right),  \tag{3.2}\\
& c_{2}=2^{-\left(1+2 \gamma_{1} / \gamma\right)} c_{1}^{\gamma_{1}} b^{-1}, \quad c_{3}=2^{-\gamma_{1} / \gamma} b^{-1}, \quad \text { and } \kappa=\min \left(c_{2}, c_{3}\right) . \tag{3.3}
\end{align*}
$$

The following proposition, based on decorrelation arguments on Cantor-type sets, is the key tool to derive Proposition 2, that provides upper bounds for the log-Laplace transform of the partial sums of the random variables $\bar{X}_{M}(i)$ for a suitable $M$.

Proposition 1 Let $\left(X_{j}\right)_{j \geq 1}$ be a sequence of centered and real valued random variables satisfying (2.6), (2.7) and (2.8). Let $A$ and $\ell$ be two positive integers such that $A 2^{-\ell} \geq\left(1 \vee 2 c_{0}^{-1}\right)$. Let $M=H^{-1}\left(a^{-1} \tau^{*}\left(c^{-1 / \gamma_{1}} A\right)\right)$. Then there exists $a$ subset $K_{A}^{(\ell)}$ of $\{1, \ldots, A\}$ with $\operatorname{Card}\left(K_{A}^{(\ell)}\right) \geq A / 2$, such that for any positive $t \leq$ $\kappa\left(A^{\gamma-1} \wedge\left(2^{\ell} / A\right)\right)^{\gamma_{1} / \gamma}$, where $\kappa$ is defined by (3.3),

$$
\begin{align*}
& \log \mathbb{E} \exp \left(t \bar{S}_{M}\left(K_{A}^{(\ell)}\right)\right) \leq t^{2} v^{2} A \\
& \quad+t^{2}\left(a b \ell(2 A)^{1+\frac{\gamma_{1}}{\gamma}}+4 b^{2} A^{\gamma}(2 A)^{\frac{2 \gamma_{1}}{\gamma}}\right) \exp \left(-\frac{1}{2}\left(\frac{c_{1} A}{2^{\ell}}\right)^{\gamma_{1}}\right) \tag{3.4}
\end{align*}
$$

with $\bar{S}_{M}\left(K_{A}^{(\ell)}\right)$ defined by (3.1) and

$$
\begin{equation*}
v^{2}=\sup _{T>0} \sup _{K \subset \mathbb{N}^{*}} \frac{1}{\operatorname{Card} K} \operatorname{Var} \sum_{i \in K} \varphi_{T}\left(X_{i}\right) \tag{3.5}
\end{equation*}
$$

(the maximum being taken over all nonempty finite sets $K$ of integers).
Remark 4 Notice that $v^{2} \leq V$ (the proof is immediate).
Proof of Proposition 1 The proof is divided in several steps.
Step 1. The construction of $K_{A}^{(\ell)}$. Let $c_{0}$ be defined by (3.2) and $n_{0}=A . K_{A}^{(\ell)}$ will be a finite union of $2^{\ell}$ disjoint sets of consecutive integers with same cardinal spaced according to a recursive "Cantor"-like construction. We first define an integer $d_{0}$ as follows:

$$
d_{0}= \begin{cases}\sup \left\{m \in 2 \mathbb{N}, m \leq c_{0} n_{0}\right\} & \text { if } n_{0} \text { is even } \\ \sup \left\{m \in 2 \mathbb{N}+1, m \leq c_{0} n_{0}\right\} & \text { if } n_{0} \text { is odd }\end{cases}
$$

It follows that $n_{0}-d_{0}$ is even. Let $n_{1}=\left(n_{0}-d_{0}\right) / 2$, and define two sets of integers of cardinal $n_{1}$ separated by a gap of $d_{0}$ integers as follows

$$
I_{1,1}=\left\{1, \ldots, n_{1}\right\}, \quad I_{1,2}=\left\{n_{1}+d_{0}+1, \ldots, n_{0}\right\}
$$

We define now the integer $d_{1}$ by

$$
d_{1}= \begin{cases}\sup \left\{m \in 2 \mathbb{N}, m \leq c_{0} 2^{-\left(\ell \wedge \frac{1}{\gamma}\right)} n_{0}\right\} & \text { if } n_{1} \text { is even } \\ \sup \left\{m \in 2 \mathbb{N}+1, m \leq c_{0} 2^{-\left(\ell \wedge \frac{1}{\gamma}\right)} n_{0}\right\} & \text { if } n_{1} \text { is odd. }\end{cases}
$$

Noticing that $n_{1}-d_{1}$ is even, we set $n_{2}=\left(n_{1}-d_{1}\right) / 2$, and define four sets of integers of cardinal $n_{2}$ by
$I_{2,1}=\left\{1, \ldots, n_{2}\right\}, I_{2,2}=\left\{n_{2}+d_{1}+1, \ldots, n_{1}\right\}, I_{2, i+2}=\left(n_{1}+d_{0}\right)+I_{2, i}$ for $i=1,2$.

Iterating this procedure $j$ times (for $1 \leq j \leq \ell$ ), we then get a finite union of $2^{j}$ sets, $\left(I_{j, k}\right)_{1 \leq k \leq 2^{j}}$, of consecutive integers, with same cardinal, constructed by induction from $\left(I_{j-1, k}\right)_{1 \leq k \leq 2^{j-1}}$ as follows: First, for $1 \leq k \leq 2^{j-1}$, we have $I_{j-1, k}=$ $\left\{a_{j-1, k}, \ldots, b_{j-1, k}\right\}$, where $1+b_{j-1, k}-a_{j-1, k}=n_{j-1}$ and

$$
1=a_{j-1,1}<b_{j-1,1}<a_{j-1,2}<b_{j-1,2}<\cdots<a_{j-1,2^{j-1}}<b_{j-1,2^{j-1}}=n_{0}
$$

Let $n_{j}=2^{-1}\left(n_{j-1}-d_{j-1}\right)$ and

$$
d_{j}= \begin{cases}\sup \left\{m \in 2 \mathbb{N}, m \leq c_{0} 2^{-\left(\ell \wedge \frac{j}{\gamma}\right)} n_{0}\right\} & \text { if } n_{j} \text { is even } \\ \sup \left\{m \in 2 \mathbb{N}+1, m \leq c_{0} 2^{-\left(\ell \wedge \frac{j}{\gamma}\right)} n_{0}\right\} & \text { if } n_{j} \text { is odd. }\end{cases}
$$

Then $I_{j, k}=\left\{a_{j, k}, a_{j, k}+1, \ldots, b_{j, k}\right\}$, where the double indexed sequences $\left(a_{j, k}\right)$ and ( $b_{j, k}$ ) are defined as follows:

$$
\begin{gathered}
a_{j, 2 k-1}=a_{j-1, k}, b_{j, 2 k}=b_{j-1, k}, b_{j, 2 k}-a_{j, 2 k}+1=n_{j} \\
\text { and } \quad b_{j, 2 k-1}-a_{j, 2 k-1}+1=n_{j} .
\end{gathered}
$$

With this selection, we then get that there is exactly $d_{j-1}$ integers between $I_{j, 2 k-1}$ and $I_{j, 2 k}$ for any $1 \leq k \leq 2^{j-1}$.

Finally we get

$$
K_{A}^{(\ell)}=\bigcup_{k=1}^{2^{\ell}} I_{\ell, k}
$$

Since $\operatorname{Card}\left(I_{\ell, k}\right)=n_{\ell}$, for any $1 \leq k \leq 2^{\ell}$, we get that $\operatorname{Card}\left(K_{A}^{(\ell)}\right)=2^{\ell} n_{\ell}$. Now notice that

$$
A-\operatorname{Card}\left(K_{A}^{(\ell)}\right)=\sum_{j=0}^{\ell-1} 2^{j} d_{j} \leq A c_{0}\left(\sum_{j \geq 0} 2^{j(1-1 / \gamma)}+\sum_{j \geq 1} 2^{-j}\right) \leq A / 2
$$

Consequently

$$
A \geq \operatorname{Card}\left(K_{A}^{(\ell)}\right) \geq A / 2 \text { and } n_{\ell} \leq A 2^{-\ell}
$$

The following notation will be useful for the rest of the proof: For any $k$ in $\{0,1, \ldots, \ell\}$ and any $j$ in $\left\{1, \ldots, 2^{\ell}\right\}$, we set

$$
\begin{equation*}
K_{A, k, j}^{(\ell)}=\bigcup_{i=(j-1) 2^{\ell-k}+1}^{j 2^{\ell-k}} I_{\ell, i} . \tag{3.6}
\end{equation*}
$$

Notice that $K_{A}^{(\ell)}=K_{A, 0,1}^{(\ell)}$ and that for any $k$ in $\{0,1, \ldots, \ell\}$

$$
\begin{equation*}
K_{A}^{(\ell)}=\bigcup_{j=1}^{2^{k}} K_{A, k, j}^{(\ell)} \tag{3.7}
\end{equation*}
$$

where the union is disjoint.
In what follows we shall also use the following notation: for any integer $j$ in $[0, \ell]$, we set

$$
\begin{equation*}
M_{j}=H^{-1}\left(a^{-1} \tau^{*}\left(c^{-1 / \gamma_{1}} A 2^{-\left(\ell \wedge \frac{j}{\gamma}\right)}\right)\right) \tag{3.8}
\end{equation*}
$$

Since $H^{-1}(y)=b(\log (e / y))^{1 / \gamma_{2}}$ for any $y \leq e$, we get that for any $x \geq 1$,

$$
\begin{equation*}
H^{-1}\left(a^{-1} \tau^{*}\left(c^{-1 / \gamma_{1}} x\right)\right)=b\left(1+x^{\gamma_{1}}\right)^{1 / \gamma_{2}} \leq b(2 x)^{\gamma_{1} / \gamma_{2}} . \tag{3.9}
\end{equation*}
$$

Consequently since for any $j$ in $[0, \ell], A 2^{-\left(\ell \wedge \frac{j}{\gamma}\right)} \geq 1$, the following bound is valid:

$$
\begin{equation*}
M_{j} \leq b\left(2 A 2^{-\left(\ell \wedge \frac{j}{\gamma}\right)}\right)^{\gamma_{1} / \gamma_{2}} \tag{3.10}
\end{equation*}
$$

Step 2. Proof of Inequality (3.4) with $K_{A}^{(\ell)}$ defined in step 1.
Consider the decomposition (3.7), and notice that for any $i=1,2, \operatorname{Card}\left(K_{A, 1, i}^{(\ell)}\right) \leq$ A/2 and

$$
\tau\left(\sigma\left(X_{i}: i \in K_{A, 1,1}^{(\ell)}\right), \bar{S}_{M_{0}}\left(K_{A, 1,2}^{(\ell)}\right) \leq A \tau\left(d_{0}\right) / 2\right.
$$

Since $\bar{X}_{M_{0}}(j) \leq 2 M_{0}$, we get that $\left|\bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)\right| \leq A M_{0}$. Consequently, by using Lemma 2 from Appendix, we derive that for any positive $t$,

$$
\left|\mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right)-\prod_{i=1}^{2} \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)\right)\right| \leq \frac{A t}{2} \tau\left(d_{0}\right) \exp \left(2 t A M_{0}\right)
$$

Since the random variables $\bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)$ and $\bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)$ are centered, their Laplace transform are greater than one. Hence applying the elementary inequality

$$
\begin{equation*}
|\log x-\log y| \leq|x-y| \quad \text { for } x \geq 1 \text { and } y \geq 1, \tag{3.11}
\end{equation*}
$$

we get that, for any positive $t$,
$\left|\log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right)-\sum_{i=1}^{2} \log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)\right)\right| \leq \frac{A t}{2} \tau\left(d_{0}\right) \exp \left(2 t A M_{0}\right)$.

The next step is to compare $\mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)\right)$ with $\mathbb{E} \exp \left(t \bar{S}_{M_{1}}\left(K_{A, 1, i}^{(\ell)}\right)\right)$ for $i=1,2$. The random variables $\bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)$ and $\bar{S}_{M_{1}}\left(K_{A, 1, i}^{(\ell)}\right)$ have values in $\left[-A M_{0}\right.$, $A M_{0}$ ], hence applying the inequality

$$
\begin{equation*}
\left|e^{t x}-e^{t y}\right| \leq|t||x-y|\left(e^{|t x|} \vee e^{|t y|}\right) \tag{3.12}
\end{equation*}
$$

we obtain that, for any positive $t$,

$$
\begin{aligned}
& \left|\mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)\right)-\mathbb{E} \exp \left(t \bar{S}_{M_{1}}\left(K_{A, 1, i}^{(\ell)}\right)\right)\right| \\
& \quad \leq t e^{t A M_{0}} \mathbb{E}\left|\bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)-\bar{S}_{M_{1}}\left(K_{A, 1, i}^{(\ell)}\right)\right| .
\end{aligned}
$$

Notice that

$$
\mathbb{E}\left|\bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)-\bar{S}_{M_{1}}\left(K_{A, 1, i}^{(\ell)}\right)\right| \leq 2 \sum_{j \in K_{A, 1, i}^{(\ell)}} \mathbb{E}\left|\left(\varphi_{M_{0}}-\varphi_{M_{1}}\right)\left(X_{j}\right)\right| .
$$

Since for all $x \in \mathbb{R},\left|\left(\varphi_{M_{0}}-\varphi_{M_{1}}\right)(x)\right| \leq M_{0} \mathbb{I}_{|x|>M_{1}}$, we get that

$$
\mathbb{E}\left|\left(\varphi_{M_{0}}-\varphi_{M_{1}}\right)\left(X_{j}\right)\right| \leq M_{0} \mathbb{P}\left(\left|X_{j}\right|>M_{1}\right) \leq a^{-1} M_{0} \tau^{*}\left(c^{-\frac{1}{\gamma_{1}}} A 2^{-\left(\ell \wedge \frac{1}{\gamma}\right)}\right)
$$

Consequently, since $\operatorname{Card}\left(K_{A, 1, i}^{(\ell)}\right) \leq A / 2$, for any $i=1,2$ and any positive $t$,

$$
\begin{aligned}
\left|\mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)\right)-\mathbb{E} \exp \left(t \bar{S}_{M_{1}}\left(K_{A, 1, i}^{(\ell)}\right)\right)\right| \leq & t A a^{-1} M_{0} e^{t A M_{0}} \tau^{*} \\
& \times\left(c^{-\frac{1}{\gamma_{1}}} A 2^{-\left(\ell \wedge \frac{1}{\gamma}\right)}\right) .
\end{aligned}
$$

Using again the fact that the variables are centered and taking into account the inequality (3.11), we derive that for any $i=1,2$ and any positive $t$,

$$
\begin{align*}
& \left|\log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A, 1, i}^{(\ell)}\right)\right)-\log \mathbb{E} \exp \left(t \bar{S}_{M_{1}}\left(K_{A, 1, i}^{(\ell)}\right)\right)\right| \\
& \quad \leq a^{-1} e^{2 t A M_{0}} \tau^{*}\left(c^{-\frac{1}{\gamma_{1}}} A 2^{-\left(\ell \wedge \frac{1}{\gamma}\right)}\right) . \tag{3.13}
\end{align*}
$$

Now for any $k=1, \ldots, \ell$ and any $i=1, \ldots, 2^{k}, \operatorname{Card}\left(K_{A, k, i}^{(\ell)}\right) \leq 2^{-k} A$. By iterating the above procedure, we then get for any $k=1, \ldots, \ell$, and any positive $t$,

$$
\begin{aligned}
& \left|\sum_{i=1}^{2^{k-1}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{k-1}}\left(K_{A, k-1, i}^{(\ell)}\right)\right)-\sum_{i=1}^{2^{k}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{k-1}}\left(K_{A, k, i}^{(\ell)}\right)\right)\right| \\
& \quad \leq 2^{k-1} \frac{t A}{2^{k}} \tau\left(d_{k-1}\right) \exp \left(\frac{2 t A M_{k-1}}{2^{k-1}}\right)
\end{aligned}
$$

and for any $i=1, \ldots, 2^{k}$,

$$
\begin{aligned}
& \left|\log \mathbb{E} \exp \left(t \bar{S}_{M_{k-1}}\left(K_{A, k, i}^{(\ell)}\right)\right)-\log \mathbb{E} \exp \left(t \bar{S}_{M_{k}}\left(K_{A, k, i}^{(\ell)}\right)\right)\right| \\
& \quad \leq a^{-1} \tau^{*}\left(c^{-\frac{1}{\gamma_{1}}} A 2^{-\left(\ell \wedge \frac{k}{\gamma}\right)}\right) \exp \left(\frac{2 t A M_{k-1}}{2^{k-1}}\right) .
\end{aligned}
$$

Hence finally, we get that for any $j=1, \ldots, \ell$, and any positive $t$,

$$
\begin{aligned}
& \left|\sum_{i=1}^{2^{j-1}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{j-1}}\left(K_{A, j-1, i}^{(\ell)}\right)\right)-\sum_{i=1}^{2^{j}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{j}}\left(K_{A, j, i}^{(\ell)}\right)\right)\right| \\
& \quad \leq \frac{t A}{2} \tau\left(d_{j-1}\right) \exp \left(2 t A M_{j-1} 2^{1-j}\right) \\
& \quad+2^{j} a^{-1} \tau^{*}\left(c^{-\frac{1}{\gamma_{1}}} A 2^{-\left(\ell \wedge \frac{j}{\gamma}\right)}\right) \exp \left(2 t A M_{j-1} 2^{1-j}\right)
\end{aligned}
$$

Set $k_{\ell}=\sup \{j \in \mathbb{N}, j / \gamma<\ell\}$, and notice that $0 \leq k_{\ell} \leq \ell-1$. Since $K_{A}^{(\ell)}=K_{A, 0,1}^{(\ell)}$, we then derive that for any positive $t$,

$$
\begin{align*}
& \left|\log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right)-\sum_{i=1}^{2^{k_{\ell}+1}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{k_{\ell}+1}}\left(K_{A, k_{\ell}+1, i}^{(\ell)}\right)\right)\right| \\
& \leq \frac{t A}{2} \sum_{j=0}^{k_{\ell}} \tau\left(d_{j}\right) \exp \left(\frac{2 t A M_{j}}{2^{j}}\right) \\
& \quad+2 a^{-1} \sum_{j=0}^{k_{\ell}-1} 2^{j} \tau^{*}\left(2^{-1 / \gamma} c^{-1 / \gamma_{1}} A 2^{-j / \gamma}\right) \exp \left(\frac{2 t A M_{j}}{2^{j}}\right) \\
& \quad+2^{k_{\ell}+1} a^{-1} \tau^{*}\left(c^{-1 / \gamma_{1}} A 2^{-\ell}\right) \exp \left(2 t A M_{k_{\ell}} 2^{-k_{\ell}}\right) . \tag{3.14}
\end{align*}
$$

Notice now that for any $i=1, \ldots, 2^{k_{\ell}+1}, S_{M_{k_{\ell}+1}}\left(K_{A, k_{\ell}+1, i}^{(\ell)}\right)$ is a sum of $2^{\ell-k_{\ell}-1}$ blocks, each of size $n_{\ell}$ and bounded by $2 M_{k_{\ell}+1} n_{\ell}$. In addition the blocks are equidistant and there is a gap of size $d_{k_{\ell}+1}$ between two blocks. Consequently, by using Lemma 2 along with inequality (3.11) and the fact that the variables are centered, we get that

$$
\begin{align*}
& \left|\log \mathbb{E} \exp \left(t \bar{S}_{M_{k_{\ell}+1}}\left(K_{A, k_{\ell}+1, i}^{(\ell)}\right)\right)-\sum_{j=(i-1) 2^{\ell-k_{\ell}-1}+1}^{i 2^{\ell-k_{\ell}-1}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{k_{\ell}+1}}\left(I_{\ell, j}\right)\right)\right| \\
& \leq t n_{\ell} 2^{\ell} 2^{-k_{\ell}-1} \tau\left(d_{k_{\ell}+1}\right) \exp \left(2 t M_{k_{\ell}+1} n_{\ell} 2^{\ell-k_{\ell}-1}\right) . \tag{3.15}
\end{align*}
$$

Starting from (3.14) and using (3.15) together with the fact that $n_{\ell} \leq A 2^{-\ell}$, we obtain:

$$
\begin{align*}
& \left|\log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right)-\sum_{j=1}^{2^{\ell}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{k_{\ell}+1}}\left(I_{\ell, j}\right)\right)\right| \\
& \leq \frac{t A}{2} \sum_{j=0}^{k_{\ell}} \tau\left(d_{j}\right) \exp \left(\frac{2 t A M_{j}}{2^{j}}\right) \\
& \quad+2 a^{-1} \sum_{j=0}^{k_{\ell}-1} 2^{j} \tau^{*}\left(2^{-1 / \gamma} c^{-1 / \gamma_{1}} A 2^{-j / \gamma}\right) \exp \left(\frac{2 t A M_{j}}{2^{j}}\right) \\
& \quad+2^{k_{\ell}+1} a^{-1} \tau^{*}\left(c^{-1 / \gamma_{1}} A 2^{-\ell}\right) \exp \left(\frac{2 t A M_{k_{\ell}}}{2^{k_{\ell}}}\right) \\
& \quad+t A \tau\left(d_{k_{\ell}+1}\right) \exp \left(t M_{k_{\ell}+1} A 2^{-k_{\ell}}\right) . \tag{3.16}
\end{align*}
$$

Notice that for any $j=0, \ldots, \ell-1$, we have $d_{j}+1 \geq\left[c_{0} A 2^{-\left(\ell \wedge \frac{j}{\gamma}\right)}\right]$ and $c_{0} A 2^{-\left(\ell \wedge \frac{j}{\gamma}\right)} \geq 2$. Whence

$$
d_{j} \geq\left(d_{j}+1\right) / 2 \geq c_{0} A 2^{-\left(\ell \wedge \frac{j}{\gamma}\right)-2}
$$

Consequently setting $c_{1}=\min \left(\frac{1}{4} c^{1 / \gamma_{1}} c_{0}, 2^{-1 / \gamma}\right)$ and using (2.6), we derive that for any positive $t$,

$$
\begin{aligned}
& \left|\log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right)-\sum_{j=1}^{2^{\ell}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{k_{\ell}+1}}\left(I_{\ell, j}\right)\right)\right| \\
& \leq \frac{t A a}{2} \sum_{j=0}^{k_{\ell}} \exp \left(-\left(c_{1} A 2^{-j / \gamma}\right)^{\gamma_{1}}+\frac{2 t A M_{j}}{2^{j}}\right) \\
& \quad+2 \sum_{j=0}^{k_{\ell}-1} 2^{j} \exp \left(-\left(c_{1} A 2^{-j / \gamma}\right)^{\gamma_{1}}+\frac{2 t A M_{j}}{2^{j}}\right) \\
& \quad+2^{k_{\ell}+1} \exp \left(-\left(A 2^{-\ell}\right)^{\gamma_{1}}+\frac{2 t A M_{k_{\ell}}}{2^{k_{\ell}}}\right) \\
& \quad+t A a \exp \left(-\left(c_{1} A 2^{-\ell}\right)^{\gamma_{1}}+t M_{k_{\ell}+1} A 2^{-k_{\ell}}\right)
\end{aligned}
$$

By (3.10), we get that $2 A M_{j} 2^{-j} \leq b 2^{\gamma_{1} / \gamma}\left(2^{-j} A\right)^{\gamma_{1} / \gamma}$ for any $0 \leq j \leq k_{\ell}$. Also, since $k_{\ell}+1 \geq \gamma \ell$ and $\gamma<1$, we have that $M_{k_{\ell}+1} \leq b\left(2 A 2^{-\ell}\right)^{\gamma_{1} / \gamma_{2}} \leq b\left(2 A 2^{-\gamma \ell}\right)^{\gamma_{1} / \gamma_{2}}$. Whence

$$
M_{k_{\ell}+1} A 2^{-k_{\ell}}=2 M_{k_{\ell}+1} A 2^{-\left(k_{\ell}+1\right)} \leq b 2^{\gamma_{1} / \gamma_{1}} A^{\gamma_{1} / \gamma_{2}} 2^{-\gamma_{1} \ell} .
$$

In addition,

$$
2 A M_{k_{\ell}} 2^{-k_{\ell}} \leq b 2^{2 \gamma_{1} / \gamma}\left(A 2^{-k_{\ell}-1}\right)^{\gamma_{1} / \gamma} \leq b 2^{2 \gamma_{1} / \gamma} A^{\gamma_{1} / \gamma_{2}} 2^{-\gamma_{1} \ell} .
$$

Hence, if $t \leq c_{2} A^{\gamma_{1}(\gamma-1) / \gamma}$ where $c_{2}=2^{-\left(1+2 \gamma_{1} / \gamma\right)} c_{1}^{\gamma_{1}} b^{-1}$, we derive that

$$
\begin{aligned}
& \left|\log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right)-\sum_{j=1}^{2^{\ell}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{k_{\ell}+1}}\left(I_{\ell, j}\right)\right)\right| \\
& \leq \frac{t A a}{2} \sum_{j=0}^{k_{\ell}} \exp \left(-\frac{1}{2}\left(c_{1} A 2^{-j / \gamma}\right)^{\gamma_{1}}\right)+2 \sum_{j=0}^{k_{\ell}-1} 2^{j} \exp \left(-\frac{1}{2}\left(c_{1} A 2^{-j / \gamma}\right)^{\gamma_{1}}\right) \\
& \quad+\left(2^{k_{\ell}+1}+t A a\right) \exp \left(-\left(c_{1} A 2^{-\ell}\right)^{\gamma_{1}} / 2\right)
\end{aligned}
$$

Since $2^{k_{\ell}} \leq 2^{\ell \gamma} \leq A^{\gamma}$, it follows that for any $t \leq c_{2} A^{\gamma_{1}(\gamma-1) / \gamma}$,

$$
\begin{align*}
& \left|\log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right)-\sum_{j=1}^{2^{\ell}} \log \mathbb{E} \exp \left(t \bar{S}_{M_{k_{\ell}+1}}\left(I_{\ell, j}\right)\right)\right| \\
& \quad \leq\left(2 a \ell t A+4 A^{\gamma}\right) \exp \left(-\frac{1}{2}\left(\frac{c_{1} A}{2^{\ell}}\right)^{\gamma_{1}}\right) \tag{3.17}
\end{align*}
$$

We bound up now the $\log$ Laplace transform of each $\bar{S}_{M_{k_{\ell}+1}}\left(I_{\ell, j}\right)$ using the following elementary fact. Let $g(x)=x^{-2}\left(e^{x}-x-1\right)$ : for any centered random variable $U$ such that $\|U\|_{\infty} \leq M$, and any positive $t$,

$$
\begin{equation*}
\mathbb{E} \exp (t U) \leq 1+t^{2} g(t M) \mathbb{E}\left(U^{2}\right) \tag{3.18}
\end{equation*}
$$

Notice that $\left\|\bar{S}_{M_{k_{\ell}+1}}\left(I_{\ell, j}\right)\right\|_{\infty} \leq 2 M_{k_{\ell}+1} n_{\ell} \leq b 2^{\gamma_{1} / \gamma}\left(A 2^{-\ell}\right)^{\gamma_{1} / \gamma}$. Since $t \leq b^{-1} 2^{-\gamma_{1} / \gamma}$ $\left(2^{\ell} / A\right)^{\gamma_{1} / \gamma}$, by using (3.5), we then get that $\log \mathbb{E} \exp \left(t \bar{S}_{M_{k_{\ell}+1}}\left(I_{\ell, j}\right)\right) \leq t^{2} v^{2} n_{\ell}$. Consequently, for any $t \leq \kappa\left(A^{\gamma_{1}(\gamma-1) / \gamma} \wedge\left(2^{\ell} / A\right)^{\gamma_{1} / \gamma}\right)$, the following inequality holds:

$$
\begin{equation*}
\log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right) \leq t^{2} v^{2} A+\left(2 a \ell t A+4 A^{\gamma}\right) \exp \left(-\left(c_{1} A 2^{-\ell}\right)^{\gamma_{1}} / 2\right) \tag{3.19}
\end{equation*}
$$

Notice now that $\left\|\bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right\|_{\infty} \leq 2 M_{0} A \leq b 2^{\gamma_{1} / \gamma} A^{\gamma_{1} / \gamma}$. Hence if $t \leq b^{-1} 2^{-\gamma_{1} / \gamma}$ $A^{-\gamma_{1} / \gamma}$, by using (3.18) together with (3.5), we derive that

$$
\begin{equation*}
\log \mathbb{E} \exp \left(t \bar{S}_{M_{0}}\left(K_{A}^{(\ell)}\right)\right) \leq t^{2} v^{2} A \tag{3.20}
\end{equation*}
$$

which proves (3.4) in this case.

Now if $b^{-1} 2^{-\gamma_{1} / \gamma} A^{-\gamma_{1} / \gamma} \leq t \leq \kappa\left(A^{\gamma_{1}(\gamma-1) / \gamma} \wedge\left(2^{\ell} / A\right)^{\gamma_{1} / \gamma}\right)$, by using (3.19), we derive that (3.4) holds, which completes the proof of Proposition 1.

We now bound up the Laplace transform of the sum of truncated random variables on $[1, A]$. Let

$$
\begin{equation*}
\mu=\left(2\left(2 \vee 4 c_{0}^{-1}\right) /(1-\gamma)\right)^{\frac{2}{1-\gamma}} \quad \text { and } \quad c_{4}=b 2^{\gamma_{1} / \gamma_{3} \gamma_{1} / \gamma_{2}} c_{0}^{-\gamma_{1} / \gamma_{2}} \tag{3.21}
\end{equation*}
$$

where $c_{0}$ is defined in (3.2). Define also

$$
\begin{equation*}
v=\left(c_{4}\left(3-2^{(\gamma-1) \frac{\gamma_{1}}{\gamma}}\right)+\kappa^{-1}\right)^{-1}\left(1-2^{(\gamma-1) \frac{\gamma_{1}}{\gamma}}\right) \tag{3.22}
\end{equation*}
$$

where $\kappa$ is defined by (3.3).
Proposition 2 Let $\left(X_{j}\right)_{j \geq 1}$ be a sequence of centered real valued random variables satisfying (2.6), (2.7) and (2.8). Let A be an integer. Let $M=H^{-1}\left(a^{-1} \tau^{*}\left(c^{-1 / \gamma_{1}} A\right)\right)$ and $\bar{X}_{M}(k)$ be defined by (3.1) for any positive $k$. Then, if $A \geq \mu$ with $\mu$ defined by (3.21), for any positive $t<\nu A^{\gamma_{1}(\gamma-1) / \gamma}$, where $v$ is defined by (3.22), we get that

$$
\begin{equation*}
\log \mathbb{E}\left(\exp \left(t \sum_{k=1}^{A} \bar{X}_{M}(k)\right)\right) \leq \frac{A V(A) t^{2}}{1-t v^{-1} A^{\gamma_{1}(1-\gamma) / \gamma}} \tag{3.23}
\end{equation*}
$$

where $V(A)=50 v^{2}+\nu_{1} \exp \left(-\nu_{2} A^{\gamma_{1}(1-\gamma)}(\log A)^{-\gamma}\right)$ and $\nu_{1}, \nu_{2}$ are positive constants depending only on $a, b, c, \gamma$ and $\gamma_{1}$, and $v^{2}$ is defined by (3.5).

Proof of Proposition 2 Let $A_{0}=A$ and $X^{(0)}(k)=X_{k}$ for any $k=1, \ldots, A_{0}$. Let $\ell$ be a fixed positive integer, to be chosen later, which satisfies

$$
\begin{equation*}
A_{0} 2^{-\ell} \geq\left(2 \vee 4 c_{0}^{-1}\right) \tag{3.24}
\end{equation*}
$$

Let $K_{A_{0}}^{(\ell)}$ be the discrete Cantor type set as defined from $\{1, \ldots, A\}$ in Step 1 of the proof of Proposition 1. Let $A_{1}=A_{0}-\operatorname{Card} K_{A_{0}}^{(\ell)}$ and define for any $k=1, \ldots, A_{1}$,

$$
X^{(1)}(k)=X_{i_{k}} \text { where }\left\{i_{1}, \ldots, i_{A_{1}}\right\}=\{1, \ldots, A\} \backslash K_{A}
$$

Now for $i \geq 1$, let $K_{A_{i}}^{\left(\ell_{i}\right)}$ be defined from $\left\{1, \ldots, A_{i}\right\}$ exactly as $K_{A}^{(\ell)}$ is defined from $\{1, \ldots, A\}$. Here we impose the following selection of $\ell_{i}$ :

$$
\begin{equation*}
\ell_{i}=\inf \left\{j \in \mathbb{N}, A_{i} 2^{-j} \leq A_{0} 2^{-\ell}\right\} . \tag{3.25}
\end{equation*}
$$

Set $A_{i+1}=A_{i}-\operatorname{Card} K_{A_{i}}^{\left(\ell_{i}\right)}$ and $\left\{j_{1}, \ldots, j_{A_{i+1}}\right\}=\left\{1, \ldots, A_{i+1}\right\} \backslash K_{A_{i+1}}^{\left(\ell_{i+1}\right)}$. Define now

$$
X^{(i+1)}(k)=X^{(i)}\left(j_{k}\right) \text { for } k=1, \ldots, A_{i+1} .
$$

Let

$$
\begin{equation*}
m(A)=\inf \left\{m \in \mathbb{N}, A_{m} \leq A 2^{-\ell}\right\} . \tag{3.26}
\end{equation*}
$$

Note that $m(A) \geq 1$, since $A_{0}>A 2^{-\ell}(\ell \geq 1)$. In addition, $m(A) \leq \ell$ since for all $i \geq 1, A_{i} \leq A 2^{-i}$.

Obviously, for any $i=0, \ldots, m(A)-1$, the sequences $\left(X^{(i+1)}(k)\right)$ satisfy (2.6), (2.7) and (3.5) with the same constants. Now we set $T_{0}=M=H^{-1}\left(a^{-1}\right.$ $\tau^{*}\left(c^{-1 / \gamma_{1}} A_{0}\right)$ ), and for any integer $j=0, \ldots, m(A), T_{j}=H^{-1}\left(a^{-1} \tau^{*}\left(c^{-1 / \gamma_{1}} A_{j}\right)\right)$. With this definition and the Notation 1, we then define for all integers $i$ and $j$,

$$
X_{T_{j}}^{(i)}(k)=\varphi_{T_{j}}\left(X^{(i)}(k)\right)-\mathbb{E} \varphi_{T_{j}}\left(X^{(i)}(k)\right)
$$

According to (3.9), for any integer $j \geq 0$,

$$
\begin{equation*}
T_{j} \leq b\left(2 A_{j}\right)^{\gamma_{1} / \gamma_{2}} . \tag{3.27}
\end{equation*}
$$

For any $j=1, \ldots, m(A)$ and $i<j$, define

$$
\begin{aligned}
Y_{i}= & \sum_{k \in K_{A_{i}}^{\left(\ell_{i}\right)}} X_{T_{i}}^{(i)}(k), Z_{i}=\sum_{k=1}^{A_{i}}\left(X_{T_{i-1}}^{(i)}(k)-X_{T_{i}}^{(i)}(k)\right) \text { for } i>0, \\
& \quad \text { and } R_{j}=\sum_{k=1}^{A_{j}} X_{T_{j-1}}^{(j)}(k) .
\end{aligned}
$$

The following decomposition holds:

$$
\begin{equation*}
\sum_{k=1}^{A_{0}} X_{T_{0}}^{(0)}(k)=\sum_{i=0}^{m(A)-1} Y_{i}+\sum_{i=1}^{m(A)-1} Z_{i}+R_{m(A)} \tag{3.28}
\end{equation*}
$$

To control the terms in the decomposition (3.28), we need the following elementary lemma.

Lemma 1 For any $j=0, \ldots, m(A)-1, A_{j+1} \geq \frac{1}{3} c_{0} A_{j}$.
Proof of Lemma 1 Notice that for any $i$ in $\left[0, m(A)\left[\right.\right.$, we have $A_{i+1} \geq\left[c_{0} A_{i}\right]-1$. Since $c_{0} A_{i} \geq 2$, we derive that $\left[c_{0} A_{i}\right]-1 \geq\left(\left[c_{0} A_{i}\right]+1\right) / 3 \geq c_{0} A_{i} / 3$, which completes the proof.

Using (3.27), a useful consequence of Lemma 1 is that for any $j=1, \ldots, m(A)$

$$
\begin{equation*}
2 A_{j} T_{j-1} \leq c_{4} A_{j}^{\gamma_{1} / \gamma} \tag{3.29}
\end{equation*}
$$

where $c_{4}$ is defined by (3.21)

A bound for the Laplace transform of $R_{m(A)}$.
The random variable $\left|R_{m(A)}\right|$ is a.s. bounded by $2 A_{m(A)} T_{m(A)-1}$. By using (3.29) and (3.26), we then derive that

$$
\begin{equation*}
\left\|R_{m(A)}\right\|_{\infty} \leq c_{4}\left(A_{m(A)}\right)^{\gamma_{1} / \gamma} \leq c_{4}\left(A 2^{-\ell}\right)^{\gamma_{1} / \gamma} . \tag{3.30}
\end{equation*}
$$

Hence, if $t \leq c_{4}^{-1}\left(2^{\ell} / A\right)^{\gamma_{1} / \gamma}$, by using (3.18) together with (3.5), we obtain

$$
\begin{equation*}
\log \mathbb{E}\left(\exp \left(t R_{m(A)}\right)\right) \leq t^{2} v^{2} A 2^{-\ell} \leq t^{2}(v \sqrt{A})^{2}:=t^{2} \sigma_{1}^{2} \tag{3.31}
\end{equation*}
$$

A bound for the Laplace transform of the $Y_{i}$ 's.
Notice that for any $0 \leq i \leq m(A)-1$, by the definition of $\ell_{i}$ and (3.24), we get that

$$
2^{-\ell_{i}} A_{i}=2^{1-\ell_{i}}\left(A_{i} / 2\right)>2^{-\ell}(A / 2) \geq\left(1 \vee 2 c_{0}^{-1}\right)
$$

Now, by Proposition 1, we get that for any $i \in\left[0, m(A)\left[\right.\right.$ and any $t \leq \kappa\left(A_{i}^{\gamma-1} \wedge\right.$ $\left.\left(2^{\ell_{i}} / A_{i}\right)\right)^{\gamma_{1} / \gamma}$ with $\kappa$ defined by (3.3),

$$
\begin{aligned}
\log \mathbb{E}\left(e^{t Y_{i}}\right) \leq & t^{2}\left(v \sqrt{A_{i}}+\left(\sqrt{a b \ell_{i}}\left(2 A_{i}\right)^{\frac{1}{2}+\frac{\gamma_{1}}{2 \gamma}}+2 b A_{i}^{\gamma / 2}\left(2 A_{i}\right)^{\gamma_{1} / \gamma}\right)\right. \\
& \left.\times \exp \left(-\frac{1}{4}\left(c_{1} A_{i} 2^{-\ell_{i}}\right)^{\gamma_{1}}\right)\right)^{2}
\end{aligned}
$$

Notice now that $\ell_{i} \leq \ell \leq A, A_{i} \leq A 2^{-i}$ and $2^{-\ell-1} A \leq 2^{-\ell_{i}} A_{i} \leq 2^{-\ell} A$. Taking into account these bounds and the fact that $\gamma<1$, we then get that for any $i$ in $[0, m(A)[$ and any $t \leq \kappa\left(\left(2^{i} / A\right)^{1-\gamma} \wedge\left(2^{\ell} / A\right)\right)^{\gamma_{1} / \gamma}$,

$$
\begin{align*}
\log \mathbb{E}\left(e^{t Y_{i}}\right) \leq & t^{2}\left(v \frac{A^{1 / 2}}{2^{i / 2}}+\left(2^{1+\frac{\gamma_{1}}{\gamma}} \sqrt{b}(\sqrt{a}+\sqrt{b}) \frac{A^{1+\frac{\gamma_{1}}{\gamma}}}{\left(2^{i}\right)^{\frac{\gamma}{2}+\frac{\gamma_{1}}{2 \gamma}}}\right)\right. \\
& \left.\times \exp \left(-\frac{c_{1}^{\gamma_{1}}}{2^{2+\gamma_{1}}}\left(\frac{A}{2^{\ell}}\right)^{\gamma_{1}}\right)\right)^{2}:=t^{2} \sigma_{2, i}^{2} . \tag{3.32}
\end{align*}
$$

A bound for the Laplace transform of the $Z_{i}$ 's.
Notice first that for any $1 \leq i \leq m(A)-1, Z_{i}$ is a centered random variable, such that

$$
\left|Z_{i}\right| \leq \sum_{k=1}^{A_{i}}\left(\left|\left(\varphi_{T_{i-1}}-\varphi_{T_{i}}\right)\left(X^{(i)}(k)\right)\right|+\mathbb{E}\left|\left(\varphi_{T_{i-1}}-\varphi_{T_{i}}\right)\left(X^{(i)}(k)\right)\right|\right) .
$$

Consequently, using (3.29) we then get that $\left\|Z_{i}\right\|_{\infty} \leq 2 A_{i} T_{i-1} \leq c_{4} A_{i}^{\gamma_{1} / \gamma}$. In addition, since $\left|\left(\varphi_{T_{i-1}}-\varphi_{T_{i}}\right)(x)\right| \leq\left(T_{i-1}-T_{i}\right) 1 \mathrm{I}_{x>T_{i}}$, and the random variables $\left(X^{(i)}(k)\right)$ satisfy (2.7), by the definition of $T_{i}$, we get that

$$
\mathbb{E}\left|Z_{i}\right|^{2} \leq\left(2 A_{i} T_{i-1}\right)^{2} a^{-1} \tau^{*}\left(c^{-1 / \gamma_{1}} A_{i}\right) \leq c_{4}^{2} A_{i}^{2 \gamma_{1} / \gamma} \exp \left(-A_{i}^{\gamma_{1}}\right)
$$

Hence applying (3.18) to the random variable $Z_{i}$, we get for any positive $t$,

$$
\mathbb{E} \exp \left(t Z_{i}\right) \leq 1+t^{2} g\left(c_{4} t A_{i}^{\gamma_{1} / \gamma}\right) c_{4}^{2} A_{i}^{2 \gamma_{1} / \gamma} \exp \left(-A_{i}^{\gamma_{1}}\right)
$$

Now, since $A_{i} \leq A 2^{-i}$, for any positive real $t$ satisfying $t \leq\left(2 c_{4}\right)^{-1}\left(2^{i} / A\right)^{\gamma_{1}(1-\gamma) / \gamma}$, we have that $c_{4} t A_{i}^{\gamma_{1} / \gamma} \leq A_{i}^{\gamma_{1}} / 2$. Since $g(x) \leq e^{x}$ for $x \geq 0$, we infer that for $t \leq\left(2 c_{4}\right)^{-1}\left(2^{i} / A\right)^{\gamma_{1}(1-\gamma) / \gamma}$,

$$
\log \mathbb{E} \exp \left(t Z_{i}\right) \leq c_{4}^{2} t^{2}\left(2^{-i} A\right)^{2 \gamma_{1} / \gamma} \exp \left(-A_{i}^{\gamma_{1}} / 2\right)
$$

By taking into account that for any $1 \leq i \leq m(A)-1, A_{i} \geq A_{m(A)-1}>A 2^{-\ell}$ (by definition of $m(A)$ ), it follows that for any $i$ in $[1, m(A)$ [ and any positive $t$ satisfying $t \leq\left(2 c_{4}\right)^{-1}\left(2^{i} / A\right)^{\gamma_{1}(1-\gamma) / \gamma}$,

$$
\begin{equation*}
\log \mathbb{E} \exp \left(t Z_{i}\right) \leq t^{2}\left(c_{4}\left(2^{-i} A\right)^{\gamma_{1} / \gamma} \exp \left(-\left(A 2^{-\ell}\right)^{\gamma_{1}} / 4\right)\right)^{2}:=t^{2} \sigma_{3, i}^{2} \tag{3.33}
\end{equation*}
$$

## End of the proof. Let

$C=c_{4}\left(\frac{A}{2^{\ell}}\right)^{\gamma_{1} / \gamma}+\frac{1}{\kappa} \sum_{i=0}^{m(A)-1}\left(\left(\frac{A}{2^{i}}\right)^{1-\gamma} \vee \frac{A}{2^{\ell}}\right)^{\gamma_{1} / \gamma}+2 c_{4} \sum_{i=1}^{m(A)-1}\left(\frac{A}{2^{i}}\right)^{\gamma_{1}(1-\gamma) / \gamma}$,
and

$$
\sigma=\sigma_{1}+\sum_{i=0}^{m(A)-1} \sigma_{2, i}+\sum_{i=1}^{m(A)-1} \sigma_{3, i}
$$

where $\sigma_{1}, \sigma_{2, i}$ and $\sigma_{3, i}$ are respectively defined in (3.31), (3.32) and (3.33).
Notice that $m(A) \leq \ell \leq 2 \log A / \log 2$. We choose now $\ell=\inf \left\{j \in \mathbb{N}: 2^{j} \geq\right.$ $\left.A^{\gamma}(\log A)^{\gamma / \gamma_{1}}\right\}$. This selection is compatible with (3.24) if

$$
\begin{equation*}
\left(2 \vee 4 c_{0}^{-1}\right)(\log A)^{\gamma / \gamma_{1}} \leq A^{1-\gamma} . \tag{3.34}
\end{equation*}
$$

Now we use the fact that for any positive $\delta$ and any positive $u, \delta \log u \leq u^{\delta}$. Hence if $A \geq 3$,

$$
\left(2 \vee 4 c_{0}^{-1}\right)(\log A)^{\gamma / \gamma_{1}} \leq\left(2 \vee 4 c_{0}^{-1}\right) \log A \leq 2(1-\gamma)^{-1}\left(2 \vee 4 c_{0}^{-1}\right) A^{(1-\gamma) / 2}
$$

which implies that (3.34) holds as soon as $A \geq \mu$ where $\mu$ is defined by (3.21). It follows that

$$
\begin{equation*}
C \leq v^{-1} A^{\gamma_{1}(1-\gamma) / \gamma} \tag{3.35}
\end{equation*}
$$

In addition

$$
\begin{aligned}
\sigma \leq & 5 v \sqrt{A}+5 \sqrt{b}(\sqrt{a}+\sqrt{b}) 2^{2 \gamma_{1} / \gamma} A^{1+\gamma_{1} / \gamma} \exp \left(-\frac{c_{1}^{\gamma_{1}}}{2^{2+\gamma_{1}}}\left(A 2^{-\ell}\right)^{\gamma_{1}}\right) \\
& +c_{4} A^{\gamma_{1} / \gamma} \exp \left(-\frac{1}{4}\left(A 2^{-\ell}\right)^{\gamma_{1}}\right)
\end{aligned}
$$

Consequently, since $A 2^{-\ell} \geq \frac{1}{2} A^{1-\gamma}(\log A)^{-\gamma / \gamma_{1}}$, there exists positive constants $\nu_{1}$ and $\nu_{2}$ depending only on $a, b, c, \gamma$ and $\gamma_{1}$ such that

$$
\begin{equation*}
\sigma^{2} \leq A\left(50 v^{2}+\nu_{1} \exp \left(-\nu_{2} A^{\gamma_{1}(1-\gamma)}(\log A)^{-\gamma}\right)\right)=A V(A) \tag{3.36}
\end{equation*}
$$

Starting from the decomposition (3.28) and the bounds (3.31), (3.32) and (3.33), we aggregate the contributions of the terms by using Lemma 3 given in the appendix. Then, by taking into account the bounds (3.35) and (3.36), Proposition 2 follows.

### 3.2 Proof of Theorem 1

Let us give the idea of the proof. We first use the classical Bernstein-type blocking arguments on partial sums of suitably truncated variables at a level $M$; namely, the index set $\{1, \ldots, n\}$ is partitioned into blocks of size $A$. Then the blocks of even indexes are approximated by independent blocks with same marginals using the exact coupling coefficient in $\mathbb{L}^{1}$ (see Lemma 5 in [14]); and similarly for the blocks of odd indexes. We then bound the log-Laplace transforms of each block of size $A$ with the help of Proposition 2. The end of the proof consists in choosing optimally both $A$ and $M$.

- If $\lambda \geq b n^{\gamma_{1} / \gamma}$, note that $\sup \left(\left|S_{1}\right|,\left|S_{2}\right|, \ldots,\left|S_{n}\right|\right) \leq\left|X_{1}\right|+\left|X_{2}\right|+\cdots+\left|X_{n}\right|$. Hence

$$
\begin{aligned}
\mathbb{P}\left(\sup _{j \leq n}\left|S_{j}\right| \geq \lambda\right) & \leq \sum_{i=1}^{n} \mathbb{P}\left(\left|X_{i}\right| \geq \lambda / n\right) \leq n e \exp \left(-\frac{\lambda^{\gamma_{2}}}{(b n)^{\gamma_{2}}}\right) \wedge 1 \\
& \leq \sqrt{n e} \exp \left(-\frac{\lambda^{\gamma_{2}}}{2(b n)^{\gamma_{2}}}\right)
\end{aligned}
$$

Now, if $\lambda \geq b n^{\gamma_{1} / \gamma}$, then $(\lambda /(n b))^{\gamma_{2}} \geq(\lambda / b)^{\gamma}$. Hence, for any $n \geq 3$,

$$
\mathbb{P}\left(\sup _{j \leq n}\left|S_{j}\right| \geq \lambda\right) \leq n \exp \left(-\frac{\lambda^{\gamma}}{2 b^{\gamma}}\right) .
$$

- Let $\zeta=\mu \vee\left(2 / \gamma_{2}\right)^{1 / \gamma_{1}}$ where $\mu$ is defined by (3.21). Assume that $a \vee b(4 \zeta)^{\gamma_{1} / \gamma} \leq$ $\lambda \leq b n^{\gamma_{1} / \gamma}$. Let $p$ be a real in [1, $\frac{n}{2}$ ], to be chosen later on. Let

$$
A=\left[\frac{n}{2 p}\right], k=\left[\frac{n}{2 A}\right] \quad \text { and } \quad M=H^{-1}\left(a^{-1} \tau^{*}\left(c^{-\frac{1}{\gamma_{1}}} A\right)\right) .
$$

For any set of natural numbers $K$, denote

$$
\bar{S}_{M}(K)=\sum_{i \in K} \bar{X}_{M}(i) \text { where } \bar{X}_{M}(k) \text { is defined by (3.1). }
$$

For $i$ integer in $[1,2 k]$, let $I_{i}=\{1+(i-1) A, \ldots, i A\}$. Let also $I_{2 k+1}=\{1+$ $2 k A, \ldots, n\}$. Set

$$
\bar{S}_{1}(j)=\sum_{i=1}^{j} \bar{S}_{M}\left(I_{2 i-1}\right) \quad \text { and } \quad \bar{S}_{2}(j)=\sum_{i=1}^{j} \bar{S}_{M}\left(I_{2 i}\right)
$$

We then get the following inequality

$$
\begin{equation*}
\sup _{j \leq n}\left|S_{j}\right| \leq \sup _{j \leq k+1}\left|\bar{S}_{1}(j)\right|+\sup _{j \leq k}\left|\bar{S}_{2}(j)\right|+2 A M+\sum_{i=1}^{n}\left|X_{i}-\bar{X}_{M}(i)\right| . \tag{3.37}
\end{equation*}
$$

Now

$$
\mathbb{P}\left(\sum_{i=1}^{n}\left|X_{i}-\bar{X}_{M}(i)\right| \geq \lambda\right) \leq \frac{1}{\lambda} \sum_{i=1}^{n} \mathbb{E}\left|X_{i}-\bar{X}_{M}(i)\right| \leq \frac{2 n}{\lambda} \int_{M}^{\infty} H(x) d x
$$

Now recall that $\log H(x)=1-(x / b)^{\gamma_{2}}$. It follows that the function $x \rightarrow \log \left(x^{2} H(x)\right)$ is nonincreasing as soon as $x \geq b\left(2 / \gamma_{2}\right)^{1 / \gamma_{2}}$. Hence, for $M \geq b\left(2 / \gamma_{2}\right)^{1 / \gamma_{2}}$,

$$
\int_{M}^{\infty} H(x) d x \leq M^{2} H(M) \int_{M}^{\infty} \frac{d x}{x^{2}}=M H(M) .
$$

Whence

$$
\begin{equation*}
\mathbb{P}\left(\sum_{i=1}^{n}\left|X_{i}-\bar{X}_{M}(i)\right| \geq \lambda\right) \leq 2 n \lambda^{-1} M H(M) \quad \text { for } M \geq b\left(2 / \gamma_{2}\right)^{1 / \gamma_{2}} \tag{3.38}
\end{equation*}
$$

Using (3.38) together with our selection of $M$, we get for all positive $\lambda$ that

$$
\mathbb{P}\left(\sum_{i=1}^{n}\left|X_{i}-\bar{X}_{M}(i)\right| \geq \lambda\right) \leq 2 n \lambda^{-1} M \exp \left(-A^{\gamma_{1}}\right) \text { for } A \geq\left(2 / \gamma_{2}\right)^{1 / \gamma_{1}}
$$

By using Lemma 5 in Dedecker and Prieur [14], we get the existence of independent random variables $\left(\bar{S}_{M}^{*}\left(I_{2 i}\right)\right)_{1 \leq i \leq k}$ with the same distribution as the random variables $\bar{S}_{M}\left(I_{2 i}\right)$ such that

$$
\begin{equation*}
\mathbb{E}\left|\bar{S}_{M}\left(I_{2 i}\right)-\bar{S}_{M}^{*}\left(I_{2 i}\right)\right| \leq A \tau(A) \leq a A \exp \left(-c A^{\gamma_{1}}\right) \tag{3.39}
\end{equation*}
$$

The same is true for the sequence $\left(\bar{S}_{M}\left(I_{2 i-1}\right)\right)_{1 \leq i \leq k+1}$. Hence for any positive $\lambda$ such that $\lambda \geq 2 A M$,

$$
\begin{aligned}
\mathbb{P}\left(\sup _{j \leq n}\left|S_{j}\right| \geq 6 \lambda\right) \leq & a \lambda^{-1} A(2 k+1) \exp \left(-c A^{\gamma_{1}}\right)+2 n \lambda^{-1} M \exp \left(-A^{\gamma_{1}}\right) \\
& +\mathbb{P}\left(\max _{j \leq k+1}\left|\sum_{i=1}^{j} \bar{S}_{M}^{*}\left(I_{2 i-1}\right)\right| \geq \lambda\right) \\
& +\mathbb{P}\left(\max _{j \leq k}\left|\sum_{i=1}^{j} \bar{S}_{M}^{*}\left(I_{2 i}\right)\right| \geq \lambda\right) .
\end{aligned}
$$

For any positive $t$, due to the independence and since the variables are centered, $\left(\exp \left(t S_{M}\left(I_{2 i}\right)\right)\right)_{i}$ is a submartingale. Hence Doob's maximal inequality entails that for any positive $t$,

$$
\mathbb{P}\left(\max _{j \leq k} \sum_{i=1}^{j} \bar{S}_{M}^{*}\left(I_{2 i}\right) \geq \lambda\right) \leq e^{-\lambda t} \prod_{i=1}^{k} \mathbb{E}\left(\exp \left(t \bar{S}_{M}\left(I_{2 i}\right)\right)\right) .
$$

To bound the Laplace transform of each random variable $\bar{S}_{M}\left(I_{2 i}\right)$, we apply Proposition 2 to the sequences $\left(X_{i+s}\right)_{i \in \mathbb{Z}}$ for suitable values of $s$. Hence we derive that, if $A \geq \mu$ then for any positive $t$ such that $t<\nu A^{\gamma_{1}(\gamma-1) / \gamma}$ (where $v$ is defined by (3.22)),

$$
\begin{equation*}
\sum_{i=1}^{k} \log \mathbb{E}\left(\exp \left(t \bar{S}_{M}\left(I_{2 i}\right)\right)\right) \leq A k t^{2} \frac{V(A)}{1-t v^{-1} A^{\gamma_{1}(1-\gamma) / \gamma}} \tag{3.40}
\end{equation*}
$$

Obviously the same inequalities hold true for the sums associated to $\left(-X_{i}\right)_{i \in \mathbb{Z}}$. Now the usual Chernoff computations for the optimization of $t$ in (3.40) lead to

$$
\mathbb{P}\left(\max _{j \leq k}\left|\sum_{i=1}^{j} \bar{S}_{M}^{*}\left(I_{2 i}\right)\right| \geq \lambda\right) \leq 2 \exp \left(-\frac{\lambda^{2}}{4 \operatorname{AkV}(A)+2 \lambda v^{-1} A^{\gamma_{1}(1-\gamma) / \gamma}}\right) .
$$

Similarly, we obtain that

$$
\mathbb{P}\left(\max _{j \leq k+1}\left|\sum_{i=1}^{j} \bar{S}_{M}^{*}\left(I_{2 i-1}\right)\right| \geq \lambda\right) \leq 2 \exp \left(-\frac{\lambda^{2}}{4 A(k+1) V(A)+2 \lambda v^{-1} A^{\gamma_{1}(1-\gamma) / \gamma}}\right)
$$

Choose now $p=n\left(b^{-1} \lambda\right)^{-\gamma / \gamma_{1}}$. It follows that $2 A \leq(\lambda / b)^{\gamma / \gamma_{1}}$ and, since $M \leq$ $b(2 A)^{\gamma_{1} / \gamma_{2}}$, we obtain that $2 A M \leq b(2 A)^{\gamma_{1} / \gamma} \leq \lambda$. Also, since $\lambda / b \geq(4 \zeta)^{\gamma_{1} / \gamma}$, $p \leq n / 4$ and consequently $A \geq n /(4 p) \geq\left(b^{-1} \lambda\right)^{\gamma / \gamma_{1}} / 4 \geq \zeta \geq \mu$. Hence we get that for $a \vee b(4 \zeta)^{\gamma_{1} / \gamma} \leq \lambda \leq b n^{\gamma_{1} / \gamma}$,

$$
\begin{aligned}
\mathbb{P}\left(\sup _{j \leq n}\left|S_{j}\right| \geq 6 \lambda\right) \leq & 1 \wedge\left(4 n \exp \left(-(c \wedge 1) \frac{\lambda^{\gamma}}{4 \gamma_{1} b^{\gamma}}\right)\right. \\
& \left.+4 \exp \left(-\frac{\lambda^{2}}{4 n B(\lambda)+2 \lambda^{2-\gamma} \nu^{-1} b^{\gamma-1}}\right)\right)
\end{aligned}
$$

with $B(\lambda)=50 v^{2}+\nu_{1} \exp \left(-\tilde{\nu}_{2} \lambda^{\gamma(1-\gamma)}(\log \lambda)^{-\gamma}\right)$, where $\tilde{\nu}_{2}$ is a positive constant depending on $a, b, c, \gamma$ and $\gamma_{1}$. The result follows from the previous bound.

- To end the proof, we mention that if $\lambda \leq a \vee b(4 \zeta)^{\gamma_{1} / \gamma}$, then

$$
\mathbb{P}\left(\sup _{j \leq n}\left|S_{j}\right| \geq \lambda\right) \leq 1 \leq e \exp \left(-\frac{\lambda^{\gamma}}{a^{\gamma} \vee b^{\gamma}(4 \zeta)^{\gamma_{1}}}\right)
$$

which is less than $n \exp \left(-\lambda^{\gamma} / C_{1}\right)$ as soon as $n \geq 3$ and $C_{1} \geq a^{\gamma} \vee b^{\gamma}(4 \zeta)^{\gamma_{1}}$.

### 3.3 Proof of Remark 3

Using the Notation 1 and setting $W_{i}=\varphi_{M}\left(X_{i}\right)$ we first bound $\left.\operatorname{Cov}\left(W_{i}, W_{i+k}\right)\right)$. Applying (4.2) of Proposition 1 in Dedecker and Doukhan [12], we derive that, for any positive $k$,

$$
\left|\operatorname{Cov}\left(W_{i}, W_{i+k}\right)\right| \leq 2 \int_{0}^{\gamma\left(\mathcal{M}_{i}, W_{i+k}\right) / 2} Q_{\left|W_{i}\right|} \circ G_{\left|W_{i+k}\right|}(u) d u
$$

where

$$
\gamma\left(\mathcal{M}_{i}, W_{i+k}\right)=\left\|\mathbb{E}\left(W_{i+k} \mid \mathcal{M}_{i}\right)-\mathbb{E}\left(W_{i+k}\right)\right\|_{1} \leq \tau(k),
$$

since $x \mapsto \varphi_{M}(x)$ is 1-Lipschitz. Now for any $j, Q_{\left|W_{j}\right|} \leq Q_{\left|X_{j}\right|} \leq Q$, implying that $G_{\left|W_{j}\right|} \geq G$, where $G$ is the inverse function of $u \rightarrow \int_{0}^{u} Q(v) d v$. Taking $j=i$ and $j=i+k$, we get that

$$
\left|\operatorname{Cov}\left(W_{i}, W_{i+k}\right)\right| \leq 2 \int_{0}^{\tau(k) / 2} Q_{\left|X_{i}\right|} \circ G(u) d u .
$$

Making the change-of-variables $u=G(v)$ we also have

$$
\begin{equation*}
\left|\operatorname{Cov}\left(W_{i}, W_{i+k}\right)\right| \leq 2 \int_{0}^{G(\tau(k) / 2)} Q_{X_{i}}(u) Q(u) d u \tag{3.41}
\end{equation*}
$$

proving the remark.

### 3.4 Proof of Theorem 2

We first use Theorem 1 allowing essentially us to reduce the proof of the MDP to the one of bounded random variables. For any $n \geq 1$, let $T=T_{n}$ where $\left(T_{n}\right)$ is a sequence of real numbers greater than 1 such that $\lim _{n \rightarrow \infty} T_{n}=\infty$, that will be specified later. We truncate the variables at the level $T_{n}$. So using the Notation 1, we consider

$$
X_{i}^{\prime}=\varphi_{T_{n}}\left(X_{i}\right)-\mathbb{E} \varphi_{T_{n}}\left(X_{i}\right), W_{i}^{\prime}=X_{i}-\varphi_{T_{n}}\left(X_{i}\right) \text { and } X_{i}^{\prime \prime}=X_{i}-X_{i}^{\prime} .
$$

Let $S_{n}^{\prime}=\sum_{i=1}^{n} X_{i}^{\prime}$ and $S_{n}^{\prime \prime}=\sum_{i=1}^{n} X_{i}^{\prime \prime}$. To prove the result, by exponentially equivalence lemma in Dembo and Zeitouni [22, Theorem 4.2.13, p. 130], it suffices to prove that for any $\eta>0$,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} a_{n} \log \mathbb{P}\left(\frac{\sqrt{a_{n}}}{\sigma_{n}}\left|S_{n}^{\prime \prime}\right| \geq \eta\right)=-\infty \tag{3.42}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\{\sigma_{n}^{-1} S_{n}^{\prime}\right\} \text { satisfies (2.9) with the rate function } I(t)=t^{2} / 2 \tag{3.43}
\end{equation*}
$$

To prove (3.42), we first notice that $\left|x-\varphi_{T}(x)\right|=(|x|-T)_{+}$. Hence $Q_{\left|W_{i}^{\prime}\right|} \leq$ $(Q-T)_{+}$and, denoting by $V_{T_{n}}^{\prime \prime}$ the upper bound for the variance of $S_{n}^{\prime \prime}$ (corresponding to $V$ for the variance of $S_{n}$ ) we have, by Remark 3,

$$
V_{T_{n}}^{\prime \prime} \leq \int_{0}^{1}\left(Q(u)-T_{n}\right)_{+}^{2} d u+4 \sum_{k>0} \int_{0}^{\tau_{W^{\prime}}(k) / 2}\left(Q\left(G_{T_{n}}(v)\right)-T_{n}\right)_{+} d v .
$$

where $G_{T}$ is the inverse function of $x \rightarrow \int_{0}^{x}(Q(u)-T)_{+} d u$ and the coefficients $\tau_{W^{\prime}}(k)$ are the $\tau$-mixing coefficients associated to $\left(W_{i}^{\prime}\right)_{i}$. Next, since $x \rightarrow x-\varphi_{T}(x)$ is 1-Lipschitz, we have that $\tau_{W^{\prime}}(k) \leq \tau_{X}(k)=\tau(k)$. Moreover, $G_{T} \geq G$, because
$(Q-T)_{+} \leq Q$. Since $Q$ is nonincreasing, it follows that

$$
V_{T_{n}}^{\prime \prime} \leq \int_{0}^{1}\left(Q(u)-T_{n}\right)_{+}^{2} d u+4 \sum_{k>0} \int_{0}^{\tau(k) / 2}\left(Q(G(v))-T_{n}\right)_{+} d v
$$

Hence

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} V_{T_{n}}^{\prime \prime}=0 \tag{3.44}
\end{equation*}
$$

The sequence ( $X_{i}^{\prime \prime}$ ) satisfies (2.6) and we now prove that it satisfies also (2.7) for $n$ large enough. With this aim, we first notice that

$$
\left|\mathbb{E}\left(\varphi_{T_{n}}\left(X_{i}\right)\right)\right| \leq \mathbb{E}\left|W_{i}^{\prime}\right| \leq \int_{T_{n}}^{\infty} H(x) d x<b \text { for } n \text { large enough. }
$$

Hence for $n$ large enough, $\left|X_{i}^{\prime \prime}\right| \leq\left(\left|X_{i}\right|-T_{n}\right)_{+}+b \leq b \vee\left|X_{i}\right|$ provided that $T_{n} \geq b$. Then for $n$ large enough, the sequence ( $X_{i}^{\prime \prime}$ ) satisfies (2.7) and we can apply Theorem 1 to the sequence ( $X_{i}^{\prime \prime}$ ): for any $\eta>0$, and $n$ large enough

$$
\begin{aligned}
\mathbb{P}\left(\sqrt{\frac{a_{n}}{\sigma_{n}^{2}}}\left|S_{n}^{\prime \prime}\right| \geq \eta\right) \leq & n \exp \left(-\frac{\eta^{\gamma} \sigma_{n}^{\gamma}}{C_{1} a_{n}^{\frac{\gamma}{2}}}\right)+\exp \left(-\frac{\eta^{2} \sigma_{n}^{2}}{C_{2} a_{n}\left(1+n V_{T_{n}}^{\prime \prime}\right)}\right) \\
& +\exp \left(-\frac{\eta^{2} \sigma_{n}^{2}}{C_{3} n a_{n}} \exp \left(\frac{\eta^{\delta} \sigma_{n}^{\delta}}{C_{4} a_{n}^{\frac{\delta}{2}}}\right)\right)
\end{aligned}
$$

where $\delta=\gamma(1-\gamma) / 2$. This proves (3.42), since $a_{n} \rightarrow 0, a_{n} n^{\gamma /(2-\gamma)} \rightarrow \infty$, $\lim _{n \rightarrow \infty} V_{T_{n}}^{\prime \prime}=0$ and $\liminf _{n \rightarrow \infty} \sigma_{n}^{2} / n>0$.

We turn now to the proof of (3.43). Let $p_{n}=\left[n^{1 /(2-\gamma)}\right]$ and $q_{n}=\delta_{n} p_{n}$ where $\delta_{n}$ is a sequence of integers tending to zero and such that $\delta_{n}^{\gamma_{1}} n^{\gamma_{1} /(2-\gamma)} / \log n \rightarrow \infty$ and $\delta_{n}^{\gamma_{1}} a_{n} n^{\gamma_{1} /(2-\gamma)} \rightarrow \infty$ (this is always possible since $\gamma_{1} \geq \gamma$ and by assumption $\left.a_{n} n^{\gamma /(2-\gamma)} \rightarrow \infty\right)$. Let now $m_{n}=\left[n /\left(p_{n}+q_{n}\right)\right]$. We divide the variables $\left\{X_{i}^{\prime}\right\}$ in big blocks of size $p_{n}$ and small blocks of size $q_{n}$, in the following way: Let us set for all $1 \leq j \leq m_{n}$,

$$
Y_{j, n}=\sum_{i=(j-1)\left(p_{n}+q_{n}\right)+1}^{(j-1)\left(p_{n}+q_{n}\right)+p_{n}} X_{i}^{\prime} \quad \text { and } \quad Z_{j, n}=\sum_{i=(j-1)\left(p_{n}+q_{n}\right)+p_{n}+1}^{j\left(p_{n}+q_{n}\right)} X_{i}^{\prime} .
$$

Then we have the following decomposition:

$$
\begin{equation*}
S_{n}^{\prime}=\sum_{j=1}^{m_{n}} Y_{j, n}+\sum_{j=1}^{m_{n}} Z_{j, n}+\sum_{i=m_{n}\left(p_{n}+q_{n}\right)+1}^{n} X_{i}^{\prime} . \tag{3.45}
\end{equation*}
$$

For any $j=1, \ldots, m_{n}$, let now $I(n, j)=\left\{(j-1)\left(p_{n}+q_{n}\right)+1, \ldots,(j-1)\right.$ $\left.\left(p_{n}+q_{n}\right)+p_{n}\right\}$. These intervals are of cardinal $p_{n}$. Let $\ell_{n}=\inf \left\{k \in \mathbb{N}^{*}, 2^{k} \geq\right.$ $\left.\varepsilon_{n}^{-1} p_{n}^{\gamma / 2} a_{n}^{-1 / 2}\right\}$, where $\varepsilon_{n}$ a sequence of positive numbers tending to zero and satisfying

$$
\begin{equation*}
\varepsilon_{n}^{2} a_{n} n^{\gamma /(2-\gamma)} \rightarrow \infty \tag{3.46}
\end{equation*}
$$

For each $j \in\left\{1, \ldots, m_{n}\right\}$, we construct discrete Cantor sets, $K_{I(n, j)}^{\left(\ell_{n}\right)}$, as described in the proof of Proposition 1 with $A=p_{n}, \ell=\ell_{n}$, and the following selection of $c_{0}$,

$$
c_{0}=\frac{\varepsilon_{n}}{1+\varepsilon_{n}} \frac{2^{(1-\gamma) / \gamma}-1}{2^{1 / \gamma}-1} .
$$

Notice that clearly with the selections of $p_{n}$ and $\ell_{n}, p_{n} 2^{-\ell_{n}} \rightarrow \infty$. In addition with the selection of $c_{0}$ we get that for any $1 \leq j \leq m_{n}$,

$$
\operatorname{Card}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)^{c} \leq \frac{\varepsilon_{n} p_{n}}{1+\varepsilon_{n}}
$$

and

$$
K_{I(n, j)}^{\left(\ell_{n}\right)}=\bigcup_{i=1}^{2^{\ell_{n}}} I_{\ell_{n}, i}\left(p_{n}, j\right)
$$

where the $I_{\ell_{n}, i}\left(p_{n}, j\right)$ are disjoint sets of consecutive integers, each of same cardinal such that

$$
\begin{equation*}
\frac{p_{n}}{2^{\ell_{n}}\left(1+\varepsilon_{n}\right)} \leq \operatorname{Card} I_{\ell_{n}, i}\left(p_{n}, j\right) \leq \frac{p_{n}}{2^{\ell_{n}}} . \tag{3.47}
\end{equation*}
$$

With this notation, we derive that

$$
\begin{equation*}
\sum_{j=1}^{m_{n}} Y_{j, n}=\sum_{j=1}^{m_{n}} S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)+\sum_{j=1}^{m_{n}} S^{\prime}\left(\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)^{c}\right) \tag{3.48}
\end{equation*}
$$

Combining (3.45) with (3.48), we can rewrite $S_{n}^{\prime}$ as follows

$$
\begin{equation*}
S_{n}^{\prime}=\sum_{j=1}^{m_{n}} S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)+\sum_{k=1}^{r_{n}} \tilde{X}_{i}, \tag{3.49}
\end{equation*}
$$

where $r_{n}=n-m_{n} \operatorname{Card} K_{I(n, 1)}^{\left(\ell_{n}\right)}$ and the $\tilde{X}_{i}$ are obtained by renumbering the $X_{i}^{\prime}$ and satisfy (2.6) and (2.7) with the same constants. Since $r_{n}=o(n)$, applying Theorem 1 and using the fact that $\lim \inf _{n \rightarrow \infty} \sigma_{n}^{2} / n>0$, we get that for any $\eta>0$,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} a_{n} \log \mathbb{P}\left(\frac{\sqrt{a_{n}}}{\sigma_{n}} \sum_{k=1}^{r_{n}} \tilde{X}_{i} \geq \eta\right)=-\infty \tag{3.50}
\end{equation*}
$$

Hence to prove (3.43), it suffices to prove that

$$
\begin{equation*}
\left\{\sigma_{n}^{-1} \sum_{j=1}^{m_{n}} S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)\right\} \text { satisfies (2.9) with the rate function } I(t)=t^{2} / 2 \tag{3.51}
\end{equation*}
$$

With this aim, we choose now $T_{n}=\varepsilon_{n}^{-1 / 2}$ where $\varepsilon_{n}$ is defined by (3.46).
By using Lemma 5 in Dedecker and Prieur [14], we get the existence of independent random variables $\left(S^{*}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)\right)_{1 \leq j \leq m_{n}}$ with the same distribution as the random variables $S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)$ such that

$$
\sum_{j=1}^{m_{n}} \mathbb{E}\left|S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)-S^{*}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)\right| \leq \tau\left(q_{n}\right) \sum_{j=1}^{m_{n}} \operatorname{Card} K_{I(n, j)}^{\left(\ell_{n}\right)}
$$

Consequently, since $\sum_{j=1}^{m_{n}} \operatorname{Card} K_{I(n, j)}^{\left(\ell_{n}\right)} \leq n$, we derive that for any $\eta>0$ and any $n \geq 2^{2-\gamma}$,

$$
\begin{aligned}
& a_{n} \log \mathbb{P}\left(\frac{\sqrt{a_{n}}}{\sigma_{n}}\left|\sum_{j=1}^{m_{n}}\left(S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)-S^{*}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)\right)\right| \geq \eta\right) \\
& \quad \leq a_{n} \log \left(\frac{a n \sqrt{a_{n}}}{\eta \sigma_{n}} \exp \left(-\frac{c \delta_{n}^{\gamma_{1}} n^{\gamma_{1} /(2-\gamma)}}{2^{\gamma_{1}}}\right)\right),
\end{aligned}
$$

which tends to $-\infty$ by the fact that $\liminf _{n} \sigma_{n}^{2} / n>0$ and the selection of $\delta_{n}$. Hence the proof of the MDP for $\left\{\sigma_{n}^{-1} \sum_{j=1}^{m_{n}} S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)\right\}$ is reduced to proving the MDP for $\left\{\sigma_{n}^{-1} \sum_{j=1}^{m_{n}} S^{*}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right)\right\}$. By Ellis Theorem, to prove (3.51) it remains then to show that, for any real $t$,

$$
\begin{equation*}
a_{n} \sum_{j=1}^{m_{n}} \log \mathbb{E} \exp \left(t S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right) / \sqrt{a_{n} \sigma_{n}^{2}}\right) \rightarrow \frac{t^{2}}{2} \text { as } n \rightarrow \infty \tag{3.52}
\end{equation*}
$$

As in the proof of Proposition 1, we decorrelate step by step. Using Lemma 2 and taking into account the fact that the variables are centered together with the inequality (3.11), we obtain, proceeding as in the proof of Proposition 1, that for any real $t$,

$$
\begin{aligned}
& \mid \sum_{j=1}^{m_{n}} \log \mathbb{E} \exp \left(t S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right) / \sqrt{a_{n} \sigma_{n}^{2}}\right) \\
& \quad-\sum_{j=1}^{m_{n}} \sum_{i=1}^{2^{\ell_{n}}} \log \mathbb{E} \exp \left(t S^{\prime}\left(I_{\ell_{n}, i}\left(p_{n}, j\right)\right) / \sqrt{a_{n} \sigma_{n}^{2}}\right) \mid \\
& \leq \frac{a|t| m_{n} p_{n}}{\sqrt{a_{n} \sigma_{n}^{2}}}\left(\exp \left(-c \frac{c_{0}^{\gamma_{1}}}{4} \frac{p_{n}^{\gamma_{1}}}{2^{\ell_{n} \gamma_{1}}}+2 \frac{|t|}{\sqrt{\varepsilon_{n} a_{n} \sigma_{n}^{2}}} \frac{p_{n}}{2^{\gamma \ell_{n}}}\right)\right. \\
& \left.\quad+\sum_{j=0}^{k_{\ell_{n}}} \exp \left(-c \frac{c_{0}^{\gamma_{1}}}{4} \frac{p_{n}^{\gamma_{1}}}{2^{j \gamma_{1} / \gamma}}+2 \frac{|t|}{\sqrt{\varepsilon_{n} a_{n} \sigma_{n}^{2}}} \frac{p_{n}}{2^{j}}\right)\right),
\end{aligned}
$$

where $k_{\ell_{n}}=\sup \left\{j \in \mathbb{N}, j / \gamma<\ell_{n}\right\}$. By the selection of $p_{n}$ and $\ell_{n}$, since $\lim \inf _{n \rightarrow \infty}$ $\sigma_{n}^{2} / n>0$ and $\varepsilon_{n}^{2} a_{n} n^{\gamma /(2-\gamma)} \rightarrow \infty$, we get that $2^{-\gamma_{1} \ell_{n}} 2^{\gamma \ell_{n}} p_{n}^{\gamma_{1}-1} \sqrt{\varepsilon_{n} a_{n} \sigma_{n}^{2}}$ tends to $\infty$ as $n$ goes to $\infty$. Consequently for $n$ large enough, there exist positive constants $K_{1}$ and $K_{2}$ depending on $c, \gamma$ and $\gamma_{1}$ such that

$$
\begin{align*}
& a_{n} \mid \sum_{j=1}^{m_{n}} \log \mathbb{E} \exp \left(t S^{\prime}\left(K_{I(n, j)}^{\left(\ell_{n}\right)}\right) / \sqrt{a_{n} \sigma_{n}^{2}}\right) \\
& \quad-\sum_{j=1}^{m_{n}} \sum_{i=1}^{2^{\ell_{n}}} \log \mathbb{E} \exp \left(t S^{\prime}\left(I_{\ell_{n}, i}\left(p_{n}, j\right)\right) / \sqrt{a_{n} \sigma_{n}^{2}}\right) \mid \\
& \leq a K_{1}|t| \sqrt{a_{n} n} \log (n) \exp \left(-K_{2}\left(\varepsilon_{n}^{2} a_{n} n^{\gamma /(2-\gamma)}\right)^{\gamma / 2} n^{\gamma(1-\gamma) /(2-\gamma)}\right), \tag{3.53}
\end{align*}
$$

which converges to zero by the selection of $\varepsilon_{n}$.
Hence (3.52) will hold if we prove that for any real $t$

$$
\begin{equation*}
a_{n} \sum_{j=1}^{m_{n}} \sum_{k=1}^{2^{\ell_{n}}} \log \mathbb{E} \exp \left(t S^{\prime}\left(I_{\ell_{n}, i}\left(p_{n}, j\right) / \sqrt{a_{n} \sigma_{n}^{2}}\right) \rightarrow \frac{t^{2}}{2} \text { as } n \rightarrow \infty\right. \tag{3.54}
\end{equation*}
$$

With this aim, we first notice that, by the selection of $\ell_{n}$ and the fact that $\varepsilon_{n} \rightarrow 0$,

$$
\begin{equation*}
\| S^{\prime}\left(I_{\ell_{n}, i}\left(p_{n}, j\right) \|_{\infty} \leq 2 T_{n} 2^{-\ell_{n}} p_{n}=o\left(\sqrt{n a_{n}}\right)=o\left(\sqrt{\sigma_{n}^{2} a_{n}}\right)\right. \tag{3.55}
\end{equation*}
$$

In addition, since $\lim _{n} V_{T_{n}}^{\prime \prime}=0$ and the fact that $\liminf _{n} \sigma_{n}^{2} / n>0$, we have $\lim _{n} \sigma_{n}^{-2} \operatorname{Var} S_{n}^{\prime}=1$. Notice that by (3.49) and the fact that $r_{n}=o(n)$,

$$
\operatorname{Var} S_{n}^{\prime}=\mathbb{E}\left(\sum_{j=1}^{m_{n}} \sum_{i=1}^{2^{\ell_{n}}} S^{\prime}\left(I_{\ell_{n}, i}\left(p_{n}, j\right)\right)\right)^{2}+o(n) \text { as } n \rightarrow \infty
$$

Also, straightforward computations as in the proof of Remark 3 show that under (2.6) and (2.7),
$\mathbb{E}\left(\sum_{j=1}^{m_{n}} \sum_{i=1}^{2^{\ell_{n}}}\left(S^{\prime}\left(I_{\ell_{n}, i}\left(p_{n}, j\right)\right)\right)\right)^{2}=\sum_{j=1}^{m_{n}} \sum_{i=1}^{2^{\ell_{n}}} \mathbb{E}\left(S^{\prime 2}\left(I_{\ell_{n}, i}\left(p_{n}, j\right)\right)\right)+o(n)$ as $n \rightarrow \infty$.
Hence

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left(\sigma_{n}\right)^{-2} \sum_{j=1}^{m_{n}} \sum_{i=1}^{2^{\ell_{n}}} \mathbb{E}\left(S^{\prime 2}\left(I_{\ell_{n}, i}\left(p_{n}, j\right)\right)\right)=1 \tag{3.56}
\end{equation*}
$$

Consequently (3.54) holds by taking into account (3.55) and (3.56) and by using Lemma 2.3 in Arcones [3].

## 4 Appendix

### 4.1 Technical lemmas

We first give the following decoupling inequality.
Lemma 2 Let $Y_{1}, \ldots, Y_{p}$ be real-valued random variables each a.s. bounded by $M$. For every $i \in[1, p]$, let $\mathcal{M}_{i}=\sigma\left(Y_{1}, \ldots, Y_{i}\right)$ and for $i \geq 2$, let $Y_{i}^{*}$ be a random variable independent of $\mathcal{M}_{i-1}$ and distributed as $Y_{i}$. Then for any real $t$,

$$
\left|\mathbb{E} \exp \left(t \sum_{i=1}^{p} Y_{i}\right)-\prod_{i=1}^{p} \mathbb{E} \exp \left(t Y_{i}\right)\right| \leq|t| \exp (|t| M p) \sum_{i=2}^{p} \mathbb{E}\left|Y_{i}-Y_{i}^{*}\right| .
$$

In particular, we have for any real $t$,

$$
\left|\mathbb{E} \exp \left(t \sum_{i=1}^{p} Y_{i}\right)-\prod_{i=1}^{p} \mathbb{E} \exp \left(t Y_{i}\right)\right| \leq|t| \exp (|t| M p) \sum_{i=2}^{p} \tau\left(\sigma\left(Y_{1}, \ldots, Y_{i-1}\right), Y_{i}\right)
$$

where $\tau$ is defined by (2.2).

Proof of Lemma 2 Set $U_{k}=Y_{1}+Y_{2}+\cdots+Y_{k}$. We first notice that

$$
\begin{equation*}
\mathbb{E}\left(e^{t U_{p}}\right)-\prod_{i=1}^{p} \mathbb{E}\left(e^{t Y_{i}}\right)=\sum_{k=2}^{p}\left(\mathbb{E}\left(e^{t U_{k}}\right)-\mathbb{E}\left(e^{t U_{k-1}}\right) \mathbb{E}\left(e^{t Y_{k}}\right)\right) \prod_{i=k+1}^{p} \mathbb{E}\left(e^{t Y_{i}}\right) \tag{4.1}
\end{equation*}
$$

with the convention that the product from $p+1$ to $p$ has value 1 . Now

$$
\begin{aligned}
& \left|\mathbb{E} \exp \left(t U_{k}\right)-\mathbb{E} \exp \left(t U_{k-1}\right) \mathbb{E} \exp \left(t Y_{k}\right)\right| \\
& \quad \leq\left\|\exp \left(t U_{k-1}\right)\right\|_{\infty}\left\|\mathbb{E}\left(e^{t Y_{k}}-e^{t Y_{k}^{*}} \mid \mathcal{M}_{k-1}\right)\right\|_{1} .
\end{aligned}
$$

Using (3.12) we then derive that

$$
\begin{equation*}
\left|\mathbb{E} \exp \left(t U_{k}\right)-\mathbb{E} \exp \left(t U_{k-1}\right) \mathbb{E} \exp \left(t Y_{k}\right)\right| \leq|t| \exp (|t| k M)\left\|Y_{k}-Y_{k}^{*}\right\|_{1} \tag{4.2}
\end{equation*}
$$

Since the variables are bounded by $M$, starting from (4.1) and using (4.2), the result follows.

One of the tools we use repeatedly is the technical lemma below, which provides bounds for the log-Laplace transform of any sum of real-valued random variables.

Lemma 3 Let $Z_{0}, Z_{1}, \ldots$ be a sequence of real valued random variables. Assume that there exist positive constants $\sigma_{0}, \sigma_{1}, \ldots$ and $c_{0}, c_{1}, \ldots$ such that, for any positive $i$ and any $t$ in $\left[0,1 / c_{i}[\right.$,

$$
\log \mathbb{E} \exp \left(t Z_{i}\right) \leq\left(\sigma_{i} t\right)^{2} /\left(1-c_{i} t\right)
$$

Then, for any positive $n$ and any $t$ in $\left[0,1 /\left(c_{0}+c_{1}+\cdots+c_{n}\right)[\right.$,

$$
\log \mathbb{E} \exp \left(t\left(Z_{0}+Z_{1}+\cdots+Z_{n}\right)\right) \leq(\sigma t)^{2} /(1-C t)
$$

where $\sigma=\sigma_{0}+\sigma_{1}+\cdots+\sigma_{n}$ and $C=c_{0}+c_{1}+\cdots+c_{n}$.
Proof of Lemma 3 Lemma 3 follows from the case $n=1$ by induction on $n$. Let $L$ be the $\log$-Laplace of $Z_{0}+Z_{1}$. Define the functions $\gamma_{i}$ by

$$
\gamma_{i}(t)=\left(\sigma_{i} t\right)^{2} /\left(1-c_{i} t\right) \text { for } t \in\left[0,1 / c_{i}\left[\text { and } \gamma_{i}(t)=+\infty \text { for } t \geq 1 / c_{i}\right.\right.
$$

For $u$ in $] 0$, $1\left[\right.$, let $\gamma_{u}(t)=u \gamma_{1}(t / u)+(1-u) \gamma_{0}(t /(1-u))$. From the Hölder inequality applied with $p=1 / u$ and $q=1 /(1-u)$, we get that $L(t) \leq \gamma_{u}(t)$ for any nonnegative $t$. Now, for $t$ in $\left[0,1 / C\left[\right.\right.$, choose $u=\left(\sigma_{1} / \sigma\right)(1-C t)+c_{1} t$ (here $C=c_{0}+c_{1}$ and $\left.\sigma=\sigma_{0}+\sigma_{1}\right)$. With this choice $1-u=\left(\sigma_{0} / \sigma\right)(1-C t)+c_{0} t$, so that $u$ belongs to ]0, $1\left[\right.$ and $L(t) \leq \gamma_{u}(t)=(\sigma t)^{2} /(1-C t)$, which completes the proof of Lemma 3.

### 4.2 Proof of Corollary 3

The proof of Corollary 3 is based on the following proposition together with Remark 5.
Proposition 3 Assume that $\left(Y_{i}\right)_{i \in \mathbb{Z}}$ belongs to $\operatorname{ARL}(C, \delta, \eta)$. Then the probability $\mu$ satisfies

$$
\begin{equation*}
\int \exp \left(\omega|x|^{\eta(1-\delta)}\right) \mu(d x)<+\infty \tag{4.3}
\end{equation*}
$$

for any positive $\omega<\lambda(8(1-\delta))^{-\eta} C^{\eta}$.
Furthermore there exist positive constants a and c such that the following bound holds for the $\tau$-mixing coefficient associated to the stationary Markov chain $\left(Y_{i}\right)_{i \in \mathbb{Z}}$ :

$$
\begin{equation*}
\tau(n) \leq a \exp \left(-c n^{\eta(1-\delta) /(\eta(1-\delta)+\delta)}\right) \tag{4.4}
\end{equation*}
$$

Remark 5 Due to the definition of $\tau$, if $h$ is a 1 -Lipschitz function, then the $\tau$-mixing coefficients $\left(\tau_{h(Y)}(n)\right)_{n \geq 1}$ associated to the sequence $\left(h\left(Y_{i}\right)\right)_{i \in \mathbb{Z}}$ also satisfy (4.4). On an other hand, according to Remark 2, (4.3) entails that if $|g(x)| \leq c\left(1+|x|^{\zeta}\right)$ the process $\left(g\left(Y_{i}\right)-\mathbb{E}\left(g\left(Y_{i}\right)\right)\right)_{i \in \mathbb{Z}}$ satisfies (2.7) for some $b>0$ and $\gamma_{2}=\eta(1-\delta) / \zeta$.

Proof of Proposition 3 We leave to the reader the case $\delta=0$, which is easy to treat. Throughout $\delta>0$. We start by proving (4.3). Let $K$ be the transition kernel of the stationary Markov chain $\left(Y_{i}\right)_{i \in \mathbb{Z}}$ belonging to $\operatorname{ARL}(C, \delta, \eta)$. For $n>0$, we write $K^{n} g$ for the function $\int g(y) K^{n}(x, d y)$.

Let $k_{0}=\inf \left\{k \in \mathbb{N}^{*}: k \eta(1-\delta) \geq 1\right\}$. To prove (4.3) it suffices to show that

$$
\begin{equation*}
\sum_{k \geq k_{0}} \frac{\omega^{k}}{k!} \int|x|^{k \eta(1-\delta)} \mu(d x)<+\infty \tag{4.5}
\end{equation*}
$$

Let $S=k \eta(1-\delta)+\delta$ and $V(x)=|x|^{S}$. Arguing as in the proof of Proposition 2 of Dedecker and Rio [15], we get that

$$
\frac{[K V(x)]^{1 / S}}{|x|} \leq 1+\frac{1}{|x|^{\delta}}\left(\frac{C}{(1-\delta)}\left[\frac{1}{|x|^{1-\delta}}-\left(1+\frac{1}{|x|}\right)^{1-\delta}+\frac{(1-\delta)\left\|\varepsilon_{0}\right\|_{S}}{C|x|^{1-\delta}}\right]\right)
$$

Then for $|x| \geq R_{S}=\left(2 C^{-1}(1-\delta)\left\|\varepsilon_{0}\right\|_{S}+2\right)^{1 /(1-\delta)}$,

$$
|x|^{-1}[K V(x)]^{1 / S} \leq 1-C(2-2 \delta)^{-1}|x|^{-\delta} .
$$

On the other hand, for $x \in\left[-R_{S}, R_{S}\right],[K V(x)]^{1 / S} \leq|f(x)|+\left\|\varepsilon_{0}\right\|_{S} \leq R_{S}+\left\|\varepsilon_{0}\right\|_{S}$. Furthermore, $\left\|\varepsilon_{0}\right\|_{S} \leq\left(\delta+(1-\delta)\left\|\varepsilon_{0}\right\|_{S}\right)^{1 /(1-\delta)} \leq R_{S}$. Therefore, for any $S \geq 1$,

$$
K V(x) \leq V(x)-C(2-2 \delta)^{-1}|x|^{S-\delta}+2^{S} R_{S}^{S}
$$

Integrating w.r.t. $\mu$ this inequality gives for $S \geq 1$,

$$
\mu(K V) \leq \mu(V)-C(2-2 \delta)^{-1} \int|x|^{S-\delta} \mu(d x)+2^{S} R_{S}^{S}
$$

Since $\mu(K V)=\mu(V)$, it follows that for $S \geq 1$,

$$
\frac{C}{2(1-\delta)} \int|x|^{S-\delta} \mu(d x) \leq 2^{S} R_{S}^{S} \leq 2^{S /(1-\delta)} R_{S}^{S}
$$

Now

$$
\begin{aligned}
R_{S}^{S} & \leq 4^{S /(1-\delta)}\left(1+\left(\frac{1-\delta}{C}\left\|\varepsilon_{0}\right\|_{S}\right)^{S /(1-\delta)}\right) \\
& \leq 4^{S /(1-\delta)}+\left(\frac{4(1-\delta)}{C}\right)^{S /(1-\delta)} \mathbb{E}\left(\left|\varepsilon_{0}\right|^{S /(1-\delta)}\right)
\end{aligned}
$$

For $k \geq k_{0}$ and $S=k \eta(1-\delta)+\delta$, we get that

$$
\frac{C}{2(1-\delta)} \int|x|^{S-\delta} \mu(d x) \leq 8^{k \eta+\beta}+\left(\frac{8(1-\delta)}{C}\right)^{k \eta+\beta} \mathbb{E}\left(\left|\varepsilon_{0}\right|^{k \eta+\beta}\right),
$$

where $\beta=\delta /(1-\delta)$. Hence

$$
\begin{aligned}
& \frac{C}{2(1-\delta)} \sum_{k \geq k_{0}} \frac{\omega^{k}}{k!} \int|x|^{k \eta(1-\delta)} \mu(d x) \\
& \leq 8^{\beta} \exp \left(\omega 8^{\eta}\right)+\left(8 C^{-1}(1-\delta)\right)^{\beta} \mathbb{E}\left(\left|\varepsilon_{0}\right|^{\beta} \exp \left(\omega\left(8 C^{-1}(1-\delta)\right)^{\eta}\left|\varepsilon_{0}\right|^{\eta}\right)\right)<\infty
\end{aligned}
$$

provided that $\omega<\lambda(8(1-\delta))^{-\eta} C^{\eta}$. Consequently (4.5) holds and so does (4.3).
We turn now to the proof of (4.4). We denote by $\left(Y_{n}^{x}\right)_{n \geq 0}$ the chain starting from $Y_{0}=x$. According to inequality (3.5) in Dedecker and Prieur [14], for $j_{k}>\cdots>$ $j_{1}>0$,

$$
\begin{equation*}
\tau\left(\sigma\left(Y_{0}\right),\left(Y_{j_{1}}, \ldots, Y_{j_{k}}\right)\right) \leq \sum_{l=1}^{k} \iint\left\|Y_{j_{l}}^{x}-Y_{j_{l}}^{y}\right\|_{1} \mu(d x) \mu(d y) . \tag{4.6}
\end{equation*}
$$

Now we use the same scheme of proof than the one of Proposition 3 in Dedecker and Rio [15]. Since $\left|Y_{n}^{x}-Y_{n}^{y}\right|=\left|f\left(Y_{n-1}^{x}\right)-f\left(Y_{n-1}^{y}\right)\right|$, we have

$$
\begin{equation*}
\left|Y_{n}^{x}-Y_{n}^{y}\right| \leq\left(1-\frac{C}{\left(1+\max \left(\left|Y_{n-1}^{x}\right|,\left|Y_{n-1}^{y}\right|\right)\right)^{\delta}}\right)\left|Y_{n-1}^{x}-Y_{n-1}^{y}\right| \tag{4.7}
\end{equation*}
$$

Set $\alpha(t)=1-C(1+t)^{-\delta}$ and $\Sigma_{k}=\left|\varepsilon_{1}\right|+\cdots+\left|\varepsilon_{k}\right|$. Noting that $\max \left(\left|Y_{n-1}^{x}\right|,\left|Y_{n-1}^{y}\right|\right) \leq$ $|x|+|y|+\Sigma_{n-1}$, and iterating (4.7) $n$ times, we get

$$
\left|Y_{n}^{x}-Y_{n}^{y}\right| \leq \alpha^{n}\left(|x|+|y|+\Sigma_{n-1}\right)|x-y| .
$$

Then setting $I_{n}(x, y):=\mathbb{E}\left(\alpha^{n}\left(|x|+|y|+\Sigma_{n-1}\right)\right)$, we get

$$
\begin{equation*}
\tau\left(\sigma\left(Y_{0}\right),\left(Y_{j_{1}}, \ldots, Y_{j_{k}}\right)\right) \leq \sum_{l=1}^{k} \iint I_{j_{l}}(x, y) \mu(d x) \mu(d y) . \tag{4.8}
\end{equation*}
$$

Setting $\Gamma_{n-1}=\Sigma_{n-1}-(n-1) \mathbb{E}\left|\varepsilon_{0}\right|$, we first write:

$$
I_{n}(x, y)=n \int_{0}^{1} \mathbb{P}\left(\Gamma_{n-1}>[C / u]^{1 / \delta}-\left[1+|x|+|y|+(n-1) \mathbb{E}\left|\varepsilon_{0}\right|\right]\right)(1-u)^{n-1} d u
$$

Set $A_{n}(x, y)=C\left[2\left(1+|x|+|y|+(n-1) \mathbb{E}\left|\varepsilon_{0}\right|\right)\right]^{-\delta}$. We have

$$
\begin{align*}
I_{n}(x, y) & \leq\left(1-A_{n}(x, y)\right)^{n}+n \int_{0}^{A_{n}(x, y)} \mathbb{P}\left(2 \Gamma_{n-1}>[C / u]^{1 / \delta}\right)(1-u)^{n-1} d u \\
& :=I_{n}^{(1)}(x, y)+I_{n}^{(2)}(x, y) \tag{4.9}
\end{align*}
$$

We first control $\iint I_{n}^{(1)}(x, y)|x-y| \mu(d x) \mu(d y)$. Set $B_{n}(x)=C[4(1+|x|+(n-1)$ $\left.\left.\mathbb{E}\left|\varepsilon_{0}\right|\right)\right]^{-\delta}$. Then we have

$$
\begin{aligned}
\int I_{n}^{(1)}(x, y)|x-y| \mu(d y) \leq & \left(1-B_{n}(x)\right)^{n} \int|x-y| \mu(d y) \\
& +n \int_{0}^{B_{n}(x)}\left[\int|x-y| \mathbb{1}_{\left(4|y|>[C / u]^{1 / \delta}\right)} \mu(d y)\right](1-u)^{n-1} d u .
\end{aligned}
$$

Let $0<\omega<\lambda(8(1-\delta))^{-\eta} C^{\eta}$. Since by (4.3), $x \rightarrow \exp \left(\omega|x|^{\eta(1-\delta)}\right)$ belongs to $\mathbb{L}^{1}(\mu)$, we have the finite upper bound

$$
\begin{aligned}
& \int|x-y| \mathbb{1}_{\left(4|y|>[C / u]^{1 / \delta}\right)} \mu(d y) \\
& \leq \exp \left(-\frac{\omega}{2}\left(\frac{C}{4^{\delta} u}\right)^{\eta(1-\delta) / \delta}\right) \int|x-y| \exp \left(\frac{\omega}{2}|y|^{\eta(1-\delta)}\right) \mu(d y)
\end{aligned}
$$

Consequently there exists $c>0$ such that, for any positive $n$,

$$
\begin{align*}
& \int I_{n}^{(1)}(x, y)|x-y| \mu(d y) \leq\left(1-B_{n}(x)\right)^{n} \int|x-y| \mu(d y) \\
& +\exp \left(-c n^{\eta(1-\delta) /(\eta(1-\delta)+\delta)}\right) \int|x-y| \exp \left(\frac{\omega}{2}|y|^{\eta(1-\delta)}\right) \mu(d y) \tag{4.10}
\end{align*}
$$

Notice that that $\left(1-B_{n}(x)\right)^{n} \leq \exp \left(-n B_{n}(x)\right)$. If $|x| \leq n\left\|\varepsilon_{0}\right\|_{1}$,

$$
\exp \left(-n B_{n}(x)\right) \leq \exp \left(-C n\left(4+8 n\left\|\varepsilon_{0}\right\|_{1}\right)^{-\delta}\right)
$$

Therefore there exists $c>0$, such that

$$
\begin{align*}
& \int_{0}^{n\left\|\varepsilon_{0}\right\|_{1}}\left(1-B_{n}(x)\right)^{n} \int|x-y| \mu(d y) \mu(d x) \\
& \quad \leq \exp \left(-c n^{(1-\delta)}\right) \int|x-y| \mu(d y) \mu(d x) . \tag{4.11}
\end{align*}
$$

On the other hand, using again the fact that $x \rightarrow \exp \left(\omega|x|^{\eta(1-\delta)}\right)$ belongs to $\mathbb{L}^{1}(\mu)$, we have the finite upper bound

$$
\begin{align*}
& \quad \int_{|x|>n\left\|\varepsilon_{0}\right\|_{1}}|x|\left(1-B_{n}(x)\right)^{n} \mu(d x) \\
& \leq \int_{n\left\|\varepsilon_{0}\right\|_{1}}^{\infty}|x| \exp \left(\frac{\omega}{2}|x|^{\eta(1-\delta)}\right) \exp \left(-\frac{\omega}{2}|x|^{\eta(1-\delta)}-C n(1+2|x|)^{-\delta}\right) \mu(d x) \\
& \leq 2 \exp \left(-c n^{\eta(1-\delta) /(\eta(1-\delta)+\delta)}\right) \int_{0}^{\infty}|x| \exp \left(\frac{\omega}{2}|x|^{\eta(1-\delta)}\right) \mu(d x) \tag{4.12}
\end{align*}
$$

where $c$ is a positive constant. Starting from (4.10) and using (4.11) and (4.12), we derive that there exist positive constants $c$ and $K$ such that

$$
\begin{equation*}
\iint I_{n}^{(1)}(x, y)|x-y| \mu(d y) \mu(d x) \leq K \exp \left(-c n^{\eta(1-\delta) /(\eta(1-\delta)+\delta)}\right) \tag{4.13}
\end{equation*}
$$

We control now $\iint I_{n}^{(2)}(x, y)|x-y| \mu(d x) \mu(d y)$. Using Borovkov inequality (1.4), there exist positive constants $c_{1}$ and $c_{2}$ depending on $C, \delta$ and $\eta$ such that

$$
\mathbb{P}\left(2 \Gamma_{n-1}>[C / u]^{1 / \delta}\right) \leq \exp \left(-c_{1} /\left(n u^{2 / \delta}\right)\right)+n \exp \left(-c_{2}(1 / u)^{\eta / \delta}\right)
$$

Consequently, there exists a positive constant $c_{3}>0$ such that

$$
\int_{0}^{1} \mathbb{P}\left(2 \Gamma_{n-1}>[C / u]^{1 / \delta}\right)(1-u)^{n-1} d u \leq 2 \exp \left(-c_{3} n^{\min \left(\frac{\eta}{n+\delta}, \frac{2-\delta}{2+\delta}\right)}\right)
$$

Since $\eta \in[0,1]$ and $\delta>0, \min \left(\frac{\eta}{\eta+\delta}, \frac{2-\delta}{2+\delta}\right) \geq \eta(1-\delta) /(\eta(1-\delta)+\delta)$. Then there exists $c>0$ such that

$$
\begin{align*}
& \iint I_{n}^{(2)}(x, y)|x-y| \mu(d y) \mu(d x) \\
& \leq \exp \left(-c n^{\eta(1-\delta) /(\eta(1-\delta)+\delta)}\right) \int|x-y| \mu(d y) \mu(d x) \tag{4.14}
\end{align*}
$$

Starting from (4.8) and combining (4.9) and (4.13) and (4.14), we get that there exist positive constants $a$ and $c$ such that

$$
\tau\left(\sigma\left(Y_{0}\right),\left(Y_{j_{1}}, \ldots, Y_{j_{k}}\right)\right) \leq a k \exp \left(-c j_{1}^{\eta(1-\delta) /(\eta(1-\delta)+\delta)}\right)
$$

which combined with the definition (2.3) ends the proof of (4.4).

Acknowledgments The authors are grateful to the referee and the associate editor for carefully reading the paper and for numerous suggestions that improved its presentation.

## References

1. Adamczak, R.: A tail inequality for suprema of unbounded empirical processes with applications to Markov chains. Electron. J. Probab. 13, 1000-1034 (2008)
2. Ango-Nzé, P.: Critères d'ergodicité de modèles markoviens. Estimation non paramétrique des hypothèses de dépendance. Thèse de doctorat d'université, Université Paris 9, Dauphine (1994)
3. Arcones, M.A.: Moderate deviations of empirical processes. Stochastic inequalities and applications. Progr. Probab. 56, 189-212. Birkhäuser, Basel (2003)
4. Bass, R.: Law of the iterated logarithm for set-indexed partial sum processes with finite variance. Z. Wahrsch. Verw. Gebiete 70, 591-608 (1985)
5. Bertail, P., Clémençon, S.: Sharp bounds for the tails of functionals of Markov chains. Teor. Veroyatnost. i Primenen. 54, 609-619 (2009)
6. Borovkov, A.A.: Estimates for the distribution of sums and maxima of sums of random variables when the Cramér condition is not satisfied. Sib. Math. J. 41(5), 811-848 (2000)
7. Borovkov, A.A.: Probabilities of large deviations for random walks with semi-exponential distributions. Sib. Math. J. 41(6), 1061-1093 (2000)
8. Bradley, R.C.: On quantiles and the central limit question for strongly mixing sequences. J. Theor. Probab. 10, 507-555 (1997)
9. Bryc, W., Dembo, A.: Large deviations and strong mixing. Ann. Inst. Henri Poincaré 32, 549569 (1996)
10. Chen, X., de Acosta, A.: Moderate deviations for empirical measures of Markov chains: upper bounds. J. Theor. Probab. 11(4), 1075-1110 (1998)
11. de Acosta, A.: Moderate deviations for empirical measures of Markov chains: lower bounds. Ann. Probab. 25(1), 259-284 (1997)
12. Dedecker, J., Doukhan, P.: A new covariance inequality and applications. Stoch. Process. Appl. 106, 63-80 (2003)
13. Dedecker, J., Merlevède, F.: Inequalities for partial sums of Hilbert-valued dependent sequences and applications. Math. Methods Stat. 15, 176-206 (2006)
14. Dedecker, J., Prieur, C.: Coupling for tau-dependent sequences and applications. J. Theor. Probab. 17, 861-885 (2004)
15. Dedecker, J., Rio, E.: On the functional central limit theorem for stationary processes. Ann. Inst. Henri Poincaré, Prob. Stat. 36, 1-34 (2000)
16. Delyon, B.: Limit theorems for mixing processes. Technical Report IRISA, Rennes 1, 546 (1990)
17. Dembo, A., Zeitouni, O.: Large Deviations Techniques and Applications, 2nd edn. Springer, New York (1998)
18. Diaconis, P., Freedman, D.: Iterated random functions. SIAM Rev. 41(1), 45-76 (1999)
19. Djellout, H., Guillin, A.: Moderate deviations for Markov chains with atom. Stoch. Process. Appl. 95, 203-217 (2001)
20. Douc, R., Fort, G., Moulines, E., Soulier, P.: Practical drift conditions for subgeometric rates of convergence. Ann. Appl. Probab. 14, 1353-1377 (2004)
21. Douc, R., Guillin, A., Moulines, E.: Bounds on regeneration times and limit theorems for subgeometric Markov chains. Ann. Inst. Henri Poincaré Prob. Stat. 44, 239-257 (2008)
22. Duflo, M.: Random Iterative Models. Applications of Mathematics. Springer, Berlin (1996)
23. Liu, Q., Watbled, F.: Exponential inequalities for martingales and asymptotic properties of the free energy of directed polymers in a random environment. Stoch. Process. Appl. 119, 3101-3132 (2009)
24. Merlevède, F., Peligrad, M., Rio, E.: Bernstein inequality and moderate deviations under strong mixing conditions. IMS Collections. High Dimensional Probability V, 273-292 (2009)
25. Rio, E.: Théorie asymptotique des processus aléatoires faiblement dépendants. Mathématiques et Applications, vol. 31. Springer, Berlin (2000)
26. Rosenblatt, M.: A central limit theorem and a strong mixing condition. Proc. Natl. Acad. Sci. USA 42, 43-47 (1956)
27. Rozanov, Y.A., Volkonskii, V.A.: Some limit theorems for random functions I. Theory Probab. Appl. 4, 178-197 (1959)
28. Tuominen, P., Tweedie, R.L.: Subgeometric rates of convergence of $f$-ergodic Markov chains. Adv. Appl. Probab. 26, 775-798 (1994)
29. Viennet, G.: Inequalities for absolutely regular sequences: application to density estimation. Probab. Theory Relat. Fields 4, 467-492 (1997)

[^0]:    M. Peligrad is supported in part by a Charles Phelps Taft Memorial Fund grant, and NSA grants, H98230-07-1-0016 and H98230-09-1-0005. E. Rio is supported in part by Centre INRIA Bordeaux Sud-Ouest \& Institut de Mathématiques de Bordeaux.
    F. Merlevède

    Laboratoire de mathématiques, Université Paris Est, UMR 8050 CNRS, Bâtiment Copernic, 5 Boulevard Descartes, 77435 Champs-Sur-Marne, France
    e-mail: florence.merlevede@univ-mlv.fr
    M. Peligrad

    Department of Mathematical Sciences, University of Cincinnati, P.O. Box 210025, Cincinnati, OH 45221-0025, USA
    e-mail: magda.peligrad@uc.edu
    E. Rio ( $\triangle$ )

    Laboratoire de mathématiques, Université de Versailles, UMR 8100 CNRS, Bâtiment Fermat, 45 Avenue des Etats-Unis, 78035 Versailles, France
    e-mail: rio@ math.uvsq.fr

