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On conditional McKean Lagrangian stochastic models

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Abstract This paper is motivated by a new class of SDEs–PDEs systems, the so called Lagrangian stochastic models which are commonly used in the simulation of turbulent flows. We study a position–velocity system which is nonlinear in the sense of McKean. As the dynamics of the velocity depends on the conditional expectation with respect to its position, the interaction kernel is singular. We prove existence and uniqueness of the solution to the system by solving a nonlinear martingale problem and showing that the corresponding interacting particle system propagates chaos.

Keywords Lagrangian stochastic model · Conditional McKean nonlinearity

Mathematics Subject Classification (2000) 60H10 · 60K35 · 65C35

1 Introduction

In this paper, we prove the well-posedness of a simplified Lagrangian stochastic model describing the time evolution of the position and velocity of a fluid-particle, and we construct an interacting particle approximation of the model. More precisely, given a finite horizon time $T > 0$, we consider a d-dimensional standard Brownian motion $(W_t;\ t \in [0, T])$, and a $\mathbb{R}^{2d}$-valued r.v. $(X_0, \mathcal{U}_0)$ independent of W. We aim to prove

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that there exists a unique solution $((X_t, \mathcal{U}_t);\ t \in [0, T])$ to the nonlinear McKean system

$$\begin{cases} X_t = X_0 + \int_0^t \mathcal{U}_s\, ds, \\ \mathcal{U}_t = \mathcal{U}_0 + \int_0^t B\left[X_s, \mathcal{U}_s; \rho_s\right] ds + \int_0^t \sigma(s, X_s, \mathcal{U}_s)\, dW_s, \\ \rho_t \text{ is the density distribution of } (X_t, \mathcal{U}_t) \quad \text{for all } t \in (0, T]. \end{cases} \tag{1.1}$$

Here, B is the mapping from $\mathbb{R}^d \times \mathbb{R}^d \times L^1(\mathbb{R}^{2d})$ to $\mathbb{R}^d$ defined by

$$B\left[x, u; \gamma\right] = \begin{cases} \dfrac{\int_{\mathbb{R}^d} b(v, u)\gamma(x, v)\, dv}{\int_{\mathbb{R}^d} \gamma(x, v)\, dv} & \text{if } \int_{\mathbb{R}^d} \gamma(x, v)\, dv \neq 0, \\ 0 \quad \text{elsewhere}, \end{cases} \tag{1.2}$$

where $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is a bounded interaction kernel. Formally, the drift component of (1.1) involves the function

$$(x, u) \ \mapsto \ \mathbb{E}\left[b\left(\mathcal{U}_t, u\right) \Big/ X_t = x\right]. \tag{1.3}$$

Such nonlinearity is typical of Lagrangian stochastic models which describe characteristics, including positions X_t and velocities $\mathcal{U}_t$, of fluid particles in a turbulent flow. Although simple, the model (1.1) actually inherits two important features of such Lagrangian stochastic models. First, due to the Langevin dynamics, the infinitesimal generator of the solution is not uniformly elliptic. Second, the drift coefficient of the velocity involves a conditional expectation w.r.t. the particle position. Because of these two features of the model, existence and uniqueness of the solution to the non classical nonlinear McKean equation (1.1) require a careful analysis.

We emphasize that our result is a first step in the analysis of Lagrangian stochastic models for the simulation of turbulent flows and the related probability density function (PDF) methods. These models and numerical methods actually have a dramatic complexity (see Sect. 2), which is not astonishing since they aim to be alternative approaches to Navier–Stokes equations for turbulent flows. Several recent works separately face some of the difficulties. For example, Bossy, Fontbona and Jabir study the Poisson partial differential equation (PDE) (2.2) and its relation with the incompressibility of the mean field velocity; Bossy and Jabir study (1.1) with a specular reflection boundary condition.

The paper is organized as follows. In Sect. 2, we present the Lagrangian stochastic models in turbulent fluid dynamics, and list some references on the models and their numerical issues. In Sect. 3, we state our main results. In Sect. 4, we prove that the system (1.1) has at most one weak solution, in the sense that a suitable nonlinear martingale problem has at most one solution. In Sect. 5, we exhibit a solution to the nonlinear martingale problem by studying the limit of solutions to smoothed systems (see Theorem 3.2). The existence of solutions to the smoothed systems is obtained by proving that corresponding interacting particle systems propagate chaos.

2 A brief description of Lagrangian stochastic models for turbulent flows

We start this section with a short reminder on the notion of statistical solutions to the Navier–Stokes equations for turbulent flows. For the sake of simplicity, we limit our presentation to monophasic flows. These statistical solutions are random fields, the velocity and the pressure, which are decomposed into their averaged and fluctuating parts. The so called Reynolds decomposition of the Eulerian velocity U is

$$U(t,x,\omega) = \langle U\rangle(t,x) + \mathbf{u}(t,x,\omega),$$

where $\langle U\rangle$ is the (ensemble) averaged part, and $\mathbf{u}$ is the fluctuating part. The Reynolds average $\langle\ \rangle$ is a linear operator applied to the random fields, which is assumed to commute with spatial and times derivatives. Formally applying the Reynolds average to the Navier–Stokes equations, one obtains the so called Reynolds Averaged Navier–Stokes (RANS) equations:

$$\begin{cases} \nabla_x \cdot \langle U\rangle = 0, \\ \partial_t \langle U^{(i)}\rangle + \langle U\rangle \cdot \nabla_x \langle U^{(i)}\rangle = -\dfrac{1}{\varrho}\nabla_x \langle \mathcal{P}^{(i)}\rangle + \nu \triangle_x \langle U^{(i)}\rangle - \partial_{x_j}\langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\rangle, \\ \langle U\rangle(0,x) = \langle U_0\rangle(x). \end{cases} \tag{2.1}$$

The averaged pressure $\langle\mathcal{P}\rangle$ solves the Poisson equation

$$-\frac{1}{\varrho}\triangle_x\langle\mathcal{P}\rangle = \partial_{x_i x_j}\langle U^{(i)}\rangle\langle U^{(j)}\rangle + \partial_{x_i x_j}\langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\rangle. \tag{2.2}$$

The Reynolds stress tensor stands for the covariance of velocity components:

$$\langle \mathbf{u}^{(i)}\mathbf{u}^{(j)}\rangle = \langle U^{(i)}U^{(j)}\rangle - \langle U^{(i)}\rangle\langle U^{(j)}\rangle.$$

These terms are not closed in Eq. (2.1). This problem has led to the introduction of closure models based on Kolmogorov's theory for turbulent flows and experimental observations. For example, the $k-\mathscr{E}$ closures consist in a set of closed equations for the turbulent kinetic k and the dissipation rate $\mathscr{E}$ defined as

$$\begin{aligned} k(t,x) &= \frac{1}{2}\langle \mathbf{u}^{(i)}\mathbf{u}^{(i)}\rangle(t,x), \\ \mathscr{E}(t,x) &= \nu\langle \partial_{x_j}\mathbf{u}^{(i)}\partial_{x_j}\mathbf{u}^{(i)}\rangle(t,x), \end{aligned}$$

(see, e.g., [11,14]).

Lagrangian stochastic models have been successfully proposed to provide an alternative approach to the numerical resolution of RANS equations combined with closure models to simulate complex flows for which PDE solvers are inefficient.

In a series of papers initiated in 1980s, Stephen B. Pope has proposed Lagrangian stochastic models to describe the position and the instantaneous velocity $(X_t, \mathcal{U}_t)$ of a fluid-particle. Depending on the flow, other Lagrangian characteristics of the turbulence are added to the model. For a fluid with constant mass density ϱ, Lagrangian and Eulerian quantities are related as follows: for all suitable measurable function $g : \mathbb{R}^d \to \mathbb{R}^d$,

$$\langle g(U)\rangle(t,x) = \mathbb{E}\left[g(\mathcal{U}_t)/X_t = x\right].$$

Assuming that $(X, \mathcal{U})$ is a diffusion process, the coefficients of its generator are designed such that the Lagrangian laws are consistent with closed RANS equations and other relevant physical laws in turbulence theory (see [13,14] for details). This methodology is known as *PDF method for turbulent flows* in the literature. The simplest model proposed by Pope is the so called simplified Langevin model (see [14]):

$$\begin{cases} X_t = X_0 + \int_0^t \mathcal{U}_s \, ds, \\ \mathcal{U}_t = \mathcal{U}_0 - \dfrac{1}{\varrho} \int_0^t \nabla_x \langle \mathcal{P}\rangle(s, X_s)\, ds + \nu \int_0^t \triangle_x \langle U\rangle(s, X_s)\, ds \\ \qquad + C_1 \int_0^t \dfrac{\mathscr{E}(s, X_s)}{k(s, X_s)} \left(\langle U\rangle(s, X_s) - \mathcal{U}_s\right) \, ds + \int_0^t \sqrt{C_2 \mathscr{E}(s, X_s)}\, dW_s. \end{cases}$$

Here, C_1 and C_2 are positive constants, and $(W_t;\ t \geq 0)$ is a standard $\mathbb{R}^3$-valued Brownian motion. The Poisson equation (2.2) provides the averaged pressure $\langle \mathcal{P}\rangle(t,x)$, and k and $\mathcal{E}$ are assumed to be known.

A less elementary model was proposed by Dreeben and Pope [7] where

$$k(t,x) = \mathbb{E}\left((\mathcal{U}_t^{(i)})^2/X_t = x\right) - \left(\mathbb{E}\left(\mathcal{U}_t^{(i)}/X_t = x\right)\right)^2,$$

and $\mathscr{E}$ is defined as $\mathscr{E}(t,x) = \langle\omega\rangle(t,x)k(t,x)$ where $\langle\omega\rangle(t,x) = \mathbb{E}\left(\omega_t/X_t = x\right)$ and $(\omega_t;\ t \geq 0)$ is the solution of the following stochastic differential equation (SDE):

$$\omega_t = \omega_0 + C_3 \int_0^t \langle\omega\rangle(s, X_s)\left(\langle\omega\rangle(s, X_s) - \omega_s\right)\, ds$$
$$- \int_0^t \omega_s S_\omega(s, X_s)\langle\omega\rangle(s, X_s)\, ds + \int_0^t \sqrt{C_4 \omega_s \left(\langle\omega\rangle(s, X_s)\right)^2}\, d\widetilde{W}_s.$$

Here, C_3, C_4 are positive constants, $\widetilde{W}$ is a one-dimensional standard Brownian motion independent of W, and

$$S_\omega(t,x) = C_{\omega 2} + C_{\omega 1} \frac{\left(\langle U^{(i)} U^{(j)}\rangle(t,x) - \langle U^{(i)}\rangle(t,x)\langle U^{(j)}\rangle(t,x)\right) \partial_{x_j} \langle U^{(i)}\rangle(t,x)}{\langle\omega\rangle(t,x)k(t,x)}.$$

A description of the numerical issues can be found in [15]. A recent application to meteorology is developed by one of the authors: see, e.g., [2].

Notation. Let $0 < T < +\infty$ be fixed.

- For all $t \in (0, T]$, we set $Q_t = (0, t) \times \mathbb{R}^{2d}$.
- For all $q \geq 1$, $\mathcal{C}([0, T]; \mathbb{R}^q)$ denotes the space of $\mathbb{R}^q$-valued continuous functions equipped with the uniform metric $\| \|_\infty$.
- Given a metric space E, $\mathcal{C}_b^k(E)$ denotes the set of real-valued bounded functions defined on E with continuous derivatives up to order k; $\mathcal{C}_c^k(E)$ denotes the set of real-valued functions with continuous derivatives up to order k and with compact support.
- Given a metric space E, $\mathcal{M}(E)$ denotes the set of probability measures defined on E, equipped with the weak convergence topology.
- In all the paper, C is a constant which may vary from line to line, but does not depend on the approximation parameters ϵ and N introduced in (3.3).

3 Main result

In the study of (1.1), difficulties arise from the dependency of the drift coefficient on the conditional expectation (1.3). Related situations have been studied by Sznitman [18], Oelschlager [12] and Dermoune [4]. Sznitman [18] has considered the one-dimensional nonlinear SDE

$$d\zeta_t = p_t(\zeta_t)\, dt + dW_t,$$

where p_t is the Lebesgue density of ζ_t. Oelschlager [10] has considered the family of models

$$d\zeta_t = F(\zeta_t, p_t(\zeta_t))\, dt + dW_t,$$

where $F : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^d$ is a bounded Lipschitz function, and

$$d\zeta_t = \nabla p_t(\zeta_t)\, dt + dW_t.$$

Dermoune [4] has studied the system

$$d\zeta_t = \mathbb{E}\left(v(\zeta_0)/\zeta_t\right)\, dt + dW_t,$$

where $v : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded continuous function. Our situation substantially differs from the above: our drift coefficient depends on a conditional density rather than the density and the infinitesimal generator of $(X, \mathcal{U})$ is not strongly elliptic.

In the sequel, we suppose that the following hypotheses hold true:

(H) • b is a bounded continuous function and the law μ_0 of $(X_0, \mathcal{U}_0)$ is such that

$$\int_{\mathbb{R}^{2d}} (|x| + |u|^2)\mu_0(dx, du) < +\infty.$$

- The velocity diffusion coefficient σ is bounded and strongly elliptic: for $a(t,x,u) := \sigma(t,x,u)\sigma^*(t,x,u)$, there exists $\lambda > 0$ such that, for all $t \in (0,T]$, $x,\ u,\ v \in \mathbb{R}^d$,

$$\frac{|v|^2}{\lambda} \leq \sum_{i,j=1}^{d} a^{(i,j)}(t,x,u)v_i v_j \leq \lambda\,|v|^2\,. \tag{3.1}$$

- For all $1 \leq i, j \leq d$, $a^{(i,j)}$ is Hölder continuous in the following sense: there exist $\alpha \in (0,1]$ and K depending only on T and d such that, for all $(s,x,u), (t,y,v) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$,

$$|a^{(i,j)}(t,x,u) - a^{(i,j)}(s,y,v)| \leq K(|t-s|^{\frac{\alpha}{2}} + |x-y-v(t-s)|^{\frac{\alpha}{3}} + |u-v|^{\alpha}). \tag{3.2}$$

Remark 3.1 The hypothesis (3.2) on the matrix a is classical in the literature on ultraparabolic PDEs, see e.g. Theorem A.1 and Sect. A.2 in the Appendix.

For fixed $N \geq 1$ and $\epsilon > 0$, we consider the interacting particle system $\{(X_t^{i,\epsilon,N}, \mathcal{U}_t^{i,\epsilon,N};\ t\in[0,T]);\ 1\leq i\leq N\}$ defined by

$$\begin{cases} X_t^{i,\epsilon,N} &= X_0^i + \int_0^t \mathcal{U}_s^{i,\epsilon,N}\,ds,\\ \mathcal{U}_t^{i,\epsilon,N} &= \mathcal{U}_0^i + \int_0^t \dfrac{\sum_{j=1}^N b(\mathcal{U}_s^{j,\epsilon,N},\mathcal{U}_s^{i,\epsilon,N})\phi_\epsilon(X_s^{i,\epsilon,N}-X_s^{j,\epsilon,N})}{\sum_{j=1}^N\left(\phi_\epsilon(X_s^{i,\epsilon,N}-X_s^{j,\epsilon,N})+\epsilon\right)}\,ds\\ &\quad + \int_0^t \sigma(s,X_s^{i,\epsilon,N},\mathcal{U}_s^{i,\epsilon,N})\,dW_s^i,\ i=1,\ldots,N. \end{cases} \tag{3.3}$$

Here, $\{(X_0^i,\mathcal{U}_0^i),\ (W_t^i;\ t\in[0,T]);\ i\geq 1\}$ are independent copies of $((X_0,\mathcal{U}_0), (W_t;\ t\in[0,T]))$, and $\{\phi_\epsilon;\ \epsilon>0\}$ denotes a family of mollifiers of the type $\phi_\epsilon(x) = \frac{1}{\epsilon^d}\phi(\frac{x}{\epsilon})$, where $\phi\in\mathcal{C}_c^1(\mathbb{R}^d)$ is such that $\phi \geq 0$ and $\int_{\mathbb{R}^d}\phi(z)\,dz = 1$. As the drift coefficient of the particle system (3.3) is uniformly bounded, the well-posedness of (3.3) follows from Proposition 4.4 (see Sect. 4.1) and Girsanov's theorem.

In Sect. 5, we prove that the particles propagate chaos. In particular, as N tends to infinity, $(X^{1,\epsilon,N},\mathcal{U}^{1,\epsilon,N})$ converges weakly to the solution of

$$\begin{cases} X_t^\epsilon = X_0 + \int_0^t \mathcal{U}_s^\epsilon\,ds,\\ \mathcal{U}_t^\epsilon = \mathcal{U}_0 + \int_0^t B_\epsilon\left[X_s^\epsilon,\mathcal{U}_s^\epsilon;\rho_s^\epsilon\right]ds + \int_0^t\sigma(s,X_s^\epsilon,\mathcal{U}_s^\epsilon)\,dW_s,\\ \rho_t^\epsilon \text{ is the density of } (X_t^\epsilon,\mathcal{U}_t^\epsilon) \quad \text{for all } t\in(0,T], \end{cases} \tag{3.4}$$

where the kernel $B_\epsilon\left[x,u;\gamma\right]$ is defined as follows: for all nonnegative $\gamma\in L^1(\mathbb{R}^{2d})$, $(x,u)\in\mathbb{R}^{2d}$,

$$B_\epsilon\left[x,u;\gamma\right] = \frac{\int_{\mathbb{R}^d} b(v,u)\phi_\epsilon\star\gamma(x,v)\,dv}{\int_{\mathbb{R}^d}\phi_\epsilon\star\gamma(x,v)\,dv+\epsilon},$$

where

$$\phi_\epsilon \star \gamma(x,u) = \int_{\mathbb{R}^d} \phi_\epsilon(x-y)\gamma(y,u)\,dy.$$

Our main result is as follows.

Theorem 3.2 *Assume* (**H**).

(i) *For all* $\epsilon > 0$*, the sequence* $\{(X^{1,\epsilon,N}, \mathcal{U}^{1,\epsilon,N});\ N \geq 1\}$ *converges weakly to a weak solution* $(X^\epsilon, \mathcal{U}^\epsilon)$ *of* (3.4)*. This solution is unique and, if* $\mathbb{P}^\epsilon$ *denotes the law of* $(X^\epsilon, \mathcal{U}^\epsilon)$*, the interacting particle system is* $\mathbb{P}^\epsilon$*-chaotic; that is, for every integer* $k \geq 2$ *and every finite family* $\{\psi_l;\ l = 1, \ldots, k\}$ *of* $\mathcal{C}_b(\mathcal{C}([0,T];\mathbb{R}^{2d}))$*,*

$$\langle \mathbb{P}^{\epsilon,N}, \psi_1 \otimes \ldots \psi_k \otimes \ldots \rangle \longrightarrow \prod_{l=1}^{k} \langle \mathbb{P}^\epsilon, \psi_l \rangle, \text{ when } N \longrightarrow +\infty.$$

(ii) *When* ϵ *tends to* 0*,* $(X^\epsilon, \mathcal{U}^\epsilon)$ *converges weakly to the unique solution* $(X, \mathcal{U})$ *of* (1.1)*.*

Weak solutions of (3.4) and (1.1) are defined by appropriate martingale problems in the next section (see Definitions 4.1 and 4.2).

The rest of the paper is organized as follows: in Sect. 4, we prove weak uniqueness results for Eqs. (3.4) and (1.1). In Sect. 5, we get existence: we show that, for all $\epsilon > 0$, the law of the particle system (3.3) converges weakly and we identify the limit as the weak solution of (3.4); we then get existence of a weak solution of (1.1) by letting ϵ decreases to 0.

In all the statements below, we implicitly assume (H) and we do not repeat it.

4 Uniqueness results

We introduce the notions of weak solutions to (1.1) and (3.4). Let $((x_t, u_t);\ t \in [0,T])$ be the canonical processes in the sample space $\mathcal{C}([0,T];\mathbb{R}^{2d})$. The martingale problem related to the smoothed particle system (3.4) is stated as follows.

Definition 4.1 A probability measure $\mathbf{P}^\epsilon$ on the canonical space $\mathcal{C}([0,T];\mathbb{R}^{2d})$ is a weak solution of (3.4), or equivalently, a solution to the martingale problem (MP_ϵ) if

(i) $\mathbf{P}^\epsilon \circ (x_0, u_0)^{-1} = \mu_0$.
(ii) For all $t \in (0,T]$, the time marginal $\mathbf{P}^\epsilon \circ (x_t, u_t)^{-1}$ has a density ρ_t^ϵ with respect to Lebesgue measure on $\mathbb{R}^{2d}$.
(iii) For all $f \in \mathcal{C}_b^2(\mathbb{R}^{2d})$, the process

$$f(x_t, u_t) - f(x_0, u_0) - \int_0^t \mathcal{A}^\epsilon_{\rho_s^\epsilon} f(s, x_s, u_s)\,ds$$

is a $\mathbf{P}^\epsilon$-martingale where, for all $\gamma \in L^1(\mathbb{R}^{2d})$, $\mathcal{A}^\epsilon_\gamma$ is the differential operator

$$\mathcal{A}^\epsilon_\gamma f(t,x,u) := u \cdot \nabla_x f(x,u) + B_\epsilon\left[x,u;\gamma\right] \cdot \nabla_u f(x,u) + \frac{1}{2}\sum_{i,j=1}^{d} a^{(i,j)}(t,x,u)\partial_{u_i u_j} f(x,u).$$

The martingale problem related to (1.1) is stated as follows.

Definition 4.2 A probability measure $\mathbf{P}$ on the canonical space $\mathcal{C}([0,T];\mathbb{R}^{2d})$ is a weak solution of (1.1), or equivalently, a solution to the martingale problem (*MP*) if:

- (i) $\mathbf{P} \circ (x_0,u_0)^{-1} = \mu_0$.
- (ii) For all $t \in (0,T]$, the time marginal $\mathbf{P} \circ (x_t,u_t)^{-1}$ has a positive density ρ_t w.r.t. Lebesgue measure on $\mathbb{R}^{2d}$.
- (iii) For all $f \in \mathcal{C}^2_b(\mathbb{R}^{2d})$, the process

$$f(x_t,u_t) - f(x_0,u_0) - \int_0^t \mathcal{A}_{\rho_s} f(s,x_s,u_s)\,ds$$

is a $\mathbf{P}$-martingale, where, for each positive $\gamma \in L^1(\mathbb{R}^{2d})$, $\mathcal{A}_\gamma$ is the differential operator

$$\mathcal{A}_\gamma f(t,x,u) := u \cdot \nabla_x f(x,u) + B\left[x,u;\gamma\right] \cdot \nabla_u f(x,u) + \frac{1}{2}\sum_{i,j=1}^{d} a^{(i,j)}(t,x,u)\partial_{u_i u_j} f(x,u).$$

We prove the following uniqueness result.

Proposition 4.3 *There is at most one weak solution to Eq.* (1.1) *and one weak solution to Eq.* (3.4).

For a weak solution $\mathbf{P}$ of (1.1), and a weak solution $\mathbf{P}^\epsilon$ of (3.4), we consider the densities ρ_t and ρ^ϵ_t as in Definitions 4.1 and 4.2. We prove that ρ_t and ρ^ϵ_t are the unique solutions of nonlinear mild equations (see Lemma 4.5) which implies Proposition 4.3. A preliminary step consists in studying the linear case ($b = 0$).

4.1 Study of a Langevin system

For $(y,v) \in \mathbb{R}^{2d}$, consider the pair of processes $\left(Y^{s,y,v}_t, V^{s,y,v}_t;\ t \geq s \geq 0\right)$ solution of the Langevin equation

$$\begin{cases} Y^{s,y,v}_t = y + \int_s^t V^{s,y,v}_\theta\,d\theta, \\ V^{s,y,v}_t = v + \int_s^t \sigma(\theta, Y^{s,y,v}_\theta, V^{s,y,v}_\theta)\,dW_\theta. \end{cases} \tag{4.1}$$

The following result is a slight extension of a theorem due to Di Francesco and Pascucci [5]. We postpone the statement of this theorem and the proof of Proposition 4.4 in Sect. A.2.

Proposition 4.4 *There exists a unique weak solution to* (4.1). *In addition, this solution admits a density* $\Gamma(s, y, v; t, x, u)$ *w.r.t. Lebesgue measure such that*:

(i) *For all* $(t, x, u) \in (0, T] \times \mathbb{R}^{2d}$, $1 \le i, \ j \le d$, *the derivatives* $\partial_{v_i}\Gamma(s, y, v; t, x, u)$ *exist and are continuous for all* $(s, y, v) \in \mathbb{R} \times \mathbb{R}^{2d}$ *such that* $(s, y, v) \neq (t, x, u)$.

(ii) *Let* $f : \mathbb{R}^{2d} \to \mathbb{R}$ *be a bounded continuous function. Then the function* $G_{t,f}$ *defined by*

$$G_{t,f}(s, y, v) = \int_{\mathbb{R}^{2d}} \Gamma(s, y, v; t, x, u) f(x, u)\, dx\, du,$$

is the unique classical solution of the Cauchy problem

$$\begin{cases} \partial_s G_{t,f} + \mathcal{L}_s G_{t,f} = 0 \quad \text{in } [0, t) \times \mathbb{R}^{2d}, \\ \lim_{s \to t^-} G_{t,f}(s, y, v) = f(y, v) \quad \text{in } \mathbb{R}^{2d}, \end{cases} \tag{4.2}$$

where

$$\mathcal{L}_s \psi(s, y, v) := v \cdot \nabla_x \psi(s, y, v) + \frac{1}{2} \sum_{i,j=1}^{d} a^{(i,j)}(s, y, v) \partial_{v_i v_j} \psi(s, y, v). \tag{4.3}$$

(iii) *There exists a constant* $C > 0$ *depending only on* T *and* λ *such that*

$$\sup_{(y,v) \in \mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |\nabla_v \Gamma(s, y, v; t, x, u)|\, dx\, du \le \frac{C}{\sqrt{t - s}}, \quad \forall\, 0 \le s < t \le T. \tag{4.4}$$

Let us now identify mild equations satisfied by $(\rho_t;\ t \in (0, T])$ and $(\rho_t^{\epsilon};\ t \in (0, T])$.

4.2 Mild equations for the densities of $(X, \mathcal{U})$ and $(X^{\epsilon}, \mathcal{U}^{\epsilon})$

Consider a weak solution $(X, \mathcal{U})$ of (1.1). For all $f \in \mathcal{C}_b(\mathbb{R}^{2d})$, since $G_{t,f}$ is a classical solution of (4.2), Itô's formula leads to

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}\left[f(X_{t\wedge\tau_M}, \mathcal{U}_{t\wedge\tau_M})\right] &= \mathbb{E}_{\mathbb{P}}\left[G_{t,f}(0, X_0, \mathcal{U}_0)\right] \\ &\quad + \mathbb{E}_{\mathbb{P}}\left[\int_0^{t\wedge\tau_M} \left(\nabla_v G_{t,f}(s, X_s, \mathcal{U}_s) \cdot B\left[X_s, \mathcal{U}_s; \rho_s\right]\right) ds\right], \end{aligned} \tag{4.5}$$

where $\{\tau_M;\ M \geq 1\}$ is the sequence of stopping times

$$\tau_M = \inf\{t > 0\,;\ \ |X_t| + |\mathcal{U}_t| \geq M\}.$$

The boundedness of b and σ implies that $\lim_{M\to+\infty} \tau_M = +\infty$, $\mathbb{P}$-a.s. By Lebesgue's Dominated Convergence theorem, the left-hand side of (4.5) converges to $\mathbb{E}_{\mathbb{P}}\left[f(X_t, \mathcal{U}_t)\right]$ as M tends to infinity. For the right-hand side, Proposition 4.4 shows that, for $s \neq t$

$$\nabla_v G_{t,f}(s, y, v) = \int_{\mathbb{R}^{2d}} \nabla_v \Gamma(s, y, v; t, x, u) f(x, u)\, dx\, du,$$

and, $\mathbb{P} -$ a.s.,

$$\sup_{M\geq 1} \int_0^{t\wedge\tau_M} \left|\nabla_v G_{t,f}(s, X_s, \mathcal{U}_s) \cdot B\left[X_s, \mathcal{U}_s; \rho_s\right]\right| ds \leq \|b\|_\infty \|f\|_\infty \int_0^t \frac{C}{\sqrt{t-s}}\, ds.$$

Letting M tends to infinity, we get

$$\begin{aligned}\int_{\mathbb{R}^{2d}} f(x, u)\rho_t(x, u)\, dx\, du &= \int_{\mathbb{R}^{2d}} G_{t,f}(0, y, v)\mu_0(dy,\, dv)\\ &\quad + \int_{Q_t} \left(\nabla_v G_{t,f}(s, y, v) \cdot \rho_s(y, v) B\left[y, v; \rho_s\right]\right) dy\, dv\, ds.\end{aligned} \tag{4.6}$$

We denote by $(S^*_{t,s};\ \ 0 \leq s < t \leq T)$ the adjoint of the transition operator of $(Y_t^{s,y,v}, V_t^{s,y,v})$, that is,

$$S^*_{t,s}(f)(x, u) = \int_{\mathbb{R}^{2d}} \Gamma(s, y, v; t, x, u) f(y, v)\, dy\, dv.$$

In view of Proposition 4.4, for all $0 \leq s < t \leq T$, $S^*_{t,s}$ is a linear operator from $\mathcal{M}(\mathbb{R}^{2d})$ to $L^1(\mathbb{R}^{2d})$. In particular, $S^*_{t,0}(\mu_0) \in L^1(\mathbb{R}^{2d})$ and the first term in the right-hand side in (4.6) can be rewritten as

$$\int_{\mathbb{R}^{2d}} S^*_{t,0}(\mu_0)(x, u) f(x, u)\, dx\, du.$$

In addition, for all $0 \leq s < t \leq T$, we define the operator $S'_{t,s} : L^1(\mathbb{R}^{2d}; \mathbb{R}^d) \to L^1(\mathbb{R}^{2d}; \mathbb{R})$ by

$$S'_{t,s}(h(\cdot))(x,u) = \int_{\mathbb{R}^{2d}} (\nabla_v \Gamma(s,y,v;t,x,u) \cdot h(y,v))\, dy\, dv.$$

In particular, Proposition 4.4 shows that

$$\int_0^t \|S'_{t,s}(h(s,\cdot))\|_{L^1(\mathbb{R}^{2d})}\, ds \leq \int_0^t \frac{C}{\sqrt{t-s}} \|h(s,\cdot)\|_{L^1(\mathbb{R}^{2d})} ds \tag{4.7}$$

for all $h \in L^\infty((0,T); L^1(\mathbb{R}^{2d}))$. Thus the second term in the right-hand side of (4.6) writes

$$\int_{\mathbb{R}^{2d}} f(x,u) \int_0^t S'_{t,s}\,(\rho_s(\cdot) B\,[\cdot\,;\rho_s])\,(x,u)\, dx\, du\, ds.$$

Therefore, the marginal distributions $(\rho_t;\ t \in (0,T])$ of the solution of (1.1) satisfy the mild equation in $L^1(\mathbb{R}^{2d})$

$$\forall\, t \in (0,T],\ \rho_t = S^*_{t,0}(\mu_0) + \int_0^t S'_{t,s}\,(\rho_s(\cdot) B\,[\cdot\,;\rho_s])\, ds. \tag{4.8}$$

The preceding calculations hold true when ρ and $B[\cdot\,;\rho_.]$ are replaced by ρ^ϵ and $B_\epsilon[\cdot\,;\rho^\epsilon_.]$. Therefore, the marginal distributions $(\rho^\epsilon_t;\ t \in (0,T])$ satisfy the mild equation in $L^1(\mathbb{R}^{2d})$

$$\forall\, t \in (0,T],\ \rho^\epsilon_t = S^*_{t,0}(\mu_0) + \int_0^t S'_{t,s}\left(\rho^\epsilon_s(\cdot) B_\epsilon\left[\cdot\,;\rho^\epsilon_s\right]\right) ds. \tag{4.9}$$

4.3 Uniqueness of the solutions to the mild Eqs. (4.8) and (4.9)

Lemma 4.5 *There exists at most one positive solution (ρ_t) to Eq. (4.8), and at most one solution (ρ^ϵ_t) to (4.9).*

Proof We start with proving uniqueness for (4.8). Let (ρ^1_t) and (ρ^2_t) be two positive solutions of (4.8). Set

$$\overline{\rho}^i_t(x) := \int_{\mathbb{R}^d} \rho^i_t(x,u)\, du.$$

For a.e. (t, x) in $(0, T] \times \mathbb{R}^d$, $\overline{\rho}_t^i(x) > 0$ for $i = 1, 2$ and so, $B[x, u; \rho_t^i]$ in (1.2) is well defined for a.e. $(t, x, u) \in (0, T] \times \mathbb{R}^{2d}$. In view of (4.8), we have

$$\|\rho_t^1 - \rho_t^2\|_{L^1(\mathbb{R}^{2d})} = \int_{\mathbb{R}^{2d}} \left| \int_0^t S'_{t,s} \left(\rho_s^1(\cdot) B\left[\cdot\,;\rho_s^1\right] - \rho_s^2(\cdot) B\left[\cdot\,;\rho_s^2\right]\right)(x, u)\, ds \right| dx\, du. \tag{4.10}$$

We aim to prove the following estimate which implies the uniqueness result by a classical singular Gronwall's lemma (see e.g. [1] or [8, Chap.7]): there exists $C > 0$ such that, for all $t \in (0, T]$,

$$\|\rho_t^1 - \rho_t^2\|_{L^1(\mathbb{R}^{2d})} \leq \int_0^t \frac{C}{\sqrt{t-s}} \|\rho_s^1 - \rho_s^2\|_{L^1(\mathbb{R}^{2d})} ds. \tag{4.11}$$

From (4.10), we have

$$\begin{aligned} \|\rho_t^1 - \rho_t^2\|_{L^1(\mathbb{R}^{2d})} &\leq \int_{Q_t} \left| S'_{t,s} \left(\left(\rho_s^1(\cdot) - \rho_s^2(\cdot)\right) B\left[\cdot\,;\rho_s^1\right]\right)(x, u) \right| dx\, du\, ds \\ &\quad + \int_{Q_t} \left| S'_{t,s} \left(\rho_s^2(\cdot) \left(B\left[\cdot\,;\rho_s^1\right] - B\left[\cdot\,;\rho_s^2\right]\right)\right)(x, u) \right| dx\, du\, ds \\ &=: A_1 + A_2. \end{aligned} \tag{4.12}$$

In view of (4.7) and the boundedness of b, we get

$$A_1 \leq \int_0^t \frac{C}{\sqrt{t-s}} \|\rho_s^1 - \rho_s^2\|_{L^1(\mathbb{R}^{2d})}\, ds. \tag{4.13}$$

We now consider A_2. As

$$\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_1 - a_2}{b_2} + \frac{a_1(b_2 - b_1)}{b_2 b_1}, \quad \forall a_1,\ a_2 \in \mathbb{R},\ b_1,\ b_2 > 0, \tag{4.14}$$

we observe that

$$\begin{aligned} B\left[y, v; \rho_s^1\right] - B\left[y, v; \rho_s^2\right] &= \frac{1}{\overline{\rho}_s^2(y)} \int_{\mathbb{R}^d} b(v', v) \left(\rho_s^1(y, v') - \rho_s^2(y, v')\right) dv' \\ &\quad + \frac{\int_{\mathbb{R}^d} b(v', v) \rho_s^1(y, v')\, dv'}{\overline{\rho}_s^2(y) \overline{\rho}_s^1(y)} \left(\overline{\rho}_s^2(y) - \overline{\rho}_s^1(y)\right). \end{aligned}$$

Hence, for all $0 \leq s < t$,

$$\begin{aligned}
&\|\rho_s^2(\cdot)\left(B\left[\cdot;\rho_s^1\right]-B\left[\cdot;\rho_s^2\right]\right)\|_{L^1(\mathbb{R}^{2d})}\\
&\leq 2\|b\|_\infty \int_{\mathbb{R}^{2d}} \frac{\rho_s^2(y,v)}{\overline{\rho}_s^2(y)} \int_{\mathbb{R}^d} \left|\rho_s^1(y,v')-\rho_s^2(y,v')\right| dv'\,dy\,dv\\
&\leq 2\|b\|_\infty \|\rho_s^1-\rho_s^2\|_{L^1(\mathbb{R}^{2d})}.
\end{aligned}$$

In view of (4.7) we thus have

$$A_2 \leq \int_0^t \frac{C}{\sqrt{t-s}} \|\rho_s^1-\rho_s^2\|_{L^1(\mathbb{R}^{2d})}\,ds. \tag{4.15}$$

It remains to gather (4.13) and (4.15) to get (4.11).

We now prove uniqueness for (4.9). First, let us observe that, by using (4.7) in Eq. (4.9), one can found $C > 0$ such that, for all solution (ρ_t^ϵ) of (4.9),

$$\|\rho_t^\epsilon\|_{L^1(\mathbb{R}^{2d})} \leq C, \quad \forall\, t \in (0,T].$$

Next, consider two nonnegative solutions $(\rho_t^{\epsilon,1})$ and $(\rho_t^{\epsilon,2})$ of (4.9). As in (4.12), we have

$$\begin{aligned}
&\|\rho_t^{\epsilon,1}-\rho_t^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})}\\
&\leq \int_{Q_t} \left|S'_{t,s}\left(\left(\rho_s^{\epsilon,1}(\cdot)-\rho_s^{\epsilon,2}(\cdot)\right)B_\epsilon\left[\cdot\,;\rho_s^{\epsilon,1}\right]\right)(x,u)\right| dx\,du\,ds\\
&\quad+\int_{Q_t} \left|S'_{t,s}\left(\rho_s^{\epsilon,2}(\cdot)\left(B_\epsilon\left[\cdot\,;\rho_s^{\epsilon,1}\right]-B_\epsilon\left[\cdot\,;\rho_s^{\epsilon,2}\right]\right)\right)(x,u)\right| dx\,du\,ds\\
&=: A_1^\epsilon + A_2^\epsilon.
\end{aligned}$$

Obviously,

$$A_1^\epsilon \leq \int_0^t \frac{C}{\sqrt{t-s}} \|\rho_s^{\epsilon,1}-\rho_s^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})}\,ds.$$

In order to estimate A_2^ϵ, we use again (4.14) and observe: for a.e. $(s,y,v) \in Q_t$,

$$\begin{aligned}
&\rho_s^{\epsilon,2}(y,v)\left(B_\epsilon[y,v;\rho_s^{\epsilon,1}]-B_\epsilon\left[y,v;\rho_s^{\epsilon,2}\right]\right)\\
&=\frac{\rho_s^{\epsilon,2}(y,v)}{\phi_\epsilon \star \overline{\rho}_s^{\epsilon,2}(y)+\epsilon} \int_{\mathbb{R}^d} b(v',v)\left(\phi_\epsilon \star \rho_s^{\epsilon,1}(y,v')-\phi_\epsilon \star \rho_s^{\epsilon,2}(y,v')\right) dv'
\end{aligned}$$

$$+\frac{\rho_s^{\epsilon,2}(y,v)\int_{\mathbb{R}^d} b(v',v)\phi_\epsilon \star \rho_s^{\epsilon,1}(y,v')\,dv'}{\left(\phi_\epsilon \star \overline{\rho}_s^{\epsilon,2}(y)+\epsilon\right)\left(\phi_\epsilon \star \overline{\rho}_s^{\epsilon,1}(y)+\epsilon\right)}\left(\phi_\epsilon \star \overline{\rho}_s^{\epsilon,2}(y) - \phi_\epsilon \star \overline{\rho}_s^{\epsilon,1}(y)\right)$$

$$\leq \frac{2\|b\|_\infty \rho_s^{\epsilon,2}(y,v)}{\left(\phi_\epsilon \star \overline{\rho}_s^{\epsilon,2}(y)+\epsilon\right)}\int_{\mathbb{R}^{2d}} \phi_\epsilon(y-y')\left|\rho_s^{\epsilon,1}(y',v')-\rho_s^{\epsilon,2}(y',v')\right| dy'\,dv'.$$

In view of (4.7), it follows that

$$A_2^\epsilon \leq \int_{Q_t} \frac{C}{\sqrt{t-s}}\|\rho_s^{\epsilon,1}-\rho_s^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})}\,ds.$$

Hence

$$\|\rho_t^{\epsilon,1}-\rho_t^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})} \leq \int_0^t \frac{C}{\sqrt{t-s}}\|\rho_s^{\epsilon,1}-\rho_s^{\epsilon,2}\|_{L^1(\mathbb{R}^{2d})}\,ds.$$

We conclude on the uniqueness result for (4.9) by applying a singular Gronwall's lemma as above. □

5 Existence results

In this section, we establish that Eqs. (3.4) and (1.1) admit a solution.

Proposition 5.1 *The martingale problem* (MP_ϵ) *stated in Definition 4.1 has a unique solution* $\mathbb{P}^\epsilon$. *Furthermore, when* ϵ *tends to* 0, $\mathbb{P}^\epsilon$ *converges to a solution of the martingale problem* (MP) *stated in Definition 4.2.*

The proof of Proposition 5.1 proceeds in two steps.

The first step consists in constructing a weak solution to (3.4) by studying the interacting system (3.3) as the number of particles tends to infinity. As in [19], we prove the relative compactness of the sequence of the empirical measures of the particles (see Lemma 5.3). We then show that the support of the limit probability measure is the set of solutions of the martingale problem (MP_ϵ) (see Lemma 5.4). Using the uniqueness result in Proposition 4.3, we then get the propagation of chaos result.

The second step consists in exhibiting a solution to the martingale problem (MP) as the limit of the solution to the martingale problem (MP_ϵ) when ϵ tends to 0.

5.1 A propagation of chaos result for the smoothed system

Throughout this section we fix $\epsilon > 0$.

Proposition 5.2 *There exists a unique probability measure* $\mathbb{P}^\epsilon$ *solution to the martingale problem* (MP_ϵ). *Moreover, the sequence of probability laws* $\{\mathbb{P}^{\epsilon,N};\ N \geq 1\}$ *of the processes* $\{(X^{i,\epsilon,N},\mathcal{U}^{i,\epsilon,N});\ 1 \leq i \leq N\}$ *is* $\mathbb{P}^\epsilon$*-chaotic.*

Let $\overline{\mu}^{\epsilon,N}$ be the empirical measure valued in $\mathcal{M}(\mathcal{C}([0,T];\mathbb{R}^{2d}))$ and defined by

$$\overline{\mu}^{\epsilon,N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\{X^{i,\epsilon,N},\mathcal{U}^{i,\epsilon,N}\}}.$$

Let $\overline{\mathbb{P}}^{\epsilon,N}$ in $\mathcal{M}(\mathcal{M}(\mathcal{C}([0,T];\mathbb{R}^{2d})))$ be the probability law of $\overline{\mu}^{\epsilon,N}$.

Lemma 5.3 *The sequence* $\{\overline{\mathbb{P}}^{\epsilon,N};\ N \geq 1\}$ *is tight.*

Proof We proceed as in [19]: since the particle systems are exchangeable, the tightness of $\overline{\mathbb{P}}^{\epsilon,N}$ is equivalent to the tightness of the probability laws of $\{(X^{1,\epsilon,N},\mathcal{U}^{1,\epsilon,N});\ N \geq 1\}$. Let $((x_t^{(i)}, u_t^{(i)});\ t \in [0,T], i = 1,\ldots,N)$ be the canonical processes in the sample space $\mathcal{C}([0,T];\mathbb{R}^{2dN})$. In view of the boundedness of b and σ,

$$\mathbb{E}_{\mathbb{P}^{\epsilon,N}}[|u_t^{(1)} - u_s^{(1)}|^4] \leq C(t-s)^2 \quad \text{and} \quad \mathbb{E}_{\mathbb{P}^{\epsilon,N}}[|x_t^{(1)} - x_s^{(1)}|^2] \leq C(t-s)^2.$$

The result follows from the Kolmogorov criterion. □

We have shown that $\{\overline{\mathbb{P}}^{\epsilon,N};\ N \geq 1\}$ is relatively compact. We still denote by $\{\overline{\mathbb{P}}^{\epsilon,N};\ N \geq 1\}$ a weakly convergent subsequence. Let $\overline{\mathbb{P}}^{\epsilon,\infty}$ be the limit of such a subsequence.

Lemma 5.4 $\overline{\mathbb{P}}^{\epsilon,\infty}$ *assigns full measure to the set of the solutions to the martingale problem* (MP_ϵ).

Proof Denote by m a sample point in $\mathcal{M}(\mathcal{C}([0,T];\mathbb{R}^{2d}))$. Since $\{(X_0^{i,\epsilon,N},\mathcal{U}_0^{i,\epsilon,N});\ 1 \leq i \leq N\}$ are μ_0—i.i.d., it is easy to check that

$$m \circ (x_0, u_0)^{-1} = \mu_0,\ \overline{\mathbb{P}}^{\epsilon,N} - \text{a.s.};$$

a similar equality holds true for $\overline{\mathbb{P}}^{\epsilon,\infty}$ in view of the weak convergence of $\overline{\mathbb{P}}^{\epsilon,N}$ to $\overline{\mathbb{P}}^{\epsilon,\infty}$, which solves the part (i) of the martingale problem (MP_ϵ).

We now prove that, $\overline{\mathbb{P}}^{\epsilon,\infty}$—a.e., m satisfies the properties (ii) and (iii) of (MP_ϵ). Define $\alpha : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{M}(\mathcal{C}([0,T];\mathbb{R}^{2d})) \to \mathbb{R}^d$ by

$$\alpha(t,\xi,v,m) := \frac{\int_{\mathcal{C}([0,T];\mathbb{R}^{2d})} b(\bar{u}_t, v)\phi_\epsilon(\xi - \bar{x}_t)m(d\bar{x},d\bar{u})}{\int_{\mathcal{C}([0,T];\mathbb{R}^{2d})} \phi_\epsilon(\xi - \bar{x}_t)m(d\bar{x},d\bar{u}) + \epsilon}.$$

For all $f \in \mathcal{C}_c^2(\mathbb{R}^{2d})$, all $0 \le t_1 \le \cdots \le t_n \le s < t \le T$, and all finite family of functions $\{\psi_j\ ;\ 1 \le j \le n\}$ in $\mathcal{C}_b(\mathbb{R}^{2d})$, we set

$$F^\epsilon(m) := \Bigg| \mathbb{E}_m \Bigg[\prod_{j=1}^n \psi_j(x_{t_j}, u_{t_j}) \Bigg(f(x_t, u_t) - f(x_s, u_s) - \int_s^t (u_\theta \cdot \nabla_x f(x_\theta, u_\theta))\, d\theta \\ - \int_s^t (\alpha(\theta, x_\theta, u_\theta, m) \cdot \nabla_u f(x_\theta, u_\theta))\, d\theta \\ + \int_s^t \frac{1}{2} \sum_{i,j=1}^d a^{(i,j)}(\theta, x_\theta, u_\theta) \partial_{u_i,u_j} f(x_\theta, u_\theta)\, d\theta \Bigg) \Bigg] \Bigg|.$$

Suppose that we have proven that $F^\epsilon = 0, \overline{\mathbb{P}}^{\epsilon,\infty} - a.s.$ Then

$$f(x_t, u_t) - f(x_0, u_0) - \int_0^t (u_\theta \cdot \nabla_x f(x_\theta, u_\theta))\; d\theta \\ - \int_0^t (\alpha(\theta, x_\theta, u_\theta, m) \cdot \nabla_u f(x_\theta, u_\theta))\, d\theta \\ + \int_0^t \frac{1}{2} \sum_{i,j=1}^d a^{(i,j)}(\theta, x_\theta, u_\theta) \partial_{u_i,u_j} f(x_\theta, u_\theta)\, d\theta$$

would be a m-martingale, $\overline{\mathbb{P}}^{\epsilon,\infty}$ − a.s. As α is bounded, by Girsanov's theorem $m \circ (x_\theta, u_\theta)^{-1}$ would have a density ρ_θ^ϵ, so that

$$f(x_t, u_t) - f(x_0, u_0) - \int_0^t \mathcal{A}^\epsilon_{\rho_\theta^\epsilon} f(\theta, x_\theta, u_\theta)\, d\theta$$

would be a m-martingale. We thus would have solved the parts (ii) and (iii) of the martingale problem (MP_ϵ). It now remains to prove that $F^\epsilon = 0, \overline{\mathbb{P}}^{\epsilon,\infty}$-a.s. From (3.3) and Cauchy–Schwarz's inequality we easily get that $\mathbb{E}_{\overline{\mathbb{P}}^{\epsilon,N}}[F^\epsilon(m)] \le C/\sqrt{N}$. Therefore it suffices to deduce that $\mathbb{E}_{\overline{\mathbb{P}}^{\epsilon,N}}[F^\epsilon(m)]$ tends to $\mathbb{E}_{\overline{\mathbb{P}}^{\epsilon,\infty}}[F^\epsilon(m)]$ from the weak convergence of $\overline{\mathbb{P}}^{\epsilon,N}$ to $\overline{\mathbb{P}}^{\epsilon,\infty}$. As the function a is bounded continuous and the function f is smooth with compact support, it actually suffices to show the continuity (for the weak convergence topology) of the function Φ defined as

$$\Phi(m) := \mathbb{E}_m \int_s^t \alpha(\theta, x_\theta, u_\theta, m)\nabla_u f(x_\theta, u_\theta)d\theta = \int_{\mathcal{C}([0,T];\mathbb{R}^{2d})} J^\epsilon(x, u, m)m(dx, du)$$

with

$$J^\epsilon(x, u, m) := \int_s^t \alpha(\theta, x_\theta, u_\theta, m)\nabla_u f(x_\theta, u_\theta)d\theta.$$

Fix m and let (m_n) be a sequence of measures in $\mathcal{M}(\mathcal{C}([0, T]; \mathbb{R}^{2d}))$ weakly converging to m. We have

$$|\Phi(m_n) - \Phi(m)| \leq \left| \int_{\mathcal{C}([0,T];\mathbb{R}^{2d})} J^\epsilon(x, u, m)m_n(dx, du) - \int_{\mathcal{C}([0,T];\mathbb{R}^{2d})} J^\epsilon(x, u, m)m(dx, du) \right| + \left| \int_{\mathcal{C}([0,T];\mathbb{R}^{2d})} \left(J^\epsilon(x, u, m_n) - J^\epsilon(x, u, m)\right) m_n(dx, du) \right|.$$

The first term of the right-hand side tends to 0 when n goes to infinity by weak convergence of (m_n). To show that the second term tends also to 0, since m_n is a probability measure for all n, it suffices to show that there exists a sequence (γ_n) tending to 0 such that

$$\sup_{(x,u)\in\mathcal{C}([0,T];\mathbb{R}^{2d})} |J^\epsilon(x, u, m_n) - J^\epsilon(x, u, m)| \leq \gamma_n. \tag{5.1}$$

Let K_f be the compact support of the function f. Notice that

$$\sup_{(x,u)\in\mathcal{C}([0,T];\mathbb{R}^{2d})} |J^\epsilon(x, u, m_n) - J^\epsilon(x, u, m)| \leq C \int_s^t \Gamma_n(\theta)d\theta,$$

where

$$\Gamma_n(\theta) := \sup_{(\xi,\nu)\in K_f} \left| \frac{\int b(\bar{u}_\theta, \nu)\phi_\epsilon(\xi-\bar{x}_\theta)m_n(d\bar{x}, d\bar{u})}{\int \phi_\epsilon(\xi-\bar{x}_\theta)m_n(d\bar{x}, d\bar{u}) + \epsilon} - \frac{\int b(\bar{u}_\theta, \nu)\phi_\epsilon(\xi-\bar{x}_\theta)m(d\bar{x}, d\bar{u})}{\int \phi_\epsilon(\xi-\bar{x}_\theta)m(d\bar{x}, d\bar{u}) + \epsilon} \right|,$$

the integrals above being computed over $\mathcal{C}([0,T];\mathbb{R}^{2d})$. We aim to prove that we may choose

$$\gamma_n := C\int_s^t \Gamma_n(\theta)d\theta.$$

By Lebesgue's Dominated Convergence theorem it suffices to show that, for all bounded continuous function b, all $\theta \in [s,t]$, all $\eta > 0$, there exists $N(\eta)$ satisfying, for all $n > N(\eta)$,

$$\sup_{(\xi,\nu)\in K_f}\left|\int b(\bar{u}_\theta,\nu)\phi_\epsilon(\xi-\bar{x}_\theta)m_n(d\bar{x},d\bar{u}) - \int b(\bar{u}_\theta,\nu)\phi_\epsilon(\xi-\bar{x}_\theta)m(d\bar{x},d\bar{u})\right| \leq \eta. \tag{5.2}$$

By weak convergence of (m_n),

$$\limsup_{n\to+\infty} m_n(\{(\bar{x},\bar{u});\ |\bar{x}_\theta|+|\bar{u}_\theta|\geq R\}) \leq m(\{(\bar{x},\bar{u});\ |\bar{x}_\theta|+|\bar{u}_\theta|\geq R\}).$$

Choosing R large enough, the right-hand side is smaller than $\frac{\eta}{2}$. Now choose a continuous function h with compact support and such that $h(y,v)=1$ when $|y|+|v|\leq R$. Finally, consider the family $F^{\xi,\nu}$ of the functions defined on K_f by

$$(y,v) \ \mapsto\ b(v,\nu)\phi_\epsilon(\xi-y)h(y,v),$$

where (ξ,ν) is in K_f. The uniform continuity on $K_f\times K_f$ of the mapping

$$(\xi,\nu,y,v)\ \mapsto\ b(v,\nu)\phi_\epsilon(\xi-y)h(y,v)$$

implies that the family $F^{\xi,\nu}$ is equicontinuous. Therefore, in view of Lemma A.4, we have

$$\sup_{(\xi,\nu)\in K_f}\left|\int b(\bar{u}_\theta,\nu)\phi_\epsilon(\xi-\bar{x}_\theta)h(\bar{x}_\theta,\bar{u}_\theta)m_n(d\bar{x},d\bar{u}) - \int b(\bar{u}_\theta,\nu)\phi_\epsilon(\xi-\bar{x}_\theta)h(\bar{x}_\theta,\bar{u}_\theta)m(d\bar{x},d\bar{u})\right| \leq \frac{\eta}{2},$$

for all n large enough. We thus have obtained (5.2). That ends the proof. □

Proposition 4.3 ensures that $\overline{\mathbb{P}}^{\epsilon,\infty}$ is reduced to a Dirac mass. Denote by $\mathbb{P}^\epsilon$ the point such that $\overline{\mathbb{P}}^{\epsilon,\infty}=\delta_{\{\mathbb{P}^\epsilon\}}$. Clearly $\mathbb{P}^\epsilon$ is the unique solution to (MP_ϵ). Notice that this implies the $\mathbb{P}^\epsilon$-chaoticity of the particle system $(X^{i,\epsilon,N},\mathcal{U}^{i,\epsilon,N})$ (see [19, Prop. 2.2]).

5.2 Convergence of the smoothed system

In this subsection, we prove that the probability measure $\mathbb{P}^\epsilon$, solution to the martingale problem (MP_ϵ), converges to the solution to the martingale problem (MP). We start with studying the probability measure $\widetilde{\mathbb{P}}^\epsilon$ defined on $\mathcal{C}([0,T];\mathbb{R}^{3d})$ by

$$\widetilde{\mathbb{P}}^\epsilon = \mathbb{P}^\epsilon \circ \left((x_t, u_t, u_t - u_0 - \int_0^t B_\epsilon[x_s, u_s; \rho_s^\epsilon]\,ds);\ t \in [0,T] \right)^{-1}.$$

As in the proof of Lemma 5.3, using (3.4), the Kolmogorov criterion implies that the sequence $\{\widetilde{\mathbb{P}}^\epsilon;\ \epsilon > 0\}$ is relatively compact in $\mathcal{C}([0,T];\mathbb{R}^{3d})$. Let $\widetilde{\mathbb{P}}^\epsilon$ be a converging subsequence and denote its limit by $\widetilde{\mathbb{P}}$. Let us characterize the support of $\widetilde{\mathbb{P}}$. To this aim, we introduce the subset $\mathcal{H}_{\|b\|_\infty}$ of $\mathcal{C}([0,T];\mathbb{R}^{3d})$ defined by

$$\mathcal{H}_{\|b\|_\infty} = \left\{ \begin{array}{l} (x,u,D) \text{ in } \mathcal{C}([0,T];\mathbb{R}^{3d}),\ \text{s.t. } x(t) = x(0) + \int_0^t u(s)\,ds,\ \text{and} \\ u(t) - u(0) - D(t) = \int_0^t \beta(s)\,ds,\ \text{for a measurable function} \\ \quad \beta : [0,T] \to \mathbb{R}^d \\ \text{s.t. } \sup_{t\in[0,T]} |\beta(t)| \le \|b\|_\infty. \end{array} \right\}$$

We now prove that

Lemma 5.5 *$\widetilde{\mathbb{P}}$ has full measure on $\mathcal{H}_{\|b\|_\infty}$.*

Proof In view of the Portemanteau theorem, the weak convergence of $\widetilde{\mathbb{P}}^\epsilon$ to $\widetilde{\mathbb{P}}$ yields that, for all closed subset F of $\mathcal{C}([0,T];\mathbb{R}^{3d})$,

$$\limsup_{\epsilon\to 0^+} \widetilde{\mathbb{P}}^\epsilon(F) \le \widetilde{\mathbb{P}}(F).$$

Since $\widetilde{\mathbb{P}}^\epsilon(\mathcal{H}_{\|b\|_\infty}) = 1$ for all $\epsilon > 0$, it suffices to show that $\mathcal{H}_{\|b\|_\infty}$ is a closed subset of $\mathcal{C}([0,T];\mathbb{R}^{3d})$. Let $\{(x_n,u_n,D_n);\ n \in \mathbb{N}\}$ be a sequence of $\mathcal{H}_{\|b\|_\infty}$ converging to (x,u,D) in $\mathcal{C}([0,T];\mathbb{R}^{3d})$. Set $A_n(t) := u_n(t) - u_n(0) - D_n(t)$ and $A(t) := u(t) - u(0) - D(t)$. By uniform convergence, it holds that

$$x(t) = x(0) + \int_0^t u(s)\,ds, \quad \forall\, t \in [0,T],$$

$$\text{and} \quad \lim_{n\to+\infty} \max_{t\in[0,T]} |A_n(t) - A(t)| = 0.$$

To prove that (x,u,D) belongs to $\mathcal{H}_{\|b\|_\infty}$, it remains to show that A is a.e. differentiable with a time derivative uniformly bounded by $\|b\|_\infty$. By the Riesz representation theorem, it is enough to prove that

$$\left| \int_0^T A(t) f'(t)\,dt \right| \le \|b\|_\infty \int_0^T |f(t)|\,dt, \quad \forall f \in \mathcal{C}_c^1([0,T];\mathbb{R}^d).$$

As $A_n(t) = \int_0^t \beta_n(s)\,ds$ for some measurable function β_n satisfying $\sup_{t\in[0,T]} |\beta_n(t)| \leq \|b\|_\infty$, an integration by parts allows us to write

$$\left|\int_0^T A(t) f'(t)\,dt\right| \leq \left|\lim_{n\to+\infty} \int_0^T \beta_n(t) f(t)\,dt\right| \leq \|b\|_\infty \int_0^T |f(t)|\,dt,$$

which ends the proof. □

Consider the marginal distribution $\mathbb{P}$ of $\widetilde{\mathbb{P}}$ on $\mathcal{C}([0,T];\mathbb{R}^{2d})$, defined by

$$\mathbb{P} = \widetilde{\mathbb{P}} \circ ((x_t, u_t); \quad t \in [0,T])^{-1}.$$

We have the following result:

Proposition 5.6 *$\mathbb{P}$ solves the martingale problem (MP) stated in Definition 4.2.*

Proof The part (i) of (*MP*) is obvious.

To solve (ii), consider $((x_t, u_t, D_t);\ t \in [0,T])$ the canonical processes of $\mathcal{C}([0,T];\mathbb{R}^{3d})$. In view of Lemma 5.5, we know that, $\widetilde{\mathbb{P}}$ − a.s., for all $t \in [0,T]$,

$$x_t = x_0 + \int_0^t u_s\,ds,$$
$$u_t = u_0 + \int_0^t \beta_s\,ds + D_t,$$

with $\sup_{t\in[0,T]} |\beta_t| \leq \|b\|_\infty$. Since a is bounded continuous, the weak convergence of $\widetilde{\mathbb{P}}^\epsilon$ to $\widetilde{\mathbb{P}}$ yields that, for all function f in $\mathcal{C}_b^2(\mathbb{R}^d)$,

$$f(D_t) - f(D_0) - \frac{1}{2}\sum_{i,j=1}^{d} \int_0^t a^{(i,j)}(s, x_s, u_s)\partial_{u_i u_j} f(D_s) ds$$

is a $\widetilde{\mathbb{P}}$-martingale. We deduce that $D_t = \int_0^t \sigma(\theta, x_\theta, u_\theta)\,dw_\theta$ for some d-dimensional Wiener process $(w_t;\ t \in [0,T])$.

In view of (3.1), Girsanov's theorem allows one to define a new probability $\mathbb{Q}$ absolutely continuous to $\widetilde{\mathbb{P}}$ on $\mathcal{C}([0,T];\mathbb{R}^{3d})$ such that $\mathbb{Q} \circ (x_t, u_t)^{-1}$ is the law of the Langevin system

$$(y_t, v_t) = \left(x_0 + \int_0^t v_s\,ds, u_0 + \int_0^t \sigma(s, y_s, v_s)\,dw_s\right).$$

In view of Proposition 4.4, for all $t \in [0,T]$ the law of (y_t, v_t) is absolutely continuous w.r.t. to Lebesgue measure. Thus the measure $\mathbb{P} \circ (x_t, u_t)^{-1}$ has also a density ρ_t which satisfies

$$\gamma_t(x,u) = \rho_t(x,u)\mathbb{E}_{\mathbb{P}}\left(Z_t \Big/ (x_t,u_t) = (x,u)\right) \quad \text{for a.e. } (x,u) \in \mathbb{R}^{2d},$$

where Z_t is the restriction to $\mathcal{C}([0,t];\mathbb{R}^{3d})$ of the density $\frac{d\mathbb{Q}}{d\widetilde{\mathbb{P}}}$, and

$$\gamma_t(x,u) := \int_{\mathbb{R}^{2d}} \Gamma(0,y,v;t,x,u)\mu_0(dy,\,dv),$$

where $\Gamma(0,y,v;t,x,u)$ is defined as in Proposition 4.4. We now recall the following estimate (see [6]): there exist $\eta > 0$ and $c > 0$, depending only on λ, T, and d such that

$$\Gamma(s,y,v;t,x,u) \geq c\,\Gamma_\eta(s,y,v;t,x,u), \quad \forall s < t < T,$$

where Γ_η is defined in A.1 in the Appendix. Hence the function $\rho_t(x,u)$ is strictly positive. We thus have solved the part (ii) of (MP).

We now solve (iii) of (MP). Observe that there exists $C > 0$ such that

$$\sup_{\epsilon>0} \mathbb{E}_{\mathbb{P}^\epsilon}\left[\max_{t\in[0,T]} |u_t|\right] \leq C.$$

Therefore it suffices to prove that, for all $f \in \mathcal{C}_c^2(\mathbb{R}^{2d})$ and all process (Ψ_s) of the form

$$\Psi_s = \Psi_s(x.,u.) := \prod_{j=1}^{n} \psi_j(x_{t_j},u_{t_j}),$$

where the ψ_j's are bounded continuous functions, one has

$$\mathbb{E}_{\mathbb{P}}\left[\Psi_s\left(f(x_t,u_t) - f(x_s,u_s) - \int_s^t \mathcal{A}_{\rho_\theta} f(x_\theta,u_\theta)d\theta\right)\right] = 0, \tag{5.3}$$

Since $\widetilde{\mathbb{P}}^\epsilon$ converges weakly to $\widetilde{\mathbb{P}}$, we have

$$\lim_{\epsilon\to0^+} \mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s\left(f(x_t,u_t) - f(x_s,u_s) - \int_s^t \mathcal{L}_\theta f(x_\theta,u_\theta)d\theta\right)\right]$$
$$= \mathbb{E}_{\mathbb{P}}\left[\Psi_s\left(f(x_t,u_t) - f(x_s,u_s) - \int_s^t \mathcal{L}_\theta f(x_\theta,u_\theta)d\theta\right)\right],$$

where $\mathcal{L}_\theta$ is defined as in (4.3). To obtain (5.3), it thus remains to show

$$\lim_{\epsilon\to 0^+} \mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(B_\epsilon\left[x_\theta, u_\theta; \rho_\theta^\epsilon\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right] = \mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(B\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right]. \tag{5.4}$$

If $\mathbb{P}^\epsilon$ were the law of a strongly elliptic diffusion process and the coefficient B_ϵ would not depend on ρ^ϵ, (5.4) would result from Stroock and Varadhan's results on limits of martingale problems: see Lemmas 9.1.15 and 11.3.2 in [17]. In our situation, we prove that (5.4) holds true by adapting Stroock and Varadhan's techniques and by taking advantage of the mild equation (4.9). Let $\xi > 0$ be a positive parameter that will be choosen below. We add and subtract to the brackets in (5.4) the terms

$$\mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(B_\xi\left[x_\theta, u_\theta; \rho_\theta^\epsilon\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right],$$

$$\text{and} \quad \mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(B_\xi\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right].$$

Then we have

$$\begin{aligned}
&\Bigg|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(B_\epsilon\left[x_\theta, u_\theta; \rho_\theta^\epsilon\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right] \\
&\quad -\mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(B\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right]\Bigg| \\
&\le \Bigg|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(B_\xi\left[x_\theta, u_\theta; \rho_\theta^\epsilon\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right] \\
&\quad - \mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(B_\xi\left[x_\theta, u_\theta; \rho_\theta\right] \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right]\Bigg| \\
&\quad + \Bigg|\mathbb{E}_{\mathbb{P}^\epsilon}\left[\Psi_s \int_s^t \left(\left(B_\epsilon\left[x_\theta, u_\theta; \rho_\theta^\epsilon\right] - B_\xi\left[x_\theta, u_\theta; \rho_\theta^\epsilon\right]\right) \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right]\Bigg| \\
&\quad + \Bigg|\mathbb{E}_{\mathbb{P}}\left[\Psi_s \int_s^t \left(\left(B_\xi\left[x_\theta, u_\theta; \rho_\theta\right] - B\left[x_\theta, u_\theta; \rho_\theta\right]\right) \cdot \nabla_u f(x_\theta, u_\theta)\right) d\theta\right]\Bigg| \\
&=: I_{\epsilon,\xi} + J_{\epsilon,\xi} + K_\xi.
\end{aligned}$$

Use the Lemma 5.7 below and let successively ξ and ϵ tend to 0: we get (5.4), which ends the resolution of the part (iii) of (MP). □

Lemma 5.7 *There holds*

$$\forall \xi > 0, \quad \lim_{\epsilon \to 0^+} I_{\epsilon,\xi} = 0, \tag{5.5}$$

and

$$\lim_{\xi \to 0^+} K_\xi = 0. \tag{5.6}$$

In addition, there exist a function $\delta_1(\epsilon)$ which does not depend on ξ, and a function $\delta_2(\xi)$ which does not depend on ϵ, such that

$$\lim_{\epsilon \to 0^+} \delta_1(\epsilon) = \lim_{\xi \to 0^+} \delta_2(\xi) = 0$$

and

$$J_{\epsilon,\xi} \leq \delta_1(\epsilon) + \delta_2(\xi). \tag{5.7}$$

The proof of this lemma is long. We split it into two parts: we prove technical results in the next subsection, and finally prove the lemma in Sect. 5.4.

5.3 Technical results

A key step to prove Lemma 5.7 is the following proposition.

Proposition 5.8 *For all $0 < t \leq T$, ρ_t^ϵ converges to ρ_t in $L^1(\mathbb{R}^{2d})$ when $\epsilon \to 0^+$.*

In view of Lemma A.2 in the Appendix, Proposition 5.8 results from the following lemma:

Lemma 5.9 *For all $t \in (0, T]$, $h, \delta \in \mathbb{R}^d$, it holds that*

$$\lim_{|h|,|\delta| \to 0} \limsup_{\epsilon \to 0^+} \int_{\mathbb{R}^{2d}} \left| \rho_t^\epsilon(x+h, u+\delta) - \rho_t^\epsilon(x,u) \right| dx\, du = 0.$$

Proof As $\mathbb{P}^\epsilon$ is the unique solution to the martingale problem (MP_ϵ), its time marginals satisfy the mild equation (4.9). Thus, for all $t \in (0, T]$,

$$\begin{aligned}
&\limsup_{\epsilon\to 0^+} \int_{\mathbb{R}^{2d}} \left|\rho_t^\epsilon(x+h, u+\delta) - \rho_t^\epsilon(x,u)\right| \, dx\, du \\
&\leq \int_{\mathbb{R}^{2d}} \left|S^*_{t,0}(\mu_0)(x+h, u+\delta) - S^*_{t,0}(\mu_0)(x,u)\right| \, dx\, du \\
&\quad + \limsup_{\epsilon\to 0^+} \int_{Q_t} \left|S'_{t,s}\left(\rho_s^\epsilon(\cdot) B_\epsilon\left[\cdot\,;\rho_s^\epsilon\right]\right)(x+h, u+\delta)\right. \\
&\quad \left. - S'_{t,s}\left(\rho_s^\epsilon(\cdot) B_\epsilon\left[\cdot\,;\rho_s^\epsilon\right]\right)(x,u)\right| \, dx\, du\, ds.
\end{aligned}$$

Since $S^*_{t,0}(\mu_0)$ belongs to $L^1(\mathbb{R}^{2d})$, Lemma A.3 implies that

$$\lim_{|h|,|\delta|\to 0} \int_{\mathbb{R}^{2d}} \left|S^*_{t,0}(\mu_0)(x+h, u+\delta) - S^*_{t,0}(\mu_0)(x,u)\right| \, dx\, du = 0.$$

In addition,

$$\begin{aligned}
&\int_{Q_t} \left|S'_{t,s}\left(\rho_s^\epsilon(\cdot) B_\epsilon\left[\cdot\,;\rho_s^\epsilon\right]\right)(x+h, u+\delta) - S'_{t,s}\left(\rho_s^\epsilon(\cdot) B_\epsilon\left[\cdot\,;\rho_s^\epsilon\right]\right)(x,u)\right| \, dx\, du\, ds \\
&\quad \leq \|b\|_\infty \int_{Q_t} \left(\int_{\mathbb{R}^{2d}} \left|\nabla_v \Gamma(s,y,v;t,x+h,u+\delta) - \nabla_v \Gamma(s,y,v;t,x,u)\right| \, dx\, du \right) \\
&\qquad \times \rho_s^\epsilon(y,v)\, dy\, dv\, ds.
\end{aligned}$$

Set

$$L_{h,\delta}(t,s,y,v) := \int_{\mathbb{R}^{2d}} \left|\nabla_v \Gamma(s,y,v;t,x+h,u+\delta) - \nabla_v \Gamma(s,y,v;t,x,u)\right| \, dx\, du.$$

As $\widetilde{\mathbb{P}}^\epsilon$ converges weakly to $\widetilde{\mathbb{P}}$, ρ_t^ϵ converges weakly to ρ_t for all $t \in [0, T]$ and

$$\lim_{\epsilon\to 0^+} \int_{Q_t} L_{h,\delta}(t,s,y,v) \rho_s^\epsilon(y,v)\, dy\, dv\, ds = \int_{Q_t} L_{h,\delta}(t,s,y,v) \rho_s(y,v)\, dy\, dv\, ds.$$

In addition, in view of (4.4), one has

$$\sup_{(y,v)\in\mathbb{R}^{2d}} L_{h,\delta}(t,s,y,v) \leq \frac{C}{\sqrt{t-s}},$$

for all $t > s$, from which

$$\int_{\mathbb{R}^{2d}} L_{h,\delta}(t,s,y,v)\rho(y,v)dydv \leq \frac{C}{\sqrt{t-s}}.$$

It then remains to apply Lebesgue's Dominated Convergence theorem. □

Below we will also use the following three elementary results.

The first result follows from Proposition 5.8 and Lebesgue's Dominated Convergence theorem (since $\|\rho^\epsilon_\theta\|_{L^1(\mathbb{R}^{2d})} = 1$, for all $\theta \in (0, T]$):

Corollary 5.10

$$\lim_{\epsilon\to 0^+} \int_0^T \|\rho^\epsilon_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta = 0.$$

Lemma 5.11 *It holds*

$$\lim_{\xi\to 0^+} \int_0^T \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})}\, d\theta = 0. \tag{5.8}$$

Proof As

$$\|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} \leq \int_{\mathbb{R}^d} \phi(y) \int_{\mathbb{R}^{2d}} |\rho_\theta(x-\xi y, v) - \rho_\theta(x,v)|\; dx\, dv\, dy,$$

the result follows from Lemma A.3 in the Appendix, and the fact that $\phi \in L^1(\mathbb{R}^d)$ which allows one to apply Lebesgue's Dominated Convergence theorem. □

Lemma 5.12 *Set*

$$\overline{\rho}_\theta(x) := \int_{\mathbb{R}^d} \rho_\theta(x,v)\, dv.$$

Then

$$\lim_{\xi\to 0^+} \int_0^T \int_{\mathbb{R}^d} \frac{\xi\overline{\rho}_\theta(x)}{\overline{\rho}_\theta(x)+\xi}\, dx\, d\theta = 0.$$

Proof For all θ the function $\overline{\rho}_\theta$ is strictly positive a.e. Notice that for all θ in $[0, T]$ and all x such that $\overline{\rho}_\theta(x) > 0$, the function

$$\xi \in [0,1] \;\mapsto\; D(\xi; x, \theta) := \frac{\xi\overline{\rho}_\theta(x)}{\overline{\rho}_\theta(x)+\xi}$$

is bounded from above by $\overline{\rho}_\theta(x)$ for all $\xi \in [0,1]$. Lebesgue's Dominated Convergence theorem implies that $\int_0^T \int_{\mathbb{R}^d} D(\xi; x, \theta) dx\, d\theta$ tends to 0 with ξ. □

5.4 Proof of Lemma 5.7

We are now in a position to prove (5.5), (5.6), and (5.7).

Proof of (5.5). Set

$$F_\xi^{\epsilon'}(x., u.) := \Psi_s \int_s^t \Big(B_\xi[x_\theta, u_\theta; \rho_\theta^{\epsilon'}] \cdot \nabla_u f(x_\theta, u_\theta) \Big)\, d\theta$$

and

$$F_\xi(x., u.) := \Psi_s \int_s^t \big(B_\xi[x_\theta, u_\theta; \rho_\theta] \cdot \nabla_u f(x_\theta, u_\theta) \big)\, d\theta.$$

We have

$$\begin{aligned} I_{\epsilon,\xi} &= |\mathbb{E}_{\mathbb{P}^\epsilon} F_\xi^\epsilon - \mathbb{E}_{\mathbb{P}} F_\xi| \\ &\le \sup_{\epsilon'>0} |\mathbb{E}_{\mathbb{P}^\epsilon} F_\xi^{\epsilon'} - \mathbb{E}_{\mathbb{P}} F_\xi^{\epsilon'}| + |\mathbb{E}_{\mathbb{P}} F_\xi^\epsilon - \mathbb{E}_{\mathbb{P}} F_\xi|. \end{aligned}$$

Now, for all fixed $\xi > 0$, $0 \le s \le t \le T$, the bounded functions $\{F_\xi^{\epsilon'};\ \epsilon' > 0\}$ defined on $\mathcal{C}([0,T]; K_f)$ are equicontinuous (this latter property results from the definition of B_ξ, the fact that f has compact support, and Proposition 5.8). Therefore, in view of Lemma A.4 in the Appendix, the first term in the right-hand side of the preceding inequality tends to 0 with ϵ. The second term tends also to 0 in view of Proposition 5.8. We thus have proven (5.5).

Proof of (5.6). We recall the notation

$$\overline{\rho}_\theta(x) := \int_{\mathbb{R}^d} \rho_\theta(x, v)\, dv.$$

Observe that

$$\begin{aligned} &\big| B_\xi\, [x, u, \rho_\theta] - B\, [x, u; \rho_\theta] \big| \\ &\quad \le \left| \frac{\int_{\mathbb{R}^d} b(v,u) \phi_\xi \star \rho_\theta(x, v)\, dv}{\phi_\xi \star \overline{\rho}_\theta(x) + \xi} - \frac{\int_{\mathbb{R}^d} b(v,u) \rho_\theta(x, v)\, dv}{\overline{\rho}_\theta(x) + \xi} \right| \\ &\qquad + \left| \frac{\int_{\mathbb{R}^d} b(v,u) \rho_\theta(x, v)\, dv}{\overline{\rho}_\theta(x) + \xi} - \frac{\int_{\mathbb{R}^d} b(v,u) \rho_\theta(x, v)\, dv}{\overline{\rho}_\theta(x)} \right|. \end{aligned}$$

In view of (4.14), we have

$$
\begin{aligned}
&\left|\frac{\int_{\mathbb{R}^d} b(v,u)\phi_\xi \star \rho_\theta(x,v)\,dv}{\phi_\xi \star \overline{\rho}_\theta(x)+\xi} - \frac{\int_{\mathbb{R}^d} b(v,u)\rho_\theta(x,v)\,dv}{\overline{\rho}_\theta(x)+\xi}\right| \\
&\quad \le \left|\frac{\int_{\mathbb{R}^d} b(v,u)\left(\phi_\xi \star \rho_\theta(x,v)-\rho_\theta(x,v)\right)\,dv}{\overline{\rho}_\theta(x)+\xi}\right| \\
&\qquad + \frac{\left|\int_{\mathbb{R}^d} b(v,u)\phi_\xi \star \rho_\theta(x,v)\,dv\right| \left|\int_{\mathbb{R}^d}\left(\phi_\xi \star \rho_\theta(x,v)-\rho_\theta(x,v)\right)\,dv\right|}{\left(\phi_\xi \star \overline{\rho}_\theta(x)+\xi\right)\left(\overline{\rho}_\theta(x)+\xi\right)} \\
&\quad \le \frac{2\|b\|_\infty}{\overline{\rho}_\theta(x)+\xi}\int_{\mathbb{R}^d}\left|\phi_\xi \star \rho_\theta(x,v)-\rho_\theta(x,v)\right|\,dv.
\end{aligned}
\tag{5.9}
$$

Thus

$$
\begin{aligned}
&\int_{Q_t}\left|B_\xi\left[x,u,\rho_\theta\right]-B\left[x,u;\rho_\theta\right]\right|\rho_\theta(x,u)\,dx\,du\,d\theta \\
&\quad \le 2\|b\|_\infty \int_0^t \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})}\,d\theta \\
&\qquad + \|b\|_\infty \int_{Q_t}\left|\frac{1}{\overline{\rho}_\theta(x)+\xi}-\frac{1}{\overline{\rho}_\theta(x)}\right|\int_{\mathbb{R}^d}\rho_\theta(x,v)\,dv\rho_\theta(x,u)\,dx\,du\,d\theta \\
&\quad \le \int_0^t \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})}\,d\theta + C\int_0^t\int_{\mathbb{R}^d}\frac{\xi\overline{\rho}_\theta(x)}{\overline{\rho}_\theta(x)+\xi}\,dx\,d\theta.
\end{aligned}
\tag{5.10}
$$

We now use the Lemmas 5.11 and 5.12. That ends the proof of (5.6).

Proof of (5.7). Observe that

$$
\begin{aligned}
J_{\epsilon,\xi} \;\le\; &\|\Psi_s\|_\infty\|\nabla_u f\|_\infty\left(\int_{Q_t}\left|B_\xi[x,u;\rho_\theta^\epsilon]-B[x,u;\rho_\theta^\epsilon]\right|\rho_\theta^\epsilon(x,u)\,dx\,du\,d\theta\right. \\
&\left. + \int_{Q_t}\left|B[x,u;\rho_\theta^\epsilon]-B_\epsilon[x,u;\rho_\theta^\epsilon]\right|\rho_\theta^\epsilon(x,u)\,dx\,du\,d\theta\right) \\
=:\; &\|\Psi_s\|_\infty\|\nabla_u f\|_\infty\left(J^1_{\epsilon,\xi}+J^2_\epsilon\right).
\end{aligned}
\tag{5.11}
$$

In order to estimate $J^1_{\epsilon,\xi}$, observe that

$$\int_{Q_t} \left| B_\xi \left[x, u, \rho^\epsilon_\theta\right] - B\left[x, u; \rho^\epsilon_\theta\right]\right| \rho^\epsilon_\theta(x,u)\,dx\,du\,d\theta$$
$$\leq \int_{Q_t} \left| \frac{\int_{\mathbb{R}^d} b(v,u)\phi_\xi \star \rho^\epsilon_\theta(x,v)\,dv}{\phi_\xi \star \overline{\rho}^\epsilon_\theta(x)+\xi} - \frac{\int_{\mathbb{R}^d} b(v,u)\rho^\epsilon_\theta(x,v)\,dv}{\overline{\rho}^\epsilon_\theta(x)+\xi}\right| \rho^\epsilon_\theta(x,u)\,dx\,du\,d\theta$$
$$+ \int_{Q_t} \left| \frac{\int_{\mathbb{R}^d} b(v,u)\rho^\epsilon_\theta(x,v)\,dv}{\overline{\rho}^\epsilon_\theta(x)+\xi} - \frac{\int_{\mathbb{R}^d} b(v,u)\rho^\epsilon_\theta(x,v)\,dv}{\overline{\rho}^\epsilon_\theta(x)}\right| \rho^\epsilon_\theta(x,u)\,dx\,du\,d\theta. \tag{5.12}$$

We now estimate each term in the right-hand side of (5.12).

Using (4.14) again, we get

$$\left| \frac{\int_{\mathbb{R}^d} b(v,u)\phi_\xi \star \rho^\epsilon_\theta(x,v)\,dv}{\phi_\xi \star \overline{\rho}^\epsilon_\theta(x)+\xi} - \frac{\int_{\mathbb{R}^d} b(v,u)\rho^\epsilon_\theta(x,v)\,dv}{\overline{\rho}^\epsilon_\theta(x)+\xi}\right|$$
$$\leq \frac{2\|b\|_\infty}{\overline{\rho}^\epsilon_\theta(x)+\xi} \int_{\mathbb{R}^d} \left|\phi_\xi \star \rho^\epsilon_\theta(x,v) - \rho^\epsilon_\theta(x,v)\right|\,dv.$$

Therefore the first term in the right-hand side of (5.12) is bounded from above by

$$C \int_{Q_t} \left|\phi_\xi \star \rho^\epsilon_\theta(x,v) - \rho^\epsilon_\theta(x,v)\right|\,dx\,dv\,d\theta,$$

and therefore by

$$C\int_0^t \|\phi_\xi \star \rho^\epsilon_\theta - \phi_\xi \star \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta + C\int_0^t \|\rho^\epsilon_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})}\,d\theta$$
$$+C\int_0^t \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta,$$

which can be bounded from above by

$$C\int_0^t \|\rho^\epsilon_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta + C\int_0^t \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta.$$

The second term in the right-hand side of (5.12) is bounded from above by

$$\|b\|_\infty \int_0^t \int_{\mathbb{R}^d} \frac{\xi \overline{\rho}_\theta^\epsilon(x)}{\overline{\rho}_\theta^\epsilon(x)+\xi}\, dx\, d\theta.$$

Insert

$$\int_0^t \int_{\mathbb{R}^d} \frac{\xi \overline{\rho}_\theta(x)}{\overline{\rho}_\theta(x)+\xi}\, dx\, d\theta,$$

and observe that, for all $\xi > 0$,

$$\left| \int_0^t \int_{\mathbb{R}^d} \frac{\xi \overline{\rho}_\theta^\epsilon(x)}{\overline{\rho}_\theta^\epsilon(x)+\xi}\, dx\, d\theta - \int_0^t \int_{\mathbb{R}^d} \frac{\xi \overline{\rho}_\theta(x)}{\overline{\rho}_\theta(x)+\xi}\, dx\, d\theta \right| \le \int_0^t \|\rho_\theta^\epsilon - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta. \tag{5.13}$$

We thus have obtained

$$J^1_{\epsilon,\xi} \le C \int_0^t \|\rho_\theta^\epsilon - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta + C \int_0^t \|\phi_\xi \star \rho_\theta - \rho_\theta\|_{L^1(\mathbb{R}^{2d})} d\theta$$
$$+ C \int_0^t \int_{\mathbb{R}^d} \frac{\xi \overline{\rho}_\theta(x)}{\overline{\rho}_\theta(x)+\xi}\, dx\, d\theta. \tag{5.14}$$

Similarly, J^2_ϵ being defined as in (5.11), we have

$$J^2_\epsilon \le C \int_0^t \|\phi_\epsilon \star \rho_\theta^\epsilon - \rho_\theta^\epsilon\|_{L^1(\mathbb{R}^{2d})}\, d\theta$$
$$+ C \int_{Q_t} \left| \frac{1}{\overline{\rho}_\theta^\epsilon(x)+\epsilon} - \frac{1}{\overline{\rho}_\theta^\epsilon(x)} \right| \left(\int_{\mathbb{R}^d} |b(v,u)|\, \rho_\theta^\epsilon(x,v)\, dv \right) \rho_\theta^\epsilon(x,u)\, dx\, du\, d\theta.$$

In view of (5.13) we deduce

$$J^2_\epsilon \le C \int_0^t \|\phi_\epsilon \star \rho_\theta^\epsilon - \rho_\theta^\epsilon\|_{L^1(\mathbb{R}^{2d})}\, d\theta + C \int_0^t \int_{\mathbb{R}^d} \frac{\xi \overline{\rho}_\theta(x)}{\overline{\rho}_\theta(x)+\xi}\, dx\, d\theta.$$

Combining this estimate with (5.14) and (5.11) and using the Lemmas 5.11 and 5.12 we obtain (5.7).

6 Conclusion and perspectives

In this paper we have studied a Lagrangian stochastic model and shown its well-posedness. We have proved that the unique weak solution is an hypoelliptic diffusion process whose dynamics depends on the conditional distribution of the velocity component knowing the position component. To our knowledge, this is the first theoretical result on the Lagrangian stochastic models modelling turbulent fluid particles. Bossy and Jabir [3] consider models with specular reflection boundary conditions. See also [9]. A lot remains to be done to study the complex models developed by Pope [14].

We also emphasize another possible extension of our result. We conjecture the following PDE analysis result: the estimate (4.4) holds true under classical Hölder conditions rather than (3.2), possibly by using Maxwellian approximations rather than using the parametrix method.

Appendix A

A.1 Di Francesco and Pascucci's estimates on fundamental solutions of ultraparabolic PDEs

Before stating the estimate on fundamental solutions of ultraparabolic PDEs which are used in this paper, we need to introduce some new notation.

In [5], Di Francesco and Pascucci consider ultraparabolic PDEs of the type

$$-\partial_s \psi + \frac{1}{2}\sum_{i,j} \overline{a}^{(i,j)} \partial_{v_i v_j} \psi + (y, v) \cdot B \nabla_{(y,v)} \psi = 0,$$

where $\overline{a}$ and B are $2d \times 2d$ matrices, and B has constant entries. The statement of their results for general matrices B require to introduce a pseudo-metric depending on B and some notational effort. We thus limit ourselves to our context where

$$B = \begin{pmatrix} 0 & 0 \\ \mathrm{Id}_{\mathbb{R}^d} & 0 \end{pmatrix}.$$

In this context, Di Francesco and Pascucci's assumption on the coefficient $\overline{a}$ writes as follows: $\overline{a}$ is a bounded function and there exist $\alpha \in (0, 1]$ and $C > 0$ such that, for all (s, x, u) , $(t, y, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, the inequality (3.2) holds true.

For all $\eta > 0$, for $s < t$, let $\Gamma_\eta(s, y, v; t, x, u)$ be the transition density function of

$$\left(y_t^{s,y,v} = y + \int_s^t v_\theta^{s,v} d\theta,\ \ v_t^{s,v} = v + \eta(W_t - W_s) \right)$$

that is,

$$\Gamma_\eta(s,y,v;t,x,u) = \frac{\sqrt{3^d}}{(\sqrt{\pi}\eta(t-s))^{2d}} \times \exp\left\{-\frac{6\,|x-y-(t-s)v|^2}{\eta(t-s)^3}+\frac{6(x-y-(t-s)v)\cdot(u-v)}{\eta(t-s)^2}-\frac{2\,|u-v|^2}{\eta(t-s)}\right\}. \tag{A.1}$$

We are in a position to state the following theorem:

Theorem A.1 [5] *Suppose that* $\overline{a}$ *satisfies* (3.1) *and* (3.2). *There exists a fundamental solution* $\overline{\Gamma}(s,y,v;\theta,x,u)$ *to the operator*

$$\overline{\mathcal{L}} = -\partial_\theta + \frac{1}{2}\sum_{i,j}\overline{a}^{(i,j)}(\theta,x,u)\partial_{u_iu_j} + u\cdot\nabla_x$$

which satisfies

(i) *for all* $(s,y,v)\in(0,T]\times\mathbb{R}^{2d}$, $1\le i\le d$, *the derivatives* $\partial_{u_i}\overline{\Gamma}(s,y,v;\theta,x,u)$ *exist and are continuous in* $(0,T]\times\mathbb{R}^{2d}\backslash\{s,y,v\}$.

(ii) *Let* $f:\mathbb{R}^{2d}\to\mathbb{R}$ *be a bounded continuous function. Then, for any* $T_0\ge 0$, *the function* $J_{T_0,T}$ *defined by*

$$J_{T_0,T}(\theta,x,u) = \int_{\mathbb{R}^{2d}}\overline{\Gamma}(T_0,y,v;\theta,x,u)f(y,v)\,dy\,dv,\quad \theta\in(T_0,T]$$

is the unique solution of the Cauchy problem

$$\begin{cases}\overline{\mathcal{L}}J_{T_0,T}=0,\\ J_{T_0,T}(T_0,y,v)=f(y,v)\ \text{in}\ \mathbb{R}^{2d}.\end{cases}$$

(iii) *For all* $\eta>\lambda$, *there exists a constant* $C>0$ *such that, for* $s<\theta<T$

$$\left|\nabla_u\overline{\Gamma}(s,y,v;\theta,x,u)\right| \le \frac{C}{\sqrt{\theta-s}}\Gamma_\eta(T-(\theta-s),x,u;T,y,v).$$

A.2 Proof of Proposition 4.4

To get the existence of a weak solution, we adapt the proof of Theorem 6.3.2 in [17]: consider a sequence $\{\sigma^n;\ n\in\mathbb{N}\}$ of $\mathbb{R}^d\times\mathbb{R}^d$-valued Lipschitz functions on $[0,T]\times\mathbb{R}^d$ such that $\lim_{n\to+\infty}\sigma^n=\sigma$ uniformly. For all $n\in\mathbb{N}$, one has existence of a strong solution $(Y_t^{n,s,y,v},\ V_t^{n,s,v};\ s\le t\le T)$ to Eq. (4.1) when one substitutes σ to σ^n. Then it is easy to check that $(Y_t^{n,s,y,v},\ V_t^{n,s,v};\ s \leq t \leq T)$ converges in distribution to a weak solution of (4.1).

The uniqueness of the weak solution and the properties (i) to (iii) result from Theorem A.1: observe that $G_{t,f}(s, y, v) = J_{0,t}(t-s, y, v)$, where we have set $\overline{a}(s, y.v) := a(t-s, y, v)$ in the definition of the operator $\overline{\mathcal{L}}$. Consequently, the density $\Gamma(s, y, v; t, x, u)$ writes

$$\Gamma(s, y, v; t, x, u) = \overline{\Gamma}(0, x, u; t-s, y, v),$$

where $\overline{\Gamma}(0, y, v; \theta, x, u)$ is the fundamental solution to $\overline{\mathcal{L}}$.

A.3 Technical lemmas

For the reader's convenience we state three technical results which played a key role in our proofs.

The first lemma can be found in [17, Lemma 11.4.1].

Lemma A.2 *Let $\{f_n;\ n \geq 1\}$ be a sequence of non-negative measurable functions such that $\int_{\mathbb{R}^q} f_n(z)\,dz = 1$ and, for all $h \in \mathbb{R}^q$,*

$$\lim_{|h|\to 0} \sup_{n\geq 1} \int_{\mathbb{R}^q} |f_n(z+h) - f_n(z)|\,dz = 0.$$

Suppose that there exists a density function f such that, for all function $\psi \in \mathcal{C}_c(\mathbb{R}^q)$,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^q} f_n(z)\psi(z)\,dz = \int_{\mathbb{R}^q} f(z)\psi(z)\,dz.$$

Then $\{f_n\}$ converges to f in $L^1(\mathbb{R}^q)$.

The next lemma can be found in [16].

Lemma A.3 *Let $1 \leq p < +\infty$. For all $f \in L^p(\mathbb{R}^q)$ and $h \in \mathbb{R}^q$ we have*

$$\lim_{|h|\to 0} \int_{\mathbb{R}^q} |f(z+h) - f(z)|^p\,dz = 0. \tag{A.2}$$

The last lemma can be found in [17, Cor.1.1.5].

Lemma A.4 *Let S be a Polish space and let $\{F_{\epsilon'},\ \epsilon' > 0\}$ be a uniformly bounded set of functions which are equicontinuous at each point of S. For all $\{\mu_\epsilon;\ \epsilon > 0\}$ and μ in $\mathcal{M}(S)$ such that $\lim_{\epsilon\to 0}\mu_\epsilon = \mu$ one has*

$$\lim_{\epsilon\to 0^+} \sup_{\epsilon'>0} \left| \int_S F_{\epsilon'}\,d\mu_\epsilon - \int_S F_{\epsilon'}\,d\mu \right| = 0.$$

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