Probab. Theory Relat. Fields (2011) 149:97–137
DOI 10.1007/s00440-009-0243-5

Structural properties of semilinear SPDEs driven by cylindrical stable processes

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Received: 24 November 2008 / Revised: 3 August 2009 / Published online: 16 September 2009

Abstract We consider a class of semilinear stochastic evolution equations driven by an additive cylindrical stable noise. We investigate structural properties of the solutions like Markov, irreducibility, stochastic continuity, Feller and strong Feller properties, and study integrability of trajectories. The obtained results are applied to semilinear stochastic heat equations with Dirichlet boundary conditions and bounded and Lipschitz nonlinearities.

Keywords Stochastic PDEs with jumps · Strong Feller property · Regularity of trajectories

Mathematics Subject Classification (2000) 60H15 · 60J75 · 47D07 · 35R60

1 Introduction

The paper is concerned with structural properties of solutions to nonlinear stochastic equations

Research of E. Priola was supported by the M.I.U.R. research projects Prin 2004 and 2006 "Kolmogorov equations".
Research of E. Priola and J. Zabczyk was supported by the Polish Ministry of Science and Education project 1PO 3A 034 29 "Stochastic evolution equations with Lévy noise" and by the European Grant "SPADE2".

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$$dX_t = AX_t dt + F(X_t)dt + dZ_t, \quad t \geq 0, \quad X_0 = x \in H, \tag{1.1}$$

in a real separable Hilbert space H driven by an infinite dimensional stable process $Z = (Z_t)$. In particular, we study Markov, irreducibility, stochastic continuity, Feller and strong Feller properties for the solutions, and investigate integrability of trajectories. The main results are gradient estimates for the associated transition semigroup (see Theorem 4.14 when $F = 0$ and Theorem 5.7 in the general case), from which we deduce the strong Feller property, and a theorem on time regularity of trajectories (see Theorem 4.4).

To cover interesting cases, we consider processes Z which take values in a Hilbert space U usually greater than H. Moreover $A : \mathrm{dom}(A) \subset H \to H$ is a linear possibly unbounded operator which generates a C_0-semigroup (e^{tA}) on H and $F : H \to H$ denotes a Lipschitz continuous and bounded function.

In the case when Z is a Wiener process the theory of equations (1.1) is well understood. The situation changes completely in the stable noise case and new phenomena appear. For instance, even in the linear case $F = 0$, it is not known when solutions of (1.1) have càdlàg trajectories (see Sect. 4.1). That lack of càdlàg regularity is possible was noted in [18, Proposition 9.4.4] in a similar situation. Another difficulty is related to the fact that general necessary and sufficient conditions for absolute continuity of stable measures on Hilbert spaces do not exist.

We restrict our considerations to SPDEs with additive noise as even in this case some new phenomena, related to the cylindrical Lévy noise, appear. We hope that the results presented here will form a proper starting point to treat general equations with multiplicative Lévy perturbations.

In this paper we consider a *cylindrical α-stable process* $Z = (Z_t)$, $\alpha \in (0, 2)$, defined by the orthogonal expansion

$$Z_t = \sum_{n \geq 1} \beta_n Z_t^n e_n, \quad t \geq 0, \tag{1.2}$$

where (e_n) is an orthonormal basis of H and (Z_t^n) are independent, real valued, normalized, symmetric α-stable processes defined on a fixed stochastic basis. Moreover, (β_n) is a given, possibly unbounded, sequence of *positive* numbers.

The results of the paper apply to stochastic heat equations with Dirichlet boundary conditions

$$\begin{cases} dX(t,\xi) = (\triangle X(t,\xi) + f(X(t,\xi)))\, dt + dZ(t,\xi), \quad t > 0, \\ X(0,\xi) = x(\xi), \quad \xi \in D, \\ X(t,\xi) = 0, \quad t > 0, \quad \xi \in \partial D, \end{cases} \tag{1.3}$$

in a given bounded domain $D \subset \mathbb{R}^d$ having Lipschitz-continuous boundary ∂D. Here $x(\xi) \in H = L^2(D)$, $f : \mathbb{R} \to \mathbb{R}$ is bounded and Lispchitz continuous and the noise Z is a cylindrical α-stable process of the form (1.2), where (e_n) is a basis of eigenfunctions for the Laplace operator $\triangle$ (with Dirichlet boundary conditions).

Irreducibility and strong Feller property can be used to establish uniqueness of an invariant measure for the solutions of (1.1) through a well known approach based on

the so called Doob theorem (see [6, Theorem 4.2.1]). However our results indicate that solutions of (1.1) with non-Gaussian noise are less regular than those with a Wiener process. Thus to cover equations with Lévy noise, having only a finite number of modes or with modes vanishing rapidly (compare with [13]) one needs an extension of the methods developed in [13–15].

After short Preliminaries, concerned with notations and basic definitions, in Sect. 3, we deal with real and Hilbert space valued α-stable random variables. The most important result here is a necessary and sufficient condition for the absolute continuity of shifts of infinite products of symmetric α-stable, one dimensional distributions (see Theorem 3.4). It is an improvement of an old result by Zinn (see [30]) with a direct proof. Section 4 is concerned with linear equations

$$dX_t = AX_t dt + dZ_t, \quad t \geq 0, \quad X_0 = x \in H \tag{1.4}$$

(see also [4,3,7,11,18,23]). We require that vectors (e_n) from the representation (1.2) are eigenvectors of A. All assumptions concerning (1.4) are gathered in Hypothesis (L).

In Proposition 4.2 we give if and only if conditions under which X, the solution of (1.4), takes values in H, and establish its measurability and markovianity. Then we deal with the time regularity of trajectories. The main result here is Theorem 4.4, which establishes stochastic continuity of the solution and integrability of its trajectories. Better regularity, like right or left continuity of trajectories is established here in very special cases and *is an open question for general equations*. Note that in [11] it is proved that trajectories of (X_t^x) are càdlàg only in some enlarged Hilbert space U containing H. This lack of time regularity introduces additional difficulties into the theory (see also [18, Proposition 9.4.4]). We establish also irreducibility of the solution. Theorem 4.12 gives conditions under which all transition laws of X are equivalent and establishes a formula for the densities. Moreover (see Theorem 4.14) under the assumptions of Theorem 4.12, the transition semigroup corresponding to X is not only strong Feller but transforms bounded measurable functions into Fréchet differentiable functions with continuous derivative. Important gradient estimates are established as well. Theorems on Ornstein-Uhlenbeck processes are based on results about stable measures established in Sect. 3.

Section 5 is devoted to nonlinear equations (1.1). We gather all assumptions on the Eq. (1.1) in Hypothesis (N). Markov property and irreducibility require special attention due to the lack of càdlàg regularity of the trajectories. They are established in Theorems 5.3 and 5.4. Then estimates of Sect. 4 are used to establish the strong Feller property of the solution to the nonlinear equation (see Theorem 5.7). The main tool here is the so called mild version of the Kolmogorov equation and Galerkin's approximation. It is proper to add that the classical approach to get strong Feller using the Bismut–Elworthy–Li formula is not available in the non-Gaussian case. A related formula, but requiring a non trivial Gaussian component in the Lévy noise, was established in finite dimensions in [21]. Finally, we mention that a preliminary expanded version of this paper is given in [22].

2 Preliminaries

H will denote a real separable Hilbert with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. By $\mathcal{L}(H)$ we denote the Banach space of all bounded linear operators from H into H endowed with the operator norm $\|\cdot\|_{\mathcal{L}(H)}$. We will fix an orthonormal basis (e_n) in H. Through the basis (e_n) we will often identify H with l^2. More generally, for a given sequence $\rho = (\rho_n)$ of real numbers, we set

$$l^2_\rho = \{(x_n) \in \mathbb{R}^\infty \ : \sum_{n\geq 1} x_n^2 \rho_n^2 < \infty\}, \tag{2.1}$$

where $\mathbb{R}^\infty = \mathbb{R}^{\mathbb{N}}$. l^2_ρ becomes a separable Hilbert space with the inner product: $\langle x, y\rangle = \sum_{n\geq 1} x_n y_n \rho_n^2$, for $x = (x_n)$, $y = (y_n) \in l^2_\rho$.

The space $C_b(H)$ (resp. $B_b(H)$) stands for the Banach space of all real, continuous (resp. Borel) and bounded functions $f : H \to \mathbb{R}$, endowed with the supremum norm: $\|f\|_0 = \sup_{x\in H} |f(x)|$.

The space $C_b^k(H)$, $k \geq 1$, is the set of all k-times differentiable functions $f : H \to \mathbb{R}$, whose Fréchet derivatives $D^i f$, $1 \leq i \leq k$, are continuous and bounded on H, up to the order k. Moreover we set $C_b^\infty(H) = \cap_{k\geq 1} C_b^k(H)$.

Let us recall that a Lévy process (Z_t) with values in H is an H-valued process defined on some stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, having stationary independent increments, càdlàg trajectories, and such that $Z_0 = 0$, $\mathbb{P}$-a.s.

One has that

$$\mathbb{E}[e^{i\langle Z_t, s\rangle}] = \exp(-t\psi(s)), \ s \in H, \tag{2.2}$$

where the exponent ψ can be expressed by the following infinite dimensional Lévy-Khintchine formula,

$$\psi(s) = \frac{1}{2}\langle Qs, s\rangle - i\langle a, s\rangle - \int_H \left(e^{i\langle s,y\rangle} - 1 - \frac{i\langle s, y\rangle}{1+|y|^2}\right)\nu(dy), \quad s \in H. \tag{2.3}$$

Here Q is a symmetric non-negative trace class operator on H, $a \in H$ and ν is the Lévy measure or the jump intensity measure associated to (Z_t) (see [18,26]).

According to Proposition 3.3 (see also Remark 4.1) our cylindrical α-stable process Z appearing in (1.2) is a Lévy process taking values in the Hilbert space $U = l^2_\rho$, see (2.1), with a properly chosen weight ρ.

Let $Z = (Z_t)$ be a real valued Lévy process. We will consider Wiener–Lévy type integrals

$$I_t = \int_0^t f(s)dZ_s,$$

where $f : [0, t] \to \mathbb{R}$ is a (deterministic) continuous function. Similarly to [2, Section 4.3.5], I_t can be defined as a limit in probability of Riemann sums. Moreover (see [26, formula (17.3), page 105], [25, Chapter 3] or [4]), we find

$$\mathbb{E}[e^{ih\int_0^t f(s)dZ_s}] = \exp\left(-\int_0^t \psi\left(f(s)h\right)ds\right), \quad h \in \mathbb{R}, \tag{2.4}$$

where ψ is the exponent of (Z_t) (see (2.2)).

3 Stable measures on Hilbert spaces

Here we gather and strengthen results on stable distributions needed in the sequel.

3.1 Stable densities

A symmetric real α-stable distribution $S(\alpha, \sigma, 0), \alpha \in (0, 2), \sigma \geq 0$, has characteristic function

$$\hat{S}(\alpha, \sigma, 0)(s) = e^{-\sigma^\alpha |s|^\alpha}, \quad s \in \mathbb{R} \tag{3.1}$$

(see [25, Chapter 1]). A symmetric random variable X is α-stable if its law is $S(\alpha, \sigma, 0)$, for some $\alpha \in (0, 2)$ and $\sigma \geq 0$. We have if $\sigma > 0$,

$$\begin{cases} \mathbb{E}|X|^p = \infty, & p \geq \alpha, \\ \mathbb{E}|X|^p = c_{\alpha,p}\,\sigma^p, & 0 < p < \alpha, \end{cases} \tag{3.2}$$

where the explicit form of $c_{p,\alpha}$ is given in [25, page 18].

Let us consider a normalized, symmetric α-stable distribution $\mu_\alpha = S(\alpha, 1, 0)$. The *density of* μ_α, with respect to Lebesgue measure, will be denoted by p_α, $\alpha \in (0, 2)$. We need to know the *precise asymptotic behaviour* of the even function p_α. We have that, for any $\alpha \in (0, 2)$, there exists $C_\alpha > 0$ such that

$$p_\alpha(x) \sim \frac{C_\alpha}{x^{\alpha+1}}, \quad \text{as } x \to \infty, \tag{3.3}$$

see [29], [26, page 88] and [10, pages 582–583]. We need two lemmas about p_α. The first one is straightforward.

Lemma 3.1 *Let p_α be the density of the one dimensional α-stable measure μ_α in (3.1). Then, for any $\alpha \in (0, 2)$, $p_\alpha \in C^\infty(\mathbb{R}) \cap C_0(\mathbb{R})$ and moreover (with $p'_\alpha(x) = \frac{dp_\alpha}{dx}$)*

$$x^2 p'_\alpha(x) \in L^\infty(\mathbb{R}). \tag{3.4}$$

Proof Let $p = p_\alpha$. It is well known that $p \in C^\infty(\mathbb{R}) \cap C_0(\mathbb{R})$ (see, for instance, [26, Chapter 1]). To get the second assertion, we use the inversion Fourier formula and integrate by parts,

$$\begin{aligned} x^2 p'(x) &= -\frac{i}{2\pi} \int_{\mathbb{R}} x^2 e^{-ixy} y e^{-|y|^\alpha} dy \\ &= \frac{x}{2\pi} \int_{\mathbb{R}} \frac{d}{dy}\left(e^{-ixy}\right) y e^{-|y|^\alpha} dy = \frac{x}{2\pi} \int_{\mathbb{R}} e^{-ixy} e^{-|y|^\alpha} \left(\alpha |y|^\alpha - 1\right) dy \\ &= \frac{-i\alpha^2}{2\pi} \int_{\mathbb{R}} e^{-ixy} e^{-|y|^\alpha} \frac{y}{|y|^{2-\alpha}} dy \\ &\quad + \frac{i\alpha}{2\pi} \int_{\mathbb{R}} e^{-ixy} e^{-|y|^\alpha} y \left(\frac{\alpha}{|y|^{2-2\alpha}} - \frac{1}{|y|^{2-\alpha}}\right) dy. \end{aligned}$$

Since, $x^2 p'(x)$ is the sum of two Fourier transforms of integrable functions we get assertion (3.4). □

Lemma 3.2 *Let us consider the function*

$$g(x) = 1 - \int_{\mathbb{R}} p_\alpha^{1/2}(z)\, p_\alpha^{1/2}(z - x)\, dz, \qquad x \in (-1, 1).$$

We have

$$g(x) \sim c_\alpha x^2 \quad as \quad x \to 0, \quad where \quad c_\alpha = \frac{1}{8} \int_{\mathbb{R}} \frac{p'_\alpha(z)^2}{p_\alpha(z)} dz. \tag{3.5}$$

Proof Let $p = p_\alpha$. Clearly $g(0) = 0$. In order to prove (3.5) we will apply Hospital's rule. To this purpose we prove that g is twice differentiable, with $g'(0) = 0$ and $g''(0) \neq 0$. We have, for $|x| < 1$,

$$g'(x) = \frac{1}{2} \int_{\mathbb{R}} p^{1/2}(z) \frac{1}{p^{1/2}(z - x)} p'(z - x)\, dz.$$

The differentiation is justified by (3.3) and (3.4), using also the fact that p is a positive function on $\mathbb{R}$. We only point out the following useful estimate: for any $M > 1$, there exists $c > 0$, such that, for any $x \in (-1, 1)$, $|z| > M$,

$$p^{1/2}(z) \frac{1}{p^{1/2}(z - x)} |p'(z - x)| \leq \frac{c}{|z|^{1/2+\alpha/2}} \frac{(|z| + 1)^{1/2+\alpha/2}}{|z - 1|^2}. \tag{3.6}$$

To check (3.6), remark that by (3.3) there exists $M > 1$ such that $\frac{C_\alpha}{2} \le |z|^{1/2+\alpha/2}$ $p^{1/2}(z) \le \frac{3C_\alpha}{2}$, for any $z \in \mathbb{R}$, $|z| > M$. Since $|z - x| \ge |z - 1| \ge M - 1$, when $|z| > M$, $x \in (-1, 1)$, we have by (3.4)

$$|p'(z - x)| \le \frac{c'}{|z - x|^2} \le \frac{c'}{|z - 1|^2}, \quad |z| > M, \ x \in (-1, 1),$$

and (3.6) follows.

We also get $g'(0) = \frac{1}{2} \int_{\mathbb{R}} p'(z)\, dz = 0$. We show now that there exists the second derivative of g. To this purpose, we write

$$g'(x) = \frac{1}{2} \int_{\mathbb{R}} p^{1/2}(z + x) \frac{1}{p^{1/2}(z)} p'(z)\, dz.$$

We have, for any $x \in (-1, 1)$,

$$g''(x) = \frac{1}{4} \int_{\mathbb{R}} \frac{p'(z + x)}{p^{1/2}(z + x)} \frac{1}{p^{1/2}(z)} p'(z)\, dz.$$

The differentiation can be done, since, for any $M > 1$, there exists $c' > 0$, such that, for any $x \in (-1, 1)$, $|z| > M$,

$$\frac{|p'(z + x)|}{p^{1/2}(z + x)} \frac{|p'(z)|}{p^{1/2}(z)} \le \frac{c'(|z| + 1)^{1+\alpha}}{|z|^2 |z - 1|^2}. \tag{3.7}$$

This estimate can be proved arguing as in the proof of (3.6). We have also that

$$g''(0) = \frac{1}{4} \int_{\mathbb{R}} \frac{p'(z)^2}{p(z)}\, dz.$$

and so (3.5) is proved. □

3.2 Supports of stable measures

Let us consider a sequence (ξ_n) of independent real random variables, having the same law μ_α and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Take nonnegative numbers q_n and consider the random variable

$$\xi = (q_1\xi_1, \ldots, q_n\xi_n, \ldots) \tag{3.8}$$

with values in $\mathbb{R}^\infty$. We start with a preliminary result, which is a special case of [17, Corollary 2.4.2]. We provide a self-contained proof for the sake of completeness.

Proposition 3.3 *For any $\alpha \in (0,2)$, the random variable ξ in (3.8) takes values in l^2, $\mathbb{P}$-a.s., if and only if*

$$\sum_{n\geq 1} q_n^\alpha < \infty. \tag{3.9}$$

If, in addition to (3.9), $q_n > 0$, $n = 1, 2, \ldots$, then the support of the law of ξ is l^2.

Proof We will use the following theorem (see, for instance [16], page 70–71): let U_n be a sequence of independent and symmetric real random variables; then the following statements are equivalent: $\sum_{n\geq 1} U_n$ converge in distribution; $\sum_{n\geq 1} U_n$ converges $\mathbb{P}$-a.s.; $\sum_{n\geq 1} U_n^2$ converges $\mathbb{P}$-a.s.

We have: $\mathbb{E}[e^{i\sum_{n=1}^N q_n\xi_n h}] = \prod_{n=1}^N \mathbb{E}[e^{iq_n\xi_n h}] = e^{-\sum_{n=1}^N q_n^\alpha |h|^\alpha}$, for any $N \in \mathbb{N}$, $h \in \mathbb{R}$.

Then it is clear that $\sum_{k=1}^N q_k\xi_k$ converges in distribution if and only if (3.9) holds. Moreover if (3.9) holds, then we have convergence in distribution to the random variable $\xi_1 \left(\sum_{k=1}^\infty q_k^\alpha\right)^{1/\alpha}$. It follows that the series $\sum_{k\geq 1} q_k\xi_k$ converges, $\mathbb{P}$-a.s., and also that

$$\sum_{k\geq 1} q_k^2\,\xi_k^2 < \infty, \quad \mathbb{P}-a.s., \tag{3.10}$$

and this proves the first part. To prove the second assertion, we fix an arbitrary ball $B \subset l^2$, $B = B(y,r)$ with center in $y = (y_k) \in l^2$ and radius $r > 0$. Using independence, we find

$$\begin{aligned}
&\mathbb{P}\left(\sum_{k\geq 1}(q_k\xi_k - y_k)^2 < r^2\right)\\
&\quad\geq \mathbb{P}\left(\sum_{k=1}^N (q_k\xi_k - y_k)^2 < \epsilon\right)\mathbb{P}\left(\sum_{k>N}(q_k\xi_k - y_k)^2 < r^2 - \epsilon\right).
\end{aligned}$$

Now we use that the one dimensional measure μ_α has a positive density on $\mathbb{R}$. This implies that, for any $N \in \mathbb{N}$, $\epsilon > 0$, $\mathbb{P}\left(\sum_{k=1}^N (q_k\xi_k - y_k)^2 < \epsilon\right) > 0$.

Since $\mathbb{P}(\sum_{k>N}(q_k\xi_k - y_k)^2 < r^2 - \epsilon) \to 1$, as $N \to \infty$, the assertion follows. □

3.3 Equivalence of shifts of stable measures

Here we give necessary and sufficient conditions in order that shifts of infinite products of one dimensional α-stable distributions, are equivalent. Our theorem on equivalence strengthen an absolute continuity result of Zinn [30] (see Remark 3.5) with a different proof which requires Lemma 3.2.

Theorem 3.4 *Let us consider the l^2-random variable ξ in (3.8) under the condition $q_k > 0$, $k \geq 1$, and $\sum_{k\geq 1} q_k^\alpha < \infty$. Take arbitrary $u, v \in l^2$ such that*

$$\sum_{k\geq 1} \frac{|u_k - v_k|^2}{q_k^2} < \infty. \tag{3.11}$$

Then the law of the random variable $\xi + u$ and the one of $\xi + v$ are equivalent.

In addition, if μ and ν denote the laws of $\xi + u$ and $\xi + v$ respectively, the density $\frac{d\mu}{d\nu}$ of μ with respect to ν is given by

$$\frac{d\mu}{d\nu} = \lim_{N\to\infty} \prod_{k=1}^{N} \frac{p_\alpha\left(\frac{z_k - u_k}{q_k}\right)}{p_\alpha\left(\frac{z_k - v_k}{q_k}\right)} \quad in \;\; L^1(\nu), \quad \alpha \in (0, 2).$$

Proof Let $p_\alpha = p$ with $\alpha \in (0, 2)$. The measures μ and ν can be seen as Borel product measures in $\mathbb{R}^\infty$, i.e.,

$$\mu = \prod_{k\geq 1} \mu^k, \qquad \nu = \prod_{k\geq 1} \nu^k, \quad \text{where } \; \mu^k, \; \nu^k \; \text{ have densities, respectively,}$$
$$\frac{1}{q_k} p\left(\frac{z_k - u_k}{q_k}\right) \;\text{ and }\; \frac{1}{q_k} p\left(\frac{z_k - v_k}{q_k}\right).$$

According to [5, Proposition 2.19], μ and ν are equivalent if and only if

$$H(\mu, \nu) = \prod_{k\geq 1} \int_{\mathbb{R}} \left(\frac{d\mu^k}{d\nu^k}\right)^{1/2} \nu^k(dz_k) = \prod_{k\geq 1} \int_{\mathbb{R}} \left(\frac{d\mu^k}{dz_k}\right)^{1/2} \left(\frac{d\nu^k}{dz_k}\right)^{1/2} (dz_k) > 0.$$

Define, for any $k \geq 1$,

$$\begin{aligned} a_k &= \int_{\mathbb{R}} \left(\frac{d\mu^k}{dz_k}(z_k)\right)^{1/2} \left(\frac{d\nu^k}{dz_k}(z_k)\right)^{1/2} dz_k \\ &= \int_{\mathbb{R}} \left[p^{1/2}\left(z_k - \frac{u_k}{q_k}\right) p^{1/2}\left(z_k - \frac{v_k}{q_k}\right)\right] dz_k \in (0, 1]. \end{aligned}$$

Note that

$$H(\mu, \nu) = \prod_{k\geq 1} a_k = \prod_{k\geq 1} (1 - (1 - a_k)) = e^{\sum_{k\geq 1} \ln(1-(1-a_k))}.$$

Hence $H(\mu,\nu) > 0$ if and only if $\sum_{k\geq 1} \ln(1-(1-a_k)) > -\infty$ and this is equivalent to

$$\sum_{k\geq 1}(1-a_k) < +\infty. \tag{3.12}$$

Let us write

$$1-a_k = 1-\int_{\mathbb{R}} p^{1/2}(z)p^{1/2}\left(z-\left(\frac{v_k}{q_k}-\frac{u_k}{q_k}\right)\right)dz$$

so that

$$1-a_k = g\left(\frac{v_k}{q_k}-\frac{u_k}{q_k}\right),$$

where the function g is considered in Lemma 3.2. Using (3.5) and (3.11), we get that (3.12) holds and this proves the first statement.

The second assertion follows from the first one, applying [5, Proposition 2.19]. □

Remark 3.5 The result agrees with [30, Corollary 8.1], which shows that the law of $\xi+u$, $u\in l^2$, is *absolutely continuous* with respect to the one of ξ if and only if

$$\sum_{k\geq 1}\frac{u_k^2}{q_k^2} < \infty.$$

We point out that in [30], there are no conditions to assure the equivalence of α-stable measures.

4 The linear stochastic PDE

We start from the linear equation

$$dX_t = AX_t dt + dZ_t, \quad x\in H. \tag{4.1}$$

The process Z is a cylindrical α-stable process, $\alpha\in(0,2)$, given by

$$Z_t = \sum_{n\geq 1}\beta_n Z_t^n e_n, \quad t\geq 0,$$

where (e_n) is the fixed reference orthonormal basis in H, (β_n) is a given sequence of *positive* numbers and (Z_t^n) are independent one dimensional α-stable processes defined on the same stochastic basis $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$, satisfying the usual assumptions. We have, for any $n\in\mathbb{N}$, $t\geq 0$,

$$\mathbb{E}[e^{iZ_t^n h}] = e^{-t|h|^\alpha}, \quad h\in\mathbb{R}.$$

Remark 4.1 Identifying, through the basis (e_n), the Hilbert space H with l^2 and using Proposition 3.3, one gets that our cylindrical Lévy process Z takes values in l^2_ρ, see (2.1), where (ρ_n) is a sequence of positive numbers such that $\sum_{n\geq 1} \beta_n^\alpha \rho_n^\alpha < \infty$; thus Z is an "honest" l^2_ρ-valued Lévy process.

We gather here all assumptions which will be made on the linear equation (4.1).
Hypothesis (L)

(i) The processes Z^n are independent and α-stable, $\alpha \in (0, 2)$.
(ii) $A : D(A) \subset H \to H$ is a self-adjoint operator such that the fixed basis (e_n) of H verifies: $(e_n) \subset D(A)$, $Ae_n = -\gamma_n e_n$ with $\gamma_n > 0$, for any $n \geq 1$, and $\gamma_n \to +\infty$.
(iii) $\displaystyle\sum_{n\geq 1} \frac{\beta_n^\alpha}{\gamma_n} < \infty$ (and $\beta_n > 0$, for any $n \geq 1$).
(iv) For any $t > 0$,

$$\sup_{n\geq 1} \frac{e^{-\gamma_n t}\, \gamma_n^{1/\alpha}}{\beta_n} = C_t < \infty. \tag{4.2}$$

Clearly, under (ii), $D(A) = \{x = (x_n) \in H : \sum_{n\geq 1} x_n^2 \gamma_n^2 < +\infty\}$. In addition A generates a compact C_0-semigroup (e^{tA}) on H such that

$$e^{tA} e_k = e^{-\gamma_k t} e_k, \quad k \in \mathbb{N}, \quad t \geq 0.$$

According to (i) and (ii) in Hypothesis (L), we may consider our equation as an infinite sequence of independent one dimensional stochastic equations, i.e.,

$$dX_t^n = -\gamma_n X_t^n dt + \beta_n dZ_t^n, \quad X_0^n = x_n, \quad n \in \mathbb{N}, \tag{4.3}$$

with $x = (x_n) \in l^2 = H$. The *unique solution* is a stochastic process $X = (X_t^x)$ which takes values in $\mathbb{R}^\infty$ with components

$$X_t^n = e^{-\gamma_n t} x_n + \int_0^t e^{-\gamma_n (t-s)} \beta_n dZ_s^n, \quad n \in \mathbb{N}, \quad t \geq 0 \tag{4.4}$$

Proposition 4.2 *Assume (i) and (ii) in Hypothesis (L). Then, for any $x \in H$, the process $X = (X_t^x)$ given in (4.4) takes values in H if and only if condition (iii) holds. Moreover, under (i), (ii) and (iii), we have*

$$X_t^x = \sum_{n\geq 1} X_t^n e_n = e^{tA} x + Z_A(t), \quad \textit{where} \tag{4.5}$$

$$Z_A(t) = \int_0^t e^{(t-s)A} dZ_s = \sum_{n\geq 1} \left(\int_0^t e^{-\gamma_n (t-s)} \beta_n dZ_s^n \right) e_n.$$

The process (X_t^x) is $\mathcal{F}_t$-adapted, $x \in H$. Moreover X is Markovian.

Proof Let us consider the stochastic convolution

$$Y_t^n = Z_A^n(t) = \int_0^t e^{-\gamma_n(t-s)}\beta_n dZ_s^n, \quad n \in \mathbb{N}, \quad t \geq 0. \tag{4.6}$$

A direct calculation shows that, for any $h \in \mathbb{R}$,

$$\mathbb{E}[e^{ihY_t^n}] = \exp\left[-\beta_n^\alpha |h|^\alpha \int_0^t e^{-\alpha\gamma_n s} ds\right] = \exp\left[-|h|^\alpha c_n^\alpha(t)\right],$$
$$\text{where } c_n(t) = \beta_n \left(\frac{1-e^{-\alpha\gamma_n t}}{\alpha\gamma_n}\right)^{1/\alpha} \tag{4.7}$$

(see (2.4)). It follows that

$$\mathbb{E}[e^{ih\,Y_t^n}] = \mathbb{E}[e^{ih\,c_n(t)\,L_n}], \quad h \in \mathbb{R}, \tag{4.8}$$

where (L_n) are independent α-stable random variables having the same law μ_α (see (3.1)). Now the first assertion follows directly from Proposition 3.3.
The property that (X_t^x) is $\mathcal{F}_t$-adapted is equivalent to the fact that each real process $\langle X_t^x, e_k\rangle$ is $\mathcal{F}_t$-adapted, for any $k \geq 1$, and this clearly holds.

To prove the Markov property, we start from the identity

$$Z_A(t+h) - e^{hA}Z_A(t) = \int_t^{t+h} e^{(t+h-s)A} dZ_s \qquad t,\, h \geq 0. \tag{4.9}$$

Introduce the bounded linear operators $R_h : B_b(H) \to B_b(H)$, $h \geq 0$; $R_h f(x) = \mathbb{E}[f(X_h^x)]$, for any $x \in H$, $f \in B_b(H)$. Using (4.9) and independence, we get, for any $f \in B_b(H)$,

$$\mathbb{E}[f\left(e^{hA}(e^{tA}x) + Z_A(t+h)\right)/\mathcal{F}_t] = \mathbb{E}\left[f\left(e^{hA}y + \int_t^{t+h} e^{(t+h-s)A} dZ_s\right)\right]_{y=e^{tA}x+Z_A(t)}.$$

Note that, for any $t, h \geq 0$, the H-valued random variable $\int_t^{t+h} e^{(t+h-s)A} dZ_s$ has the same law of $Z_A(h)$. Indeed both have the same characteristic function. Therefore (see (4.5))

$$\mathbb{E}[f\left(X_{t+h}^x\right)/\mathcal{F}_t] = \mathbb{E}[e^{hA}y + Z_A(h)]_{y=e^{tA}x+Z_A(t)} = R_h(X_t^x).$$

The proof is complete. □

Example 4.3 Consider the following linear stochastic heat equation on $D = [0, \pi]^d$ with Dirichlet boundary conditions (see also (1.3))

$$\begin{cases} dX(t,\xi) = \triangle X(t,\xi)\,dt + dZ(t,\xi), \quad t > 0, \\ X(0,\xi) = x(\xi), \quad \xi \in D, \\ X(t,\xi) = 0, \quad t > 0, \quad \xi \in \partial D, \end{cases} \tag{4.10}$$

where Z is a cylindrical α-stable process with respect to the basis of eigenfunctions of the Laplacian $\triangle$ in $H = L^2(D)$ (with Dirichlet boundary conditions). The eigenfunctions are

$$e_j(\xi_1, \ldots, \xi_d) = (\sqrt{2/\pi})^d \sin(n_1\xi_1) \cdots \sin(n_d\xi_d), \quad \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d,$$

$j = (n_1, \ldots, n_d) \in \mathbb{N}^d$. The corresponding eigenvalues are $-\gamma_j$, where $\gamma_j = (n_1^2 + \cdots + n_d^2)$. The operator $A = \triangle$ with $D(A) = H^2(D) \cap H_0^1(D)$ verifies condition (ii) in Hypothesis (L). Moreover (see [28, Section 4.4.3]) we have

$$D((-A)^{p/2}) = \begin{cases} H^p(D) \cap H_0^1(D) & \text{if } 1 < p \le 2, \\ H_0^p(D) & \text{if } 1/2 < p \le 1, \\ H^p(D) & \text{if } 0 < p \le 1/2. \end{cases}$$

If we identify H with l^2 then $D((-A)^{p/2})$ can be identified with the weighted space l^2_ρ (see (2.1)) where $\rho = (\rho_j)$ and $\rho_j = \gamma_j^{p/2}$. This follows from the spectral decomposition of self-adjoint operators, taking into account that $-A$ is a positive definite self-adjoint operator with compact resolvent.

The corresponding dual spaces can be identified with $l^2_{1/\rho}$ or with Sobolev spaces of distributions $H^{-p}(D)$. We only sketch the argument. Given any $x \in l^2_{1/\rho}$, one defines $T_x : l^2_\rho \to \mathbb{R}$, $T_x(y) = \sum_{n\ge1} x_n y_n$, $y \in l^2_\rho$, and so $T_x \in (l^2_\rho)'$. Viceversa, given $T \in (l^2_\rho)'$, by the Riesz theorem, there exists $z \in l^2_\rho$ such that $T(y) = \langle z, y\rangle_{l^2_\rho}$, $y \in l^2_\rho$. Setting $x = (z_n\rho_n^2) \in l^2_{1/\rho}$, one has $T = T_x$.

By considering sequences (β_j) of the form $(\beta_j) = (\gamma_j^\delta)$ one easily indicates Sobolev spaces of distributions in which the cylindrical Lévy process Z evolves and, at the same time, the Ornstein-Uhlenbeck process X has trajectories in $L^2(D)$.

For instance, assume that Z is a standard cylindrical α-stable process, that is $\beta_j = 1$, for any $j = (n_1, \ldots, n_d) \in \mathbb{N}^d$. If $p > 0$ and $\rho_j = \gamma_j^{p/2}$, then, by Proposition 3.3,

$$\sum_{j\in\mathbb{N}^d} (Z_t^j)^2 \gamma_j^{-p} < \infty, \quad t > 0, \quad \mathbb{P} - a.s.,$$

if and only if $\sum_{j\in\mathbb{N}^d} \gamma_j^{-\alpha p/2} < \infty$ and thus if and only if $\alpha p > d$. Consequently, $Z_t \in H^{-p}(D)$, $t > 0$, if and only if $p > \frac{d}{\alpha}$.

4.1 Time regularity of trajectories

If the cylindrical Lévy process Z in (4.1) takes values in the Hilbert space H then, by the Kotelenez regularity result (see [18, Theorem 9.20]) trajectories of the process X which solves (4.1) are càdlàg with values in H. However $Z_t \in H$, for any $t > 0$, if and only if

$$\sum_{k\geq 1} \beta_k^\alpha < \infty, \tag{4.11}$$

and this is a very restrictive assumption. We conjecture that the càdlàg property holds under much weaker conditions but, at the moment, we are able to establish a weaker time regularity of the solutions.

Theorem 4.4 *Assume (i), (ii) and (iii) in Hypothesis (L). Then the Ornstein-Uhlenbeck process $X = (X_t^x)$ satisfies*:

(i) for any $x \in H$, X is stochastically continuous;
(ii) for any $x \in H$, $T > 0$, X has trajectories in $L^p(0, T; H)$, for any $0 < p < \alpha$, $\mathbb{P}$-a.s.

Proof Let $0 < p < \alpha$. We set $Y_t = Z_A(t)$, $t \geq 0$, and first show that

$$\mathbb{E}|Y_t|^p \leq \tilde{c}_p \left(\sum_{n\geq 1} |\beta_n|^\alpha \frac{(1 - e^{-\alpha\gamma_n t})}{\alpha\gamma_n} \right)^{p/\alpha}, \tag{4.12}$$

where the constant $\tilde{c}_p$ depends on p and α. Recall that (X_t^x) and (Y_t) are defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. Consider a new probability space $(\Omega', \mathcal{F}', \mathbb{P}')$ where a Rademacher sequence (r_n) is defined (i.e., $r_n : \Omega' \to \{1, -1\}$ are independent and identically distributed with $\mathbb{P}'(r_n = 1) = \mathbb{P}'(r_n = -1) = 1/2$).

The following Khintchine inequality holds, for arbitrary real numbers $c_1, \ldots, c_n$, for any $p > 0$,

$$\left(\sum_{n\geq 1} c_n^2 \right)^{1/2} \leq c_p \left(\mathbb{E}' \left| \sum_{n\geq 1} r_n c_n \right|^p \right)^{1/p}$$

(see, for instance, [12]), where the constant c_p depends only on p (for $p = 1$, we have $c_1 = \sqrt{2}$) and $\mathbb{E}'$ indicates the expectation with respect to $\mathbb{P}'$.

We fix $\omega \in \Omega$, $t \geq 0$, and write

$$\left(\sum_{n\geq 1} |Y_t^n(\omega)|^2 \right)^{1/2} \leq c_p \left(\mathbb{E}' \left| \sum_{n\geq 1} r_n Y_t^n(\omega) \right|^p \right)^{1/p}.$$

Integrating with respect to ω and using the Fubini theorem on the product space $\Omega \times \Omega'$, we find

$$\mathbb{E}|Y_t|^p \le c_p^p\, \mathbb{E}\left[\mathbb{E}'\left|\sum_{n\ge1} r_n Y_t^n\right|^p\right] = c_p^p\, \mathbb{E}'\left[\mathbb{E}\left|\sum_{n\ge1} r_n Y_t^n\right|^p\right]$$
$$= c_p^p\, \mathbb{E}'\left[\mathbb{E}\left|\sum_{n\ge1} r_n \int_0^t e^{-\gamma_n(t-s)}\beta_n dZ_s^n\right|^p\right]. \tag{4.13}$$

For any $t \ge 0$, $\lambda \in \mathbb{R}$ (using that $|r_n| = 1$, $n \ge 1$, and (2.4)),

$$\mathbb{E}[e^{i\lambda \sum_{n\ge1} r_n Y_t^n}] = e^{-|\lambda|^\alpha \sum_{n\ge1} |\beta_n|^\alpha \int_0^t e^{-\alpha\gamma_n(t-s)}ds}.$$

Now we get assertion (4.12), taking into account (3.2).

(i) It is enough to show that, for any $\epsilon > 0$,

$$\lim_{h\to0^+} \sup_{t\ge0} \mathbb{P}(|Y_{t+h} - Y_t| > \epsilon) = 0. \tag{4.14}$$

Let us choose $p \in (0, \alpha)$. We have, using (4.9),

$$\mathbb{P}(|Y_{t+h} - Y_t| > \epsilon) \le \mathbb{P}\left(\left|e^{hA}Y_t - Y_t\right| > \frac{\epsilon}{2}\right) + \mathbb{P}\left(\left|\int_t^{t+h} e^{(t+h-s)A}dZ_s\right| > \frac{\epsilon}{2}\right)$$
$$\le 2^p \frac{\mathbb{E}|e^{hA}Y_t - Y_t|^p}{\epsilon^p} + 2^p \frac{\mathbb{E}|\int_0^h e^{sA}dZ_s|^p}{\epsilon^p} = I_1(t,h) + I_2(h).$$

But (see (4.12))

$$\mathbb{E}|Y_t|^p \le \tilde{c}_p \left(\sum_{n\ge1} |\beta_n|^\alpha \frac{(1 - e^{-\alpha\gamma_n t})}{\alpha\gamma_n}\right)^{p/\alpha}$$

and so

$$[I_2(h)]^{\alpha/p} \to 0, \quad \text{as } h \to 0^+.$$

Concerning I_1, we find, using again the Khintchine inequality,

$$|e^{hA}Y_t - Y_t| = \left(\sum_{n\ge1}\left|(e^{-\gamma_n h} - 1)Y_t^n\right|^2\right)^{1/2} \le c_p \left(\mathbb{E}'\left|\sum_{n\ge1} r_n(e^{-\gamma_n h} - 1)Y_t^n\right|^p\right)^{1/p}$$

and, reasoning as in (4.13) with β_n replaced by $(1-e^{-\gamma_n h})\beta_n$,

$$\begin{aligned}\mathbb{E}|e^{hA}Y_t - Y_t|^p &\le c_p^p\,\mathbb{E}'\mathbb{E}\left|\sum_{n\ge1} r_n(e^{-\gamma_n h}-1)Y_t^n\right|^p\\ &\le C_p\left(\sum_{n\ge1}|(1-e^{-\gamma_n h})\beta_n|^\alpha\frac{(1-e^{-\alpha\gamma_n t})}{\alpha\gamma_n}\right)^{p/\alpha}\\ &\le \frac{C_p}{\alpha^{p/\alpha}}\left(\sum_{n\ge1}\frac{|(1-e^{-\gamma_n h})\beta_n|^\alpha}{\gamma_n}\right)^{p/\alpha},\end{aligned}$$

$t\ge0$. Since

$$\lim_{h\to0^+}\left(\sum_{n\ge1}\frac{|(1-e^{-\gamma_n h})\beta_n|^\alpha}{\gamma_n}\right)^{p/\alpha}=0,$$

we get

$$\lim_{h\to0^+}\sup_{t\ge0} I_1(t,h)=0$$

and so assertion (4.14) is proved.

(ii) It is enough to show that

$$\mathbb{E}\int_0^T\left(\sum_{n\ge1}|Y_t^n|^2\right)^{p/2}dt<\infty, \tag{4.15}$$

where $Y_t = Z_A(t)$, $t\ge0$. Using (4.12) we get

$$\begin{aligned}\int_0^T\mathbb{E}|Y_t|^p dt &\le \tilde{c}_p\int_0^T\left(\sum_{n\ge1}|\beta_n|^\alpha\frac{(1-e^{-\alpha\gamma_n t})}{\alpha\gamma_n}\right)^{p/\alpha}dt\\ &\le C_{p,\alpha}\,T\left(\sum_{n\ge1}\frac{|\beta_n|^\alpha}{\gamma_n}\right)^{p/\alpha}<+\infty.\end{aligned}$$

The proof is complete. □

Remark 4.5 In the limiting Gaussian case of $\alpha=2$, the previous proof allows to get the well known result that trajectories of X are in $L^2(0,T;H)$, for any $T>0$.

Using that $X=(X_t^x)$, $x\in H$, is stochastically continuous and $\mathcal{F}_t$-adapted (see Theorem 4.4 and Proposition 4.10) we can apply [5, Proposition 3.6] and obtain

Corollary 4.6 *For any $x \in H$, the process (X_t^x) has a predictable version.*

For $p \in (0,1)$, $L^p(0,T;H)$ is a linear complete and separable metric space with respect to the distance $d_p(f,g) = \int_0^T |f(t) - g(t)|^p dt$, $f, g \in L^p(0,T;H)$. From Theorem 4.4 it is straightforward to obtain

Corollary 4.7 *Assume (i), (ii) and (iii) in Hypothesis (L). Then, for any $T > 0$, $x \in H$, $\mathbb{P}$-a.s., the Ornstein-Uhlenbeck process $X = (X_t^x)_{t\in[0,T]}$ is a random variable with values in $L^p(0,T;H)$, for any $0 < p < \alpha$.*

4.2 Support

We start with a preliminary one dimensional result.

Proposition 4.8 *Let $L = (L_t)$ be a one dimensional α-stable process, $\alpha \in (0,2)$. Let $\gamma \in \mathbb{R}$ and set*

$$K(t) = \int_0^t e^{-\gamma(t-s)} dL_s, \qquad t \geq 0. \tag{4.16}$$

Let $T > 0$ and consider $K = \{K(t)\}_{t\in[0,T]}$ as a random variable with values in $L^p(0,T)$. Then, for any $p > 0$, the random variable $(K, K(T))$ has full support in the product space $L^p(0,T) \times \mathbb{R}$.

The proposition is a consequence of the next result. This lemma could be deduced from results in [27, Section 2.1] on the support of Lévy processes with respect to the Skorokhod topology. However, we provide a self-contained, direct proof of independent interest.

Lemma 4.9 *Let $L = (L_t)$ be a real valued Lévy process with intensity measure ν (see (2.3)). Suppose that there exists $R > 0$ such that the support of ν contains interval $[-R, R]$. Then, for any $p > 0$, $T > 0$, the random variable (L, L_T) has full support in $L^p(0,T) \times \mathbb{R}$.*

Proof The proof is divided into some steps.

Step I. First, we set $\nu = 1_{[-R,R]}\nu + 1_{\mathbb{R}\setminus[-R,R]}\nu = \mu_0 + \mu_1$, where $\mu_0 = 1_{[-R,R]}\nu$ denotes the Borel measure such that $\mu_0(A) = \nu(A \cap [-R,R])$, for any Borel set $A \subset \mathbb{R}$. Then, we write $\nu = \nu_0 + \nu_1$, where

$$\nu_0 = \mu_1 + \Big(1 - (1 \wedge |x|^2)\Big)\mu_0 \quad \text{and} \quad \nu_1 = (1 \wedge |x|^2)\mu_0.$$

Note that ν_0 and ν_1 are both positive measures and moreover ν_1 is a finite measure with support which contains $[-R, R]$.

Let us introduce two independent Lévy processes $L^0 = (L_t^0)$ and $L^1 = (L_t^1)$. The exponent ψ of L^0 (see (2.3)) is the same as that of L but with the jump intensity measure ν replaced by ν_0. The exponent of L^1 is given in (2.3) with $Q = 0$, $a = 0$

and ν replaced by ν_1, i.e., L^1 is a compound Poisson process with intensity measure ν_1. By using the characteristic function and independence, we obtain that the process

$$\tilde{L} = L^1 + L^0, \quad \text{i.e.,} \;\; \tilde{L}_t = L^1_t + L^0_t, \quad t \geq 0,$$

has the same law of the initial Lévy process L. It follows that the law of (L, L_T) is the convolution of the laws of (L^0, L^0_T) and (L^1, L^1_T). Our assertion will follow from the fact that (L^1, L^1_T) has full support in $L^p(0, T) \times \mathbb{R}$.

Step II. We have to show that for fixed $\phi \in L^p(0, T)$, $a \in \mathbb{R}$ and $\epsilon > 0$,

$$\mathbb{P}\left(\int_0^T |L^1_t - \phi(t)|^p dt \; + \; |L^1_T - a| < \epsilon \right) > 0. \tag{4.17}$$

Since continuous functions, vanishing at $t = 0$ and taking value a at $t = T$, are dense in $L^p(0, T)$, for any $p > 0$, it follows that also piece-wise constant functions ϕ such that $0 = \phi(0^+) = \lim_{s \to 0^+} \phi(s)$ and $a = \phi(T^-) = \lim_{s \to T^-} \phi(s)$, and having jumps in absolute value less than R (i.e., for any $t \in (0, T)$, $|\triangle\phi(t)| < R$ with $\triangle\phi(t) = \phi(t^+) - \phi(t^-)$) are dense in $L^p(0, T)$, for any $p > 0$. Therefore, we may assume that ϕ in (4.17) satisfies the previous conditions.

Let the number of jumps of ϕ in the interval $[0, T]$ be k_0 and assume that the jumps occur exactly at $0 < t_1 < t_1 + t_2 < \cdots < t_1 + \cdots + t_{k_0} < T$. Let $x_1, \ldots, x_{k_0} \in \mathbb{R}$ be the sizes of jumps of ϕ, i.e., $x_i = \triangle\phi(t_1 + \cdots + t_i)$, $i = 1, \ldots, k_0$; we have $|x_i| < R$, $i = 1, \ldots, k_0$, and $\sum_{i=1}^{k_0} x_i = a$.

Step III. Let ϕ be as in Step II. For arbitrary $\delta > 0$ denote by $\mathcal{M}(\phi, \delta)$ the set of all piece-wise constant functions g, taking value 0 at $t = 0$, with k_0 moments of jumps occurring at $0 < s_1 < s_1 + s_2 < \cdots < s_1 + \cdots + s_{k_0} < T$, and with sizes of jumps $y_1, \ldots, y_{k_0} \in \mathbb{R}$, respectively, such that

$$|t_i - s_i| < \delta, \quad |x_i - y_i| < \delta, \quad i = 1, \ldots, k_0.$$

Let

$$M(\phi, \delta) = \sup_{g \in \mathcal{M}(\phi, \delta)} \left\{ \int_0^T |g(t) - \phi(t)|^p dt \right\}$$

One can easily show that $\lim_{\delta \to 0^+} M(\phi, \delta) = 0$.

Step IV. Let us prove (4.17) for ϕ as in Step II. Clearly, $t \mapsto L^1_t(\omega)$, $\mathbb{P}-$a.s., is a (càdlàg) piece-wise constant function on $[0, T]$, taking value 0 at $t = 0$. Let $0 < \tau_1 < \ldots$ be the consecutive moments of jumps for the process L^1. Set $\tau_0 = 0$ and

$$\xi_i = \tau_i - \tau_{i-1}, \quad Y_i = L^1_{\tau_i} - L^1_{\tau_i^-}, \quad i \geq 1.$$

Note that (ξ_i) and (Y_i) are independent random variables and moreover $\mathbb{P}(\xi_i \in (b, c)) > 0$, $\mathbb{P}(Y_i \in (d, e)) > 0$, for arbitrary $0 < b < c$, $-R \leq d < e \leq R$.

We consider $\delta > 0$ small enough such that $M(\phi, \delta) < \epsilon/2$ and $k_0\delta < \epsilon/2$. Then by Step III

$$\begin{aligned}
&\mathbb{P}\left(\int_0^T |L_t^1 - \phi(t)|^p dt \ + \ |L_T^1 - a| < \epsilon\right) \\
&\geq \mathbb{P}\left(\int_0^T |L_t^1 - \phi(t)|^p dt \ < \epsilon/2, \ \ |L_T^1 - a| < \epsilon/2\right) \\
&\geq \mathbb{P}(|\xi_i - t_i| < \delta, \ \ |Y_i - x_i| < \delta, \quad i = 1, \ldots, k_0, \ \xi_{k_0+1} > T) \\
&= \left[\prod_{i=1}^{k_0} \mathbb{P}(|\xi_i - t_i| < \delta)\, \mathbb{P}(|Y_i - x_i| < \delta)\right] \mathbb{P}(\xi_{k_0+1} > T) \ > 0.
\end{aligned}$$

The proof is complete. □

Proof (Proposition 4.8) We consider $\gamma \neq 0$ (the case $\gamma = 0$ follows from Lemma 4.9). Using [24, Theorem 3.1], we know that there exists an α-stable process $Z = (Z_t)$ such that

$$\int_0^t e^{\gamma s} dL_s = Z(h(t)), \quad \text{where } \ h(t) = \frac{e^{\alpha\gamma t} - 1}{\alpha\gamma}, \quad t \geq 0.$$

Consequently, $K(t) = e^{-\gamma t} Z(h(t))$ (see (4.16)). Assume $\gamma > 0$ (the case of $\gamma < 0$ can be treated similarly). Since continuous functions are dense in $L^p(0, T)$, for any $p > 0$, we fix a continuous function $\phi : [0, T] \to \mathbb{R}$, $a \in \mathbb{R}$ and $\epsilon > 0$ and prove that

$$b_\epsilon = \mathbb{P}\left(\int_0^T |e^{-\gamma t} Z(h(t)) - \phi(t)|^p dt \ + \ |e^{-\gamma T} Z(h(T)) - a| < \epsilon\right) > 0. \tag{4.18}$$

We have

$$\begin{aligned}
b_\epsilon \ &\geq \mathbb{P}\left(\int_0^T |Z(h(t)) - e^{\gamma t}\phi(t)|^p dt \ + \ |Z(h(T)) - e^{\gamma T} a| < \epsilon\right) \\
&= \mathbb{P}\left(\int_0^{h(T)} \frac{|Z(u) - (1 + u\alpha\gamma)^{1/\alpha}\, \phi(h^{-1}(u))|^p}{1 + u\alpha\gamma}\, du + |Z(h(T)) - e^{\gamma T} a| < \epsilon\right) \\
&\geq \mathbb{P}\left(\int_0^{h(T)} |Z(u) - (1 + u\alpha\gamma)^{1/\alpha}\, \phi(h^{-1}(u))|^p\, du + |Z(h(T)) - e^{\gamma T} a| < \epsilon\right)
\end{aligned}$$

and the last probability is positive, since by Lemma 4.9 the random variable $(Z, Z(h(T)))$ has full support in $L^p(0, h(T)) \times \mathbb{R}$. □

Theorem 4.10 *Assume (i), (ii) and (iii) in Hypothesis (L) and fix $T > 0$, $x \in H$ and $p \in (0, \alpha)$. Consider the Ornstein-Uhlenbeck process $X = (X_t^x)_{t\in[0,T]}$, solving (4.1). The support of the random variable $(X, X_T^x) : \Omega \to L^p(0, T; H) \times H$ is $L^p(0, T; H) \times H$.*

Proof Let us denote the components of X_t^x by X_t^n, $n \geq 1$ (see (4.5)). It is enough to prove that, for any $\epsilon > 0$, and for any $(\phi, a) \in L^p(0, T; H) \times H$, one has

$$\mathbb{P}\left(\int_0^T \left(\sum_{n\geq 1} |X_t^n - \phi_n(t)|^2\right)^{p/2} dt < \epsilon, \ \sum_{n\geq 1} |X_T^n - a_n|^2 < \epsilon\right) > 0.$$

By using a standard density argument, we may assume that (ϕ, a) is of the form

$$\phi(t) = \sum_{k=1}^N \phi_k(t) e_k, \quad a = \sum_{k=1}^N a_k e_k,$$

for some $N \in \mathbb{N}$. Using that $\left(\sum_{n=1}^N |X_t^n - \phi_n(t)|^2\right)^{p/2} \leq \sum_{n=1}^N |X_t^n - \phi_n(t)|^p, \quad t \in [0, T]$, since $p/2 < 1$, we write

$$\begin{aligned}
&\mathbb{P}\left(\int_0^T \left(\sum_{n=1}^N \left|X_t^n - \phi_n(t)\right|^2\right)^{p/2} dt < \epsilon, \ \sum_{n=1}^N \left|X_T^n - a_n\right|^2 < \epsilon\right) \\
&\geq \mathbb{P}\left(\int_0^T \sum_{n=1}^N \left|X_t^n - \phi_n(t)\right|^p dt < \epsilon, \ \sum_{n=1}^N \left|X_T^n - a_n\right|^2 < \epsilon\right) \\
&\geq \mathbb{P}\left(\int_0^T \left|X_t^1 - \phi_1(t)\right|^p dt < \epsilon/N, \ \left|X_T^1 - a_1\right|^2 < \epsilon/N\right) \cdots \\
&\cdots \mathbb{P}\left(\int_0^T \left|X_t^N - \phi_N(t)\right|^p dt < \epsilon/N, \ \left|X_T^N - a_N\right|^2 < \epsilon/N\right),
\end{aligned}$$

by independence. Thanks to Proposition 4.8 we know that the previous product of probabilities is positive. The proof is complete. □

By considering the second component of the random variable (X, X_T^x) appearing in Theorem 4.10 we get

Corollary 4.11 *Under (i), (ii) and (iii) in Hypothesis (L), for any $x \in H$, the OU process (X_t^x) is irreducible, i.e., for any open ball $B \subset H$, $t > 0$, we have $\mathbb{P}(X_t^x \in B) > 0$.*

4.3 Equivalence of transition probabilities

Here we will assume all conditions in Hypothesis (L).

Theorem 4.12 *Assume Hypothesis (L). Then the laws μ_t^x and μ_t^y of X_t^x and X_t^y, respectively, are equivalent, for any $t > 0$, $x, y \in H$, $\alpha \in (0, 2)$. Moreover the density $\frac{d\mu_t^x}{d\mu_t^y}$ of μ_t^x with respect to μ_t^y is given by*

$$\frac{d\mu_t^x}{d\mu_t^y} = \lim_{N\to\infty} \prod_{k=1}^{N} \frac{p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)}{p_\alpha\left(\frac{z_k - e^{-\gamma_k t} y_k}{c_k(t)}\right)} \quad in \quad L^1(\mu_t^y),$$

where $c_k(t) = \beta_k \left(\frac{1-e^{-\alpha\gamma_k t}}{\alpha\gamma_k}\right)^{1/\alpha}$ and p_α is the density of the one dimensional α-stable measure considered in (3.1).

If (4.2) does not hold then, for some $x \in H$, μ_t^x is not absolutely continuous with respect to μ_t^0 (the law of $X_t^0 = Z_A(t)$, see(4.5)).

Remark 4.13 If we assume (i)–(iii) in Hypothesis (L), then condition (iv) is sharp in the limiting Gaussian case of $\alpha = 2$. Indeed, under (i)–(iii) and $\alpha = 2$, condition (iv) is equivalent to each of the following facts:

(i) the laws of X_t^x and X_t^y are equivalent, for any $t > 0$, $x, y \in H$;
(ii) the Gaussian Ornstein-Uhlenbeck semigroup (R_t) associated to (X_t^x) is strong Feller (see [5, Section 9.4.1]).

In addition, under (i)–(iii) and $\alpha = 2$, the following regularizing property:

$$R_t f \in C_b^\infty(H), \quad t > 0, \quad f \in B_b(H),$$

holds if and only if $(e^{-\gamma_n t}\sqrt{\frac{\gamma_n}{\beta_n^2}})$ is a bounded sequence.

Proof (Theorem 4.12) Fix $x = (x_n)$ and $y = (y_n) \in l^2$. Let $Y_t = Z_A(t)$ and $p = p_\alpha$. Consider formulas (4.4), (4.6) and (4.8). The density of the random variable Y_t^k is clearly $\frac{1}{c_k(t)} p\left(\frac{z_k}{c_k(t)}\right)$ so that the density of X_t^k is $\frac{1}{c_k(t)} p\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)$. The measures μ_t^x and μ_t^y can be seen as Borel product measures in $\mathbb{R}^\infty$, i.e.,

$$\mu_t^x = \prod_{k\geq 1} \mu_t^{x_k}, \qquad \mu_t^y = \prod_{k\geq 1} \mu_t^{y_k}, \quad \text{where } \mu_t^{x_k},\ \mu_t^{y_k} \text{ have densities, respectively,}$$
$$\frac{1}{c_k(t)} p\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right) \quad \text{and} \quad \frac{1}{c_k(t)} p\left(\frac{z_k - e^{-\gamma_k t} y_k}{c_k(t)}\right).$$

To get the assertion we will apply Theorem 3.4. To this purpose, one checks that

$$\sum_{k\geq 1} \frac{e^{-2\gamma_k t}|x_k - y_k|^2}{c_k(t)^2} < \infty.$$

This follows easily from (4.2). If (4.2) does not hold, for some $t > 0$, then it is easy to see that there exists $\hat{x} = (\hat{x}_n) \in H$ such that

$$\sum_{k\geq 1} \frac{e^{-2\gamma_k t}\, \hat{x}_k^2}{c_k(t)^2} = +\infty.$$

According to Remark 3.5, this condition means that $\mu_t^{\hat{x}}$, the law of $X_t^{\hat{x}}$, is not absolutely continuous with respect to μ_t^0. □

4.4 Smoothing effect

We now consider the transition Markov semigroup (R_t) associated to (X_t^x), i.e. $R_t : B_b(H) \to B_b(H)$,

$$R_t f(x) = \mathbb{E}[f(X_t^x)], \quad x \in H, \quad f \in B_b(H), \quad t \geq 0.$$

The next result shows that (R_t) has a smoothing effect and that gradient estimates hold for it. In particular, this will imply the strong Feller property for (R_t). Recall that a Markov semigroup (P_t) acting on $B_b(H)$ is said to be *strong Feller,* if $P_t f \in C_b(H)$, for any $t > 0$ and $f \in B_b(H)$.

Theorem 4.14 *Assume Hypothesis (L). Then, for any $t > 0$, the transition semigroup (R_t) maps Borel and bounded functions into $C_b^1(H)$−functions. Moreover, for any $k \in H$ with $|k| \leq 1$, $f \in B_b(H)$, $t > 0$, we have*

$$\sup_{x\in H} |\langle DR_t f(x), k\rangle| \leq 8c_\alpha\, C_t \|f\|_0, \quad \textit{where} \quad C_t = \sup_{n\geq 1} \frac{e^{-\gamma_n t}\, \gamma_n^{1/\alpha}}{\beta_n} \tag{4.19}$$

(c_α *is defined in* (3.5)). *Finally, for any $t > 0$, $f \in C_b(H)$, $x = (x_n)$, $h = (h_n) \in H$, we have*

$$\langle DR_t f(x), h\rangle = \int_H f(e^{tA}x + y) \sum_{k\geq 1} \frac{p'_\alpha(\frac{y_k}{c_k(t)})}{p_\alpha(\frac{y_k}{c_k(t)})} \frac{e^{-\gamma_k t} h_k}{c_k(t)}\, \mu_t^0(dy), \tag{4.20}$$

where μ_t^0 is the law of $X_t^0 = Z_A(t)$.

Proof We fix $t > 0$. The proof is divided into some steps. By the first three steps, we will show that, for any $f \in C_b(H)$, $R_t f$ is Gâteaux differentiable at any $x \in H$ and moreover that equality (4.20) holds.
Step. I We assume that $f \in C_b(H)$ is cylindrical, i.e., it depends only on a finite numbers of coordinates. Identifying H with l^2 through the basis (e_n), we have

$$f(x) = \tilde{f}(x_1, \ldots, x_j), \quad x \in H, \tag{4.21}$$

for some $j \geq 1$, and $\tilde{f} : \mathbb{R}^j \to \mathbb{R}$ continuous and bounded. In this first step we also assume that $\tilde{f}$ has bounded support in $\mathbb{R}^j$.

Fix arbitrary $x, h \in H$. We want to show that there exists $D_h R_t f(x)$, the directional derivative of $R_t f$ at x, along the direction h. Set $h_N = \sum_{k=1}^N h_k e_k$ so that $h_N \to h$ in H. Since f is cylindrical and $\prod_{k\geq 1} \mu_t^{x_k}$ is a product measure, we get, for $m \geq \max(j, N)$,

$$R_t f(x) = \int_H f(y) \prod_{k\geq 1} \mu_t^{x_k}(dy) = \int_{\mathbb{R}^m} \tilde{f}(z) \prod_{k=1}^m p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right) \frac{1}{c_k(t)}\, dz_k.$$

Using our assumptions on $\tilde{f}$, it is not difficult to show that there exists

$$\begin{aligned} D_{h_N} R_t f(x) &= -\int_{\mathbb{R}^m} \tilde{f}(z) \left(\sum_{k=1}^N \frac{p'_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)}{p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \right) \cdot \\ &\quad \cdot \prod_{k=1}^m p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right) \frac{1}{c_k(t)}\, dz_k \\ &= -\int_H f(z) \left(\sum_{k=1}^N \frac{p'_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)}{p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \right) \prod_{k\geq 1} \mu_t^{x_k}(dz_k), \quad N \in \mathbb{N}. \end{aligned}$$

In order to pass to the limit, as $N \to \infty$, we show that

$$g_N(t, x) = \sum_{k=1}^N \frac{p'_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)}{p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \quad \text{converges in } L^2(\mu_t^x). \tag{4.22}$$

Note that, for $j \neq k$,

$$\begin{aligned} &\frac{e^{-\gamma_k t} h_k}{c_k(t)} \frac{e^{-\gamma_j t} h_j}{c_j(t)} \int_{\mathbb{R}^2} \frac{p'_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)}{p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)} \frac{p'_\alpha\left(\frac{z_j - e^{-\gamma_j t} x_j}{c_j(t)}\right)}{p_\alpha\left(\frac{z_j - e^{-\gamma_j t} x_j}{c_j(t)}\right)} \cdot \\ &\quad \cdot p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right) p_\alpha\left(\frac{z_j - e^{-\gamma_j t} x_j}{c_j(t)}\right) dz_k dz_j = 0. \end{aligned}$$

Indeed p'_α is odd since by (3.1) we deduce that p_α is even. We get, for $N, p \in \mathbb{N}$,

$$\int_H \left| \sum_{k=N}^{N+p} \frac{p'_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)}{p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \right|^2 \mu_t^x(dz)$$

$$= \int_{\mathbb{R}^{p+1}} \left| \sum_{k=N}^{N+p} \frac{p'_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)}{p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \right|^2 \prod_{k=N}^{N+p} p_\alpha\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right) \frac{1}{c_k(t)} dz_k$$

$$= \int_H \sum_{k=N}^{N+p} \frac{(p'_\alpha)^2\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)}{p_\alpha^2\left(\frac{z_k - e^{-\gamma_k t} x_k}{c_k(t)}\right)} \frac{e^{-2\gamma_k t} h_k^2}{c_k^2(t)} \mu_t^x(dz)$$

$$= \sum_{k=N}^{N+p} \frac{e^{-2\gamma_k t} h_k^2}{c_k^2(t)} \int_{\mathbb{R}} \frac{p_\alpha'^2(y_k)}{p_\alpha(y_k)} dy_k \le 8 c_\alpha C_t^2 \sum_{k=N}^{N+p} h_k^2,$$

where $8c_\alpha = \int_{\mathbb{R}} \frac{p_\alpha'^2(y)}{p_\alpha(y)} dy$ (see (3.5)). This proves (4.22).

Note that, for any $N \in \mathbb{N}$,

$$D_{h_N} R_t f(x) = -\int_H f(z + e^{tA} x) \left(\sum_{k=1}^{N} \frac{p'_\alpha\left(\frac{z_k}{c_k(t)}\right)}{p_\alpha\left(\frac{z_k}{c_k(t)}\right)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \right) \mu_t^0(dz).$$

Up to now we have showed that

$$\frac{R_t f(x + s h_N) - R_t f(x)}{s} = \frac{1}{s} \int_0^s D_{h_N} R_t f(x + r h_N) dr, \quad s \in (-1, 1). \tag{4.23}$$

Using also (4.22), it is not difficult to show that, for any $r \in (-1, 1)$, $N \in \mathbb{N}$,

$$\lim_{N \to \infty} D_{h_N} R_t f(x + r h_N) = -\int_H f(z + e^{tA}(x + rh)) \left(\sum_{k=1}^{\infty} \frac{p'_\alpha\left(\frac{z_k}{c_k(t)}\right)}{p_\alpha\left(\frac{z_k}{c_k(t)}\right)} \frac{e^{-\gamma_k t} h_k}{c_k(t)} \right) \mu_t^0(dz). \tag{4.24}$$

Moreover, $|D_{h_N} R_t f(x + r h_N)| \le 8 c_\alpha C_t |h| \|f\|_0$, for any $r \in (-1, 1)$. Thus we can pass to the limit, as $N \to \infty$, in (4.23) and get

$$\frac{R_t f(x + sh) - R_t f(x)}{s} = \frac{1}{s} \int_0^s u(t, x + rh)\, dr, \quad s \in (-1, 1), \tag{4.25}$$

where $u(t, x + rh)$ is the right-hand side of (4.24). This shows that $R_t f$ is Gâteaux differentiable at $x \in H$ along the direction h and moreover that (4.20) holds.

Step II. We consider $f \in C_b(H)$ which is only cylindrical (i.e., f is given by (4.21) but the function $\tilde{f}$ is not assumed to have bounded support in $\mathbb{R}^j$).

Define $\tilde{f}_n(y) = \tilde{f}(y)\phi(\frac{|y|}{n})$, for any $y \in \mathbb{R}^j$, where $\phi : [0, +\infty[\to \mathbb{R}_+$ is a continuous function such that, $\phi(s) = 1$, $s \in [0, 1]$, $\phi(s) = 0$, $s \geq 2$.

We have that $\|\tilde{f}_n\|_0 \leq \|\tilde{f}\|_0$, $n \in \mathbb{N}$, and moreover $\tilde{f}_n(y) \to \tilde{f}(y)$, as $n \to \infty$, for any $y \in \mathbb{R}^j$.

Let $f_n : H \to \mathbb{R}$, $f_n(x) = \tilde{f}_n(x_1, \ldots, x_j)$, for any $x \in H$, $n \in \mathbb{N}$.

We find by the previous step, for any $n \in \mathbb{N}$ and $x \in H$,

$$\frac{R_t f_n(x + sh) - R_t f_n(x)}{s} = \frac{1}{s} \int_0^s D_h R_t f_n(x + rh) dr, \quad s \in (-1, 1). \quad (4.26)$$

Passing to the limit, as $n \to \infty$, it is easy to see that (4.25) holds for f. This shows the Gâteaux differentiability of $R_t f$ on H and also the equality (4.20).

Step. III We consider an arbitrary $f \in C_b(H)$. Let us introduce the cylindrical functions g_n, $g_n(x) = f\left(\sum_{k=1}^n x_k e_k\right)$, $n \in \mathbb{N}$, $x \in H$.

It is clear that $\|g_n\|_0 \leq \|f\|_0$, $n \in \mathbb{N}$, and moreover $g_n(x) \to f(x)$, for any $x \in H$. Repeating the argument of the previous step, with f_n replaced by g_n, and passing to the limit, we get that the assertion of the previous step holds even for any $f \in C_b(H)$.

Step IV. Let $f \in C_b(H)$ and consider the Gâteaux derivative of $R_t f$ in $x \in H$

$$DR_t f(x) = \int_H f(e^{tA}x + y) \sum_{k \geq 1} \frac{p'_\alpha(\frac{y_k}{c_k(t)})}{p_\alpha(\frac{y_k}{c_k(t)})} \frac{e^{-\gamma_k t}}{c_k(t)} e_k \, \mu_t^0(dy).$$

By the dominated convergence theorem, $DR_t f : H \to H$ is continuous. This gives that *$R_t f$ is Fréchet differentiable at any $x \in H$.* Moreover, we have the required gradient estimate

$$\|DR_t f\|_0 \leq 8 c_\alpha C_t \|f\|_0$$

Step V. To finish the proof, take $g \in B_b(H)$. A well known argument (see [6, Lemma 7.1.5]) shows that $R_t g$ is Lipschitz continuous on H, for any $t > 0$. Then by the semigroup law: $R_t g = R_{t/2} R_{t/2} g$ and therefore $R_t g \in C_b^1(H)$, for any $t > 0$. The proof is complete. □

Remark 4.15 Under the assumptions of Theorem 4.14, one could show the following regularizing property $R_t f \in C_b^\infty(H)$, $t > 0$, $f \in B_b(H)$. This generalizes the well known smoothing property of the Gaussian Ornstein-Uhlenbeck semigroup (see Remark 4.13).

Remark 4.16 Theorem 4.12 can be also deduced from Theorem 4.14 and Corollary 4.11 if one applies the Hasminkii theorem (see [6, Proposition 4.1.1]).

5 Nonlinear stochastic PDEs

We pass now to nonlinear SPDEs of the form

$$dX_t = AX_t dt + F(X_t)dt + dZ_t, \qquad X_0 = x \in l^2 = H, \tag{5.1}$$

where $Z = (Z_t)$ is a cylindrical α-stable Lévy process. To study (5.1) we will assume Hypothesis (L) and also
Hypothesis (N)

(i) $F : H \to H$ is Lipschitz continuous and bounded.
(ii) There exists $\gamma \in (0, 1)$ and $C_1 > 0$ such that

$$\beta_n \geq C_1 \, \gamma_n^{(\frac{1}{\alpha} - \gamma)}, \quad n \geq 1. \tag{5.2}$$

Condition (ii) is only needed to get the strong Feller property for the associated Markov semigroup. This assumption is stronger than (iv) in Hypothesis (L). Indeed, assuming (i)–(iii) in Hypothesis (L), the present condition (ii) holds if and only there exist $\hat{c} > 0$, $\gamma \in (0, 1)$, such that

$$\sup_{n \geq 1} \frac{e^{-\gamma_n t} \, \gamma_n^{1/\alpha}}{\beta_n} \leq \frac{\hat{c}}{t^\gamma}, \qquad t > 0. \tag{5.3}$$

To see this let us first assume (5.3). By choosing $t = \frac{\gamma}{\gamma_n}$, we find (5.2), with $C_1 = \frac{\gamma^\gamma}{\hat{c}\, e^\gamma}$. Viceversa, assume (5.2). Since $t \mapsto t^\gamma e^{-\gamma_n t}$ has a maximum over $\mathbb{R}_+$ at $t_n = \frac{\gamma}{\gamma_n}$, formula (5.3) is equivalent to $t_n^\gamma e^{-\gamma_n t_n} \leq \frac{\hat{c}\, \beta_n}{\gamma_n^{1/\alpha}}$ and this is precisely (5.3) with $\hat{c} = \frac{C_1 \gamma^\gamma}{e^\gamma}$.

5.1 Existence, uniqueness and Markov property

We say that a predictable H-valued stochastic process $X = (X_t^x)$, depending on $x \in H$, is a *mild solution* to Eq. (5.1) if, for any $t \geq 0$, $x \in H$, it holds ($\mathbb{P} - a.s.$):

$$X_t^x = e^{tA} x + \int_0^t e^{(t-s)A} F(X_s^x) ds + Z_A(t), \quad \text{with } Z_A(t) = \int_0^t e^{(t-s)A} dZ_s, \tag{5.4}$$

see (4.5). In formula (5.4) we are considering a predictable version of the process $(Z_A(t))$ according to Corollary 4.6.

Note that, since F is bounded, the deterministic integral in (5.4) is a well defined continuous process. Moreover, as far as the regularity of trajectories is concerned, the mild solution will have the same regularity as $(Z_A(t))$. In particular, according to Theorem 4.4, any mild solution X will be stochastically continuous.

To show existence and uniqueness we need the following deterministic result which is not standard in the case $p \in (0, 1)$.

Recall that, for an arbitrary C_0-semigroup (e^{tA}) on H, there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|e^{tA}\|_{\mathcal{L}(H)} \leq Me^{\omega t}$, $t \geq 0$. Moreover, in the sequel, $Lip(F)$ will denote the Lipschitz constant of F.

Proposition 5.1 *Let $F : H \to H$ be Lipschitz continuous and bounded and $f \in L^p(0, T; H)$, for some $p > 0$. Let $A : D(A) \subset H \to H$ be the generator of a C_0-semigroup (e^{tA}).*

(i) For any $x \in H$, the equation

$$y(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(y(s) + f(s))ds \tag{5.5}$$

has a unique continuous solution $y : [0, T] \to H$.

(ii) There exists a constant $C = C(p, \omega, M, Lip(F), \|F\|_0) > 0$ such that for solutions y and $z \in C([0, T]; H)$, corresponding respectively to functions f, $g \in L^p(0, T; H)$ and to the same $x \in H$, we have the estimates

$$(a) \quad \|y - z\|_{C([0,T];H)} \leq C \left(\int_0^T |f(t) - g(t)|^p dt \right)^{1/p}, \quad p \geq 1;$$

$$(b) \quad \|y - z\|_{C([0,T];H)} \leq C \int_0^T |f(t) - g(t)|^p dt, \quad p \in (0, 1).$$

Proof Assertion (i) follows easily by a fixed point argument. Let us consider (ii). The proof of (ii) when $p \geq 1$ is an easy application of the Gronwall lemma. Thus we only prove (b).

We consider a family of equivalent norms $\|\cdot\|_\lambda$ on the Banach space $E = C([0, T]; H)$, for $\lambda \geq 0$,

$$\|h\|_\lambda = \sup_{t \in [0,T]} e^{-\lambda t} |h(t)|, \quad h \in E$$

(for $\lambda = 0$ we get the usual sup norm). For a fixed $f \in L^p(0, T; H)$, let us define the operator $K_f : E \to E$,

$$(K_f y)(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(y(s) + f(s))ds, \quad y \in E, \quad t \in [0, T].$$

We find for $\lambda > \omega$, for any $y, z \in E$,

$$\|K_f y - K_f z\|_\lambda \le C \sup_{t\in[0,T]} e^{-\lambda t} \int_0^t e^{\omega(t-s)} |y(s) - z(s)| ds$$
$$\le C\|y - z\|_\lambda \sup_{t\in[0,T]} \int_0^t e^{-(\lambda-\omega)(t-s)} ds \le \frac{C}{\lambda - \omega} \|y - z\|_\lambda.$$

Let us choose λ_0 large enough such that $c_0 = \frac{C}{\lambda_0 - \omega} < 1$. We have

$$\|K_f y - K_f z\|_{\lambda_0} \le c_0 \|y - z\|_{\lambda_0}, \quad y, z \in E. \tag{5.6}$$

Let now f and $g \in L^p(0, T; H)$. We get, for any $t \in [0, T]$, $y \in E$,

$$|(K_f y)(t) - (K_g y)(t)| \le M \int_0^t e^{\omega(t-s)} |F(y(s) + f(s)) - F(y(s) + g(s))| ds.$$

Since F is bounded and Lipschitz continuous, it is also Hölder continuous of order $p \in (0, 1)$ and we find

$$\|K_f y - K_g y\|_{\lambda_0} \le c M e^{\omega T} \int_0^T |f(s) - g(s)|^p ds.$$

If we have solutions y and z corresponding to f and g, then $y = K_f y$ and $z = K_g z$. We get

$$\|y - z\|_{\lambda_0} = \|K_f y - K_g z\|_{\lambda_0}$$
$$\le \|K_f y - K_f z\|_{\lambda_0} + \|K_f z - K_g z\|_{\lambda_0} \le C_T \int_0^T |f(s) - g(s)|^p ds + c_0 \|y - z\|_{\lambda_0}$$

and the assertion follows since $c_0 \in (0, 1)$. □

Remark 5.2 Clearly the previous result holds when F is only Lipschitz continuous and $f \in L^p(0, T; H)$ with $p \ge 1$.

Theorem 5.3 *Assume (i)–(iii) in Hypothesis (L) and (i) in Hypothesis (N). Then there exists a unique mild solution* (X_t^x) *to the Eq.* (5.1). *Moreover* (X_t^x) *is a Markov-Feller process.*

Proof Step 1. Existence and uniqueness. Uniqueness follows by the Gronwall lemma. Let us prove existence. By using Proposition 5.1 and Theorem 4.10 we find that, for

any $x \in H$, there exists a continuous $\mathcal{F}_t$-adapted process $(Y_t) = (Y_t^x)$ with values in H which solves $\mathbb{P}$-a.s.

$$Y_t = e^{tA}x + \int_0^t e^{(t-s)A} F(Y_s + Z_A(s))ds, \quad t \geq 0.$$

Let us define

$$X_t^x = Y_t^x + Z_A(t), \quad t \geq 0, \ x \in H$$

(where $Z_A(t)$ is given in (4.5)). Since $Z_A(t)$ is predictable it follows that $X = (X_t^x)$ is predictable as well. Clearly (X_t^x) is the unique mild solution.
Step 2. Markov property. The proof of the Markov property is more involved. Since the solution may not have càdlàg trajectories, we have to proceed differently from [5, Theorem 7.10].

For any measurable function $\psi : [0, T] \to H$, let $y(t)$ be the unique continuous function with values in H which solves the equation

$$y(t) = \int_0^t e^{(t-s)A} F(y(s) + \psi(s))ds.$$

Set $y(t) = y(t, \psi)$, $t \in [0, T]$, to stress the dependence on ψ. We have

$$\begin{aligned} y(t+h, \psi) &= \int_0^{t+h} e^{(t+h-s)A} F(y(s, \psi) + \psi(s))ds \\ &= e^{hA} \int_0^t e^{(t-s)A} F(y(s, \psi) + \psi(s))ds \\ &\quad + \int_t^{t+h} e^{(t+h-s)A} F(y(s, \psi) + \psi(s))ds \\ &= e^{hA}[y(t, \psi)] + \int_0^h e^{(h-s)A} F(y(t+s, \psi) \\ &\quad + \psi(t+s))ds, \quad t, t+h \in [0, T]. \end{aligned}$$

Define a new function on $[0, T-t]$,

$$v(\cdot, \psi) := y(t + \cdot, \psi) - e^{(\cdot)A}[y(t, \psi)].$$

We have

$$v(h,\psi)=\int_0^h e^{(h-s)A}F(v(s,\psi)+e^{sA}[y(t,\psi)]+\psi(t+s))ds,\quad h\in[0,T-t].$$

By uniqueness, $v(h,\psi)=y(h,e^{(\cdot)A}[y(t,\psi)]+\psi(t+\cdot))$, for any $h\in[0,T-t]$. Thus we get, for $t,t+h\in[0,T]$,

$$y\left(h,e^{(\cdot)A}[y(t,\psi)]+\psi(t+\cdot)\right)+e^{hA}[y(t,\psi)]=y(t+h,\psi). \tag{5.7}$$

Defining $u(t,\psi)=y(t,\psi)+\psi(t)$, $t\in[0,T]$, we find

$$\begin{aligned}u(t+h,\psi)-\psi(t+h)=u\left(h,e^{(\cdot)A}[u(t,\psi)-\psi(t)]+\psi(t+\cdot)\right)\\ -e^{hA}[u(t,\psi)-\psi(t)]-\psi(t+h)+e^{hA}[u(t,\psi)-\psi(t)]\end{aligned}$$

and so we get

$$u(t+h,\psi)=u(h,e^{(\cdot)A}[u(t,\psi)-\psi(t)]+\psi(t+\cdot))$$

which is the same formula at the end of [5, page 256]. Now the Markov property follows arguing as in [5, page 257].
Step 3. Feller property. Fix $x,y\in H$. From the estimate,

$$|X_t^x-X_t^y|\le\|e^{tA}\|_{\mathcal{L}(H)}\,|x-y|+\int_0^t\|e^{(t-s)A}\|_{\mathcal{L}(H)}\,|F(X_s^x)-F(X_s^y)|ds, \tag{5.8}$$

using the Lipschitz continuity of F and the Gronwall lemma, we find that, for any $T>0$, $|X_t^x-X_t^y|\le M_T|x-y|$, $t\in[0,T]$, $x,y\in H$, $\mathbb{P}$-a.s.. The Feller property (i.e., the mapping $x\mapsto\mathbb{E}[f(X_t^x)]$ is continuous on H, for any $f\in C_b(H)$, $t\ge 0$) follows easily. □

5.2 Irreducibility

We establish now irreducibility of the solutions to (5.1).

Theorem 5.4 *Assume (i)–(iii) in Hypothesis (L) and (i) in Hypothesis (N). Then, for any $x\in H$, the mild solution $X=(X_t^x)$ to the Eq. (5.1) is irreducible.*

Proof Fix $x\in H$, $T>0$, and denote by $X=(X_t)$ the solution to (5.1) starting from x. Set

$$Y_t=X_t-Z_A(t),\quad t\in[0,T],$$

where

$$\begin{cases} dZ_A(t) = AZ_A(t)dt + dZ_t, \\ Z_A(0) = 0, \quad t \geq 0. \end{cases} \tag{5.9}$$

Note that

$$Y_t = e^{tA}x + \int_0^t e^{(t-s)A} F(Y_s + Z_A(s))ds.$$

Let z^u and $y^{u,x}$ be the solutions, driven by a control function $u \in L^2(0, T; H)$, of the following control systems, respectively,

$$\begin{cases} \dfrac{dz}{dt} = Az(t) + u(t), \\ z(0) = 0, \quad t \in [0, T], \end{cases} \qquad \begin{cases} \dfrac{dy}{dt} = Ay(t) + F(y(t)) + u(t), \\ y(0) = x \in H, \quad t \in [0, T]. \end{cases} \tag{5.10}$$

Thus

$$z^u(t) = \int_0^t e^{(t-s)A} u(s)ds, \quad t \in [0, T], \tag{5.11}$$

and $y^{u,x}$ is the solution of the following integral equation

$$y(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(y(s))ds + z^u(t), \quad t \in [0, T].$$

By Theorem 7.4.2 of [6] we know that the second system in (5.10) is approximately controllable at time $T > 0$ in the sense that, for any $x, a \in H$ and for any $\epsilon > 0$, there exists a control function $u \in L^2(0, T; H)$ such that $|y^{u,x}(T) - a| < \epsilon$.

Let

$$\bar{y}(t) = y^{u,x}(t) - z^u(t), \quad t \in [0, T].$$

Note that

$$\bar{y}(t) = e^{tA}x + \int_0^t e^{(t-s)A} F(\bar{y}(s) + z^u(s))ds.$$

Take $p < \alpha$ with $p \in (0, 1)$. By estimate (b) in Proposition 5.1 we get, $\mathbb{P}$-a.s.,

$$\sup_{t\in[0,T]} |Y_t - \bar{y}(t)| \le C \int_0^T |Z_A(t) - z^u(t)|^p dt.$$

and so $|Y_T - \bar{y}(T)| \le C \int_0^T |Z_A(t) - z^u(t)|^p dt$ or, equivalently,

$$|X_T - Z_A(T) - y^{u,x}(T) + z^u(T)| \le C \int_0^T |Z_A(t) - z^u(t)|^p dt.$$

We write, for any $a \in H$,

$$\begin{aligned} |X_T - a| &\le |X_T - Z_A(T) - y^{u,x}(T) + z^u(T)| + |Z_A(T) + y^{u,x}(T) - z^u(T) - a| \\ &\le C \int_0^T |Z_A(t) - z^u(t)|^p dt + |y^{u,x}(T) - a| + |Z_A(T) - z^u(T)| = I_1 + I_2 + I_3. \end{aligned}$$

For a given $\epsilon > 0$, let us fix a control function u such that $I_2 = |y^{u,x}(T) - a| < \epsilon/3$. Using Proposition 4.10, we get with positive probability that $I_1 < \epsilon/3$ and $I_3 < \epsilon/3$. The result follows. □

5.3 Strong Feller property

Let (P_t) be the Markov semigroup associated to $X = (X_t^x)$, i.e. $P_t : B_b(H) \to B_b(H)$,

$$P_t f(x) = \mathbb{E}[f(X_t^x)], \quad x \in H, \quad f \in B_b(H), \quad t \ge 0. \tag{5.12}$$

To show the strong Feller property of (P_t), we will assume all conditions in Hypotheses (L) and (N). Before stating our theorem on the strong Feller property, we discuss a motivating example.

Example 5.5 Consider the following non-linear version of the stochastic heat equation on $D = [0, \pi]^d$ treated in Example 4.3:

$$\begin{cases} dX(t,\xi) = \triangle X(t,\xi)\, dt + f(X(t,\xi))dt + dZ(t,\xi), \quad t > 0, \\ X(0,\xi) = x(\xi), \quad \xi \in D, \\ X(t,\xi) = 0, \quad t > 0, \quad \xi \in \partial D, \end{cases} \tag{5.13}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a bounded and Lipschitz continuous function and $Z = (Z_t)$ is a cylindrical α-stable process $Z_t = \sum_{j=(n_1,\dots,n_d)\in\mathbb{N}^d} \beta_j Z_t^j e_j, \quad t \ge 0$, where

(e_j) is the basis of eigenfunctions of the Laplacian Δ in $H = L^2(D)$ (with Dirichlet boundary conditions). Thus if

$$\sum_{j\in\mathbb{N}^d} \beta_j^\alpha < +\infty \quad \text{and} \quad \beta_{(n_1,\dots,n_d)} \geq c(n_1^2 + \cdots + n_d^2)^{\frac{1}{\alpha}-\gamma}, \quad (n_1,\dots,n_d) \in \mathbb{N}^d, \tag{5.14}$$

for some constants $c > 0$ and $\gamma \in (0, 1)$, then, by Theorems 5.4 and 5.7, the solution to (5.13) is irreducible and strong Feller. By the Doob theorem (see [6, Theorem 4.2.1]) if there exists an invariant measure for (5.13) then it must be unique. In particular, if $\beta_{(n_1,\dots,n_d)} = (n_1^2 + \cdots + n_d^2)^{1/\alpha-\gamma}, \quad (n_1,\dots,n_d) \in \mathbb{N}^d$, then (5.14) is equivalent to

$$\sum_{(n_1,\dots,n_d)\in\mathbb{N}^d} (n_1^2 + \cdots + n_d^2)^{1-\alpha\gamma} < +\infty.$$

This holds if and only if $\alpha\gamma > \frac{d}{2} + 1$. Thus if $d = 1$, one requires that $\alpha\gamma > \frac{3}{2}$.

Remark 5.6 The above example shows that the strong Feller property holds in rather special situation. It seems thus of great interest to develop the concept of asymptotic strong Feller property in the case of SPDEs with Lévy noise (compare with [13–15]).

Theorem 5.7 *Assume that Hypotheses (L) and (N) hold. Then, for any $t > 0$, the transition semigroup (P_t) (see (5.12)) maps Borel and bounded functions into Lipschiz continuous functions. Moreover, there exists $\tilde{C} = \tilde{C}(\gamma, c_\alpha, C_1, \|F\|_0) > 0$, such that, for any $x, y \in H$, we have*

$$|P_t f(x) - P_t f(y)| \leq \tilde{C}\|f\|_0 \frac{1}{t^\gamma \wedge 1}|x - y|, \qquad t > 0, \quad f \in B_b(H). \tag{5.15}$$

Recall that $t^\gamma \wedge 1 = \min(t^\gamma, 1)$. Note that it is enough to prove estimate (5.15), for any $t \in (0, 1]$. Indeed, when $t > 1$, we can replace $|P_t f(x) - P_t f(y)|$ in (5.15) with $|P_1(P_{t-1} f)(x) - P_1(P_{t-1} f)(y)|$.

To prove the result we first investigate generalised solutions to the Kolmogorov equation associated to (P_t) (or to (X_t^x)) as in [5, Section 9.4.2].

Note that the generator $\mathcal{A}_0$ of (P_t) is formally given by

$$\mathcal{A}_0 f(x) = \langle Ax + F(x), Df(x)\rangle + \sum_{n\geq 1} \beta_n^\alpha \int_{\mathbb{R}} (f(x + e_n z) - f(e_n z)) \frac{1}{|z|^{1+\alpha}} dz, \tag{5.16}$$

for regular and cylindrical functions $f : H \to \mathbb{R}$. The associated Kolmogorov equation is

$$\begin{cases} \partial_t u(t, x) = \mathcal{A}_0 u(t, \cdot)(x), \quad t > 0, \ x \in H, \\ u(0, x) = f(x), \quad x \in H. \end{cases} \tag{5.17}$$

Let us fix $T > 0$ and consider the space

$$\Lambda(0,T) = \{u \in C(]0,T]; C_b^1(H)) \quad \sup_{t\in]0,T]} t^\gamma \|u(t,\cdot)\|_1 < \infty\},$$

where $\|u(t,\cdot)\|_1 = \|u(t,\cdot)\|_0 + \|D_x u(t,\cdot)\|_0$ and $\gamma \in (0,1)$ is fixed in (ii) of Hypothesis (N). According to [5, Section 9.4] a *mild solution* to the Kolmogorov equation (5.17) (on $[0,T]$ with initial datum $f \in B_b(H)$) is a function $u \in \Lambda(0,T)$ such that

$$u(t,x) = R_t f(x) + \int_0^t R_{t-s}\left(\langle F(\cdot), Du(s,\cdot)\rangle\right)(x)\, ds, \quad t \in [0,T], \ x \in H, \tag{5.18}$$

where $D = D_x$ and (R_t) is the transition semigroup determined by the linear equation (4.1). To stress the dependence on f, we will also write

$$u = u(t,x) = u^f(t,x), \quad t \in [0,T], \quad x \in H.$$

Note that using Theorem 4.14 and (5.3), we get, for any $f \in B_b(H)$,

$$\|DR_t f\|_0 \le \frac{C_0}{t^\gamma}\|f\|_0, \quad t > 0, \quad \text{where } C_0 = 8c_\alpha\,\hat{c}. \tag{5.19}$$

Thanks to (5.19), we can adapt the proof of [5, Theorem 9.24] and obtain that the mapping $S : \Lambda(0,T) \to \Lambda(0,T)$,

$$S(u)(t,x) = R_t f(x) + \int_0^t R_{t-s}\left(\langle F(\cdot), Du(s,\cdot)\rangle\right)(x) ds, \quad u \in \Lambda(0,T), \tag{5.20}$$

is a contraction for T small enough. Therefore, we obtain

Proposition 5.8 *For any $f \in B_b(H)$, $T > 0$, there exists a unique mild solution $u = u^f$ to (5.17). Moreover, for any $t \ge 0$, we may define*:

$$\tilde{P}_t f(\cdot) := u^f(t,\cdot), \quad f \in B_b(H).$$

It turns out that $(\tilde{P}_t)$ is a semigroup of bounded linear operators on $B_b(H)$.

Proof We only note that the semigroup property follows from an argument based on uniqueness of solutions (see [5, page 270]). □

In the proof of the next lemma, we will use the following Gronwall type lemma. Let a, b, γ be non-negative constants, with $\gamma < 1$. Let $T > 0$. For any integrable function $v : [0, T] \to \mathbb{R}$,

$$0 \le v(t) \le at^{-\gamma} + b\int_0^t (t-s)^{-\gamma} v(s)ds, \;\; t \in [0, T[\text{ a.e., implies } \;\; v(t) \le aMt^{-\gamma}, \tag{5.21}$$

$t \in [0, T[$, a.e.. (where $M = M(b, \gamma, T)1 + b\, k_\gamma\, T^{1-\gamma}$).

Lemma 5.9 *For any $T > 0$, there exists $c = c(\gamma, c_\alpha, C_1, \|F\|_0, T) > 0$ such that, for any $f \in B_b(H), t \in]0, T]$,*

$$\|D\tilde{P}_t f\|_0 \le \frac{c}{t^\gamma}\|f\|_0.$$

Proof We have

$$Du(t, x) = DR_t f(x) + \int_0^t DR_{t-s}\left(\langle F(\cdot), Du(s, \cdot)\rangle\right)(x)\, ds, \quad x \in H.$$

By using (5.19) and the previous Gronwall lemma, we get

$$\|Du(t, \cdot)\|_0 \le \frac{C_0 M}{t^\gamma}\|f\|_0, \;\; t \in]0, T], \quad M = M(\gamma, c_\alpha, \hat{c}, \|F\|_0, T) > 0.$$

□

Galerkin's approximation. To show the regularizing effect of (P_t), according to [5, Theorem 9.27], it would be enough to prove that (P_t) and $(\tilde{P}_t)$ coincide. However the proof of [5, Theorem 9.27] is not complete and we are unable to fill the gap in our situation. We therefore resort to Galerkin's approximations and we will only identify suitable finite-dimensional semigroups which approximate (P_t) and $(\tilde{P}_t)$ respectively. Let us consider orthogonal projections $\pi_n : H \to H_n$, $n \in \mathbb{N}$, where H_n is the subspace of H generated by $\{e_1, \ldots, e_n\}$. For any $n \in \mathbb{N}$, $x \in H$, define the H_n-valued process $(Y_t^n) = (Y_t^n(x))$ as the unique mild solution to

$$Y_t^n = e^{tA_n}x + \int_0^t e^{(t-s)A_n}(\pi_n \circ F \circ \pi_n)(Y_s^n)ds + Z_{A_n}(t), \tag{5.22}$$

where $A_n = \pi_n \circ A$. Let $F_n = \pi_n \circ F \circ \pi_n$. Note that, for any $n \in \mathbb{N}$, it holds:

$$\|F_n\|_0 \le \|F\|_0, \quad Lip(F_n) \le Lip(F), \tag{5.23}$$

where $Lip(F_n)$ denotes the Lipschitz constant of F_n.

Consider the mild solution u_n to the Kolmogorov equation corresponding to Y_t^n, i.e.,

$$u_n(t,x) = u_n^f(t,x) = R_t^n f(x) + \int_0^t R_{t-s}^n \left(\langle F_n(\cdot), Du_n(s,\cdot)\rangle\right)(x)ds, \quad x \in H,$$
$$\text{where } \; R_t^n f(x) = \mathbb{E}[f(e^{tA_n}x + \pi_n Z_A(t))] = \int_H f(e^{tA_n}x + \pi_n y)\mu_t^0(dy). \tag{5.24}$$

Define the following two approximating semigroups on $B_b(H)$ (see (5.22) and (5.24)):

$$P_t^n f(x) = \mathbb{E}[f(Y_t^n(x))], \quad \tilde{P}_t^n f(x) = u_n^f(t,x), \quad f \in B_b(H), \tag{5.25}$$

Lemma 5.10 *For any function $f \in B_b(H)$, $n \in \mathbb{N}$, we have*

$$P_t^n f = \tilde{P}_t^n f, \quad t \geq 0.$$

Proof We fix $n \in \mathbb{N}$. It is enough to prove the assertion for any cylindrical function $f \in B_b(H)$, which depends only on the first n-coordinates. Identifying (P_t^n) and $(\tilde{P}_t^n)$ with the corresponding semigroups acting on $B_b(\mathbb{R}^n)$ (F_n with the corresponding function from $\mathbb{R}^n$ into $\mathbb{R}^n$), we have to check that

$$P_t^n f = \tilde{P}_t^n f, \quad f \in B_b(\mathbb{R}^n), \quad t \geq 0. \tag{5.26}$$

First note that (P_t^n) is a strongly continuous semigroup on the space $C_0(\mathbb{R}^n)$ of real, continuous functions on $\mathbb{R}^n$, vanishing at infinity and endowed with the supremum norm. We could not find a precise reference for this result and so we decided to give a sketch of the proof (note that [2, Theorem 6.7.2] requires boundedness of the drift term). To prove that $P_t^n(C_0(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$, let us fix $f \in C_0(\mathbb{R}^n)$ and $t > 0$. We have by the dominated convergence theorem (since F_n is a bounded function)

$$\lim_{x\in\mathbb{R}^n,\, x\to\infty} P_t^n f(x) = \lim_{x\to\infty} \mathbb{E}\left[f\left(e^{tA_n}x + \int_0^t e^{(t-s)A_n} F_n(Y_s^n)ds + Z_{A_n}(t) \right)\right] = 0.$$

To show that, for any $f \in C_0(\mathbb{R}^n)$, $P_t^n f$ tends to f in $C_0(\mathbb{R}^n)$ as $t \to 0^+$, we write

$$P_t^n f(x) - f(x) = a_t(x) + b_t(x),$$

where $a_t(x) = P_t^n f(x) - f(e^{tA_n}x)$ and $b_t(x) = f(e^{tA_n}x) - f(x)$. One checks that $b_t(x) \to f(x)$, as $t \to 0^+$, uniformly in $x \in \mathbb{R}^n$ (using that f vanishes at infinity).

Moreover, for any $x \in \mathbb{R}^n$,

$$|a_t(x)| \leq \mathbb{E}\left| f\left(e^{tA_n}x + \int_0^t e^{(t-s)A_n} F_n(Y_s^n)ds + Z_{A_n}(t)\right) - f(e^{tA_n}x)\right|.$$

Using the uniform continuity of f, the boundedness of F_n and the stochastic continuity of $(Z_{A_n}(t))$, one obtains that $a_t(x) \to 0$, as $t \to 0^+$, uniformly in $x \in \mathbb{R}^n$. This proves the assertion.

Let us consider now $(\tilde{P}_t^n)$. We start to show that $\tilde{P}_t^n(C_0(\mathbb{R}^n)) \subset C_0(\mathbb{R}^n)$, $t \geq 0$. Fix $T > 0$ and let $f \in C_0(\mathbb{R}^n)$ and $t \in]0, T]$; we will use an inductive argument to prove that $\tilde{P}_t f \in C_0(\mathbb{R}^n)$. By (5.20), we know that

$$\tilde{P}_t^n f = \lim_{m\to\infty} S^m(0) = \lim_{m\to\infty} (S \circ \cdots \circ S)(0) \quad \text{in}$$
$$\Lambda(0, T) = \{u \in C(]0, T]; C_b^1(\mathbb{R}^n)) : \sup_{t\in]0,T]} t^\gamma \|u(t, \cdot)\|_1 < \infty\}.$$

We prove that, for any $m \in \mathbb{N}$, $S^m(0)(t, \cdot)$ and $D_x S^m(0)(t, \cdot) \in C_0(\mathbb{R}^n)$. We have (for $m = 1$) $S^1(0)(t, \cdot)(x) = R_t f(x)$, and so

$$D_x S^1(0)(t, \cdot)(x) = DR_t^n f(x) = \int_{\mathbb{R}^n} f(e^{tA_n}x + y)\, U_n(y, t)\, \mu_t^n(dy), \quad x \in \mathbb{R}^n, \text{ where}$$

$$\mu_t^n \text{ has density } \prod_{k=1}^n p_\alpha\left(\frac{y_k}{c_k(t)}\right)\frac{1}{c_k(t)}$$

$$\text{and } U_n(y, t) = \sum_{k=1}^n \frac{p'_\alpha(\frac{y_k}{c_k(t)})}{p_\alpha(\frac{y_k}{c_k(t)})}\frac{e^{-\gamma_k t}}{c_k(t)} e_k \in L^2(\mu_t^n; \mathbb{R}^n).$$

It follows easily that $S^1(0)(t, \cdot)$ and $D_x S^1(0)(t, \cdot) \in C_0(\mathbb{R}^n)$. Assume that the assertion holds for an arbitrary $m \in \mathbb{N}$. Since

$$S^{m+1}(0)(t, \cdot)(x) = R_t^n f(x) + \int_0^t R_{t-s}^n \left(\langle F_n(\cdot), DS^m(0)(s, \cdot)\rangle\right)(x)\, ds,$$

$$D_x S^{m+1}(0)(t, \cdot)(x) = DR_t^n f(x) + \int_0^t DR_{t-s}^n \left(\langle F_n(\cdot), DS^m(0)(s, \cdot)\rangle\right)(x)\, ds\, DR_t^n f(x)$$
$$+ \int_0^t ds \int_{\mathbb{R}^n} \langle F_n(e^{(t-s)A_n}x + y), DS^m(0)$$
$$(s, e^{(t-s)A_n}x + y)\rangle\, U_n(y, t-s)\mu_{t-s}^n(dy),$$

$x \in \mathbb{R}^n$, we have easily that the assertion holds also for $m + 1$.

Using Lemma 5.9, we get that $(\tilde{P}_t^n)$ is a strongly continuous semigroup of bounded linear operators on $C_0(\mathbb{R}^n)$.

We will prove (5.26) when $f \in C_0(\mathbb{R}^n)$. Indeed, by a standard argument (see [9, Chapter 4]) this is enough to get (5.26).

By Ito formula $D_0 = C_0^2(\mathbb{R}^n) = \{f \in C_0(\mathbb{R}^n) \ : \ Df \ and \ D^2 f \in C_0(\mathbb{R}^n)\}$ is invariant for (P_t^n) (compare with [2, Theorem 6.7.4]). Moreover, $D_0 \subset$dom $(\mathcal{A}_n)$, where $\mathcal{A}_n$ is the generator of (P_t^n). By a well known result, D_0 is a core for (P_t^n) (see [8, page 52]). Note that

$$\mathcal{A}_n f(x) = \langle A_n x + F_n(x), Df(x)\rangle + \sum_{k=1}^{n} \beta_k^\alpha \int_{\mathbb{R}} (f(x+e_k z) - f(e_k z)) \frac{1}{|z|^{1+\alpha}} dz,$$

$f \in D_0$. Let us consider $(\tilde{P}_t^n)$. If $f \in D_0$, we can solve (by the contraction principle)

$$u(t,x) = R_t^n f(x) + \int_0^t R_{t-s}^n \left(\langle F_n(\cdot), Du(s,\cdot)\rangle\right)(x) ds, \quad x \in \mathbb{R}^n,$$

in the space $C([0,T]; C_0^2(\mathbb{R}^n))$ and get that D_0 is also invariant for $(\tilde{P}_t^n)$. A straightforward calculation, shows that $D_0 \subset$dom $(\tilde{\mathcal{A}}_n)$, where $\tilde{\mathcal{A}}_n$ is the generator of $(\tilde{P}_t^n)$. Thus D_0 is a core also for $(\tilde{P}_t^n)$. Moreover, $\tilde{\mathcal{A}}_n$ coincides with $\mathcal{A}_n$ on D_0. It follows that (P_t^n) and $(\tilde{P}_t^n)$ coincide on $C_0(\mathbb{R}^n)$ and this finishes the proof. □

Proof (Theorem 5.7)
I Step. Using (5.24), Lemmas 5.9 and 5.10 and the semigroup property, there exists $\tilde{C} = \tilde{C}(\gamma, c_\alpha, C_1, \|F\|_0) > 0$ such that, for any $f \in C_b(H)$,

$$\begin{aligned} &|u_n(t,x) - u_n(t,y)| = |P_t^n f(x) - P_t^n f(y)| \le |R_t^n f(x) - R_t^n f(y)| \\ &+ \int_0^t |R_{t-s}^n \left(\langle F_n(\cdot), Du_n(s,\cdot)\rangle\right)(x) - R_{t-s}^n \left(\langle F_n(\cdot), Du_n(s,\cdot)\rangle\right)(y)| ds \\ &\le \tilde{C} \|f\|_0 \frac{1}{t^\gamma \wedge 1} |x-y|, \quad x, y \in H, \quad n \in \mathbb{N}, \ t > 0. \end{aligned} \tag{5.27}$$

II Step. For any $f \in C_b(H)$, we have:

$$\lim_{n\to\infty} P_t^n f(x) = P_t f(x), \quad x \in H, \quad t \ge 0.$$

Recall that $P_t^n f(x) = \mathbb{E}[f(Y_t^n(x))]$ (see (5.22)). The assertion will follow by proving that

$$\lim_{n\to\infty} Y_t^n(x) = X_t^x, \quad x \in H, \quad t \ge 0, \quad \mathbb{P} - a.s. \tag{5.28}$$

To show (5.28), we fix $x \in H$ and write

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A} F_n(X_s^x)ds + Z_A(t) + f_n(t), \quad \text{where}$$

$$f_n(t) = \int_0^t e^{(t-s)A}[F(X_s^x) - F_n(X_s^x)]ds, \quad t \geq 0, \ n \in \mathbb{N}.$$

(see the notation in (5.22)). Defining $U_t^n = X_t^x - Z_A(t) - f_n(t)$, we have:

$$U_t^n = e^{tA}x + \int_0^t e^{(t-s)A} F_n(U_s^n + Z_A(s) + f_n(s))ds.$$

Note that

$$Y_t^n(x) = e^{tA}x + \int_0^t e^{(t-s)A} F_n(Y_s^n(x))ds + Z_{A_n}(t) + g_n(t), \quad \text{where}$$

$$g_n(t) = e^{tA_n}x - e^{tA}x, \quad t \geq 0, \quad n \in \mathbb{N}.$$

Introducing $V_t^n = Y_t^n(x) - Z_{A_n}(t) - g_n(t)$, we find

$$V_t^n = e^{tA}x + \int_0^t e^{(t-s)A} F_n(V_s^n + Z_{A_n}(s) + g_n(s))ds.$$

Now to estimate $|U_t^n - V_t^n|$, we use (b) in Proposition 5.1. Let us choose $p \in (0, \alpha) \cap (0, 1)$ and fix any $T > 0$. We have

$$\sup_{t\in[0,T]} |U_t^n - V_t^n| \leq C \int_0^T |Z_A(s) + f_n(s) - Z_{A_n}(s) - g_n(s)|^p ds. \tag{5.29}$$

Note that $Z_{A_n}(t) = \pi_n Z_A(t)$, $n \in \mathbb{N}$. Moreover, it is easy to see that

$$\lim_{n\to\infty} (|f_n(t)| + |g_n(t)|) = 0,$$

for any $t \geq 0$. Applying the dominated convergence theorem in (5.29), we infer

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |U_t^n - V_t^n| = 0.$$

Using the inequality

$$|Y_t^n(x) - X_t^x| \le |Y_t^n(x) - Z_{A_n}(t) - g_n(t) - X_t^x + Z_A(t) + f_n(t)| \\ +|Z_{A_n}(t) + g_n(t) - Z_A(t) - f_n(t)|,$$

$t \ge 0, n \in \mathbb{N}$, and passing to the limit as $n \to \infty$, we get assertion (5.28).
III Step. By the previous steps we know that, for any $f \in C_b(H)$,

$$|P_t f(x) - P_t f(y)| \le \tilde{C} \|f\|_0 \frac{1}{t^\gamma \wedge 1} |x - y|, \quad x, y \in H, \quad t > 0.$$

Now we get the assertion, using that

$$\mathrm{Var}\,[p_t(x, \cdot) - (p_t(y, \cdot)] = \sup_{f \in C_b(H),\ \|f\|_0 \le 1} |P_t f(x) - P_t f(y)|,$$

for any $t > 0, x, y \in H$, where $p_t(x, \cdot)$ denotes the kernel of P_t and *Var* the total variation (see the proof of [5, Theorem 9.28] or [6, Lemma 7.1.5]). □

Acknowledgments The authors wish to thank professor Kwapien for providing the proof of part (ii) in Theorem 4.4, which significantly improved our initial result. They also thank the referees of the paper for their careful reading and helpful suggestions.

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