

Number of distinct sites visited by a random walk with internal states

Péter Nándori

Received: 1 September 2009 / Revised: 4 February 2010 / Published online: 11 March 2010
© Springer-Verlag 2010

Abstract In the classical paper of Dvoretzky and Erdős (Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probability, pp 353–367, 1951), asymptotics for the expected value and the variance of the number of distinct sites visited by a Simple Symmetric Random Walk were calculated. Here, these results are generalized for Random Walks with Internal States. Moreover, both weak and strong laws of large numbers are proved. As a tool for these results, the error term of the local limit theorem in Krámli and Szász (Zeitschrift Wahrscheinlichkeitstheorie verw Gebiete 63:85–95, 1983) is also estimated.

Keywords Random walk with internal states · Markov chain · Visited points · Local limit theorem · Law of large numbers

Mathematics Subject Classification (2000) 60J10 · 82C41

1 Introduction

The model of a random walk with internal states (or, alternatively, random walk with internal degrees of freedom; briefly RWwIS) was introduced by Sinai in 1981 in his Kyoto talk [16]. His aim was to get an efficient tool for examining the Lorentz process (in this context, internal states would represent the elements of the Markov partition or of a Markov sieve). Beside the Lorentz process, however, several other motivations and applications have appeared, among others, in some models of queueing systems, cf. [6] as for an extensive treatment of other motivations. Nevertheless, the investiga-

P. Nándori (✉)
Institute of Mathematics, Budapest University of Technology and Economics,
Egry József u. 1., Budapest 1111, Hungary
e-mail: nandori@math.bme.hu

tion of this model is important for its own sake, as it is a manifest generalization of a gem of probability theory: the simple symmetric random walk. Let us begin with the definition of RWwIS with the notation in [10] and [11] (or of [12], where RWwIS served as a model of Fourier law of heat conduction).

Definition 1 Let E be a finite set. On the set $H = \mathbb{Z}^d \times E$ ($d = 1, 2, \dots$), the Markov chain $\xi_n = (\eta_n, \varepsilon_n)$ is a random walk with internal states (RWwIS), if for $\forall x_n, x_{n+1} \in \mathbb{Z}^d, j_n, j_{n+1} \in E$

$$P(\xi_{n+1} = (x_{n+1}, j_{n+1}) | \xi_n = (x_n, j_n)) = p_{x_{n+1}-x_n, j_n, j_{n+1}}.$$

In fact, E could be countable, as well, but we will consider only the finite case. We will denote $s = \#E$.

There are some basic assumptions which will throughout be supposed. These are the following:

- (i) $(\varepsilon_0, \varepsilon_1, \dots)$ —obviously a Markov chain—is irreducible and aperiodic (its stationary distribution will be denoted by μ)
- (ii) the arithmetics are trivial, with the notation in [10], $L = \mathbb{Z}^d$
- (iii) the expectation of one step is zero provided that ε_0 is distributed according to its unique stationary measure
- (iv) the covariance matrix, which is exactly defined in Sect. 2, exists and is nonsingular.

In general, we will assume that $\eta_0 = 0$. Let $L_d(n)$ denote the number of distinct sites visited by $(\eta_k)_k$ up to n steps. The expectation of $L_d(n)$ is $E_d(n)$, and the variance is $V_d(n)$. $\{e_j\}_{j=1, \dots, s}$ is the standard basis in \mathbb{R}^s , and $\underline{1} = (1, 1, \dots, 1)^T$. Our aim is to find asymptotics of $E_d(n)$, further, by using bounds on $V_d(n)$, we want to prove weak and strong laws of large numbers. Similar results in terms of simple symmetric random walks (which will later on be referred to as SSRW) are found in [4]. Recently, in the case of two dimensional Lorenz process, Pène discussed the same question in [15]. There are numerous fairly new papers on $L_d(n)$ for random walks with independent steps (see [1] and references therein).

This paper is organized as follows: in Sect. 2 the main theorem of [10] is generalized. Namely, a remainder term of the local limit theorem is computed, as it will be necessary for estimating $E_2(n)$. A further refinement of the local limit theorem will also be given as it will be useful when proving the strong law of large numbers in the plane. Although these results are used in the forthcoming Sections, they can be interesting in their own rights. In Sect. 3, the number of visited points in the high dimensional case, i.e. when $d \geq 3$, is dealt with. We prove asymptotics for $E_d(n)$, and estimate $V_d(n)$, from which we can prove both the weak and strong laws of large numbers. In this Section, we will not use the result of Sect. 2, Theorem 5.2. in [10] will be enough for our purposes. In Sect. 4 the $d = 2$ case is discussed. For $E_2(n)$, same asymptotics ($const \frac{n}{\log n}$) is found as in [4], but with some different constant. $V_2(n)$ is also estimated, and the weak law of large numbers is also proved. The proof of the strong law in the plane is a little bit cumbersome calculation, so it is postponed to Sect. 5. In Sect. 6, the one dimensional settings are considered. This case requires a

little bit different approach from the previous ones (and is not treated in [4]), so the application of a Tauberian theorem will be very useful. Section 7 gives some remarks.

2 Preliminaries

2.1 Local limit theorem with remainder term

In this subsection, we calculate a remainder term for Theorem 5.2. in [10]. Furthermore, another refinement of this theorem will be proved, as it will be used when proving the strong law of large numbers in the plane. First, we reformulate the mentioned theorem. We have to start with some definitions. Denote

$$\begin{aligned}
 A_y &= (p_{y,j,k})_{j,k=1,\dots,s} : \mathbb{C}^s \rightarrow \mathbb{C}^s, \\
 Q &= \sum_{y \in \mathbb{Z}^d} A_y, \\
 M_l &= \sum_{y \in \mathbb{Z}^d} y_l A_y, \\
 \Sigma_{l,m} &= \sum_{y \in \mathbb{Z}^d} y_l y_m A_y.
 \end{aligned}$$

So, the transition matrix of the Markov chain $(\varepsilon_0, \varepsilon_1, \dots)$ is Q and its unique stationary measure is μ .

Theorem 1 [10] *Consider a RWwIS in \mathbb{Z}^d and assume that the matrix $\sigma = (\sigma_{l,m})_{1 \leq l,m \leq d}$ whose elements are*

$$\sigma_{l,m} = \langle \mu, \Sigma_{l,m} \mathbf{1} \rangle - \langle \mu, M_l (Q - 1)^{-1} M_m \mathbf{1} \rangle - \langle \mu, M_m (Q - 1)^{-1} M_l \mathbf{1} \rangle$$

(which can be called a covariance matrix) is positive definite, then

$$\sum_{(x,k) \in H} \left| P(\xi_n = (x, k) | \xi_0 = (0, j)) - n^{-d/2} \mu_k g_\sigma \left(\frac{x}{\sqrt{n}} \right) \right| \rightarrow 0$$

as $n \rightarrow \infty$, where $g_\sigma(x)$ denotes the density of a Gaussian distribution with mean 0 and covariance matrix σ .

Of course, the condition concerning the positive definiteness of the matrix in one dimension means $\sigma > 0$. We omit the proof, it can be found in [4]. In fact, there is a typo in [4] as they write $n^{-1/2}$ instead of $n^{-d/2}$ but it is easy to correct it even in the proof.

Our calculation will be similar to the one of [10]. The main point is that while in [10] it is sufficient to consider the Taylor expansion of the largest eigenvalue up to the quadratic term, now, we have to calculate the third term, as well.

Define the Fourier transform

$$\alpha(t) = \sum_{y \in \mathbb{Z}^d} \exp(i \langle t, y \rangle) A_y, \quad t \in [-\pi, \pi]^d.$$

Now, we have to consider the Taylor expansion of the largest eigenvalue of $\alpha(t)$, which is denoted by $\lambda(t)$, up to the third term.

Let us first assume that $d = 1$. From our basic assumptions it follows that $M = \sum_{y \in \mathbb{Z}} y A_y$ and $\Sigma = \sum_{y \in \mathbb{Z}} y^2 A_y$ are convergent series. But now, we also suppose the absolute convergence of

$$\Xi = \sum_{y \in \mathbb{Z}} y^3 A_y. \tag{1}$$

The existence of M, Σ and Ξ implies

$$\alpha(t) = Q + itM - \frac{t^2}{2} \Sigma - \frac{it^3}{6} \Xi + o(t^3) \quad (t \rightarrow 0). \tag{2}$$

Now, by perturbation theoretic means (i.e. the straightforward extension of Theorem 5.11. of Chapter II. in [9]) it can be easily proved that

$$\lambda(t) = 1 + r_1 t + \frac{r_2}{2} t^2 + \frac{r_3}{6} t^3 + o(t^3) \quad (t \rightarrow 0). \tag{3}$$

From [10] we know that $r_1 = 0$ and $r_2 = -\langle \Sigma \mathbf{1}, \mu \rangle + 2\langle M(Q - 1)^{-1} M, \mu \rangle$.

Using the notation $\sigma^2 = -r_2$ we can now formulate our theorem:

Theorem 2 *For a one dimensional RWwIS the existence of (1) imply*

$$P(\xi_n = (x, k) | \xi_0 = (0, j)) - \mu_k \frac{1}{\sqrt{2\pi n \sigma}} \exp\left(-\frac{x^2}{2n\sigma^2}\right) \left[1 - \frac{ir_3}{6} x \left(3\sigma^2 n - x^2\right) \frac{1}{\sigma^6} \frac{1}{n^2}\right] = o\left(\frac{1}{n}\right),$$

where the small order is uniform in x .

Proof The proof is similar to the one of Theorem 2.1. in [10]. In the neighborhood of the origin, we have $\alpha^n(t) = \lambda^n(t)p(t) + b_n(t)$, where p is the projector to the eigenspace associated to $\lambda(t)$, and $b_n(t)$ is the contribution of the other eigenvalues. The term $b_n(t)$ is in $O(\alpha^n)$ for some $\alpha \in (0, 1)$.

Because of (3) we have

$$\alpha^n(t) = \left(\mathbf{1}\mu^T + tp'(0) + O(t^2)\right) \left(1 - \frac{\sigma^2 t^2}{2} + \frac{r_3}{6} t^3 + o(t^3)\right)^n + b_n(t). \tag{4}$$

Elementary calculations show that

$$\left(1 - \frac{\sigma^2 s^2}{2n} + \frac{r_3}{6} \frac{s^3}{n^{\frac{3}{2}}} + o\left(\frac{s^3}{n^{\frac{3}{2}}}\right)\right)^n = \exp\left(-\frac{\sigma^2 s^2}{2}\right) \left(1 + \frac{r_3}{6} s^3 \frac{1}{\sqrt{n}} + o\left(\frac{s^3}{\sqrt{n}}\right)\right) \tag{5}$$

holds uniformly for $|s| < n^\epsilon$ with $0 < \epsilon < 1/6$. In order to prove the statement, we use the Fourier transforms and the usual estimations

$$\begin{aligned} & \left\| \sqrt{n} \int_{-\pi}^{\pi} \exp(-ixt) e_j^T \alpha^n(t) dt \right. \\ & \quad \left. - \mu^T \frac{\sqrt{2\pi}}{\sigma} \exp\left(-\frac{x^2}{2n\sigma^2}\right) \left[1 - \frac{ir_3}{6} x (3\sigma^2 n - x^2) \frac{1}{\sigma^6} \frac{1}{n^2}\right] \right\| \\ & \leq \int_{|s| < n^\epsilon} \left\| e_j^T p(0) \lambda^n\left(\frac{s}{\sqrt{n}}\right) - \mu^T \exp\left(-\frac{\sigma^2 s^2}{2}\right) \left(1 + \frac{r_3}{6} \frac{s^3}{\sqrt{n}}\right) \right\| ds + o\left(\frac{1}{\sqrt{n}}\right) \\ & \quad + c \|\mu\| \int_{|s| > n^\epsilon} (1 + s^3) \exp\left(-\frac{\sigma^2 s^2}{2}\right) ds + \int_{n^\epsilon < |s| < \gamma\sqrt{n}} \left\| e_j^T \alpha^n\left(\frac{s}{\sqrt{n}}\right) \right\| ds \\ & \quad + \int_{\gamma\sqrt{n} < |s| < \pi\sqrt{n}} \left\| e_j^T \alpha^n\left(\frac{s}{\sqrt{n}}\right) \right\| ds \\ & = I_1 + o\left(\frac{1}{\sqrt{n}}\right) + I_2 + I_3 + I_4, \end{aligned}$$

where $0 < \epsilon < \frac{1}{6}$ is arbitrary. The term $o\left(\frac{1}{\sqrt{n}}\right)$ is the contribution of the terms $\frac{s}{\sqrt{n}} p'(0) + O\left(\frac{s^2}{n}\right)$ in (4), as we can see that

$$\begin{aligned} & \int_{|s| < n^\epsilon} \lambda^n\left(\frac{s}{\sqrt{n}}\right) \frac{s}{\sqrt{n}} p'(0) ds \\ & = \int_{|s| < n^\epsilon} \exp\left(-\frac{\sigma^2 s^2}{2}\right) \frac{s}{\sqrt{n}} p'(0) ds + O\left(\int_{|s| < n^\epsilon} \exp\left(-\frac{\sigma^2 s^2}{2}\right) \frac{s^3}{\sqrt{n}} \frac{s}{\sqrt{n}} p'(0) ds\right), \end{aligned}$$

which is $0 + o\left(\frac{1}{\sqrt{n}}\right)$, and

$$\int_{|s| < n^\epsilon} \left| \lambda^n\left(\frac{s}{\sqrt{n}}\right) \right| \frac{s^2}{n} ds = o\left(\frac{1}{\sqrt{n}}\right).$$

It is clear that proving $I_j = o(\frac{1}{\sqrt{n}})$, $j = 1, 2, 3, 4$ is enough for our purposes. (5) yields that the integrand in I_1 is equal to $\frac{\delta(n)}{n^{1/2}}s^3 \exp(-\frac{\sigma^2s^2}{2})$, where $\delta(n) \rightarrow 0$ uniformly in s . Thus we have $I_1 = o(\frac{1}{\sqrt{n}})$. It is clear that $I_2 = o(\frac{1}{\sqrt{n}})$, and I_4 converges exponentially fast to zero. Finally, if $\gamma > 0$ is small enough, then

$$I_3 = \int_{n^\epsilon < |s| < \gamma\sqrt{n}} \left\| e_j^T \alpha^n \left(\frac{s}{\sqrt{n}} \right) \right\| ds \leq \int_{n^\epsilon < |s| < \gamma\sqrt{n}} \exp \left(-\frac{\sigma^2s^2}{4} \right) ds.$$

So we have $I_3 = o(\frac{1}{\sqrt{n}})$, too. □

Remark 1 In Theorem 2 for the expression subtracted from the appropriate probability we have:

$$\begin{aligned} & \mu_k \frac{1}{\sqrt{2\pi n\sigma}} \exp \left(-\frac{x^2}{2n\sigma^2} \right) \left[1 - \frac{ir_3}{6} x \left(3\sigma^2n - x^2 \right) \frac{1}{\sigma^6} \frac{1}{n^2} \right] \\ & = \mu_k \frac{1}{\sqrt{2\pi n\sigma}} \exp \left(-\frac{y^2}{2} \right) + \mu_k \frac{1}{\sqrt{n}} \frac{1}{\sigma} \frac{q_1(y)}{\sqrt{n}}, \end{aligned}$$

where $y = \frac{x}{\sqrt{n\sigma}}$, and the $q_1(y)$ is the function defined in [13, Chapter VI. (1.14)]. In this sense, the local limit theorem concerning RWwIS is analogous to the one of Simple Symmetric Random Walk (see [13, Chapter VII. Theorem 13]).

The extension of Theorem 2 to the multidimensional case is straightforward. Analogously to (3), we have:

$$\lambda(t) = 1 - \frac{1}{2}t^T \sigma t + f(t) + o(|t|^3) \quad (|t| \rightarrow 0),$$

where $f(t) = \sum_{i=1}^d \sum_{j=1}^d \sum_{k=1}^d r_{3,i,j,k} t_i t_j t_k$ is the third term of the Taylor expansion. Denote

$$\Omega = n^{d/2} P(\xi_n = (x, \cdot) | \xi_0 = (0, j)) = \frac{n^{d/2}}{(2\pi)^d} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} \exp(-i(x, t)) e_j^T \alpha^n(t) dt.$$

So the analogue of the expression subtracted from the appropriate probability in Theorem 2 (multiplied by $\frac{n^{d/2}}{(2\pi)^d}$) is

$$I^{(n)} := \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp \left(-\frac{s\sigma s}{2} - i \left\langle x, \frac{s}{\sqrt{n}} \right\rangle \right) \frac{f(s)}{\sqrt{n}} ds.$$

Using Lebesgue’s Theorem, it is easy to see that $I^{(n)} = O(n^{-1/2})$. One can estimate I_1, I_2, I_3, I_4 the same way, as it was done in the proof of Theorem 2 (see [10, Section 5] for more details). So we have arrived at

Proposition 1 *Supposing that (1) exists, for a d dimensional RWwIS*

$$P(\xi_n = (x, k) | \xi_0 = (0, j)) = \frac{1}{n^{d/2}} \mu_k g_\sigma \left(\frac{x}{\sqrt{n}} \right) + O \left(n^{-(d+1)/2} \right)$$

holds, where $g_\sigma(x)$ denotes the density of a Gaussian distribution with mean 0 and covariance matrix σ and the great order is uniform in x .

A further refinement of the local limit theorem will be useful in the sequel. Now, we would like to go further in the asymptotic expansion, and apply our techniques in the two dimensional case. Nevertheless, we are interested only in an estimation, not in the exact result which will simplify the calculation. Just like previously, let us begin with the one dimensional case. Assume the convergence of the series

$$\Upsilon = \sum_{y \in \mathbb{Z}} y^4 A_y. \tag{6}$$

Now, just like previously, we may write

$$\alpha(t) = Q + itM - \frac{t^2}{2} \Sigma - \frac{it^3}{6} \Xi + \frac{t^4}{24} \Upsilon + o(t^4) \quad (t \rightarrow 0)$$

for the Fourier transform, and

$$\lambda(t) = 1 + r_1 t - \frac{\sigma^2}{2} t^2 + \frac{r_3}{6} t^3 + O(t^4) \quad (t \rightarrow 0) \tag{7}$$

for the largest eigenvalue of $\alpha(t)$. As previously, we have

$$\begin{aligned} & \left(1 - \frac{\sigma^2 s^2}{2n} + \frac{r_3 s^3}{6 n^{3/2}} + O \left(\frac{s^4}{n^2} \right) \right)^n \\ &= \exp \left(-\frac{\sigma^2 s^2}{2} \right) \left(1 + \frac{r_3 s^3}{6} \frac{1}{\sqrt{n}} + O \left(\frac{s^4 + s^6}{n} \right) \right) \end{aligned}$$

uniformly for $|s| < n^\epsilon$. A very similar argument to the previous one (with $I_j = o(\frac{1}{n})$, $j = 1, 2, 3, 4$) leads to

$$\begin{aligned} & P(\xi_n = (x, k) | \xi_0 = (0, j)) \\ &= \mu_k \frac{1}{\sqrt{2\pi n \sigma}} \exp \left(-\frac{x^2}{2n\sigma^2} \right) \left[1 - \frac{ir_3}{6} x \left(3\sigma^2 n - x^2 \right) \frac{1}{\sigma^6 n^2} \right] + O \left(\frac{1}{n^{3/2}} \right), \end{aligned} \tag{8}$$

where the great order on the right hand side is uniform in x .

Now our aim is to formulate an assertion similar to (8) in two dimensions. Applying the one dimensional proof to the two dimensional case it is easily seen that

$$P(\xi_n = (x, k) | \xi_0 = (0, j)) - \mu_k \frac{1}{n} g_\sigma \left(\frac{x}{\sqrt{n}} \right) + F(x, n) = O \left(\frac{1}{n^2} \right),$$

where the great order is again uniform in x , and $F(x, n)$ is equal to

$$\frac{1}{2\pi n^{3/2}} \sum_{i_1=1}^2 \sum_{i_2=1}^2 \sum_{i_3=1}^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left(-\frac{s^T \sigma s}{2} - i \left\langle x, \frac{s}{\sqrt{n}} \right\rangle \right) r_{3, i_1, i_2, i_3} s_{i_1} s_{i_2} s_{i_3} ds.$$

We estimate $F(x, n)$ just like it was done in [14]. Observe that with the notation

$$\Psi(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left(-\frac{s^T \sigma s}{2} - i \langle x, s \rangle \right) ds = \frac{2\pi}{\sqrt{|\sigma|}} \exp \left(-\frac{x^T \sigma^{-1} x}{2} \right),$$

we have with an appropriate C_1 constant

$$|F(x, n)| < C_1 \frac{1}{n^{3/2}} \max_{i_1, i_2, i_3} \left| \frac{\partial^3 \Psi}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} \left(\frac{x}{\sqrt{n}} \right) \right|.$$

Further, observe that

$$\left| \frac{\partial^3 \Psi}{\partial x_{i_1} \partial x_{i_2} \partial x_{i_3}} (x) \right| < C_2 \left(\|x\| + \|x\|^3 \right) \exp \left(-\frac{x^T \sigma^{-1} x}{2} \right).$$

So we have arrived at

Proposition 2 *Assume that for a two dimensional RWwIS (6) exists. Then there is a C constant, such that for every $x \in \mathbb{R}^2$ and for every $1 \leq j, k \leq s$ the following estimation holds*

$$\begin{aligned} & \left| P(\xi_n = (x, k) | \xi_0 = (0, j)) - \mu_k \frac{1}{n} g_\sigma \left(\frac{x}{\sqrt{n}} \right) \right| \\ & \leq C \left(\frac{1}{n^{3/2}} \left(\frac{\|x\|}{n^{1/2}} + \frac{\|x\|^3}{n^{3/2}} \right) \exp \left(-\frac{x^T \sigma^{-1} x}{2n} \right) + \frac{1}{n^2} \right). \end{aligned}$$

By an elementary argument (see, for instance in [7, Theorem 4.2.2]), using Proposition 2 one can easily deduce

Corollary 1 *Under the conditions of Proposition 2*

$$\sum_{x \in \mathbb{Z}^2} \left| P(\eta_n = x | \eta_0 = 0) - \frac{1}{n} g_\sigma \left(\frac{x}{\sqrt{n}} \right) \right| = O \left(n^{-1/4} \right).$$

2.2 Reversed walks

The so-called reversed walk will be important in the sequel. If a RWwIS is given with the appropriate $(p_{y,i,j})$ probabilities, then we define the $(q_{y,i,j})$ reversed random walk for which

$$q_{y,i,j} = \frac{\mu_j p_{-y,j,i}}{\mu_i}. \tag{9}$$

Obviously, the stationary measure of the reversed walk is also μ . As we would like to apply the local limit theorem for the reversed walk, we need

Proposition 3 *If the primary RWwIS fulfills our basic assumptions, then the reversed walk fulfills them as well. Furthermore, the so-called covariance matrix of the reversed walk is the same as the one of the primary walk.*

Proof Basic assumptions (i)–(iii) are fulfilled obviously. So it suffices to prove the second statement. Let us introduce some notations

$$\begin{aligned} \tilde{A}_y &= (q_{y,j,k})_{j,k=1,\dots,s}, \\ \tilde{Q} &= \sum_{y \in \mathbb{Z}^d} \tilde{A}_y, \\ \tilde{M}_l &= \sum_{y \in \mathbb{Z}^d} y_l \tilde{A}_y, \\ \tilde{\Sigma}_{l,m} &= \sum_{y \in \mathbb{Z}^d} y_l y_m \tilde{A}_y, \end{aligned}$$

and a new inner product

$$\begin{aligned} (\cdot, \cdot) &: \mathbb{R}^s \times \mathbb{R}^s \rightarrow \mathbb{R}, \\ (u, v) &= \sum_{i=1}^s \mu_i u_i v_i. \end{aligned}$$

Let us denote by A^* the adjoint of the linear operator A , i.e. $(u, Av) = (A^*u, v)$ for all $u, v \in \mathbb{R}^s$. Elementary calculations show that $\tilde{Q} = Q^*$, $\tilde{A}_y = (A_{-y})^*$, $\tilde{M}_l = -(M_l)^*$, $\tilde{\Sigma}_{l,m} = (\Sigma_{l,m})^*$ for all $y \in \mathbb{Z}^d$, $1 \leq l, m \leq s$. Now, for an arbitrary element $\tilde{\sigma}_{l,m}$ of the “covariance matrix” defined for the reversed walk

$$\begin{aligned} \tilde{\sigma}_{l,m} &= (1, \tilde{\Sigma}_{l,m} 1) - (1, \tilde{M}_l (\tilde{Q} - 1)^{-1} \tilde{M}_m 1) - (1, \tilde{M}_m (\tilde{Q} - 1)^{-1} \tilde{M}_l 1) \\ &= (\Sigma_{l,m} 1, 1) - (M_m (Q - 1)^{-1} M_l 1, 1) - (M_l (Q - 1)^{-1} M_m 1, 1) \\ &= \sigma_{l,m}. \end{aligned}$$

Hence the statement. □

3 Visited points in high dimensions

In the high dimensional case, we find that $E_d(n)$ grows fast, i.e. linearly in n , as we could have conjectured it from the transiency of the RWwIS. In Theorem 3 we prove this fact and compute remainder terms, too. Our approach is based on the one of [4], but there are some main differences. First, we have to consider the reversed random walk which is trivial in the case of [4]. After it, the renewal equation is written with matrices and vectors, which is more technical than in the case of [4]. Moreover, there will be a technical difficulty, namely we will have to consider the case, when the distribution of ε_0 is arbitrary. This will be treated separately in Proposition 4. After it, we will be able to estimate $V_d(n)$. In fact, $o(n^2)$ is enough for proving weak law of large numbers, and $O(n^{2-\delta})$ for strong law of large numbers, but our estimations will be sharper. Nevertheless, these estimations are weaker than the ones of [4] because a symmetry argument, used in [4], fails here. That is why the computation is longer and it uses Proposition 4, too. Let us see the details.

Theorem 3 *Let $d \geq 3$. Assuming that ε_0 is distributed according to its unique stationary measure, we have*

$$\begin{aligned} E_3(n) &= n\gamma_3 + O(\sqrt{n}) \\ E_4(n) &= n\gamma_4 + O(\log n) \\ E_d(n) &= n\gamma_d + \beta_d + O(n^{2-d/2}) \text{ for } d \geq 5 \end{aligned}$$

with some constants γ_d, β_d , depending on the RWwIS.

Proof Fix some dimension $d \geq 3$. For the sake of simplicity, we skip the index d and denote $E_d(n) = \sum_{k=1}^n \gamma(k)$. Consider an $\{\xi_k = (\eta_k, \varepsilon_k), 0 \leq k\}$ RWwIS fulfilling our assumptions. Let $\{\tilde{\xi}_k = (\tilde{\eta}_k, \tilde{\varepsilon}_k), 0 \leq k\}$ be the reversed walk, i.e. for which the transition probabilities are defined by (9). Put $\eta_0 = 0, \gamma(0) = 1$ and define

$$\gamma(n) = P(\eta_n \notin \{\eta_0, \dots, \eta_{n-1}\})$$

which is just the probability that the walk visits a new point at step n . Obviously

$$\begin{aligned} \gamma(n) &= P(\eta_i \neq \eta_n \quad i = 0, \dots, n-1) \\ &= P(\eta_n - \eta_i \neq 0 \quad i = 0, \dots, n-1) \\ &= P(\tilde{\eta}_{n-i} \neq 0 \quad i = 0, \dots, n-1) \\ &= P(\tilde{\eta}_j \neq 0 \quad j = 1, \dots, n). \end{aligned}$$

It is clear that we have to examine the reversed walk.

Define $U_k \in \mathbb{R}^{s \times s}$ with

$$(U_k)_{i,j} = P(\tilde{\xi}_k = (0, j) | \tilde{\xi}_0 = (0, i))$$

and $R_k \in \mathbb{R}^s$ with

$$(R_k)_j = P(0 \notin \{\tilde{\eta}_1, \dots, \tilde{\eta}_k\} | \tilde{\xi}_0 = (0, j)).$$

Obviously, we have:

$$\sum_{k=0}^n U_k \cdot R_{n-k} = \underline{1}.$$

We are interested in $\langle R_n, \mu \rangle = \gamma(n)$. From the definition of R_k , for $n_1 > n_2$ we have $R_{n_2} - R_{n_1} \geq 0$, which means that all the components of the vector are non-negative.

We know from Proposition 3 and [10, Theorem 5.2.] that $(U_k)_{i,j} = c_j k^{-\frac{d}{2}} + o_{i,j}(k^{-\frac{d}{2}})$. Here we have $c_j = c\mu_j$, but this fact will not be used. So we have

$$\left(\sum_{k=0}^n U_k\right)_{i,j} = \tilde{c}_{i,j} + O\left(n^{1-\frac{d}{2}}\right).$$

Using the monotonicity of R_k we infer

$$\underline{1} \geq \left(\sum_{k=0}^n U_k\right) \cdot R_n.$$

Defining \hat{c}_j the following way

$$\left(\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{k=0}^n U_k\right)\right)_j = \frac{1}{s} \sum_{i=1}^s (\tilde{c}_{i,j} + O\left(n^{1-\frac{d}{2}}\right)) = \hat{c}_j + O\left(n^{1-\frac{d}{2}}\right),$$

we have

$$1 \geq \left\langle \left(\hat{c}_1 + O\left(n^{1-\frac{d}{2}}\right), \dots, \hat{c}_s + O\left(n^{1-\frac{d}{2}}\right)\right), R_n \right\rangle. \tag{10}$$

For all j , $(R_n)_j$ has a limit in n , being a decreasing non-negative sequence. So write $(R_n)_j = R^j + a_n^j$, where $a_n^j \searrow 0$. It will be enough to estimate the order of a_n^j , because $\gamma(n) = \sum_{j=1}^s \mu_j (R^j + a_n^j)$.

For the estimation of the other direction let $k < n$. We have:

$$\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=0}^k U_i\right) \cdot R_{n-k} + \left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{i=k+1}^n U_i\right) \cdot \underline{1} \geq 1.$$

Since $(U_k)_{i,j} \geq 0$ for all k, i, j , we have $(\frac{1}{s}\mathbf{1})^T \cdot (\sum_{i=0}^k U_i) \leq (\widehat{c}_1, \dots, \widehat{c}_s)$. On the other hand, $(\frac{1}{s}\mathbf{1})^T \cdot (\sum_{i=k+1}^n U_i) \cdot \mathbf{1} = o(1)$, as $k \rightarrow \infty$, thus

$$\langle (\widehat{c}_1, \dots, \widehat{c}_s), R_{n-k} \rangle \geq 1 + o(1). \tag{11}$$

So if we let $n \rightarrow \infty, k \rightarrow \infty, n - k \rightarrow \infty$, (11) together with (10) yields

$$\widehat{c}_1 R^1 + \dots + \widehat{c}_s R^s = 1.$$

Substituting to (10) we have:

$$\sum_{j=1}^s \left[\widehat{c}_j a_n^j + O\left(n^{1-\frac{d}{2}}\right) R^j + O\left(n^{1-\frac{d}{2}}\right) a_n^j \right] \leq 0,$$

whence

$$\sum_{j=1}^s \widehat{c}_j a_n^j \leq O\left(n^{1-\frac{d}{2}}\right).$$

Since $\widehat{c}_j > 0$ and $a_n^j \geq 0$, we conclude that $a_n^j = O(n^{1-\frac{d}{2}})$ for $1 \leq j \leq s$. This yields $\gamma(n) = \sum_{j=1}^s \mu_j (R^j + a_n^j) = \gamma + O(n^{1-\frac{d}{2}})$. Hence the statement (just like in [4]). □

Proposition 4 *The assertion of Theorem 3 remains true when the distribution of ε_0 is arbitrary.*

Proof With the notation $\gamma(n) = \gamma + h(n)$ we already know that $h(n) = O(n^{1-\frac{d}{2}})$. Define $\gamma^{ej}(n) = P(\eta_n \notin \{\eta_0, \dots, \eta_{n-1}\} | \varepsilon_0 = j)$ and $\gamma^{ej}(n) = \gamma + h^j(n)$ for $j = 1, \dots, s$. As in the previous proof, it would be sufficient to prove $h^j(n) = O(n^{1-\frac{d}{2}})$ for all j .

For the present, let K be a fixed, great natural number, and

$$\mu_k + b_k^j(K) = P(\varepsilon_K = k | \varepsilon_0 = j) \quad j, k = 1, \dots, s.$$

We know from the ergodic theorem of Markov chains that $b_k^j(K)$ tends to zero exponentially fast in K .

Denote by $p(K, n)$ the probability of visiting such a site at time n that was visited during the first K steps, but was not visited in the following $(n - K - 1)$ steps, provided that $\varepsilon_0 = j$. We know from [10, Theorem 5.2.] that $p(K, n) = O(K \cdot (n - K)^{-\frac{d}{2}})$, whence

$$\gamma^{ej}(n) = \sum_{k=1}^s \left[(\mu_k + b_k^j(K)) \gamma^{ek}(n - K) \right] + O\left(K \cdot (n - K)^{-\frac{d}{2}}\right). \tag{12}$$

Recall $\gamma^{e_j}(n) = \gamma + h^j(n)$ to infer that $h^j(n)$ is equal to

$$\begin{aligned} & \sum_{k=1}^s \mu_k h^k(n - K) + \sum_{k=1}^s b_k^j(K) h^k(n - K) + O\left(K \cdot (n - K)^{-\frac{d}{2}}\right) \\ & =: I + II + III. \end{aligned} \tag{13}$$

Now, put $K = K(n) = \lfloor n^\alpha \rfloor$ with arbitrary $0 < \alpha < 1$. It is clear that I is equal to $h(n - K)$, so the proof of Theorem 3 yields $I = O((n - n^\alpha)^{1-\frac{d}{2}}) \leq O(n^{1-\frac{d}{2}})$. Since $b_k^j(K)$ tends to zero exponentially fast in K we have $II \leq O(n^{1-\frac{d}{2}})$. Finally, $III = O(n^\alpha(n - n^\alpha)^{-\frac{d}{2}}) \leq O(n^{1-\frac{d}{2}})$. Hence the statement. \square

Now, let us see the estimation of $V_d(n)$.

Theorem 4 For $d \geq 3$ assuming that $\varepsilon_0 \sim \mu$ we have

$$V_d(n) = O\left(n^{1+\frac{2}{d}}\right).$$

Proof Let $\gamma(n, m)$ denote the probability that the RWwIS visits new points in both the n th and the m th step under the condition that $\varepsilon_0 \sim \mu$, and let $A = \{\eta_i \neq \eta_m, i = 0, \dots, m - 1\}$. Obviously, $\gamma_d(n, m) = \gamma_d(m, n)$, so, when estimating $\gamma(n, m)$ one can assume $n > m$.

$$\begin{aligned} \gamma(m, n) &= P(A \ \& \ \eta_j \neq \eta_n, \quad j = 0, \dots, n - 1) \\ &\leq P(A \ \& \ \eta_j \neq \eta_n, \quad j = m, \dots, n - 1) \\ &= \gamma(n)P(\eta_j \neq \eta_n, \quad j = m, \dots, n - 1|A). \end{aligned}$$

Here, $P(\eta_j \neq \eta_n, i = m, \dots, n - 1|A)$ is the probability that the RWwIS visits a new point in the $(n - m)$ th step, assuming that the distribution of ε_0 is some $\mu(n)$. So the condition A is involved in $\mu(n)$, and because of the Markov property, it has no other contribution. The probability of this event is denoted by $\gamma_d^{\mu(n)}(n - m)$. Because of Proposition 4 we know that $\gamma_d^{\mu(n)}(n - m) \rightarrow \gamma_d$, as $(n - m) \rightarrow \infty$, and it is easy to see that this convergence is uniform in $\mu(n)$. So we know that for $\forall \delta > 0 \exists N = N(\delta)$, such that for $\forall n - m > N$ the following estimation holds.

$$\gamma_d^{\mu(n)}(n - m) = \sum_{j=1}^s \mu(n)_j \gamma_d^{e_j}(n - m) < (1 + \delta)\gamma_d(n - m).$$

In addition, using Proposition 4, one can estimate $N(\delta)$, which will be done a little bit later. Now, let us see the estimation of $V_d(n)$

$$\begin{aligned}
 V_d(n) &= \sum_{i,j=0}^n \gamma_d(i, j) - \sum_{i=0}^n \gamma_d(i) \sum_{j=0}^n \gamma_d(j) \\
 &\leq 2 \sum_{0 \leq i \leq j \leq n} (\gamma_d(i, j) - \gamma_d(i) \gamma_d(j)) \\
 &\leq 2 \sum_{0 \leq i < i+K \leq j \leq n} (\gamma_d(i, j) - \gamma_d(i) \gamma_d(j)) + 2 \sum_{\substack{0 \leq i \leq n \\ i \leq j < i+K}} \gamma_d(i, j) \\
 &=: S_1 + S_2.
 \end{aligned}$$

Let K be big enough, such that for $n - m > K$ one would have $\gamma_d^v(n - m) < (1 + \delta)\gamma_d(n - m)$ for arbitrary v . Estimating S_1 and S_2 separately, we get

$$\begin{aligned}
 \frac{S_1}{2} &= \sum_{i=0}^{n-K} \sum_{j=i+K}^n \gamma_d(i, j) - \sum_{i=0}^{n-K} \sum_{j=i}^n \gamma_d(i) \gamma_d(j) + \sum_{i=0}^{n-K} \sum_{j=i}^{n-K+i+K} \gamma_d(i) \gamma_d(j) \\
 &\leq \sum_{i=0}^{n-K} \gamma_d(i) \max_{0 \leq i \leq n-K} \left(\sum_{j=i}^n (1 + \delta) \gamma_d(j - i) - \sum_{j=i}^n \gamma_d(j) \right) \\
 &\quad + \sum_{i=0}^{n-K} \gamma_d(i) \sum_{j=i}^{i+K} \gamma_d(j),
 \end{aligned}$$

which can be bounded by

$$\begin{aligned}
 &\leq \sum_{i=0}^{n-K} \gamma_d(i) \left[\delta E_d(n) + E_d\left(n - \left\lfloor \frac{n}{2} \right\rfloor\right) - E_d(n) + E_d\left(\left\lfloor \frac{n}{2} \right\rfloor\right) \right] \\
 &\quad + \sum_{i=0}^{n-K} \gamma_d(i) K.
 \end{aligned}$$

On the other hand,

$$S_2 \leq 2 \sum_{\substack{0 \leq i \leq n \\ i \leq j < i+K}} \gamma(i) \leq 2K E_d(n).$$

From the proof of Proposition 4, one can easily deduce that for k large enough

$$\gamma_d^v(k) < \left(1 + O(k^{1-\frac{d}{2}})\right) \gamma_d(k),$$

uniformly in ν . So replacing K to $K(n)$ in the above argument, one can change δ to $O(K(n)^{1-\frac{d}{2}})$, thus

$$\begin{aligned} V_3(n) &\leq O(n) \left[O\left(K(n)^{1-\frac{d}{2}}\right) O(n) + O(\sqrt{n}) \right] + K(n) O(n) \\ V_4(n) &\leq O(n) \left[O\left(K(n)^{1-\frac{d}{2}}\right) O(n) + O(\log n) \right] + K(n) O(n) \\ V_d(n) &\leq O(n) \left[O\left(K(n)^{1-\frac{d}{2}}\right) O(n) + O(1) \right] + K(n) O(n) \quad d \geq 5. \end{aligned}$$

Now, the choice $K(n) = \lfloor n^{\frac{2}{d}} \rfloor$ completes the proof. □

Proposition 5 *The assertion of Theorem 4 remains true when the distribution of ε_0 is some arbitrary ν . Moreover, the great order is uniform in ν .*

Proof Let us introduce the notation $E^\nu[\cdot]$ for the expectation when $\varepsilon_0 \sim \nu$. For convenience, we also write $E_d^\nu(n)$ and $V_d^\nu(n)$ for the expectation and variance of $L_d(n)$ when $\varepsilon_0 \sim \nu$. Obviously,

$$V_d^\nu(n) = E^\nu \left[(L_d(n))^2 \right] - (E_d^\nu(n))^2. \tag{14}$$

On the other hand,

$$\sum_{j=1}^s \nu_j V_d^{e_j}(n) = \sum_{j=1}^s \nu_j E^{e_j} \left[(L_d(n))^2 \right] - \sum_{j=1}^s \nu_j (E_d^{e_j}(n))^2. \tag{15}$$

Since $E^\nu[(L_d(n))^2] = \sum_{j=1}^s \nu_j E^{e_j}[(L_d(n))^2]$, subtracting (15) from (14), we conclude

$$V_3^\nu(n) - \sum_{j=1}^s \nu_j V_3^{e_j}(n) = O\left(n^{3/2}\right), \tag{16}$$

$$V_d^\nu(n) - \sum_{j=1}^s \nu_j V_d^{e_j}(n) = O(n \log n) \quad d \geq 4. \tag{17}$$

It is clear that the great order on the right hand side is uniform in ν . In the sense of (16) and (17) it is enough to prove the statement for $\nu = e_j$, ($j = 1, \dots, s$). To do so, substitute $\mu = \nu$ to (16) and (17) and use Theorem 4 to infer

$$\sum_{j=1}^s \mu_j V_3^{e_j}(n) = O\left(n^{1+\frac{d}{2}}\right), \quad d \geq 3.$$

Since for all d, j and n μ_j and $V_d^{e_j}(n)$ are non negative, we have proved the statement for all e_j . □

Corollary 2 For RWwIS in $d \geq 3$ the weak law of large numbers holds, namely

$$P(|L_d(n) - E_d(n)| > \varepsilon E_d(n)) \rightarrow 0$$

for $\forall \varepsilon > 0$.

Proof Since $V_d(n) = o(n^2)$ Chebyshev’s inequality applies (just like in [4]). □

From Theorem 4 one can deduce even strong law of large numbers:

Theorem 5 For RWwIS in $d \geq 3$ strong law of large numbers holds, namely

$$P\left(\lim_{n \rightarrow \infty} \frac{L_d(n)}{E_d(n)} = 1\right) = 1.$$

Theorem 5 can be proved almost the same way as it was done in [4]. The difference is that if we have $V_d(n) = O(n^\tau)$ with some $\tau < 2$, then we have to choose parameters α and β to fulfill

$$\begin{aligned} \frac{1 + \tau}{3} < \alpha < 1 \\ \frac{1}{2\alpha - \tau} < \beta < \frac{1}{1 - \alpha}. \end{aligned}$$

After it, the argument of [4] works. So the main point is that we should have some $\tau < 2$ such that $V_d(n) = O(n^\tau)$ as it was mentioned at the beginning of the Section.

Identifying the constant γ is an interesting question, though we cannot give a closed formula in the general case.

We only know that for the constant γ we have

$$\gamma = P(\eta_k \neq 0 : k \geq 1 | \varepsilon_0 \sim \mu). \tag{18}$$

To see this, first, observe that the constant γ is the same for the primary and the reversed walk. We have seen that

$$\gamma(n) = P(\tilde{\eta}_j \neq 0 \quad j = 1, \dots, n).$$

Taking $n \rightarrow \infty$ (18) follows.

4 Visited points in two dimensions

In this section we calculate $E_2(n)$ and estimate $V_2(n)$. The arguments (assuming that $\varepsilon_0 \sim \mu$) are similar to the ones of Theorem 3 and 4, or [4, Theorem 1 and Theorem 2]. The computations are longer than in [4]. We have to write the renewal equation in terms of vectors and matrices, which is a new idea, and we use the above proved Proposition 1 because it is essential that the remainder term of the probability of

returning to the origin should be summable, which was trivial in the case of [4]. We have to consider the case of arbitrary initial distribution, separately, just like in Sect. 3. In this case, we formulate the fact that after some steps the distribution of ε will be very close to μ .

Theorem 6 *Let $d = 2$. Assuming that $\varepsilon_0 \sim \mu$ and that (1) exists, we have*

$$E_2(n) = \frac{2\pi\sqrt{|\sigma|}n}{\log n} + O\left(\frac{n \log \log n}{\log^2 n}\right).$$

Proof As in the proof of Theorem 3, we examine the reversed RWwIS and write the renewal equation

$$\sum_{k=0}^n U_k \cdot R_{n-k} = \underline{1}. \tag{19}$$

Proposition 1 yields

$$(U_k)_{i,j} = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \frac{1}{k} + O\left(k^{-3/2}\right),$$

thus

$$\left(\sum_{k=0}^n U_k\right)_{i,j} = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log(c_{i,j}n) + O\left(n^{-1/2}\right). \tag{20}$$

Our purpose is to estimate $\langle R_n, \mu \rangle = \gamma(n)$. Exactly as in the high dimensional case, R_n is decreasing, so (19) yields

$$\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=0}^k U_l\right) \cdot R_{n-k} + \left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=k+1}^n U_l\right) \cdot \underline{1} \geq 1. \tag{21}$$

Let $k \rightarrow \infty, n \rightarrow \infty$. The relation between k and n will be fixed later. From (20) it follows that

$$\left[\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=0}^k U_l\right)\right]_j = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log(\widehat{c}_j k) + O\left(k^{-1/2}\right) \tag{22}$$

for some \widehat{c}_j . So we have for $k < n$

$$\left[\left(\frac{1}{s}\underline{1}\right)^T \cdot \left(\sum_{l=k+1}^n U_l\right)\right]_j = \frac{1}{2\pi\sqrt{|\sigma|}}\mu_j \log \frac{n}{k} + O\left(k^{-1/2}\right). \tag{23}$$

Substituting (22) and (23) to the left hand side of (21) we get

$$\sum_{j=1}^s \left[\frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log(\widehat{c}_j k) + O(k^{-1/2}) \right] (R_{n-k})_j + \sum_{j=1}^s \frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log \frac{n}{k} + O(k^{-1/2}). \tag{24}$$

Put $k = \lfloor n - \frac{n}{\log n} \rfloor$. This yields $\log k \sim \log(n - k)$. Using the fact $\gamma(n - k) = \sum_{j=1}^s \mu_j (R_{n-k})_j$, (24) can be written as

$$\gamma(n - k) \left[\frac{1}{2\pi\sqrt{|\sigma|}} \log k \right] + \sum_{j=1}^s \left[\frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log \widehat{c}_j + O(k^{-1/2}) \right] (R_{n-k})_j + C \log \frac{n}{k} + O(k^{-1/2}). \tag{25}$$

Since $\log \frac{n}{k} \rightarrow 0$, and $(R_{n-k})_j \rightarrow 0$, as $n - k \rightarrow \infty$ (the latter is the recurrence property of the two dimensional RWwIS, which is proved in [17]), it follows that

$$\gamma(n - k) \geq \frac{2\pi\sqrt{|\sigma|}}{\log k} + o\left(\frac{1}{\log k}\right). \tag{26}$$

Hence, by the choice of k ,

$$\gamma(n - k) \geq \frac{2\pi\sqrt{|\sigma|}}{\log(n - k)} + o\left(\frac{1}{\log(n - k)}\right). \tag{27}$$

Now let us give an upper estimation to $\gamma(n)$. From (19) it follows that

$$\left(\sum_{k=0}^n U_k \right) \cdot R_n \leq \mathbf{1}.$$

Multiplying by the vector $\frac{1}{s}\mathbf{1}$, we get

$$\sum_{j=1}^s \left[\frac{1}{2\pi\sqrt{|\sigma|}} \mu_j \log(\widehat{c}_j n) + O(n^{-1/2}) \right] (R_n)_j \leq 1,$$

thus

$$S_1 + S_2 + S_3 := \frac{1}{2\pi\sqrt{|\sigma|}} \sum_{j=1}^s \mu_j (R_n)_j \log n + \frac{1}{2\pi\sqrt{|\sigma|}} \sum_{j=1}^s \mu_j (R_n)_j \log \widehat{c}_j + \sum_{j=1}^s O\left(n^{-1/2}\right) (R_n)_j \leq 1.$$

Since $(R_n)_j \rightarrow 0$, it follows that $S_2 + S_3 = o(1)$. So we have the upper estimation

$$\gamma(n) \leq \frac{2\pi\sqrt{|\sigma|}}{\log n} + o\left(\frac{1}{\log n}\right). \tag{28}$$

From (27) and (28) we get

$$\gamma(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + o\left(\frac{1}{\log n}\right). \tag{29}$$

Unfortunately, the estimation (29) is not good enough for our purposes (but observe that we have not really used (1) yet). Now (28) yields $(R_n)_j = O\left(\frac{1}{\log n}\right)$ for all $1 \leq j \leq s$. Hence, with the previous notation, $S_2 = O\left(\frac{1}{\log n}\right)$. Obviously $S_3 = O\left(\frac{1}{\log n}\right)$. Thus we arrived at

$$\gamma(n) \leq \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{1}{\log^2 n}\right). \tag{30}$$

This estimation will be sharp enough.

Now, we have to improve our lower estimation. From (29) and (25) it follows that

$$\gamma(n - k) \left[\frac{1}{2\pi\sqrt{|\sigma|}} \log k + O(1) \right] + C \log \frac{n}{k} + O\left(k^{-\frac{1}{2}}\right) \geq 1,$$

thus

$$\gamma(n - k) \log(n - k) \geq \left(2\pi\sqrt{|\sigma|} - C2\pi\sqrt{|\sigma|} \log \frac{n}{k} + O\left(k^{-\frac{1}{2}}\right)\right) \frac{\log(n - k)}{\log k + O(1)}.$$

Now, similarly to the case of [4], it follows that

$$\gamma(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{\log \log n}{\log^2 n}\right). \tag{31}$$

Now, an elementary calculation completes the proof. □

As in the high dimensional case, the initial distribution does not influence the asymptotic behavior. More precisely

Proposition 6 *The assertion of Theorem 6 remains true when the distribution of ε_0 is arbitrary.*

Proof The proof is very similar to the one of Proposition 4. We know that

$$\gamma(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + O\left(\frac{\log \log n}{\log^2 n}\right).$$

With the notation $\gamma^{e_j}(n) = \frac{2\pi\sqrt{|\sigma|}}{\log n} + h^j(n)$ our aim is to prove $h^j(n) = O\left(\frac{\log \log n}{\log^2 n}\right)$. The analogue of (12) is

$$\begin{aligned} & \frac{2\pi\sqrt{|\sigma|}}{\log n} + h^j(n) \\ &= \sum_{k=1}^s \left[(\mu_k + b_k^j(K)) \left(\frac{2\pi\sqrt{|\sigma|}}{\log(n-K)} + h^k(n-K) \right) \right] + O\left(K \cdot (n-K)^{-1}\right), \end{aligned}$$

and the analogue of (13) is

$$\begin{aligned} h^j(n) &= \sum_{k=1}^s \mu_k h^k(n-K) + \sum_{k=1}^s b_k^j(K) h^k(n-K) + O\left(K \cdot (n-K)^{-1}\right) \\ &+ \left(\frac{2\pi\sqrt{|\sigma|}}{\log(n-K)} - \frac{2\pi\sqrt{|\sigma|}}{\log n} \right) \\ &=: I + II + III + IV. \end{aligned}$$

With the choice $K(n) = \lfloor \sqrt{n} \rfloor$ elementary calculations show that $I + II + III + IV \leq O\left(\frac{\log \log n}{\log^2 n}\right)$. □

Now let us see the estimation of the variance.

Theorem 7 *If (1) exists, then we have with arbitrary ν distribution of ε_0*

$$V_2(n) = O\left(\frac{n^2 \log \log n}{\log^3 n}\right).$$

Moreover, the great order is uniform in ν .

Proof First, suppose $\varepsilon_0 \sim \mu$. The beginning of the proof of this case is the same as in Theorem 4. The difference is that when we change K to $K(n)$, we can write $O\left(\frac{\log \log K(n)}{\log K(n)}\right)$ instead of δ in the sense of Proposition 6. From now, just like in the proof of Theorem 4, it is not difficult to deduce that

$$O\left(\frac{n}{\log n}\right) \left[\frac{\log \log K(n)}{\log K(n)} O\left(\frac{n}{\log n}\right) + O\left(\frac{n \log \log n}{\log^2 n}\right) \right] + K(n) O\left(\frac{n}{\log n}\right)$$

is an upper bound for $V_2(n)$. Taking $K(n) = \lfloor \frac{n}{\log^2 n} \rfloor$ proves the statement. For the case of arbitrary initial distribution, one can repeat the proof of Proposition 5. \square

Corollary 3 *For a RWwIS in $d = 2$ dimension weak law of large numbers holds.*

Proof Since $O(\frac{n^2 \log \log n}{\log^3 n}) < O(\frac{n^2}{\log^2 n})$, Chebyshev’s inequality applies. \square

The proof of the strong law of large numbers is quite complicated, so we treat it in a different section.

5 Law of large numbers in the plane

This section is dedicated to the strong law in $d = 2$.

Theorem 8 *For any RWwIS in $d = 2$, for which (6) exists, strong law of large numbers holds, namely*

$$P \left(\lim_{n \rightarrow \infty} \frac{L_2(n)}{E_2(n)} = 1 \right) = 1.$$

Almost the whole proof in [4] can be easily generalized to our case with the observation that since our estimations for $E_2(n)$ and $V_2(n)$ are uniform in the initial distribution, the computations, used in [4], can be repeated. That is why we write here the only non-trivial part (i.e. formulae corresponding to (5.13) and (5.15) in [4]) of the generalization. In fact, there is apparently a gap in the argument in [4], as it was already remarked in [8]. What we represent here is a simplified version of a proof in [15]. For the other parts of the proof the reader is referred to [4].

Proof Denote

$$K = \lfloor \log \log n \rfloor,$$

and let $M_{ij} (1 \leq i, j \leq K)$ be the number of lattice points which are common in path parts M_i and M_j , where M_i denotes the set of points which are visited between $\lfloor (i - 1)n/K \rfloor + 1$ and $\lfloor in/K \rfloor (1 \leq i \leq K)$. First, we would like to prove the formula corresponding to [4, (5.13)]:

$$\sup_{i < j} E(M_{ij}) = O \left(\frac{n \log \log n}{K \log^2 n} \right). \tag{32}$$

If it is done, then for every ϑ with $0 < \vartheta < 1$ we will have

$$\sup_{i < j} P \left(M_{ij} > \frac{n \log \log n}{K \log^{1+\vartheta} n} \right) = O \left(\frac{1}{\log^{1-\vartheta} n} \right). \tag{33}$$

Let C_{ij} denote the event whose probability is estimated in (33). As (32) yields

$$\sup_j E(M_{1j}) = O\left(\frac{n \log \log n}{K \log^2 n}\right)$$

for arbitrary ν initial distribution of internal states, and under the condition C_{ij} the probability of $C_{i'j'}$ with $1 \leq i < j < i' < j' \leq K$ is only affected via the distribution of $\varepsilon_{i'}$, we conclude

$$\sup_{1 \leq i < j < i' < j' \leq K} P(C_{i,j} \cap C_{i',j'}) = O\left(\frac{1}{\log^{2-2\vartheta} n}\right). \tag{34}$$

If we were able to prove

$$\sup_{i,j,i',j'} E(M_{ij}M_{i'j'}) = O\left(\frac{n^2 \log^2 \log n}{K^2 \log^4 n}\right), \tag{35}$$

where the supremum is taken over indices for which $\#\{i, j, i', j'\} = 4$ and either $1 \leq i < i' < j' < j \leq K$ or $1 \leq i < i' < j < j' \leq K$ holds, then using

$$P\left(M_{ij} > \frac{n \log \log n}{K \log^{1+\vartheta} n}, M_{i'j'} > \frac{n \log \log n}{K \log^{1+\vartheta} n}\right) < P\left(M_{ij}M_{i'j'} > \frac{n^2 \log^2 \log n}{K^2 \log^{2+2\vartheta} n}\right)$$

and (34) we could infer that the probability that two events C_{ij} and $C_{i'j'}$ with $\#\{i, j, i', j'\} = 4$ occur is

$$O\left(\frac{K^4}{\log^{2-2\vartheta} n}\right), \tag{36}$$

which is the formula corresponding to [4, (5.15)].

So our aim is to prove (32) and (35). The idea of [15] is that in order to prove (32) and (35) it is useful to cut down the points of M_i which are visited in the extreme $\lceil n/\log^2 n \rceil$ steps. The number of these points can be roughly estimated, while the others are visited in steps quite far from each other and this will be enough for us. However, the precise arguments need some awkward computations.

Proof of (32) We introduce the notations

$$\alpha_{a,b} = \{\forall t = a, \dots, b - 1 : \eta_t \neq \eta_b\} \quad \text{and} \quad \beta_{a,b} = \{\forall t = a + 1, \dots, b : \eta_a \neq \eta_t\}$$

which will be useful in the sequel. Following [15], we define

$$n_{(i,-)} = \lfloor (i - 1)n/K \rfloor + \lceil n/\log^2 n \rceil \quad \text{and} \quad n_{(i,+)} = \lfloor in/K \rfloor - \lceil n/\log^2 n \rceil.$$

A point, which is common in the paths $\eta_{n_{(i,-)}}, \dots, \eta_{n_{(i,+)}}$ and $\eta_{n_{(j,-)}}, \dots, \eta_{n_{(j,+)}}$ and not visited in the extreme $\lceil n/\log^2 n \rceil$ steps of M_i and M_j , has a pair of indices (k, l) ,

$k \in n_{(i,-)}, \dots, n_{(i,+)}, l \in n_{(j,-)}, \dots, n_{(j,+)}$, such that it is visited at steps k and l , and it is not visited during steps $\lfloor (i-1)n/K \rfloor + 1, \dots, k-1$, and steps $l+1, \dots, \lfloor jn/K \rfloor$. So we have

$$\begin{aligned} E(M_{ij}) &\leq \frac{3n}{\log^2 n} + \sum_{k=n_{(i,-)}}^{n_{(i,+)}} \sum_{l=n_{(j,-)}}^{n_{(j,+)}} P(\alpha_{\lfloor (i-1)n/K \rfloor + 1, k} \cap \{\eta_k = \eta_l\} \cap \beta_{l, \lfloor jn/K \rfloor}) \\ &\leq \frac{3n}{\log^2 n} \\ &\quad + C_1 \sum_{k=n_{(i,-)}}^{n_{(i,+)}} \sum_{l=n_{(j,-)}}^{n_{(j,+)}} \frac{1}{\log(k - \lfloor (i-1)n/K \rfloor)} \frac{1}{l - k} \frac{1}{\log(\lfloor jn/K \rfloor - l)} \\ &\leq \frac{3n}{\log^2 n} + C_2 \frac{n}{K} \frac{\log n - \log(n/\log^2 n)}{\log^2 n} = O\left(\frac{n \log \log n}{K \log^2 n}\right). \end{aligned}$$

Note that we have used our estimations for the probability of avoiding the origin in some steps, visiting a new point, and returning to the origin, and these estimations are uniform in the initial distribution (with an appropriate C_1). Because the events whose intersection’s probability is estimated above are dependent only via the internal states, it is obvious that the great order is uniform in i and j . So we arrived at (32). \square

Proof of (35) Let us prove

$$\sup_{1 \leq i < i' < j' < j \leq K} E(M_{ij} M_{i'j'}) = O\left(\frac{n^2 (\log \log n)^2}{K^2 \log^4 n}\right).$$

Let us introduce the notation \mathcal{L} for the set of (k, k', l', l) such that

$$\begin{aligned} n_{(i,-)} \leq k \leq n_{(i,+)}, \quad n_{(i',-)} \leq k' \leq n_{(i',+)}, \\ n_{(j',-)} \leq l' \leq n_{(j',+)}, \quad n_{(j,-)} \leq l \leq n_{(j,+)}. \end{aligned}$$

As it was mentioned before, we estimate the number of pair of points one of which is visited in either extreme $\lceil n/\log^2 n \rceil$ steps of $M_i, M_j, M_{i'}$ or $M_{j'}$ in a very obvious manner. The other pairs of lattice points (x and y , say) have a (k, k', l', l) element of \mathcal{L} , such that x is visited at step k but not visited during $\lfloor (i-1)n/K \rfloor + 1, \dots, k-1$, and it is visited again at step l but not visited during $l+1, \dots, \lfloor jn/K \rfloor$; while y is visited at step k' but not visited during $k'+1, \dots, \lfloor i'n/K \rfloor$, and it is visited again at step l' but not visited during $\lfloor (j'-1)n/K \rfloor + 1, \dots, l'-1$. So we have

$$E(M_{ij} M_{i'j'}) = O\left(\frac{n^2 \log \log n}{K \log^4 n}\right) + \sum_{M \in \mathbb{Z}^2} \sum_{(k,k',l',l) \in \mathcal{L}} P(\mathcal{A}), \tag{37}$$

where

$$\mathcal{A} = \alpha_{\lfloor(i-1)n/K\rfloor+1,k} \cap \{\eta_{k'} - \eta_k = M\} \cap \beta_{k',\lfloor i'n/K\rfloor} \cap \{\eta_{l'} - \eta_{k'} = (0, 0)\} \\ \cap \alpha_{\lfloor(j'-1)n/K\rfloor+1,l'} \cap \{\eta_l - \eta_{l'} = -M\} \cap \beta_{l,\lfloor jn/K\rfloor},$$

Denote the seven events, whose intersection is \mathcal{A} , by $\mathcal{A}_1, \dots, \mathcal{A}_7$. Observe that for every $2 \leq m \leq 7$ the probability of \mathcal{A}_m under the condition $\mathcal{A}_1 \cap \dots \cap \mathcal{A}_{m-1}$ is just the probability of \mathcal{A}_m with an appropriate initial distribution of ε . As we have uniform estimations in the initial distribution, we will be able to use them.

In the first step, let us estimate the part of the sum in (37) corresponding to $M \in [-n, n]^2$. Proposition 2 yields the existence of $a > 0$ (which depends only on the RWwIS), such that

$$P(\eta_k - \eta_0 = M) < C_3 \exp\left(-\frac{a}{2k} M^T M\right) \left[\frac{1}{k} + \frac{1}{k^{3/2}}\right] + \frac{C_3}{k^2} \\ < C_4 \left(\frac{1}{k} \exp\left(-\frac{a}{2k} M^T M\right) + \frac{1}{k^2}\right).$$

So the formula

$$C_5 \frac{1}{\log n} \left(\frac{1}{k' - k} \exp\left(-\frac{a}{2(k' - k)} M^T M\right) + \frac{1}{(k' - k)^2}\right), \tag{38}$$

is an upper bound for $P(\mathcal{A}_1 \cap \mathcal{A}_2)$, and the formula

$$C_5 \frac{1}{\log n} \left(\frac{1}{l - l'} \exp\left(-\frac{a}{2(l - l')} M^T M\right) + \frac{1}{(l - l')^2}\right). \tag{39}$$

is an upper bound for $P(\mathcal{A}_6 \cap \mathcal{A}_7 | \mathcal{A}_1 \cap \dots \cap \mathcal{A}_5)$.

Consider the following factorization

$$P(\mathcal{A}) = P(\mathcal{A}_1 \cap \mathcal{A}_2) P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2) P(\mathcal{A}_6 \cap \mathcal{A}_7 | \mathcal{A}_1 \cap \dots \cap \mathcal{A}_5), \tag{40}$$

and observe that

$$\sum_{k',l'} P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2) < C_6 E(M_{i'j'}) = O\left(\frac{n \log \log n}{K \log^2 n}\right). \tag{41}$$

So we have to take the product of the expressions in (38), (39) and $P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)$ and sum them up in all of the four indices to estimate (37). First, let us consider the product of the first terms in (38) and (39). We have to estimate

$$\sum \sum \exp\left(-\frac{a}{2} \left(\frac{1}{k' - k} + \frac{1}{l - l'}\right) M^T M\right) \frac{P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)}{(k' - k)(l - l') \log^2 n},$$

where the to sums are taken over $M \in [-n, n]^2$ and $(k, k', l', l) \in \mathcal{L}$, respectively. Using the fact

$$\sup_{d>a} \frac{1}{d} \sum_{M \in \mathbb{Z}^2} \exp\left(-\frac{a}{2d} M^T M\right) < +\infty \tag{42}$$

it suffices to estimate

$$\begin{aligned} & \sum_{(k,k',l',l) \in \mathcal{L}} \frac{(k' - k)(l - l')}{(l - l') + (k' - k)} \frac{P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)}{(k' - k)(l - l') \log^2 n} \\ & \leq \frac{1}{\log^2 n} \sum_{(k,k',l',l) \in \mathcal{L}} \frac{P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2)}{(l - n_{(j',+)} + (n_{(i',-)} - k))}. \end{aligned}$$

Using (41), it remains to estimate

$$\frac{1}{\log^2 n} E(M_{ij}) \sum_{k,l} \frac{1}{(l - n_{(j,-)} + (n_{(i,+)} - k) + 2n / \log^2 n - 1)}$$

and it is just

$$O\left(\frac{n^2 (\log \log n)^2}{K^2 \log^4 n}\right)$$

uniformly in i and j , by an elementary computation.

Now, let us consider the product of the first term in (38) and the second term in (39) (the product of the second term in (38) and the first term in (39) can be estimated equivalently). In this case the easier estimation

$$P(\mathcal{A}_3 \cap \mathcal{A}_4 \cap \mathcal{A}_5 | \mathcal{A}_1 \cap \mathcal{A}_2) < C_8 \frac{1}{l' - k'} \tag{43}$$

will be enough. Thus our aim is to estimate

$$\frac{1}{\log^2 n} \sum_{(k,k',l',l) \in \mathcal{L}} \sum_{M \in [-n,n]^2} \exp\left(-\frac{a}{2(k' - k)} M^T M\right) \frac{1}{(k' - k)(l - l')^2(l' - k')}.$$

As above, we use (42) to handle the exponential terms. So the following estimation is enough for our purposes

$$\begin{aligned} \frac{1}{\log^2 n} \sum_{(k,k',l',l) \in \mathcal{L}} \frac{1}{(l - l')^2(l' - k')} & \leq \frac{1}{\log^2 n} \frac{n^4 \log^4 n \log^2 n}{K^4 n^2 n} \\ & = O\left(\frac{n^2 (\log \log n)^2}{K^2 \log^4 n}\right). \end{aligned}$$

Our last task is to estimate the product of the second term in (38) and (39). The previous estimation (43) and

$$\begin{aligned} & \frac{1}{\log^2 n} \sum_{(k,k',l,l') \in \mathcal{L}} \sum_{M \in [-n,n]^2} \frac{1}{(k' - k)^2 (l - l')^2 (l' - k')} \\ & \leq n^6 \frac{1}{\log^2 n} \frac{\log^4 n}{n^2} \frac{\log^4 n}{n^2} \frac{\log^2 n}{n} = O\left(\frac{n^2 (\log \log n)^2}{K^2 \log^4 n}\right) \end{aligned}$$

yield the required estimation.

In the second step, we estimate the part of the sum in (37) corresponding to $M \in \mathbb{Z}^2 \setminus [-n, n]^2$. Corollary 1 implies that

$$\sup_{(k,k',l,l') \in \mathcal{L}} \sum_{M \in \mathbb{Z}^2 \setminus [-n,n]^2} P(\mathcal{A}_6 | \mathcal{A}_1 \cap \dots \cap \mathcal{A}_5) < \frac{1}{n^{1/4-\varepsilon}}$$

for small $\varepsilon > 0$. Thus

$$\begin{aligned} & \sup_{k',l'} \sum_{k,l} \sup_{M \in \mathbb{Z}^2 \setminus [-n,n]^2} P(\mathcal{A}_1 \cap \mathcal{A}_2) \sum_{M \in \mathbb{Z}^2 \setminus [-n,n]^2} P(\mathcal{A}_6 \cap \mathcal{A}_7 | \mathcal{A}_1 \cap \dots \cap \mathcal{A}_5) \\ & < C_7 n^2 \frac{1}{\log n} \frac{\log^2 n}{n} \frac{1}{\log n} \frac{1}{n^{1/4-\varepsilon}}. \end{aligned}$$

The above estimation together with (41) yield the required error term.

A modified version of the proof presented above can be repeated for indices $1 \leq i < i' < j < j' \leq K$. So we have finished the proof of formula (35). \square

6 Visited points in one dimension

Investigating the one dimensional case is not as important as the higher dimensions, as Lorentz processes used to be examined mainly in higher dimensions. However, one dimension is also interesting, as we will see some new features. We need some different means from the previous ones to prove asymptotics for $E_1(n)$, namely Tauberian arguments. Let us see the details.

Proposition 7 *For a one dimensional RWwIS with $\varepsilon_0 \sim \mu$ we have*

$$\gamma_1(n) \sim \sqrt{\frac{2|\sigma|}{\pi}} * n^{-1/2}$$

Proof Just like in the higher dimensional cases we consider the renewal equation for the reversed walk

$$\sum_{k=0}^n U_k \cdot R_{n-k} = \underline{1}.$$

Now, from row i we obtain

$$\sum_{j=1}^s \sum_{k=0}^n (U_k)_{i,j} x^k (R_{n-k})_j x^{n-k} = x^n. \tag{44}$$

Let us introduce the notations

$$\begin{aligned} \sum_{k=0}^{\infty} (U_k)_{i,j} x^k &= \alpha_{ij}(x) \\ \sum_{k=0}^{\infty} (R_k)_j x^k &= \beta_j(x) \\ \sum_{k=0}^{\infty} x^k &= \omega(x). \end{aligned}$$

Obviously, these power series are convergent for $0 \leq x < 1$. In these terms, (44) means

$$\sum_{j=1}^s \alpha_{ij} \beta_j = \omega. \tag{45}$$

In order to obtain the order of the coefficients of $\gamma_1(n) = \sum_{j=1}^s \mu_j (R_n)_j$ we use a Tauberian theorem which may be found in [5, Theorem 5 of XIII.5]. According to this we have

$$\omega(x) \sim \frac{1}{1-x}, \quad x \rightarrow 1-. \tag{46}$$

For the coefficients of α_{ij}

$$\sum_{k=0}^n (U_k)_{i,j} \sim 2 \frac{1}{\sqrt{2\pi |\sigma|}} \mu_j n^{1/2}.$$

So, using the Tauberian theorem, we infer

$$\alpha_{ij}(x) \sim 2 \frac{1}{\sqrt{2\pi |\sigma|}} \mu_j \Gamma\left(\frac{3}{2}\right) * \frac{1}{(1-x)^{1/2}}, \quad x \rightarrow 1-. \tag{47}$$

From (45) we obtain

$$\sum_{j=1}^s \frac{\alpha_{ij}}{\alpha_{ii}} \beta_j = \frac{\omega}{\alpha_{ii}}. \tag{48}$$

Now, (47) yields

$$\frac{\alpha_{ij}(x)}{\alpha_{ii}(x)} \rightarrow \frac{\mu_j}{\mu_i}, \quad x \rightarrow 1-. \tag{49}$$

Whence

$$\sum_{j=1}^s \mu_j \beta_j(x) \sim \frac{\sqrt{2\pi|\sigma|}}{2\Gamma(\frac{3}{2})} * \frac{1}{(1-x)^{1/2}}, \quad x \rightarrow 1-.$$

Since $\sum_{j=1}^s \mu_j (R_k)_j$ is monotonic in k , using the mentioned Tauberian theorem we conclude

$$\gamma_1(n) = \sum_{j=1}^s \mu_j (R_k)_j \sim \frac{\sqrt{2\pi|\sigma|}}{2\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})} n^{-1/2} = \sqrt{\frac{2|\sigma|}{\pi}} n^{-1/2}$$

□

Proposition 8 *With arbitrary distribution of ε_0 the following holds*

$$E_1(n) \sim \sqrt{\frac{8|\sigma|}{\pi}} n^{1/2}.$$

Proof From Proposition 7 the assertion immediately follows in the case of $\varepsilon_0 \sim \mu$. However, the case of arbitrary initial distribution requires a little care. Analogously to (12), we have

$$\gamma^{e_j}(n) = \sum_{k=1}^s \mu_k \gamma^{e_k}(n-K) + \sum_{k=1}^s b_k^j(K) \gamma^{e_k}(n-K) + p(K, n). \tag{50}$$

But now, the rough estimation of $p(K, n)$ used in higher dimensions is not enough, as the local limit theorem provides a term of order $n^{-1/2}$ and our aim is to prove $o(n^{-1/2})$. Nevertheless, because of the definition of $p(K, n)$, we have to estimate the probability of the first return to some place after m steps. In particular, if we proved that this probability is $O(m^{-3/2})$, then taking $K = \lfloor \sqrt{n} \rfloor$ and multiplying (50) by \sqrt{n} we would find that the right hand side converges to $\sqrt{\frac{2|\sigma|}{\pi}}$ as $n \rightarrow \infty$. So, in order to finish our proof, we need the following lemma. □

Lemma 1 *For a one dimensional RWwIS fulfilling our basic assumptions with arbitrary ν distribution of ε_0*

$$f^\nu(n) = O\left(n^{-3/2}\right), \tag{51}$$

where $f^\nu(n)$ denotes the probability of the event that the random walker starting from the origin with $\varepsilon_0 \sim \nu$ returns to the origin at time n for the first time.

Proof First of all, observe that proving the statement for $\nu = \mu$ would be enough as since our basic assumption (i) all component of μ are positive. In the proof, we generalize an argument in [2]. Define

$$Q^n(x, i, y, j) = P(\xi_n = (y, j), \eta_k \neq 0, \forall 1 \leq k < n | \xi_0 = (x, i)).$$

Let $n = 3m$ and $1 \leq i \leq m$. The cases $n = 3m \pm 1$ can be treated the same way.

$$\begin{aligned} f^{e_i}(n) &= \sum_{l=1}^s Q^n(0, i, 0, l) \\ &= \sum_{y,z \neq 0} \sum_{j,k,l=1}^s Q^m(0, i, y, j) Q_d^m(y, j, z, k) Q^m(z, k, 0, l) \\ &\leq \sup_{y,z,j,k} Q^m(y, j, z, k) P(\eta_k \neq 0, \forall 1 \leq k < m | \xi_0 = (0, i)) \sum_{z \neq 0k, l=1}^s Q^m(z, k, 0, l) \end{aligned}$$

From the local limit theorem it follows that

$$\sup_{y,z,j,k} Q^m(y, j, z, k) = O(m^{-1/2}).$$

Proposition 7 yields $P(\eta_k \neq 0, \forall 1 \leq k < m | \xi_0 = (0, i)) = O(m^{-1/2})$. So, it suffices to prove

$$\sum_{z \neq 0k, l=1}^s Q^m(z, k, 0, l) = O(m^{-1/2}). \tag{52}$$

In order to prove (52) we use the reversed walk, again. (9) yields that for all $((0, i_1), (y_1, i_2), (y_1 + y_2, i_3), \dots, (y_1 + y_2 + \dots + y_{m-1}, i_m))$ trajectories

$$\begin{aligned} &\mu_{i_1} P_{y_1, i_1, i_2} \mu_{i_2} P_{y_2, i_2, i_3} \dots \mu_{i_{m-1}} P_{y_{m-1}, i_{m-1}, i_m} \\ &= \mu_{i_2} q_{-y_1, i_2, i_1} \mu_{i_3} q_{-y_2, i_3, i_2} \dots \mu_{i_m} q_{-y_{m-1}, i_m, i_{m-1}}, \end{aligned}$$

where the factors $\mu_{i_2}, \dots, \mu_{i_{m-1}}$ drop out. Thus

$$\sum_{z \neq 0k, l=1}^s Q^m(z, k, 0, l) \leq \max_{1 \leq i, j \leq s} \frac{\mu_i}{\mu_j} \sum_{z \neq 0k, l=1}^s \tilde{Q}^m(0, l, z, k), \tag{53}$$

where \tilde{Q} is the same object as Q defined for the reversed walk. The right hand side of (53) can be bounded by some constant times the probability of the event that the stationary reversed walk does not return to the origin in the first m steps, which is $O(m^{-1/2})$. Thus we arrived at (52). \square

So, we have ascertained the asymptotic behavior of $E_d(n)$ in each dimension. While strong law of large numbers holds in $d \geq 2$, even the weak law of large numbers for one dimensional SSRW fails to hold, which is a consequence of, for instance, Theorem 1 in [3].

7 Final remarks

1. Our asymptotic investigations show that RWwIS behaves like the simple symmetric random walk in an asymptotic sense. The main features are very similar, only the involved constants differ. The results showing that the asymptotic behavior is independent from the initial distribution on the internal states (e.g. Propositions 4 and 6) are intuitively trivial as after some steps ε will be very close to μ . Nevertheless, these assertions need formal proofs as well, especially as they are used in the sequel. Of course, this similarity to the simple symmetric random walk could change if the generalization were carried further, for instance, if a countable set of internal states was allowed. This model is not yet discussed, it must need some more involved technics.
2. Our basic assumption (ii) is not essential. The above theorems could be generalized to the case of dropping basic assumption (ii), as the limit theorem in [10] is proved for this case, as well. Only the computations would become longer. The other three assumptions are essential.

Acknowledgments I am grateful to Domokos Szász for his constant support and invaluable comments before and during writing this paper. My research work was partially supported by OTKA (Hungarian National Research Fund) grant NK63066. I thank for the kind hospitality of Erwin Schrödinger Institute (Vienna), where a part of this work was done. I am also grateful to the referee for his careful reading of this paper and for his pertinent remarks.

References

1. Bass, R.F., Rosen, J.: An almost sure invariance principle for the range of random walks. *Ann. Probab.* **33**, 1856–1885 (2005)
2. Bender, E.A., Lawler, G.F., Pemantle, R., Wilf, H.S.: Irreducible compositions and the first return to the origin of a random walk. *Seminaire Lotharingien de Combinatoire* **50**, B50h, pp. 13 (2004)
3. Chen, X.: Moderate and small deviations for the ranges of one-dimensional random walks. *J. Theor. Probab.* **19**, 721–739 (2006)
4. Dvoretzky, A., Erdős, P.: Some problems on random walk in space. *Proceedings of the 2nd Berkeley Symposium on Mathematical Statistics and Probability*, pp. 353–367 (1951)
5. Feller, W.: *An introduction to Probability Theory and Its Applications*, 2nd edn, vol. 2. John Wiley and Sons, New York (1971)
6. Hughes, B.D.: *Random Walks and Random Environments: Random walks*, Oxford University Press, Oxford (1995)
7. Ibragimov, I.A., Linnik, Yu.V.: *Independent and Stationarily Depending Variables* (in Russian). Nauka, Moscow (1965)
8. Jain, N.C., Pruitt, W.E.: The Range of Recurrent Random Walk in the Plane. *Zeitschrift Wahrscheinlichkeitstheorie Verw. Gebiete* **16**, 279–292 (1970)
9. Kato, T.: *Perturbation Theory for Linear Operators*, 2nd edn. Springer, Berlin (1980)
10. Krámlí, A., Szász, D.: Random walks with internal degrees of freedom I. *Zeitschrift Wahrscheinlichkeitstheorie Verw. Gebiete* **63**, 85–95 (1983)

11. Krámlı, A., Szász, D.: Random walks with internal degrees of freedom II. *Zeitschrift Wahrscheinlichkeitstheorie Verw. Gebiete* **68**, 53–64 (1984)
12. Krámlı, A., Simányi, N., Szász, D.: Random walks with internal degrees of freedom III. *Zeitschrift Wahrscheinlichkeitstheorie Verw. Gebiete* **72**, 603–617 (1986)
13. Petrov, V.V.: *Sums of Independent Random Variables*. Akademie Verlag, Berlin (1975)
14. Pène, F.: Planar Lorentz gas in a random scenery. *Annales de l'Institut Henri Poincaré Probabilités Et Statistiques* **45**, 818–839 (2009)
15. Pène, F.: Asymptotic of the number of obstacles visited by a planar Lorentz process. *Discrete Cont. Dyn. Syst. Ser. A* **24**, 567–588 (2009)
16. Sinai, Ya.G.: Random walks and some problems concerning Lorentz gas. *Proceedings of the Kyoto Conference*, pp. 6–17 (1981)
17. Telcs, A.: Random walks with internal states (in Hungarian). *MTA Sztaki Tanulmányok* **145**, 1–60 (1983)