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Boundary proximity of SLE

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Abstract This paper examines how close the chordal SLE_κ curve gets to the real line asymptotically far away from its starting point. In particular, when $\kappa \in (0, 4)$, it is shown that if $\beta > \beta_\kappa := 1/(8/\kappa - 2)$, then the intersection of the SLE_κ curve with the graph of the function $y = x/(\log x)^\beta$, $x > e$, is a.s. bounded, while it is a.s. unbounded if $\beta = \beta_\kappa$. The critical SLE_4 curve a.s. intersects the graph of $y = x^{-(\log \log x)^\alpha}$, $x > e^e$, in an unbounded set if $\alpha \leq 1$, but not if $\alpha > 1$. Under a very mild regularity assumption on the function $y(x)$, we give a necessary and sufficient integrability condition for the intersection of the SLE_κ path with the graph of y to be unbounded. When the intersection is bounded a.s., we provide an estimate for the probability that the SLE_κ path hits the graph of y. We also prove that the Hausdorff dimension of the intersection set of the SLE_κ curve and the real axis is $2 - 8/\kappa$ when $4 < \kappa < 8$.

Keywords SLE · Hausdorff dimension

Mathematics Subject Classification (2000) 60D05 · 28A80

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1 Introduction

The stochastic Loewner evolution paths (SLE) are random curves in the plane that are obtained by running Loewner's differential equation with a scaled Brownian motion as the driving parameter. They have been shown to describe several critical statistical physics systems, and have been useful in the analysis of these systems. This has been proved for critical site-percolation on the triangular lattice [5,14], loop erased random walks and uniform spanning tree Peano paths [9], the level lines of the discrete Gaussian free field [12], the interfaces of the random cluster model associated with the Ising model [15], as well as a few other systems. For further background, the reader is advised to consult the surveys [6–8,16].

In order to understand the corresponding disordered systems well, it is then natural to investigate the properties of SLE. In [10], the basic topological and geometric properties of SLE were investigated. In [2] the Hausdorff dimension of the SLE_6 curve and its outer boundary were determined. Several years later it was proved [3] that the Hausdorff dimension of the SLE curves is $\min(1 + \kappa/8, 2)$.

There are several different versions of SLE. If B_t is a one-dimensional Brownian motion starting at 0, the chordal SLE_κ in the upper half plane $\mathbb{H}$ from 0 to ∞ with parameter κ is the solution of the differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \qquad g_0(z) = z, \tag{1.1}$$

where $z \in \mathbb{H}$ and $W_t = \sqrt{\kappa} B_t$. It can be shown [9,10] that a.s. g_t^{-1} extends continuously to $\overline{\mathbb{H}}$ for every $t \geq 0$ and $\gamma(t) := g_t^{-1}(W_t)$ is a continuous path. This is the SLE path, and the domain of definition of g_t is the unbounded connected component H_t of $\mathbb{H} \setminus \gamma[0, t]$. We shall denote by K_t the closure of the complement of H_t in $\mathbb{H}$.

It is known [10] that when $\kappa \geq 8$ a.s. $\gamma \cap \mathbb{R} = \mathbb{R}$ and when $\kappa \in [0, 4]$ a.s. $\gamma \cap \mathbb{R} = \{0\}$. In this paper, we will study the boundary behavior of SLE curves. More precisely, given the graph of a function $h : [r, \infty) \to (0, \infty)$ we will discuss whether the intersection set of the SLE_κ curve γ and the graph of $h(x)$ is bounded or not. Clearly, this intersection is a.s. unbounded when $\kappa > 4$, since γ swallows every point of $\mathbb{H}$ a.s. when $4 < \kappa < 8$ and $\gamma = \overline{\mathbb{H}}$ when $\kappa \geq 8$. The only non-trivial case is $\kappa \in (0, 4]$.

For a function $h : [r, \infty) \to (-\infty, \infty)$, let Γ^h denote its graph; that is,

$$\Gamma^h := \{x + i\, h(x) : x \geq r\}\,.$$

Set

$$s_\kappa := 8/\kappa - 1\,,$$

and

$$\Lambda_\kappa^h(x) := \begin{cases} h(x)^{s_\kappa - 1} & \kappa < 4\,, \\ 1/\log\left(\frac{x}{h(x)} \vee 2\right) & \kappa = 4\,. \end{cases}$$

Our main theorem is the following.

Theorem 1.1 *Let* $\kappa \in (0, 4]$, *and let* γ *be the chordal* SLE_κ *path. Fix* $r > 1$, *and suppose that* $h : [r, \infty) \to (0, \infty)$ *is continuous and satisfies*

$$\sup\left\{\Lambda_\kappa^h(x)/\Lambda_\kappa^h(y) : r \le x \le y \le 2x\right\} < \infty\,. \tag{1.2}$$

If

$$\int_r^\infty \frac{\Lambda_\kappa^h(x)}{x^{s_\kappa}}\,dx < \infty\,, \tag{1.3}$$

then $\gamma \cap \Gamma^h$ *is bounded a.s. Conversely, if the integral in* (1.3) *is infinite, then* $\gamma \cap \Gamma^h$ *is unbounded a.s.*

To illustrate the theorem, we note that if $\kappa < 4$ and $h(x) = x\,(\log x)^{-\beta}$, then $\gamma \cap \Gamma^h$ is bounded a.s. if $\beta > (8/\kappa - 2)^{-1}$ and unbounded a.s. if $\beta = (8/\kappa - 2)^{-1}$.

The case $\kappa = 4$ is critical for SLE to hit the boundary, and it is therefore not entirely surprising that its behavior is different. In that case, if $h(x) = x^{-(\log\log x)^\alpha}$, then $\gamma \cap \Gamma^h$ is a.s. unbounded when $\alpha = 1$, but bounded a.s. if $\alpha > 1$.

Now suppose instead that h is continuous in $[0, 1]$ and $h(0) = 0$. One can ask if 0 is in the closure of the intersection of $\{x + i\,h(x) : x \in (0, 1]\}$ and γ. Using reversibility of SLE [17], this translates to the type of question addressed by Theorem 1.1. Alternatively, the proof of Theorem 1.1 can be easily adapted to also handle this question.

Another natural question related to Theorem 1.1 is to estimate the probability that γ hits Γ^h. Actually, it is not too hard to modify the proof of Theorem 1.1 to show that when $\kappa \le 4$

$$P\left(\gamma \cap \Gamma^h \ne \emptyset\right) \eqsim 1 \wedge \int_r^\infty \frac{\Lambda_\kappa^h(x)}{x^{s_\kappa}}\,dx\,, \tag{1.4}$$

where $\eqsim$ denotes equivalence up to a multiplicative constant that depends only on κ and the left hand side in (1.2). Likewise, the proof of Theorem 1.1 easily gives the following estimate for the probability that γ hits the set $A_\epsilon = \{x + i\,y : 1 \le x \le 2,\ 0 \le y \le \epsilon\}$:

$$P(\gamma \cap A_\epsilon \ne \emptyset) \asymp \begin{cases} \epsilon^{s_\kappa - 1} & \kappa < 4\,, \\ |\log\epsilon|^{-1} & \kappa = 4\,, \end{cases} \tag{1.5}$$

where $\epsilon \in (0, 1/2)$ and the constants implied by $\asymp$ depend only on κ. Somewhat related results in the setting of discrete models appear in [11, Theorem 10.7] and in [4].

We also make use of the machinery developed for the proof of Theorem 1.1 to obtain the Hausdorff dimension of $\gamma \cap \mathbb{R}$ when $\kappa \in (4, 8)$.

Theorem 1.2 *If* $4 < \kappa < 8$, *then with probability one,*

$$\dim_H(\gamma \cap \mathbb{R}) = 2 - 8/\kappa\,.$$

A proof of this result based on Beffara's argument should be possible, but our proof is different and simpler. In fact, one may hope that the argument we present would generalize to give a simpler proof of Beffara's theorem, but so far we were not able to achieve this. An alternative and independent proof of Theorem 1.2 can be found in [1].

The paper is organized as follows. In Sect. 2, we consider for each $x > 0$ a local martingale M_t^x and relate its behavior to the geometry of the path near x. We also derive an estimate for the probability that both M_t^x and M_t^y become large, as a function of the positions of the points x, y. In Sect. 3, we prove Theorem 1.1 using the first and second moment methods. The Hausdorff dimension proof is given in Sect. 4.

2 The local martingale and its properties

2.1 Basic properties

We assume throughout this paper that $\kappa \in (0, 8)$. Let $x > 0$ and set

$$t_x := \sup\{t \geq 0 : x \notin K_t\}\,.$$

Then we have from [10] that $t_x = \infty$ a.s. if $\kappa \leq 4$ and $t_x < \infty$ a.s. if $\kappa > 4$. Define for $t \in (0, t_x)$,

$$M_t^x := \left(\frac{g_t'(x)}{g_t(x) - W_t}\right)^{s_\kappa}\,.$$

Also, for $\epsilon > 0$ set

$$\tau_x = \tau_x^\epsilon := \inf\left\{t \in (0, t_x) : M_t^x \geq \epsilon^{-s_\kappa}\right\}$$

and

$$C_\epsilon := \{x > 0 : \tau_x < t_x\}\,. \tag{2.6}$$

As usual, we use the convention that $\inf \emptyset = \infty$.

Write $\mathcal{F}_t := \sigma(B_s, 0 \leq s \leq t)$. Then $\{M_t^x, \mathcal{F}_t, t \in (0, t_x)\}$ is a local martingale by Theorem 6 and Remark 7 in [13] (this is, of course, easily verified using Itô's formula). The reason for our interest in M_t^x is the following lemma.

Lemma 2.1 *If* $x \in C_\epsilon$, *then the distance from* γ *to* x *is at most* $4\,\epsilon$.

Proof Suppose that $x > 0$, $t > 0$, and $x \notin K_t$. Set $\bar{K}_t := \{\bar{z} : z \in K_t\}$, and let G denote the extension of g_t to $\mathbb{C} \setminus (K_t \cup \bar{K}_t)$, which is obtained by Schwarz reflection.

Let $d_t = d_t^x$ denote the distance from x to $\gamma[0,t]$. Then W_t is not in $G\,(B(x,d_t))$, and therefore the Koebe 1/4 theorem gives

$$G'(x)\,d_t/4 \le G(x) - W_t\,.$$

This translates to

$$M_t^x \le (4/d_t)^{s_\kappa}\,, \tag{2.7}$$

and the lemma immediately follows. □

Next, we prove that in some situations the inequality (2.7) may be reversed.

Lemma 2.2 *Let $x > 0$, $t > 0$, $x_0 := \operatorname{Re}\gamma(t)$ and $y_0 := \operatorname{Im}\gamma(t)$. Suppose that $x \notin K_t$, $x - x_0 \ge y_0$, and $\gamma[0,t)$ does not intersect the line segment $[x_0, \gamma(t)]$. Then*

$$M_t^x \ge (c\,d_t)^{-s_\kappa}\,,$$

where $0 < c < \infty$ is a universal constant.

Proof Let G be as in the proof of Lemma 2.1. Set $r := G(x) - G(x_0)$. Then the inverse of G is defined in the ball $B\,(G(x), r)$. Therefore, the Koebe 1/4 theorem gives

$$\frac{r}{4}\,G'(x)^{-1} \le d_t\,.$$

It therefore suffices to prove a positive lower bound on

$$\frac{r}{G(x) - W_t} = \frac{G(x) - G(x_0)}{G(x) - W_t}\,. \tag{2.8}$$

Every path in H_t going from $[x,\infty)$ to the union of $[0,x_0]$ and the right hand side of $\gamma[0,t]$ must intersect the line segment $[x_0,\gamma(t)]$. Since we may consider the Euclidean metric on the square of sidelength $2\,y_0$ centered at $\gamma(t)$, normalized to have area 1, the extremal length of this collection of paths is bounded away from zero. By conformal invariance of extremal distance, it follows that the extremal distance from $[G(x),\infty)$ to $[W_t, G(x_0)]$ in $\mathbb{H}$ is likewise bounded away from zero. This implies the required lower bound on (2.8), and completes the proof. □

For a given point $x > 0$, we are interested in the probability that $x \in C_\epsilon$.

Proposition 2.3 *Let $0 < \kappa < 8$, $x > 0$ and $\epsilon > 0$. Then*

$$P(x \in C_\epsilon) = (\epsilon/x)^{s_\kappa} \wedge 1\,. \tag{2.9}$$

The proof is dependent on the properties of the local martingale M_t^x as $t \nearrow t_x$. Write $T_x = t_x \wedge \tau_x$. If $0 < \kappa \le 4$, then $t_x = \infty$ a.s. and $T_x = \tau_x$. We use $I(\mathcal{A})$ for the indicator function of an event $\mathcal{A}$.

Proof Since $M^x_{t\wedge T_x}$ is a bounded local martingale on $t \in (0, T_x)$, it is a martingale and the limit $M^x_{T_x} := \lim_{t\nearrow T_x} M^x_t$ exists. On the event $0 < \tau_x < \infty$, we have $M_{T_x} = \epsilon^{-s_\kappa}$. Hence, the optional sampling theorem gives

$$M^x_0 = E(M^x_{T_x}) = P(x \in C_\epsilon)\,\epsilon^{-s_\kappa} + E\left(M^x_{T_x}\, I(\tau_x = \infty)\right).$$

Therefore, the proof is complete once we prove that

$$P\left(M^x_{T_x} \neq 0,\ \tau_x = \infty\right) = 0. \tag{2.10}$$

Consider first the case $\kappa \in (4, 8)$. In this case a.s. $t_x < \infty$ and $x_1 := \gamma(t_x) \in (x, \infty)$. Suppose that this is indeed the case. Let $r > 0$ be much smaller than the distance from x to x_1, and let s be the first time t at which $|\gamma(t) - x_1| = r$. Let G denote the Schwarz reflection of g_s with respect to the real line, let $a = a_s := \sup(K_s \cap \mathbb{R})$ and $a' = a'_s := \inf\{G(x') : x' > a\}$. Then a' is not in $G\,(B(x, d_s))$. Therefore, the Koebe $1/4$ theorem implies

$$G'(x)\, d_s/4 \le G(x) - a'.$$

That is,

$$\frac{g'_s(x)}{g_s(x) - a'} \le 4/d_s.$$

Since $d_{t_x} > 0$ a.s., it therefore suffices to prove that

$$\lim_{r\searrow 0} \frac{g_s(x) - W_s}{g_s(x) - a'} = \infty. \tag{2.11}$$

Consider the extremal distance in H_s from (a, x) to the union of $(-\infty, 0)$ and the left hand side of $\gamma[0, s]$. This extremal distance is clearly at least as large as the extremal distance from the circle of radius $|x - x_1|$ about x_1 to the circle of radius r about x_1, which is at least a constant times $\log\,(|x - x_1|/r)$. By conformal invariance of extremal distance, it follows that the extremal distance in $\mathbb{H}$ from $[a', g_s(x)]$ to $(-\infty, W_s]$ goes to infinity as $r \searrow 0$, which proves (2.11) and completes the proof in the case $\kappa \in (4, 8)$.

The argument in the case $\kappa \in (0, 4]$ is similar. We choose $R > 0$ large, and let s be the first time at which $|\gamma(s)| = R$. The extremal distance in H_s from $(0, x]$ to the union of the left hand side of $\gamma[0, s]$ with $(-\infty, 0)$ is then at least a constant times $\log(R/x)$, which implies (2.11) in the same way. □

Observe that the proposition implies that given $x > 0$ there is a.s. some $\epsilon > 0$ such that $x \notin C_\epsilon$. Therefore, (2.10) gives

$$M^x_{t_x} := \lim_{t\nearrow t_x} M^x_t = 0 \quad \text{a.s.} \tag{2.12}$$

2.2 Correlation estimate

Let $0 < x < y$, $\epsilon_x, \epsilon_y > 0$, $\tau_x := \tau_x^{\epsilon_x}$, $\tau_y := \tau_y^{\epsilon_y}$, $T_x := t_x \wedge \tau_x$ and $T_y := t_y \wedge \tau_y$. Define

$$Z_t := \frac{g_t(x) - W_t}{g_t(y) - W_t}, \qquad T := T_x \wedge T_y.$$

A simple but tedious calculation via Itô's formula implies that $u(Z_t)\, M_t^x\, M_t^y$ is a local martingale while $t \in (0, T)$, where

$$u(z) := (1-z)^{-s_\kappa}\, {}_2F_1(1 - 8/\kappa, 4/\kappa, 8/\kappa; 1 - z)\,.$$

Euler's integral representation of hypergeometric functions shows that

$${}_2F_1(1 - 8/\kappa, 4/\kappa, 8/\kappa; z) = \frac{\Gamma(8/\kappa)}{\Gamma(4/\kappa)^2} \int_0^1 t^{4/\kappa - 1}(1-t)^{4/\kappa-1}(1-zt)^{8/\kappa-1}\, dt,$$

where Γ is the gamma function. This implies that $u(z) > 0$ when $z \in (0, 1)$. Since $8/\kappa - 4/\kappa - (1 - 8/\kappa) > 0$, we have

$${}_2F_1(1 - 8/\kappa, 4/\kappa, 8/\kappa; 1) = \frac{\Gamma(8/\kappa)\Gamma(12/\kappa - 1)}{\Gamma(16/\kappa - 1)\Gamma(4/\kappa)}\,.$$

Hence $q_1 := \inf_{z \in (0,1)} u(z)$ and $q_2 := \sup_{z\in(0,1)} (1-z)^{s_\kappa}\, u(z)$ are both finite and positive. It follows from (2.10) that

$$P(x \in C_{\epsilon_x}, y \in C_{\epsilon_y}) = P\left(M_{T_x}^x = \epsilon_x^{-s_\kappa},\ M_{T_y}^y = \epsilon_y^{-s_\kappa}\right) = (\epsilon_x\, \epsilon_y)^{s_\kappa}\, E\left(M_{T_x}^x\, M_{T_y}^y\right).$$

Recall that $T = T_x \wedge T_y$. If $T = T_x < \infty$, then we have that $M_{t\wedge T_x}^x$ is constant in the range $t \in [T_x, T_y)$, while $M_{t\wedge T_y}^y$ is a martingale. The symmetric statement also holds when we exchange x and y. It should be clear that this implies

$$E\left(M_{T_x}^x\, M_{T_y}^y\right) = E\left(M_T^x\, M_T^y\right), \tag{2.13}$$

but for the sake of completeness, we prove this. First, since $I(T = T_x)\, M_{T_x}^x$ is $\mathcal{F}_T$-measurable, we have

$$\begin{aligned} E\left(I(T = T_x)\, M_{T_x}^x\, M_{T_y}^y \mid \mathcal{F}_T\right) &= I(T = T_x)\, M_{T_x}^x\, E\left(M_{T_y}^y \mid \mathcal{F}_T\right) \\ &= I(T = T_x)\, M_T^x\, M_T^y\,. \end{aligned}$$

Second, on the complement of the event $T = T_x$, we have $T = T_y$. Hence, $I(T \neq T_x)\, M^y_{T_y}$ is also $\mathcal{F}_T$-measurable, and we get in the same way

$$E\left(I(T \neq T_x)\, M^x_{T_x}\, M^y_{T_y} \,\Big|\, \mathcal{F}_T\right) = I(T \neq T_x)\, M^x_T\, M^y_T.$$

Summing the above and taking expectations, we obtain (2.13).

Since $u(Z_{t\wedge T})\, M^x_{t\wedge T}\, M^y_{t\wedge T}$ is a non-negative local martingale, it is also a supermartingale. This justifies the second inequality in the following estimate:

$$\begin{aligned} P(x \in C_{\epsilon_x}, y \in C_{\epsilon_y}) &= (\epsilon_x\, \epsilon_y)^{s_\kappa}\, E(M^x_T\, M^y_T) \\ &\le (\epsilon_x\, \epsilon_y)^{s_\kappa}\, E\left(u(Z_T)\, M^x_T\, M^y_T\right)/q_1 \\ &\le (\epsilon_x\, \epsilon_y)^{s_\kappa}\, u(Z_0)\, M^x_0\, M^y_0/q_1 \\ &= (\epsilon_x\, \epsilon_y)^{s_\kappa}\, \frac{u(x/y)}{q_1\, x^{s_\kappa}\, y^{s_\kappa}} \le \frac{q_2}{q_1}\, (\epsilon_x\, \epsilon_y)^{s_\kappa}\, x^{-s_\kappa}\, (y-x)^{-s_\kappa}. \end{aligned}$$

Hence, we obtain the following proposition.

Proposition 2.4 *Let* $0 < x < y$, $0 < \kappa < 8$, $\epsilon_x, \epsilon_y > 0$. *Then*

$$P\left(x \in C_{\epsilon_x}, y \in C_{\epsilon_y}\right) \le c_\kappa\, (\epsilon_x\, \epsilon_y)^{s_\kappa}\, x^{-s_\kappa}\, (y-x)^{-s_\kappa}, \tag{2.14}$$

where c_κ *is a constant depending only on* κ. □

3 Proximity estimates

3.1 Bounded intersection

In this subsection, we assume (1.3), as well as the other assumptions in Theorem 1.1, and prove that $\gamma \cap \Gamma^h$ is bounded a.s.

Let $\rho : [r, \infty) \to (0, \infty)$ be a function such that

$$\lim_{x\to\infty} \rho(x)/\Lambda^h_\kappa(x) = \infty, \tag{3.15}$$

but ρ satisfies (1.2) and (1.3) in place of $\Lambda^h_\kappa(x)$, namely,

$$\int_r^\infty \frac{\rho(x)}{x^{s_\kappa}}\, dx < \infty, \tag{3.16}$$

and

$$\sup\{\rho(x)/\rho(y) : r \le x \le y \le 2x\} < \infty. \tag{3.17}$$

In the following, we let $\asymp$ mean equivalence up to positive multiplicative constants that may depend on h, ρ and κ. Likewise, $a \lesssim b$ means that there is some $a' \leq b$ such that $a' \asymp a$.

Define

$$Z_t := \int_r^\infty \rho(x)\, M_t^x\, dx\,.$$

Since $M_0^x = x^{-s_\kappa}$, it follows from (3.16) that $Z_0 < \infty$. As M_t^x is a supermartingale for each $x > 0$, it follows that Z_t is a supermartingale.

By (1.2), for every $x \geq r$ such that $h(x) \geq x/2$, the contribution to the integral in (1.3) from the interval $[x, 2\,x]$ is bounded from zero. Since the integral in (1.3) is finite, we conclude that there is a finite $R_0 > r$ such that $h(x) < x/2$ for $x \geq R_0$. Fix an $R > R_0$, and let A be the set $A := \{x + i\,y : x \geq R,\ y \leq h(x)\}$. Let $T_A := \inf\{t \geq 0 : \gamma_t \in A\}$, and on the event $T_A < \infty$ set $x_0 := \operatorname{Re}\gamma(T_A)$, $y_0 := \operatorname{Im}\gamma(T_A)$. From our choice of R, we have $y_0 \leq x_0/2$. By Lemma 2.2, $M_{T_A}^x \gtrsim (x - x_0)^{-s_\kappa}$ holds for every $x > x_0 + y_0$. Therefore, on the event $T_A < \infty$,

$$\begin{aligned} Z_{T_A} &\gtrsim \int_{x_0+y_0}^\infty \rho(x)\,(x - x_0)^{-s_\kappa}\,dx \overset{(3.17)}{\gtrsim} \rho(x_0) \int_{x_0+y_0}^{2x_0} (x - x_0)^{-s_\kappa}\,dx \\ &\gtrsim \frac{\rho(x_0)}{\Lambda_\kappa^h(x_0)}\,. \end{aligned}$$

Since Z_t is a supermartingale, the optional sampling theorem gives

$$Z_0 \geq E\left(Z_{T_A}\, I(T_A < \infty)\right) \gtrsim P(T_A < \infty)\, \inf_{x \geq R} \frac{\rho(x)}{\Lambda_\kappa^h(x)}\,.$$

By (3.15), we conclude that $\lim_{R\to\infty} P(T_A < \infty) = 0$. Thus, $\gamma \cap \Gamma^h$ is bounded a.s., as required. □

3.2 Unbounded intersection

In this subsection, we assume that the integral in (1.3) is infinite, and prove that $\gamma \cap \Gamma^h$ is unbounded a.s., thus completing the proof of Theorem 1.1. In the following, $\asymp$ denotes equivalence up to multiplicative constants that may depend on κ and h, and similarly for $\lesssim$.

Suppose that we prove that the intersection of γ with $\Theta_+^h := \{x + i\,y : y \leq h(x),\ x \geq r\}$ is a.s. unbounded. Symmetry then implies that the intersection of γ with $\Theta_-^h := \{-x + i\,y : y \leq h(x),\ x \geq r\}$ is a.s. unbounded as well. For every $R \geq \sqrt{r^2 + h(r)^2}$, the set $\Gamma^h \setminus B(0, R)$ separates $\Theta_+^h \setminus B(0, R)$ from $\Theta_-^h \setminus B(0, R)$ in $\mathbb{H} \setminus B(0, R)$. Since γ is a.s. transient, it follows that $\gamma \cap \Gamma^h$ must be a.s. unbounded, as required.

Now observe that $\tilde{h}(x) := h(x) \wedge (x/2)$ satisfies the same assumptions as we have for h. The above then implies that it suffices to prove the claim for $\tilde{h}$. Thus, we assume with no loss of generality that $h(x) \leq x/2$ holds for every $x \geq r$.

Define $\rho(x) := \Lambda_\kappa^h(x)$. Let $a > r$ and let $b > a$ satisfy

$$\int_a^b \frac{\rho(x)}{x^{s_\kappa}}\, dx = 1\,. \tag{3.18}$$

Define $X := \{x \geq r : x \in C_{h(x)}\}$. We will show that $\sup X = \infty$ a.s. Set

$$Q_a := \int_a^b \frac{\rho(x)}{h(x)^{s_\kappa}}\, I(x \in X)\, dx\,.$$

Then by (3.18) and Proposition 2.3, we have

$$E(Q_a) = 1\,.$$

We will now prove that $E(Q_a^2)$ is bounded by some constant independent of a. First, observe that

$$E(Q_a^2) = \int_a^b \int_a^b \frac{\rho(x)\rho(y)}{h(x)^{s_\kappa} h(y)^{s_\kappa}}\, P(x, y \in X)\, dx\, dy\,.$$

Let $F(x, y)$ denote the integrand. Set $S := [a, b]^2$. Let S_1 be the set of pairs $(x, y) \in S$ such that $y \in [x - h(x), x]$, let S_2 be the set of pairs $(x, y) \in S$ such that $y \in [x/2, x - h(x)]$, and let S_3 be the set of pairs $(x, y) \in S$ such that $y \leq x/2$. Then since S_1, S_2 and S_3 tile the set $\{(x, y) \in S : y \leq x\}$, we have

$$E(Q_a^2) = 2 \int_{S_1 \cup S_2 \cup S_3} F\, dx\, dy\,.$$

To estimate F on S_1, we use the bound

$$P(x, y \in X) \leq P(x \in X) \wedge P(y \in X) \overset{(2.9)}{=} \frac{h(x)^{s_\kappa}}{x^{s_\kappa}} \wedge \frac{h(y)^{s_\kappa}}{y^{s_\kappa}}\,.$$

Since $x/2 \leq y \leq x$ on S_1, this is bounded by $h(y)^{s_\kappa}\, y^{-s_\kappa} \lesssim h(y)^{s_\kappa}\, x^{-s_\kappa}$. Hence,

$$\int_{S_1} F(x,y) \lesssim \int_a^b \int_{x-h(x)}^x \frac{\rho(x)\rho(y)}{h(x)^{s_\kappa} x^{s_\kappa}}\, dy\, dx \overset{(1.2)}{\lesssim} \int_a^b \int_{x-h(x)}^x \frac{\rho(x)^2}{h(x)^{s_\kappa} x^{s_\kappa}}\, dy\, dx$$
$$= \int_a^b \frac{\rho(x)^2\, h(x)^{1-s_\kappa}}{x^{s_\kappa}}\, dx$$

By the definition of Λ_κ, we have $\rho(x)\, h(x)^{1-s_\kappa} \lesssim 1$. Thus, (3.18) implies that $\int_{S_1} F \lesssim 1$.

For S_2, we use the estimate (2.14), the fact that $y \asymp x$ when $y \in [x/2, x - h(x)]$ and (1.2), to get

$$\int_{S_2} F \lesssim \int_a^b \int_{x/2}^{x-h(x)} \frac{\rho(x)}{x^{s_\kappa}} \frac{\rho(y)}{(x-y)^{s_\kappa}}\, dy\, dx$$
$$\lesssim \int_a^b \frac{\rho(x)^2}{x^{s_\kappa}} \int_{x/2}^{x-h(x)} \frac{1}{(x-y)^{s_\kappa}}\, dy\, dx$$
$$\lesssim \int_a^b \frac{\rho(x)^2}{x^{s_\kappa}}\, \Lambda_\kappa^h(x)^{-1}\, dx = \int_a^b \frac{\rho(x)}{x^{s_\kappa}}\, dx = 1\,.$$

On the set S_3, the estimate (2.14) gives

$$P(x, y \in X) \lesssim \frac{h(x)^{s_\kappa} h(y)^{s_\kappa}}{x^{s_\kappa} y^{s_\kappa}}\,.$$

Hence,

$$\int_{S_3} F \lesssim \int_a^b \int_a^b \frac{\rho(x)\rho(y)}{x^{s_\kappa} y^{s_\kappa}}\, dx\, dy = \left(\int_a^b \frac{\rho(x)}{x^{s_\kappa}}\, dx \right)^2 \overset{(2.9)}{=} 1\,.$$

Thus, we conclude that $E(Q_a^2) \lesssim 1 = E(Q_a)^2$. The Paley–Zygmund inequality therefore gives

$$P\left(Q_a \geq E Q_a/2\right) \gtrsim 1\,. \tag{3.19}$$

Note that $Q_a > 0$ implies $\sup X \geq a$. But since a can be arbitrarily large and the constant implied in (3.19) does not depend on a, it follows from (3.19) that $P(\sup X = \infty) \gtrsim 1$.

Now fix some $t \in (0, \infty)$. We will show that a.s.

$$P(\sup X = \infty \mid \mathcal{F}_t) \gtrsim 1\,.$$

Define $X_t := \{x \geq r : \tau_x^{h(x)} \leq t\}$. Suppose that $\sup X_t < \infty$. Let $x > y > r \vee \sup X_t$. Then we have by the Markov property of SLE and (2.9) that

$$P(x \in X \mid \mathcal{F}_t) = \left(\frac{g_t'(x)\, h(x)}{g_t(x) - W_t}\right)^{s_\kappa} . \tag{3.20}$$

The same reasoning, but this time with (2.14), shows that

$$P(x, y \in X \mid \mathcal{F}_t) \lesssim \left(\frac{g_t'(x) g_t'(y) h(x) h(y)}{(g_t(y) - W_t)(g_t(x) - g_t(y))}\right)^{s_\kappa} . \tag{3.21}$$

Since g_t has a power series expansion near ∞ of the form

$$g_t(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots ,$$

we have in particular that $\lim_{x\to\infty} g_t(x) - x = 0$ and $\lim_{x\to\infty} g_t'(x) = 1$. Therefore, on the event $\sup X_t < \infty$ there is some random $a' > a \vee \sup X_t$, which is $\mathcal{F}_t$-measurable, such that for all $x \geq a'$ we have $g_t'(x) \in [1/2, 2]$ and $(g_t(x) - W_t)/x \in [1/2, 2]$. Therefore, for $x > y > a'$ we have that the estimates in (3.20) and (3.21) are within a constant multiplicative factor (which depends only on κ) from their values at $t = 0$. Consequently, our proof above with a replaced by a' and with probabilities and expectations replaced by conditional probabilities and conditional expectations given $\mathcal{F}_t$ implies that on the event $\sup X_t < \infty$,

$$P(\sup X > a' \mid \mathcal{F}_t) \gtrsim 1. \tag{3.22}$$

But since $X \supset X_t$, this holds even if $\sup X_t = \infty$. Because (3.22) a.s. holds for every $t > 0$, we get $P(\sup X > a) = 1$, and since a was arbitrary, we get $\sup X = \infty$ a.s. Lemma 2.1 implies therefore that the set of $x \geq r$ such that $\inf_t d_t^x \leq 4\, h(x)$ is a.s. unbounded.

Condition (1.2) implies that for some finite constant $A > 1$ and x, y satisfying $r \leq x \leq y \leq 2x$, we have

$$A\, h(y) \geq h(x) \quad \text{if } \kappa < 4 ,$$

and

$$\frac{h(y)}{y} \geq \left(\frac{h(x)}{x}\right)^A \quad \text{if } \kappa = 4 .$$

Define

$$H(x) := \begin{cases} (4A)^{-1} h(2x/3) & \kappa < 4 , \\ 4^{-1} (2x/3)^{1-A} h(2x/3)^A & \kappa = 4 . \end{cases}$$

Since the function $H(x)$ satisfies the same assumptions we have for $h(x)$, we conclude that a.s.

$$\sup\left\{x \geq 3r/2 : \inf_t d_t^x \leq 4H(x)\right\} = \infty\,.$$

As the balls $B\,(x, 4H(x))$ lie below the graph of $y = h(x)$ when $x > 2r$, it follows that $\gamma \cap \{x + i\,y : y \leq h(x),\ x \geq r\}$ is a.s. unbounded. As we have seen, this implies that $\gamma \cap \Gamma^h$ is a.s. unbounded. The proof is thus complete. □

4 Hausdorff dimension

In this part, we will prove Theorem 1.2. The usual strategy of deriving Theorem 1.2 is to estimate the two probabilities

$$P(x \in \gamma_\epsilon), \quad P(x \in \gamma_\epsilon,\ y \in \gamma_\epsilon),$$

where $\gamma_\epsilon := \{x \in \mathbb{R} : \mathrm{dist}(x, \gamma) \leq \epsilon\}$, and then prove some 0–1 law to show that the Hausdorff dimension is an a.s. constant.

In this paper, instead of γ_ϵ, we consider C_ϵ. Let

$$C := \bigcap_{\epsilon > 0} C_\epsilon\,.$$

Then Lemma 2.1 gives

$$C \subset \gamma \cap \mathbb{R}\,. \tag{4.23}$$

Proposition 4.1 *Assume that* $\kappa \in (4, 8)$. *Then for any* $\delta > 0$,

$$P\left(\dim_H C \geq 1 - s_\kappa - \delta\right) > 0\,.$$

Proof The proof follows the standard Frostman measure argument. We introduce random measures μ_ϵ defined on the Borel σ-field of the interval $[1, 2]$ by

$$\mu_\epsilon([1, x]) := \epsilon^{-s_\kappa} \int_1^x I(x_1 \in C_\epsilon)\, dx_1$$

for $0 < \epsilon < 1$ and $x \in [1, 2]$. The $(1 - s_\kappa - \delta)$-energy of μ_ϵ is

$$\mathcal{E}(\mu_\epsilon) = \int_1^2 \int_1^2 \frac{1}{|y - x|^{1 - s_\kappa - \delta}}\, d\mu_\epsilon(x)\, d\mu_\epsilon(y).$$

Its expectation is

$$\begin{aligned} E\left(\mathcal{E}(\mu_\epsilon)\right) &= 2\,\epsilon^{-2s_\kappa} \int_1^2 \int_x^2 \frac{P(x\in C_\epsilon,\, y\in C_\epsilon)}{|y-x|^{1-s_\kappa-\delta}}\, dy\, dx \\ &\le 2\,\epsilon^{-2s_\kappa} \int_1^2 \int_x^{x+\epsilon} \frac{P(x\in C_\epsilon)}{|y-x|^{1-s_\kappa-\delta}}\, dy\, dx \\ &\quad + 2\,\epsilon^{-2s_\kappa} \int_1^{2-\epsilon} \int_{x+\epsilon}^2 \frac{P(x\in C_\epsilon,\, y\in C_\epsilon)}{|y-x|^{1-s_\kappa-\delta}}\, dy\, dx \\ &=: E_1 + E_2\,. \end{aligned} \tag{4.24}$$

For E_1, Proposition 2.3 gives that

$$E_1 \le 2\,\epsilon^{-s_\kappa} \int_1^2 \int_x^{x+\epsilon} (y-x)^{s_\kappa-1}\, dy\, dx = 2\, s_\kappa{}^{-1}. \tag{4.25}$$

For E_2, Proposition 2.4 gives that

$$E_2 \le 2\, c_\kappa \int_1^{2-\epsilon} \int_{x+\epsilon}^2 (y-x)^{-1+\delta}\, dy\, dx \le 2\, c_\kappa\, \delta^{-1}. \tag{4.26}$$

Combining (4.24), (4.25) and (4.26), we obtain that

$$E\left(\mathcal{E}(\mu_\epsilon)\right) \le 2\, s_\kappa{}^{-1} + 2\, c_\kappa\, \delta^{-1}.$$

Noting that

$$E|\mu_\epsilon| = \epsilon^{-s_\kappa} \int_1^2 (\epsilon/x)^{s_\kappa}\, dx = (1-s_\kappa)^{-1}(2^{1-s_\kappa}-1) > 0, \tag{4.27}$$

and $E\left(|\mu_\epsilon|^2\right) \le E\left(\mathcal{E}(\mu_\epsilon)\right)$, the Paley–Zygmund inequality implies that there is a $\lambda > 0$, which does not depend on ϵ, such that with probability at least λ, $|\mu_\epsilon| > \lambda$ and $\mathcal{E}(\mu_\epsilon) < 1/\lambda$. With probability at least λ this will hold for a sequence of positive ϵ tending to 0. On this event, we can take a subsequential limit μ supported on C and satisfying $|\mu| > \lambda$ and $\mathcal{E}(\mu) < 1/\lambda$. Frostman's lemma therefore implies that $P\left(\dim_H(C\cap[1,2]) > 1-s_\kappa-\delta\right) > \lambda$, which concludes the proof. □

The following proposition tells us that $\dim_H(\gamma\cap\mathbb{R}) \le 1-s_\kappa$ a.s.

Proposition 4.2 *Let $x \in [1, 2)$, and $\kappa \in (4, 8)$. Then for $\epsilon \in (0, 1)$,*

$$P(\gamma \cap [x, x+\epsilon] \neq \emptyset) \asymp \left(\frac{\epsilon}{x}\right)^{s_\kappa},$$

where the constants implied by $\asymp$ depend only on κ.

Proof Lemma 6.6 of [10] gives

$$\begin{aligned} &P(\gamma \cap [x, x+\epsilon] \neq \emptyset) \\ &= 1 - \frac{4^{(\kappa-4)/\kappa} \sqrt{\pi}\, {}_2F_1(1-4/\kappa, 2-8/\kappa, 2-4/\kappa; 1/q) q^{(4-\kappa)/\kappa}}{\Gamma(2-4/\kappa)\Gamma(4/\kappa-1/2)}, \end{aligned} \tag{4.28}$$

where $q := (x+\epsilon)/x$. Using $\Gamma(2\theta)\,\Gamma(1/2) = 2^{2\theta-1}\,\Gamma(\theta)\,\Gamma(\theta+1/2)$, Euler's integral representation of ${}_2F_1$ and a change of variable, we have

$$\begin{aligned} (4.28) &= 1 - \frac{\Gamma(4/\kappa)}{\Gamma(1-4/\kappa)\Gamma(8/\kappa-1)} \int_0^{\frac{x}{x+\epsilon}} t^{-4/\kappa}(1-t)^{8/\kappa-2}\, dt \\ &= \frac{\Gamma(4/\kappa)}{\Gamma(1-4/\kappa)\Gamma(8/\kappa-1)} \int_{\frac{x}{x+\epsilon}}^{1} t^{-4/\kappa}(1-t)^{8/\kappa-2}\, dt \\ &\asymp \left(\frac{\epsilon}{x}\right)^{s_\kappa}. \end{aligned}$$

□

Lemma 4.3 *There is a constant $d = d_\kappa$ such that $\dim_H(\gamma \cap \mathbb{R}) = d$ a.s.*

Proof For all $n \in \mathbb{Z}$ let $D_n := \dim_H(\gamma[0, 2^n] \cap \mathbb{R})$. Then $D_{n+1} \geq D_n$. In addition, D_n and D_{n+1} have the same distribution, by scale invariance. Therefore, $D_m = D_n$ a.s. for all $m, n \in \mathbb{Z}$. Hence, $\dim_H(\gamma \cap \mathbb{R}) = \sup_{n\in\mathbb{Z}} D_n$ is $\mathcal{F}_{2^n}$-measurable for all $n \in \mathbb{Z}$, which implies that $\dim_H(\gamma \cap \mathbb{R})$ is $\mathcal{F}_{0+}$-measurable. By Blumenthal's 0-1 law, the σ-field $\mathcal{F}_{0+}$ is trivial. □

Proof of Theorem 1.2 First note that $\dim_H(\gamma \cap \mathbb{R}) = \dim_H(\gamma \cap \mathbb{R}_+) = \dim_H(\gamma \cap \mathbb{R}_-)$ a.s. by the symmetry property of SLE curves. Proposition 4.2 implies $\dim_H\left(\gamma \cap \mathbb{R}^+\right) \leq 1 - s_\kappa$ a.s. On the other hand, (4.23) and Proposition 4.1 give

$$P\left(\dim_H(\gamma \cap \mathbb{R}) \geq 1 - s_\kappa - \delta\right) > 0$$

for every $\delta > 0$. Therefore $\dim_H(\gamma \cap \mathbb{R}) = 1 - s_\kappa$ a.s. by Lemma 4.3. □

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