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Stochastic aspects of easy quantum groups

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Abstract We consider several orthogonal quantum groups satisfying the "easiness" assumption axiomatized in our previous paper. For each of them we discuss the computation of the asymptotic law of $\mathrm{Tr}(u^k)$ with respect to the Haar measure, u being the fundamental representation. For the classical groups O_n, S_n we recover in this way some well-known results of Diaconis and Shahshahani.

Keywords Random matrices · Quantum group · Noncrossing partition

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Introduction

The present paper is a continuation of our previous work [6] on easy quantum groups. We will present a concrete application of our formalism: a unified approach plus quantum group extension of some results of Diaconis–Shahshahani [14].

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The objects of interest will be the compact quantum groups satisfying $S_n \subset G \subset O_n^+$. Here O_n^+ is the free analogue of the orthogonal group, constructed by Wang in [21], and for the compact quantum groups we use Woronowicz's formalism in [23].

As in [6], we restrict attention to the "easy" case. The easiness assumption, essential to our considerations, roughly states that the tensor category of G should be spanned by certain partitions, coming from the tensor category of S_n. This might look like a quite technical condition, but in our opinion this provides a good framework for understanding certain probabilistic and representation theory aspects of orthogonal quantum groups.

There are 14 natural examples of easy quantum groups found in [6]. The list is as follows:

(1) Groups: $O_n, S_n, H_n, B_n, S_n', B_n'$.
(2) Free versions: $O_n^+, S_n^+, H_n^+, B_n^+, S_n'^+, B_n'^+$.
(3) Half-liberations: O_n^*, H_n^*.

The 4 "primed" versions above are rather trivial modifications of their "unprimed" versions, corresponding to taking a product with a copy of $\mathbb{Z}_2$. We will focus then on the remaining 10 examples in this paper. In addition to the 14 examples listed above, there are two infinite series $H_n^{(s)}$ and $H_n^{[s]}$, described in [4], which are related to the complex reflection groups $H_n^s = \mathbb{Z}_s \wr S_n$.

Our motivating belief, already present in [6], is that "any result which holds for O_n, S_n should have a suitable extension to all the easy quantum groups". This is of course a quite vague statement, its precise target being actually formed by a number of questions at the borderline between representation theory and probability theory.

It was suggested in [6] that a first such application might come from the results of Diaconis of Shahshahani in [14], regarding the groups O_n, S_n. We will show in this paper that this is indeed the case:

(1) The problematics makes indeed sense for all easy quantum groups.
(2) There is a global approach to it, by using partitions and cumulants.
(3) The new computations lead to a number of interesting conclusions.

As a first example, consider the orthogonal group O_n, with fundamental representation denoted u. The results in [14], that we will recover as well by using our formalism, state that the asymptotic variables $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ are real Gaussian and independent, with variance k and mean 0 or 1, depending on whether k is odd or even.

In the case of O_n^+, however, the situation is quite different: the variables u_k are free, as one could expect, but they are semicircular at $k = 1, 2$, and circular at $k \geq 3$.

Summarizing, in the orthogonal case we have the following table:

Variable	O_n	O_n^+
u_1	Real Gaussian	Semicircular
u_2	Real Gaussian	Semicircular
u_k $(k \geq 3)$	Real Gaussian	Circular

In the symmetric case the situation is even more surprising, with the Poisson variables from the classical case replaced by several types of variables:

Variable	S_n	S_n^+
u_1	Poisson	Free Poisson
$u_2 - u_1$	Poisson	Semicircular
$u_k - u_1$ $(k \geq 3)$	Sum of Poissons	Circular

We will present as well similar computations for the groups H_n, B_n, for their free analogues H_n^+, B_n^+, for the half-liberated quantum groups O_n^*, H_n^*, as well as for the series $H_n^{(s)}$. The calculations in the latter case rely essentially on Diaconis–Shahshahani type results for the complex reflection groups $H_n^s = \mathbb{Z}_s \wr S_n$.

The challenging question, that will eventually be left open, is to find a formal "eigenvalue" interpretation for all the quantum group results.

The paper is organized as follows. After a short Sect. 0 with notational remarks, we recall the basic definitions and facts about easy quantum groups in Sect. 1. In Sect. 2, we recall the Weingarten formula for our easy quantum groups and use it to derive a formula for the moments of traces of powers. In Sect. 3, this is refined to a formula for corresponding cumulants, in the classical and the free cases. In Sects. 4–7, this will then be used to study the orthogonal, bistochastic, symmetric, and hyperoctahedral classical and quantum groups, respectively. Section 8 deals with the half-liberated quantum groups O_n^* and H_n^*. The results in these cases will rely on the observation that these half-liberated quantum groups are in some sense orthogonal versions of classical groups, U_n for O_n^* and H^∞ for H_n^*. The main calculations will take place for these classical groups. The same ideas work actually for the half-liberated series $H_n^{(s)}$, by considering those as orthogonal versions of the complex reflection groups H_n. One of the main results in Sect. 8 is a Diaconis–Shahshahani type result for those classical reflection groups. In Sect. 9, we finish with some concluding remarks and open problems.

0 Notation

0.1 Quantum groups

As in our previous work [6], the basic object under consideration will be a compact quantum group G. The concrete examples of such quantum groups include the usual compact groups G, and, to some extent, the duals of discrete groups $\widehat{\Gamma}$. In the general case, however, G is just a fictional object, which exists only via its associated Hopf C^*-algebra of "complex continuous functions", generically denoted A.

The fact that G itself doesn't exist is not really an issue, because many advanced tools coming from algebra, analysis and geometry are available. In fact, to the well-known criticism stating that "quantum groups do not exist", our answer would be that "classical groups exist, indeed, but is their existence property the most important?".

For simplicity of notation, we will rather use the quantum group G instead of the Hopf algebra A. For instance we will write integrals of the following type:

$$\int_G u_{i_1 j_1} \ldots u_{i_k j_k} \, du.$$

The value of this integral is of course the complex number obtained by applying the Haar functional $\varphi : A \to \mathbb{C}$ to the well-defined quantity $u_{i_1 j_1} \dots u_{i_k j_k} \in A$.

We will use the quantum group notation depending on the setting: in case where this can lead to confusion, we will rather switch back to the Hopf algebra notation.

0.2 Partitions

For the notations and basic facts around the set of all and non-crossing partitions we refer to [17] and our previous papers [4–6]. We will in particular use the following notations.

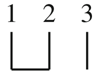
P_k denotes the set of partitions of the set $\{1, \dots, k\}$. 1_k denotes the maximal element in P_k, which consists only of one block. For a partition $\pi \in P_k$ we denote by $|\pi|$ the number of blocks of π.

With $\mathbf{i}$ we will usually denote multi-indices $\mathbf{i} = (i_1, \dots, i_k)$. Often, the constraints in sums for such indices are given in terms of their kernel, $\ker \mathbf{i} = \ker(i_1, \dots, i_k)$. This is the partition in P_k determined as follows:

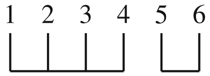
$$\text{s and t are in the same block of } \ker \mathbf{i} \iff i_s = i_t.$$

For given $k_1, \dots, k_r \in \mathbb{N}$, $k := \sum k_i$, let $\gamma \in S_k$ be the permutation with cycles $(1, \dots, k_1)$, $(k_1 + 1, \dots, k_1 + k_2)$, …,$(k - k_s + 1, \dots, k)$. If we have, in addition, a partition $\sigma \in P_r$, then σ^γ will denote the canonical lift of σ from P_r to P_k, associated to γ. Thus, σ^γ is that partition which we get from σ by replacing each $j \in \{1, \dots, r\}$ by the jth cycle of γ, i.e., $\sigma^\gamma \geq \gamma$ and the ith and the jth cycle of γ are in the same block of σ^γ if and only if i and j are in the same block of σ.

As an example, let $\gamma = (1)(2, 3, 4)(5, 6)$. Consider now $\sigma = \{(1, 2), (3)\} \in P_3$:



Then σ^γ is given by making the replacements $1 \to 1$, $2 \to 2, 3, 4$ and $3 \to 5, 6$,



thus $\sigma^\gamma = \{(1, 2, 3, 4), (5, 6)\} \in P_6$.

Note that in [17] the notation $\hat{\sigma}$ was used for σ^γ.

1 Easy quantum groups

In this section and in the next one we briefly recall some notions and results from [6,4].

Consider first a compact group satisfying $S_n \subset G \subset O_n$. That is, $G \subset O_n$ is a closed subgroup, containing the subgroup $S_n \subset O_n$ formed by the permutation matrices.

Let u, v be the fundamental representations of G, S_n. By functoriality we have $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) \subset \mathrm{Hom}(v^{\otimes k}, v^{\otimes l})$, for any k, l. On the other hand, the Hom-spaces for v are well-known: they are spanned by certain explicit operators T_p, with p belonging to $P(k, l)$, the set of partitions between k points and l points. More precisely, if $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{C}^n$, the formula of T_p is as follows:

$$T_p(e_{i_1} \otimes \cdots \otimes e_{i_k}) = \sum_{j_1,\dots,j_l} \delta_p \begin{pmatrix} i_1 & \dots & i_k \\ j_1 & \dots & j_l \end{pmatrix} e_{j_1} \otimes \cdots \otimes e_{j_l}.$$

Here the δ symbol on the right is 0 or 1, depending on whether the indices “fit” or not, i.e., $\delta = 1$ if all blocks of p contains equal indices, and $\delta = 0$ if not.

We conclude from the above discussion that the space $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l})$ consists of certain linear combinations of operators of type T_p, with $p \in P(k, l)$.

We call G “easy” if its tensor category is spanned by partitions.

Definition 1.1 A compact group $S_n \subset G \subset O_n$ is called *easy* if there exist sets $D(k, l) \subset P(k, l)$ such that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p | p \in D(k, l))$, for any k, l.

It follows from the axioms of tensor categories that the collection of sets $D(k, l)$ must be closed under certain categorical operations, namely the vertical and horizontal concatenation, and the upside-down turning. The corresponding algebraic structure formed by the sets $D(k, l)$, axiomatized in [6], is called “category of partitions”.

We denote by $H_n = \mathbb{Z}_2 \wr S_n$ the hyperoctahedral group, formed by the monomial (i.e., permutation-like) matrices having ± 1 nonzero entries. The bistochastic group, $B_n \simeq O_{n-1}$, is by definition formed by the matrices in O_n having sum 1 on each row and each column. Finally, the modified symmetric and bistochastic groups are by definition $S_n' = \mathbb{Z}_2 \times S_n$ and $B_n' = \mathbb{Z}_2 \times B_n$, both viewed as subgroups of O_n. See [6].

Theorem 1.2 *There are exactly six easy orthogonal groups, namely:*

(1) O_n*: the orthogonal group.*
(2) S_n*: the symmetric group.*
(3) H_n*: the hyperoctahedral group.*
(4) B_n*: the bistochastic group.*
(5) S_n'*: the modified symmetric group.*
(6) B_n'*: the modified bistochastic group.*

Proof As explained in [6], this follows from a sixfold classification result for the corresponding categories of partitions, which are as follows:

(1) P_o: all pairings.
(2) P_s: all partitions.
(3) P_h: partitions with blocks of even size.
(4) P_b: singletons and pairings.
(5) $P_{s'}$: all partitions (even part).
(6) $P_{b'}$: singletons and pairings (even part). □

Let us discuss now the free analogue of the above results. Let O_n^+, S_n^+ be the free orthogonal and symmetric quantum groups, corresponding to the Hopf algebras $A_o(n)$, $A_s(n)$ constructed by Wang in [21,22]. Here, and in what follows, we use Woronowicz's compact quantum group formalism in [23], cf. Sect. 0 above.

We have $S_n \subset S_n^+$, so by functoriality the Hom-spaces for S_n^+ appear as subspaces of the corresponding Hom-spaces for S_n. The Hom-spaces for S_n^+ have in fact a very simple description: they are spanned by the operators T_p, with $p \in NC(k,l)$, the set of noncrossing partitions between k upper points and l lower points.

We have the following "free analogue" of Definition 1.1.

Definition 1.3 A compact quantum group $S_n^+ \subset G \subset O_n^+$ is called *free* if there exist sets $D(k,l) \subset NC(k,l)$ such that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p|p \in D(k,l))$, for any k, l.

In this definition, the word "free" has of course a quite subtle meaning, to be fully justified later on. For the moment, let us just record the fact that the passage from Definition 1.1 to Definition 1.3 is basically done by "restricting attention to the noncrossing partitions", which, according to [19], should indeed lead to freeness.

As in the classical case, the sets of partitions $D(k,l)$ must be stable under certain categorical operations, coming this time from the axioms in [24]. The corresponding algebraic structure, axiomatized in [6], is called "category of noncrossing partitions".

We denote by H_n^+ the hyperoctahedral quantum group, constructed in [2], and by B_n^+, $S_n'^+$, $B_n'^+$ the free analogues of the groups B_n, S_n', B_n', constructed in [6].

Theorem 1.4 *There are exactly six free orthogonal quantum groups, namely:*

(1) O_n^+*: the orthogonal quantum group.*
(2) S_n^+*: the symmetric quantum group.*
(3) H_n^+*: the hyperoctahedral quantum group.*
(4) B_n^+*: the bistochastic quantum group.*
(5) $S_n'^+$*: the modified symmetric quantum group.*
(6) $B_n'^+$*: the modified bistochastic quantum group.*

Proof As explained in [6], this follows from a sixfold classification result for the corresponding categories of noncrossing partitions, which are as follows:

(1) NC_o: all noncrossing pairings.
(2) NC_s: all noncrossing partitions.
(3) NC_h: noncrossing partitions with blocks of even size.
(4) NC_b: singletons and noncrossing pairings.
(5) $NC_{s'}$: all noncrossing partitions (even part).
(6) $NC_{b'}$: singletons and noncrossing pairings (even part). □

Observe the symmetry between Theorem 1.2 and Theorem 1.4: this corresponds to the "liberation" operation for orthogonal Lie groups, further investigated in [6].

Consider now the general situation where we have a compact quantum group satisfying $S_n \subset G \subset O_n^+$. Once again, we can ask for the tensor category of G to be spanned by certain partitions, coming from the tensor category of S_n.

Definition 1.5 A compact quantum group $S_n \subset G \subset O_n^+$ is called *easy* if there exist sets $D(k,l) \subset P(k,l)$ such that $\mathrm{Hom}(u^{\otimes k}, u^{\otimes l}) = \mathrm{span}(T_p | p \in D(k,l))$, for any k, l.

As a first remark, this definition generalizes at the same time Definition 1.1 and Definition 1.3. In fact, the easy quantum groups $S_n \subset G \subset O_n^+$ satisfying the extra assumption $G \subset O_n$ are precisely the easy groups, and those satisfying the extra assumption $S_n^+ \subset G$ are precisely the free quantum groups.

Once again, the sets of partitions $D(k,l)$ must be stable under certain categorical operations, coming from the axioms in [24]. The corresponding algebraic structure, axiomatized in [6], is called "full category of partitions".

We already know that the easy orthogonal quantum groups include the six easy groups in Theorem 1.2, and the six free quantum groups in Theorem 1.4. In [6], two more canonical examples of easy quantum groups were found. These extra two examples are the quantum groups O_n^*, H_n^*, obtained as "half-liberations" of O_n, H_n. The idea is as follows: instead of removing the commutativity relations of type $ab = ba$ from the standard presentation of $C(G)$, which would produce the algebra $C(G^+)$, we replace these commutativity relations by the weaker relations $abc = cba$, which produce by definition the algebra $C(G^*)$. See [6], where also the following theorem is proved.

Theorem 1.6 *The following are easy orthogonal quantum groups:*

(1) O_n^**: the half-liberated orthogonal group.*
(2) H_n^**: the half-liberated hyperoctahydral group.*

These correspond to the following categories of partitions:

(1) E_o*: pairings with each pair connecting an odd and an even number.*
(2) E_h*: partitions with each block having the same number of odd and even legs.*

In addition to the 14 natural examples defined above, there are also two infinite "hyperoctahedral" series $H_n^{(s)}$ and $H_n^{[s]}$. These are introduced in [4], where we give also some partial classification results for easy quantum groups, with the conjectural conclusion that the easy quantum groups consists of the 14 natural examples, and a multi-parameter "hyperoctahedral" series unifying $H_n^{(s)}$ and $H_n^{[s]}$. In the present paper we will mainly consider the natural easy quantum groups. Since the modified permutation and bistochastic groups S_n', B_n' and their free versions $S_n'^+$, $B_n'^+$ are somewhat trivial modifications of their "unprimed" versions, we will not consider them any further, and thus restrict our attention to the easy groups O_n, S_n, H_n, B_n, the free quantum groups O_n^+, S_n^+, H_n^+, B_n^+, and the half-liberated quantum groups O_n^*, H_n^*. Since it turns out that also the series $H_n^{(s)}$ (which includes H_n and H_n^* for $s = 2$ and $s = \infty$, respectively) can be treated by the same methods as H_n and H_n^*, we will also include this series in our considerations. The series $H_n^{[s]}$, on the other side, is quite elusive at the moment, and it seems that one needs new tools to address them. We plan to return to this question after completing the full classification of all easy quantum groups.

Let us finally describe also the quantum groups $H_n^{(s)}$ in terms of their category of partitions. For more details, as well as the proof of the following theorem, see [4].

Theorem 1.7 *For $s \in \{2, 3, 4, \ldots, \infty\}$, $H_n^{(s)}$ is an easy quantum group, and its associated category E_h^s is that of the "s-balanced" partitions, i.e., partitions satisfying the following conditions:*

(1) *The total number of legs is even.*
(2) *In each block, the number of odd legs equals the number of even legs, modulo s. (For $s = \infty$, this means that the number of odd legs equals the number of even legs.)*

2 Moments of powers

In this section we discuss the computation of the asymptotic joint distribution of the variables $\mathrm{Tr}(u^k)$, generalizing the fundamental character $\chi = \mathrm{Tr}(u)$.

Let us first recall some general results from [6]. Let G be an easy orthogonal quantum group, and denote by $D_k \subset P(0, k)$ the corresponding sets of diagrams, having no upper points. We define the Gram matrix to be $G_{kn}(p, q) = n^{|p \vee q|}$, where $|p|$ denotes the number of blocks of the partition p. The Weingarten matrix is by definition its inverse, $W_{kn} = G_{kn}^{-1}$. In order for this inverse to exist, n has to be big enough, and the assumption $n \geq k$ is sufficient. See [6].

We use the notation for integrals from Sect. 0 above.

Theorem 2.1 *The Haar integration over G is given by*

$$\int_G u_{i_1 j_1} \ldots u_{i_k j_k} \, du = \sum_{\substack{p,q \in D_k \\ p \leq \ker \mathbf{i} \\ q \leq \ker \mathbf{j}}} W_{kn}(p, q).$$

Proof This is proved in [6], the idea being that the integrals on the left, taken altogether, form the orthogonal projection on $\mathrm{Fix}(u^{\otimes k}) = \mathrm{span}(D_k)$. □

The above formula can be used for computing the asymptotic moments, cumulants and spectral densities of the truncated characters $\chi_t = \sum_{i=1}^{[tn]} u_{ii}$. Without getting into details, let us just mention that the laws of truncated characters are given as follows (see [4–6] for notions and proofs):

(1) For O_n, S_n, H_n, B_n we get the Gaussian, Poisson, Bessel and shifted Gaussian laws, which form convolution semigroups.
(2) For $O_n^+, S_n^+, H_n^+, B_n^+$ we get the semicircular, free Poisson, free Bessel and shifted semicircular laws, which form free convolution semigroups.
(3) For $S_n', H_n', S_n'^+, H_n'^+$ we get the symmetrized versions of the corresponding laws in (1,2), which do not form convolution or free convolution semigroups, because the canonical copy of $\mathbb{Z}_2$ gives rise to a correlation.
(4) For O_n^*, H_n^* we get squeezed versions of the complex Gaussian and Bessel measure, which form "half-independent" convolution semigroups.

We turn now to our main problem: the computation of the asymptotic laws of powers $\mathrm{Tr}(u^k)$ with $k \in \mathbb{N}$, generalizing the usual characters $\chi = \mathrm{Tr}(u)$. In the classical case these laws, computed by Diaconis and Shahshahani in [14], can be of course understood in terms of the asymptotic behavior of the eigenvalues of the random matrices $u \in G$.

As in [14], we will be actually interested in the more general problem consisting in computing the joint asymptotic law of the variables $\mathrm{Tr}(u^k)$, with $k \in \mathbb{N}$ varying. In order to deal with these joint laws, it is convenient to use the following definition.

Definition 2.2 Associated to $k_1, \ldots, k_s \in \mathbb{N}$ is the *trace permutation* $\gamma \in S_k$, with $k = \Sigma k_i$, having as cycles $(1, \ldots, k_1), (k_1+1, \ldots, k_1+k_2), \ldots, (k-k_s+1, \ldots, k)$.

Our first general result concerns the joint moments of the variables $\mathrm{Tr}(u^k)$, and is valid for any easy quantum groups.

We denote by $\gamma(q)$ the partition given by $i \sim_q j$ iff $\gamma(i) \sim_{\gamma(q)} \gamma(j)$.

Theorem 2.3 *Let G be an easy quantum group. Consider $s \in \mathbb{N}$, $k_1, \ldots, k_s \in \mathbb{N}$, $k := \sum_{i=1}^{s} k_i$, and denote by $\gamma \in S_k$ the trace permutation associated to $k_1, \ldots, k_s$. Then we have, for any n such that G_{nk} is invertible,*

$$\int_G \mathrm{Tr}(u^{k_1}) \cdots \mathrm{Tr}(u^{k_s})\, du = \#\{p \in D_k | p = \gamma(p)\} + O(1/n). \tag{1}$$

If G is a classical easy group, then (1) is exact, without any lower order corrections in n.

Proof We denote by I the integral to be computed. According to the definition of γ, we have the following formula:

$$\begin{aligned} I &= \int_G \mathrm{Tr}(u^{k_1}) \cdots \mathrm{Tr}(u^{k_s})\, du \\ &= \sum_{i_1 \cdots i_k} \int_G \left(u_{i_1 i_2} \cdots u_{i_{k_1} i_1}\right) \cdots \left(u_{i_{k-k_s+1} i_{k-k_s+2}} \cdots u_{i_k i_{k-k_s+1}}\right) \\ &= \sum_{i_1 \cdots i_k} \int_G u_{i_1 i_{\gamma(1)}} \cdots u_{i_k i_{\gamma(k)}}. \end{aligned}$$

We use now the Weingarten formula from Theorem 2.1. We get:

$$\begin{aligned} I &= \sum_{i_1 \ldots i_k = 1}^{n} \sum_{\substack{p,q \in D_k \\ p \leq \ker \mathbf{i}, q \leq \ker \mathbf{i} \circ \gamma}} W_{kn}(p, q) \\ &= \sum_{i_1 \ldots i_k = 1}^{n} \sum_{\substack{p,q \in D_k \\ p \leq \ker \mathbf{i}, \gamma(q) \leq \ker \mathbf{i}}} W_{kn}(p, q) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{p,q\in D_k} n^{|p\vee\gamma(q)|} W_{kn}(p,q)\\
&= \sum_{p,q\in D_k} n^{|p\vee\gamma(q)|} n^{|p\vee q|-|p|-|q|}(1+O(1/n)).
\end{aligned}
$$

The leading order of $n^{|p\vee\gamma(q)|+|p\vee q|-|p|-|q|}$ is n^0, which is achieved if and only if $q \geq p$ and $p \geq \gamma(q)$, or equivalently $p = q = \gamma(q)$. This gives the formula (1).

In the classical case, instead of using the approximation for $W_{nk}(p,q)$, we can write $n^{|p\vee\gamma(q)|}$ as $G_{nk}(\gamma(q), p)$. (Note that this only makes sense if we know that $\gamma(q)$ is also an element in D_k; and this is only the case for the classical partition lattices.) Then one can continue as follows:

$$
I = \sum_{p,q\in D_k} G_{nk}(\gamma(q), p) W_{kn}(p,q) = \sum_{q\in D_k} \delta(\gamma(q), q) = \#\{q \in D_k | q = \gamma(p)\}.
$$

□

We discuss now the computation of the asymptotic joint $*$-distribution of the variables $\mathrm{Tr}(u^k)$. Observe that this is of relevance only in the non-classical context, where the variables $\mathrm{Tr}(u^k)$ are in general (for $k \geq 3$) not self-adjoint.

If c is a cycle we use the notation $c^1 = c$, and c^*= cycle opposite to c.

Definition 2.4 Associated to any $k_1, \ldots, k_s \in \mathbb{N}$ and any $e_1, \ldots, e_s \in \{1, *\}$ is the *trace permutation* $\gamma \in S_k$, with $k = \Sigma k_i$, having as cycles $(1, \ldots, k_1)^{e_1}$, $(k_1 + 1, \ldots, k_1 + k_2)^{e_2}, \ldots, (k - k_s + 1, \ldots, k)^{e_s}$.

Observe that with $e_1, \ldots, e_s = 1$ we recover the permutation in Definition 2.2. With this notation, we have the following slight generalization of Theorem 2.3.

Theorem 2.5 *Let G be an easy quantum group. Consider $s \in \mathbb{N}$, $k_1, \ldots, k_s \in \mathbb{N}$, $e_1, \ldots, e_s \in \{1, *\}$, $k := \sum_{i=1}^{s} k_i$, and denote by $\gamma \in S_k$ the trace permutation associated to $k_1, \ldots, k_s$ and $e_1, \ldots, e_s$. Then we have, for any n such that G_{nk} is invertible,*

$$
\int_G \mathrm{Tr}(u^{k_1})^{e_1} \cdots \mathrm{Tr}(u^{k_s})^{e_s}\, du = \#\{p \in D_k | p = \gamma(p)\} + O(1/n).
$$

If G is a classical easy group, then this formula is valid without any lower order corrections.

Proof This is similar to the proof of Theorem 2.3. □

3 Cumulants of powers

The formula for the moments of the variables $\mathrm{Tr}(u^k)$ contains in principle all information about their distribution. However, in order to specify this more explicitly, in particular, to recognize independence/freeness between those (or suitable modifications), it is more advantageous to look on the cumulants of these variables. For this we restrict, in this section, to the classical and free case. We will calculate the classical cumulants (denoted by c_r) for the classical easy groups and the free cumulants (denoted by κ_r) for the free easy groups. Actually we will restrict to the cases

(1) Classical groups: O_n, S_n, H_n, B_n.
(2) Free quantum groups: $O_n^+, S_n^+, H_n^+, B_n^+$.

The reason for this is that we need some kind of multiplicativity for the underlying partition lattice in our calculations, as specified in the next proposition.

Proposition 3.1 *Assume that G is one of the easy quantum groups* O_n, S_n, H_n, B_n *or* $O_n^+, S_n^+, H_n^+, B_n^+$ *and denote by* D_k *the corresponding category of partitions. Then we have the following property: let* $p \in D_k$ *be a partition, and let* $q \in P_l$ *with* $l \leq k$ *be a partition arising from p by deleting some blocks. Then* $q \in D_l$.

Proof This follows from the explicit description of the full categories of partitions for the various easy quantum groups, given in Sect. 1. □

Theorem 3.2 1) *Let G be one of the easy classical groups* O_n, S_n, H_n, B_n *with* D_k *as corresponding category of partitions. Consider* $r \in \mathbb{N}$, $k_1, \ldots, k_r \in \mathbb{N}$, $k := \sum_{i=1}^r k_i$ *and* $e_1, \ldots, e_r \in \{1, *\}$, *and denote by* $\gamma \in S_k$ *the trace permutation associated to* $k_1, \ldots, k_r$ *and* $e_1, \ldots, e_r$. *Then we have, for any n such that* G_{nk} *is invertible, the classical cumulants*

$$c_r(\mathrm{Tr}(u^{k_1})^{e_1}, \ldots, \mathrm{Tr}(u^{k_r})^{e_r}) = \#\{p \in D_k | p \vee \gamma = 1_k,\ p = \gamma(p)\}.$$

2) *Let G be one of the easy free groups* $O_n^+, S_n^+, H_n^+, B_n^+$ *with* D_k *as corresponding category of non-crossing partitions. Consider* $r \in \mathbb{N}$, $k_1, \ldots, k_r \in \mathbb{N}$, $k := \sum_{i=1}^r k_i$ *and* $e_1, \ldots, e_r \in \{1, *\}$, *and denote by* $\gamma \in S_k$ *the trace permutation associated to* $k_1, \ldots, k_r$ *and* $e_1, \ldots, e_r$. *Then we have, for any n such that* G_{nk} *is invertible, the free cumulants*

$$\kappa_r(\mathrm{Tr}(u^{k_1})^{e_1}, \ldots, \mathrm{Tr}(u^{k_r})^{e_r}) = \#\{p \in D_k | p \vee \gamma = 1_k,\ p = \gamma(p)\} + O(1/n).$$

Proof 1) Let us denote by c_r the considered cumulant. For $\sigma \in P_r$, we write

$$D_\sigma := \{p \in P_k \mid p|_v \in D_{|v|}\, \forall v \in \sigma^\gamma\}$$

for those partitions p in P_k such that the restriction of p to a block of σ^γ is an element in the corresponding set $D_{|v|}$. Clearly, one has that a $p \in D_\sigma$ is in D_k and must satisfy $p \leq \sigma^\gamma$. Our exclusion of the primed classical groups guarantees, by Proposition 3.1, that this is actually a characterization, i.e., we have

$$D_\sigma = \{p \in D_k \mid p \leq \sigma^\gamma\}. \tag{2}$$

Then, by the definition of the classical cumulants via Möbius inversion of the moments, we get from (1):

$$\begin{aligned}
c_r &= \sum_{\sigma \in P_r} \mu(\sigma, 1_r) \cdot \#\{p \in D_\sigma : p = \gamma(p)\} \\
&= \sum_{\sigma \in P_r} \mu(\sigma, 1_r) \cdot \#\{p \in D_k : p \leq \sigma^\gamma, p = \gamma(p)\} \\
&= \sum_{\sigma \in P_r} \mu(\sigma, 1_r) \sum_{\substack{p \in D_k \\ p \leq \sigma^\gamma, p = \gamma(p)}} 1.
\end{aligned}$$

In order to exchange the two summations, we first have to replace the summation over $\sigma \in P_r$ by a summation over $\tau := \sigma^\gamma \in P_k$. Note that the condition on the latter is exactly $\tau \geq \gamma$ and that we have $\mu(\sigma, 1_r) = \mu(\sigma^\gamma, 1_k)$. Thus:

$$c_r = \sum_{\substack{\tau \in P_k \\ \tau \geq \gamma}} \mu(\tau, 1_k) \sum_{\substack{p \in D_k \\ p \leq \tau, p = \gamma(p)}} 1 = \sum_{\substack{p \in D_k \\ p = \gamma(p)}} \sum_{\substack{\tau \in P_k \\ p \vee \gamma \leq \tau}} \mu(\tau, 1_k).$$

The definition of the Möbius function [see (10.11) in [17]] gives for the second summation

$$\sum_{\substack{\tau \in P_k \\ p \vee \gamma \leq \tau}} \mu(\tau, 1_k) = \begin{cases} 1, & p \vee \gamma = 1_k \\ 0, & \text{otherwise} \end{cases}$$

and the assertion follows.

2) In the free case, the proof runs in the same way, by using free cumulants and the corresponding Möbius function on non-crossing partitions. Note that we have the analogue of (2) in this case only for non-crossing σ. □

4 The orthogonal case

In this section we discuss what Theorem 3.2 implies for the asymptotic distribution of traces in the case of the orthogonal quantum groups. For the classical orthogonal group we will in this way recover the theorem of Diaconis and Shahshahani [14].

Theorem 4.1 *The variables* $u_k = \lim_{n \to \infty} \mathrm{Tr}(u^k)$ *are as follows:*

(1) *For* O_n*, the* u_k *are real Gaussian variables, with variance* k *and mean* 0 *or* 1*, depending on whether* k *is odd or even. The* u_k*'s are independent.*

(2) *For* O_n^+*, at* $k = 1, 2$ *we get semicircular variables of variance* 1 *and mean* 0 *for* u_1 *and mean* 1 *for* u_2*, and at* $k \geq 3$ *we get circular variables of mean* 0 *and covariance* 1*. The* u_k*'s are* $*$*-free.*

Proof (1) In this case D_k consists of all pairings of k elements. We have to count all pairings p with the properties that $p \vee \gamma = 1_k$ and $p = \gamma(p)$.

Note that if p connects two different cycles of γ, say c_i and c_j, then the property $p = \gamma(p)$ implies that each element from c_i must be paired with an element from c_j; thus those cycles cannot be connected to other cycles and they must contain the same number of elements. This means that for $s \geq 3$ there are no p with the required properties. Thus all cumulants of order 3 and higher vanish asymptotically and all traces are asymptotically Gaussian.

Since in the case $s = 2$ we only have permissible pairings if the two cycles have the same number of elements, i.e., both powers of u are the same, we also see that the covariance between traces of different powers vanishes and thus different powers are asymptotically independent. The variance of u_k is given by the number of matchings between $\{1, \dots, k\}$ and $\{k+1, \dots, 2k\}$ which are invariant under rotations. Since such a matching is determined by the partner of the first element 1, for which we have k possibilities, the variance of u_k is k. For the mean, if k is odd there is clearly no pairing at all, and if $k = 2p$ is even then the only pairing of $\{1, \dots, 2p\}$ which is invariant under rotations is $(1, p+1), (2, p+2), \dots, (p, 2p)$. Thus the mean of u_k is zero if k is odd and 1 if k is even.

(2) In the quantum case D_k consists of non-crossing pairings. We can essentially repeat the arguments from above but have to take care that only non-crossing pairings are counted. We also have to realize that for $k \geq 3$, the u_k are not selfadjoint any longer, thus we have to consider also u_k^* in these cases. This means that in our arguments we have to allow cycles which are rotated "backwards" under γ.

By the same reasoning as before we see that free cumulants of order three and higher vanish. Thus we get a (semi)circular family. The pairing which gave mean 1 in the classical case is only in the case $k = 2$ a non-crossing one, thus the mean of u_2 is 1, all other means are zero. For the variances, one has again that different powers allow no pairings at all and are asymptotically $*$-free. For the matchings between $\{1, \dots, k\}$ and $\{k+1, \dots, 2k\}$ one has to observe that there is only one non-crossing possibility, namely $(1, 2k), (2, 2k-1), \dots, (k, k+1)$ and this satisfies $p = \gamma(p)$ only if γ rotates both cycles in different directions.

For $k = 1$ and $k = 2$ there is no difference between both directions, but for $k \geq 3$ this implies that we get only a non-vanishing covariance between u_k and u_k^* (with value 1). This shows that u_1 and u_2 are semicircular, whereas the higher u_k are circular. □

5 The bistochastic case

In the bistochastic case we have the following version of Theorem 4.1.

Theorem 5.1 *The variables* $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ *are as follows:*

(1) *For B_n, the u_k are real Gaussian variables, with variance k and mean 1 or 2, depending on whether k is odd or even. The u_k's are independent.*

(2) *For B_n^+, at $k = 1, 2$ we get semicircular variables of variance 1 and mean 1 for u_1 and mean 2 for u_2, and at $k \geq 3$ we get circular variables of mean 1 and covariance 1. The u_k's are $*$-free.*

Proof When replacing O_n and O_n^+ by B_n and B_n^+, we also have to allow singletons in p. Note however that the condition $p = \gamma(p)$ implies that if p has a singleton, then the corresponding cycle of γ must consist only of singletons of p , which means in particular that this cycle cannot be connected via p to other cycles. Thus singletons are not allowed for permissible p, unless we only have one cycle of γ, i.e., we are looking on the mean. In this case there is one additional p, consisting just of singletons, which makes a contribution. So the results for B_n and B_n^+ are the same as those for O_n and O_n^+, respectively, with the only exception that all means are shifted by 1. □

6 The symmetric case

Let us now consider the case of the symmetric groups. In this case we have to consider all partitions instead of just pairings and the arguments are getting a bit more involved. Nevertheless one can treat these cases still in a quite straightforward way. For the classical permutation groups, one recovers in this way the corresponding result of Diaconis and Shahshahani [14].

Proposition 6.1 *The cumulants of* $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ *are as follows:*

(1) *For* S_n*, the classical cumulants are given by:*

$$c_r(u_{k_1}, \ldots, u_{k_r}) = \sum_{q|k_i \forall i=1,\ldots,r} q^{r-1}.$$

(2) *For* S_n^+*, the free cumulants are given by:*

$$c_r(u_{k_1}^{e_1}, \ldots, u_{k_r}^{e_r}) = \begin{cases} 2, & r = 1,\ k_1 \geq 2 \\ 2, & r = 2,\ k_1 = k_2,\ e_1 = e_2^* \\ 2, & r = 2,\ k_1 = k_2 = 2 \\ 1, & \textit{otherwise.} \end{cases}$$

Proof (1) Now D_k consists of all partitions. We have to count partitions p which have the properties that $p \vee \gamma = 1_k$ and $p = \gamma(p)$.

Consider the block of a partition p which connects different cycles of γ. Consider the restriction of this block of p to one cycle. Let k be the number of elements in this cycle and t be the number of the points in the restriction. Then the orbit of those t points under γ must give a partition of that cycle; this means that t is a divisor of k and that the t points are equally spaced. The same must be true for all cycles of γ which are connected via p, and the ratio between t and k is the same for all those cycles. This means that if one block of p connects some cycles then the orbit under γ of this block connects exactly those cycles and exhausts all points of those cycles. So if we want to connect all cycles of γ then this can only happen in the way that we have one (and thus all) block of p intersecting each of the cycles of γ. To be more precise, let us consider $c_r(u_{k_1}, \ldots, u_{k_r})$. We have then to look for a common divisor q of all $k_1, \ldots, k_r$; a contributing p is then one the blocks of which are of the following form: k_1/q points in the first cycle (equally spaced), $\ldots k_r/q$ points in the last cycle (equally spaced). We can specify this by saying to which points in the other cycles the first point in the first cycle is connected. There are q^{r-1} possibilities for such choices. Thus:

$$c_r(u_{k_1}, \ldots, u_{k_r}) = \sum_{q|k_i \forall i=1,\ldots,r} q^{r-1}.$$

(2) In the quantum permutation case we have to consider non-crossing partitions instead of all partitions. Most of the contributing partitions from the classical case are crossing, so do not count for the quantum case. Actually, whenever a restriction of a block to one cycle has two or more elements then the corresponding partition is crossing, unless the restriction exhausts the whole group. This is the case $q = 1$ from the considerations above (corresponding to the partition which has only one block), giving a contribution 1 to each cumulant $c_r(u_{k_1}, \ldots, u_{k_r})$. For cumulants of order 3 or higher there are no other contributions. For cumulants of second order one might also have contributions coming from pairings (where each restriction of a block to a cycle has one element). This is the same problem as in the O_n^+ case; i.e., we only get an additional contribution for the second order cumulants $c_2(u_k, u_k^*)$. For first order cumulants, singletons can also appear and make an additional contribution. Taking this all together gives the formula in the statement. □

In contrast to the two previous cases, the different traces are now not independent/free any more. Actually, one knows in the classical case that some more fundamental random variables, counting the number of different cycles, are independent. We can recover this result, and its free analogue, from Proposition 6.1 in a straightforward way.

Theorem 6.2 *The variables $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ are as follows:*

(1) *For S_n we have a decomposition of type*

$$u_k = \sum_{l|k} l C_l$$

with the variables C_k being Poisson of parameter $1/k$, and independent.

(2) *For S_n^+ we have a decomposition of the type*

$$u_1 = C_1, \quad u_k = C_1 + C_k \quad (k \geq 2)$$

where the variables C_l are $*$-free; C_1 is free Poisson, whereas C_2 is semicircular and C_k, for $k \geq 3$, are circular.

Let us first note that the first statement is the result of Diaconis and Shahshahani in [14]. Indeed, the matrix coefficients for S_n are given by $u_{ij} = \chi(\sigma|\sigma(j) = i)$, and it follows that the variable C_l defined by the decomposition of u_k in the statement is nothing but the number of l-cycles. For a direct proof for the fact that these variables C_k are indeed independent and Poisson of parameter $1/k$, see [14]. In what follows we present a global proof for (1) and (2), by using Proposition 6.1.

Proof (1) Let C_k be the number of cycles of length k. Instead of writing this in terms of traces of powers of u, it clearer to do it the other way round. We have $u_k = \sum_{l|k} lC_l$. We are claiming now that the C_k are independent and each is a Poisson variable of parameter $1/k$, i.e., that $c_r(C_{l_1}, \ldots, C_{l_r})$ is zero unless all the l_i's are the same, say $= l$, in which case it is $1/l$ (independent of r). This is compatible with the cumulants for the u_k, according to:

$$c_r(u_{k_1}, \ldots, u_{k_r}) = \sum_{l_1|k_1} \cdots \sum_{l_r|k_r} l_1 \cdots l_r c_r(C_{l_1}, \ldots, C_{l_r}) = \sum_{l|k_i \forall i} l^r \frac{1}{l}.$$

Since the C_k's are uniquely determined by the u_k's, via some kind of Möbius inversion, this shows that also the other way round the formula for the cumulants of the u_k's implies the above stated formula for the cumulants of the C_k's; i.e., we get the result that the C_k are independent and C_k is Poisson with parameter $1/k$.

(2) This follows easily from Proposition 6.1. □

Remark 6.3 1) In the classical case the random variable C_l can be defined by

$$C_l = \frac{1}{l} \sum_{\substack{i_1,\ldots,i_l \\ \text{distinct}}} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}. \tag{3}$$

Note that we divide by l because each term appears actually l-times, in cyclically permuted versions (which are all the same because our variables commute).

Note that, by using commutativity and the monomial condition, in general the expression $u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_k i_1}$ has to be zero unless the indices $(i_1, \ldots, i_k)$ are of the form $(i_1, \ldots, i_l, i_1, \ldots, i_l, \ldots)$ where l divides k and $i_1, \ldots, i_l$ are distinct. This yields then the relation

$$\mathrm{Tr}(u^k) = \sum_{i_1,\ldots,i_l=1}^{n} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1} = \sum_{l|k} \sum_{\substack{i_1,\ldots,i_l \\ \text{distinct}}} (u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1})^{k/l} = \sum_{l|k} lC_l,$$

which we used before to define the C_l. [Note that each $u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}$ is an idempotent, thus the power k/l does not matter.] This explicit form (3) of the C_l in terms of the u_{ij} can be used to give a direct proof, by using the Weingarten formula, of the fact that the C_l are independent and Poisson. We will not present this calculation here, but will come back to this approach in the case of the hyperoctahedral group in the next section.

2) In the free case we define the "cycle" C_l by requiring neighboring indices to be different,

$$C_l = \sum_{i_1 \neq i_2 \neq \cdots \neq i_l \neq i_1} u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}. \tag{4}$$

Note that if two adjacent indices are the same in $u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}$ then, because of the relation $u_{ij} u_{ik} = 0$ for $j \neq k$, all must be the same or the term vanishes. For the case where all indices are the same we have

$$\sum_i u_{ii}u_{ii}\cdots u_{ii} = \sum_i u_{ii} = C_1.$$

This gives then the relation

$$\mathrm{Tr}(u^k) = C_k + C_1.$$

Again, the C_l are uniquely determined by the $\mathrm{Tr}(u^k)$ and thus our calculations also show that the C_l defined by (4) are $*$-free and have the distributions as stated.

7 The hyperoctahedral case

The methods in the previous section apply, modulo a grain of salt, as well to the hyperoctahedral case.

Proposition 7.1 *The cumulants of $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ are as follows:*

(1) *For H_n, the classical cumulants are given by:*

$$c_r(u_{k_1}, \ldots, u_{k_r}) = \sum_{\substack{q|k_i \forall i=1,\ldots,r \\ 2|(\Sigma k_i/q)}} q^{r-1}.$$

(2) *For H_n^+, the free cumulants are given by:*

$$c_r(u_{k_1}^{e_1}, \ldots, u_{k_r}^{e_r}) = 2, \quad \text{if } r = 2, k_1 = k_2, e_1 = e_2^* \text{ or if } r = 2, k_1 = k_2 = 2$$

and otherwise by

$$c_r(u_{k_1}^{e_1}, \ldots, u_{k_r}^{e_r}) = \begin{cases} 1, & \sum_l k_l \text{ even} \\ 0, & \sum_l k_l \text{ odd.} \end{cases}$$

Proof This follows similarly as the proof of Proposition 6.1, by taking into account that we have to restrict attention to the partitions having even blocks only. □

Theorem 7.2 *The variables $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ are as follows:*

(1) *For H_n we have a decomposition of type*

$$u_k = \sum_{l|k} l[C_l^+ + (-1)^{k/l} C_l^-]$$

with the variables C_l^+ and C_l^- being Poisson of parameter $1/2l$, and all C_l^+, C_l^- ($l \in \mathbb{N}$) being independent.

(2) *For H_n^+ we have a decomposition of type*

$$u_1 = C_1^+ - C_1^-, \quad u_k = C_1^+ + (-1)^k C_1^- + C_k \quad (k \geq 2)$$

where C_1^+, C_1^- and C_k ($k \geq 2$) are $*$-free and C_1^+, C_1^- are free Poisson elements of parameter $1/2$, C_2 is a semicircular element, and C_k ($k \geq 3$) are circular elements.

Proof (1) This follows in the same way as in the proof of Theorem 6.2.

(2) This follows by direct computation. □

In the classical case the random variables C_l^+ and C_l^- should count the number of positive cycles of length l and the number of negative cycles of length l, respectively, and should be given by $C_l^+ = Z_l^+$ and $C_l^- = Z_l^-$ with

$$Z_l^+ = \frac{1}{l} \sum_{\substack{i_1,\ldots,i_l \\ \text{distinct}}} 1_{\{1\}}(u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}) \tag{5}$$

and

$$Z_l^- = -\frac{1}{l} \sum_{\substack{i_1,\ldots,i_l \\ \text{distinct}}} 1_{\{-1\}}(u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}). \tag{6}$$

Note that $u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}$ is either $-1, 0, 1$. $1_{\{1\}}$ denotes the characteristic function on 1 and $1_{\{-1\}}$ the characteristic function on -1. As in Remark 6.3 it follows that one has the decomposition

$$\mathrm{Tr}(u^k) = \sum_{l|k} l \left[Z_l^+ + (-1)^{k/l} Z_l^- \right]. \tag{7}$$

Again one expects that the random variables defined by (5) and (6) are all independent and are Poisson, but because now the Z_l^+, Z_l^- are not uniquely determined by the relations (7), we cannot argue that the C_l^+ and C_l^- showing up in the decomposition in Theorem 7.2 are the same as Z_l^+ and Z_l^- defined by (5) and (6). In order to see that this is actually the case, we will now, in the following, calculate the cumulants of the random variables defined by (5) and (6).

Note first that we can replace the characteristic functions in the following way:

$$2Z_l^+ = \frac{1}{l} \sum_{\substack{i_1,\ldots,i_l \\ \text{distinct}}} (u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1})^2 + u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1} \tag{8}$$

and

$$2Z_l^- = \frac{1}{l} \sum_{\substack{i_1,\ldots,i_l \\ \text{distinct}}} (u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1})^2 - u_{i_1 i_2} u_{i_2 i_3} \cdots u_{i_l i_1}. \tag{9}$$

Furthermore, we have

$$\sum_{\substack{i_1,\dots,i_l\\ \text{distinct}}} u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1} = \sum_{\ker \mathbf{i}=1_l} u_{i_1i_2}\cdots u_{i_li_1}$$

and

$$\sum_{\substack{i_1,\dots,i_l\\ \text{distinct}}} (u_{i_1i_2}u_{i_2i_3}\cdots u_{i_li_1})^2 = \sum_{\ker \mathbf{i}=\tau_l^{2l}} u_{i_1i_2}\cdots u_{i_{2l}i_1},$$

where $\tau_l^{2l} \in P(2l)$ is the pairing $\{(1, l+1), (2, l+2), \dots, (l, 2l)\}$.

So what we need are cumulants for general variables of the form

$$Z(\sigma) := \sum_{\ker \mathbf{i}=\sigma} u_{i_1i_2}\cdots u_{i_li_1},$$

for arbitrary $l \in \mathbb{N}$ and $\sigma \in P_l$. Note that $Z(\sigma)$ can only be different from zero if σ is of the form $\sigma = \tau_l^k$ for $k, l \in \mathbb{N}$ with $l|k$, where

$$\tau_l^k = \{(1, l+1, 2l+1, \dots, k-l+1), (2, l+2, 2l+2, \dots, k-l+2), \dots, (l, 2l, \dots, k)\}.$$

For $k = l$, we have $\tau_l^l = 1_l$.

Theorem 7.3 *For all $s \in \mathbb{N}$, $k_1, \dots, k_s \in \mathbb{N}$, $\sigma_1 \in P_{k_1}, \dots, \sigma_s \in P_{k_s}$ we have*

$$c_r[Z(\sigma_1), \dots, Z(\sigma_r)] = \#\{q \in D_k \mid q = \gamma(q), q \vee \gamma = 1_k,\\ q \text{ restricted to the } i\text{th cycle of } \gamma \text{ is } \sigma_i (i = 1, \dots, r)\},$$

where $k = \sum_{i=1}^r k_i$ and γ is the trace permutation associated to $k_1, \dots, k_r$

Note that also the right hand side of the equation is, by the condition $q = \gamma(q)$, zero unless all σ are of the form τ_l^k.

Proof Let us first calculate the corresponding moment. For this we note that one has

$$\sum_{\ker \mathbf{i}\geq\pi} u_{i_1i_2}\cdots u_{i_li_1} = \sum_{\substack{\sigma\in P_l\\ \sigma\geq\pi}} Z(\sigma),$$

and thus, by Möbius inversion on P_l

$$Z(\sigma) = \sum_{\substack{\pi\in P_l\\ \pi\geq\sigma}} \mu(\sigma, \pi) \sum_{\ker \mathbf{i}\geq\pi} u_{i_1i_2}\cdots u_{i_li_1}.$$

With this we can calculate

$$\begin{aligned}
\int_{H_n} Z(\sigma_1)\cdots Z(\sigma_r)du &= \sum_{\pi_1,\dots,\pi_r} \mu(\sigma_1,\pi_1)\cdots\mu(\sigma_r,\pi_r) \sum_{\pi_1\circ\cdots\circ\pi_r\le \ker \mathbf{i}} \int_{H_n} u_{i_1 i_{\gamma(1)}}\cdots u_{i_k i_{\gamma(k)}} \\
&= \sum_{\pi_1,\dots,\pi_r} \mu(\sigma_1,\pi_1)\cdots\mu(\sigma_r,\pi_r) \sum_{\pi_1\circ\cdots\circ\pi_r\le \ker \mathbf{i}} \sum_{\substack{q,p\in D_k\\ p\le \ker \mathbf{i},\gamma(q)\le\ker\mathbf{i}}} W_{kn}(p,q) \\
&= \sum_{\pi_1,\dots,\pi_r} \mu(\sigma_1,\pi_1)\cdots\mu(\sigma_r,\pi_r) \sum_{q,p\in D_k} G_{kn}(\gamma(q)\vee\pi_1\circ\cdots\circ\pi_r,p)W_{kn}(p,q) \\
&= \sum_{\pi_1,\dots,\pi_r} \mu(\sigma_1,\pi_1)\cdots\mu(\sigma_r,\pi_r) \sum_{q\in D_k} \delta(\gamma(q)\vee\pi_1\circ\cdots\circ\pi_r,q).
\end{aligned}$$

In the third line, it looks as if we might have a problem because $\pi_1\circ\cdots\circ\pi_r$ is in P_k, but not necessarily in D_k. However, our category D_k has the nice property that, for $\pi\in P_k$ and $\sigma\in D_k$, $\pi\ge\sigma$ implies that also $\pi\in D_k$. Thus in particular, $\pi\vee\sigma\in D_k$ for any $\pi\in P_k$ and $\sigma\in D_k$, and we have in our case that always $\gamma(q)\vee\pi_1\circ\cdots\circ\pi_r\in D_k$. Now note further that $\gamma(q)\vee\pi_1\circ\cdots\circ\pi_r=q$ is actually equivalent to $\gamma(q)=q$ and $\pi_1\circ\cdots\circ\pi_r\le q$. One direction is clear, for the other one has to observe that $\gamma(q)\le q$ implies $\gamma(q)=q$. With this, we get finally

$$\begin{aligned}
&\int_{H_n} Z(\sigma_1)\cdots Z(\sigma_r)du \\
&\quad = \#\{q\in D_k \mid q=\gamma(q), q \text{ restricted to the } i\text{th cycle of } \gamma \text{ is } \sigma_i\,(i=1,\dots,r)\}.
\end{aligned} \tag{10}$$

From this, we get for the cumulants

$$\begin{aligned}
&c_r[Z(\sigma_1),\dots,Z(\sigma_r)] \\
&\quad = \sum_{\pi\in P_r} \mu(\pi,1_r)\cdot\#\{q\in D_k \mid q=\gamma(q), q\vee\gamma\le\pi^\gamma, \\
&\qquad q \text{ restricted to the } i\text{th cycle of } \gamma \text{ is } \sigma_i\,(i=1,\dots,r)\}.
\end{aligned}$$

The result follows then from Möbius inversion, as in the proof of Theorem 3.2. □

This shows in particular that $c_r[Z(\sigma_1),\dots,Z(\sigma_r)]$ vanishes unless all $\sigma_1,\dots,\sigma_r$ have the same number of blocks. This implies that the sets $\{Z_l^+, Z_l^-\}$ are independent for different l. For fixed l, we have for all $e_1,\dots,e_r\in\mathbb{N}$:

$$c_r[Z(\tau_l^{e_1 l}),\dots,Z(\tau_l^{e_r l})] = \begin{cases} l^{r-1}, & \text{if } \sum_i e_i \text{ is even} \\ 0, & \text{otherwise.}\end{cases}$$

Thus we get in particular

$$\begin{aligned}&c_r[Z(\tau_{l_1}^{2l_1}) \pm Z(1_{l_1}), \dots, Z(\tau_{l_r}^{2l_r}) \pm Z(1_{l_r})]\\ &\quad = \begin{cases} d_{r,l}, & \text{if } l_1 = \cdots = l_r = l \text{ and all signs are either } + \text{ or all are } - \\ 0, & \text{otherwise} \end{cases}\end{aligned}$$

where

$$d_{r,l} = l^{r-1} \sum_{\substack{t=0\\ t \text{ even}}}^{r} \binom{r}{t} = l^{r-1} 2^{r-1}.$$

This shows that also Z_l^+ and Z_l^- are independent, and each of them is Poisson of parameter $1/(2l)$.

Corollary 7.4 *In H_n the random variables Z_l^+, Z_l^- ($l \in \mathbb{N}$), defined by* (5) *and* (6)*, are independent Poisson variables of parameter* $1/(2l)$.

Remark 7.5 In the free case H_n^+, the variables u_{ii} have also spectrum $\{-1, 0, 1\}$ and we can consider a positive/negative decomposition for C_1, i.e.,

$$C_1^+ = \sum_i 1_{\{1\}}(u_{ii})$$

and

$$C_1^- = -\sum_i 1_{\{-1\}}(u_{ii});$$

the other $C_l, l \geq 2$, are just as in the case of S_n^+. Similarly as for H_n, one can show that these variables are the ones showing up in the decomposition for H_n^+ in Theorem 7.2.

8 The half-liberated cases

The half-liberated quantum groups O_n^* and $H_n^{(s)}$ are neither classical nor free groups, so both classical and free cumulants are inadequate tools for getting information on the distribution of their traces. In [5], we introduced half-liberated cumulants to deal with half-independence, but one has to realize that we do not get an analogue of Theorem 3.2 for them, because the underlying "balanced" partition lattices do not share the multiplicativity property from Proposition 3.1. In order to investigate the distribution of traces in the half-liberated cases we will thus have to proceed via another route. The key insight is here that the half-liberated situations are actually "orthogonal" versions of classical unitary groups and that the main computations can be done over these unitary groups instead.

8.1 The half-liberated orthogonal group O_n^*

Let $u = (u_{ij})_{i,j=1}^n$ be the fundamental representation of O_n^*, and let $v = (v_{ij})_{i,j=1}^n$ be the fundamental representation of the unitary group U_n. Then we can "orthogonalize" U_n by considering

$$w_{ij} := \begin{pmatrix} 0 & v_{ij} \\ \overline{v_{ij}} & 0 \end{pmatrix}. \tag{11}$$

Then $w = (w_{ij})_{i,j=1}^n$ is an orthogonal matrix and a simple calculation shows that the w_{ij} half-commute. It is also easy to see [by invoking the Weingarten formula for U_n, see below, Eq. (13)], that under this map the Haar state on O_n^* goes to $\int_{U_n} \otimes \mathrm{tr}_2$. Since the Haar state on O_n^* is faithful [7] , the mapping $u_{ij} \mapsto w_{ij}$ is actually an isomorphism.

So we have

$$\mathrm{Tr}(u^{2k+1}) = \begin{pmatrix} 0 & \mathrm{Tr}\left((v\bar{v})^k v\right) \\ \mathrm{Tr}\left((\bar{v}v)^k \bar{v}\right) & 0 \end{pmatrix}$$

and

$$\mathrm{Tr}(u^{2k}) = \begin{pmatrix} \mathrm{Tr}\left((v\bar{v})^k\right) & 0 \\ 0 & \mathrm{Tr}\left((\bar{v}v)^k\right) \end{pmatrix}.$$

So what we need to understand is the distribution of the variables

$$v_{2k+1} := \lim_{n\to\infty} \mathrm{Tr}\left((v\bar{v})^k v\right), \quad v_{2k} := \lim_{n\to\infty} \mathrm{Tr}\left((v\bar{v})^k\right). \tag{12}$$

Proposition 8.1 *Let $(v_k)_{k\geq 1}$ be as in (12), where $v = (v_{ij})_{i,j=1}^n$ are the coordinates of the classical unitary group U_n. Then we have: the variables $(v_k)_{k\geq 1}$ are independent; for k even, v_k is a real Gaussian with mean 0 or 1, depending on whether $k/2$ is odd or even, and variance equal to $k/2$; for k odd, v_k is a complex Gaussian with mean 0 and variance 1.*

Proof For $\epsilon = (e_1, \ldots, e_k)$ a string of 1's and $*$'s, let us denote by $P_2(\epsilon)$ the pairings in P_k such that each block joins a 1 and a $*$. Then the Weingarten formula for U_n [11] says that with the notation

$$u_{ij}^e := \begin{cases} u_{ij}, & \text{if } e = 1 \\ \overline{u_{ij}}, & \text{if } e = * \end{cases}$$

we have

$$\int_{U_n} u_{i_1 j_1}^{e_1} \cdots u_{i_k j_k}^{e_k} du = \sum_{\substack{p,q \in P_2(\epsilon) \\ p \leq \ker \mathbf{i}, q \leq \ker \mathbf{i}}} W(p,q), \tag{13}$$

where $\epsilon = (e_1, \ldots, e_k)$.

As in the proof of Theorem 2.3 this implies that

$$\int v_{k_1}^{e_1} \cdots v_{k_s}^{e_s} = \#\{p \in P_2(\epsilon) \mid p = \gamma(p)\} + O(1/n). \tag{14}$$

The condition $p = \gamma(p)$ implies that the pairing p cannot join two cycles of different lengths, which shows that such an expectation factorizes according to the cycle lengths, which implies the independence of the v_k. The statements on the distribution of v_k follow also immediately from (14). □

Transferring these results from the v_k to

$$u_{2k+1} = \begin{pmatrix} 0 & v_{2k+1} \\ \overline{v_{2k+1}} & 0 \end{pmatrix}, \quad u_{2k} = \begin{pmatrix} v_{2k} & 0 \\ 0 & \overline{v_{2k}} \end{pmatrix} \tag{15}$$

and noting that the distribution of u_{2k+1} is equal to that of $\sqrt{|v_{2k+1}|^2}$ (which is a symmetrized Rayleigh distribution) yields then the following result. (See [5] for the notion of "half-independence".)

Theorem 8.2 *For O_n^*, the variables $u_k = \lim_{n\to\infty} \mathrm{Tr}(u^k)$ are as follows. The sets $\{u : k \mid k \text{ odd}\}$ and $\{u_k \mid k \text{ even}\}$ are independent; for k even, the u_k are independent real Gaussian of mean 0 or 1, depending on whether $k/2$ is even or odd, and variance $k/2$; for k odd, the v_k are half-independent symmetrized Rayleigh variables with variance 1.*

8.2 The hyperoctahedral series $H_n^{(s)}$

The hyperoctahedral series $H_n^{(s)}$ (for $s = 2, 3, \ldots, \infty$) is determined by the partition lattice of all s-balanced partitions, see Theorem 1.7. This series includes the classical hyperoctrahedral group for $s = 2$, $H_n^{(2)} = H_n$, and the half-liberated hyperoctahedral group for $s = \infty$, $H_n^{(\infty)} = H_n^*$. As for O_n^*, these groups can be considered as orthogonal versions of classical unitary groups. Namely, let $H_n^s = \mathbb{Z}_s \wr S_n$ be the complex reflection group consisting of monomial matrices having the s-roots of unity as nonzero entries. (Note that for $s = 2$, $H_n^{(2)} = H_n^2$.) Then the relation between $H_n^{(s)}$ and H_n^s is the same as the one between O_n^* and U_n, i.e., we can represent the coordinates u_{ij} of $H_n^{(s)}$ by w_{ij} according to (11), where v_{ij} are the coordinates of H_n^s. So again, we can realize the asymptotic traces u_k of $H_n^{(s)}$ in the form (15), where the v_k are now the asymptotic traces in H_n^s according to (12). So our main task will be the determination of the distribution of these v_k.

Actually, we can treat more generally asymptotic traces with arbitrary pattern of the conjugates. So let us consider for an arbitrary string $\epsilon = (e_1, \ldots, e_k)$ of 1 and $*$ the variable

$$v(\epsilon) := \lim_{n\to\infty} \mathrm{Tr}(v^{e_1} \cdots v^{e_k}).$$

For $\epsilon = (e_1, \ldots, e_k)$ a string of 1's and $*$'s, we denote by $P^s(\epsilon)$ the partitions in P_k such that each block joins the same number, modulo s, of 1 and $*$. Then the Weingarten formula for H^s says that with the notation

$$v_{ij}^e := \begin{cases} v_{ij}, & \text{if } e = 1 \\ \overline{v_{ij}}, & \text{if } e = * \end{cases}$$

we have for the coordinate functions $v = (v_{ij})_{i,j=1}^n$ of H_n^s that

$$\int_{H_n^s} v_{i_1 j_1}^{e_1} \cdots v_{i_k j_k}^{e_k} du = \sum_{\substack{\pi,\sigma \in P^s(\epsilon) \\ \pi \leq \ker \mathbf{i} \\ \sigma \leq \ker \mathbf{i}}} W_{\epsilon,n}(\pi, \sigma),$$

where $\epsilon = (e_1, \ldots, e_k)$ and $W_{\epsilon,n}$ is the inverse of the Gram matrix $G_{\epsilon,n} = (n^{|p \vee q|})_{p,q \in P^s(\epsilon)}$. The leading order in n of the Weingarten function $W_{\epsilon,n}$ is given by

$$W_{\epsilon,n}(p,q) = n^{|p \vee q| - |p| - |q|} \left(1 + O(1/n)\right).$$

Theorem 8.3 *Fix $s \in \{2, 3, \ldots, \infty\}$ and consider H^s. Consider $r \in \mathbb{N}$, $k_1, \ldots, k_r \in \mathbb{N}$, and denote by $\gamma \in S_k$ the trace permutation associated to $k_1, \ldots, k_r$. Then, for any strings $\epsilon_1, \ldots, \epsilon_r$ of respective lengths $k_1, \ldots, k_r$ we have the classical cumulants*

$$c_r(v(\epsilon_1), \ldots, v(\epsilon_r)) = \#\{p \in P^s(\epsilon_1 \cdots \epsilon_r) | p \vee \gamma = 1_k,\ p = \gamma(p)\},$$

where the product of strings is just given by their concatenation.

Proof As in the proof of Theorem 2.3 one gets for the moments

$$\int_{H_n^s} \mathrm{Tr}(v(\epsilon_1)) \cdots \mathrm{Tr}(v(\epsilon_r)) dv = \#\{p \in P^s(\epsilon_1 \cdots \epsilon_r) |\ p = \gamma(p)\} + O(1/n).$$

Note that γ does not necessarily map $P^s(\epsilon_1 \cdots \epsilon_r)$ into itself, and thus we only get the asymptotic version with lower order corrections.

We can then repeat the proof of Theorem 3.2. Let us write ϵ for $\epsilon_1 \cdots \epsilon_r$; then we only have to note that $P^s(\epsilon|_v)$ (for a block $v \in \sigma$) records the information about the original positions of the 1 and $*$ in ϵ, and thus the multiplicativity issue which prevented us from extending Theorem 3.2 to all easy classical groups, is not a problem here. Indeed, we have the analogue of (2),

$$P^s(\epsilon)_\sigma := \{p \in P_k \mid p|_v \in P^s(\epsilon|_v)\, \forall v \in \sigma\} = \{p \in P^s(\epsilon) \mid p \leq \sigma^\gamma\}.$$

□

Again, one can reduce the traces to more basic "cycle" variables. As before, we denote, for $l|k$, by $\tau_l^k \in P_k$ the partition

$$\tau_l^k = \{(1, l+1, 2l+1, \dots, k-l+1), (2, l+2, 2l+2, \dots, k-2+2), \dots, (l, 2l, \dots, k)\}.$$

Then we have

$$v(\epsilon) = \sum_{l|k} Z(\tau_l^k, \epsilon), \tag{16}$$

where

$$Z(\tau_l^k, \epsilon) := \sum_{\ker \mathbf{i} = \tau_l^k} v_{i_1 i_2}^{e_1} \cdots v_{i_k i_1}^{e_k}, \tag{17}$$

for arbitrary $l, k \in \mathbb{N}$ with $l|k$, and $\epsilon = (e_1, \dots, e_k)$.

As in the proof of Theorem 7.3 we can show

$$\begin{aligned} &c_r[Z(\tau_{l_1}^{k_1}, \epsilon_1), \dots, Z(\tau_{l_r}^{k_r}, \epsilon_r)] \\ &\quad = \#\{p \in P^s(\epsilon_1 \cdots \epsilon_r) \mid p = \gamma(p), p \vee \gamma = 1_k, \\ &\qquad p \text{ restricted to the } i\text{th cycle of } \gamma \text{ is } \tau_{l_i}^{k_i} \, (i = 1, \dots, r)\}. \end{aligned} \tag{18}$$

Clearly, this is only different from zero if $l_1 = \cdots = l_r$.

Let us define random variables $C(\tau_l^k, \epsilon)$ by specifying their distribution as

$$\begin{aligned} &c_r[C(\tau_{l_1}^{k_1}, \epsilon_1), \dots, C(\tau_{l_r}^{k_r}, \epsilon_r)] \\ &\quad = \begin{cases} 1/l, & \text{if } l_1 = \cdots = l_r = l \text{ and } p(\tau_l^{k_1}, \dots, \tau_l^{k_r}) \in P^s(\epsilon_1 \cdots \epsilon_r) \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{19}$$

where $p(\tau_l^{k_1}, \dots, \tau_l^{k_r})$ is that partition in P_k whose ith block consists of the union of the ith blocks of all the τ's, i.e., it is equal to

$$\tau_l^{k_1} \circ \cdots \circ \tau_l^{k_r} \vee \{(1, k_1+1, k_1+k_2+1, \dots, k-k_r+1), \dots, (l, k_1+l, k_1+k_2 \\ + l, \dots, k-k_r+l)\}.$$

Then we can express our variables $Z(\tau_l^k, \epsilon)$ in terms of the $C(\tau_l^k, \epsilon)$ by

$$Z(\tau_l^k, \epsilon) = \sum_{t=1}^{l} C(\tau_l^k, \epsilon^{(t)}),$$

where $\epsilon^{(t)}$ is the t-fold cyclic shift of the string ϵ, i.e.,

$$\epsilon^{(t)} = (e_{t+1}, e_{t+2}, \dots, e_t).$$

The definition (19), on the other hand, shows that the variables $C(\tau_l^k, \epsilon)$ are compound Poisson elements, which are independent for different l. Namely, we can associate to $C(\tau_l^k, \epsilon)$ a random variable

$$a(\tau_l^k, \epsilon) = \left(\prod_{\substack{i_1 \text{ in first} \\ \text{block of } \tau_l^k}} \omega^{e_{i_1}} \right) \otimes \left(\prod_{\substack{i_2 \text{ in second} \\ \text{block of } \tau_l^k}} \omega^{e_{i_2}} \right) \otimes \cdots \otimes \left(\prod_{\substack{i_l \text{ in } l\text{th} \\ \text{block of } \tau_l^k}} \omega^{e_{i_l}} \right) \in C(\mathbb{T})^{\otimes l}.$$

Then we have

$$c_r \left[C\left(\tau_{l_1}^{k_1}, \epsilon_1\right), \ldots, C\left(\tau_{l_r}^{k_r}, \epsilon_r\right) \right] = \frac{1}{l_1} \psi \left(a\left(\tau_{l_1}^{k_1}, \epsilon_1\right) \cdots a\left(\tau_{l_r}^{k_r}, \epsilon_r\right) \right) \tag{20}$$

where ψ is $\bigoplus_l \varphi^{\otimes l}$ on $\bigoplus_l C(\mathbb{T})^{\otimes l}$, with φ denoting integration with respect to the Haar measure on $\mathbb{Z}_s$ (where the latter is being embedded into the unit circle $\mathbb{T}$).

The equation (20) shows that the cumulants of the variables C are given, up to some factor, as the corresponding moments of some variables a; this is the characterizing property of compound Poisson variables.

If we put now

$$\epsilon_k := (\underbrace{1, *, 1, *, \ldots}_{k}),$$

then we have for our asymptotic traces the decomposition

$$v_k = \sum_{l|k} Z\left(\tau_l^k, \epsilon_k\right) = \sum_{l|k} \sum_{t=1}^{l} C\left(\tau_l^k, \epsilon_k^{(t)}\right). \tag{21}$$

Thus we have written the v_k as a sum of compound Poisson variables. These are independent for different l; however, for fixed l, the relation between the C for various k is more complicated, according to the ϵ-strings. For k even this reduces again essentially to a sum of independent Poisson variables, whereas for k odd the situation is getting more involved. As we do not see a nice more explicit description, we refrain from working out the details in this case.

9 Concluding remarks

We have seen in this paper that the original philosophy suggested in [6], namely the fact that "any result which holds for S_n, O_n should have an extension to easy quantum groups", has indeed a first illustration in the context of the fundamental stochastic eigenvalue computations of Diaconis and Shahshahani in [14].

A first natural question is about the possible "eigenvalue" interpretations of our results. The point is that in the classical case the law of $\mathrm{Tr}(u^k)$ is of course a function of the eigenvalues of the random matrix $u \in G$, but in the quantum case such a simple interpretation is lacking. One technical problem is for instance the fact that the variables $\mathrm{Tr}(u^k)$ are not self-adjoint in the quantum case. So, as a first conclusion of our study, we would like to point out the fact that the following question is still open: *What are the eigenvalues of a random quantum group matrix?*

Remark 9.1 The half-liberated cases are, as we have seen in the last section, quite close to the classical world, and the representation in terms of 2 × 2-matrices over classical unitary groups suggests an answer to the above question in this case: If λ_i are the eigenvalues of U_n, then

$$\begin{pmatrix} 0 & \lambda_i \\ \bar{\lambda}_i & 0 \end{pmatrix}$$

should be the corresponding eigenvalues of O_n^*. The same for H^s and $H^{(s)}$.

For the free quantum groups, however, the situation is less clear and we have no suggestion for a possible candidate for eigenvalues.

A second natural question is about what happens in the unitary case. It is known indeed since [1] that the fundamental character $\chi = \mathrm{Tr}(u)$ is asymptotically circular in the sense of Voiculescu [20] for the quantum group U_n^+, so the results in [14] about U_n should probably have some kind of "free version". However, the general study and classification of the easy quantum groups in the unitary case seems to be a quite difficult combinatorial problem, and we do not have so far concrete results in this direction. We would like to refer here to the concluding section in our previous paper [6], which contains a brief description of the whole problematics in the unitary case.

A third question concerns the relationship of the present results with the Bercovici-Pata bijection [9]. This bijection makes a correspondence between classical measures and their "free versions", and a key problem in free probability is to find concrete models for it. A general random matrix model, providing a full answer to the question, comes from the work of Benaych-Georges [8] and Cabanal-Duvillard [10]. As for the representation theory implementations, meant to be "finer", these concern so far the laws of truncated characters for O_n, S_n, H_n, B_n, as explained in [6]. So, the question that we would like to raise here is as follows: is it possible to unify the truncated character computations in [6] with the Diaconis-Shahshahani type computations from the present paper? This is definitely possible in the classical case, where several "truncation" procedures are available in the general context of stochastic eigenvalue analysis. See e.g., [18].

A fourth fundamental problem is about what happens when n is fixed. In the classical case the subject is of course quite technical, but the results in this sense abound. In the quantum case the situation is definitely more complicated, because the main tool that we have so far, namely the Weingarten formula, cannot properly handle the problem. In the O_n^+ case the law of u_{11}, which can be regarded as a truncated character, was computed only recently, in [3]. We do not know if the techniques developed there, which are new, can be applied to the variables investigated in the present paper.

Finally, we have the general question of trying to apply our "S_n, O_n philosophy" to some new, totally different situations. The first thought here goes to the various de Finetti type theorems, available for S_n, O_n and other classical groups from [15], and for S_n^+, O_n^+ from [16] and [12,13]. A global approach to the problem, by using easy quantum groups, is developed in our paper [5].

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