Probab. Theory Relat. Fields (2011) 151:153–172
DOI 10.1007/s00440-010-0295-6

Strong Central Limit Theorem for isotropic random walks in $\mathbb{R}^d$

Piotr Graczyk · Jean-Jacques Loeb · Tomasz Żak

Received: 8 December 2009 / Revised: 6 March 2010 / Published online: 29 April 2010

Abstract We prove an optimal Gaussian upper bound for the densities of isotropic random walks on $\mathbb{R}^d$ in spherical case ($d \geq 2$) and ball case ($d \geq 1$). We deduce the strongest possible version of the Central Limit Theorem for the isotropic random walks: if $\tilde{S}_n$ denotes the normalized random walk and Y the limiting Gaussian vector, then $\mathbb{E} f(\tilde{S}_n) \to \mathbb{E} f(Y)$ for all functions f integrable with respect to the law of Y. We call such result a "Strong CLT". We apply our results to get strong hypercontractivity inequalities and strong Log-Sobolev inequalities.

Keywords Random walks · Central Limit Theorem · Gaussian estimates · Logarithmic Sobolev inequality

Mathematics Subject Classification (2000) 60G50 · 60F05 · 60B10 · 47D06

This research was partially supported by grants MNiSW N N201 373136 and ANR-09-BLAN-0084-01.

P. Graczyk (✉) · J.-J. Loeb
Laboratoire de Mathématiques LAREMA, Université d'Angers, 2 boulevard Lavoisier,
49045 Angers Cedex 01, France
e-mail: graczyk@univ-angers.fr

J.-J. Loeb
e-mail: loeb@univ-angers.fr

T. Żak
Institute of Mathematics and Computer Science, Wroclaw University of Technology,
Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland
e-mail: Tomasz.Zak@pwr.wroc.pl

1 Introduction

Let σ_d be the normalized Lebesgue measure on the unit sphere S_1^{d-1}, $d \geq 2$. Let $X_1, X_2, \ldots, X_n$ be independent random vectors with the distribution σ_d. The sequence $(S_n)_n$, where $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, describes an isotropic random walk in $\mathbb{R}^d$. Evidently, the distribution of $X_1 + \cdots + X_n$ is given by σ_d^{*n} (written shortly σ_d^n).

Similarly, we will consider μ_d, the uniform measure of the unit ball B_d in $\mathbb{R}^d$, $d \geq 1$ and the related isotropic random walk.

Isotropic random walks were thoroughly examined; the history, useful facts and formulae concerning them can be found e.g. in the monograph of Hughes [17].

One of important applications of isotropic random walks is that in $\mathbb{R}^d$, $d \geq 1$, we can approximate the standard Gaussian measure by (normalized) convolution powers of σ_d or μ_d. If $\tilde{S}_n = \frac{S_n}{\sqrt{n}}$ denotes the normalized random walk, by the Central Limit Theorem, the sequence $\tilde{S}_n$ converges weakly to a Gaussian random vector Y. In Sect. 2 of this article we show that for $d \geq 2$ the densities of $\tilde{S}_n$ are bounded by const $\times\, g_Y$ where g_Y is the density of the limiting Gaussian distribution (Theorems 2.1 and 2.4) in both isotropic cases σ_d and μ_d, as well as in some other related cases, see Subsection 2.3. Gaussian bounds of convolutions of compactly supported symmetric measures with bounded densities have been proved by Hebisch and Saloff-Coste [14] but our optimal estimate by the limiting density g_Y cannot be deduced from [14]. As a corollary of this optimal estimate we prove that

$$\mathbb{E} f(\tilde{S}_n) \to \mathbb{E} f(Y) \quad \text{for all } f \in L^1(g_Y dy), \tag{1}$$

i.e. the convergence holds for any function f integrable with respect to the law of Y. It is an essential strengthening of the classical Central Limit Theorem for the sequence X_n and its strongest possible version. We call our limit theorem a “Strong Central Limit Theorem”. It should not be confused with the “strong approximation” results, i.e. the construction of sums of random vectors together with approximating Gaussian vectors on the same probability space, e.g. the strong KMT approximation [18].

For the considered uniform measures, our Strong Central Limit Theorems generalize to d dimensions a strengthened CLT proved on $\mathbb{R}$ by Fomin [9], see Remark 2.9.

In Sect. 3 we discuss the 1-dimensional case of symmetric Bernoulli distributions $\sigma_1 = \frac{1}{2}(\delta_{-1} + \delta_1)$. The distributions of $\tilde{S}_n = \frac{S_n}{\sqrt{n}}$ are discrete measures and it is natural to compare their tails with the Gaussian tail. An optimal Gaussian upper bound for the tails $P(\tilde{S}_n > x)$, see Proposition 3.2, is a special case of results of Pinelis [19], solving the Eaton’s conjecture.

We can thus say that in Sect. 2 we have generalized Pinelis’ estimate to the d-dimensional case and uniform measures on spheres and balls, for which comparing the densities with the limiting Gaussian density is more adequate.

The research on the tail estimates of sums of random variables and processes has been very active in recent 20 years and there is a rich literature on this subject (see e.g. [6,15,16,20]). However estimates of corresponding densities are not considered in this literature.

From the optimal Gaussian bound for the tails, we deduce in Theorem 3.4 a 1-dimensional version of the strong Central Limit Theorem:

$$\mathbb{E}f(\tilde{S}_n) \to \mathbb{E}f(Y) \quad \text{for all monotonic } f \in L^1(g_Y dy). \tag{2}$$

We also show an example that without some additional assumption (such as monotonicity) the convergence in (2) may fail.

As an application, in Sect. 4 we show how to use our strong Central Limit Theorems to obtain direct proofs of strong hypercontractivity and strong logarithmic Sobolev inequalities for log-subharmonic functions and the Gaussian measure in $\mathbb{R}^d$, $d \geq 1$. This approach mirrors, to some extent, Gross's proof of the Gaussian log-Sobolev inequality in [12].

2 Isotropic random walks in $\mathbb{R}^d$ for $d \geq 2$

2.1 Sphere case σ_d

If $X = (X^{(1)}, X^{(2)}, \ldots, X^{(d)})$ is a random vector with the distribution σ_d, then $\mathbb{E}(X) = (0, 0, \ldots, 0)$ and $Cov(X^{(i)}, X^{(j)}) = \frac{1}{d}\delta_{ij}$, that is, the covariance matrix of X is $\Sigma = \frac{1}{d}I$; in order to justify the last assertion observe that if $Y = (Y_1, \ldots, Y_d)$ is a standard Gaussian vector in $\mathbb{R}^d$, then $\frac{Y}{|Y|}$ has the uniform distribution on S_1^{d-1}.

Define $\gamma_d = N(0, \frac{1}{d}I)$, the Gaussian measure with the density $g_d(y) = (\frac{d}{2\pi})^{d/2}e^{-\frac{d|x|^2}{2}}$ on $\mathbb{R}^d$.

Let $X_1, X_2, \ldots, X_n$ be independent random vectors with the distribution σ_d. By the Central Limit Theorem the distribution of normalized sum $\tilde{S}_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$ tends weakly to γ_d. In the language of convolutions this means that after normalization, σ_d^n tend weakly to γ_d.

It turns out that for $n > d+2$ the measures σ_d^n are absolutely continuous with respect to the Lebesgue measure. Indeed, the characteristic function of σ_d^n equals

$$(\hat{\sigma}_d(y))^n = \left(\Gamma\left(\frac{d}{2}\right)\left(\frac{2}{|y|}\right)^{d/2-1} J_{d/2-1}(|y|)\right)^n,$$

where $J_{d/2-1}$ is a Bessel function of the first kind (see e.g. [17, (2.30)]). When $n > d+2$ this characteristic function is absolutely integrable, which implies that the density of σ_d^n exists and is bounded and continuous. For fixed d let us denote by $f_n(x)$ the density of σ_d^n. We notice that for $n \leq d+2$, the density f_n can exist and be unbounded, see [17].

By the Local Central Limit Theorem [4, Th.19.1], the densities $f_{\tilde{S}_n}$ of the normalized sums $\tilde{S}_n = \frac{S_n}{\sqrt{n}}$, i.e. the functions $(\sqrt{n})^d f_n(\sqrt{n}\, x)$, tend uniformly to the Gaussian density $g_d(y) = (\frac{d}{2\pi})^{d/2}e^{-\frac{d|x|^2}{2}}$. In the main theorem of this section we show that the quotient of these densities and of the limiting Gaussian density is bounded.

Theorem 2.1 (Optimal Gaussian Bound for σ_d^n). *Let $d \geq 2$. There exists a constant C_d such that for all $x \in \mathbb{R}^d$ and $n > d+2$ there holds*

$$f_{\tilde{S}_n}(x) = (\sqrt{n})^d f_n(\sqrt{n}\, x) \leq C_d \left(\frac{d}{2\pi}\right)^{d/2} e^{-\frac{d\,|x|^2}{2}}. \tag{3}$$

Remark 2.2 For the random walk $S_n = X_1 + \cdots + X_n$ the inequality (3) reads as follows: for any $x \in \mathbb{R}^d$ and $n > d+2$,

$$f_n(x) \leq C_d \left(\frac{d}{2\pi\, n}\right)^{d/2} e^{-\frac{d\,|x|^2}{2n}}. \tag{4}$$

The measures σ_d and γ_d are rotationally invariant and so are their densities. Denote $\tilde{f}_n(r) = f_n(|x|)$ for $|x| = r$. The estimates (3) and (4) are equivalent to the following inequality

$$\tilde{f}_n(r) \leq \frac{C_d}{n^{d/2}} e^{-\frac{d\,r^2}{2n}}. \tag{5}$$

To simplify the notation we write C_d for a modified constant. In order to prove Theorem 2.1, we will justify (5).

Proof From the Local Central Limit Theorem it follows that for any $a > 0$ there exists some $c_a > 0$ such that (3) holds for all x with $|x| \leq a$ and $n > d+2$, that is

$$(\sqrt{n})^d f_n(\sqrt{n}\, x) \leq c_a \left(\frac{d}{2\pi}\right)^{d/2} e^{-\frac{d\,|x|^2}{2}}.$$

Equivalently, for all $n > d+2$

$$\tilde{f}_n(r) \leq c_a \left(\frac{d}{2\pi\, n}\right)^{d/2} e^{-\frac{d\,r^2}{2n}}, \quad r \leq a\sqrt{n}.$$

Hence it is enough to prove (5) for $r \geq a\sqrt{n}$.

The maximal distance of S_n to the origin after n steps of the walk is less or equal to n so that $f_n(x) = 0$ for $|x| > n$. Consequently, when $n = d+3$, the inequality (5) holds for all r and a constant C.

Let $a = 3\sqrt{d}$ and $C_d = \max(c_a, C)$. We will show by induction that (5) holds for this C_d. As we have noticed above, (5) is true for $n = d+3$.

We have

$$f_{n+1}(x) = \int_{S_1^{d-1}} f_n(x-u)\, d\sigma_d(u). \tag{6}$$

The function f_n depends only on $|x-u| = (r^2+1-2r\cos(x,u))^{1/2}$. The function f_{n+1} is also radial, so $f_{n+1}(x) = f_{n+1}(0,\ldots,|x|)$. For $x' = (0,\ldots,|x|)$ we have $\cos(x',u) = u_d$. We obtain

$$\tilde{f}_{n+1}(r) = \int_{S_1^{d-1}} \tilde{f}_n\left(\sqrt{r^2+1-2ru_d}\right) d\sigma_d(u).$$

Taking spherical coordinates in $\mathbb{R}^d$ with $u_d = R\cos\phi$ (cf. [17, p.61]) we get

$$\tilde{f}_{n+1}(r) = \frac{|S_1^{d-2}|}{|S_1^{d-1}|}\int_0^\pi \tilde{f}_n\left(\sqrt{r^2+1-2r\cos\phi}\right)\sin^{d-2}\phi\, d\phi, \tag{7}$$

where $|S_1^{d-1}| = 2\pi^{d/2}/\Gamma(d/2)$ is the measure of the unit sphere in $\mathbb{R}^d$. Note that formula (7) is also true for $d=2$ with $|S_1^0| = 2$.

Suppose that for some $n > d+2$ and all $0 \le r \le n$ the inequality (5) is true. Then, by this assumption, we have

$$\begin{aligned}
\tilde{f}_{n+1}(r) &= \frac{|S_1^{d-2}|}{|S_1^{d-1}|}\int_0^\pi \tilde{f}_n\left(\sqrt{r^2+1-2r\cos\phi}\right)\sin^{d-2}\phi\, d\phi \\
&\le \frac{C_d}{n^{d/2}}\frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)}\int_0^\pi e^{-\frac{d(r^2+1-2r\cos\phi)}{2n}}\sin^{d-2}\phi\, d\phi \\
&= \frac{C_d}{n^{d/2}}\frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)}e^{-\frac{d(r^2+1)}{2n}}\int_0^\pi e^{\frac{dr\cos\phi}{n}}\sin^{d-2}\phi\, d\phi \\
&= \frac{C_d}{n^{d/2}}\frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)}e^{-\frac{d(r^2+1)}{2n}}\sqrt{\pi}2^{d/2-1}\Gamma((d-1)/2) \\
&\quad\times\left(\frac{n}{dr}\right)^{d/2-1} I_{d/2-1}\left(\frac{dr}{n}\right) \\
&= \frac{C_d}{n^{d/2}}\Gamma\left(\frac{d}{2}\right)e^{-\frac{d(r^2+1)}{2n}}\left(\frac{2n}{dr}\right)^{d/2-1} I_{d/2-1}\left(\frac{dr}{n}\right),
\end{aligned}$$

because, by [2, (9.6.18)], we have

$$\int_0^\pi e^{x\cos\theta}\sin^{d-2}\theta\, d\theta = \sqrt{\pi}2^{d/2-1}\Gamma((d-1)/2)x^{1-d/2}I_{d/2-1}(x), \tag{8}$$

where I_α is a modified Bessel function of order α.

We want to show that for $r \ge a\sqrt{n+1}$ the last estimate for $\tilde{f}_{n+1}(r)$ is less than $\frac{C_d}{(n+1)^{d/2}}e^{-\frac{dr^2}{2(n+1)}}$, i.e. to prove that for $r \ge a\sqrt{n+1}$

$$\Gamma\left(\frac{d}{2}\right)\left(\frac{2n}{dr}\right)^{d/2-1} I_{d/2-1}\left(\frac{dr}{n}\right) \le \left(\frac{n}{n+1}\right)^{d/2} e^{\frac{d(r^2+1)}{2n}-\frac{dr^2}{2(n+1)}}.$$

But

$$\frac{d(r^2+1)}{2n} - \frac{dr^2}{2(n+1)} = \frac{dr^2}{2n(n+1)} + \frac{d}{2n},$$

hence it is enough to show that

$$\Gamma\left(\frac{d}{2}\right)\left(\frac{2n}{dr}\right)^{d/2-1} I_{d/2-1}\left(\frac{dr}{n}\right) \le \left(\frac{n}{n+1}e^{1/n}\right)^{d/2} e^{\frac{dr^2}{2n(n+1)}}.$$

Now

$$\frac{n}{n+1}e^{1/n} = \frac{n}{n+1}\left(1+\frac{1}{n}+\cdots\right) > 1,$$

hence it is enough to prove that for $r \ge a\sqrt{n+1}$

$$\Gamma\left(\frac{d}{2}\right)\left(\frac{2n}{dr}\right)^{d/2-1} I_{d/2-1}\left(\frac{dr}{n}\right) \le e^{\frac{dr^2}{2n(n+1)}}. \tag{9}$$

For this, we use Taylor expansions of both sides. By the well-known formula [2, (9.6.10)]

$$\frac{I_\alpha(x)}{(x/2)^\alpha} = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k!\Gamma(\alpha+k+1)} \tag{10}$$

and thus the left-hand side of (9) is equal to

$$\text{LHS} = \Gamma\left(\frac{d}{2}\right)\sum_{k=0}^{\infty}\frac{(\frac{dr}{2n})^{2k}}{k!\Gamma(d/2+k)} = \Gamma\left(\frac{d}{2}\right)\sum_{k=0}^{\infty}\frac{(\frac{r}{n})^{2k}(\frac{d}{2})^{2k}}{k!\Gamma(d/2+k)}.$$

The right-hand side of (9) is equal to

$$\text{RHS} = e^{\frac{dr^2}{2n(n+1)}} = e^{(\frac{r}{n})^2\cdot\frac{d}{2}\cdot\frac{n}{n+1}} = \sum_{k=0}^{\infty}\frac{(\frac{r}{n})^{2k}(\frac{d}{2})^k(\frac{n}{n+1})^k}{k!},$$

hence the difference

$$\text{RHS} - \text{LHS} = \sum_{k=0}^{\infty}\frac{(\frac{r}{n})^{2k}(\frac{d}{2})^k}{k!}\left[\left(\frac{n}{n+1}\right)^k - \frac{\Gamma(d/2)}{\Gamma(d/2+k)}\left(\frac{d}{2}\right)^k\right].$$

Denote the quantity in the square bracket: $A_{n,k} = (\frac{n}{n+1})^k - \frac{\Gamma(d/2)}{\Gamma(d/2+k)}(\frac{d}{2})^k$. It is easy to check that $A_{n,0} = 0$ and $A_{n,1} = -\frac{1}{n+1}$. Now we show that for $k = 2, 3, \ldots$, we have $A_{n,k} > 0$. Indeed, for $k = 2$

$$A_{n,2} = \left(\frac{n}{n+1}\right)^2 - \frac{\Gamma(d/2)}{\Gamma(d/2+2)}\left(\frac{d}{2}\right)^2 = \left(\frac{n}{n+1}\right)^2 - \frac{1}{\frac{d}{2}(\frac{d}{2}+1)}\left(\frac{d}{2}\right)^2$$
$$= \left(\frac{n}{n+1}\right)^2 - \frac{d/2}{\frac{d}{2}+1} = \left(\frac{n}{n+1}\right)^2 - \frac{d}{d+2} > 0, \quad \text{if } n \geq d+2.$$

For $k \geq 3$ we estimate:

$$A_{n,k} = \left(\frac{n}{n+1}\right)^k - \frac{\Gamma(d/2)}{\Gamma(d/2+k)}\left(\frac{d}{2}\right)^k = \left(\frac{n}{n+1}\right)^k - \frac{(d/2)^{k-1}}{(\frac{d}{2}+1)\ldots(\frac{d}{2}+k-1)} > 0,$$

because for $k = 2$ we chose n such that $(\frac{n}{n+1})^2 > \frac{d}{d+2}$, and for $k \geq 3$ this implies

$$\frac{(d/2)^{k-1}}{(\frac{d}{2}+1)\ldots(\frac{d}{2}+k-1)} \leq \left(\left(\frac{n}{n+1}\right)^2\right)^{k-1} < \left(\frac{n}{n+1}\right)^k.$$

Finally,

$$\text{RHS} - \text{LHS} > -\frac{1}{n+1}\cdot\left(\frac{r}{n}\right)^2\frac{d}{2} + \frac{1}{2!}\left(\frac{r}{n}\right)^4\left(\frac{d}{2}\right)^2\left[\left(\frac{n}{n+1}\right)^2 - \frac{d}{d+2}\right]$$
$$= \left(\frac{r}{n}\right)^2\frac{d}{2}\left[\frac{1}{2}\left(\frac{r}{n}\right)^2\frac{d}{2}\left(\left(\frac{n}{n+1}\right)^2 - \frac{d}{d+2}\right) - \frac{1}{n+1}\right].$$

Now, if $r > a\sqrt{n+1} > a\sqrt{n}$ then

$$\frac{1}{2}\left(\frac{r}{n}\right)^2\frac{d}{2}\left(\left(\frac{n}{n+1}\right)^2 - \frac{d}{d+2}\right) > \frac{1}{n}\frac{da^2}{4}\left(\left(\frac{n}{n+1}\right)^2 - \frac{d}{d+2}\right).$$

When $d \geq 2$ and $n \geq d+2$ we have $((\frac{n}{n+1})^2 - \frac{d}{d+2})^{-1} < 2d^2$. Thus for $a \geq 3\sqrt{d}$ the following inequality holds

$$\frac{1}{n}\frac{da^2}{4}\left(\left(\frac{n}{n+1}\right)^2 - \frac{d}{d+2}\right) > \frac{1}{n+1}.$$

□

We can now deduce the following strengthened Central Limit Theorem.

Theorem 2.3 (Strong CLT for σ_d) *Let $d \geq 2$, Y be a random Gaussian vector with law γ_d and $f \in L^1(\gamma_d)$. Then* $\lim_n \mathbb{E} f(\tilde{S}_n) = \mathbb{E} f(Y)$.

Proof Let $k_n(x) = \sqrt{n} f_n(\sqrt{n}\,x)$ be the density of the normalized sum $\tilde{S}_n = \frac{X_1+\cdots+X_n}{\sqrt{n}}$. By the Local Central Limit Theorem, $\lim_n f k_n = f g_d$. By Theorem 2.1, $|f k_n| \le C_d |f g_d| \in L^1(\mathbb{R}^d)$. By the Dominated Convergence Theorem $\int f(x) k_n(x)\,dx \to \int g(x) d\gamma_d(x)$. □

2.2 Ball case μ_d

Let $(X_1, X_2, \ldots, X_d)$ be a random vector with the uniform distribution μ_d on the unit ball B_d in $\mathbb{R}^d$. When $d \ge 2$, we have in polar coordinates (for $0 \le r \le 1$ and $s \in S_1^{d-1}$)

$$d\mu_d(r, s) = dr^{d-1}\,dr \frac{ds}{|S_1^{d-1}|}$$

Now $\sigma_d(ds) = \frac{ds}{|S_1^{d-1}|}$ is a probability measure on the unit sphere and so is $dr^{d-1}\,dr$ on the unit interval.

Because $\mathbb{E}(X_1^2 + \cdots + X_d^2) = \int_0^1 r^2 \cdot dr^{d-1}\,dr = d\int_0^1 r^{d+1}\,dr = \frac{d}{d+2}$, hence $\mathbb{E}(X_i^2) = \frac{1}{d+2}$ and the covariance matrix of $(X_1, X_2, \ldots, X_d)$ equals $\frac{1}{d+2} I$. Observe that for $d = 1$ the formula $\mathbb{E}(X_1^2) = \frac{1}{d+2}$ also holds.

The weak limit of normalized sums $\tilde{S}_n = \frac{X_1+\cdots+X_n}{\sqrt{n}}$ is $\nu_d = N(0, \frac{1}{d+2} I)$, the Gaussian measure with the density $h_d(y) = (\frac{d+2}{2\pi})^{d/2} e^{-\frac{(d+2)\,|x|^2}{2}}$ on $\mathbb{R}^d$. The Gaussian measure approximating μ^n, the law of $S_n = X_1 + \cdots + X_n$, has the radial part $g(R) = (\frac{d+2}{2\pi n})^{d/2} e^{-\frac{(d+2)R^2}{2n}}$. Denote by f_n the density of μ_d^n.

Theorem 2.4 (Optimal Gaussian Bound for μ_d^n). *Let $d \ge 1$. There exists a constant C_d such that for all $x \in \mathbb{R}^d$ and $n \ge 1$ there holds*

$$f_{\tilde{S}_n}(x) = (\sqrt{n})^d f_n(\sqrt{n}\,x) \le C_d \left(\frac{d+2}{2\pi}\right)^{d/2} e^{-\frac{(d+2)\,|x|^2}{2}}. \tag{11}$$

Proof Similarly as in the proof of Theorem 2.1, by the Local CLT we see that it is sufficient to prove that there exist constants $C, a > 0$ such that

$$\tilde{f}_n(R) \le \frac{C}{n^{d/2}} e^{-\frac{(d+2)R^2}{2n}}, \quad R > a\sqrt{n}, n \in \mathbb{N}. \tag{12}$$

We set $a = 2$. The proof of the inequality (12) will proceed by induction. The starting point of induction n_0 will be specified later. There is a constant $C = C(n_0)$ such that (12) holds for all $R > 0$ and $n = 1, \ldots, n_0$.

Analogously to (6) we have

$$f_{n+1}(x) = \int_{B_d} f_n(x - u) d\sigma_d(u). \tag{13}$$

Here we divide the proof into two cases $d \ge 2$ and $d = 1$.

Case $d \geq 2$. Applying the polar coordinates, similarly as the formula (7), we obtain

$$\tilde{f}_{n+1}(R) = \frac{|S_1^{d-2}|}{|S_1^{d-1}|} \int_0^1 \int_0^\pi f_n\left(\sqrt{r^2 + R^2 - 2rR\cos\phi}\right) dr^{d-1} \sin^{d-2}\phi \, d\phi dr.$$

Using the assumption $\tilde{f}_n(R) \leq \frac{C_d}{n^{d/2}} e^{-\frac{(d+2)R^2}{2n}}$ and the formula (8) we get

$$\tilde{f}_{n+1}(R) \leq \frac{C_d}{n^{d/2}} \int_0^1 dr^{d-1} e^{-\frac{(d+2)(r^2+R^2)}{2n}} \Gamma\left(\frac{d}{2}\right) \frac{I_{d/2-1}\left(\frac{(d+2)rR}{n}\right)}{\left(\frac{(d+2)rR}{2n}\right)^{d/2-1}} dr.$$

We want to prove that for n sufficiently big and all $R > a\sqrt{n}$ there holds the following inequality

$$\Gamma\left(\frac{d}{2}\right) \int_0^1 dr^{d-1} e^{-\frac{(d+2)(r^2+R^2)}{2n}} \frac{I_{d/2-1}\left(\frac{(d+2)rR}{n}\right)}{\left(\frac{(d+2)rR}{2n}\right)^{d/2-1}} dr \leq \left(\frac{n}{n+1}\right)^{d/2} e^{-\frac{(d+2)R^2}{2(n+1)}}.$$

This is equivalent to the following

$$\Gamma\left(\frac{d}{2}\right) \int_0^1 dr^{d-1} e^{-\frac{(d+2)r^2}{2n}} \frac{I_{d/2-1}\left(\frac{(d+2)rR}{n}\right)}{\left(\frac{(d+2)rR}{2n}\right)^{d/2-1}} dr \leq \left(\frac{n}{n+1}\right)^{d/2} e^{\frac{(d+2)R^2}{2n(n+1)}}.$$

When $0 < r < 1$ and $n > d + 2$, we have $0 < \frac{(d+2)r^2}{2n} < \frac{1}{2}$ so that for such r and n there holds

$$e^{-\frac{(d+2)r^2}{2n}} \leq 1 - \frac{(d+2)r^2}{2n} + \frac{1}{2}\left(\frac{(d+2)r^2}{2n}\right)^2 \leq 1 - \left(1 - \frac{d+2}{4n}\right) \frac{(d+2)r^2}{2n}.$$

Thus, by positivity of the function $I_{d/2+1}$, it is enough to prove that

$$\Gamma\left(\frac{d}{2}\right) \int_0^1 dr^{d-1} \left(1 - \frac{\left(1 - \frac{d+2}{4n}\right)(d+2)r^2}{2n}\right) \frac{I_{d/2-1}\left(\frac{(d+2)rR}{n}\right)}{\left(\frac{(d+2)rR}{2n}\right)^{d/2-1}} dr$$
$$\leq \left(\frac{n}{n+1}\right)^{d/2} e^{\frac{(d+2)R^2}{2n(n+1)}}. \tag{14}$$

We have to evaluate two integrals involving Bessel functions. By [11, (5.52.1), page 624] $\int x^{p+1} I_p(x)\,dx = x^{p+1} I_{p+1}(x)$, hence

$$\int_0^1 r^{d/2} I_{\frac{d}{2}-1}\left(\frac{(d+2)rR}{n}\right) dr = \frac{n I_{\frac{d}{2}}\left(\frac{(d+2)R}{n}\right)}{(d+2)R}.$$

Integrating by parts and using identity $xI_{p-1}(x) - xI_{p+1}(x) = 2pI_p(x)$ [11, (8.486.1), page 918], we get

$$\int_0^1 r^{d/2+2} I_{\frac{d}{2}-1}\left(\frac{(d+2)rR}{n}\right) dr = \frac{n\left[dnI_{\frac{d}{2}+1}\left(\frac{(d+2)R}{n}\right) + (d+2)RI_{\frac{d}{2}+2}\left(\frac{(d+2)R}{n}\right)\right]}{(d+2)^2R^2}.$$

Consequently, the left-hand side of (14) is

$$\frac{\Gamma(\frac{d}{2}+1)}{8n^2}\left[\left(8n^2 - 4n(d+2) + (d+2)^2\right)\frac{I_{\frac{d}{2}}\left(\frac{(d+2)R}{n}\right)}{\left(\frac{(d+2)R}{2n}\right)^{d/2}} + (d+2)(4n-d-2)\frac{I_{\frac{d}{2}+1}\left(\frac{(d+2)R}{n}\right)}{\left(\frac{(d+2)R}{2n}\right)^{d/2+1}}\right].$$

Write $g_1 = \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2}+k+1)}$ and $g_2 = \frac{\Gamma(\frac{d}{2}+1)}{\Gamma(\frac{d}{2}+k+2)}$. By the formula (10) the last quantity is equal

$$\sum_{k=0}^{\infty} \frac{\left(\frac{(d+2)R}{2n}\right)^{2k}}{k!}\left(g_1\frac{8n^2 - 4n(d+2) + (d+2)^2}{8n^2} + g_2\frac{(d+2)(4n-d-2)}{8n^2}\right).$$

The right-hand side of (14) has the following expansion:

$$\left(\frac{n}{n+1}\right)^{d/2} e^{\frac{(d+2)R^2}{2n(n+1)}} = \sum_{k=0}^{\infty} \frac{\left(\frac{(d+2)R}{2n}\right)^{2k}}{k!}\left(\frac{n}{n+1}\right)^{\frac{d}{2}+k}\left(\frac{2}{d+2}\right)^k.$$

Thus inequality (14) is equivalent to the following inequality for expansions:

$$\sum_{k=0}^{\infty} B_{n,k}\frac{\left(\frac{(d+2)R}{2n}\right)^{2k}}{k!} \geq 0, \qquad (15)$$

where for fixed $d \geq 2$ we define the coefficients

$$B_{n,k} = \left(\frac{n}{n+1}\right)^{\frac{d}{2}+k} \left(\frac{2}{d+2}\right)^k - g_1 \frac{8n^2 - 4n(d+2) + (d+2)^2}{8n^2} - g_2 \frac{(d+2)(4n-d-2)}{8n^2}.$$

We have

$$B_{n,0} = \left(\frac{n}{n+1}\right)^{d/2} - 1 + \frac{4nd - d(d+2)}{8n^2} \sim -\frac{d(d+2)(d+4)}{48n^3}, \quad \text{as } n \to \infty.$$

We will show that $B_{n,k} > 0$ for all $k \geq 3$ and that for $R > a\sqrt{n}$ there holds

$$B_{n,0} + B_{n,1} \left(\frac{(d+2)R}{2n}\right)^2 + \frac{1}{2} B_{n,2} \left(\frac{(d+2)R}{2n}\right)^4 > 0. \tag{16}$$

We have

$$B_{n,1} = \frac{2}{d+2} \left(\frac{n}{n+1}\right)^{d/2+1} - \frac{2}{d+2} \left(1 - \frac{4n(d+2) - (d+2)^2}{8n^2}\right) - \frac{4n-d-2}{2n^2(d+4)}$$

and if we put $R = a\sqrt{n}$ in the second term of the inequality (16), then we get

$$B_{n,1} \left(\frac{(d+2)a\sqrt{n}}{2n}\right)^2 \sim -\frac{a^2(d+2)^2}{2(d+4)n^2}, \quad \text{as } n \to \infty.$$

Similarly, the third term of the inequality (16) for $R = a\sqrt{n}$ satisfies

$$\frac{1}{2} B_{n,2} \left(\frac{(d+2)a\sqrt{n}}{2n}\right)^4 \sim \frac{a^4(d+2)^2}{4(d+4)n^2}, \quad \text{as } n \to \infty.$$

In order that inequality (16) holds for $n > n_0(d)$, the parameter a must fulfill the condition

$$\frac{1}{n^2} \left(-\frac{d(d+2)(d+4)}{48n} - \frac{a^2(d+2)^2}{2(d+4)} + \frac{a^4(d+2)^2}{4(d+4)}\right) > 0,$$

which is true for $n > n_0(d)$, if only

$$\frac{a^4(d+2)^2}{4(d+4)} > \frac{a^2(d+2)^2}{2(d+4)},$$

and this is true for $a > \sqrt{2}$. In the bi-squared polynomial $p(x) = Ax^4 + Bx^2 + C$ from (16), we have $A > 0$ and $p(0) = C < 0$, hence $p(a\sqrt{n}) > 0$ implies $p(R) > 0$ for all $R > a\sqrt{n}$ and the inequality (16) holds for such R.

Finally let us show that $B_{n,k} > 0$ if $k \geq 3$. Let $\varepsilon > 0$. We have $B_{n,k} = (\frac{2}{d+2})^k[(\frac{n}{n+1})^{\frac{d}{2}+k} - E]$ where

$$E = \frac{(d+2)^{k-1}}{(d+4)\dots(d+2k)}\left[1 + \frac{(4n-d-2)}{4n}\cdot\frac{d+2}{n}\cdot\frac{d+2k+4}{d+2k+2}\right] < (1+\varepsilon)\frac{(d+2)^{k-1}}{(d+4)\dots(d+2k)},$$

because $\frac{d+2}{n}$ can be as small as we want, if n is big enough. This implies that it is sufficient to prove that for some small ε and n big enough there holds

$$\left(\frac{n}{n+1}\right)^{\frac{d}{2}+k} > (1+\varepsilon)\left(\frac{d+2}{d+4}\right)^{k-2}\frac{d+2}{d+2k} > (1+\varepsilon)\frac{(d+2)^{k-1}}{(d+4)\dots(d+2k)}.$$

Choose $\varepsilon > 0$ such that $(1+\varepsilon)\frac{d+2}{d+4} < 1$. Then for n big enough and $k \geq 3$ there holds

$$\left(\frac{n}{n+1}\right)^{\frac{d}{2}+2} > (1+\varepsilon)\frac{d+2}{d+4} > (1+\varepsilon)\frac{d+2}{d+2k} \quad \text{and} \quad \frac{n}{n+1} > \frac{d+2}{d+4}.$$

The inequality $B_{n,k} > 0$ for $k \geq 3$ is proved.
Case $d = 1$. In this case μ_1 is the uniform distribution on the interval $[-1, 1]$ with the density $f(x) = \frac{1}{2}\mathbf{1}_{[-1,1]}(x)$. We will show that for a constant C_1

$$f_n(x) = f^{*n}(x) \leq \frac{C_1}{\sqrt{2\pi n}}e^{-\frac{3x^2}{2n}}.$$

Suppose that for all $x \in \mathbb{R}$ and some $n \in \mathbb{N}$ there holds $f_n(x) \leq \frac{C_1}{\sqrt{2\pi n}}e^{-\frac{3x^2}{2n}}$. Then, by (6)

$$f_{n+1}(x) = \int_{-1}^{1} f_n(x-u)\frac{du}{2} \leq \int_{-1}^{1}\frac{C_1}{\sqrt{2\pi n}}e^{-\frac{3(x-u)^2}{2n}}\frac{du}{2}.$$

We have to prove that

$$\int_{-1}^{1}\frac{C_1}{\sqrt{2\pi n}}e^{-\frac{3(x-u)^2}{2n}}\frac{du}{2} \leq \frac{C_1}{\sqrt{2\pi(n+1)}}e^{-\frac{3x^2}{2(n+1)}},$$

which is equivalent to

$$\int_{-1}^{1} e^{-\frac{3(x-u)^2}{2n}}\frac{du}{2} \leq \sqrt{\frac{n}{n+1}}e^{-\frac{3x^2}{2(n+1)}}.$$

But the left-hand side of the above is equal to

$$\int_{-1}^{1} e^{-\frac{3(x-u)^2}{2n}} \frac{du}{2} = e^{-\frac{3x^2}{2n}} \int_{-1}^{1} e^{\frac{3xu}{n}} e^{-\frac{3u^2}{2n}} \frac{du}{2}.$$

For $-1 \leq u \leq 1$ and $n \geq 2$ there holds $0 \leq \frac{3u^2}{2n} \leq \frac{3}{4}$ and we can estimate $e^{-\frac{3u^2}{2n}}$ from above by three first terms of its Taylor expansion. This gives the following inequality

$$\int_{-1}^{1} e^{\frac{3xu}{n}} e^{-\frac{3u^2}{2n}} \frac{du}{2} \leq \int_{-1}^{1} e^{\frac{3xu}{n}} \left(1 - \left(1 - \frac{3}{4n}\right) \frac{3u^2}{2n}\right) \frac{du}{2}$$
$$= \frac{\sinh\left(\frac{3x}{n}\right)}{\frac{3x}{n}} - \left(1 - \frac{3}{4n}\right) \left[\frac{2n^2 + 9x^2}{18x^3} \sinh\left(\frac{3x}{n}\right) - \frac{nx}{3x^3} \cosh\left(\frac{3x}{n}\right)\right].$$

Now we have to show that for $x > a\sqrt{n}$ there holds

$$\frac{\sinh\left(\frac{3x}{n}\right)}{\frac{3x}{n}} - \left(1 - \frac{3}{4n}\right) \left[\frac{2n^2 + 9x^2}{18x^3} \sinh\left(\frac{3x}{n}\right) - \frac{nx}{3x^3} \cosh\left(\frac{3x}{n}\right)\right] \leq \sqrt{\frac{n}{n+1}} e^{\frac{3x^2}{2n(n+1)}}.$$

We expand both sides in Taylor series and get

$$\text{RHS} = \sum_{k=0}^{\infty} \frac{\left(\frac{3x}{n}\right)^{2k}}{6^k k!} \left(\frac{n}{n+1}\right)^{k+\frac{1}{2}},$$
$$\text{LHS} = \sum_{k=0}^{\infty} \frac{\left(\frac{3x}{n}\right)^{2k} \left[1 - \frac{3}{2n} + \frac{3}{n(2k+3)} + \frac{9}{8n^2} - \frac{9}{4n^2(2k+3)}\right]}{(2k+1)!}$$

so that RHS $-$ LHS $= \sum_{k=0}^{\infty} B_{n,k} \frac{(\frac{3x}{n})^{2k}}{k!}$, where

$$B_{n,k} = \left(\frac{n}{n+1}\right)^{k+\frac{1}{2}} \frac{1}{6^k} - \frac{k!}{(2k+1)!} \left(1 - \frac{3}{2n} + \frac{3}{n(2k+3)} + \frac{9}{8n^2} - \frac{9}{4n^2(2k+3)}\right).$$

In particular, $B_{n,0} = (\frac{n}{n+1})^{\frac{1}{2}} - 1 + \frac{1}{2n} - \frac{3}{8n^2}$, $B_{n,1} = \frac{1}{6}(\frac{n}{n+1})^{\frac{3}{2}} - \frac{1}{6}(1 - \frac{9}{10n} + \frac{27}{40n^2})$ and $B_{n,2} = \frac{1}{36}(\frac{n}{n+1})^{\frac{5}{2}} - \frac{2!}{5!}(1 - \frac{15}{14n} + \frac{45}{56n^2})$. We observe that $B_{n,0} \sim -5/(16n^3)$, $B_{n,1} \sim -1/(10n)$ and $B_{n,2} \sim 1/90$.

The rest of the proof is similar to the proof of the case $d \geq 2$. Put $x = a\sqrt{n}$ and consider the fourth degree polynomial of a

$$w_n(a) = B_{n,0} + B_{n,1} \left(\frac{3a\sqrt{n}}{n}\right)^2 + \frac{1}{2} B_{n,2} \left(\frac{3a\sqrt{n}}{n}\right)^4.$$

For large n we have

$$w_n(a) \sim -\frac{5}{16n^3} - \frac{9a^2}{10n^2} + \frac{9a^4}{10n^2} = \frac{1}{n^2}\left(-\frac{5}{16n} - \frac{9a^2}{10} + \frac{9a^4}{10}\right).$$

For all n, if $a \geq \sqrt{2}$, then $w_n(a) > 0$.

It remains to prove that $B_{n,k} > 0$ for $k \geq 3$. Observe first that

$$-\frac{3}{2n}+\frac{3}{n(2k+3)}+\frac{9}{8n^2}-\frac{9}{4n^2(2k+3)}=\frac{3}{2n}\left[-1+\frac{1}{k+3/2}+\frac{3}{2n}\left(\frac{1}{2}-\frac{1}{2k+3}\right)\right]<0$$

for $k \geq 3$ and $n \geq 2$. Hence it is enough to show that for $k \geq 3$

$$\frac{1}{6^k}\left(\frac{n}{n+1}\right)^{k+1/2} - \frac{k!}{(2k+1)!} > 0.$$

We must show that for $n > n_0(d)$ and all $k \geq 3$

$$\left(\frac{n}{n+1}\right)^{k+1/2} > \frac{6^k k!}{(2k+1)!} = \frac{6^k}{(k+1)(k+2)\dots(2k+1)}.$$

We check by simple calculations that this is true for $k = 3, 4, 5$. For instance for $k = 3$ and n big enough $(\frac{n}{n+1})^{3+1/2} > \frac{6^3}{4\cdot5\cdot6\cdot7} = \frac{18}{70}$. When $k \geq 6$, we have

$$\frac{6^k}{(k+1)(k+2)\dots(2k+1)} < \left(\frac{6}{7}\right)^k \frac{1}{2k+1}.$$

We notice that $\frac{n}{n+1} > \frac{6}{7}$ for $n > 6$ and $(\frac{n}{n+1})^{\frac{1}{2}} > \frac{1}{2k+1}$ for $n \geq 1$. This ends the proof. □

In the same way as the Theorem 2.3 we prove

Theorem 2.5 (Strong CLT for μ_d) *Let $d \geq 1$. Let Y be a random Gaussian vector with law ν_d and $f \in L^1(\nu_d)$. Then* $\lim_n \mathbb{E}f(\tilde{S}_n) = \mathbb{E}f(Y)$.

Remark 2.6 The Fourier transform of the uniform distribution on the sphere σ_d is well known, cf. Folland [8, p.248], Stein–Weiss [21, Chapter 4], and is expressed by a Bessel function $J_{(d/2)-1}$. Consequently the Fourier transform of σ_d^n is also explicitly known. However, using the inverse Fourier transform in order to estimate the density of σ_d^n does not seem feasible, due to the oscillations of Bessel functions.

Similar remarks may be done for the measures μ_d.

2.3 Convolutions

The Optimal Gaussian Bound is inherited by convolutions of measures having this property. Below we write $\tilde{S}_n(X) = (X_1 + \cdots + X_n)/\sqrt{n}$.

Proposition 2.7 *Let (X_i), (Y_j) be independent random variables with laws μ and ν, with zero means and covariance matrices A and B respectively. Suppose that μ and ν satisfy the Optimal Gaussian Bound inequality*:

$$f_{\tilde{S}_n(X)} \leq C_1 f_{N(0,A)}, \quad f_{\tilde{S}_n(Y)} \leq C_2 f_{N(0,B)}.$$

Then

$$f_{\tilde{S}_n(X+Y)} \leq C_1 C_2 f_{N(0,A+B)},$$

i.e. the measure $\mu * \nu$ satisfies the Optimal Gaussian Bound inequality.

Proof We have $\tilde{S}_n(X+Y) = \tilde{S}_n(X) + \tilde{S}_n(Y)$, so

$$f_{\tilde{S}_n(X+Y)} = f_{\tilde{S}_n(X)} * f_{\tilde{S}_n(Y)} \leq C_1 C_2 f_{N(0,A)} * f_{N(0,B)}$$

and the Proposition follows. □

It is clear that Theorems 2.1 and 2.4 hold for uniform measures $\sigma_d(r)$ and $\mu_d(r)$ on spheres and balls of radius $r > 0$, respectively. By Proposition 2.8 we obtain

Corollary 2.8 *Finite convolutions of measures $\sigma_d(r)$ (when $d \geq 2$) and $\mu_d(s)$ satisfy the Optimal Gaussian Bound inequality and the Strong Central Limit Theorem.*

Remark 2.9 In [9] Fomin studies on $\mathbb{R}$ a Central Limit Theorem strengthened in the following way. Let γ be the standard normal law with density $g(x) = d\gamma(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. Let $\mathcal{H}_\gamma$ be a Hilbert space of Borel functions on $\mathbb{R}$, equipped with the norm

$$\| u \| = \left(\int_{-\infty}^{\infty} u^2(x) e^{x^2/2}\, dx \right)^{1/2}.$$

In the main Theorem of [9] sufficient conditions are given for a probability density $p(x)$ on $\mathbb{R}$ in order that $p_n(x) = \sqrt{n} p^{*n}(\sqrt{n}x)$ converges in $\mathcal{H}_\gamma$ to the standard Gaussian density $g(x)$. It is shown that the uniform density on $[-\sqrt{3}, \sqrt{3}]$ verifies this property.

By the Local CLT and the dominated convergence theorem, it is easy to see that if p^{*n} satisfies an Optimal Gaussian Bound inequality, then $p_n(x)$ converges in $\mathcal{H}_\gamma$ to the limiting Gaussian density γ. Thus Theorems 2.1, 2.4 and 2.8 imply that the densities of σ_d^n, μ_d^n as well as of their finite convolutions converge in $\mathcal{H}_\gamma$ to the limiting Gaussian density γ. Theorem 2.4 implies the main Theorem of [9] for the uniform symmetric measures (without giving the rate of convergence).

On the other hand, in Proposition 13 of [9], a class of symmetric bounded densities with compact support is indicated for which p_n do not converge in $\mathcal{H}_\gamma$. To belong to this class it is enough that $\int_T^\infty p(x)dx > \exp(-T^2/4)$ for some $T > 0$. Consequently, for such densities p there is no Optimal Gaussian Bound inequality.

It would be interesting to characterize the class of probability densities on $\mathbb{R}^d$ for which an Optimal Gaussian Bound inequality holds. This question seems however difficult. An analogous question for the convergence of p_n in $\mathcal{H}_\gamma$ was not solved in [9].

3 Isotropic random walk in $\mathbb{R}^1$

Let $X_1, X_2, \ldots$ be independent random variables with the same symmetric Bernoulli distribution, $P(X_n = 1) = P(X_n = -1) = \frac{1}{2}$, and put $S_n = X_1 + \cdots + X_n$. By μ_n we denote the distribution of the normalized sum $\tilde{S}_n = \frac{S_n}{\sqrt{n}}$. Theorem 3.1 is a result of Pinelis [19, Cor.2.6], obtained in answer to the Eaton's conjecture. Another proof of Theorem 3.1 was given by Bobkov et al. [5].

Let Y be the standard normal $N(0, 1)$ random variable. If Φ is the distribution function of Y, then the tail $\Psi(x) = 1 - \Phi(x)$.

Theorem 3.1 *Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $xx^T = 1$. There exists a constant $C > 0$ such that for any $u > 0$ and $n \in \mathbb{N}$*

$$P\left(|x_1 X_1 + \cdots + x_n X_n| > u\right) \leq C(1 - \Phi(u)) = C\Psi(u).$$

Using Theorem 3.1 with $x_i = 1/\sqrt{n}$, $i = 1, \ldots, n$, we obtain

Proposition 3.2 (Optimal Gaussian Bound for σ_1^n) *There exists $C > 0$ such that for all $x > 0$ and all $n \in \mathbb{N}$*

$$\Psi_n(x) = P(\tilde{S}_n > x) \leq C\, P(Y > x) = C\Psi(x).$$

Observe that a weaker form of Proposition 3.2 was obtained by Talagrand [22, (1.3)], as a strengthening of the Hoeffding inequality see e.g. [7, Prop.1.3.5].

In order to prove a 1-dimensional strengthened Central Limit Theorem, we need the following Proposition.

Proposition 3.3 *If $g \in L^1(\gamma)$ is in $\mathcal{C}^1([0, \infty))$ and g is strictly increasing on $[x_0, \infty)$ for an $x_0 \geq 0$ then*

$$\int_{x_0}^{\infty} g d\gamma = g(x_0)\Psi(x_0) + \int_{x_0}^{\infty} g' \Psi dx. \tag{17}$$

In particular, $g'\Psi \in L^1(Leb)$.

The formula (17) is also true with Bernoulli-type measures μ_n in the place of the Gaussian law γ

$$\int_{x_0}^{\infty} g d\mu_n = g(x_0)\Psi_n(x_0) + \int_{x_0}^{\infty} g' \Psi_n dx, \quad n \in \mathbb{N}. \tag{18}$$

Proof In order to prove (17), we define Y^{x_0} as a bounded and positive random variable with law $\gamma|_{[x_0,\infty)}/\gamma([x_0,\infty))$. By Fubini theorem we write

$$\frac{1}{\gamma([x_0,\infty))}\int_{x_0}^{\infty} g d\gamma = \mathbb{E}g(Y^{x_0}) = \int_0^{\infty} P(g(Y^{x_0}) > x)dx$$
$$= \left(\int_0^{g(x_0)} + \int_{g(x_0)}^{\infty}\right) P(g(Y^{x_0}) > x)dx = g(x_0) + \int_{g(x_0)}^{\infty} P(g(Y^{x_0}) > x)dx.$$

The function g is a $\mathcal{C}^1$ bijection of $[x_0,\infty)$ and $[g(x_0), G)$, where $G = \lim_{x\to\infty} g(x)$. In the last integral we change the variables $u = g^{-1}(x)$ and we obtain

$$\int_{g(x_0)}^{\infty} P(g(Y^{x_0}) > x)dx = \int_{g(x_0)}^{G} P(g(Y^{x_0}) > x)dx = \int_{x_0}^{\infty} P(Y^{x_0} > u)g'(u)du$$

and (17) follows. The proof for symmetric binomial measures μ_n is analogous. □

Theorem 3.4 (Strong CLT for σ_1) *If $g \in L^1(\gamma)$ is in $\mathcal{C}^1([0,\infty))$ and g is strictly monotonous on $[x_0,\infty)$ for an $x_0 \geq 0$, then the DeMoivre–Laplace CLT holds for g:*

$$\int_0^{\infty} g d\mu_n \to \int_0^{\infty} g d\gamma, \quad n \to \infty.$$

Proof In the proof we suppose that g is strictly increasing on $[x_0,\infty)$. By the Central Limit Theorem $\int_0^{x_0} g d\mu_n \to \int_0^{x_0} g d\gamma$ and $\Psi_n(x) \to \Psi(x)$, $n \to \infty$. In order to establish the convergence of integrals on $[x_0,\infty)$, write the formula (18). The convergence of the term $\int_{x_0}^{\infty} g'\Psi_n dx$ to $\int_{x_0}^{\infty} g'\Psi dx$ follows by the Dominated Convergence Theorem if we use the Proposition 3.2 and the integrability of $g'\Psi$ with respect to the Lebesgue measure on $\mathbb{R}^+$. An application of (17) ends the proof. □

In order to get a strong Central Limit Theorem from Proposition 3.1 in dimension one, some additional assumptions (such as monotonicity) must be put on f. This is showed by the following example.

Counterexample on the real line. For $n = 4k^2$ we have

$$\mu_{4k^2}(\{2k\}) = P\left(\frac{S_{4k^2} - 2k^2}{k} = 2k\right) = P\left(S_{4k^2} = 4k^2\right) = \left(\frac{1}{2}\right)^{4k^2}.$$

Consider the following continuous function $f : \mathbb{R} \to \mathbb{R}$: for $k = 1, 2, \ldots$ put $f(k) = 3^{4k^2}$ and $f(k \pm 4^{-4k^2}) = 0$. Now let f be linear and continuous on intervals $(k - 4^{-4k^2}, k)$ and $(k, k + 4^{-4k^2})$. Finally put $f(x) = 0$ on the complement of $\cup_{k=1}^{\infty}[k - 4^{-4k^2}, k + 4^{-4k^2}]$.

Let γ_1 denote the standard Gaussian distribution on the real line. Because f is nonnegative and the standard Gaussian density on $\mathbb{R}$ is bounded by $\frac{1}{\sqrt{2\pi}}$, we have the following estimate:

$$0 < \int_{-\infty}^{\infty} f(x)\, d\gamma_1(x) = \sum_{k=1}^{\infty} \int_{k-4^{-4k^2}}^{k+4^{-4k^2}} f(x)\, d\gamma_1(x) < \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \int_{k-4^{-4k^2}}^{k+4^{-4k^2}} f(x)\, dx$$
$$= \frac{1}{\sqrt{2\pi}} \sum_{k=1}^{\infty} \frac{3^{4k^2}}{4^{4k^2}} < \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k = 3.$$

On the other hand we have

$$\int_{-\infty}^{\infty} f(x)\, d\mu_{4k^2}(x) \geq f(k)\mu_{4k^2}(\{2k\}) = \frac{3^{4k^2}}{2^{4k^2}} \longrightarrow \infty, \quad \text{as } k \to \infty.$$

Of course, after a simple modification we can construct f of class C^∞, having the same integration properties.

4 Applications: strong hypercontractivity and Log-Sobolev inequalities

Applications of the Central Limit Theorem in studying hypercontractivity and Logarithmic Sobolev Inequalities (LSI) have always been important and useful, see [1, 12, 13]. In this section we show how the strengthened Central Limit Theorems we obtained, allow to deduce strong hypercontractivity and Euler type LSI inequalities from analogous properties proved for symmetric Bernoulli measures on $\mathbb{R}$ or for uniform spherical measures on $\mathbb{R}^d$, $d \geq 2$.

Denote $f_r(x) = f(rx)$. In [10] an elementary and short proof was given, of the following strong hypercontractivity inequality for a symmetric Bernoulli measure and for log–convex functions (f is log–convex if $f \geq 0$ and $\log f$ is convex).

Proposition 4.1 *If* $m = \frac{1}{2}(\delta_1 + \delta_{-1})$ *then one has*

$$\|f_r\|_{1/r^2,m} \leq \|f\|_{1,m} \tag{19}$$

for every $r \in (0, 1]$ *and any log–convex function* f.

In [10] we proved the following strong hypercontractivity (SHC) inequality for a Gaussian measure, using the classical Nelson's hypercontractivity inequality and properties of the Mehler kernel. We give here a direct proof, using Proposition 4.1 and Theorem 3.4.

Corollary 4.2 *Let* γ *be the standard Gaussian measure* $N(0, 1)$ *on* $\mathbb{R}$. *Then one has*

$$\|f_r\|_{1/r^2,\gamma} \leq \|f\|_{1,\gamma} \tag{20}$$

for every $r \in (0, 1]$ *and any log-convex function* f.

Proof We use the notation μ_n from Sect. 3. From [10] we know that (19) holds for all Bernoulli-type measures μ_n:

$$\left(\int f_r(x)^{1/r^2} d\mu_n(x)\right)^{r^2} \le \int f(x) d\mu_n(x).$$

It is sufficient to show that the integrals $\int_0^\infty h\, d\mu_n$ converge to $\int_0^\infty h\, d\gamma$ for h equal to log-convex functions f and f_r^{1/r^2}. As a positive convex function, h is bounded on $\mathbb{R}^+$ or strictly increasing on an interval $[x_0, \infty)$. The convergence $\int h\, d\mu_n \to \int h\, d\gamma$ follows respectively from the usual CLT or from Theorem 3.4. □

A function $f \ge 0$ on $\mathbb{R}^d$ is called log-subharmonic if $\log f$ is subharmonic. In [10] it was shown by an elementary classical argument of norm differentiation that a (SHC) inequality

$$\|f_{e^{-t}}\|_{q,\mu} \le \|f\|_{p,\mu} \quad \text{for} \quad t \ge c \cdot \frac{1}{2} \log \frac{q}{p}, \quad 0 < p \le q < \infty, \tag{21}$$

for log-subharmonic functions f and a compactly supported measure μ implies a strong LSI inequality for log-subharmonic functions f,

$$\int f^2 \log f^2 d\mu - \|f\|_{2,\mu}^2 \log \|f\|_{2,\mu}^2 \le c \int f E f\, d\mu \tag{22}$$

where E is the Euler operator $E = x \cdot \nabla$ and $f \in \mathcal{C}^1$ is such that all the integrals in (22) converge. Recall that E is the generator of the dilation semigroup $T_t f = f_{e^{-t}}$. In correspondence to the terminology "*strong* hypercontractivity", we called the inequality (22) *strong* LSI or Euler type LSI.

The strengthened CLT's allow us to deduce a strong LSI for a Gaussian measure from the LSI proved for spherical measures.

Corollary 4.3 *The strong LSI (22) holds with $c = 1$ for Gaussian measures γ_d, $d \ge 1$.*

Proof The SHC inequality (21) holds with $c = 1$ for the measures σ_d and for their normalized convolution powers. For $d = 1$ this is Proposition 4.1; for $d \ge 2$ this follows from results of Beckner [3] who proved the classical Nelson's hypercontractivity for the Poisson semigroup and the uniform spherical measures σ_d, $d \ge 2$. The strong hypercontractivity for σ_d and log-subharmonic functions follows from the classical hypercontractivity in a similar way as in the proof of (SHC) for Gaussian measures in [10].

Consequently, we get the inequality (22) for $\mu_n = (\sigma_d^n)\tilde{}$, where $\tilde{}$ denotes the standard normalization. An application of Theorem 2.3 ends the proof of the Corollary for $d \ge 2$. When $d = 1$, we show, similarly as in the proof of Corollary 4.2, that the usual CLT or its strengthened version given in Theorem 3.4 apply to all terms of (22) with $\mu = \mu_n$, for $n \to \infty$. □

Acknowledgments We thank the referees for calling our attention to Pinelis' and Fomin's papers and for many valuable remarks that enriched the paper in a significant way.

References

1. Ané, C., et al.: Sur les inégalités de Sobolev logarithmiques. Panoramas et Synthèses, vol. 10. Société mathématique de France (2000)
2. Abramowitz, M., Stegun, I.: Handbook of mathematical functions. National Bureau of Standards Applied Mathematics Series, Washington, D.C. (1964)
3. Beckner, W.: Sobolev inequalities, the Poisson semigroup, and analysis on the sphere S^n. Proc. Natl. Acad. Sci. U.S.A. **89**, 4816–4819 (1992)
4. Bhattacharya, R., Rao, R.R.: Normal Approximation and Asymptotic Expansions. Wiley, New York (1976)
5. Bobkov, S.G., Goetze, F., Houdre, C.: On Gaussian and Bernoulli covariance representations. Bernoulli **7**(3), 439–451 (2001)
6. Breton, J.C., Houdre, C., Privault, N.: Dimension free and infinite variance tail estimates on Poisson space. Acta Appl. Math. **95**, 151–203 (2007)
7. Dudley, R.M.: Uniform Central Limit Theorems. Cambridge Studies in Advanced Mathematics, vol. 63. Cambridge University Press, Cambridge (1999)
8. Folland, G.B.: Fourier analysis and its applications. The Wadsworth & Brooks/Cole Mathematics Series, Pacific Grove, CA (1992)
9. Fomin, S.V.: The central limit theorem: convergence in the norm $\| u \|= (\int_{-\infty}^{\infty} u^2(x)e^{x^2/2}\,dx)^{1/2}$. Problems of the theory of probability distribution, VII. Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **119**, 218–229, 242, 245 (1982)
10. Graczyk, P., Loeb, J.J., Kemp, T.: Hypercontractivity for log-subharmonic functions. J. Funct. Anal. (2010, to appear)
11. Gradshteyn, I.S., Ryzhik, I.M.: Table of Integrals, Series, and Products, 7th edn. Elsevier/Academic Press, Amsterdam (2007)
12. Gross, L.: Logarithmic Sobolev inequalities. Am. J. Math. **97**, 1061–1083 (1975)
13. Gross, L.: Logarithmic Sobolev inequalities and contractivity properties of semigroups. Dirichlet forms, Varenna, pp. 54–88 (1992)
14. Hebisch, W., Saloff-Coste, L.: Gaussian estimates for Markov chains and random walks on groups. Ann. Probab. **21**, 673–709 (1993)
15. Houdre, C.: Remarks on deviation inequalities for functions of infinitely divisible random vectors. Ann. Probab. **30**, 1223–1237 (2002)
16. Houdre, C., Privault, N.: A concentration inequality on Riemannian path space, In: Houdre C, Privault N. (eds.) Stochastic Inequalities and Applications. Progress in Probability, vol. 56, pp. 15–21 (2003)
17. Hughes, B.D.: Random walks in random environments, vol. 1. Oxford University Press, Oxford (1995)
18. Komlós, J., Major, P., Tusnády, G.: An approximation of partial sums of independent RV's and the sample DF I. Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **32**, 111–131 (1975)
19. Pinelis, I.: Extremal probabilistic problems and Hotelling's T2 test under a symmetry condition. Ann. Stat. **22**, 357–368 (1994)
20. Pinelis, I.: Exact inequalities for sums of asymmetric random variables, with applications. Prob. Theory Relat. Fields **139**, 605–635 (2007)
21. Stein, E.M., Weiss, G.: Introduction to Fourier analysis on Euclidean spaces. Princeton Mathematical Series, No. 32. Princeton University Press, Princeton (1971)
22. Talagrand, M.: The missing factor in Hoeffding's inequalities. Annales de l'I.H.P. Probabilités Et Statistiques **31**, 689–702 (1995)