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Two-sided heat kernel estimates for censored stable-like processes

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Abstract In this paper, we study the precise behavior of the transition density functions of censored (resurrected) α-stable-like processes in $C^{1,1}$ open sets in $\mathbb{R}^d$, where $d \geq 1$ and $\alpha \in (1, 2)$. We first show that the semigroup of the censored α-stable-like process in any bounded Lipschitz open set is intrinsically ultracontractive. We then establish sharp two-sided estimates for the transition density functions of a large class of censored α-stable-like processes in $C^{1,1}$ open sets. We further obtain sharp two-sided estimates for the Green functions of these censored α-stable-like processes in bounded $C^{1,1}$ open sets.

Keywords Fractional Laplacian · Censored stable process · Censored stable-like process · Symmetric α-stable process · Symmetric stable-like process · Heat kernel · Transition density · Transition density function · Green function · Exit time · Lévy system · Boundary Harnack principle · Parabolic Harnack principle · Intrinsic ultracontractivity

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1 Introduction

There are close relationships between second order elliptic differential operators and diffusion processes. For a large class of second order elliptic differential operators $\mathcal{L}$ on $\mathbb{R}^d$ that satisfy the maximum principle, there is a diffusion process X on $\mathbb{R}^d$ associated with it so that $\mathcal{L}$ is the infinitesimal generator of X. A prototype is the celebrated interplay between Laplacian $\frac{1}{2}\Delta$ on $\mathbb{R}^d$ and Brownian motion on $\mathbb{R}^d$. The fundamental solution of $\partial_t u = \mathcal{L}u$ (also called the heat kernel of $\mathcal{L}$) is the transition density function $p(t, x, y)$ of X. Thus obtaining sharp two-sided estimates for $p(t, x, y)$ is a fundamental problem in both analysis and probability theory. In fact, two-sided heat kernel estimates for diffusions in $\mathbb{R}^d$ have a long history and many beautiful results have been established. See [10,12] and the references therein. But, due to the complication near the boundary, two-sided estimates for the transition density functions of killed diffusions in a domain D (equivalently, the Dirichlet heat kernels) have been established only recently. See [11–13] for upper bound estimates and [21] for the lower bound estimates of the Dirichlet heat kernels in bounded $C^{1,1}$ domains.

Markov processes with discontinuous sample paths constitute an important family of stochastic processes in probability theory and they have been widely used in various applications. One of the most important and most widely used family of discontinuous Markov processes is the family of (rotationally) symmetric α-stable process on $\mathbb{R}^d$, $0 < \alpha < 2$. A (rotationally) symmetric α-stable process $Y = \{Y_t, \mathbb{P}_x\}$ on $\mathbb{R}^d$ is a Lévy process such that

$$\mathbb{E}_x\left[e^{i\xi\cdot(Y_t - Y_0)}\right] = e^{-t|\xi|^\alpha} \quad \text{for every } x \in \mathbb{R}^d \text{ and } \xi \in \mathbb{R}^d.$$

The infinitesimal generator of a symmetric α-stable process Y in $\mathbb{R}^d$ is the fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$, which is a prototype of nonlocal operators. The fractional Laplacian can be written in the form

$$\Delta^{\alpha/2}u(x) = \lim_{\varepsilon\downarrow 0} \int_{\{y\in\mathbb{R}^d:\, |y-x|>\varepsilon\}} (u(y) - u(x)) \frac{\mathcal{A}(d,\alpha)}{|x-y|^{d+\alpha}}\, dy \quad \text{for } u \in C_c^\infty(\mathbb{R}^d),$$

where

$$\mathcal{A}(d,\alpha) := \frac{\alpha\, \Gamma(\frac{d+\alpha}{2})}{2^{1-\alpha}\, \pi^{d/2}\Gamma(1-\frac{\alpha}{2})}. \tag{1.1}$$

In a recent paper [5], we succeeded in establishing sharp two-sided estimates for the heat kernel of the fractional Laplacian $\Delta^{\alpha/2}$ with zero exterior condition on D^c (or equivalently, the transition density function of the killed α-stable process) in any $C^{1,1}$ open set.

Another important family of discontinuous Markov processes is the family of censored α-stable-like processes studied in [3] (see Sect. 2 for the precise definition). For any open subset D of $\mathbb{R}^d$, a censored α-stable-like process X in D is a strong Markov process whose infinitesimal generator is given by

$$\mathcal{L}_D^\alpha u(x) := \lim_{\varepsilon \downarrow 0} \int_{\{y \in D : |y-x| > \varepsilon\}} (u(y) - u(x)) \frac{\mathcal{C}(x, y)}{|x-y|^{d+\alpha}} \, dy \quad \text{for } u \in C_c^2(D),$$

where $\mathcal{C}(x, y)$ is a measurable symmetric function on $D \times D$ that is bounded between two positive constants. When $\mathcal{C}(x, y) = \mathcal{A}(d, \alpha)$, X is called the censored α-stable process in D.

The objective of this paper is to investigate the precise behavior of the transition density functions $p_D(t, x, y)$ of censored α-stable-like processes. We first discuss the intrinsic ultracontractivity of the semigroups of censored stable-like processes. Intrinsic ultracontractivity was introduced by Davies and Simon in [13]. It is concerned with the "boundary" behavior of the transition density function of the semigroup when the semigroup has discrete spectrum. The intrinsic ultracontractivity gives sharp two-sided estimates of the transition density function for each fixed $t > 0$. The intrinsic ultracontractivity of semigroups of killed jump processes was first considered in [8], where it was shown that the semigroup of the killed symmetric α-stable process on a bounded $C^{1,1}$ domain is intrinsically ultracontractive. In [19] it was shown that the semigroup of the killed symmetric α-stable process on any bounded open set is intrinsically ultracontractive. In this paper, we show that, when D is an open d-set in $\mathbb{R}^d$ with finite Lebesgue measure and ∂D has positive r-dimensional Hausdorff measure for some $r > d - \alpha$, the semigroup of a censored stable-like process in D is intrinsically ultracontractive. In particular, for $\alpha \in (1, 2)$, the semigroup of a censored α-stable-like process in any bounded Lipschitz open set is intrinsically ultracontractive.

The main goal of this paper is to establish sharp two-sided estimates for the transition density functions $p_D(t, x, y)$ (as functions of (t, x, y)) of a large class of censored α-stable-like processes in every $C^{1,1}$ open set $D \subset \mathbb{R}^d$ for $d \geq 1$ and $\alpha \in (1, 2)$. A precise definition of $C^{1,1}$ open set in $\mathbb{R}^d$ will be given in Sect. 3. The transition density function $p_D(t, x, y)$ is also the heat kernel of the operator $\mathcal{L}_D^\alpha$ with zero boundary condition on the boundary D, i.e., for any bounded continuous function f on D, $u(t, x) := \int_D p(t, x, y) f(y) dy$ is the solution to $\mathcal{L}_D^\alpha u = \partial_t u$, $u(0, x) = f(x)$ on D and $u = 0$ on ∂D. Note that in contrast to the killed symmetric α-stable processes, a censored α-stable-like process X in a $C^{1,1}$ open set D with $d \geq 1$ and $\alpha \in (1, 2)$ approaches the boundary ∂D in a continuous way [3, Theorem 1.1] so its infinitesimal generator has zero Dirichlet boundary condition as opposed to zero exterior condition. This indicates that censored processes are natural and important for boundary problems in analysis [16,17].

Now we state the main result of this paper. We assume the censored stable-like processes under consideration enjoy the dilation invariant boundary Harnack principle (**BHP**) (see Sect. 3 for the precise statement). This assumption is automatically satisfied for any censored stable process in a $C^{1,1}$ open set, and it is also satisfied for censored stable-like processes in $C^{1,1}$ open sets when $\mathcal{C}(x, y)$ satisfies certain regularity conditions; see Sect. 3 for details. It is an open problem to find the minimal condition on $\mathcal{C}(x, y)$ so that (**BHP**) holds for the corresponding censored stable-like process in every $C^{1,1}$ open sets.

Theorem 1.1 *Suppose that $d \geq 1$, $\alpha \in (1, 2)$ and D is a $C^{1,1}$ open subset of $\mathbb{R}^d$. Let $\delta_D(x)$ be the Euclidean distance between x and D^c. Suppose that the censored stable-like process X satisfies* (**BHP**) (*see Sect.* 3 *for a precise definition and sufficient conditions for it to be true*).

(i) *For every $T > 0$, on $(0, T] \times D \times D$*

$$p_D(t, x, y) \asymp t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}.$$

(ii) *Suppose in addition that D is bounded. For every $T > 0$, there exist positive constants $c_1 < c_2$ such that for all $(t, x, y) \in [T, \infty) \times D \times D$,*

$$c_1 e^{-\lambda_1 t} \delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1} \leq p_D(t, x, y) \leq c_2 e^{-\lambda_1 t} \delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1},$$

where $-\lambda_1 < 0$ is the largest eigenvalue of $\mathcal{L}_D^\alpha$.

Here and in the sequel, for two non-negative functions f and g, the notation $f \asymp g$ means that there are positive constants c_1 and c_2 so that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ in the common domain of definition for f and g. For $a, b \in \mathbb{R}$, $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$.

By integrating the above two-sided heat kernel estimates in Theorem 1.1 with respect to t, one can easily obtain the following sharp two-sided estimate on the Green function $G_D(x, y) = \int_0^\infty p_D(t, x, y)dt$ of a censored stable-like process in a bounded $C^{1,1}$ open set D.

Corollary 1.2 *Suppose that $d \geq 1$, $\alpha \in (1, 2)$ and D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$. Assume that the censored stable-like process X satisfies* (**BHP**). *Then on $D \times D$, we have*

$$G_D(x, y) \asymp \begin{cases} \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha-1} & \text{when } d \geq 2, \\ (\delta_D(x)\delta_D(y))^{(\alpha-1)/2} \wedge \left(\frac{\delta_D(x)\delta_D(y)}{|x-y|}\right)^{\alpha-1} & \text{when } d = 1. \end{cases}$$

Sharp two-sided estimates of the Green function are very important in understanding deep potential theoretic properties of Markov processes. Such two-sided estimates for the Green functions of symmetric stable processes were obtained in [9,19]. In [4], sharp two-sided estimates for the Green functions of censored stable processes (i.e. when $\mathcal{C}(x, y)$ is a constant) in bounded $C^{1,1}$ connected open sets in $\mathbb{R}^d$ were obtained for $d \geq 2$ and $\alpha \in (1, 2)$. Corollary 1.2 is a significant generalization of the Green function estimates in [4] in that (1) $\mathcal{C}(x, y)$ needs not be constant, (2) the $C^{1,1}$-open set D here does not need to be connected, and (3) $d = 1$ is allowed. We emphasize here that the Green function estimates for censored stable processes obtained in [4] will not be used in this paper.

Theorem 1.1(i) will be established through Theorems 3.5 and 4.9, which give the upper bound and lower bound estimates, respectively. Theorem 1.1(ii) is an easy consequence of Theorem 1.1(i) and the intrinsic ultracontractivity of X in a bounded $C^{1,1}$

open set D, which will be established in Sect. 2. The proofs of Theorem 1.1(ii) and Corollary 1.2 will be given in Sect. 5.

The approach of this paper is adapted from that of [5], which deals with two-sided sharp heat kernel estimates for symmetric α-stable processes killed upon exiting a $C^{1,1}$ open set. In [5], the following domain monotonicity for the killed symmetric stable processes is used in a crucial way. Let Z be a symmetric α-stable process and Z^D be the subprocess of Z killed upon leaving an open set D. If U is an open subset of D, then Z^U is a subprocess of Z^D killed upon leaving U. However censored stable-like processes do not have this kind of domain monotonicity. This lack of domain monotonicity produces new difficulties, which can be seen, for example, from the proofs of the estimate given in Lemma 4.5 of this paper and its exact analog in [5, Lemma 3.6] for symmetric α-stable processes. The proof of [5, Lemma 3.6], which is a key step in deriving the sharp lower bound estimate for the killed symmetric α-stable process in a bounded $C^{1,1}$-open set D, is established by comparing with a suitably chosen interior ball. But such an approach breaks down even for the censored α-stable process. We use a new probabilistic approach together with a crucial application of (**BHP**) to establish the estimate in Lemma 4.5. The intrinsic ultracontractivity of the censored α-stable-like process is also used in our proof.

Another tool that we use in this paper is the reflected stable-like process $\overline{X}$ on $\overline{D}$, whose subprocess killed upon leaving D is the censored stable-like process X. The reflected α-stable-like processes have been studied in [3] and [6]. In particular, two-sided heat kernel estimates have been obtained in [6] for reflected stable-like processes on open d-sets (including globally Lipschitz open sets) in $\mathbb{R}^d$—see (2.4) below. When D is a globally Lipschitz open set, it is proved in [3] that the censored α-stable-like process in D coincides with the corresponding reflected α-stable-like process if (and only if) $\alpha \in (0, 1]$. That is why we focus on the case of $\alpha \in (1, 2)$ in this paper.

The approach of this paper is mainly probabilistic. It is based on the following four key ingredients:

(i) Lévy system of X that describes how the process jumps—see (2.3) below;
(ii) the two-sided heat kernel estimates (2.4) for the reflected α-stable process $\overline{X}$ on $\overline{D}$ obtained in [6] and a scaling property of X—see (3.2) below;
(iii) the boundary Harnack principle of X in $C^{1,1}$ open sets (see Sect. 3) and the parabolic Harnack principle of X obtained in [6];
(iv) inequality (2.9) and the intrinsic ultracontractivity of X in bounded open sets—established in Theorem 2.2 below.

Even though the intrinsic ultracontractivity gives sharp two-sided estimates of the transition density function $p(t, x, y)$ for each fixed $t > 0$, the estimates are far from sharp as a function of (t, x, y). But the inequality (2.9), which implies the intrinsic ultracontractivity, plays an important role in our approach.

Throughout this paper, unless otherwise specified, we assume $d \geq 1$. The Euclidean distance between x and y will be denoted as $|x - y|$. For any open set D, $\delta_D(x) := \mathrm{dist}(x, D^c)$. We will use dx to denote the Lebesgue measure in $\mathbb{R}^d$. Throughout this paper, we use $c_1, c_2, \ldots$ to denote generic constants, whose exact values are not important and can change from one appearance to another. The labeling of the constants

$c_1, c_2, \ldots$ starts anew in the statement of each result. The values of the constants $M_1, M_2, \ldots$ will remain the same throughout this paper and the dependence of the constant c on the dimension d and the constants $M_1, M_2, \ldots$ will not be mentioned explicitly. We will use ":=" to denote a definition, which is read as "is defined to be". We will use ∂ to denote a cemetery point and for every function f, we extend its definition to ∂ by setting $f(\partial) = 0$. For a Borel set $A \subset \mathbb{R}^d$, we also use $|A|$ to denote the Lebesgue measure of A.

2 Censored stable-like process and intrinsic ultracontractivity

Censored α-stable-like processes in open subsets of $\mathbb{R}^d$ were studied by Bogdan et al. [3] (see also [18]). Fix an open set D in $\mathbb{R}^d$ with $d \geq 1$. Define a bilinear form $\mathcal{E}$ on $C_c^\infty(D)$ by

$$\mathcal{E}(u, v) := \frac{1}{2} \int_D \int_D (u(x) - u(y))(v(x) - v(y)) \frac{\mathcal{C}(x, y)}{|x - y|^{d+\alpha}} dx\, dy, \quad u, v \in C_c^\infty(D), \tag{2.1}$$

where $\mathcal{C}(x, y)$ is a measurable symmetric function on $D \times D$ satisfying

$$M_1 \leq \mathcal{C}(x, y) \leq M_2 \tag{2.2}$$

for some positive constants M_1 and M_2. Using Fatou's lemma, it is easy to check that the bilinear form $(\mathcal{E}, C_c^\infty(D))$ is closable in $L^2(D, dx)$. Let $\mathcal{F}$ be the closure of $C_c^\infty(D)$ under the Hilbert inner product $\mathcal{E}_1 := \mathcal{E} + (\cdot, \cdot)_{L^2(D,dx)}$. As noted in [3], $(\mathcal{E}, \mathcal{F})$ is Markovian and hence a regular symmetric Dirichlet form on $L^2(D, dx)$, and therefore there is an associated symmetric Hunt process $X = \{X_t, t \geq 0, \mathbb{P}_x, x \in D\}$ taking values in D (cf. Theorem 3.1.1 of [14]). The process X is called a censored α-stable-like process in D.

We fix an arbitrary symmetric measurable extension of $\mathcal{C}(\cdot, \cdot)$ onto $\mathbb{R}^d \times \mathbb{R}^d$ satisfying (2.2) and we still denote it by $\mathcal{C}(\cdot, \cdot)$. It is well known (see, for instance, [6]) that the bilinear form $(\mathcal{Q}, \mathcal{F}^{\mathbb{R}^d})$ defined by

$$\mathcal{Q}(u, v) = \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (u(x) - u(y))(v(x) - v(y)) \frac{\mathcal{C}(x, y)}{|x - y|^{d+\alpha}} dx\, dy,$$

$$\mathcal{F}^{\mathbb{R}^d} = \left\{ u \in L^2(\mathbb{R}^d) : \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(u(x) - u(y))^2}{|x - y|^{d+\alpha}} dx\, dy < \infty \right\},$$

is a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d, dx)$ and hence there is an associated symmetric Hunt process $Y = \{Y_t, \mathbb{P}_x\}$ on $\mathbb{R}^d$. The process Y is called an α-stable-like process in $\mathbb{R}^d$, which is studied in [6]. Among other things, it is shown in [6] that Y is conservative and has a Hölder continuous transition density function. The latter in particular implies that Y can be modified to start from every point $x \in \mathbb{R}^d$ and the modified process is a Feller process on $\mathbb{R}^d$. Note that if $\mathcal{C}(x, y)$ is equal to the constant $\mathcal{A}(d, \alpha)$, Y is the symmetric α-stable process on $\mathbb{R}^d$.

For any open subset D of $\mathbb{R}^d$, we use Y^D to denote the subprocess of Y killed upon exiting from D. The following result gives two other ways of constructing a censored α-stable-like process.

Theorem 2.1 [3, Theorem 2.1 and Remark 2.4] *The following processes have the same distribution:*

(i) *the symmetric Hunt process X associated with the regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(D, dx)$;*
(ii) *the strong Markov process X obtained from the killed symmetric α-stable-like process Y^D in D through the Ikeda–Nagasawa–Watanabe piecing together procedure;*
(iii) *the process X obtained from Y^D through the Feynman–Kac transform $e^{\int_0^t \kappa_D(Y_s^D)ds}$ with*

$$\kappa_D(x) := \int_{D^c} \frac{\mathcal{C}(x, y)}{|x-y|^{d+\alpha}} dy.$$

The Ikeda–Nagasawa–Watanabe piecing together procedure mentioned in (ii) goes as follows. Let $X_t(\omega) = Y_t^D(\omega)$ for $t < \tau_D(\omega)$. If $Y^D_{\tau_D -}(\omega) \notin D$, set $X_t(\omega) = \partial$ for $t \geq \tau_D(\omega)$. If $Y^D_{\tau_D -}(\omega) \in D$, let $X_{\tau_D}(\omega) = Y^D_{\tau_D -}(\omega)$ and glue an independent copy of Y^D starting from $Y^D_{\tau_D -}(\omega)$ to $X_{\tau_D}(\omega)$. Iterating this procedure countably many times, we obtain a process on D which is a version of the strong Markov process X; the procedure works for every starting point in D. Because of this procedure, a censored stable-like process is also called a resurrected stable-like process.

By (2.1), the jump function $J(x, y)$ of the censored α-stable-like process X is given by

$$J(x, y) = \frac{\mathcal{C}(x, y)}{|x-y|^{d+\alpha}} \quad \text{for } x, y \in D.$$

It determines a Lévy system for X, which describes the jumps of the process X: for any non-negative measurable function f on $\mathbb{R}_+ \times D \times D$, $t \geq 0$, $x \in D$ and stopping time T (with respect to the filtration of X),

$$\mathbb{E}_x \left[\sum_{s \leq T} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^T \left(\int_D f(s, X_s, y) J(X_s, y) dy \right) ds \right], \tag{2.3}$$

(see, for example [7, Appendix A]).

Recall that an open set $D \subset \mathbb{R}^d$ is said to be a d-set if there exist two positive constants c_1, c_2 so that for every $x \in D$ and $0 < r \leq 1$,

$$c_1 r^d \leq |D \cap B(x, r)| \leq c_2 r^d.$$

Clearly any globally Lipschitz open set in $\mathbb{R}^d$ is a d-set. See [3] for examples of non-smooth open d-sets in $\mathbb{R}^d$.

For any open d-set D in $\mathbb{R}^d$, define

$$\mathcal{F}^{\text{ref}} := \left\{ u \in L^2(D) : \int_D \int_D \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}} dx\, dy < \infty \right\}$$

and

$$\mathcal{E}^{\text{ref}}(u,v) := \frac{1}{2} \int_D \int_D (u(x)-u(y))(v(x)-v(y)) \frac{\mathcal{C}(x,y)}{|x-y|^{d+\alpha}} dx\, dy, \quad u, v \in \mathcal{F}^{\text{ref}}.$$

It is shown in [3, Remark 2.1] that the bilinear form $(\mathcal{E}^{\text{ref}}, \mathcal{F}^{\text{ref}})$ is a regular symmetric Dirichlet form on $L^2(\overline{D}, dx)$. The process $\overline{X}$ on $\overline{D}$ associated with $(\mathcal{E}^{\text{ref}}, \mathcal{F}^{\text{ref}})$ is called a reflected α-stable-like process on $\overline{D}$. It is shown in [6, Theorem 1.1] that $\overline{X}$ has a Hölder continuous transition density function $\bar{p}(t,x,y)$ on $(0,\infty) \times \overline{D} \times \overline{D}$ and for every $T_0 > 0$, there are positive constants c_1, c_2 so that for $t \in (0, T_0]$ and $x, y \in \overline{D}$,

$$c_1\, t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{d+\alpha} \le \bar{p}(t,x,y) \le c_2\, t^{-d/\alpha} \left(1 \wedge \frac{t^{1/\alpha}}{|x-y|} \right)^{d+\alpha}. \quad (2.4)$$

The Hölder continuity of $p(t,x,y)$ in particular implies that $\overline{X}$ can be refined to start from every point x in $\overline{D}$ and the refined process is a Feller process on $\overline{D}$. When D is an open d-set in $\mathbb{R}^d$, the censored α-stable-like process X can be realized as a subprocess of $\overline{X}$ killed upon leaving D, see [3, Remark 2.1].

In the remainder of this paper, we will fix an open d-set in $\mathbb{R}^d$ and a symmetric measurable function $\mathcal{C}(\cdot,\cdot)$ on $D \times D$ satisfying (2.2) and a symmetric measurable extension of it onto $\mathbb{R}^d \times \mathbb{R}^d$. Unless explicitly mentioned otherwise, whenever we speak of a censored α-stable-like process X we mean the symmetric Hunt process associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ above on $L^2(D, dx)$, and whenever we speak of an α-stable-like process Y on $\mathbb{R}^d$ (resp. a reflected α-stable-like process $\overline{X}$ on $\overline{D}$) we mean the symmetric Hunt process associated with the Dirichlet form $(\mathcal{Q}, \mathcal{F})$ above on $L^2(\mathbb{R}^d, dx)$ (resp. $(\mathcal{E}^{\text{ref}}, \mathcal{F}^{\text{ref}})$ above on $L^2(\overline{D}, dx)$).

We will use $\{P_t, t \ge 0\}$ to denote the transition semigroup of X. Since X is the subprocess of $\overline{X}$ killed upon exiting D, X has a transition density function $p_D(t,x,y)$ with respect to the Lebesgue measure on D, which is also called the heat kernel of X. It follows from (2.4) that for every $T_0 > 0$, there is a constant $c > 0$ so that

$$p_D(t,x,y) \le c \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \quad \text{on } (0, T_0] \times D \times D. \quad (2.5)$$

For any open set $U \subset D$, we define $\tau_U := \inf\{t > 0 : X_t \notin U\}$ and we will use X^U to denote the subprocess of X killed upon exiting U. Let $\{P_t^U : t \ge 0\}$ be the transition semigroup of X^U and $p_D^U(t,x,y)$ be the transition density function of X^U. We will use G_D^U to denote the Green function of X^U:

$$G_D^U(x,y) := \int_0^\infty p_D^U(t,x,y)dt.$$

When $U = D$, $G_D^D(x,y)$ will simply be denoted by $G_D(x,y)$ and called the Green function of X.

We now show that for any bounded open subset U of D that has the property

$$\mathbb{P}_x(\tau_U < \infty) = 1 \quad \text{for every } x \in U, \tag{2.6}$$

the semigroup $\{P_t^U, t > 0\}$ is intrinsically ultracontractive. Note that condition (2.6) is satisfied if

- (i) $D\backslash U$ has positive Lebesgue measure in view of (2.4) and the strong Markov property of X; or
- (ii) $U = D$ is a bounded Lipschitz open set and $\alpha \in (1,2)$ in view of [3, Theorem 1.1].

The intrinsic ultracontractivity for the case $U = D$ when D is a bounded $C^{1,1}$ open set will be used to derive Theorem 1.1(ii) and the intrinsic ultracontractivity for the case $U \neq D$ will be used to derive Theorem 1.1(i).

By (2.5), we know that for any bounded open subset U of D, the semigroup $\{P_t^U, t > 0\}$ is a semigroup of Hilbert–Schmidt operators and hence is compact. Let $-\lambda_1^U < 0$ be the largest eigenvalue of the generator of X^U and let $\phi_1^U(x)$ be the positive eigenfunction of P_1^U corresponding to $e^{-\lambda_1^U}$ with $\|\phi_1^U\|_{L^2(U)} = 1$. When D is bounded and $U = D$, λ_1^U and ϕ_1^U will be denoted as λ_1 and ϕ_1, respectively. The semigroup $\{P_t^U, t > 0\}$ is said to be intrinsically ultracontractive if for any $t > 0$ there exists a positive constant $C_t > 1$ such that

$$p_D^U(t,x,y) \le C_t\, \phi_1^U(x)\phi_1^U(y) \quad \text{for } x, y \in U. \tag{2.7}$$

It follows from [13, Theorem 3.2] that if $\{P_t^U, t > 0\}$ is intrinsically ultracontractive then for any $t > 0$ there exists a positive constant $c_t > 1$ such that

$$p_D^U(t,x,y) \ge c_t^{-1}\phi_1^U(x)\phi_1^U(y) \quad \text{for } x, y \in U. \tag{2.8}$$

The proof of the following result is adapted from an argument given in [20].

Theorem 2.2 *Suppose that D is an open d-set in $\mathbb{R}^d$ and U is a bounded open subset of D satisfying condition (2.6). Then the semigroup $\{P_t^U, t > 0\}$ is intrinsically ultracontractive. Moreover, for every $B(x_0, 2r) \subset U$ there exists a constant $c = c(\alpha, r, diam(U)) > 0$ which is independent of D and depends on the function $\mathcal{C}(\cdot,\cdot)$*

only via the constants M_1, M_2 in (2.2) such that

$$\mathbb{E}_x\left[\int_0^{\tau_U} \mathbf{1}_{B(x_0,r)}(X_t^U)dt\right] \geq c\,\mathbb{E}_x\left[\tau_U\right] \quad \textit{for every } x \in U. \tag{2.9}$$

Proof Fix a ball $B(x_0, 2r) \subset U$ and put

$$B_0 := B(x_0, r/4), \quad C_1 := \overline{B(x_0, r/2)} \quad \text{and} \quad B_2 := B(x_0, r).$$

Let $\{\theta_t, t > 0\}$ be the time-shift operators of X and we define stopping times S_n and T_n recursively by

$$\begin{aligned} S_1(\omega) &:= 0, \\ T_n(\omega) &:= S_n(\omega) + \tau_{U\setminus C_1} \circ \theta_{S_n}(\omega) \quad \text{for } S_n(\omega) < \tau_U \\ \text{and} \quad S_{n+1}(\omega) &:= T_n(\omega) + \tau_{B_2} \circ \theta_{T_n}(\omega) \quad \text{for } T_n(\omega) < \tau_U. \end{aligned}$$

Clearly $S_n \leq \tau_U$. Let $S := \lim_{n\to\infty} S_n \leq \tau_U$. On $\{S < \tau_U\}$, we must have $S_n < T_n < S_{n+1}$ for every $n \geq 0$. Using (2.6) and the quasi-left continuity of X^U, we have $\mathbb{P}_x(S < \tau_U) = 0$. Therefore, for every $x \in U$,

$$\mathbb{P}_x\left(\lim_{n\to\infty} S_n = \lim_{n\to\infty} T_n = \tau_U\right) = 1. \tag{2.10}$$

We claim that there exists a constant $c_1 = c_1(\alpha, r) > 0$ depending on the function $\mathcal{C}(\cdot,\cdot)$ only via the constants M_1 and M_2 in (2.2) such that

$$\mathbb{E}_x[\tau_{B_2}] \geq c_1 \quad \text{for every } x \in C_1. \tag{2.11}$$

In fact, for any $x \in C_1$, we have

$$\mathbb{E}_x[\tau_{B_2}] \geq \mathbb{E}_x[\tau_{B(x,r/2)}] \geq \mathbb{E}_x[\tau^Y_{B(x,r/2)}] \geq c_1,$$

where in the second inequality above, we used Theorem 2.1 and in the third inequality above, we used [6, Proposition 4.1]. Here Y denotes the symmetric α-stable-like process in $\mathbb{R}^d$ (corresponding to a fixed symmetric measurable extension of $\mathcal{C}(\cdot,\cdot)$ satisfying (2.2)) and $\tau^Y_{B(x,r/2)}$ the exit time from the ball $B(x, r/2)$ by Y. Now it follows from the strong Markov property that

$$\begin{aligned} \mathbb{E}_x\left[S_{n+1} - T_n\right] &= \mathbb{E}_x\left[\mathbb{E}_{X^U_{T_n}}[\tau_{B_2}];\ T_n < \tau_U\right] \geq c_1\mathbb{P}_x(X^U_{T_n} \in B_0) \\ &= c_1\mathbb{E}_x\left[\mathbb{P}_{X^U_{S_n}}(X^U_{\tau_{U\setminus C_1}} \in B_0)\right]. \end{aligned}$$

Note that for any $x \in U \backslash B_2$, by the Lévy system of X in (2.3), we have

$$\begin{aligned}
\mathbb{P}_x\left(X^U_{\tau_{U\backslash C_1}} \in B_0\right) &= \int\limits_{U\backslash C_1} G_D^{U\backslash C_1}(x,y)\int\limits_{B_0}\left(\frac{\mathcal{C}(y,z)}{|y-z|^{d+\alpha}}dz\right)dy \\
&\geq M_1 \int\limits_{U\backslash C_1} G_D^{U\backslash C_1}(x,y)\int\limits_{B_0}\left(\frac{dz}{(\mathrm{diam}(U))^{d+\alpha}}\right)dy \\
&= c_2\mathbb{E}_x[\tau_{U\backslash C_1}]
\end{aligned}$$

for some constant $c_2 = c_2(\alpha, r, \mathrm{diam}(U)) > 0$. It follows then

$$\mathbb{E}_x\left[S_{n+1} - T_n\right] \geq c_1c_2\mathbb{E}_x\left[\mathbb{E}_{X^U_{S_n}}[\tau_{U\backslash C_1}]\right] = c_1c_2\mathbb{E}_x[T_n - S_n]. \tag{2.12}$$

Since $X^U_t \in B_2$ for $T_n < t < S_{n+1}$, we have

$$\begin{aligned}
\mathbb{E}_x\left[\int\limits_0^{\tau_U}\mathbf{1}_{B_2}(X^U_t)dt\right] &= \mathbb{E}_x\left[\sum_{n=1}^{\infty}\left(\int\limits_{S_n}^{T_n}\mathbf{1}_{B_2}(X^U_t)dt + \int\limits_{T_n}^{S_{n+1}}\mathbf{1}_{B_2}(X^U_t)dt\right)\right] \\
&\geq \mathbb{E}_x\left[\sum_{n=1}^{\infty}\left(\int\limits_{T_n}^{S_{n+1}}\mathbf{1}_{B_2}(X^U_t)dt\right)\right] \\
&= \mathbb{E}_x\left[\sum_{n=1}^{\infty}(S_{n+1} - T_n)\right].
\end{aligned}$$

Using (2.12) and noting that $X^U_t \notin U\backslash B_2$ for $t \in [T_n, S_{n+1})$, we get

$$\begin{aligned}
\mathbb{E}_x\left[\int\limits_0^{\tau_U}\mathbf{1}_{B_2}(X^U_t)dt\right] &\geq c_1c_2\mathbb{E}_x\left[\sum_{n=1}^{\infty}(T_n - S_n)\right] \\
&\geq c_1c_2\mathbb{E}_x\left[\sum_{n=1}^{\infty}\left(\int\limits_{S_n}^{T_n}\mathbf{1}_{U\backslash B_2}(X^U_t)dt + \int\limits_{T_n}^{S_{n+1}}\mathbf{1}_{U\backslash B_2}(X^U_t)dt\right)\right] \\
&= c_1c_2\mathbb{E}_x\left[\int\limits_0^{\tau_U}\mathbf{1}_{U\backslash B_2}(X^U_t)dt\right].
\end{aligned}$$

Thus

$$\mathbb{E}_x\left[\int\limits_0^{\tau_U}\mathbf{1}_{B_2}(X^U_t)dt\right] \geq \frac{c_1\,c_2}{1 + c_1\,c_2}\mathbb{E}_x\left[\tau_U\right].$$

Since $\phi_1^U = e^{\lambda_1^U} P_1^U \phi_1^U$, it follows that ϕ_1^U is strictly positive and continuous in U [19]. The above inequality implies that

$$\mathbb{E}_x[\tau_U] \le c_3 \int_{B_2} G_D^U(x,z)\phi_1^U(z)dz \le c_3 \int_U G_D^U(x,z)\phi_1^U(z)dz = \frac{c_3}{\lambda_1^U}\phi_1^U(x). \tag{2.13}$$

By the semigroup property and (2.5),

$$\begin{aligned} p_D^U(t,x,y) &= \int_U p_D^U(t/3,x,z)\int_U p_D^U(t/3,z,w)p_D^U(t/3,w,y)dwdz \\ &\le c_4 t^{-d/\alpha}\int_U p_D^U(t/3,x,z)dz \int_U p_D^U(t/3,w,y)dw \\ &= c_4 t^{-d/\alpha}\,\mathbb{P}_x(\tau_U > t/3)\,\mathbb{P}_y(\tau_U > t/3) \\ &\le \left(9c_4/t^2\right) t^{-d/\alpha}\,\mathbb{E}_x[\tau_U]\,\mathbb{E}_y[\tau_U]. \end{aligned} \tag{2.14}$$

This together with (2.13) establishes the intrinsic ultracontractivity of X^U. □

Remark 2.3 (i) When $U = D$, sufficient conditions for (2.6) to hold can be found in [3, Theorem 2.4 and Theorem 2.7]. In particular, we know from there that if D is an open d-set in $\mathbb{R}^d$ with finite Lebesgue measure and ∂D has positive r-dimensional Hausdorff measure, then condition (2.6) holds when $\alpha > d - r$. In this case, by Theorem 2.2, the semigroup of the censored α-stable-like process in D is intrinsically ultracontractive. Clearly the latter assertion holds for any bounded Lipschitz domain $D \subset \mathbb{R}^d$ and $\alpha \in (1,2)$.

(ii) By considering $D = \mathbb{R}^d$, we get the intrinsic ultracontractivity of the killed symmetric α-stable-like process Y^U for every bounded open subset U, first proved in [20].

3 Upper bound estimate

In this section, we establish sharp upper bound heat kernel estimates for X in a $C^{1,1}$ open subset $D \subset \mathbb{R}^d$.

Recall that an open set D in $\mathbb{R}^d$ (when $d \ge 2$) is said to be a $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda_0 > 0$ such that for every $z \in \partial D$, there exist a $C^{1,1}$-function $\phi = \phi_z : \mathbb{R}^{d-1} \to \mathbb{R}$ satisfying $\phi(0) = \nabla\phi(0) = 0$, $\|\nabla\phi\|_\infty \le \Lambda_0$, $|\nabla\phi(x) - \nabla\phi(z)| \le \Lambda_0|x-z|$, and an orthonormal coordinate system CS_z: $y = (y_1, \dots, y_{d-1}, y_d) := (\widetilde{y}, y_d)$ with its origin at z such that

$$B(z, r_0) \cap D = \{y \in CS_z : |y| < r_0,\ y_d > \phi(\widetilde{y})\}.$$

By a $C^{1,1}$ open set in $\mathbb{R}$ we mean an open set which can be written as the union of disjoint intervals so that the minimum of the lengths of all these intervals is positive and the minimum of the distances between these intervals is positive. It is well known that any $C^{1,1}$ open set D satisfies the uniform interior and exterior ball conditions: there exists $r_0 < R_0$, that depends only on (R_0, Λ_0), such that (1) for any $x \in D$ with $\delta_D(x) \le r_0$, there is a unique $z_x \in \partial D$ such that $|x - z_x| = \delta_D(x)$ and (2) for any $z \in \partial D$ and $r \in (0, r_0]$ there exist two balls B_1^z and B_2^z of radius r such that $B_1^z \subset D$, $B_2^z \subset \mathbb{R}^d \setminus \overline{D}$ and $\partial B_1^z \cap \partial B_2^z = \{z\}$. For simplicity, in this paper we call the pair (r_0, Λ_0) the characteristics of the $C^{1,1}$ open set D. Note that for a $C^{1,1}$ open set D with characteristics (r_0, Λ_0), for every $T > 0$ and every $\lambda \in (0, T]$, $\lambda^{-1} D$ is a $C^{1,1}$ open set with (uniform) characteristics $(r_0/T, T\Lambda_0)$. This trivial but important fact will be used several times in this paper.

When D is a bounded Lipschitz open set in $\mathbb{R}^d$, by [3, Theorem 1.1] the censored α-stable-like process X in D is recurrent if and only if $\alpha \le 1$. In this case as well as the case $D = \mathbb{R}^d$, X is the same as the reflected α-stable-like process $\overline{X}$, and so the sharp two-sided estimates (2.4) holds for the transition density function of X.

In the remainder of this section, we assume $\alpha \in (1, 2)$. In this case, every censored α-stable-like process in a $C^{1,1}$ open proper subset of $\mathbb{R}^d$ is transient by [3, Theorem 2.7 and Remark 2.4]. The following scaling property will be used several times in the rest of this paper: If $\{X_t, t \ge 0\}$ is a censored α-stable-like process in D with the jump function

$$J(x, y) = \frac{\mathcal{C}(x, y)}{|x - y|^{d+\alpha}}, \quad x, y \in D,$$

then $\{X_t^{(\lambda)}, t \ge 0\} := \{\lambda^{-1} X_{\lambda^\alpha t}, t \ge 0\}$ is a censored α-stable-like process in $\lambda^{-1} D$ with jump function

$$J^{(\lambda)}(x, y) := \frac{\mathcal{C}(\lambda x, \lambda y)}{|x - y|^{d+\alpha}} \quad \text{for } x, y \in \lambda^{-1} D. \tag{3.1}$$

For any $\lambda > 0$, we define

$$p_{\lambda^{-1} D}(t, x, y) := \lambda^d p_D(\lambda^\alpha t, \lambda x, \lambda y) \quad \text{for } t > 0 \text{ and } x, y \in \lambda^{-1} D. \tag{3.2}$$

Clearly $p_{\lambda^{-1} D}(t, x, y)$ is the transition density function of the censored α-stable-like process $\{X_t^{(\lambda)}, t \ge 0\}$ with the jump function $J^{(\lambda)}(x, y)$. We shall denote the lifetime of $X^{(\lambda)}$ by $\zeta^{(\lambda)}$.

A key ingredient in proving our main result is a scale invariant boundary Harnack principle. We formulate this as an assumption and then we will discuss when it is satisfied. Recall that a nonnegative function u defined on D is said to be harmonic in $U \subset D$ with respect to X if $u(x) = \mathbb{E}_x[u(X_{\tau_B})]$ for every $x \in B$ and every open set B whose closure is a compact subset of U.

The next result is proved in [3, Theorem 1.2].

Theorem 3.1 *Let D be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (r_0, Λ_0) and X the censored α-stable process in D. Then there exists a positive constant $c = c(\alpha, \Lambda_0)$*

such that for $r \in (0, r_0]$, $Q \in \partial D$ and any nonnegative function u in D which is harmonic in $D \cap B(Q, r)$ with respect to X and vanishes continuously on $\partial D \cap B(Q, r)$, we have

$$\frac{u(x)}{u(y)} \le c \frac{\delta_D(x)^{\alpha-1}}{\delta_D(y)^{\alpha-1}} \quad \textit{for every } x, y \in D \cap B(Q, r/2).$$

If X is the censored α-stable process in $C^{1,1}$ open set D with characteristics (r_0, Λ_0), then for every $T > 0$ and every $\lambda \in (0, T]$, $\{X_t^{(\lambda)}, t \ge 0\} := \{\lambda^{-1} X_{\lambda^\alpha t}, t \ge 0\}$ is a censored α-stable process in $\lambda^{-1}D$, which is a $C^{1,1}$ open set with characteristics $(r_0/T, T\Lambda_0)$. Thus Theorem 3.1 is applicable with the comparison constant invariant under the domain dilation $\lambda^{-1}D$ for every $\lambda \le T$. To prove Theorem 1.1 for the censored α-stable-like process X, we need the following version of the boundary Harnack principle with the comparison constant invariant under the domain dilation $\lambda^{-1}D$ for $\lambda \le T$.

(**BHP**) : *For any $C^{1,1}$ open set D in $\mathbb{R}^d$ with characteristics (r_0, Λ_0) and every $T > 0$, there exists a positive constant $c = c(\alpha, \Lambda_0, T, \mathcal{C})$ independent of λ such that for $\lambda \in (0, T]$, $r \in (0, r_0/\lambda]$, $Q \in \partial(\lambda^{-1}D)$ and any nonnegative function u in $\lambda^{-1}D$ that is harmonic in $(\lambda^{-1}D) \cap B(Q, r)$ with respect to $X_t^{(\lambda)}$ and vanishes continuously on $\partial(\lambda^{-1}D) \cap B(Q, r)$, we have*

$$\frac{u(x)}{u(y)} \le c \frac{\delta_D(x)^{\alpha-1}}{\delta_D(y)^{\alpha-1}} \quad \textit{for every } x, y \in (\lambda^{-1}D) \cap B(Q, r/2).$$

As we discussed above, censored α-stable processes have the above property. Under some assumptions on $\mathcal{C}(x, y)$, censored stable-like processes also have this property. We now present some sufficient condition for (**BHP**) to hold.

Assume that the (symmetric) function $\mathcal{C}(x, y)$ satisfies the following conditions: there exist positive bounded functions $\psi_1, \psi_2 \in C^1(\overline{D} \times \overline{D})$ and positive constants c and $\delta < r_0$ such that for every $x, y \in \{z \in D : \delta_D(z) < \delta\}$

$$\left| \mathcal{C}(x, y) - \psi_1(x, y) - \psi_2(x, y) \frac{|x-y|^{d+\alpha}}{|x-\overline{y}|^{d+\alpha}} \right| \le c|x-y| \tag{3.3}$$

and

$$|\mathcal{C}(x, y) - \mathcal{C}(x, x)| \le c|x-y| \quad \text{for every } x, y \in \{z \in D : \delta_D(z) > \delta\}. \tag{3.4}$$

Here $\overline{y} := 2z_y - y$ is the reflection of y with respect to ∂D; more precisely, $z_y \in \partial D$ is the unique point such that $\delta_D(y) = |y - z_y|$. Put

$$M_3 := c + \sup_{x, y \in D, |x-y| < r_0} \left(|\nabla_y \psi_1(x, y)| + |\nabla_y \psi_2(x, y)| \right) \tag{3.5}$$

with c being the constant in (3.3) and (3.4). It is proved in [15] that under the assumptions (3.3) and (3.4), the boundary Harnack principle holds for the censored stable-like process X in a $C^{1,1}$ open set D with the characteristics (r_0, Λ_0) and the comparison constant depends only on α, Λ_0, M_1, M_2, M_3 and d.

In particular, if $\mathcal{C}(x, y)$ is in $C^1(\overline{D} \times \overline{D})$ with bounded derivatives, the assumptions (3.3) and (3.4) hold with $\psi_1(x, y) = \mathcal{C}(x, y)$ and $\psi_2(x, y) \equiv 0$. The reason for the general form of (3.3) is to cover cases where the derivatives of $\mathcal{C}$ are not bounded. For example, let $D = \mathbb{R}^d_+$ be the upper half space and $\mathcal{C}(x, y) = \mathcal{A}(d, \alpha)\left(1 + \frac{|x-y|^{d+\alpha}}{|x-\overline{y}|^{d+\alpha}}\right)$. Then the conditions (3.3) and (3.4) are satisfied.

Recall the fact that for a $C^{1,1}$ open set D with characteristics (r_0, Λ_0), for every $T > 0$ and every $\lambda \in (0, T]$, $\lambda^{-1}D$ is a $C^{1,1}$ open set with characteristics $(r_0/T, T\Lambda_0)$. Thus it is easy to see that under the assumptions (3.3) and (3.4) on $\mathcal{C}(x, y)$, such a censored α-stable-like process enjoys (**BHP**) with a comparison constant c independent of $\lambda \in (0, T]$. Recall that that the dependence of the constant c on $\mathcal{C}$ will not be shown in notation.

The next lemma and its proof are similar to [2, Lemma 6] and its proof.

Lemma 3.2 *Suppose that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (r_0, Λ_0) and X is the censored α-stable-like process in $D \subset \mathbb{R}^d$ where $d \geq 1$ and $\alpha \in (0, 2)$. For every $r \leq r_0$, $z \in \partial D$ and $U := D \cap B(z, r)$*

$$\mathbb{P}_x(\tau_U < \zeta \text{ and } X_{\tau_U} \in \partial U) = 0 \quad \text{for every } x \in U.$$

Proof Let Y be the symmetric stable-like process in $\mathbb{R}^d$ with the jump function

$$J_Y(x, y) = \mathcal{C}(x, y)|x - y|^{-d-\alpha} \quad \text{for } x, y \in \mathbb{R}^d.$$

For any open set $V \subset \mathbb{R}^d$, let $\tau^Y_V := \inf\{t > 0 : Y_t \notin V\}$. By Theorem 2.1(iii), we have for every $x \in D$,

$$\mathbb{P}_x(\tau_U < \zeta \text{ and } X_{\tau_U} \in \partial U) = \mathbb{E}_x\left[\exp\left(\int_0^{\tau^Y_U} \kappa_D(Y_s)ds\right); \tau^Y_U < \tau^Y_D \text{ and } Y_{\tau^Y_U} \in \partial U\right].$$

Thus it suffices to show that $\mathbb{P}_x(Y_{\tau^Y_U} \in \partial U) = 0$ for every $x \in U$.

For each $x \in U$, let $B_x := B(x, \delta_U(x)/3)$. By the Lévy system for Y, we have

$$\mathbb{P}_x\left(Y_{\tau^Y_{B_x}} \in U^c\right) = \int_{B_x} G^Y_{B_x}(x, y)\left(\int_{U^c} \frac{\mathcal{C}(y, z)}{|y - z|^{d+\alpha}}\, dz\right) dy,$$

where $G^Y_{B_x}$ is the Green function of Y^{B_x}. By the changes of variables $a = y/\delta_D(x)$ and $b = z/\delta_D(x)$,

$$\mathbb{P}_x\left(Y_{\tau^Y_{B_x}} \in U^c\right) = \int_{B(\delta_U(x)^{-1}x,1/3)} G^Y_{B_x}(x, \delta_U(x)a) \times \left(\int_{(\delta_U(x)^{-1}U)^c} \delta_U(x)^{d-\alpha} \frac{\mathcal{C}(\delta_U(x)a, \delta_U(x)b)}{|a-b|^{d+\alpha}} db \right) da. \tag{3.6}$$

Let $\widehat{Y}_t := \delta_U(x)^{-1} Y_{\delta_U(x)^\alpha t}$, which is the symmetric stable-like process with the jump function $\widehat{J}(a, b) := \mathcal{C}(\delta_U(x)a, \delta_U(x)b)|a-b|^{-d-\alpha}$. Since

$$\widehat{G}^{\widehat{Y}}_{B(\delta_U(x)^{-1}x,1/3)}(w, a) := \delta_U(x)^{d-\alpha}\, G^Y_{B_x}(\delta_U(x)w, \delta_U(x)a)$$

is the Green function of the subprocess of $\widehat{Y}$ killed upon exiting $B(\delta_U(x)^{-1}x, 1/3)$, we have by (3.6)

$$\begin{aligned} \mathbb{P}_x\left(Y_{\tau^Y_{B_x}} \in U^c\right) &= \int_{B(\delta_U(x)^{-1}x,1/3)} \widehat{G}^{\widehat{Y}}_{B(\delta_U(x)^{-1}x,1/3)}(\delta_U(x)^{-1}x, a) \\ &\quad \times \left(\int_{(\delta_U(x)^{-1}U)^c} \frac{\mathcal{C}(\delta_U(x)a, \delta_U(x)b)}{|a-b|^{d+\alpha}} db \right) da \\ &\geq M_1 \int_{B(\delta_U(x)^{-1}x,1/3)} \widehat{G}^{\widehat{Y}}_{B(\delta_U(x)^{-1}x,1/3)}(\delta_U(x)^{-1}x, a) \\ &\quad \times \left(\int_{(\delta_U(x)^{-1}U)^c} \frac{1}{|a-b|^{d+\alpha}} db \right) da. \end{aligned} \tag{3.7}$$

Let $z_x \in \partial U$ be such that $\delta_U(x) = |x - z_x|$. Since D is $C^{1,1}$, there exists $\eta > 0$ such that, under an appropriate coordinate system, we have $z_x + \widehat{C} \subset (\delta_U(x)^{-1}U)^c$ where

$$\widehat{C} := \left\{ y = (y_1, \ldots, y_d) \in \mathbb{R}^d : 0 < y_d < \eta \text{ and } \sqrt{y_1^2 + \cdots + y_{d-1}^2} < \eta y_d \right\}.$$

Thus there is a constant $c_1 > 0$ such that

$$\int_{(\delta_U(x)^{-1}U)^c} \frac{1}{|a-b|^{d+\alpha}} db \geq c_1 > 0 \quad \text{for every } a \in B(\delta_U(x)^{-1}x, 1/3).$$

So we deduce from (3.7)

$$\inf_{x\in U} \mathbb{P}_x\left(Y_{\tau^Y_{B_x}} \in U^c\right) \geq M_1 c_1 \quad \inf_{w\in\mathbb{R}^d} \mathbb{E}_w\left[\tau^{\widehat{Y}}_{B(w,1/3)}\right] \geq c_2 > 0. \tag{3.8}$$

In the second inequality above, we used [6, Proposition 4.1]. On the other hand, by the Lévy system for Y,

$$\mathbb{P}_x\left(Y_{\tau^Y_{B_x}} \in \partial U\right) = 0 \quad \text{for every } x \in U.$$

So

$$\mathbb{P}_x\left(Y_{\tau^Y_U} \in \partial U\right) = \mathbb{E}_x\left[\mathbb{P}_{Y_{\tau^Y_{B_x}}}(Y_{\tau^Y_U} \in \partial U);\, Y_{\tau^Y_{B_x}} \in U\right].$$

Inductively, we have

$$\mathbb{P}_x\left(Y_{\tau^Y_U} \in \partial U\right) = \lim_{k\to\infty} p_k(x),$$

where

$$p_0(x) := \mathbb{P}_x\left(Y_{\tau^Y_U} \in \partial U\right) \quad \text{and} \quad p_k(x) := \mathbb{E}_x\left[p_{k-1}(Y_{\tau^Y_{B_x}});\, Y_{\tau^Y_{B_x}} \in U\right] \text{ for } k \geq 1.$$

By (3.8),

$$\sup_{x\in U} p_{k+1}(x) \leq (1-c_2) \sup_{x\in U} p_k(x) \leq (1-c_2)^{k+1} \to 0.$$

Therefore

$$\mathbb{P}_x\left(Y_{\tau^Y_U} \in \partial U\right) = 0 \quad \text{for every } x \in U.$$

□

The goal of the rest of this section is to prove the upper bound in Theorem 1.1(i). [6, Theorem 1.1] and the fact that, for every $\lambda \in (0, T]$, $\lambda^{-1}D$ is a $C^{1,1}$ open set with characteristics $(r_0/T, T\Lambda_0)$ imply that, for every $T, T_1 > 0$, there exists a constant $c = c(\alpha, r_0, T, T_1) > 0$ such that for every $\lambda \in (0, T]$,

$$p_{\lambda^{-1}D}(t,x,y) \leq c\left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \quad \text{on } (0,T_1]\times(\lambda^{-1}D)\times(\lambda^{-1}D). \tag{3.9}$$

For the rest of this paper, we put $r_1 = r_0/10$.

Lemma 3.3 *Suppose that $\alpha \in (1,2)$ and that D is a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (r_0, Λ_0). For every $T > 0$, there is a constant $c = c(r_0, \alpha, \Lambda_0, T, r) > 0$ such that for all $\lambda \in (0, T]$, $t \in (0, T]$ and all $x, y \in \lambda^{-1}D$ with $\delta_{\lambda^{-1}D}(x) < r_1/(4T)$ and $|x - y| \geq 10r_1/T$,*

$$p_{\lambda^{-1}D}(t, x, y) \leq c\, \frac{\delta_{\lambda^{-1}D}(x)^{\alpha-1}}{|x-y|^{d+\alpha}}.$$

Proof Fix $T > 0, \lambda \in (0, T]$ and $t \in (0, T]$. Let $x, y \in \lambda^{-1}D$ be such that $\delta_{\lambda^{-1}D}(x) < r_1/(4T)$ and $|x - y| \geq 10r_1/T$, and choose $z_x \in \partial(\lambda^{-1}D)$ such that $\delta_{\lambda^{-1}D}(x) = |x - z_x|$. Define $U := (\lambda^{-1}D) \cap B(z_x, r_1/(2T))$ and let $p^U_{\lambda^{-1}D}(t, x, y)$ denote the transition density function of the subprocess $X^{\lambda,U}$ of $X^{(\lambda)}$ killed upon exiting U. By the strong Markov property,

$$p_{\lambda^{-1}D}(t, x, y) = \mathbb{E}_x \left[p_{\lambda^{-1}D}\left(t - \tau_U^{(\lambda)}, X^{(\lambda)}_{\tau_U^{(\lambda)}}, y\right) : \tau_U^{(\lambda)} < t < \zeta^{(\lambda)} \right] \tag{3.10}$$

where $\tau_U^{(\lambda)} := \inf\{t > 0 : X_t^{(\lambda)} \notin U\}$. Define $V_1 := \{w \in \lambda^{-1}D : r_1/(2T) < |w - z_x| \leq 3|x - y|/4\}$ and $V_2 := \{w \in \lambda^{-1}D : |w - z_x| > 3|x - y|/4\}$. It follows from (2.3), (3.10) and Lemma 3.2 that

$$\begin{aligned}
&p_{\lambda^{-1}D}(t, x, y)\\
&= \int_0^t \left(\int_U p^U_{\lambda^{-1}D}(s, x, z) \left(\int_{\{w\in\lambda^{-1}D:|w-z_x|>r_1/(2T)\}} J^{(\lambda)}(z, w) p_{\lambda^{-1}D}(t-s, w, y) dw \right) dz \right) ds\\
&= \int_0^t \left(\int_U p^U_{\lambda^{-1}D}(s, x, z) \left(\int_{V_1} J^{(\lambda)}(z, w) p_{\lambda^{-1}D}(t-s, w, y) dw \right) dz \right) ds\\
&\quad + \int_0^t \left(\int_U p^U_{\lambda^{-1}D}(s, x, z) \left(\int_{V_2} J^{(\lambda)}(z, w) p_{\lambda^{-1}D}(t-s, w, y) dw \right) dz \right) ds.\\
&= I + II.
\end{aligned} \tag{3.11}$$

Note that for $w \in V_1$,

$$|w - y| \geq |y - x| - |w - z_x| - |x - z_x| \geq \frac{|x-y|}{4} - \frac{r_1}{4T} \geq \frac{3|x-y|}{20}. \tag{3.12}$$

By (3.9) and (3.12), there exist positive constants $c = c(\alpha, r_0, T)$ and $c_1 = c_1(\alpha, r_0, T)$ such that

$$
\begin{aligned}
I &\le \int_0^t \left(\int_U p^U_{\lambda^{-1}D}(s, x, z) \left(\int_{V_1} J^{(\lambda)}(z, w) \frac{c\,T}{|w-y|^{d+\alpha}} dw \right) dz \right) ds \\
&\le \frac{c_1 T}{|x-y|^{d+\alpha}} \int_0^t \left(\int_U p^U_{\lambda^{-1}D}(s, x, z) \left(\int_{V_1} J^{(\lambda)}(z, w) dw \right) dz \right) ds \\
&= \frac{c_1 T}{|x-y|^{d+\alpha}} \mathbb{P}_x \left(X^{(\lambda)}_{\tau^{(\lambda)}_U} \in V_1 \text{ and } \tau^{(\lambda)}_U < t \right) \\
&\le \frac{c_1 T}{|x-y|^{d+\alpha}} \mathbb{P}_x \left(X^{(\lambda)}_{\tau^{(\lambda)}_U} \in V_1 \right).
\end{aligned}
$$

Let $\mathbf{n}(z_x)$ be the unit inward normal of $\lambda^{-1}D$ at the point z_x. Put $x_0 = z_x + \frac{r_1}{4T}\mathbf{n}(z_x)$. Note that $x_0 \in (\lambda^{-1}D) \cap B(z_x, r_1/(4T)) \subset U$ and $\delta_{\lambda^{-1}D}(x_0) = r_1/(4T)$. It follows from (**BHP**) that there exists a constant $c_2 = c_2(r_0, \alpha, T, \Lambda_0) > 0$ such that

$$
\mathbb{P}_x \left(X^{(\lambda)}_{\tau^{(\lambda)}_U} \in V_1 \right) \le c_2\, \mathbb{P}_{x_0} \left(X^{(\lambda)}_{\tau^{(\lambda)}_U} \in V_1 \right) \frac{\delta_{\lambda^{-1}D}(x)^{\alpha-1}}{\delta_{\lambda^{-1}D}(x_0)^{\alpha-1}} \le c_2\, \delta_{\lambda^{-1}D}(x)^{\alpha-1}.
$$

Thus we have

$$
I \le c_3\, (T \vee 1) \frac{\delta_{\lambda^{-1}D}(x)^{\alpha-1}}{|x-y|^{d+\alpha}} \tag{3.13}
$$

for some $c_3 = c_3(r_0, \alpha, T, \Lambda_0) > 0$. On the other hand, for $z \in U$ and $w \in V_2$,

$$
|z-w| \ge |w-z_x| - |z-z_x| \ge \frac{3|x-y|}{4} - \frac{r_1}{2T} \ge \frac{7|x-y|}{20}.
$$

Thus by the symmetry of $p_{\lambda^{-1}D}(t-s, w, y)$ in (w, y) and (2.9) of $X^{\lambda, U}$, we have

$$
\begin{aligned}
II &\le \int_0^t \left(\int_U p^U_{\lambda^{-1}D}(s, x, z) \left(\int_{V_2} \frac{c_4}{|x-y|^{d+\alpha}} p_{\lambda^{-1}D}(t-s, y, w) dw \right) dz \right) ds \\
&\le \frac{c_4}{|x-y|^{d+\alpha}} \int_0^\infty \left(\int_U p^U_{\lambda^{-1}D}(s, x, z) dz \right) ds \\
&\le \frac{c_5}{|x-y|^{d+\alpha}} \mathbb{E}_x \left[\int_0^{\tau^{(\lambda)}_U} \mathbf{1}_{B(x_0, r_1/(16T))}(X^{(\lambda)}_s) ds \right]
\end{aligned}
$$

for some positive constants c_4 and $c_5 = c_5(r_0, \alpha)$. Take $x_1 = z_x + \frac{r_1}{16T}\mathbf{n}(z_x)$. By (**BHP**), the last expectation above is bounded by

$$c_6 \mathbb{E}_{x_1}\left[\int_0^{\tau_U^{(\lambda)}} \mathbf{1}_{B(x_0, r_1/(16T))}(X_s^{(\lambda)})ds\right] \frac{\delta_{\lambda^{-1}D}(x)^{\alpha-1}}{\delta_{\lambda^{-1}D}(x_1)^{\alpha-1}}$$

for some $c_6 = c_6(r_0, \alpha, T, \Lambda_0) > 0$.

To bound the expectation in the last display, let $(\mathcal{E}^{(\lambda)}, \mathcal{F}^{(\lambda)})$ be the Dirichlet form of $X^{(\lambda)}$ and $(\mathcal{E}^{(\lambda)}, \mathcal{F}_U^{(\lambda)})$ be the Dirichlet form of the subprocess $X^{\lambda,U}$. The transition semigroup of the subprocess $X^{\lambda,U}$ will be denoted as $\{P_t^{\lambda,U}, t \geq 0\}$. The killing density of this subprocess is given by

$$\kappa_U(x) := \int_{(\lambda^{-1}D)\setminus U} \frac{\mathcal{C}(\lambda x, \lambda y)}{|x-y|^{d+\alpha}}dy, \quad x \in U.$$

By the $C^{1,1}$ assumption on D, there is a constant $c_7 = c_7(d, \alpha, r_0, T) > 0$ independent of $\lambda > 0$ and x such that $\kappa_U \geq 2c_7 > 0$ on U. Then for every $u \in \mathcal{F}_U^{(\lambda)}$,

$$\mathcal{E}_{-c_7}^{(\lambda)}(u, u) \geq \frac{1}{2}\mathcal{E}^{(\lambda)}(u, u) \geq c_8 \left(\int_{U\times U} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}dxdy + \int_U u(x)^2dx\right)$$

for some $c_8 = c_8(d, \alpha, r_0, T) > 0$ independent of λ, where

$$\mathcal{E}_{-c_7}^{(\lambda)}(u, u) := \mathcal{E}^{(\lambda)}(u, u) - c_7 \int_U u(x)^2dx.$$

It is known (see, for instance, [6, Section 3] or [7]) that there is a constant $c_9 > 0$ independent of λ such that for every $u \in \mathcal{F}_U^{(\lambda)}$ with $\|u\|_{L^1(U)} = 1$,

$$\|u\|_{L^2(U)}^{2+2\alpha/2} \leq c_9 \left(\int_{U\times U} \frac{(u(x)-u(y))^2}{|x-y|^{d+\alpha}}dxdy + \int_U u(x)^2dx\right).$$

So we have for every $u \in \mathcal{F}_U^{(\lambda)}$ with $\|u\|_{L^1(U)} = 1$,

$$\|u\|_{L^2(U)}^{2+2\alpha/2} \leq c_{10}\, \mathcal{E}_{-c_7}^{(\lambda)}(u, u).$$

Observe that $(\mathcal{E}_{-c_7}^{(\lambda)}, \mathcal{F}_U^{(\lambda)})$ is the quadratic form for the semigroup $\{e^{c_7 t}P_t^{\lambda,U}, t \geq 0\}$. Thus by [12, Theorem 2.4.6], there exists a positive constant independent of λ, such that

$$e^{c_7 t} p^U_{\lambda^{-1}D}(t,x,y) \le c_{11} t^{-d/\alpha} \quad \text{for every } t>0.$$

Therefore

$$\mathbb{E}_{x_1}\left[\int_0^{\tau_U^{(\lambda)}} \mathbf{1}_{B(x_0,r_1/(16T))}(X_s^{(\lambda)})ds\right] \le 1+\int_1^\infty c_{11}e^{-c_7 t}dt\,|B(0,r_1/(16T))|<\infty.$$

The proof of the lemma is now complete. □

Lemma 3.4 *Let D be a $C^{1,1}$ open set in $\mathbb{R}^d$ with characteristics (r_0,Λ_0). For every $T>0$, there is a constant $c=c(r_0,\Lambda_0,T,\alpha)>0$ such that for every $\lambda\in(0,T]$ and $x,y\in\lambda^{-1}D$,*

$$p_{\lambda^{-1}D}(1,x,y)\le c\left(1\wedge|x-y|^{-d-\alpha}\right)\delta_{\lambda^{-1}D}(x)^{\alpha-1}.$$

Proof Note that for every $\lambda\in(0,T]$, $\lambda^{-1}D$ is a $C^{1,1}$ open set with characteristics $(r_0/T, T\Lambda_0)$. Take $x,y\in\lambda^{-1}D$. In view of (3.9), it suffices to prove the theorem for $x\in\lambda^{-1}D$ with $\delta_{\lambda^{-1}D}(x)<r_1/(4T)$. When $\delta_{\lambda^{-1}D}(x)<r_1/(4T)$ and $|x-y|\ge 10r_1/T$, by Lemma 3.3, there is a constant $c_1=c_1(r_0,T,\alpha,\Lambda_0)>0$ such that

$$p_{\lambda^{-1}D}(t,x,y)\le c_1\frac{\delta_{\lambda^{-1}D}(x)^{\alpha-1}}{|x-y|^{d+\alpha}} \quad \text{for every } t\in(0,1]. \tag{3.14}$$

So it remains to show that when $\delta_{\lambda^{-1}D}(x)<r_1/(4T)$ and $|x-y|<10r_1/T$,

$$p_{\lambda^{-1}D}(1,x,y)\ \le\ c_2\,\delta_{\lambda^{-1}D}(x)^{\alpha-1} \tag{3.15}$$

for some positive constant $c_2=c_2(r_0,T,\alpha,\Lambda_0)>0$. Define $U:=(\lambda^{-1}D)\cap B(x,8r_1/T)$. Note that $x,y\in U$ and $\delta_U(x)=\delta_{\lambda^{-1}D}(x)$. Let $p^U_{\lambda^{-1}D}(t,z,w)$ be the transition density function of the subprocess $X^{\lambda,U}$ of $X^{(\lambda)}$ killed upon leaving U and let $p_{\lambda^{-1}D}(t,x,y)$ be the transition density function of $X^{(\lambda)}$. By the strong Markov property of $X^{(\lambda)}$ and the symmetry of $p_{\lambda^{-1}D}(1,x,y)$ in x and y, we have

$$p_{\lambda^{-1}D}(1,x,y)=p^U_{\lambda^{-1}D}(1,x,y)+\mathbb{E}_y\left[p_{\lambda^{-1}D}(1-\tau_U^{(\lambda)},X^{(\lambda)}_{\tau_U^{(\lambda)}},x);\ \tau_U^{(\lambda)}<1<\zeta^{(\lambda)}\right]$$

where $\tau_U^{(\lambda)}:=\inf\{t>0: X_t^{(\lambda)}\notin U\}$. Let $z_x\in\partial(\lambda^{-1}D)$ be such that $|x-z_x|=\delta_{\lambda^{-1}D}(x)$ and let $\mathbf{n}(z_x)$ be unit inward normal vector of $\lambda^{-1}D$ at z_x. Put $x_0=z_x+$

$(r_1/T)\mathbf{n}(z_x)$. By the semigroup property, (3.9) and (2.9),

$$\begin{aligned} p^U_{\lambda^{-1}D}(1,x,y) &= \int\limits_U p^U_{\lambda^{-1}D}(1/2,x,z)p^U_{\lambda^{-1}D}(1/2,z,y)dz \\ &\le \|p_{\lambda^{-1}D}(1/2,\cdot,\cdot)\|_\infty\, \mathbb{P}_x\left(\tau_U^{(\lambda)} > 1/2\right) \\ &\le c_3\, \mathbb{E}_x\left[\tau_U^{(\lambda)}\right] \\ &\le c_4 \mathbb{E}_x\left[\int\limits_0^{\tau_U^{(\lambda)}} \mathbf{1}_{B(x_0,r_1/(4T))}(X_s^{(\lambda)})ds\right] \end{aligned}$$

for some positive constants $c_i = c_i(\alpha, r_0, T)$, $i = 3, 4$. Put $x_1 = z_x + \frac{r_1}{4T}\mathbf{n}(z_x)$. By (**BHP**) and the last part of the proof of Lemma 3.3, the above is bounded by

$$c_5 \mathbb{E}_{x_1}\left[\int\limits_0^{\tau_U^{(\lambda)}} \mathbf{1}_{B(x_0,r_1/(4T))}(X_s^{(\lambda)})ds\right] \frac{\delta_D(x)^{\alpha-1}}{\delta_D(x_1)^{\alpha-1}} \le c_6 \delta_D(x)^{\alpha-1}$$

for some positive constants $c_i = c_i(\alpha, r_0, \Lambda_0, T)$ with $i = 5, 6$.

On the other hand, $X^{(\lambda)}_{\tau_U^{(\lambda)}} \in (\lambda^{-1}D) \setminus U$ on $\{\tau_U^{(\lambda)} < 1 < \zeta^{(\lambda)}\}$ and so

$$\left|X^{(\lambda)}_{\tau_U^{(\lambda)}} - x\right| \ge 7r_1/T \quad \text{on } \left\{\tau_U^{(\lambda)} < 1 < \zeta^{(\lambda)}\right\}.$$

Consequently by (3.14) for $p_{\lambda^{-1}D}(1-\tau_U^{(\lambda)}, X^{(\lambda)}_{\tau_U^{(\lambda)}}, x)$,

$$\begin{aligned} &\mathbb{E}_y\left[p_{\lambda^{-1}D}(1-\tau_U, X^{(\lambda)}_{\tau_U}, x);\ \tau_U^{(\lambda)} < 1 < \zeta^{(\lambda)}\right] \\ &\le \mathbb{E}_y\left[c_1 \frac{\delta_{\lambda^{-1}D}(x)^{\alpha-1}}{|X^{(\lambda)}_{\tau_U} - x|^{d+\alpha}};\ \tau_U^{(\lambda)} < 1 < \zeta^{(\lambda)}\right] \\ &\le c_7\, \delta_{\lambda^{-1}D}(x)^{\alpha-1}\, \mathbb{P}_y\left(\tau_U^{(\lambda)} < 1 < \zeta^{(\lambda)}\right) \le c_7\, \delta_{\lambda^{-1}D}(x)^{\alpha-1} \end{aligned}$$

for some positive constant $c_7 = c_7(\alpha, r_0, \Lambda_0, T)$. This completes the proof for (3.15) and hence the theorem. □

Theorem 3.5 *Let D be a $C^{1,1}$ open set with characteristics (r_0, Λ_0). For every $T > 0$, there exists a positive constant $c = c(T, r_0, \alpha, \Lambda_0)$ such that for $t \in (0, T]$ and $x, y \in D$,*

$$p_D(t,x,y) \le c \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{3.16}$$

Proof Fix $T > 0$. By Lemma 3.4 there exists a positive constant $c_1 = c_1(T, r_0, \alpha, \Lambda_0)$ such that for every $\lambda \in (0, T^{1/\alpha}]$,

$$p_{\lambda^{-1}D}(1,x,y) \le c_1 \left(1 \wedge |x-y|^{-d-\alpha}\right) \delta_{\lambda^{-1}D}(x)^{\alpha-1}. \tag{3.17}$$

Thus by (3.2) and (3.17), for every $t \le T$,

$$\begin{aligned} p_D(t,x,y) &= t^{-d/\alpha} p_{t^{-1/\alpha}D}\left(1, t^{-1/\alpha}x, t^{-1/\alpha}y\right) \\ &\le c_1\, t^{-d/\alpha} \left(1 \wedge |t^{-1/\alpha}(x-y)|^{-d-\alpha}\right) \delta_{t^{-1/\alpha}D}\left(t^{-1/\alpha}x\right)^{\alpha-1} \\ &= c_1 \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \frac{\delta_D(x)^{\alpha-1}}{t^{1-1/\alpha}} \\ &\le c_2\, p_{\mathbb{R}^d}(t,x,y) \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \end{aligned} \tag{3.18}$$

for some positive constant $c_2 = c_2(T, r_0, \alpha, \Lambda_0)$. Here $p_{\mathbb{R}^d}(t,x,y)$ is the transition density function of the symmetric α-stable process in $\mathbb{R}^d$ and it is known [1,6] that

$$p_{\mathbb{R}^d}(t,x,y) \asymp \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d. \tag{3.19}$$

By symmetry, the inequality (3.18) for $p_D(t,x,y)$ holds with role of x and y interchanged. Using the Chapman–Kolmogorov's equation and (3.18), for $t \le T$,

$$\begin{aligned} p_D(t,x,y) &= \int_D p_D(t/2,x,z)p_D(t/2,z,y)dz \\ &\le c_3 \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} \int_D p_{\mathbb{R}^d}(t/2,x,z)p_{\mathbb{R}^d}(t/2,z,y)dz \\ &\le c_3 \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} p_{\mathbb{R}^d}(t,x,y) \end{aligned} \tag{3.20}$$

for some positive constant $c_2 = c_2(T, r_0, \alpha, \Lambda_0)$. Combining (3.19) and (3.20), we prove the upper bound (3.16) by noting that

$$(1 \wedge a)(1 \wedge b) = \min\{1, a, b, ab\} \quad \text{for } a, b > 0.$$

□

4 Lower bound estimate

The goal of this section is to prove the lower bound for the heat kernel of X. We start with the following result for a general open d-set in $\mathbb{R}^d$.

Lemma 4.1 *Suppose that $d \geq 1$ and $\alpha \in (0, 2)$. Let D be an open d-set in $\mathbb{R}^d$ and X the censored α-stable-like process in D. For any positive constants c and a, there exists $c_1 = c_1(c, a, \alpha, d) > 0$ such that for every $z \in D$ and $\lambda > 0$ with $B(z, 2c\lambda^{1/\alpha}) \subset D$,*

$$\inf_{\substack{y \in D \\ |y-z| \leq c\lambda^{1/\alpha}}} \mathbb{P}_y \left(\tau_{B(z,2c\lambda^{1/\alpha})} > a\lambda\right) \geq c_1 > 0.$$

Proof Let $Y = \{Y_t, t \geq 0\}$ be the symmetric α-stable-like process in $\mathbb{R}^d$ (corresponding to a fixed symmetric measurable extension of $\mathcal{C}(\cdot, \cdot)$ satisfying (2.2)). For any open set $U \subset \mathbb{R}^d$, let $\tau_U^Y := \inf\{t > 0 : Y_t \notin U\}$. Then by Theorem 2.1(iii)

$$\begin{aligned}\inf_{\substack{y \in D \\ |y-z| \leq c\lambda^{1/\alpha}}} \mathbb{P}_y \left(\tau_{B(z,2c\lambda^{1/\alpha})} > a\lambda\right) &\geq \inf_{\substack{y \in D \\ |y-z| \leq c\lambda^{1/\alpha}}} \mathbb{P}_y \left(\tau^Y_{B(z,2c\lambda^{1/\alpha})} > a\lambda\right) \\ &\geq \inf_{y \in \mathbb{R}^d} \mathbb{P}_y \left(\tau^Y_{B(y,c\lambda^{1/\alpha})} > a\lambda\right).\end{aligned}$$

By [6, Proposition 4.1], there exists $\varepsilon > 0$ such that

$$\inf_{y \in \mathbb{R}^d} \mathbb{P}_y \left(\tau^Y_{B(y,c\lambda^{1/\alpha}/2)} > \varepsilon\lambda\right) \geq \frac{1}{2}.$$

Let $p_U^Y(t, x, y)$ be the transition density function of Y^U. Suppose $a > \varepsilon$. Then by the parabolic Harnack principle in [6, Proposition 4.3]

$$c_1\, p^Y_{B(y,c\lambda^{1/\alpha})}(\varepsilon\lambda, y, w) \leq p^Y_{B(y,c\lambda^{1/\alpha})}(a\lambda, y, w) \quad \text{for } w \in B(y, c\lambda^{1/\alpha}/2)$$

where the constant $c_1 > 0$ is independent of y and λ. Thus

$$\begin{aligned}\mathbb{P}_y \left(\tau^Y_{B(y,c\lambda^{1/\alpha})} > a\lambda\right) &= \int_{B(y,c\lambda^{1/\alpha})} p^Y_{B(y,c\lambda^{1/\alpha})}(a\lambda, y, w)dw \\ &\geq \int_{B(y,c\lambda^{1/\alpha}/2)} p^Y_{B(y,c\lambda^{1/\alpha})}(a\lambda, y, w)dw \\ &\geq c_1 \int_{B(y,\varepsilon\lambda^{1/\alpha}/2)} p^Y_{B(y,\varepsilon\lambda^{1/\alpha}/2)}(\varepsilon\lambda, y, w)dw \\ &\geq c_1/2.\end{aligned}$$

This proves the lemma. □

Proposition 4.2 *Assume that $d \ge 1$ and $\alpha \in (0, 2)$. Let D be an open d-set in $\mathbb{R}^d$, X the censored α-stable-like process in D and $p_D(t, x, y)$ the transition density function of X. Suppose $(t, x, y) \in (0, \infty) \times D \times D$ with $\delta_D(x) \ge t^{1/\alpha} \ge 2|x - y|$. Then there exists a positive constant $c = c(\alpha, r_0)$ such that*

$$p_D(t, x, y) \ge ct^{-d/\alpha}. \tag{4.1}$$

Proof This proof is the same as that for [5, Proposition 3.3]. We reproduce it here for reader's convenience. Let $t > 0$ and $x, y \in D$ with $\delta_D(x) \ge t^{1/\alpha} \ge 2|x - y|$. By the parabolic Harnack principle in [6, Proposition 4.3],

$$p_D(t/2, x, w) \le c_1\, p_D(t, x, y) \quad \text{for } w \in B(x, 2t^{1/\alpha}/3),$$

where the constant $c_1 > 0$ is independent of x, y and t. This together with Lemma 4.1 yields that

$$\begin{aligned} p_D(t, x, y) &\ge \frac{1}{c_1\,|B(x, t^{1/\alpha}/2)|} \int\limits_{B(x,t^{1/\alpha}/2)} p_D(t/2, x, w)dw \\ &\ge c_2 t^{-d/\alpha} \int\limits_{B(x,t^{1/\alpha}/2)} p_{B(x,t^{1/\alpha}/2)}(t/2, x, w)dw \\ &= c_2 t^{-d/\alpha}\, \mathbb{P}_x\left(\tau_{B(x,t^{1/\alpha}/2)} > t/2\right) \\ &\ge c_3\, t^{-d/\alpha}, \end{aligned}$$

where $c_i = c_i(r_0, \alpha) > 0$ for $i = 2, 3$. □

Lemma 4.3 *Assume that $d \ge 1$ and $\alpha \in (0, 2)$. Let D be an open d-set in $\mathbb{R}^d$, X the censored α-stable-like process in D. Suppose that $(t, x, y) \in (0, \infty) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \ge t^{1/\alpha}$ and $|x - y| \ge 2^{-1}t^{1/\alpha}$. There exists a constant $c = c(\alpha, d) > 0$, independent of $t > 0$ and x and y, such that*

$$\mathbb{P}_x\left(X_t \in B\left(y, 2^{-1}t^{1/\alpha}\right)\right) \ge c\frac{t^{d/\alpha+1}}{|x - y|^{d+\alpha}}.$$

Proof The proof is a simple modification of that of Proposition 4.11 in [7]. For reader's convenience, we spell out the details here.

By Lemma 4.1, starting at $z \in B(y, 4^{-1}t^{1/\alpha})$, with probability at least $c_1 = c_1(\alpha) > 0$ the process X does not move more than $6^{-1}t^{1/\alpha}$ by time t. Thus, it is sufficient to show for some constant $c_2 = c_2(\alpha, d) > 0$,

$$\mathbb{P}_x\left(X \text{ hits the ball } B(y,\ 4^{-1}t^{1/\alpha}) \text{ by time } t\right) \ge c_2\frac{t^{d/\alpha+1}}{|x - y|^{d+\alpha}} \tag{4.2}$$

for all $|x - y| \ge 2^{-1}t^{1/\alpha}$ and $t > 0$. Now with $B_x := B(x,\ 6^{-1}t^{1/\alpha})$, $B_y := B(y,\ 6^{-1}t^{1/\alpha})$ and $\tau_x := \tau_{B_x}$, it follows from Lemma 4.1, there exists $c_3 = c_3(\alpha, d) > 0$

such that

$$\mathbb{E}_x\left[t \wedge \tau_x\right] \geq \frac{t}{2}\mathbb{P}_x\left(\tau_x \geq t/2\right) \geq c_3 t, \quad \text{for } t > 0. \tag{4.3}$$

Thus by using the Lévy system of X in (2.3),

$$\begin{aligned}
&\mathbb{P}_x\Big(X \text{ hits the ball } B(y,\ 4^{-1}t^{1/\alpha}) \text{ by time } t\Big)\\
&\geq \mathbb{P}_x\Big(X_{t\wedge\tau_x} \in B(y,\ 4^{-1}t^{1/\alpha}) \text{ and } t\wedge\tau_x \text{ is a jumping time }\Big)\\
&\geq \mathbb{E}_x\left[\int_0^{t\wedge\tau_x}\int_{B_y}\frac{M_1}{|X_s-u|^{d+\alpha}}du\,ds\right]\\
&\geq c_4\mathbb{E}_x\left[t\wedge\tau_x\right]\int_{B_y}\frac{1}{|x-y|^{d+\alpha}}\,du\\
&\geq c_5\, t\,|B_y|\,|x-y|^{-d-\alpha}\\
&\geq c_6\frac{t^{d/\alpha+1}}{|x-y|^{d+\alpha}},
\end{aligned}$$

for some positive constants $c_i = c_i(\alpha, d)$, $i = 4, 5, 6$. Here in the fourth inequality, we used (4.3). The lemma is now proved. □

Proposition 4.4 *Assume that $d \geq 1$ and $\alpha \in (0, 2)$. Let D be an open d-set in $\mathbb{R}^d$, X the censored α-stable-like process in D and $p_D(t, x, y)$ the transition density function of X. Suppose that $(t, x, y) \in (0, \infty) \times D \times D$ with $\delta_D(x) \wedge \delta_D(y) \geq (t/2)^{1/\alpha}$ and $|x - y| \geq 2^{-1}(t/2)^{1/\alpha}$. Then there exists a constant $c = c(\alpha, r_0, \Lambda_0) > 0$ such that*

$$p_D(t, x, y) \geq c\frac{t}{|x-y|^{d+\alpha}}. \tag{4.4}$$

Proof By the semigroup property, Proposition 4.2 and Lemma 4.3, there exist positive constants $c_1 = c_1(\alpha, r_0, \Lambda_0)$ and $c_2 = c_3(\alpha, r_0, \Lambda_0)$ such that

$$\begin{aligned}
p_D(t, x, y) &= \int_D p_D(t/2, x, z)p_D(t/2, z, y)dz\\
&\geq \int_{B(y,\,2^{-1}(t/2)^{1/\alpha})} p_D(t/2, x, z)p_D(t/2, z, y)dz\\
&\geq c_1 t^{-d/\alpha}\mathbb{P}_x\left(X_{t/2} \in B(y, 2^{-1}(t/2)^{1/\alpha})\right)\\
&\geq c_2\frac{t}{|x-y|^{d+\alpha}}.
\end{aligned}$$

□

In the remainder of this section, we assume that D is a $C^{1,1}$ open subset in $\mathbb{R}^d$ with characteristics (r_0, Λ_0) and X is the censored α-stable-like process in D with $d \geq 1$ and $\alpha \in (1, 2)$. Let

$$T_0 := \left(\frac{r_0}{16}\right)^{\alpha}. \tag{4.5}$$

We will first establish the lower bound for the heat kernel of X for $t \leq T_0$.

The next lemma is a key step in deriving the precise boundary decay rate for the transition density function $p_D(t, x, y)$.

Lemma 4.5 *Suppose that $(t, x) \in (0, T_0] \times D$ with $\delta_D(x) \leq 3t^{1/\alpha} < r_0/4$ and $\kappa \in (0, 1)$. Let $z_x \in \partial D$ be such that $|z_x - x| = \delta_D(x)$ and let B be a ball of radius $3t^{1/\alpha}$ such that $B \subset D$ and $\partial B \cap \partial D = \{z_x\}$. Suppose $B(x_0, 2\kappa t^{1/\alpha}) \subset B \setminus \{x\}$. Then for any $a > 0$, there exists a constant $c_1 = c_1(\kappa, \alpha, r_0, \Lambda_0, a) > 0$ such that*

$$\mathbb{P}_x\left(X_{at} \in B(x_0, \kappa t^{1/\alpha})\right) \geq c_1 \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}. \tag{4.6}$$

Proof Let $0 < \kappa_1 \leq \kappa$ and assume first that $2^{-4}\kappa_1 t^{1/\alpha} < \delta_D(x) \leq 3t^{1/\alpha}$. Note that $\delta_D(x) \wedge \delta_D(x_0) > 2^{-4}\kappa_1 t^{1/\alpha}$. By the convexity of the ball B, every point on the line segment $l_{x_0,x}$ joining x_0 to x is at least of distance $2^{-4}\kappa_1 t^{1/\alpha}$ away from the boundary of D. For $a > 0$, denote by k the smallest integer that is larger than $\max\left\{(36^\alpha/a)^{1/(\alpha-1)},\ 6 \cdot 2^7/\kappa_1,\ a(2^7/(7\kappa_1))^\alpha\right\}$. Let $x_0, x_1, \ldots, x_k = x$ be $(k+1)$ equally spaced points on $l_{x_0,x}$, and set $r := |x_1 - x_0|$. Since $2\kappa t^{1/\alpha} \leq |x - x_0| \leq 6t^{1/\alpha}$, by our choice of k, we have

$$2\kappa t^{1/\alpha}/k \leq r \leq 6t^{1/\alpha}/k \leq 2^{-7}\kappa_1 t^{1/\alpha} \quad \text{and} \quad 6r \leq (at/k)^{1/\alpha} \leq 7 \cdot 2^{-7}\kappa_1 t^{1/\alpha}.$$

Since the above inequalities imply that for every $i = 0, \ldots, k-1$, $z \in B(x_i, r)$ and $w \in B(x_{i+1}, r)$

$$2|z - w| \leq 6r \leq (at/k)^{1/\alpha} \leq 7 \cdot 2^{-7}\kappa_1 t^{1/\alpha} \leq \delta_D(z) \wedge \delta_D(w),$$

by Proposition 4.2 and the semigroup property,

$$\begin{aligned}
\mathbb{P}_x\left(X_{at} \in B(x_0, \kappa t^{1/\alpha})\right) &\geq \int_{B(x_0,r)} p_D(at, x, y)dy \\
&\geq \int_{B(x_{k-1},r)} \int_{B(x_{k-2},r)} \cdots \int_{B(x_0,r)} p_D(at/k, x, y_{k-1}) p_D(at/k, y_{k-1}, y_{k-2}) \\
&\qquad \cdots p_D(at/k, y_1, y)\, dy dy_1 \cdots dy_{k-1} \\
&\geq c_1^k \left((at/k)^{-d/\alpha} r^d\right)^k \geq c_2 > 0.
\end{aligned} \tag{4.7}$$

By taking $\kappa_1 = \kappa$, this shows that (4.6) holds for every $a > 0$ and for every $x \in D$ with $2^{-4}\kappa t^{1/\alpha} < \delta_D(x) \le 3t^{1/\alpha}$. So it suffices to consider the case that $\delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$. We now show that there is some $a_0 > 1$ so that (4.6) holds for every $a \ge a_0$ and $\delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$. For simplicity, we assume without loss of generality that $x_0 = 0$ and let $\widehat{B} := B(x_0, \kappa t^{1/\alpha})$. By the scaling property for censored α-stable-like processes (see (3.2) and the line following it),

$$\mathbb{P}_x(X_{at} \in \widehat{B}) = \mathbb{P}_{t^{-1/\alpha}x}\left(Z_a \in t^{-1/\alpha}\widehat{B}\right) = \mathbb{P}_{t^{-1/\alpha}x}\left(Z_a \in B(0,\kappa)\right), \tag{4.8}$$

where Z is the censored α-stable-like process in $t^{-1/\alpha}D$ with jumping function $J^{(t^{-1/\alpha})}$ of (3.1), and, by a slight abuse of notation, the law of Z starting from a point $z \in t^{-1/\alpha}D$ is also denoted as $\mathbb{P}_z$. Let $B_0 := B(t^{-1/\alpha}z_x, \kappa/2) \cap (t^{-1/\alpha}D)$. Observe that since $B(0, 2\kappa) \subset t^{-1/\alpha}(B\backslash\{x\}) \subset t^{-1/\alpha}(D\backslash\{x\})$,

$$\kappa/2 \le |y - z| \le 6 \quad \text{for } y \in B_0 \text{ and } z \in B(0,\kappa). \tag{4.9}$$

By the strong Markov property of Z at the first exit time τ_{B_0} from B_0 and Lemma 4.1,

$$\begin{aligned}
&\mathbb{P}_{t^{-1/\alpha}x}\left(Z_a \in B(0,\kappa)\right) \\
&\quad \ge \mathbb{P}_{t^{-1/\alpha}x}\Big(\tau^Z_{B_0} < a,\ Z_{\tau_{B_0}} \in B(0,\kappa/2) \text{ and } |Z_t - Z_{\tau_{B_0}}| < \kappa/2 \\
&\qquad\qquad \text{for } t \in \left[\tau^Z_{B_0}, \tau^Z_{B_0} + a\right]\Big) \\
&\quad \ge c_3\, \mathbb{P}_{t^{-1/\alpha}x}\left(\tau^Z_{B_0} < a \text{ and } Z_{\tau_{B_0}} \in B(0,\kappa/2)\right).
\end{aligned} \tag{4.10}$$

Here, $\tau^Z_{B_0}$ denotes the first exit time from B_0 by Z.

Let $z_1 := t^{-1/\alpha}z_x \in \partial(t^{-1/\alpha}D)$ and set $y_1 := z_1 + 2^{-2}\kappa\, \mathbf{n}(z_1)$, where $\mathbf{n}(z_1)$ denotes the unit inward normal vector at z_1 for $t^{-1/\alpha}D$. Note that $t^{-1/\alpha}D$ is a $C^{1,1}$-open set with characteristics $(T_0^{-1/\alpha}r_0, T_0^{1/\alpha}\Lambda_0)$. So by (**BHP**), the Lévy system of Z and (4.9),

$$\begin{aligned}
&\mathbb{P}_{t^{-1/\alpha}x}\left(Z_{\tau_{B_0}} \in B(0,\kappa/2)\right) \\
&\quad \ge c_4 \frac{\delta_{t^{-1/\alpha}D}(t^{-1/\alpha}x)^{\alpha-1}}{\delta_{t^{-1/\alpha}D}(y_1)^{\alpha-1}} \mathbb{P}_{y_1}\left(Z_{\tau_{B_0}} \in B(0,\kappa/2)\right) \\
&\quad \ge c_4 \left(\frac{4}{\kappa}\right)^{\alpha-1} \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \\
&\qquad \times \int_0^\infty \left(\int_{B_0} p^Z_{B_0}(t, y_1, y) \left(\int_{B(0,\kappa/2)} \frac{\mathcal{C}(t^{-1/\alpha}y, t^{-1/\alpha}z)}{|y-z|^{d+\alpha}} dz\right) dy\right) dt \\
&\quad \ge c_5 \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \mathbb{E}_{y_1}\left[\tau^Z_{B_0}\right].
\end{aligned}$$

It follows from Theorem 2.1(iii), and [6, Proposition 4.1],

$$\mathbb{E}_{y_1}\left[\tau^Z_{B_0}\right] \geq \mathbb{E}_{y_1}\left[\tau^Y_{B_0}\right] \geq \mathbb{E}_{y_1}\left[\tau^Y_{B(y_1,\kappa/4)}\right] \geq c_5$$

where Y is the α-stable-like process in $t^{-1/\alpha}D$ with jumping function $J^{(t^{-1/\alpha})}$ of (3.1), and so

$$\mathbb{P}_{t^{-1/\alpha}x}\left(Z_{\tau_{B_0}} \in B(0,\kappa/2)\right) \geq c_5c_6\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}. \tag{4.11}$$

The above constants c_k, $k=4,\ldots,6$ do not depend on a. On the other hand, by Theorem 2.2 and (**BHP**)

$$\begin{aligned}\mathbb{P}_{t^{-1/\alpha}x}\left(\tau^Z_{B_0} \geq a\right) &\leq a^{-1}\,\mathbb{E}_{t^{-1/\alpha}x}\left[\tau^Z_{B_0}\right]\\ &\leq a^{-1}c_7\,\mathbb{E}_{t^{-1/\alpha}x}\left[\int_0^{\tau_{B_0}} \mathbf{1}_{B(y_1,\kappa/8)}(Z_s)ds\right]\\ &\leq a^{-1}c_8\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\mathbb{E}_{y_2}\left[\int_0^{\tau_{B_0}} \mathbf{1}_{B(y_1,\kappa/8)}(Z_s)ds\right],\end{aligned}$$

where $y_2 := z_1 + 2^{-4}\kappa\,\mathbf{n}(z_1)$. Now by the same argument as in last part of the proof of Lemma 3.3, we have

$$\mathbb{P}_{t^{-1/\alpha}x}\left(\tau^Z_{B_0} \geq a\right) \leq a^{-1}c_9\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}, \tag{4.12}$$

where constant c_9 does not depend on a. Define $a_0 = 2c_9/(c_5c_6)$. We have by (4.8) and (4.10)–(4.12) that for $a \geq a_0$,

$$\begin{aligned}\mathbb{P}_x\left(X_{at} \in \widehat{B}\right) &\geq c_2\left(\mathbb{P}_{t^{-1/\alpha}x}\left(Z_{\tau_{B_0}} \in B(0,\kappa/2)\right) - \mathbb{P}_{t^{-1/\alpha}x}\left(\tau^Z_{B_0} \geq a\right)\right)\\ &\geq c_2\,(c_5c_6/2)\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}. \end{aligned}\tag{4.13}$$

(4.7) and (4.13) show that (4.6) holds for every $a \geq a_0$ and for every $x \in D$ with $\delta_D(x) \leq 3t^{1/\alpha}$.

Now we deal with the case $0 < a < a_0$ and $\delta_D(x) \leq 2^{-4}\kappa t^{1/\alpha}$. If $\delta_D(x) \leq 3(at/a_0)^{1/\alpha}$, we have from (4.6) for the case of $a = a_0$ that

$$\begin{aligned}\mathbb{P}_x\left(X_{at} \in B(x_0,\kappa t^{1/\alpha})\right) &\geq \mathbb{P}_x\left(X_{a_0(at/a_0)} \in B\left(x_0,\ \kappa(at/a_0)^{1/\alpha}\right)\right)\\ &\geq c_{10}\left(\frac{\delta_D(x)}{(at/a_0)^{1/\alpha}}\right)^{\alpha-1} = c_{11}\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}.\end{aligned}$$

If $3(at/a_0)^{1/\alpha} < \delta_D(x) \le 2^{-4}\kappa t^{1/\alpha}$ (in this case $\kappa > 3\cdot 2^4(a/a_0)^{1/\alpha}$), we get (4.6) from (4.7) by taking $\kappa_1 = (a/a_0)^{1/\alpha}$. The proof of the lemma is now complete. □

The next three propositions and their proofs are similar to [5, Propositions 3.7–3.9] and their proofs, we give the details for readers' convenience.

Proposition 4.6 *Suppose that* $(t,x,y)\in(0,T_0]\times D\times D$ *with* $|x-y|\le t^{1/\alpha}$ *and* $\delta_D(x)\le 2t^{1/\alpha}$. *Then there exists a constant* $c=(\alpha,r_0,\Lambda_0)>0$ *such that*

$$p_D(t,x,y)\ \ge\ ct^{-d/\alpha}\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}. \tag{4.14}$$

Proof For $z\in\partial D$, let $\mathbf{n}(z)$ be the unit inward normal vector of ∂D at the point z. By the assumptions,

$$\delta_D(y)\le|x-y|+\delta_D(x)\le 3t^{1/\alpha}<r_0/5.$$

So there are unique points $z_x,\ z_y\in\partial D$ such that $\delta_D(x)=|x-z_x|$ and $\delta_D(y)=|y-z_y|$. Let

$$x_0=z_x+4t^{1/\alpha}\mathbf{n}(z_x)\quad\text{and}\quad y_0=z_y+4t^{1/\alpha}\mathbf{n}(z_y).$$

Observe that

$$\delta_D(x_0)=\delta_D(y_0)=4t^{1/\alpha}\quad\text{and}\quad |x-x_0|,\ |y-y_0|\in[t^{1/\alpha},4t^{1/\alpha}).$$

Define $B:=B(x_0,4^{-1}t^{1/\alpha})$ and $\widetilde{B}:=B(y_0,4^{-1}t^{1/\alpha})$. Observe that $x\notin B(x_0,2^{-1}t^{1/\alpha})$ and $y\notin B(y_0,2^{-1}t^{1/\alpha})$. By the semigroup property,

$$\begin{aligned}
p_D(t,x,y)&=\int_D p_D(t/3,x,z)\left(\int_D p_D(t/3,z,w)p_D(t/3,w,y)dw\right)dz\\
&\ge\int_B p_D(t/3,x,z)\left(\int_{\widetilde{B}} p_D(t/3,z,w)p_D(t/3,w,y)dw\right)dz\\
&\ge\left(\inf_{(z,w)\in B\times\widetilde{B}}p_D(t/3,z,w)\right)\left(\int_B p_D(t/3,x,z)dz\right)\\
&\quad\times\left(\int_{\widetilde{B}}p_D(t/3,w,y)dw\right).
\end{aligned}$$

Since for $z\in B$ and $w\in\widetilde{B}$,

$$\delta_D(z)\ge\delta_D(x_0)-|x_0-z|\ge t^{1/\alpha},\quad \delta_D(w)\ge\delta_D(y_0)-|y_0-w|\ge t^{1/\alpha}$$

and

$$|z-w| \leq |z-x_0| + |x_0-x| + |x-y| + |y-y_0| + |y_0-w| < 10t^{1/\alpha},$$

by combining Propositions 4.2 and 4.4, we have that there exists $c_1 = c_1(\alpha, r_0, \Lambda_0) > 0$ such that

$$\inf_{(z,w)\in B\times\widetilde{B}} p_D(t/3, z, w) \geq c_1 t^{-d/\alpha}.$$

Since $\delta_D(x) \leq 2t^{1/\alpha} < r_0/8$ and $\delta_D(y) \leq 3t^{1/\alpha}$, we deduce from Lemma 4.5

$$p_D(t,x,y) \geq c_2 t^{-d/\alpha} \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}$$

for some positive constant $c_2 = c_2(\alpha, r_0, \Lambda_0)$. □

Proposition 4.7 *Suppose that* $(t,x,y) \in (0,T_0] \times D \times D$ *with* $\delta_D(x) \leq t^{1/\alpha}$ *and* $(t/2)^{1/\alpha} \leq \delta_D(y)$ *and* $|x-y| \geq t^{1/\alpha}$. *Then there exists a constant* $c = c(\alpha, r_0, \Lambda_0) > 0$ *such that*

$$p_D(t,x,y) \geq c\, \frac{t}{|x-y|^{d+\alpha}} \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}. \tag{4.15}$$

Proof Recall that for $z \in \partial D$, $\mathbf{n}(z)$ is the unit inward normal vector of ∂D at point z. Since $\delta_D(x) \leq t^{1/\alpha} \leq r_0/16$, there is a unique $z_x \in \partial D$ such that $\delta_D(x) = |x - z_x|$. Let $z_0 = z_x + 2t^{1/\alpha}\mathbf{n}(z_x)$. Now choose x_0 in $B(z_0, 2t^{1/\alpha})$ and $\kappa = \kappa(\alpha) \in (0,1)$ such that

$$B\left(x_0, 2\kappa t^{1/\alpha}\right) \subset B\left(z_0, (2-2^{-2/\alpha})t^{1/\alpha}\right) \cap B\left(x, (1-2^{-1-2/\alpha})t^{1/\alpha}\right).$$

Such a ball $B(x_0, 2\kappa t^{1/\alpha})$ always exists because

$$2 < \left(2-2^{-1}\right) + \left(1-2^{-2}\right) < \left(2-2^{-2/\alpha}\right) + \left(1-2^{-1-2/\alpha}\right).$$

Note that $x \notin B(x_0, 2\kappa t^{1/\alpha})$ and

$$\delta_D(z) \geq (t/4)^{1/\alpha} \quad \text{and} \quad |y-z| \geq 2^{-1}(t/4)^{1/\alpha} \qquad \text{for every } z \in B(x_0, \kappa t^{1/\alpha}).$$

On the other hand, for every $z \in B(x_0, \kappa t^{1/\alpha})$,

$$|z-y| \leq |z-x| + |x-y| \leq \left(1-2^{-1-2/\alpha}\right) t^{1/\alpha} + |x-y| < 2|x-y|.$$

Thus by the semigroup property and Proposition 4.4, there exist positive constants $c_i = c_i(\alpha, r_0, \Lambda_0)$, $i = 1, 2$, such that

$$\begin{aligned} p_D(t,x,y) &= \int_D p_D(t/2,x,z)p_D(t/2,z,y)dz \\ &\geq \int_{B(x_0,\kappa t^{1/\alpha})} p_D(t/2,x,z)p_D(t/2,z,y)dz \\ &\geq c_1 \int_{B(x_0,\kappa t^{1/\alpha})} p_D(t/2,x,z)\frac{t}{|z-y|^{d+\alpha}}dz \\ &\geq c_2 \frac{t}{|x-y|^{d+\alpha}} \int_{B(x_0,\kappa t^{1/\alpha})} p_D(t/2,x,z)dz \\ &= c_2 \frac{t}{|x-y|^{d+\alpha}} \mathbb{P}_x\left(X_{t/2} \in B(x_0,\kappa t^{1/\alpha})\right). \end{aligned}$$

Applying Lemma 4.5, we arrive at the conclusion of the proposition. □

Proposition 4.8 *Suppose that* $(t, x, y) \in (0, T_0] \times D \times D$ *with*

$$\delta_D(x) \vee \delta_D(y) \leq (t/2)^{1/\alpha} \leq |x-y|.$$

Then there exists a constant $c = c(\alpha, r_0, \Lambda_0) > 0$ *such that*

$$p_D(t,x,y) \geq c\,\frac{t}{|x-y|^{d+\alpha}}\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}. \tag{4.16}$$

Proof As in the first paragraph of the proof of Proposition 4.6, let $z_x \in \partial D$ so that $|x - z_x| = \delta_D(x)$ and set $x_0 := z_x + 3t^{1/\alpha}\mathbf{n}(z_x)$. Let $\kappa := 1 - 2^{-1/\alpha}$. Note that we have $\delta_D(z) \geq 2(t/2)^{1/\alpha}$ and $|y - z| \geq \delta_D(z) - \delta_D(y) \geq (t/2)^{1/\alpha}$ for every $z \in B(x_0, \kappa t^{1/\alpha})$.

On the other hand, for every $z \in B(x_0, \kappa t^{1/\alpha})$,

$$\begin{aligned} |z-y| &\leq |z-x_0| + |x_0-x| + |x-y| \leq \kappa t^{1/\alpha} + 3t^{1/\alpha} + |x-y| \\ &\leq \left(2^\alpha(\kappa+3)+1\right)|x-y|. \end{aligned}$$

Thus, by the semigroup property and Proposition 4.7, there exist positive constants $c_i = c_i(\alpha, r_0, \Lambda_0)$, $i = 1, 2$, such that

$$\begin{aligned}
p_D(t, x, y) &= \int_D p_D(t/2, x, z) p_D(t/2, z, y) dz \\
&\geq \int_{B(x_0, \kappa t^{1/\alpha})} p_D(t/2, x, z) p_D(t/2, z, y) dz \\
&\geq c_1 \int_{B(x_0, \kappa t^{1/\alpha})} p_D(t/2, x, z) \frac{t}{|z-y|^{d+\alpha}} \left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} dz \\
&\geq c_2 \frac{t}{|x-y|^{d+\alpha}} \left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} \int_{B(x_0, \kappa t^{1/\alpha})} p_D(t/2, x, z) dz \\
&= c_2 \frac{t}{|x-y|^{d+\alpha}} \left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} \mathbb{P}_x \left(X_{t/2} \in B(x_0, \kappa t^{1/\alpha})\right).
\end{aligned}$$

Applying Lemma 4.5, we arrive at the conclusion of the proposition. □

Now we are ready to prove the main result of this section.

Theorem 4.9 *For every $T > 0$ there exists a positive constant $c = c(\alpha, r_0, \Lambda_0, T)$ such that for all $(t, x, y) \in (0, T] \times D \times D$,*

$$p_D(t, x, y) \geq c \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right). \tag{4.17}$$

Proof Assume first that $t \leq T_0$.

1. We first consider the case $|x - y| \leq t^{1/\alpha}$. We claim that in this case

$$p_D(t, x, y) \geq c t^{-d/\alpha} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}. \tag{4.18}$$

This will be proved by considering the following two possibilities.

(a) $\max\{\delta_D(x), \delta_D(y), |x-y|\} \leq t^{1/\alpha}$: Proposition 4.6 and symmetric yield (4.18)

(b) $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha} \geq |x-y|$:
If $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha} \geq 2|x-y|$, (4.18) follows from Proposition 4.2.
If $\min\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$ and $|x-y| \leq t^{1/\alpha} < 2|x-y|$,

$$\frac{t}{|x-y|^{d+\alpha}} \asymp t^{-d/\alpha} \left(1 \wedge \frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}.$$

If $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$, $\min\{\delta_D(x), \delta_D(y)\} < t^{1/\alpha}$ and $|x-y| \leq t^{1/\alpha} < 2|x-y|$,

$$\left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} \asymp \left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(1\wedge\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}$$

Thus by combining Propositions 4.4 and 4.6, we get (4.18) for the case of $\max\{\delta_D(x), \delta_D(y)\} \geq t^{1/\alpha}$ and $|x-y| \leq t^{1/\alpha} < 2|x-y|$.

2. Now we consider the case $|x-y| \geq t^{1/\alpha}$ and claim that

$$p_D(t,x,y) \geq c\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(1\wedge\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}\left(\frac{t}{|x-y|^{d+\alpha}}\right). \tag{4.19}$$

(a) $\min\{\delta_D(x), \delta_D(y)\} \leq (t/2)^{1/\alpha}$ and $|x-y| \geq t^{1/\alpha}$: By symmetry we can assume $\delta_D(x) \leq (t/2)^{1/\alpha}$. Thus Combining Propositions 4.7 and 4.8, we have (4.19) for this case.

(b) $\min\{\delta_D(x), \delta_D(y)\} \geq (t/2)^{1/\alpha}$ and $|x-y| \geq t^{1/\alpha}$. In this case, clearly

$$\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(1\wedge\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1} \asymp \left(\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}.$$

Thus Proposition 4.4 yields (4.19).

We have arrived at the conclusion of Theorem 4.9 for $t \leq T_0$.

We now consider $t > T_0$ case: let

$$q_D(t,x,y) := \left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(1\wedge\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right).$$

First we observe that for any $t > 0$ and $x, y \in D$,

$$q_D(t,x,y) \asymp q_D(t/2,x,y). \tag{4.20}$$

Then by using the semigroup property and (4.20) twice we get, for any $(t,x,y) \in (0,T_0] \times D \times D$,

$$\begin{aligned} p_D(2t,x,y) &= \int_D p_D(t,x,z)p_D(t,z,y)dz \\ &\geq c_1 \int_D q_D(t,x,z)q_D(t,z,y)dz \\ &\geq c_2 \int_D q_D(t/2,x,z)q_D(t/2,z,y)dz \end{aligned}$$

$$\geq c_3 \int_D p_D(t/2, x, z) p_D(t/2, z, y) dz$$
$$= c_3\, p_D(t, x, y) \;\geq\; c_4\, q_D(t, x, y) \;\geq\; c_5\, q_D(2t, x, y)$$

for some positive constants c_i, $i = 1, \ldots, 5$. Here in the first and fourth inequalities we used Theorem 4.9 for $t \leq T_0$ and in the third inequality we used Theorem 3.5. □

5 Large time heat kernel estimates and Green function estimates

In this section, we present proofs for Theorem 1.1 (ii) and Corollary 1.2. Throughout this section, we assume that $\alpha \in (1, 2)$ and that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^d$.

Proof of Theorem 1.1 (ii) By Theorem 2.2, the semigroup $\{P_t^D, t > 0\}$ is intrinsically ultracontractive. It follows from Theorem 4.2.5 of [12] that there exists $T_1 > 0$ such that for all $(t, x, y) \in [T_1, \infty) \times D \times D$,

$$\frac{1}{2} e^{-\lambda_1 t} \phi_1(x)\phi_1(y) \leq p_D(t, x, y) \leq \frac{3}{2} e^{-\lambda_1 t} \phi_1(x)\phi_1(y).$$

Since $\phi_1 = e^{\lambda_1} P_1 \phi_1$, we have from Theorem 1.1(i) that on D,

$$\phi_1(x) \asymp \left(1 \wedge \delta_D(x)^{\alpha-1}\right) \int_D \left(1 \wedge \delta_D(y)^{\alpha-1}\right) \left(1 \wedge \frac{1}{|x-y|^{d+\alpha}}\right) \phi_1(y) dy$$
$$\asymp \delta_D(x)^{\alpha-1}. \tag{5.1}$$

Thus there exist positive constants c_6, c_7 such that for all $(t, x, y) \in [T_1, \infty) \times D \times D$,

$$c_6 e^{-\lambda_1 t} \delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1} \leq p_D(t, x, y) \leq c_7 e^{-\lambda_1 t} \delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1}.$$

If $T < T_1$, by Theorem 1.1(i), there exist positive constants c_8, c_9 such that for $(t, x, y) \in [T, T_1] \times D \times D$,

$$c_8 \delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1} \leq p_D(t, x, y) \leq c_9 \delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1}.$$

This gives the conclusion of Theorem 1.1(ii). □

Proof of Corollary 1.2 First note that by Theorem 1.1(i), we have

$$\int_T^\infty p_D(t, x, y) \asymp \delta_D(x)^{\alpha-1} \delta_D(y)^{\alpha-1}. \tag{5.2}$$

Let $\operatorname{diam}(D)$ be the diameter of D and $T := \operatorname{diam}(D)^\alpha$. By a change of variable $u = \frac{|x-y|^\alpha}{t}$, we have

$$\int_0^T t^{-d/\alpha}\left(1\wedge\frac{t^{1/\alpha}}{|x-y|}\right)^{d+\alpha}\left(1\wedge\frac{\delta_D(x)}{t^{1/\alpha}}\right)^{\alpha-1}\left(1\wedge\frac{\delta_D(y)}{t^{1/\alpha}}\right)^{\alpha-1}dt$$
$$=\frac{1}{|x-y|^{d-\alpha}}\int_{\frac{|x-y|^\alpha}{T}}^{\infty}\left(u^{\frac{d}{\alpha}-2}\wedge u^{-3}\right)\left(1\wedge\frac{u^{1/\alpha}\delta_D(x)}{|x-y|}\right)^{\alpha-1}\left(1\wedge\frac{u^{1/\alpha}\delta_D(y)}{|x-y|}\right)^{\alpha-1}du. \tag{5.3}$$

Note that

$$\frac{1}{|x-y|^{d-\alpha}}\int_1^{\infty}\left(u^{\frac{d}{\alpha}-2}\wedge u^{-3}\right)\left(1\wedge\frac{u^{1/\alpha}\delta_D(x)}{|x-y|}\right)^{\alpha-1}\left(1\wedge\frac{u^{1/\alpha}\delta_D(y)}{|x-y|}\right)^{\alpha-1}du$$
$$\geq\frac{1}{|x-y|^{d-\alpha}}\int_1^{\infty}u^{-3}\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right)^{\alpha-1}\left(1\wedge\frac{\delta_D(y)}{|x-y|}\right)^{\alpha-1}du$$
$$=\frac{1}{2|x-y|^{d-\alpha}}\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right)^{\alpha-1}\left(1\wedge\frac{\delta_D(y)}{|x-y|}\right)^{\alpha-1}, \tag{5.4}$$

while

$$\frac{1}{|x-y|^{d-\alpha}}\int_1^{\infty}\left(u^{\frac{d}{\alpha}-2}\wedge u^{-3}\right)\left(1\wedge\frac{u^{1/\alpha}\delta_D(x)}{|x-y|}\right)^{\alpha-1}\left(1\wedge\frac{u^{1/\alpha}\delta_D(y)}{|x-y|}\right)^{\alpha-1}du$$
$$=\frac{1}{|x-y|^{d-\alpha}}\int_1^{\infty}u^{-1-2/\alpha}\left(u^{-1/\alpha}\wedge\frac{\delta_D(x)}{|x-y|}\right)^{\alpha-1}\left(u^{-1/\alpha}\wedge\frac{\delta_D(y)}{|x-y|}\right)^{\alpha-1}du$$
$$\leq\frac{1}{|x-y|^{d-\alpha}}\int_1^{\infty}u^{-1-2/\alpha}\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right)^{\alpha-1}\left(1\wedge\frac{\delta_D(y)}{|x-y|}\right)^{\alpha-1}du$$
$$=\frac{\alpha}{2}\frac{1}{|x-y|^{d-\alpha}}\left(1\wedge\frac{\delta_D(x)}{|x-y|}\right)^{\alpha-1}\left(1\wedge\frac{\delta_D(y)}{|x-y|}\right)^{\alpha-1}. \tag{5.5}$$

(i) Assume that $d \geq 2$. Observe that

$$
\begin{aligned}
&\frac{1}{|x-y|^{d-\alpha}} \int_{\frac{|x-y|^\alpha}{T}}^{1} \left(u^{\frac{d}{\alpha}-2} \wedge u^{-3}\right) \left(1 \wedge \frac{u^{1/\alpha}\delta_D(x)}{|x-y|}\right)^{\alpha-1} \left(1 \wedge \frac{u^{1/\alpha}\delta_D(y)}{|x-y|}\right)^{\alpha-1} du \\
&\leq \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha-1} \int_0^1 u^{\frac{d}{\alpha}-2} du \\
&\leq \frac{\alpha}{d-\alpha} \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha-1}. \qquad (5.6)
\end{aligned}
$$

So by (5.2)–(5.6), we have

$$
\begin{aligned}
G_D(x,y) &= \int_0^T p_D(t,x,y)dt + \int_T^\infty p_D(t,x,y)dt \\
&\asymp \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha-1} + \delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1} \\
&\asymp \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)}{|x-y|}\right)^{\alpha-1} \left(1 \wedge \frac{\delta_D(y)}{|x-y|}\right)^{\alpha-1}.
\end{aligned}
$$

In the last estimate, we used the fact that D is bounded. Since $\delta_D(x) \leq \delta_D(y) + |x-y|$ for every $x, y \in D$, it is easy to see that for every $r \in (0, 1]$,

$$
\begin{aligned}
\left(1 \wedge \frac{r\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{r\delta_D(y)}{|x-y|}\right) &\leq 1 \wedge \frac{r^2\delta_D(x)\delta_D(y)}{|x-y|^2} \\
&\leq 2\left(1 \wedge \frac{r\delta_D(x)}{|x-y|}\right) \left(1 \wedge \frac{r\delta_D(y)}{|x-y|}\right). \qquad (5.7)
\end{aligned}
$$

So on $D \times D$,

$$
G_D(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(1 \wedge \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha-1}.
$$

(ii) Now we consider the case $d = 1 < \alpha < 2$ and let

$$
u_0 := \frac{\delta_D(x)\delta_D(y)}{|x-y|^2}. \qquad (5.8)
$$

Clearly

$$
u_0^{-\alpha/2} \geq \frac{|x-y|^\alpha}{\text{diam}(D)^\alpha} = \frac{|x-y|^\alpha}{T}.
$$

By (5.7) and (5.8),

$$
\begin{aligned}
&\frac{1}{|x-y|^{d-\alpha}} \int_{\frac{|x-y|^\alpha}{T}}^{1} \left(u^{\frac{d}{\alpha}-2} \wedge u^{-3}\right) \left(1 \wedge \frac{u^{1/\alpha}\delta_D(x)}{|x-y|}\right)^{\alpha-1} \left(1 \wedge \frac{u^{1/\alpha}\delta_D(y)}{|x-y|}\right)^{\alpha-1} du \\
&\asymp \frac{1}{|x-y|^{1-\alpha}} \int_{\frac{|x-y|^\alpha}{T}}^{1} u^{(1/\alpha)-2} \left(1 \wedge \frac{u^{2/\alpha}\delta_D(x)\delta_D(y)}{|x-y|^2}\right)^{\alpha-1} du \\
&= \frac{1}{|x-y|^{1-\alpha}} \left(\int_{\frac{|x-y|^\alpha}{T}}^{1} u^{(1/\alpha)-2} \mathbf{1}_{\{u \ge u_0^{-\alpha/2}\}} du + \int_{\frac{|x-y|^\alpha}{T}}^{1} u_0^{\alpha-1} u^{-1/\alpha} \mathbf{1}_{\{u < u_0^{-\alpha/2}\}} du \right) \\
&= \frac{1}{|x-y|^{1-\alpha}} \left(\frac{\alpha}{\alpha-1} \left((u_0^{\alpha/2} \vee 1)^{1-(1/\alpha)} - 1 \right) \right. \\
&\qquad \left. + \frac{\alpha}{\alpha-1} u_0^{\alpha-1} \left((u_0 \vee 1)^{-(\alpha-1)/2} - \left(\frac{|x-y|^\alpha}{T}\right)^{(\alpha-1)/\alpha} \right) \right).
\end{aligned}
$$

So by (5.2)–(5.5), (5.7) and (5.8) and the last display, we have

$$
\begin{aligned}
&G_D(x,y) \\
&= \int_T^\infty p_D(t,x,y) + \int_0^T p_D(t,x,y)dt \\
&\asymp \delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1} + \frac{1}{|x-y|^{1-\alpha}} (1 \wedge u_0)^{\alpha-1} \\
&\quad + \frac{1}{|x-y|^{1-\alpha}} \left(\left((u_0 \vee 1)^{(\alpha-1)/2} - 1 \right) \right. \\
&\quad \left. + u_0^{\alpha-1} \left((u_0 \vee 1)^{-(\alpha-1)/2} - \left(\frac{|x-y|^\alpha}{T}\right)^{(\alpha-1)/\alpha} \right) \right) \\
&\asymp \delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1} + \frac{1}{|x-y|^{1-\alpha}} \left(u_0^{\alpha-1} \wedge u_0^{(\alpha-1)/2} \right) \\
&= \delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1} \\
&\quad + \frac{1}{|x-y|^{1-\alpha}} \left(\frac{\delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1}}{|x-y|^{2\alpha-2}} \wedge \frac{\delta_D(x)^{(\alpha-1)/2}\delta_D(y)^{(\alpha-1)/2}}{|x-y|^{\alpha-1}} \right) \\
&\asymp (\delta_D(x)\delta_D(y))^{(\alpha-1)/2} \wedge \frac{\delta_D(x)^{\alpha-1}\delta_D(y)^{\alpha-1}}{|x-y|^{\alpha-1}}.
\end{aligned}
$$

In the last estimate, we used the fact that D is bounded. This proves the corollary. □

Remark 5.1 As in [4], estimates of the Green functions can be used to show that the Martin boundaries and minimal Martin boundaries of a large class of censored stable-like processes can all be identified with the Euclidean boundary ∂D of D. Sharp two-sided estimates for the Martin kernel is easy consequence of our estimates of the Green functions.

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