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On the Itô–Wentzell formula for distribution-valued processes and related topics

N. V. Krylov

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Abstract We prove the Itô–Wentzell formula for processes with values in the space of generalized functions by using the stochastic Fubini theorem and the Itô–Wentzell formula for real-valued processes, appropriate versions of which are also proved.

Keywords Itô–Wentzell formula · Stochastic Fubini theorem

Mathematics Subject Classification (2000) 60H05 · 60H15

1 Introduction and main result

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space with an increasing filtration $\{\mathcal{F}_t, t \geq 0\}$ of complete with respect to $(\mathcal{F}, P)$ σ-fields $\mathcal{F}_t \subset \mathcal{F}$. Denote by $\mathcal{P}$ the predictable σ-field in $\Omega \times (0, \infty)$ associated with $\{\mathcal{F}_t\}$ and let τ be a stopping time with respect to $\{\mathcal{F}_t, t \geq 0\}$. Let $w_t^k, k = 1, 2, \ldots,$ be independent one-dimensional Wiener processes with respect to $\{\mathcal{F}_t\}$. Let $\mathcal{D}$ be the space of generalized functions on the Euclidean d-dimensional space $\mathbb{R}^d$ of points $x = (x^1, \ldots, x^d)$.

The following are just versions of Definitions 4.4 and 4.6 of [4]. Set $\mathbb{R}_+ = [0, \infty)$. Recall that for any $v \in \mathcal{D}$ and $\phi \in C_0^\infty = C_0^\infty(\mathbb{R}^d)$ the function $(v, \phi(\cdot - x))$ is infinitely differentiable with respect to x, so that the sup in (1.1) below is predictable.

Definition 1.1 Denote by $\mathfrak{D}$ the set of all $\mathcal{D}$-valued functions u (written as $u_t(x)$ in a common abuse of notation) on $\Omega \times \mathbb{R}_+$ such that, for any $\phi \in C_0^\infty$, the restriction

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N. V. Krylov (✉)
University of Minnesota,
127 Vincent Hall, Minneapolis, MN 55455, USA
e-mail: krylov@math.umn.edu

of the function (u_t, ϕ) on $\Omega \times (0, \infty)$ is $\mathcal{P}$-measurable and (u_0, ϕ) is $\mathcal{F}_0$-measurable. For $p = 1, 2$ denote by $\mathfrak{D}^p$ the subset of $\mathfrak{D}$ consisting of u such that for any $\phi \in C_0^\infty$ and $T, R \in \mathbb{R}_+$, we have

$$\int_0^T \sup_{|x| \le R} |(u_t, \phi(\cdot - x))|^p \, dt < \infty \quad \text{(a.s.)}. \tag{1.1}$$

In the same way, considering ℓ_2-valued distributions g on C_0^∞, that is linear ℓ_2-valued functionals such that (g, ϕ) is continuous as an ℓ_2-valued function with respect to the standard convergence of test functions, we define $\mathfrak{D}(\ell_2)$ and $\mathfrak{D}^2(\ell_2)$ replacing $|\cdot|$ in (1.1) with $p = 2$ by $|\cdot|_{\ell_2}$.

Observe that if $g \in \mathfrak{D}^2(l_2)$, then for any $\phi \in C_0^\infty$ and $T \in \mathbb{R}_+$

$$\sum_{k=1}^\infty \int_0^T \left(g_t^k, \phi\right)^2 dt = \int_0^T |(g_t, \phi)|_{\ell_2}^2 \, dt < \infty \quad \text{(a.s.)},$$

which, by well known theorems about convergence of series of martingales, implies that the series in (1.3) below converges uniformly on $[0, T]$ in probability for any $T \in \mathbb{R}_+$.

Definition 1.2 Let $f, u \in \mathfrak{D}$, $g \in \mathfrak{D}(l_2)$. We say that the equality

$$du_t(x) = f_t(x)\, dt + g_t^k(x)\, dw_t^k, \quad t \le \tau, \tag{1.2}$$

holds *in the sense of distributions* if $f I_{(0,\tau]\!]} \in \mathfrak{D}^1$, $g I_{(0,\tau]\!]} \in \mathfrak{D}^2(l_2)$ and for any $\phi \in C_0^\infty$, with probability one we have for all $t \in \mathbb{R}_+$

$$(u_{t\wedge\tau}, \phi) = (u_0, \phi) + \int_0^t I_{s\le\tau}(f_s, \phi)\, ds + \sum_{k=1}^\infty \int_0^t I_{s\le\tau}\left(g_s^k, \phi\right) dw_s^k. \tag{1.3}$$

Let x_t be an $\mathbb{R}^d$-valued stochastic process given by

$$x_t^i = \int_0^t b_s^i \, ds + \sum_{k=1}^\infty \int_0^t \sigma_s^{ik} \, dw_s^k,$$

where $b_t = (b_t^i)$, $\sigma_t^k = (\sigma_t^{ik})$ are predictable $\mathbb{R}^d$-valued processes such that for all ω and $s, T \in \mathbb{R}_+$ we have $\operatorname{tr} a_s < \infty$ and

$$\int_0^T (|b_t| + \operatorname{tr} a_t)\, dt < \infty, \tag{1.4}$$

where $a_t = (a_t^{ij})$ and $2a_t^{ij} = (\sigma_t^{i\cdot}, \sigma_t^{j\cdot})_{\ell_2}$, so that

$$2\mathrm{tr}\, a_t = \sum_{i=1}^{d} \sum_{k=1}^{\infty} \left|\sigma_t^{ik}\right|^2 .$$

Finally, before stating our main result we remind the reader that for a generalized function v, and any $\phi \in C_0^\infty$ the function $(v, \phi(\cdot - x))$ is infinitely differentiable and for any derivative operator D of order n with respect to x we have

$$D(v, \phi(\cdot - x)) = (-1)^n (v, (D\phi)(\cdot - x)) =: (Dv, \phi(\cdot - x)) =: ((Dv)(\cdot + x), \phi) \tag{1.5}$$

implying that $Du \in \mathfrak{D}$ if $u \in \mathfrak{D}$.

Here is our main result, that is a version of Lemma 4.7 of [4]. In case $b \equiv 0$ a proof of this lemma is provided in [4] without giving any precise indication as to which version of the Itô–Wentzell formula is used. We will fill this gap here. Set

$$D_i = \frac{\partial}{\partial x^i}, \quad D_{ij} = D_i D_j .$$

Theorem 1.1 *Let $f, u \in \mathfrak{D}$, $g \in \mathfrak{D}(l_2)$. Introduce*

$$v_t(x) = u_t(x + x_t)$$

and assume that (1.2) *holds* (*in the sense of distributions*). *Then*

$$\begin{aligned} dv_t(x) = \Big[f_t(x + x_t) + a_t^{ij} D_{ij} v_t(x) + b_t^i D_i v_t(x) + \Big(D_i g_t(x + x_t), \sigma_t^{i\cdot}\Big)_{\ell_2} \Big]\, dt \\ + \Big[g_t^k(x + x_t) + D_i v_t(x) \sigma_t^{ik} \Big]\, dw_t^k, \quad t \le \tau \end{aligned} \tag{1.6}$$

(*in the sense of distributions*).

The reader understands that the summation convention over the repeated indices $i, j = 1, \ldots, d$ (and $k = 1, 2, \ldots$) is enforced here and throughout the article. The fact that (1.6) makes sense and indeed holds is proved in Sect. 4. Our proof is outlined in [4] and is based on the stochastic Fubini theorem and the Itô–Wentzell formula for real-valued processes. We prove a version of the stochastic Fubini theorem in Sect. 2. The Itô–Wentzell formula for real-valued processes in the form we need is proved in Sect. 3.

There is a quite extensive literature on the stochastic Fubini theorem (see, for instance, [6,8] and the references therein). It is worth saying that with some effort by using estimates like (2.4) we could obtain our version of the theorem in a somewhat weaker form from probably the first one given in [2] or from more sophisticated versions in [8]. In this case we would work with stochastic integrals depending on the parameter x as in

$$\sum_{k=1}^{\infty} \int_0^t I_{s\le\tau} \left(g_s^k, \phi(\cdot - x)\right) dw_s^k \tag{1.7}$$

and for each t choose a jointly measurable function of (ω, x), which is equal to (1.7) (a.s.) for almost any x. However, there is a much better modification working for all x and t, which in the case of one-dimensional semimartingales is described in the corollary of Theorem IV.63 of [6] and obtained by using a method introduced by Doléans–Dade. This modification allows one also to investigate the continuous dependence on t of the integral of (1.7) with respect to x, which in the case of one driving semimartingale is proved in Theorems IV.64 and IV.65 of [6]. Our basic tools are: an analog of Theorem IV.62 of [6] (see Lemma 2.1), leading to conclusions similar to the corollary of Theorem IV.63 of [6], and Theorem IV.64 of [6], and are much more elementary than rather involved arguments in [8], where the authors treat a very general situation, which is not within the scope of the present article, by using γ-radonifying operators and the fact that L_1-spaces possess the UMD$^-$ property.

We prove and use the stochastic Fubini theorem only for functions given on $\mathbb{R}^d$ with Lebesgue measure. Its generalization for arbitrary σ-finite measure spaces is straightforward, and can be used, as in [2], to transform conditional expectations of stochastic integrals. This comment is appropriate, because, actually, for the purpose of proving Theorem 1.1 one does not need our stochastic Fubini theorem since $(u, \phi(\cdot - x))$ is an infinitely differentiable function of x and one could just approximate the integrals with respect to x by Riemann sums and then pass to the limit. This would prove the integral form of (1.6) as in (1.3) for each fixed t (a.s.) and then an additional effort based on our Corollary 2.4 would still be needed to show that the integral form holds (a.s.) for all t at once.

Passing to the discussion of the Itô–Wentzell formula for real-valued processes notice that our Theorem 3.1 is somewhat close to Theorem 3.3.1 of [5], which requires two derivatives of $F_t(x)$ in x to be continuous in (t, x). Even if $F_t(x)$ is nonrandom, when the Itô–Wentzell formula becomes just Itô's formula, our result is more general than standard versions of Itô's formula. For instance, at those instances of time when $a_t = 0$ we do not need the second derivative of $F_t(x)$ to exist.

Finally, it is worth pointing out that our results are also true when there is only finitely many Wiener processes. Considering infinitely many of them becomes indispensable in the applications of the theory of SPDEs to super-diffusions (see, for instance, [3]).

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2 A version of the stochastic Fubini theorem

If E is a Borel subset of a Euclidean space, by $\mathcal{B}(E)$ we denote the σ-field of Borel subsets of E. Let Γ be a Borel subset of $\mathbb{R}^d$ with nonzero finite Lebesgue measure.

Definition 2.1 Let $B_t(x)$ be a real-valued function on $\Omega \times \mathbb{R}_+ \times \Gamma$. We say that it is a regular field on Γ if:

(a) It is measurable with respect to $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\Gamma)$;
(b) For each $x \in \Gamma$, there is an event Ω_x such that $P(\Omega_x) = 1$ and for any $\omega \in \Omega_x$, the function $B_t(\omega, x)$ is a continuous function of t on $\mathbb{R}_+$;
(c) It is $\mathcal{F}_t$-measurable for each $x \in \Gamma$ and $t \in \mathbb{R}_+$.

We call it a regular martingale field on Γ if in addition

(d) For each $x \in \Gamma$ the process $B_t(x)$ is a local $\mathcal{F}_t$-martingale on $\mathbb{R}_+$ starting at zero.

We are going to use the following version of Theorem IV.62 of [6].

Lemma 2.1 *Let $B^n_t(x)$, $n = 1, 2, \dots$, be regular fields on Γ and let $B_t(x)$ be a real-valued function on $\Omega \times \mathbb{R}_+ \times \Gamma$. Assume that for each x we have $B^n_t(x) \to B_t(x)$ uniformly on finite time intervals in probability as $n \to \infty$. Then there exists a regular field $A_t(x)$ on Γ such that, for each x, with probability one $A_t(x) = B_t(x)$ for all t and*

(b′) For each $\omega \in \Omega$ and $x \in \Gamma$ the function $A_t(x)$ is continuous on $\mathbb{R}_+$.

Proof First, assume that for each $n \geq 1$, $\omega \in \Omega$, and $x \in \Gamma$ the function $B^n_t(x)$ is continuous on $\mathbb{R}_+$.

Let ρ be the set of rationals in $\mathbb{R}_+$ and for real-valued functions x_t on $\mathbb{R}_+$ set

$$|x_\cdot|_C = \sum_{n=1}^{\infty} 2^{-n} \arctan \sup_{\rho \cap [0,n]} |x_t|.$$

For $n = 1, 2, \dots, x \in \Gamma$, and $\varepsilon > 0$ consider the functions

$$p^n_\varepsilon(x) = \sup_{k \geq n} P\left(\left| B^n_\cdot(x) - B^k_\cdot(x) \right|_C \geq \varepsilon \right).$$

Observe that $|B^n_\cdot(x) - B^k_\cdot(x)|_C$ are $\mathcal{F} \otimes \mathcal{B}(\Gamma)$-measurable with respect to (ω, x), so that by a standard result p^n_ε are $\mathcal{B}(\Gamma)$-measurable. Furthermore, p^n_ε are decreasing functions of n and we are given that $p^n_\varepsilon(x) \to 0$ for each x and ε as $n \to \infty$. Therefore,

$$n(\varepsilon, x) := \inf\left\{ n \geq 1 : p^n_\varepsilon(x) \leq \varepsilon \right\}, \quad m(k, x) := \sum_{i=1}^{k} n(2^{-i}, x)$$

are finite, $\mathcal{B}(\Gamma)$-measurable functions and

$$p^n_\varepsilon(x) \leq \varepsilon \quad \text{for } n \geq n(\varepsilon, x).$$

Obviously, $B^{m(k,x)}_t(x)$ are $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\Gamma)$-measurable. Moreover, $m(k, x)$ are increasing in k and $m(k, x) \geq n(2^{-k}, x)$, so that by the above $p^{m(k,x)}_{2^{-k}}(x) \leq 2^{-k}$. In particular,

$$P\left(\left|B_{\cdot}^{m(k+1,x)}(x)-B_{\cdot}^{m(k,x)}(x)\right|_{C}\geq 2^{-k}\right)\leq 2^{-k}. \tag{2.1}$$

Introduce

$$\begin{aligned}\Theta &= \left\{(\omega,x): x\in\Gamma, \sum_{k=1}^{\infty}\left|B_{\cdot}^{m(k+1,x)}(x)-B_{\cdot}^{m(k,x)}(x)\right|_{C}<\infty\right\},\\ A_t(x)&=\lim_{k\to\infty}B_t^{m(k,x)}(x)\end{aligned} \tag{2.2}$$

if $(\omega,x)\in\Theta$ and $t\in\mathbb{R}_+$ and set $A_t(x)\equiv 0$ if $(\omega,x)\notin\Theta$. Recall that by the additional assumption for each $n\geq 1$, $\omega\in\Omega$, and $x\in\Gamma$ the function $B_t^n(x)$ is continuous on $\mathbb{R}_+$ and observe, that for $(\omega,x)\in\Theta$ the functions $B_t^{m(k,x)}(x)$ converge to $A_t(x)$ uniformly on finite time intervals. Therefore, $A_t(x)$ is continuous with respect to t for $(\omega,x)\in\Theta$. It is trivially continuous if $(\omega,x)\notin\Theta$.

Since $|B_{\cdot}^n(x)-B_{\cdot}^k(x)|_C$ are $\mathcal{F}\otimes\mathcal{B}(\Gamma)$-measurable, $\Theta\in\mathcal{F}\otimes\mathcal{B}(\Gamma)$ and it follows that, for each $t\in\mathbb{R}_+$ the function $A_t(x)$ is $\mathcal{F}\otimes\mathcal{B}(\Gamma)$-measurable which along with its continuity in t implies that $A_t(x)$ is $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\Gamma)$-measurable.

In light of (2.1) and the Borel–Cantelly lemma, for each $x\in\Gamma$ and almost any ω we have that $(\omega,x)\in\Theta$. Hence for each $x\in\Gamma$ and $t\in\mathbb{R}_+$ the convergence in (2.2) holds almost surely, implying that $A_t(x)$ is $\mathcal{F}_t$-measurable. Since for any x the convergence in (2.2) is uniform on finite time intervals (a.s.), we have $A_t(x)=B_t(x)$ for all t with probability one for any x. This proves the lemma under the above additional assumption.

We now show that the additional assumption does not restrict generality. To do that it suffices to prove that for each fixed n there exists a regular field $\bar{B}_t^n(x)$ such that for each $n\geq 1$, $\omega\in\Omega$, and $x\in\Gamma$ the function $\bar{B}_t^n(x)$ is continuous on $\mathbb{R}_+$ and for each x we have $|\bar{B}_{\cdot}^n-B_{\cdot}^n(x)|_C=0$ (a.s.). Indeed, if this is possible, then we can just apply the first part of the proof to $\bar{B}_t^n(x)$ in place of $B_{\cdot}^n(x)$.

Fix an n and for $m=1,2,\dots$, define $B_t^{nm}(x)$ as

$$(k+1-mt)B_{k/m}(x)+(mt-k)B_{(k+1)/m}(x)$$

for $k\leq mt\leq k+1$, $k=0,1,\dots$. Then $B_t^{nm}(x)$ are $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\Gamma)$-measurable and continuous in t for all ω and x. For each x, they are also $\mathcal{G}_t$-adapted, where $\mathcal{G}_t\equiv\mathcal{F}$. Finally, for each x with probability one $B_t^{nm}(x)\to B_t^n(x)$ as $m\to\infty$ uniformly on finite intervals of time (a.s.). By the above there exists a function $\bar{B}_t^n(x)$, which is $\mathcal{F}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(\Gamma)$-measurable, continuous in t for all ω and x, and such that for each x with probability one $\bar{B}_t^n(x)=B_t^n(x)$ for all t. The latter implies that $\bar{B}_t^n(x)$ is $\mathcal{F}_t$-measurable for each t and x and this brings the proof of the lemma to an end.

Definition 2.2 If a regular field on Γ possesses property (b$'$) of Lemma 2.1, then we call it strongly regular.

The argument in the last part of the proof of Lemma 2.1 proves the following.

Lemma 2.2 *If $B_t(x)$ is a regular field on Γ, then there exists a strongly regular field $A_t(x)$ on Γ such that, for each x, with probability one $A_t(x) = B_t(x)$ for all t.*

Lemma 2.3 *Let $p \in (0, \infty)$ and let $m_t(x)$ be a regular martingale field on Γ. Then there exists a nonnegative strongly regular field $A_t(x)$ on Γ such that, for each $x \in \Gamma$, with probability one $A_t(x) = \langle m(x)\rangle_t$ for all $t \in \mathbb{R}_+$.*

Moreover, if $A_t(x)$ is a function with the above described properties and such that

(i) *It is $\mathcal{F}_t \otimes \mathcal{B}(\Gamma)$-measurable for each $t \in \mathbb{R}_+$;*
(ii) *Almost surely*

$$\int_\Gamma \sup_{t\in\mathbb{R}_+} A_t^{p/2}(x)\, dx < \infty, \tag{2.3}$$

then for any countable set $\rho \subset \mathbb{R}_+$ with probability one

$$\int_\Gamma \sup_{t\in\rho} |m_t(x)|^p\, dx < \infty \tag{2.4}$$

and for any $\varepsilon, \delta > 0$ we have

$$P\left(\int_\Gamma \sup_{t\in\rho} |m_t(x)|^p\, dx \geq \delta\right) \leq P(C_\infty \geq \varepsilon) + \frac{N}{\delta} E(\varepsilon \wedge C_\infty) \tag{2.5}$$

where the constant N depends only on p and

$$C_t := \int_\Gamma \sup_{s\leq t} A_s^{p/2}(x)\, dx.$$

Proof To prove the first assertion, define $\kappa_n(t) = n^{-1}[nt]$, $n = 1, 2, \dots$, and observe that for each x with probability one

$$\int_0^t m_{\kappa_n(s)}(x)\, dm_s(x) = \sum_{k=0}^{\infty} m_{k/n}(x)(m_{t\wedge(k+1/n)}(x) - m_{t\wedge(k/n)}(x)) =: B_t^n(x)$$

for all t. Notice that $B_t^n(x)$ is a regular field. Furthermore, for each $x \in \Gamma$ and $T \in \mathbb{R}_+$

$$\int_0^T |m_{\kappa_n(s)}(x) - m_s(x)|^2\, d\langle m(x)\rangle_s \to 0 \quad (a.s.)$$

since $|\kappa_n(s) - s| \leq 1/n$ and $m_t(x)$ is continuous in t (a.s.). It follows that, for any $x \in \Gamma$,

$$B_t^n(x) \to \int_0^t m_s(x)\, dm_s(x)$$

uniformly on finite time intervals in probability as $n \to \infty$. By Lemma 2.1 there exists a strongly regular martingale field $B_t(x)$ on Γ such that, for each $x \in \Gamma$, with probability one

$$B_t(x) = \int_0^t m_t(x)\, dm_t(x)$$

for all t. Taking a strongly regular modification $n_t(x)$ of $m_t^2(x)$, which exists by Lemma 2.2 and letting $A_t(x) = |n_t(x) - 2B_t(x)|$ yields a function we are looking for.

To prove the second assertion, observe that the process C_t is $\mathcal{F}_t$-adapted and, with probability one, is continuous in $t \in \mathbb{R}_+$ owing to condition (2.3) and the dominated convergence theorem. Therefore

$$\tau := \inf\{t \geq 0 : C_t \geq \varepsilon\}$$

is a stopping time. Now

$$P\left(\int_\Gamma \sup_{t\in\rho} |m_t(x)|^p\, dx \geq \delta\right) \leq P(C_\infty \geq \varepsilon)$$
$$+P\left(\int_\Gamma \sup_{t\in\rho} |m_t(x)|^p\, dx \geq \delta, \tau = \infty\right),$$

where the last term is less than

$$P\left(\int_\Gamma \sup_{t\in\rho} |m_{t\wedge\tau}(x)|^p\, dx \geq \delta\right) \leq \frac{1}{\delta} E \int_\Gamma \sup_{t\in\rho} |m_{t\wedge\tau}(x)|^p\, dx,$$

which, in turn, by the Burkholder-Davis-Gundy inequalities is dominated by

$$\frac{1}{\delta}\int_\Gamma E \sup_t |m_{t\wedge\tau}(x)|^p\, dx \leq \frac{N}{\delta}\int_\Gamma E A_\tau^{p/2}(x)\, dx \leq \frac{N}{\delta} E C_\tau \leq \frac{N}{\delta} E(\varepsilon \wedge C_\infty).$$

This proves (2.5) which implies (2.4) if one first lets $\delta \to \infty$ and then $\varepsilon \to \infty$. The lemma is proved.

Remark 2.1 The existence of strongly regular $B_t(x)$ could be obtained from the corollary of Theorem IV.63 of [6]. However, this corollary is stated somewhat differently, so that what we need follows from its proof and the arguments leading to the corollary. We decided to give a direct proof also because the way to use the corollary of Theorem IV.63 of [6], which was presented in the original article, brought about some serious objections of one of the referees.

For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_1, \ldots, \alpha_d \in \{0, 1, \ldots\}$, define

$$D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}, \quad |\alpha| = |\alpha_1| + \ldots + |\alpha_d|.$$

In the following corollary Γ is a ball, $p \in [1, \infty)$, and n is an integer. We denote $\lambda = n - d/p$ and assume that either $p > 1$ and $\lambda \in (0, 1)$ or $p = 1$ and $\lambda = 1$, so that $n = d + 1$.

Corollary 2.4 (i) *Let $m_t(x)$ be a regular martingale field on Γ and assume that for each $\omega \in \Omega$ and $t \in \mathbb{R}_+$ it is n times continuously differentiable in x.*

(ii) *Suppose that, for each multi-index α with $|\alpha| \le n$, $D^\alpha m_t(x)$ is also a regular martingale field on Γ.*

(iii) *Finally, assume that for each multi-index α with $|\alpha| \le n$ (including $\alpha = 0$) on Γ there exists a nonnegative strongly regular field $A_t^\alpha(x)$ possessing the properties (i) and (ii) of Lemma 2.3 and such that, for each $x \in \Gamma$, with probability one $A_t^\alpha(x) = \langle D^\alpha m(x) \rangle_t$ for all $t \in \mathbb{R}_+$.*

Then there is a (finite) random variable ν such that with probability one for all $x, y \in \Gamma$ and $t \in \mathbb{R}_+$ we have

$$|m_t(x) - m_t(y)| \le \nu |x - y|^\lambda. \tag{2.6}$$

Furthermore, with probability one $m_t(x)$ is continuous with respect to (t, x) on $\mathbb{R}_+ \times \Gamma$.

Proof Take ρ as the set of rational numbers on $\mathbb{R}_+$, and observe that, owing to (2.4), there is an event Ω' of full probability and such that for any $\omega \in \Omega'$ we have

$$\sup_{t \in \rho} \sum_{|\alpha| \le n} \int_\Gamma \left| D^\alpha m_t(x) \right|^p \, dx =: \nu_0 < \infty.$$

By the Sobolev embedding theorems (see, for instance, Theorem 5.4 of [1]), for each ω and t, for which

$$\sum_{|\alpha| \le n} \int_\Gamma \left| D^\alpha m_t(x) \right|^p \, dx < \infty,$$

there exists a continuous function $v(x)$ on Γ such that $v(x) = m_t(x)$ for almost all $x \in \Gamma$ and

$$|v(x) - v(y)| \le N |x - y|^\lambda \sum_{|\alpha| \le n} \int_\Gamma \left| D^\alpha m_t(x) \right|^p \, dx \quad \forall x, y \in \Gamma,$$

where N depends only on d, p, and Γ. Of course, $v(x) = m_t(x)$ for all $x \in \Gamma$, since $m_t(x)$ is assumed to be continuous in x. Therefore, (2.6) holds with $\nu = N\nu_0$ for all rational t, $\omega \in \Omega'$, and $x, y \in \Gamma$.

Let $\mathbb{X}$ be the set of points with rational coordinates in Γ and for each $x \in \mathbb{X}$ let Ω_x be the event of full probability such that for each $\omega \in \Omega_x$ the function $m_t(x)$ is continuous in t. Then for any

$$\omega \in \Omega'' := \Omega' \bigcap_{x \in \mathbb{X}} \Omega_x$$

and $x, y \in \mathbb{X}$ we have (2.6) for all rational, and hence, for all t. Since $m_t(x)$ is assumed to be continuous in x, in (2.6) one can take arbitrary $x, y \in \Gamma$ and $t \in \mathbb{R}_+$ as long as $\omega \in \Omega''$. For those ω and any $x \in \Gamma$ it holds that $m_t(x_n) \to m_t(x)$ uniformly in t if $x_n \to x$. By taking $x_n \in \mathbb{X}$, so that $m_t(x_n)$ are continuous in t, we conclude that $m_t(x)$ is continuous in t for any $\omega \in \Omega''$ and $x \in \Gamma$. Since it is also uniformly continuous in x uniformly with respect to t, it is jointly continuous with respect to (t, x) for $\omega \in \Omega''$. It only remains to observe that obviously $P(\Omega'') = 1$ and this proves the corollary.

Remark 2.2 The above corollary is close in spirit to Theorem 3.1.1 of [5]. However, in the applications to the integrals like (1.7) we have in mind (see, for instance, Lemma 4.1) it is much easier to use the corollary than Theorem 3.1.1 of [5].

Corollary 2.5 *By taking $q \in (0, 1)$, substituting $\delta^{1/q}$ in (2.5) in place of δ and ε, and then integrating the result with respect to δ over $(0, \infty)$, we obtain*

$$E\left(\int_\Gamma \sup_t |m_t(x)|^p \, dx\right)^q \le NE\left(\int_\Gamma \sup_t A_t^{p/2}(x) \, dx\right)^q,$$

where the constant N depends only on p and q.

Estimate (2.5) allows us to improve in our particular case Theorem IV.65 of [6], in which in condition (2.8) below the power 1/2 is replaced with 1.

Lemma 2.6 *Let $f_t(x)$ be a real-valued function on $\Omega \times (0, \infty) \times \Gamma$ which is $\mathcal{P} \otimes \mathcal{B}(\Gamma)$-measurable and such that*

$$\int_0^\infty f_t^2(x) \, dt < \infty$$

for each $x \in \Gamma$ and ω. Then there exists a strongly regular martingale field $m_t(x)$ on Γ such that for each $x \in \Gamma$ with probability one

$$m_t(x) = \int_0^t f_s(x) \, dw_s \tag{2.7}$$

for all t. Furthermore, if

$$\int_{\Gamma}\left(\int_0^{\infty} f_t^2(x)\,dt\right)^{1/2} dx < \infty \quad (a.s.), \tag{2.8}$$

then for any function $m_t(x)$ with the properties described above

$$\int_0^{\infty}\left(\int_{\Gamma} f_s(x)\,dx\right)^2 ds < \infty, \quad \int_{\Gamma} \sup_t |m_t(x)|\,dx < \infty \quad (a.s.), \tag{2.9}$$

the stochastic integral

$$\int_0^t \left(\int_{\Gamma} f_s(x)\,dx\right) dw_s \tag{2.10}$$

is well defined, and with probability one

$$\int_{\Gamma} m_t(x)\,dx = \int_0^t \left(\int_{\Gamma} f_s(x)\,dx\right) dw_s \tag{2.11}$$

for all t.

Proof If for each x the function $f_t(x)$ is continuous in t (a.s.), then the existence of $m_t(x)$ with the claimed properties is proved as in the beginning of the proof of Lemma 2.3. In the general case we set $f_t(x) = 0$ for $t \le 0$, take a smooth function $\zeta(t)$ with support in $(-1, 0)$, set $\zeta_t^n = n\zeta(nt)$, $n = 1, 2, \ldots$, and introduce $f_t^n(x) = \zeta_t^n * f_t(x)$ with the convolution performed with respect to t. As is easy to see $f_t^n(x)$ are smooth in t, $\mathcal{P} \otimes \mathcal{B}(\Gamma)$-measurable, and as is well known

$$\int_0^{\infty} \left|f_t - f_t^n(x)\right|^2 (x)\,dt \to 0$$

for each $x \in \Gamma$ and ω. This implies the convergence of the corresponding local martingales uniformly on finite time intervals in probability and after that it only remains to use Lemma 2.1.

Furthermore, the function

$$A_t(x) = \int_0^t f_s^2(x)\,ds$$

is certainly a strongly regular field on Γ such that, for any $x \in \Gamma$, with probability one $A_t(x) = \langle m(x) \rangle_t$ for all t. Also, $A_t(x)$ possesses property (i) of Lemma 2.3 and property (ii) with $p = 1$ if condition (2.8) is satisfied. Under this condition, which we assume in the rest of the proof, the first inequality in (2.9) follows from (2.8) by Minkowski's inequality and implies that (2.10) is well defined indeed. Also, (2.4) with $p = 1$ yields the second inequality in (2.9).

Equality (2.11) follows from Theorem IV.64 of [6] if f is bounded. In the general case for $n = 1, 2, \dots$ define $\chi_n(s) = (-n) \vee s \wedge n$, $f_t^n = \chi_n(f_t)$, and let $m_t^n(x)$ be a strongly regular martingale field on Γ such that for each $x \in \Gamma$ with probability one

$$m_t^n(x) = \int_0^t f_s^n(x)\,dw_s$$

for all t. By the above with probability one

$$\int_\Gamma m_t^n(x)\,dx = \int_0^t \left(\int_\Gamma f_s^n(x)\,dx \right) dw_s \tag{2.12}$$

for all t. By (2.5) and the dominated convergence theorem for any $\varepsilon, \delta > 0$

$$\begin{aligned} &P\left(\int_\Gamma \sup_t |m_t^n(x) - m_t(x)|\,dx \geq \delta \right) \\ &\quad \leq P\left(\int_\Gamma \left(\int_0^\infty \left| f_t^n(x) - f_t(x) \right|^2 dt \right)^{1/2} dx \geq \varepsilon \right) + N\varepsilon/\delta \to N\varepsilon/\delta \end{aligned}$$

as $n \to \infty$. Also by Minkowski's inequality

$$\int_0^\infty \left(\int_\Gamma \left| f_t^n(x) - f_t(x) \right| dx \right)^2 dt \leq \left(\int_\Gamma \left(\int_0^\infty \left| f_t^n(x) - f_t(x) \right|^2 dt \right)^{1/2} dx \right)^2 \to 0$$

as $n \to \infty$ for almost any ω. Hence both sides of (2.12) converge in probability to the corresponding sides of (2.11) uniformly in t implying that (2.11) holds with probability one for all t and the lemma is proved.

Remark 2.3 In contrast with [2,7,8] we assert that Eq. (2.11) holds for all t at once with probability one and not for each fixed t with probability one. Also we are dealing not with a function, say $n_t(x)$ which for a fixed t and almost all x equals the right-hand side of (2.7) (a.s.) but with $m_t(x)$ with much better properties. In particular, the second inequality in (2.9) seems to be new.

Remark 2.4 Below we are going to use "local" versions of Lemmas 2.3 and 2.6 when all processes will be considered on $[0, T]$ with a $T \in \mathbb{R}_+$. These versions are obtained by replacing $m_t(x)$ and $f_t(x)$ with $m_{t\wedge T}$ and $f_t I_{t<T}$ respectively.

Here is a version of the stochastic Fubini theorem.

Lemma 2.7 *Let $T \in \mathbb{R}_+$ and let $G_t(x)$ be real-valued and $H_t(x) = (H_t^k(x), k = 1, 2, \ldots)$ be ℓ_2-valued functions defined on $\Omega \times (0, T] \times \Gamma$ and possessing the following properties*:

(i) *The functions $G_t(x)$ and $H_t(x)$ are $\mathcal{P}_T \otimes \mathcal{B}(\Gamma)$-measurable, where $\mathcal{P}_T$ is the restriction of $\mathcal{P}$ to $\Omega \times (0, T]$;*

(ii) *There is an event Ω' of full probability such that for each $\omega \in \Omega'$ and $x \in \Gamma$ we have*

$$\int_0^T \left(|G_t(x)| + |H_t(x)|_{\ell_2}^2\right) dt < \infty;$$

(iii) *We have (a.s.)*

$$\int_0^T \int_\Gamma |G_t(x)|\, dxdt + \int_\Gamma \left(\int_0^T |H_t(x)|_{\ell_2}^2\, dt\right)^{1/2} dx < \infty.$$

Under these assumptions we claim that

(a) *There is a function $F_t(x)$ on $\Omega \times [0, T] \times \Gamma$, which is $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\Gamma)$-measurable, continuous in t, and such that for any $x \in \Gamma$ with probability one we have*

$$F_t(x) = \int_0^t G_s(x)\, ds + \sum_{k=1}^\infty \int_0^t H_s^k(x)\, dw_s^k \tag{2.13}$$

for all $t \in [0, T]$, where the series converges uniformly on $[0, T]$ in probability;

(b) *For any $k = 1, 2, \ldots,$ the stochastic integrals (no summation in k)*

$$\int_0^t \int_\Gamma H_s^k(x)\, dxdw_s^k$$

are well defined for $t \in [0, T]$;

(c) *If we are given a function $F_t(x)$ on $\Omega \times [0, T] \times \Gamma$ with somewhat weaker properties, namely, such that*

(iv) *For each $t \in [0, T]$ the function $F_t(x)$ is measurable in (ω, x) with respect to the completion $\overline{\mathcal{F} \otimes \mathcal{B}(\Gamma)}$ of $\mathcal{F} \otimes \mathcal{B}(\Gamma)$ with respect to the product measure;*

(v) *For each $t \in [0, T]$ and $x \in \Gamma$ Eq. (2.13) holds almost surely,* *then for any countable subset* ρ *of* $[0, T]$

$$\int_\Gamma \sup_{t\in\rho} |F_t(x)|\,dx < \infty \quad (a.s.), \tag{2.14}$$

and for each $t \in [0, T]$ *almost surely*

$$\int_\Gamma F_t(x)\,dx = \int_0^t \int_\Gamma G_s(x)\,dxds + \sum_{k=1}^\infty \int_0^t \int_\Gamma H_s^k(x)\,dxdw_s^k, \tag{2.15}$$

where the series converges uniformly on $[0, T]$ *in probability.*

(d) *If for a function* $F_t(x)$ *as in* (c)*, for almost all* (ω, x)*,* $F_t(x)$ *is continuous in* t *on* $[0, T]$ *(like the one from assertion (a)), then with probability one* (2.15) *holds for all* $t \in [0, T]$.

Proof Obviously, replacing G and H with $GI_{\Omega'}$ and $HI_{\Omega'}$, respectively, will not affect anything and therefore we may suppose that assumption (ii) holds with $\Omega' = \Omega$. The fact that for each x the series in (2.13) converges uniformly on $[0, T]$ in probability due to condition (ii) is discussed after Definition 1.1. As there, the fact that, by Minkowski's inequality

$$\left(\sum_{k=1}^\infty \int_0^T \left(\int_\Gamma H_t^k(x)\,dx\right)^2 dt\right)^{1/2} \le \int_\Gamma \left(\sum_{k=1}^\infty \int_0^T |H_t^k(x)|^2\,dt\right)^{1/2} dx,$$

where the latter is finite (a.s.) due to (iii), implies that the series in (2.15) converges uniformly on $[0, T]$ in probability.

By Lemma 2.6 (see also Remark 2.4) for each k, on $\Omega \times [0, T] \times \Gamma$ there exists an $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\Gamma)$-measurable function $m_t^k(x)$, which is continuous in t for each $x \in \Gamma$ and ω and such that for any $x \in \Gamma$ with probability one

$$m_t^k(x) = \int_0^t H_s^k(x)\,dw_s^k, \quad \int_\Gamma \sup_{s\le t} |m_s^k(x)|\,dx < \infty$$

for all $t \in [0, T]$. Furthermore, by Lemma 2.6 we also have that with probability one

$$\int_\Gamma m_t^k(x)\,dx = \int_0^t \int_\Gamma H_s^k(x)\,dx\,dw_s^k \tag{2.16}$$

for all $t \in [0, T]$. Now introduce

$$M_t^k(x) = \sum_{j=1}^{k} m_t^j(x).$$

As we have pointed out in the beginning of the proof, for each x, the processes $M_t^k(x)$ converge uniformly on $[0, T]$ in probability as $k \to \infty$. By Lemma 2.1 there exists an $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}(\Gamma)$-measurable function $m_t(x)$, which is continuous in t for all ω and x and such that for any $x \in \Gamma$ we have $M_t^k(x) \to m_t(x)$ uniformly on $[0, T]$ in probability as $k \to \infty$. Of course, for each $x \in \Gamma$ with probability one

$$m_t(x) = \sum_{k=1}^{\infty} \int_0^t H_s^k(x)\, dw_s^k$$

for all $t \in [0, T]$ and this certainly proves assertion (a). Assertion (b) is proved above.

Next, condition (v) means that for each $t \in [0, T]$ and $x \in \Gamma$ we have (a.s.)

$$F_t(x) = \int_0^t G_s(x)\, ds + m_t(x). \tag{2.17}$$

Furthermore, for each x the process $m_t(x)$ is a continuous local martingale and

$$\langle m(x) \rangle_t = \int_0^t |H_s(x)|_{\ell_2}^2\, ds$$

for all $t \in [0, T]$ (a.s.). The right-hand side here can be taken as $A_t(x)$ in Lemma 2.3 and this $A_t(x)$ is strongly regular by the stipulation made in the beginning of the proof, satisfies condition (i) of that lemma and also satisfies its condition (ii) with $p = 1$ due to condition (iii) of the present lemma. By Lemma 2.3 we have

$$\int_\Gamma \sup_{t \le T} |m_t(x)|\, dx < \infty \tag{2.18}$$

(a.s.). Furthermore, in light of (2.17) for each x

$$\sup_{t \in \rho} |F_t(x) - \int_0^t G_s(x)\, ds - m_t(x)| = 0 \tag{2.19}$$

(a.s.). Here the left-hand side is a $\overline{\mathcal{F} \otimes \mathcal{B}(\Gamma)}$-measurable. Therefore, for almost any ω Eq. (2.19) holds for almost all x. This, (2.18), and condition (iii) imply (2.14).

Also, for the local martingale $m_t(x) - M_t^k(x)$ the process $\langle m(x) - M^k(x) \rangle_t$ can be taken to be

$$A_t^k(x) = \sum_{j=k+1}^{\infty} \int_0^t \left(H_s^j(x)\right)^2 ds,$$

so that by Lemma 2.3 for any $\varepsilon, \delta > 0$

$$P\left(\int_\Gamma \sup_{t\le T} \left|m_t(x) - M_t^k(x)\right| dx \ge \delta\right) \le P\left(\int_\Gamma \left(A_T^k(x)\right)^{1/2} dx \ge \varepsilon\right) + \frac{N\varepsilon}{\delta}.$$

After letting first $k \to \infty$ and then $\varepsilon \to 0$ we conclude from assumption (iii) that

$$\int_\Gamma \sup_{t\le T} \left|m_t(x) - M_t^k(x)\right| dx \to 0$$

as $k \to \infty$ in probability. In particular, for each $t \in [0, T]$

$$\sum_{i=1}^{k} \int_\Gamma m_t^k(x)\, dx = \int_\Gamma M_t^k(x)\, dx \to \int_\Gamma m_t(x)\, dx$$

as $k \to \infty$ in probability which is to say that (a.s.)

$$\sum_{i=1}^{\infty} \int_\Gamma m_t^k(x)\, dx = \int_\Gamma m_t(x)\, dx. \tag{2.20}$$

Now for $t \in [0, T]$ being fixed and for almost any ω we have that Eq. (2.17) holds for almost all x. We integrate it over Γ and using (2.20) and (2.16) conclude that (2.15) indeed holds (a.s.). This finishes the proof of assertion (c).

Observe that the right-hand side of (2.15) is continuous in t on $[0, T]$ with probability one. If for almost all (ω, x), $F_t(x)$ is continuous in $[0, T]$, then on the set of full probability for almost any $x \in \Gamma$ the function $F_t(x)$ is continuous on $[0, T]$ and owing to the dominated convergence theorem and (2.14) the left-hand side of (2.15) is also continuous in t on $[0, T]$ with probability one. This implies assertion (d) of the lemma, which is thus proved.

3 A real-valued version of the Itô–Wentzell formula

For $\gamma \in (0, \infty)$ set $B_\gamma = \{x \in \mathbb{R}^d : |x| < \gamma\}$. Also introduce

$$L_t v = a_t^{ij} D_{ij} v + b_t^i D_i v, \quad \Lambda_t^k v = \sigma_t^{ik} D_i v,$$

and recall that the process x_t is introduced after Definition 1.2.

Theorem 3.1 *Let $T \in \mathbb{R}_+$ and $\gamma \in (0, \infty)$. Set $\eta_t(x) = I_{B_\gamma}(x - x_t)$. Let real-valued $G_t(x)$ and ℓ_2-valued $H_t(x) = (H_t^k(x), k = 1, 2, \dots)$ be some functions on $\Omega \times (0, T] \times \mathbb{R}^d$ satisfying assumption (i) of Lemma 2.7 for any ball Γ and let real-valued $F_t(x)$ be a function on $\Omega \times [0, T] \times \mathbb{R}^d$ such that:*

(i) *For each x the restriction of the function $F_t(x)$ to $\Omega \times (0, T]$ is $\mathcal{P}_T$-measurable and $F_0(x)$ is $\mathcal{F}_0$-measurable;*
(ii) *For any $\omega \in \Omega$ and $t \in [0, T]$ the function $F_t(x)$ is continuous in x;*
(iii) *For almost any $(\omega, t) \in \Omega \times [0, T]$ and $k = 1, 2, \dots$*
   (a) *the functions $G_t(x)$, $H_t^k(x)$, and $|H_t(x)|_{\ell_2}$ are continuous in x,*
   (b) *the generalized functions $L_t F_t(x)$, $\Lambda_t^k F_t(x)$, and $\Lambda_t^k H_t^k(x)$ are continuous functions of x as well as the functions $|\Lambda_t F_t(x)|_{\ell_2}$ and*

$$\sum_{r=1}^{\infty} \left| \Lambda_t^r H_t^r(x) \right|;$$

(iv) *There is an event Ω' of full probability such that for each $x \in \mathbb{R}^d$ and $\omega \in \Omega'$ we have*

$$\int_0^T \Big(\eta_t(x)|F_t(x)|(|b_t| + \operatorname{tr} a_t) + \eta_t(x) F_t^2(x) \operatorname{tr} a_t + |G_t(x)| + |H_t(x)|_{\ell_2}^2 \Big)\, dt < \infty$$

and, for each $x \in \mathbb{R}^d$, with probability one equation

$$F_t(x) = F_0(x) + \int_0^t G_s(x)\, ds + \sum_{k=1}^{\infty} \int_0^t H_s^k(x)\, dw_s^k \tag{3.1}$$

holds for all $t \in [0, T]$;
(v) *We have*

$$\int_{\mathbb{R}^d} \int_0^T \eta_t(x)|F_t(x)|(|b_t| + \operatorname{tr} a_t)\, dx dt + \int_{\mathbb{R}^d} \left(\int_0^T \eta_t(x)|F_t(x)|^2 \operatorname{tr} a_t\, dt \right)^{1/2} dx < \infty \quad (a.s.), \tag{3.2}$$

$$\int_0^T \sup_{|x - x_t| \le \gamma} \Big(|G_t(x)| + |L_t F_t(x)| + |\Lambda_t F_t(x)|_{\ell_2}^2 + |H_t(x)|_{\ell_2}^2 + \sum_{k=1}^{\infty} |\Lambda_t^k H_t^k(x)| \Big)\, dt < \infty \quad (a.s.). \tag{3.3}$$

Then for any $t \in [0, T]$ with probability one

$$F_t(x_t) = F_0(0) + \sum_{k=1}^{\infty} \int_0^t \Big(H_s^k(x_s) + \Lambda_s^k F_s(x_s) \Big)\, dw_s^k$$
$$+ \int_0^t \left(G_s(x_s) + L_s F_s(x_s) + \sum_{k=1}^{\infty} \Lambda_s^k H_s^k(x_s) \right) ds. \qquad (3.4)$$

Furthermore, if $F_t(x_t)$ is continuous in t on $[0, T]$ for almost all ω, then with probability one (3.4) holds for all $t \in [0, T]$.

Proof First, observe that the series of stochastic integrals in (3.4) converges uniformly on $[0, T]$ in probability and its limit is a local martingale since

$$\sum_{k=1}^{\infty} \int_0^T \Big| H_s^k(x_s) + \Lambda_s^k F_s(x_s) \Big|^2 \, ds \le 2 \sum_{k=1}^{\infty} \int_0^T \Big| H_s^k(x_s) \Big|^2 \, ds$$
$$+ 2 \sum_{k=1}^{\infty} \int_0^T \Big| \Lambda_s^k F_s(x_s) \Big|^2 \, ds = 2 \int_0^T |H_s(x_s)|_{\ell_2}^2 \, ds + 2 \int_0^T |\Lambda_s F_s(x_s)|_{\ell_2}^2 \, ds < \infty$$

for almost all ω due to (3.3). It is seen that, in light of (3.3), the right-hand side of (3.4) is continuous in t (a.s.) and hence the second assertion of the theorem follows from the first one.

To prove the first assertion, for $R \in (0, \infty)$ introduce τ_R as the first exit time of x_t from B_R. Notice that if we take $x_{t \wedge \tau_R}$, $\sigma_t I_{t < \tau_R}$, $b_t(x) I_{t < \tau_R}$, $F_{t \wedge \tau_R}(x)$, $G_t(x) I_{t < \tau_R}$, and $H_t(x) I_{t < \tau_R}$ instead of the original ones, then the conditions (i)–(v) will be preserved. If we have (3.4) for the new objects, then we can send $R \to \infty$ and easily obtain (3.4) as is because $\tau_R \uparrow \infty$, so that for any ω there exists R such that $t \wedge \tau_R = t$. We conclude that without loss of generality we may assume that, for an $R \in [0, \infty)$, we have $|x_t| \le R$ for all t.

After that, by taking $\xi \in C_0^\infty$ such that it equals one on $B_{R+\gamma}$ and replacing F, G, H with $\xi F, \xi G, \xi H$, respectively, we see that the assumptions of the theorem will still be satisfied and assertion (3.4) will be unaffected. Therefore, without loss of generality we may assume that there exists and $R < \infty$ such that F, G, H vanish outside B_R. As in the proof of Lemma 2.7 we may assume that $\Omega' = \Omega$ in condition (iv).

After these reductions we take a $\zeta \in C_0^\infty$, which is nonnegative, radially symmetric, with unit integral and support in B_γ. Then for any $x \in \mathbb{R}^d$ Itô's formula yields that with probability one

$$F_t(x)\zeta(x - x_t) - F_0(x)\zeta(x) = \int_0^t \hat{G}_s(x)\, ds + \sum_{k=1}^{\infty} \int_0^t \hat{H}_s^k(x)\, dw_s^k \qquad (3.5)$$

for all $t \in [0, T]$, where

$$\hat{H}^k_s(x) := \zeta(x - x_s)H^k_s(x) - F_s(x)(\Lambda^k_s\zeta)(x - x_s),$$
$$\hat{G}_s(x) := \zeta(x - x_s)G_s(x) + F_s(x)(a^{ij}_s D_{ij}\zeta - b^i_s D_i\zeta)(x - x_s) - \sum_{k=1}^{\infty} H^k_s(x)(\Lambda^k_s\zeta)(x - x_s).$$

We want to apply Lemma 2.7 to (3.5).

First, observe that $\hat{G}$ and $\hat{H}$ satisfy assumption (i) of Lemma 2.7 for any ball Γ owing to the imposed measurability assumption on G and H and conditions (i) and (ii). Then, for each $x \in \mathbb{R}^d$ and $\omega \in \Omega$

$$\int_0^T |\hat{G}_t(x)|\,dt \le N\int_0^T \eta_t(x)\,(|G_t(x)| + |F_t(x)|(\mathrm{tr}\,a_t + |b_t|))\,dt + N\int_0^T \eta_t(x)\sum_{k=1}^{\infty}|H^k_t(x)|\,|\sigma^k_t|\,dt,$$

where the constants N are independent of ω, x. Regarding the last term notice that by Hölder's inequality

$$I(x) := \int_0^T \eta_t(x)\sum_{k=1}^{\infty}|H^k_t(x)|\,|\sigma^k_t|\,dt \le \left(\int_0^T \eta_t(x)|H_t(x)|^2_{\ell_2}\,dt\right)^{1/2}\left(2\int_0^T \mathrm{tr}\,a_t\,dt\right)^{1/2},$$

which is finite for all ω and x owing to condition (iv) and (1.4). We see that by condition (iv)

$$\int_0^T |\hat{G}_t(x)|\,dt < \infty$$

for all ω, x. Furthermore,

$$I(x) \le \left(\int_0^T \sup_y(\eta_t(y)|H_t(y)|^2_{\ell_2})\,dt\right)^{1/2}\left(2\int_0^T \mathrm{tr}\,a_t\,dt\right)^{1/2}$$

which is finite (a.s.) due to assumption (3.3). The last expression here is independent of x and the first one vanishes for $|x| \geq R$. Therefore, (a.s.)

$$\int_0^T \int_{\mathbb{R}^d} \eta_t(x) \sum_{k=1}^{\infty} \left| H_t^k(x) \right| \left| \sigma_t^k \right| dxdt < \infty.$$

Similarly,

$$\int_0^T \int_{\mathbb{R}^d} \eta_t(x) |G_t(x)| \, dxdt < \infty$$

(a.s.). By combining this with assumption (3.2) we see that (a.s.)

$$\int_0^T \int_{\mathbb{R}^d} |\hat{G}_t(x)| \, dxdt < \infty.$$

Furthermore, condition (iv) implies that for each $x \in \mathbb{R}^d$ and $\omega \in \Omega$

$$\int_0^T |\hat{H}_t(x)|_{\ell_2}^2 \, dt < \infty$$

and condition (v) implies that (a.s.)

$$\int_{\mathbb{R}^d} \left(\int_0^T |\hat{H}_t(x)|_{\ell_2}^2 \, dt \right)^{1/2} dx < \infty.$$

We see that the assumptions of Lemma 2.7 are satisfied for any ball Γ and recalling that F, G, H vanish for $|x| \geq R$ we conclude that for each $t \in [0, T]$ with probability one

$$\int_{\mathbb{R}^d} F_t(x)\zeta(x - x_t) \, dx = \int_{\mathbb{R}^d} F_0(x)\zeta(x) \, dx + \int_0^t \int_{\mathbb{R}^d} \hat{G}_s(x) \, dxds$$
$$+ \sum_{k=1}^{\infty} \int_0^t \int_{\mathbb{R}^d} \hat{H}_s^k(x) \, dxdw_s^k.$$

We fix $t \in [0, T]$ and use this formula with $\zeta_\varepsilon(x) := \varepsilon^{-d}\zeta(x/\varepsilon)$, $\varepsilon > 0$, in place of ζ and integrate by parts in the integrals of $\hat{G}$ and $\hat{H}$ with respect to x (that is, use the definition of generalized derivatives). Then by using the notation $u^{(\varepsilon)} = u * \zeta_\varepsilon$ we find that with probability one

$$F_t^{(\varepsilon)}(x_t) = F_0^{(\varepsilon)}(0) + \sum_{k=1}^{\infty} \int_0^t \left(H_s^{(\varepsilon)k}(x_s) + \Lambda_s^k F_s^{(\varepsilon)}(x_s) \right) dw_s^k + \int_0^t \left(G_s^{(\varepsilon)}(x_s) + L_s F_s^{(\varepsilon)}(x_s) + \sum_{k=1}^{\infty} \Lambda_s^k H_s^{(\varepsilon)k}(x_s) \right) ds. \tag{3.6}$$

We now let $\varepsilon \downarrow 0$ in (3.6). Since $F_t(x)$ is continuous in x we have $F_t^{(\varepsilon)}(x_t) \to F_t(x_t)$ (for all ω). Furthermore, by assumption for almost any ω, for almost all $s \in [0, t]$ the function $|\Lambda_s F_s(x)|_{\ell_2}$ is continuous and also $\Lambda_s^k F_s(x)$ are continuous. It follows by Dini's theorem that for the above ω and s

$$\sum_{k=n}^{\infty} \left| \Lambda_s^k F_s(x) \right|^2 \downarrow 0 \tag{3.7}$$

as $n \to \infty$ uniformly on compact sets in $\mathbb{R}^d$. Similar argument shows that (a.s.) for almost any $s \in [0, t]$

$$\sum_{k=n}^{\infty} \left(|H_s^k|^2 + |\Lambda_s^k H_s^k| \right)(x) \to 0,$$

as $n \to \infty$ uniformly on compact sets in $\mathbb{R}^d$. This implies that (a.s.) for almost any $s \in [0, t]$ as $\varepsilon \downarrow 0$,

$$\left(|(\Lambda_s F_s)^{(\varepsilon)} - \Lambda_s F_s|_{\ell_2}^2 + |H_t^{(\varepsilon)} - H_t|_{\ell_2}^2 + \sum_{k=1}^{\infty} \left| \Lambda_s^k H_s^{(\varepsilon)k} - \Lambda_s^k H_s^k \right| \right)(x_s) \to 0.$$

Hence, in light of (3.3), by the dominated convergence theorem (a.s.) we have as $\varepsilon \downarrow 0$ that

$$\int_0^t \left(|(\Lambda_s F_s)^{(\varepsilon)} - \Lambda_s F_s|_{\ell_2}^2 + \left| H_t^{(\varepsilon)} - H_t \right|_{\ell_2}^2 + \sum_{k=1}^{\infty} \left| \Lambda_s^k H_s^{(\varepsilon)k} - \Lambda_s^k H_s^k \right| \right)(x_s)\, ds \to 0.$$

This allows us to assert that part of the terms in (3.6) converges in probability to what we need.

Convergence of the remaining terms in (3.6) is proved in like manner. Thus, passing to the limit in (3.6) yields (3.4) and this brings the proof of the theorem to an end.

Remark 3.1 The assumptions of this theorem are substantially weaker than the ones usually imposed (see, for instance, Theorem 3.3.1 of [5] or Theorem 1.4.9 of [7]).

In particular, we are dealing with the generalized functions $L_t F_t(x)$, $\Lambda_t^k F_t(x)$, and $\Lambda_t^k H_t^k(x)$, which always exist and are continuous in x (just equal zero) at those (t, ω) at which $a_t = b_t = 0$. Furthermore, at those (t, ω), at which $a_t = 0$, no differentiability assumption is imposed on H.

Also, observe that if F is nonrandom (so that $H \equiv 0$), then the Itô–Wentzell formula becomes just Itô's formula and our theorem gives the proof of it under substantially weaker assumptions than the ones usually imposed (like twice continuously differentiability for all x).

4 Proof of Theorem 1.1

Here we suppose that the assumptions of Theorem 1.1 are satisfied and will base our proof on Theorem 3.1.

Note that (1.6) involves the values of x_t only for $t < \tau$. Therefore, without losing generality we may assume that $b_t^i = \sigma_t^{ik} = 0$ for $t \geq \tau$ for all i, k. Also obviously we may assume that τ is bounded. These additional assumptions are supposed to hold throughout the section.

Lemma 4.1 *Take a $\phi \in C_0^\infty$ and set*

$$F_t(x) = (u_{t\wedge\tau}(\cdot + x), \phi).$$

Then with probability one $F_t(x)$ and, for any multi-index α, $D^\alpha F_t(x)$ are continuous functions of $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$.

Proof By multiplying, if necessary, u, f, g by the indicator of an event of full probability (perhaps depending on ϕ) we may assume that for any $\omega \in \Omega$, multi-index α, and $T, R \in \mathbb{R}_+$, we have

$$\int_0^T \sup_{|x|\leq R} |(f_t, D^\alpha \phi(\cdot - x))| I_{t\leq\tau}\, dt < \infty,$$

$$\int_0^T \sup_{|x|\leq R} |(g_t, D^\alpha \phi(\cdot - x))|_{\ell_2}^2 I_{t\leq\tau}\, dt < \infty.$$

Set

$$G_t(x) = (f_t(\cdot + x), \phi) I_{t\leq\tau}, \quad H_t(x) = (g_t(\cdot + x), \phi) I_{t\leq\tau}$$

and observe that for each $x \in \mathbb{R}^d$ Eq. (3.1) holds with probability one for all t due to (1.3). Therefore,

$$m_t(x) := F_t(x) - \int_0^t G_s(x)\, ds \tag{4.1}$$

for every x with probability one satisfies

$$m_t(x) = \sum_{k=1}^{\infty} \int_0^t H_s^k(x)\, dw_s^k$$

for all t. Furthermore, for each ω and t the functions $F_t(x)$, $G_t(x)$, and $H_t^k(x)$ are infinitely differentiable in x. If $|x| \le R$ and α is a multi-index, then

$$|D^\alpha G_s(x)| \le \sup_{|y| \le R} |(f_s(\cdot + y), D^\alpha \phi)| I_{s \le \tau}$$

and by assumption the latter is locally integrable on $\mathbb{R}_+$ (for any ω). This shows that the integral in (4.1) is also infinitely differentiable in x and one can perform differentiating the integral by differentiating the integrand. Hence $m_t(x)$ is infinitely differentiable in x. By replacing ϕ with $D^\alpha \phi$ in the above argument we now see that for every x and any multi-index α with probability one

$$D^\alpha m_t(x) = \sum_{k=1}^{\infty} \int_0^t D^\alpha H_s^k(x)\, dw_s^k$$

for all t. The quadratic variation of the sum on the right can be taken to be

$$A_t^\alpha(x) = \sum_{k=1}^{\infty} \int_0^t I_{s \le \tau} \big(g_s^k, D^\alpha \phi(\cdot - x)\big)^2 \, ds.$$

Moreover, for each $x \in \mathbb{R}^d$ the function $D^\alpha m_t(x)$ is $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)$-measurable (see Definition 1.1) and for each (ω, t) continuous in x. Hence it is a regular martingale field on $\mathbb{R}^d$. Finally, by definition for any ball Γ (recall that τ is bounded)

$$\left(\int_\Gamma \left| A_\infty^\alpha(x) \right|^{1/2} dx \right)^2 \le |\Gamma|^2 \sup_{x \in \Gamma} \int_0^\tau |(g_s, D^\alpha \phi(\cdot - x))|_{\ell_2}^2 \, ds$$
$$\le |\Gamma|^2 \int_0^\tau \sup_{x \in \Gamma} |(g_s, D^\alpha \phi(\cdot - x))|_{\ell_2}^2 \, ds < \infty,$$

where $|\Gamma|$ is the volume of Γ. Now our assertion about F follows immediately from Corollary 2.4. The functions $D^\alpha F$ are taken care of by replacing ϕ with $D^\alpha \phi$. The lemma is proved.

Remark 4.1 Naturally, Eq. (1.6) is understood in the sense of Definition 1.2. Therefore, it is important to explain that the terms on the right in (1.6) belong to the right class of functions. Notice that for any $\phi \in C_0^\infty$ and $T, R \in \mathbb{R}_+$ we have

$$\int_0^T I_{t\leq\tau} \sup_{|x|\leq R} |(D_1 v_t(\cdot + x), \phi)\sigma_t^{1\cdot}|_{\ell_2}^2\, dt = \int_0^T I_{t\leq\tau} \sup_{|x|\leq R} |(v_t(\cdot + x), D_1\phi)|^2 a_t^{11}\, dt$$
$$\leq \sup_{|x|\leq R, t\leq T} |(u_{t\wedge\tau}(\cdot + x_{t\wedge\tau} + x, D_1\phi)|^2 \int_0^\tau a_t^{11}\, dt$$
$$\leq \sup_{|x|\leq R+N, t\leq T} |(u_{t\wedge\tau}(\cdot + x, D_1\phi)|^2 \int_0^\tau a_t^{11}\, dt < \infty \quad \text{(a.s.)},$$

where $N = \sup_t |x_{t\wedge\tau}|$ is a finite random variable and the last inequality follows from Lemma 4.1 and assumption (1.4). Similarly one treats the remaining terms in (1.6).

Proof of Theorem 1.1 We take ϕ, F, G, and H from Lemma 4.1. By definition, $G_t(x)$ and $H_t^k(x)$ are predictable for each x. In addition, for each ω and t these functions are infinitely differentiable with respect to x (in the strong sense in ℓ_2 in the case of $H_t(x)$). Therefore, these functions satisfy the measurability condition (i) of Lemma 2.7 for any T and ball Γ and satisfy condition (iii) (a) of Theorem 3.1. Similarly $F_t(x)$ satisfies conditions (i) and (ii) of Theorem 3.1 for any $T \in \mathbb{R}_+$.

Furthermore, not only $H_t^k(x)$ and $|H_t(x)|_{\ell_2}$ are continuous, the same is true for the derivatives of $H_t(x)$ with respect to x. In particular, as in the case of (3.7), for $i = 1, 2, \ldots, d$

$$\sum_{k=n}^{\infty} \left|D_i H_t^k(x)\right|^2 \downarrow 0$$

as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^d$ for any ω and t. The estimate

$$\sum_{k=n}^{\infty} \left|\Lambda_t^k H_t^k(x)\right| \leq (2\operatorname{tr} a_t)^{1/2} \left(\sum_{i=1}^{d} \sum_{k=n}^{\infty} \left|D_i H_t^k(x)\right|^2\right)^{1/2} \tag{4.2}$$

obtained by Hölder's inequality, shows that the left-hand side goes to zero as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}^d$ and hence its value for $n = 1$ is a continuous function of x for any ω and t. Also, the mean value theorem and the continuity of the second order derivatives of $F_t(x)$ in x easily yield the continuity of $|\Lambda_t F_t(x)|_{\ell_2}$.

Thus, assumptions (ii) and (iii) of Theorem 3.1 are satisfied for any $T \in \mathbb{R}_+$. In assumption (iv) Eq. (3.1) holds for each x with probability one for all t due to (1.3).

Furthermore, by assumption for each $T, R \in \mathbb{R}_+$, ω, and $|x| \leq R$ we have

$$\int_0^T |G_t(x)|\, dt \leq \int_0^T \sup_{|y|\leq R} |G_t(y)|\, dt < \infty. \tag{4.3}$$

Inequalities like (4.2) and (4.3), Lemma 4.1, and the fact that the support of $\sup_{t\le T} I_{B_\gamma}(x-x_t)$ is bounded for each ω imply that that assumptions (iv) and (v) of Theorem 3.1 are satisfied and we can apply it.

Now by Lemma 4.1 and Theorem 3.1 we have that with probability one (3.4) holds for all $t\in[0,T]$ and, actually, by the arbitrariness of T, for all t. We rewrite (3.4) in terms of u_t and v_t, use that $b^i_t=\sigma^{ik}_t=0$ for $t\ge\tau$ for all i,k, and see that with probability one

$$(v_{t\wedge\tau},\phi)=(v_0,\phi)+\sum_{k=1}^{\infty}\int_0^t I_{s\le\tau}\left(\left(g^k_s(\cdot+x_s),\phi\right)+\left(\sigma^{ik}_s D_i v_s,\phi\right)\right)dw^k_s$$
$$+\int_0^t I_{s\le\tau}\left[(f_s(\cdot+x_s)+L_s v_s,\phi)+\sum_{k=1}^{\infty}\sigma^{ik}_s\left(D_i g^k_s(\cdot+x_s),\phi\right)\right]ds \tag{4.4}$$

for all t. Here for each ω and s (recall the definition of the limit of distributions)

$$\sum_{k=1}^{\infty}\sigma^{ik}_s\left(D_i g^k_s(\cdot+x_s),\phi\right)=\left(\sum_{k=1}^{\infty}\sigma^{ik}_s D_i g^k_s(\cdot+x_s),\phi\right)$$

since

$$\left(\sum_{k=1}^{\infty}\left|\sigma^{ik}_s\right|\,\left|\left(D_i g^k_s(\cdot+x_s),\phi\right)\right|\right)^2\le 2\operatorname{tr}a_s\sum_{i=1}^{d}|(D_i g_s(\cdot+x_s),\phi)|^2_{\ell_2}<\infty.$$

This shows that (4.4) implies (1.6). The theorem is proved.

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