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Bernoulli coding map and almost sure invariance principle for endomorphisms of $\mathbb{P}^k$

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Abstract Let f be an holomorphic endomorphism of $\mathbb{P}^k$ and μ be its measure of maximal entropy. We prove an almost sure invariance principle for the systems $(\mathbb{P}^k, f, \mu)$. Our class $\mathcal{U}$ of observables includes the Hölder functions and unbounded ones which present analytic singularities. The proof is based on a geometric construction of a Bernoulli coding map $\omega : (\Sigma, s, \nu) \to (\mathbb{P}^k, f, \mu)$. We obtain the invariance principle for an observable ψ on $(\mathbb{P}^k, f, \mu)$ by applying Philipp–Stout's theorem for $\chi = \psi \circ \omega$ on (Σ, s, ν). The invariance principle implies the central limit theorem as well as several statistical properties for the class $\mathcal{U}$. As an application, we give a *direct* proof of the absolute continuity of the measure μ when it satisfies Pesin's formula. This approach relies on the central limit theorem for the unbounded observable $\log \mathsf{Jac}\, f \in \mathcal{U}$.

Keywords Holomorphic dynamics · Bernoulli coding map · Almost sure invariance principle

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1 Introduction

Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an holomorphic endomorphism of algebraic degree $d \geq 2$. Its equilibrium measure μ is the limit of the probability measures $d_t^{-n}(f^n)^*\eta^k$, where $d_t := d^k$ is the topological degree of f and η^k is the standard volume form on $\mathbb{P}^k$. We refer to the survey article of Sibony [38] for an introduction to the dynamical systems

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$(\mathbb{P}^k, f, \mu)$. Fornaess–Sibony proved that μ is mixing [22] and Briend–Duval that μ is the unique measure of maximal entropy [7].

Przytycki et al. [36] introduced coding techniques for $(\mathbb{P}^1, f, \mu)$. This allowed them to prove the almost sure invariance principle (ASIP) for Hölder and singular observables, like $\log|f'|$. In the present article, we extend the coding techniques to $(\mathbb{P}^k, f, \mu)$ and obtain the ASIP for observables which allow analytic singularities. As an application, we obtain a direct proof of the absolute continuity of μ when it satisfies Pesin's formula. We review our results in Sects. 1.1, 1.2, 1.3 and 1.4, Sect.1.5 is devoted to related results.

1.1 Bernoulli coding maps

Let us endow $\Sigma := \{1, \ldots, d_t\}^{\mathbb{N}}$ with the natural product measure $\nu := \otimes_{n=0}^{\infty} \bar{\nu}$, where $\bar{\nu}$ is equidistributed on $\{1, \ldots, d_t\}$. We denote by $\tilde{\alpha}$ the elements of Σ and by s the left shift acting on Σ. Let $\mathcal{J}$ be the support of μ. The following theorem yields coding maps $\omega : \Sigma \to \mathcal{J}$ up to zero measure sets. The set $\mathcal{S} \subset \mathbb{P}^k$ will be defined in Sect. 4, it has zero Lebesgue measure.

Theorem A *Let $z \in \mathbb{P}^k \backslash \mathcal{S}$. There exist an s-invariant set $\Sigma' \subset \Sigma$ of full ν-measure and an f-invariant set $\mathcal{J}' \subset \mathcal{J}$ of full μ-measure satisfying the following properties. For any $\tilde{\alpha} \in \Sigma'$, the point $\omega(\tilde{\alpha}) := \lim_{n\to\infty} z_n(\tilde{\alpha}) \in \mathcal{J}'$ is well defined. We have $\omega_*\nu = \mu$ and the following diagram commutes:*

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\;s\;} & \Sigma' \\ \omega \downarrow & & \downarrow \omega \\ \mathcal{J}' & \xrightarrow{\;f\;} & \mathcal{J}' \end{array}$$

Moreover there exist $\theta, \epsilon > 0$, $n_z \geq 1$ and $\tilde{n} : \Sigma' \to \mathbb{N}$ larger than n_z such that:

1. $d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon\, d^{-\epsilon n}$ *for every* $\tilde{\alpha} \in \Sigma'$ *and* $n \geq \tilde{n}(\tilde{\alpha})$,
2. $\nu(\{\tilde{n} \leq q\}) \geq 1 - c_\theta\, d^{-\theta q}$ *for every* $q \geq n_z$.

We note that Σ', $\mathcal{J}'$ and ω depend on $z \in \mathbb{P}^k \backslash \mathcal{S}$, but $\omega_*\nu = \mu$ holds true for any such z. Observe also that ω is not necessarily injective. The proof of Theorem A (see Sect. 4) is based on the construction of a geometric coding tree, following the approach of Przytycki et al. [36] for $(\mathbb{P}^1, f, \mu)$. The point z is the *root* of the tree, and the set $\{z_n(\tilde{\alpha}), \tilde{\alpha} \in \Sigma\}$ is a suitable enumeration of the d_t^{n+1} points of $f^{-(n+1)}(z)$, these are *vertices* of the tree. The convergence of $(z_n(\tilde{\alpha}))_n$ for a generic $\tilde{\alpha} \in \Sigma$ is obtained by constructing d_t good paths joining z to $w \in f^{-1}(z)$, whose inverse images decrease exponentially. In the context of $(\mathbb{P}^1, f, \mu)$, that property was obtained in [36] by using Koebe distortion theorem. The difficulty in higher dimensions is to substitute this argument. We establish for that purpose a quantified version of a theorem of Briend–Duval (see Sect. 3).

1.2 The class $\mathcal{U}$ and approximation by cylinders

Definition An observable $\psi : \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ belongs to the class $\mathcal{U}$ if:

- e^{ψ} is h-Hölder for some $h > 0$,
- $\mathcal{N}_\psi := \{\psi = -\infty\}$ is a (possibly empty) proper algebraic set of $\mathbb{P}^k$,
- $\psi \geq \log d(\cdot, \mathcal{N}_\psi)^\rho$ for some $\rho > 0$.

For instance, the Hölder functions are in $\mathcal{U}$, as well as the unbounded function $\log \mathrm{Jac}\, f$. We will show that $\mathcal{U} \subset L^p(\mu)$ for any $1 \leq p < +\infty$ (see Sect. 2.2).

Theorem B *Let $\psi \in \mathcal{U}$ be a μ-centered observable and ω be a coding map provided by Theorem A. Let $\chi := \psi \circ \omega$ and $1 \leq p < +\infty$. We denote by $\mathbb{E}(\chi|\mathcal{C}_n)$ the conditional expectation of χ with respect to the $(n+1)$-cylinders.*

1. *there exist $\hat{c}_p, \lambda_p > 0$ such that $\| \chi - \mathbb{E}(\chi|\mathcal{C}_n) \|_p \leq \hat{c}_p\, e^{-n\lambda_p}$ for every $n \geq 0$.*
2. *$R_j(\chi) := \int_\Sigma \chi \cdot \chi \circ s^j\, d\nu$ satisfies $|R_j(\chi)| \leq 2 \| \chi \|_2\, \hat{c}_2\, e^{-(j-1)\lambda_2}$ for every $j \geq 1$.*

The proof occupies Sect. 5, it is based on the regularity properties of ω (namely the points 1, 2 of Theorem A) and on the fact that μ is a Monge–Ampère mass with Hölder potentials. Theorem B allows us to prove Theorem C.

1.3 Almost sure invariance principle

Let $\psi \in L^2(\mu)$ be a μ-centered observable and $S_n(\psi) := \sum_{j=0}^{n-1} \psi \circ f^j$. We say that ψ satisfies the ASIP if there exist, on an extended probability space, a sequence of random variables $(\mathcal{S}_n)_{n\geq 0}$ together with a Brownian motion $\mathcal{W}$ such that for some $\gamma > 0$:

- $\mathcal{S}_n = \mathcal{W}(n) + o(n^{1/2-\gamma})$ almost everywhere,
- $(S_0(\psi), \ldots, S_n(\psi))$ and $(\mathcal{S}_0, \ldots, \mathcal{S}_n)$ have the same distribution for any $n \geq 0$.

We shall denote σ-ASIP to specify the variance of Brownian motion.

Theorem C *For every μ-centered observable $\psi \in \mathcal{U}$, we have:*

1. *$\sigma := \lim_{n\to\infty} \frac{1}{\sqrt{n}} \| S_n(\psi) \|_2$ exists, and $\sigma^2 = \int_{\mathbb{P}^k} \psi^2\, d\mu + 2 \sum_{j\geq 1} \int_{\mathbb{P}^k} \psi \cdot \psi \circ f^j\, d\mu$.*
2. *If $\sigma = 0$, then $\psi = u - u \circ f$ holds μ-a.e. for some $u \in L^2(\mu)$.*
3. *If $\sigma > 0$, then ψ satisfies the σ-ASIP.*

The ASIP implies classical limit theorems related to Brownian motion: the central limit theorem (CLT), the Law of Iterated Logarithm, Kolmogorov integral tests (see [12,35]). The ASIP also implies the almost sure version of the CLT, meaning that $\frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \delta_{\frac{1}{\sqrt{k}} S_k(\psi)(x)}$ converges μ-a.e. to the normal law $\mathcal{N}(0, \sigma^2)$ (see [10,26]).

Let us outline the proof of Theorem C (see Sect. 6). Let $\omega : \Sigma \to \mathbb{P}^k$ be a coding map provided by Theorem A and $\psi \in \mathcal{U}$. Since ω satisfies $f \circ \omega = \omega \circ s$ and $\omega_* \nu = \mu$, we are reduced to prove the assertions for $\chi = \psi \circ \omega$ on (Σ, s, ν). The points 1 and 2 follow from Theorem B(2) and classical arguments. The point 3 is a consequence of Theorem B(1) and Philipp–Stout's theorem ([35, Sect. 7]). That result relies on an approximation of the partial sums of $(\chi \circ s^j)_{j\geq 0}$ by a sequence of martingale differences defined with respect to the increasing filtration $(\mathcal{C}_n)_{n\geq 0}$.

1.4 An application to smooth ergodic theory

Let $\chi_1 \leq \cdots \leq \chi_k$ be the Lyapunov exponents of μ. Briend and Duval [6] proved that they are larger than or equal to $\log d^{1/2}$. Since μ has entropy $\log d^k$, Pesin's formula $h(\mu) = 2(\chi_1 + \cdots + \chi_k)$ holds if and only if these exponents are minimal. We proved in a previous article that μ is then absolutely continuous with respect to Lebesgue measure [21]. We there followed the classical approach of Sinai–Pesin–Ledrappier, based on the construction of a suitable invariant partition which is dilated and realizes entropy (see [27,33]). We propose in Sect. 7 a new proof, based on the CLT for the unbounded μ-centered observable $J := \log \mathsf{Jac}\, f - 2(\chi_1 + \cdots + \chi_k) \in \mathcal{U}$. We obtain the following result, where $\sigma_J := \lim_{n\to\infty} \frac{1}{\sqrt{n}} \parallel S_n(J) \parallel_2$.

Theorem D *If the Lyapunov exponents are minimal equal to* $\log d^{1/2}$*, then* $\sigma_J = 0$*, and* μ *is absolutely continuous with respect to Lebesgue measure.*

A crucial fact for the proof is that for any holomorphic endomorphism of $\mathbb{P}^k$ and any μ-generic point $x \in \mathbb{P}^k$, the minimal dilation rate of f^n at x (i.e. $\parallel (d_x f^n)^{-1} \parallel^{-1}$) is bounded below by $d^{n/2}$ up to the multiplicative factor $1/n$. In other words, the usual $e^{-n\epsilon}$-correction, due to the non-uniform hyperbolicity of $(\mathbb{P}^k, f, \mu)$, can be replaced here by $1/n$. This was proved by Berteloot and Dupont [2], using a pluripotential result of Briend and Duval [6] and the fact that μ is a Monge–Ampère mass. In particular, the product of the dilation rates satisfies $\mathsf{Jac}\, f^n(x) \geq \parallel (d_x f^n)^{-1} \parallel^{-2k} \geq (d^{n/2}/n)^{2k} = d^{kn}/n^{2k}$. Now if we assume $\sigma_J > 0$, then the function $\log \mathsf{Jac}\, f^n$ would present non-trivial oscillations around its mean value $\log d^{kn}$, due to the CLT. More precisely, it would imply $\log \mathsf{Jac}\, f^n \leq \log d^{kn} - \sigma_J \sqrt{n}$ on a subset of μ-measure $\simeq \int_{-\infty}^{-1} e^{-u^2/2}$. That contradicts the preceding estimate, hence $\sigma_J = 0$. We deduce the absolute continuity of μ from the cocycle property $J = u - u \circ f$ μ-a.e. and a linearization property of the dynamics along typical negative orbits [2].

1.5 Related results

The systems $(\mathbb{P}^k, f, \mu)$ and (Σ, s, ν) are actually conjugated by a bimeasurable map up to zero measure subsets, that property was proved by Briend [5]. However, the regularity of the conjugacy seems difficult to handle. Let us also mention that finite-to-one coding maps $(\mathbb{P}^k, f, \mu) \to (\Sigma, s, \nu)$ were constructed by Buzzi [8] by means of suitable partitions of $\mathbb{P}^k$.

The ASIP has been proved for many dynamical systems: for piecewise monotonic maps by Hofbauer and Keller [24], for Anosov maps by Denker and Philipp [13] and for partially and non-uniformly hyperbolic systems by Dolgopyat [20] and Melbourne and Nicol [32]. We refer to the survey articles of Chernov [11] and Denker [12] for limit theorems and statistical properties concerning dynamical systems.

The ASIP implies the CLT. Nevertheless, the latter can be directly proved via coding techniques and Ibragimov's theorem [25]. That method was employed by Sinai [39] and Ratner [37] for the geodesic flow in negative curvature, and by Bowen [4] for Anosov maps. In the present article, Ibragimov's condition is fulfilled by Theorem B.

The Gordin's theorem provides another method for proving the CLT (see [23,28]). It relies on an approximation of $(\psi \circ f^j)_{j\geq 0}$ by a sequence of reverse martingale differences. In our context, this can be done if $\sum_{n\geq 0} \| \Lambda^n \psi \|_2$ (denoted $(\star)$) converges, where Λ denotes the Ruelle–Perron–Frobenius operator (we have $\Lambda^n \psi(z) = \frac{1}{d_t^n} \sum_{y \in f^{-n}(z)} \psi(y)$ for every $z \in \mathbb{P}^k$). Let us note that the reverse martingale mentioned is defined with respect to the decreasing filtration $(f^{-n}\mathcal{B})_{n\geq 0}$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathbb{P}^k$.

The exponential decay of correlations ensures the convergence of $(\star)$. This was proved in the context of $(\mathbb{P}^k, f, \mu)$ by Fornaess and Sibony [22] for C^2 observables and by Dinh–Sibony for Hölder observables [18]. Dinh–Nguyen–Sibony have recently extended that property for differences of quasi-plurisubharmonic functions (the so-called *dsh* functions) [17]. The proof relies on exponential estimates for plurisubharmonic functions with respect to μ. They also obtained in that article a Large Deviations Theorem for bounded dsh and Hölder observables. In [16], Dinh–Nguyen–Sibony proved the local CLT for $(\mathbb{P}^1, f, \mu)$ by using the theory of perturbed operators.

Denker et al. [14] employed a geometric method to prove the convergence of $(\star)$ for $(\mathbb{P}^1, f, \mu)$ and Hölder observables. The idea was to compare $\Lambda^n \psi(z)$ to $\Lambda^n \psi(z')$ by using the contraction of most of the inverse branches of f^n. The cornerstone is a precise analysis of the dynamics near the critical points in the support of μ. Cantat and Leborgne [9] extended this approach to $(\mathbb{P}^k, f, \mu)$. A crucial ingredient was a polynomial estimate for the μ-measure of postcritical neighbourhoods (lemma 5.7 of [9]). The original proof of that lemma contains a gap, the authors have recently proposed another one. Cantat–Leborgne also established in [9] a quantified version of the Briend–Duval theorem. Our version is similar, but we shall give a different proof.

The systems $(\mathbb{P}^k, f, \mu)$ whose measure μ is absolutely continuous with respect to Lebesgue measure were characterized by Berteloot, Dupont and Loeb [2,3]. In that case, f is semi-conjugated to an affine dilation on a complex torus, these maps are the so-called *Lattès examples*. We note that Theorem D characterizes these maps by the minimality of the Lyapunov exponents. Another characterization of Lattès examples involves the *Hausdorff dimension* of μ, defined as the infimum of the Hausdorff dimension of Borel sets with full μ-measure (see Pesin's book [34]): Dinh and Dupont [15] proved that $\dim_{\mathcal{H}}(\mu) = 2k$ if and only if the exponents are minimal. In the context of $(\mathbb{P}^1, f, \mu)$, Mañé [30] proved that $\log d = \dim_{\mathcal{H}}(\mu) \cdot \chi$, where χ denotes the Lyapunov exponent of μ. In particular, the function $L := \log d - \dim_{\mathcal{H}}(\mu) \cdot \log |f'|$ is a μ-centered observable. Zdunik [40] proved that $\sigma_L = 0$ if and only if f is a Lattès example, a Tchebychev polynomial or a power $z^{\pm d}$. The proof relies on the classification of critically finite fractions with parabolic Thurston's orbifold.

2 Generalities

2.1 The holomorphic systems $(\mathbb{P}^k, f, \mu)$

We introduce in this section the systems $(\mathbb{P}^k, f, \mu)$. We refer to the articles [6,7,22,38] for definitions and properties. Here $\mathbb{P}^k$ denotes the complex projective space of dimension k. We denote by η the Fubini-Study form on $\mathbb{P}^k$. This is a (1, 1)-form defined in homogeneous coordinates by $\frac{i}{2\pi}\partial\bar{\partial}\log\| z \|^2$. It induces the standard metric on $\mathbb{P}^k$, the volume of $\mathbb{P}^k$ with respect to this metric is equal to 1. The form η induces on every complex line $L \subset \mathbb{P}^k$ the spherical metric with area 1. Let f be an holomorphic endomorphism of $\mathbb{P}^k$ with algebraic degree $d \geq 2$. It is defined in homogeneous coordinates by $[P_0 : \cdots : P_k]$ where the P_i are homogeneous polynomials of degree d (without common zero except the origin). The topological degree of f is $d_t := d^k$. An inverse branch of f^n on $U \subset \mathbb{P}^k$ is an injective holomorphic map g_n satisfying $f^n \circ g_n = \mathrm{Id}_U$. We let $\mathsf{Per} f := \cup_{n\geq 1}\{x \in \mathbb{P}^k,\ f^n(x) = x\}$, this set is at most countable. Let $\mathcal{C}$ be the critical set of f, $\mathcal{V} := \cup_{i=0}^{\infty} f^i(\mathcal{C})$ and $\mathcal{V}_n := \cup_{i=1}^{n} f^i(\mathcal{C})$. The degree of $\mathcal{V}_n$, denoted τ_n, is equal to $(d + \cdots + d^n)\deg\mathcal{C}$ counted with multiplicity.

The equilibrium measure μ is defined as the limit of $\mu_{n,z} := \frac{1}{d_t^n}\sum_{f^n(y)=z}\delta_y$, where δ_y denotes the Dirac mass at y. In that definition, z has to be taken outside a totally invariant algebraic set $\mathcal{E} \subset \mathcal{V}$, the so-called exceptional set of f. We denote by $\mathcal{J}$ the support of μ. The measure μ is mixing and satisfies $\mu(f(B)) = d_t\,\mu(B)$ whenever f is injective on B. It is the unique measure of maximal entropy (equal to $\log d_t$). The Lyapunov exponents $\chi_1 \leq \cdots \leq \chi_k$ of μ are larger than or equal to $\log d^{1/2}$. They satisfy the classical formula $\int_{\mathbb{P}^k}\log \mathsf{Jac}\, f\, d\mu = 2(\chi_1 + \cdots + \chi_k)$, where $\mathsf{Jac}\, f$ is the non-negative $\mathcal{C}^\infty$ function on $\mathbb{P}^k$ satisfying $f^*\eta^k = \mathsf{Jac}\, f \cdot \eta^k$. The latter is the real jacobian of f, it vanishes on the critical set $\mathcal{C}$ of f.

The measure μ can also be defined *via* pluripotential theory: we have $\mu = T^k$, where T is the Green current of f. The latter is a closed positive (1, 1) current on $\mathbb{P}^k$ with Hölder potentials. In particular, for any algebraic subset $A \subset \mathbb{P}^k$, there exist $c, \gamma > 0$ such that the r-neighbourhood of A satisfies $\mu(A[r]) \leq c\, r^\gamma$ for any $r > 0$ (see [19, Prop. 2.3.7]). For any $\delta > 0$ and $\tilde{c} > 0$, we set $c_\delta := (1 - d^{-\delta})^{-1}$ and $\tilde{c}_\delta := \tilde{c}(1 - d^{-\delta})^{-1}$. In the sequel, $c > 0$ is a constant independent of n, it may differ from a line to another.

2.2 The class $\mathcal{U}$

Let us recall the definition of the class $\mathcal{U}$ (see Sect. 1.2).

Definition 2.1 Let $\mathcal{U}$ be the set of functions $\psi : \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ satisfying:

- e^ψ is h-Hölder on $\mathbb{P}^k$ for some $h > 0$,
- $\mathcal{N}_\psi := \{\psi = -\infty\}$ is a (possibly empty) proper algebraic set of $\mathbb{P}^k$,
- $\psi \geq \log d(\cdot, \mathcal{N}_\psi)^\rho$ on $\mathbb{P}^k$ for some $\rho > 0$.

The Hölder functions belong to $\mathcal{U}$. Examples of unbounded observables are:

- the functions $\psi = \log|Q| - q\log\|\cdot\|$, where Q is a q-homogeneous polynomial on $\mathbb{C}^{k+1}$. Here the algebraic subset $\mathcal{N}_\psi$ is the zero set of Q.
- the functions $\psi = \log\|\Lambda^j d_x f\|$ $(1 \le j \le k)$, where $\Lambda^j d_x f$ is the j-exterior power of the differential $d_x f$. In particular, $\log \mathsf{Jac}\, f \in \mathcal{U}$ (take $j = k$).

The conditions of Definition 2.1 are easy to verify for these functions, the last one is a consequence of Lojasiewicz's inequality (see [29], Sect. 4.7). We prove below that $\psi \in L^p(\mu)$ for any $\psi \in \mathcal{U}$ and $1 \le p < +\infty$. Actually, we establish an estimate for $\int_{\mathcal{N}_\psi[r]} |\psi|^p$, useful to prove Theorem B. We recall that $\mu\left(\mathcal{N}_\psi[r]\right) \le c\, r^\gamma$ for some $c, \gamma > 0$ (see Sect. 2.1).

Proposition 2.2 *Let $\psi \in \mathcal{U}$ and $1 \le p < +\infty$. There exists $\kappa > 0$ such that:*

$$\forall\, 0 < r < 1/2, \quad \int\limits_{\mathcal{N}_\psi[r]} |\psi|^p \, d\mu \le \kappa\, r^{\gamma/2}.$$

In particular $\psi \in L^p(\mu)$.

Proof Let $\psi \in \mathcal{U}$ and $\mathcal{N} := \mathcal{N}_\psi$. We may assume that $0 \le e^\psi \le 1$ by adding some constant to ψ. Let $r < 1/2$ and $\mathcal{Q}_j := \mathcal{N}[r/2^j] \backslash \mathcal{N}[r/2^{j+1}]$. Since $e^\psi \ge (r/2^{j+1})^\rho$ on $\mathcal{Q}_j$, we obtain:

$$\int\limits_{\mathcal{N}[r]} |\psi|^p \, d\mu = \sum_{j\ge 0} \int\limits_{\mathcal{Q}_j} |\log e^\psi|^p \, d\mu \le \sum_{j \ge 0} \left| \rho \log\left(\frac{r}{2^{j+1}}\right)\right|^p \cdot \mu(\mathcal{Q}_j).$$

The inequalities $\mu(\mathcal{Q}_j) \le c(r/2^j)^\gamma$ and $|\log \frac{r}{2^{j+1}}| = (j+1)\log 2 + \log\frac{1}{r} \le (j+2)\log\frac{1}{r}$ yield:

$$\int\limits_{\mathcal{N}[r]} |\psi|^p \, d\mu \le \left[c\,\rho^p \sum_{j \ge 0} \frac{(j+2)^p}{2^{\gamma j}} \right] \left(\log\frac{1}{r}\right)^p r^\gamma = M_{\rho,\gamma} \cdot \left(\log \frac{1}{r}\right)^p r^{\gamma/2} \cdot r^{\gamma/2}.$$

The lemma follows with $\kappa := M_{\rho,\gamma} \cdot \sup_{0<r<1/2} \left(\log\frac{1}{r}\right)^p r^{\gamma/2}$. □

2.3 The Bernoulli space (Σ, s, ν)

We endow $\mathcal{A} := \{1, \ldots, d_t\}$ with the equidistributed probability measure $\bar{\nu}$. We set $\Sigma := \mathcal{A}^{\mathbb{N}}$, $s : \Sigma \to \Sigma$ the left shift and $\nu := \otimes_{n=0}^{\infty} \bar{\nu}$. We denote by $\tilde{\alpha} := (\alpha_n)_{n \ge 0}$ the elements of Σ, by $\mathcal{C}_n$ the set of cylinders of length $n+1$, and by $\pi_n : \Sigma \to \mathcal{A}^{n+1}$ the projection $\pi_n(\tilde{\alpha}) := (\alpha_0, \ldots, \alpha_n)$. For any $\tilde{\alpha} \in \Sigma$, we set $C_n(\tilde{\alpha}) := \pi_n^{-1}(\alpha_0, \ldots, \alpha_n)$. We denote by $\mathbb{E}(\chi|\mathcal{C}_n)$ the conditional expectation of $\chi \in L^2(\nu)$ with respect to $\mathcal{C}_n$. If $\mathcal{L} = \{A_1, \ldots, A_p\} \subset \mathcal{C}_n$, we set $\mathcal{L}^* := \cup_{1 \le j \le p} A_j$.

2.4 Almost sure invariance principle

Let (X, g, m) be either (Σ, s, ν) or $(\mathbb{P}^k, f, \mu)$. For any observable $\varphi \in L^2(m)$, we set $S_n(\varphi) := \sum_{j=0}^{n-1} \varphi \circ g^j$ and $R_j(\varphi) := \int_X \varphi \cdot \varphi \circ g^j \, dm$. We say that φ is m-centered if $\int_X \varphi \, dm = 0$ and that φ is a cocycle if $\varphi = u - u \circ g$ m-a.e. for some $u \in L^2(m)$.

An observable φ on (X, g, m) satisfies the ASIP if there exist on a probability space $(\tilde{X}, \tilde{m})$ a sequence of random variables $(\mathcal{S}_n)_{n\geq 0}$ and a Brownian motion $\mathcal{W}$ such that:

- $\mathcal{S}_n = \mathcal{W}(n) + o(n^{1/2-\gamma})$ $\tilde{m}$-a.e. for some $\gamma > 0$,
- $(S_0(\psi), \ldots, S_n(\psi))$ and $(\mathcal{S}_0, \ldots, \mathcal{S}_n)$ have the same distribution for any $n \geq 0$.

We denote σ-ASIP to specify the variance of Brownian motion. The σ-ASIP implies the σ-central limit theorem (σ-CLT), meaning that:

$$\forall t \in \mathbb{R}, \quad \lim_{n\to\infty} m\left(\frac{S_n(\varphi)}{\sigma\sqrt{n}} \leq t\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-u^2/2} \, du.$$

Remark 2.3 Suppose that $\omega : \Sigma \to \mathbb{P}^k$ is a coding map provided by Theorem A. Since $\omega_* \nu = \mu$ and $f \circ \omega = \omega \circ s$, a μ-centered observable $\psi \in L^2(\mu)$ satisfies the σ-ASIP if and only if the ν-centered observable $\chi := \psi \circ \omega \in L^2(\nu)$ satisfies the σ-ASIP.

We shall use Philipp–Stout's theorem ([35, Section 7]) to prove the ASIP for $\chi := \psi \circ \omega$ on the Bernoulli space (Σ, s, ν). The version below comes from the original one by using the s-invariance of ν and the independence of the random process $(\xi_n)_{n\geq 0}$ defined by $\xi_n(\tilde{\alpha}) = \alpha_n$.

Theorem (Philipp–Stout) *Let χ be a ν-centered observable on Σ satisfying:*

1. $\chi \in L^{2+\delta}(\nu)$ *for some* $\delta > 0$,
2. $\| \chi - \mathbb{E}(\chi | \mathcal{C}_n) \|_{2+\delta} \leq c\, \beta^n$ *for some* $c > 0$ *and* $\beta < 1$.

Then the sequence $\frac{1}{\sqrt{n}} \| S_n(\chi) \|_2$ has a limit σ. If $\sigma > 0$, then χ satisfies the σ-ASIP.

Let us compare that result with Ibragimov's theorem (see [25, Theorem 2.1]), which only requires moments of order 2 and a summability condition:

Theorem (Ibragimov) *Let χ be a ν-centered observable on Σ satisfying:*

$$\sum_{n\geq 0} \| \chi - \mathbb{E}(\chi | \mathcal{C}_n) \|_2 < \infty.$$

Then the sequence $\frac{1}{\sqrt{n}} \| S_n(\chi) \|_2$ has a limit σ. If $\sigma > 0$, then χ satisfies the σ-CLT.

3 A quantified version of Briend–Duval theorem

This section is devoted to the proof of Theorem 3.2 (see Sect. 3.2). That result will be crucial to establish Theorem A.

3.1 Briend–Duval theorem

We recall that $\mathcal{V}_l = \cup_{i=1}^{l} f^i(\mathcal{C})$, $\mathcal{V} = \cup_{i=0}^{\infty} f^i(\mathcal{C})$ and that $d_t = d^k$ is the topological degree of f (see Sect. 2.1). We set $\tau_* := 2 \deg \mathcal{V}_1/(1-1/d)$.

Theorem (Briend and Duval [7]) *Let $\eta > 0$ and $l \geq 1$ be such that $\tau_*/d^l < \eta$. Let L be a complex line in $\mathbb{P}^k$ not contained in $\mathcal{V}$, and $\Delta \Subset \tilde{\Delta}$ be topological discs in $L \backslash \mathcal{V}_l$. Then, for any $n \geq l$, there exist $(1-\eta)d_t^n$ inverse branches g_n on Δ satisfying:*

$$\mathsf{diam}\, g_n(\Delta) \leq \frac{\tilde{c}\, d^{-n/2}}{\eta^{1/2}\ \mathsf{mod}(\tilde{\Delta}\backslash\Delta)^{1/2}},$$

where $\tilde{c}$ is a universal constant, and $\mathsf{mod}(\tilde{\Delta}\backslash\Delta)$ is the modulus of the annulus $\tilde{\Delta}\backslash\Delta$.

Let us recall the definition of the modulus (see Ahlfors book [1], chapters 1 and 2). Let Λ denote the family of curves joining the boundary components of $A := \tilde{\Delta}\backslash\Delta$. For any conformal metric ρ on A, we respectively denote by area_ρ and by l_ρ the area and the length with respect to ρ. We denote by $\mathsf{conf}(A)$ the set of conformal metrics giving finite area to A. The modulus of the annulus A is then defined by:

$$\mathsf{mod}(A) := \sup_{\rho \in \mathsf{conf}(A)} \frac{l_\rho(\Lambda)^2}{\mathsf{area}_\rho(A)},$$

where $l_\rho(\Lambda) := \inf_{\lambda\in\Lambda} l_\rho(\lambda)$.

3.2 Statement of the quantified version

We begin with some notations. Let $0 < \theta < 1$ and $\theta_n := [\theta n + \frac{\log \tau_*}{\log d}] + 1$. We introduce this integer in view of applying Briend–Duval theorem with $\eta = d^{-\theta n}$ and $l = \theta_n$ (indeed, $\tau_*/d^{\theta_n} < d^{-\theta n}$). Since the degree of $\mathcal{V}_{\theta_n} = \cup_{i=1}^{\theta_n} f^i(\mathcal{C})$ is at most $\tau_{\theta_n} = (d + \cdots + d^{\theta_n}) \deg \mathcal{C}$, we have $\tau_{\theta_n} < d^{\theta_n}$ up to a multiplicative constant.

We let $0 < \theta < \theta' < 1$ and consider $n_0 \geq 1$ satisfying:

$$\forall n \geq n_0, \quad \theta_n < \theta' n \quad \text{and} \quad \tau_{\theta_n} < d^{\theta' n}. \tag{1}$$

Let us recall that $\mathcal{V}_{\theta_n}[\delta]$ is the δ-neighbourhood of $\mathcal{V}_{\theta_n}$ in $\mathbb{P}^k$. We fix $\theta'/2 < \zeta < 1$ and define $\mathcal{D} := \limsup_{n\geq n_0} \mathcal{V}_{\theta_n}[d^{-\zeta n}]$.

Proposition 3.1 *The set $\mathcal{D}$ satisfies* $\mathrm{Vol}(\mathcal{D}) = 0$.

The proof is postponed to Sect. 3.5. We now state the quantified version. The constant $\tilde{c}$ has been introduced in the statement of Briend–Duval's theorem, and we denote by L the complex line containing z and w.

Theorem 3.2 *There exists $\epsilon > 0$ such that for every distinct points $(z, w) \notin \mathcal{D} \cup \mathcal{V}$, there exist an injective smooth path $\gamma : [0, 1] \to L\backslash\mathcal{V}$ joining z and w, a decreasing family of topological discs $(\Delta_n)_n \subset L$ and an integer $n_{z,w}$ such that for any $n \geq n_{z,w}$:*

1. $\gamma[0,1] \subset \Delta_n \subset L \backslash \mathcal{V}_{\theta_n}$,
2. *there exist* $(1-d^{-\theta n})d_t^n$ *inverse branches of* f^n *on* Δ_n,
3. *these branches satisfy* $\mathsf{diam}\, g_n(\Delta_n) \le \tilde{c}\, d^{-\epsilon n}$.

We note that θ, ϵ and $\tilde{c}$ do not depend on $(z,w) \in \mathbb{P}^k \backslash (\mathcal{D} \cup \mathcal{V})$.

3.3 Construction of good paths in the complex line $L \subset \mathbb{P}^k$

Let (z,w) be distinct points in $\mathbb{P}^k \backslash (\mathcal{D} \cup \mathcal{V})$. We identify the complex line L containing z and w with the 2-dimensional sphere. We recall that the Fubini-Study metric induces on L the standard spherical metric s with area 1. We assume with no loss of generality that z and w are the North and South pole of L. Let E be the equator of L. For any $y \in E$, we denote by M_y the meridian containing y, and by $M_y\{\delta\}$ the δ-neighbourhood of M_y in L for the spherical metric. The constants $0 < \theta < \theta' < 2\zeta$ have been defined in Sect. 3.2. Now we let $0 < \zeta < \zeta' < \zeta'' < 1$ satisfying:

$$\theta' < \zeta'' - \zeta' \quad \text{and} \quad \theta + \zeta'' < 1. \tag{2}$$

We may take for $(\theta, \theta', \zeta, \zeta', \zeta'')$ suitable multiples of a small $\theta > 0$. The second inequality of (2) will be used in next subsection. The integer n_0 has been defined in Sect. 3.2.

Proposition 3.3 *Let* (z,w) *be distinct points in* $\mathbb{P}^k \backslash (\mathcal{D} \cup \mathcal{V})$. *With the above notations, there exists a subset* $F \subset E$ *of full Lebesgue measure satisfying the following properties. For any* $y \in F$, *there exists* $n_{z,w}(y) \ge n_0$ *such that:*

1. *the meridian* M_y *does not intersect* $\mathcal{V}$,
2. *the neighbourhood* $M_y\{d^{-\zeta'' n}\}$ *does not intersect* $\mathcal{V}_{\theta_n}$ *for any* $n \ge n_{z,w}(y)$.

Let us now prove Proposition 3.3. We start with some notations. Let H^+ and H^- be the (open) North and South hemispheres of L, these sets induce a partition $L = H^+ \sqcup E \sqcup H^-$. We denote by Leb the Lebesgue measure on E and by p_1 (resp. p_2) the spherical projection from z (resp. w) to E. For any $y \in E$ and $\delta > 0$, let $\mathcal{I}(y,\delta)$ be the interval in E centered at y with length 2δ. We also denote by $D(c,\delta) \subset L$ the disc with center c and radius δ. We define $p_\kappa(c) := p_1(c)$ if $c \in H^+ \cup E$ and $p_\kappa(c) := p_2(c)$ if $c \in H^-$. The same convention holds for the projection of $D(c,\delta)$ to E: we use p_1 or p_2 depending on $c \in H^+ \cup E$ or $c \in H^-$.

Let $\{c_i,\ 1 \le i \le l_{\theta_n}\} := \mathcal{V}_{\theta_n} \cap L$, where $l_{\theta_n} \le \deg(\mathcal{V}_{\theta_n}) \le \tau_{\theta_n}$. Since the Fubini-Study metric induces s on L, the set $\mathcal{L}_{\theta_n} := \cup_{i=1}^{l_{\theta_n}} D(c_i, d^{-\zeta n})$ is a subset of $\mathcal{V}_{\theta_n}[d^{-\zeta n}]$. We recall that $\mathcal{D} = \limsup_{n \ge n_0} \mathcal{V}_{\theta_n}[d^{-\zeta n}]$ and that $(z,w) \notin \mathcal{D}$. Thus there exists $n_1 \ge n_0$ depending on (z,w) such that:

$$\forall n \ge n_1, \quad (z,w) \notin \mathcal{V}_{\theta_n}[d^{-\zeta n}]. \tag{3}$$

In particular $(z,w) \notin \mathcal{L}_{\theta_n}$. Since $\zeta < \zeta' < \zeta''$, we may increase n_1 so that $d^{-\zeta' n} + d^{-\zeta'' n} < d^{-\zeta n}$ for any $n \ge n_1$. We have therefore, for $\rho = z$ or w:

$$\forall\, 1 \le i \le l_{\theta_n}, \quad \forall n \ge n_1, \quad D(\rho, d^{-\zeta' n}) \cap D(c_i, d^{-\zeta'' n}) = \emptyset.$$

This implies, with $e_i := p_\kappa(c_i) \in E$ and c a positive constant:

$$\forall\, 1 \le i \le l_{\theta_n}, \quad p_\kappa\left(D(c_i, d^{-\zeta'' n})\right) \subset \mathcal{I}_i := \mathcal{I}(e_i,\, c\, d^{-\zeta'' n} \cdot d^{\zeta' n}). \tag{4}$$

Hence $\mathcal{I}(\theta_n) := \cup_{i=1}^{l_{\theta_n}} \mathcal{I}_i$ satisfies $\mathsf{Leb}\, \mathcal{I}(\theta_n) \le \tau_{\theta_n} \cdot c\, d^{-(\zeta'' - \zeta')n} \le c\, d^{(\theta' - (\zeta'' - \zeta'))n}$. Since $\sum_n \mathsf{Leb}\, \mathcal{I}(\theta_n) < \infty$ (see 2), the Borel–Cantelli lemma yields, for every y in a full Lebesgue measure subset $F' \subset E$, an integer $n_{z,w}(y) \ge n_1$ satisfying:

$$y \notin \bigcup_{n \ge n_{z,w}(y)} \mathcal{I}(\theta_n). \tag{5}$$

Let us prove the point 2 of Proposition 3.3 (the point 1 will be proved below, F is a subset of F'). Let $y \in F'$ and $\mathcal{I} := \mathcal{I}(y, d^{-(\zeta'' - \zeta')n})$. Since the intervals $\mathcal{I}_i$ defining $\mathcal{I}(\theta_n)$ are centered at $e_i = p_\kappa(c_i)$, the set $p_1^{-1}(\mathcal{I})$ does not intersect any point $c_i \in H^+ \cup E$. The same property holds for $p_2^{-1}(\mathcal{I})$ with the $c_i \in H^-$. This implies that $M_y\{d^{-\zeta'' n}\}$ does not intersect $\mathcal{V}_{\theta_n} \cap L$ for any $n \ge n_{z,w}(y)$, and yields the point 2.

For the point 1, it suffices to verify that $p_\kappa(\mathcal{V} \cap L)$ has zero Lebesgue measure. Let $\mathcal{W} := \mathcal{V} \cap L$. Since $(z, w) \in L$ and $(z, w) \notin \mathcal{V} = \cup_{i=0}^{\infty} f^i(\mathcal{C})$, the complex line L is not an algebraic subset of the hypersurface $f^i(\mathcal{C})$ for any $i \ge 0$. In particular, $\mathcal{W}_i := f^i(\mathcal{C}) \cap L$ is finite for every $i \ge 0$. Hence $\mathcal{W} = \cup_{i \ge 0} \mathcal{W}_i$ satisfies $\mathsf{Leb}(p_\kappa(\mathcal{W})) = 0$. We finally set $F := F' \backslash p_\kappa(\mathcal{W})$, that completes the proof of Proposition 3.3.

3.4 Proof of Theorem 3.2

We set $\epsilon := \frac{1}{2}(1 - (\theta + \zeta'')) > 0$ (see 2). Let (z, w) be distinct points in $\mathbb{P}^k \backslash (\mathcal{D} \cup \mathcal{V})$ and consider some $y \in F$ provided by Proposition 3.3: the meridian M_y does not intersect $\mathcal{V}$ and its neighbourhood $M_y\{d^{-\zeta'' n}\}$ in L does not intersect $\mathcal{V}_{\theta_n}$ for every $n \ge n_{z,w}(y)$.

We set $n_{z,w} := n_{z,w}(y)$ and denote $M := M_y$ for sake of simplicity. Let $\gamma : [0, 1] \to L$ be the natural parametrization of M. We define $\Delta_n := M\{d^{-\zeta'' n}/2\}$ and $\tilde{\Delta}_n := M\{d^{-\zeta'' n}\}$. Let us apply Briend–Duval's theorem with $\eta = d^{-\theta n}$, $l = \theta_n$ and $\Delta_n \Subset \tilde{\Delta}_n \subset L \backslash \mathcal{V}_{\theta_n}$. Since $n > \theta' n \ge \theta_n = l$ and $\tau_* / d^{\theta_n} < d^{-\theta n}$ (see 1), there exist $(1 - d^{-\theta n}) d_t^n$ inverse branches on the disc Δ_n satisfying:

$$\mathsf{diam}\, g_n(\Delta_n) \le \tilde{c}\, d^{-n/2} \left(d^{-\theta n} \,\mathsf{mod}\, \left[\tilde{\Delta}_n \backslash \Delta_n\right]\right)^{-1/2}. \tag{6}$$

It remains to bound the modulus of $A_n := \tilde{\Delta}_n \backslash \Delta_n$. Let Λ_n be the set of curves joining the boundary components of A_n. We denote by area_s and by l_s the area and the length in L with respect to the spherical metric s. The following estimates hold up to multiplicative constants. We have $l_\mathsf{s}(\lambda) \ge d^{-\zeta'' n}$ for any $\lambda \in \Lambda_n$, hence

$l_{\mathsf{s}}(\Lambda_n) = \inf_{\lambda\in\Lambda_n} l_{\mathsf{s}}(\lambda) \geq d^{-\zeta''n}$. The inequalities $\mathsf{area}_{\mathsf{s}}(A_n) \leq \mathsf{area}_{\mathsf{s}}(\tilde{\Delta}_n) \leq d^{-\zeta''n}$ then imply:

$$\mathsf{mod}\,(A_n) = \sup_{\rho\in\mathsf{conf}\,A_n} \frac{l_\rho(\Lambda)^2}{\mathsf{area}_\rho(A_n)} \geq \frac{l_{\mathsf{s}}(\Lambda_n)^2}{\mathsf{area}_{\mathsf{s}}(A_n)} \geq \frac{d^{-2\zeta''n}}{d^{-\zeta''n}} = d^{-\zeta''n}. \quad (7)$$

From (6), (7) and $\epsilon = \frac{1}{2}(1-(\theta+\zeta''))$, we deduce that $\mathsf{diam}\, g_n(\Delta_n) \leq \tilde{c}\, d^{-\epsilon n}$. That completes the proof of Theorem 3.2.

3.5 Volume of neighbourhoods

This subsection is devoted to the proof of Proposition 3.1: we want to show Vol $(\mathcal{D}) = 0$, where $\mathcal{D} = \bigcap_{n\geq n_0}\bigcup_{p\geq n}\mathcal{V}_{\theta_p}[d^{-\zeta p}]$. We recall that $\mathcal{V}_{\theta_p}[d^{-\zeta p}]$ is the $d^{-\zeta p}$-neighbourhood of $\cup_{i=1}^{\theta_p} f^i(\mathcal{C})$ and that $\zeta > \theta'/2$. The proof is based on the following lemma (see [19, lemma 2.3.8]).

Lemma 3.4 *Let $X \subset \mathbb{P}^k$ be an algebraic subvariety of dimension m and degree q. Then* Vol $X[\delta] \leq q\,\delta^{2(k-m)}$ *for any $\delta > 0$, up to a multiplicative constant independent of X.*

We deduce Vol $(\mathcal{D}) = 0$ as follows. We set $p \geq n \geq n_0$ and apply Lemma 3.4 with $X = \mathcal{V}_{\theta_p}$ and $\delta = d^{-\zeta p}$ (here $k-m = 1$ and $q = \deg \mathcal{V}_{\theta_p} \leq \tau_{\theta_p}$). We obtain with $\tau_{\theta_p} \leq d^{\theta' p}$ (see 1): Vol $\mathcal{V}_{\theta_p}[d^{-\zeta p}] \leq \tau_{\theta_p}(d^{-\zeta p})^2 \leq d^{-(2\zeta-\theta')p}$. Hence:

$$\forall n \geq n_0, \quad \mathrm{Vol}\,(\mathcal{D}) \leq \mathrm{Vol} \bigcup_{p\geq n} \mathcal{V}_{\theta_p}[d^{-\zeta p}] \leq c_{2\zeta-\theta'}\, d^{-(2\zeta-\theta')n}.$$

This yields Vol $(\mathcal{D}) = 0$ when n tends to infinity.

Proof of Lemma 3.4 The argument is based on Lelong's inequality. Let $\mathcal{E}$ be a *maximal* δ-separated set in X for the ambient metric: this means that $d(a,b) \geq \delta$ for any pair of distinct elements of $\mathcal{E}$, and that for any $x \in X$ there exists $a \in \mathcal{E}$ satisfying $d(a,x) < \delta$. Since $X[\delta] \subset \cup_{a\in\mathcal{E}} B_a(2\delta)$, we get up to a multiplicative constant:

$$\mathrm{Vol}\, X[\delta] \leq (2\delta)^{2k}\, \mathsf{Card}\, \mathcal{E}. \quad (8)$$

We now give an upper bound for $\mathsf{Card}\,\mathcal{E}$. Observe that Vol X is equal to the degree of X, and that the balls $(B_a(\delta/2))_{a\in\mathcal{E}}$ are mutually disjoint. Thus:

$$q = \mathrm{Vol}\, X \geq \sum_{a\in\mathcal{E}} \mathrm{Vol}\ (X \cap B_a(\delta/2)).$$

Now Lelong's inequality asserts that Vol $(X \cap B_a(\delta/2)) \geq \delta^{2m}$ for any $a \in \mathcal{E}$, up to a multiplicative constant. Hence $\mathsf{Card}\,\mathcal{E} \leq q\,\delta^{-2m}$, as desired. ◻

4 Proof of Theorem A

We set $\mathcal{S} := \mathcal{V} \cup \mathcal{D} \cup f(\mathcal{D}) \cup \mathrm{Per}(f)$, where $\mathcal{D}$ is defined in Sect. 3.2. We have $\mathrm{Vol}\,(\mathcal{S}) = 0$ since $\mathrm{Vol}\,(\mathcal{D}) = 0$. Let us recall the statement of Theorem A.

Theorem A *Let $z \in \mathbb{P}^k \backslash \mathcal{S}$. There exist an s-invariant set $\Sigma' \subset \Sigma$ of full ν-measure and an f-invariant set $\mathcal{J}' \subset \mathcal{J}$ of full μ-measure satisfying the following properties. For any $\tilde{\alpha} \in \Sigma'$, the point $\omega(\tilde{\alpha}) := \lim_{n\to\infty} z_n(\tilde{\alpha}) \in \mathcal{J}'$ is well defined. We have $\omega_*\nu = \mu$ and the following diagram commutes:*

$$\begin{array}{ccc} \Sigma' & \xrightarrow{\ s\ } & \Sigma' \\ {\scriptstyle\omega}\downarrow & & \downarrow{\scriptstyle\omega} \\ \mathcal{J}' & \xrightarrow{\ f\ } & \mathcal{J}' \end{array}$$

Moreover there exist $\theta, \epsilon > 0$, $n_z \geq 1$ and $\tilde{n} : \Sigma' \to \mathbb{N}$ larger than n_z such that:

1. *$d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon\, d^{-\epsilon n}$ for every $\tilde{\alpha} \in \Sigma'$ and $n \geq \tilde{n}(\tilde{\alpha})$,*
2. *$\nu(\{\tilde{n} \leq q\}) \geq 1 - c_\theta\, d^{-\theta q}$ for every $q \geq n_z$.*

We shall use Theorem 3.2 and the method of coding trees introduced in [36] for $(\mathbb{P}^1, f, \mu)$. We recall that $\mathcal{A} = \{1, \ldots, d_t\}$. Let $z \notin \mathcal{S}$ and $\{w_\alpha,\ \alpha \in \mathcal{A}\} := f^{-1}(z)$. By the very definition of $\mathcal{S}$, the cardinal of $f^{-1}(z)$ is equal to d_t and $w_\alpha \neq z$, $w_\alpha \notin \mathcal{V} \cup \mathcal{D}$ for every $\alpha \in \mathcal{A}$. We denote by L_α the projective line in $\mathbb{P}^k$ containing (z, w_α) and apply Theorem 3.2: let γ_α be an injective smooth path joining (z, w_α) and $(\Delta_n(\alpha))_n \subset L_\alpha$ be a decreasing sequence of discs containing γ_α provided by that theorem. We set $n_z := \max\{n_{z,w_\alpha},\ \alpha \in \mathcal{A}\}$.

Let us fix $\tilde{\alpha} = (\alpha_n)_{n\geq 0} \in \Sigma$. We define inductively injective smooth paths $\gamma_n(\tilde{\alpha}) : [0, 1] \to \mathbb{P}^k \backslash \mathcal{V}$ and points $z_n(\tilde{\alpha}) \in \mathbb{P}^k \backslash \mathcal{V}$. We first set $\gamma_0(\tilde{\alpha}) := \gamma_{\alpha_0}$. This path joins $z = \gamma_0(\tilde{\alpha})(0)$ and $w_{\alpha_0} = \gamma_0(\tilde{\alpha})(1) =: z_0(\tilde{\alpha})$. Assume that the paths $\gamma_j(\tilde{\alpha})$ and the points $z_j(\tilde{\alpha})$ have been defined for $0 \leq j \leq n-1$. We let $\gamma_n(\tilde{\alpha})$ to be the lift of γ_{α_n} by f^n with starting point $\gamma_n(\tilde{\alpha})(0) = z_{n-1}(\tilde{\alpha})$. This path is well defined since γ_{α_n} does not intersect $\mathcal{V}$. We finally let $z_n(\tilde{\alpha}) := \gamma_n(\tilde{\alpha})(1)$.

We note that $z_{n-1}(\tilde{\alpha})$ and $z_n(\tilde{\alpha})$ are the endpoints of $\gamma_n(\tilde{\alpha})$ and that $z_n(\Sigma) = f^{-(n+1)}(z)$ has cardinal d_t^{n+1}. The reader will easily check the relation $f \circ z_n(\tilde{\alpha}) = z_{n-1} \circ s(\tilde{\alpha})$. Observe also that $\gamma_n(\tilde{\alpha})$ and $z_n(\tilde{\alpha})$ depend only on $\pi_n(\tilde{\alpha}) = (\alpha_0, \ldots, \alpha_n)$. The following lemma is a consequence of Theorem 3.2 and the fact that $\gamma_\alpha[0, 1] \subset \Delta_n(\alpha)$.

Lemma 4.1 *For every $\alpha \in \mathcal{A}$ and $n \geq n_z$, there exist at least $(1 - d^{-\theta n})d_t^n$ elements $(\alpha_0, \ldots, \alpha_{n-1}) \in \mathcal{A}^n$ such that $\mathrm{diam}\,\gamma_n(\alpha_0, \ldots, \alpha_{n-1}, \alpha) \leq \tilde{c}\, d^{-\epsilon n}$.*

Let $\Omega_n := \{\tilde{\alpha} \in \Sigma,\ \mathrm{diam}\,\gamma_n(\tilde{\alpha}) > \tilde{c}\, d^{-\epsilon n}\}$ and $\mathcal{B}_n$ be the collection of $(n+1)$-cylinders $\{C_n(\tilde{\alpha}),\ \tilde{\alpha} \in \Omega_n\}$. We have $\Omega_n = \mathcal{B}_n^*$. Let us also define:

$$\Omega(n) := \bigcup_{p\geq n} \Omega_p = \bigcup_{p\geq n} \mathcal{B}_p^*.$$

Lemma 4.2 *For any $n \geq n_z$, we have:*

1. $\mathsf{Card}\,(\mathcal{B}_n) \leq d_t^{n+1} d^{-\theta n}$.
2. $\nu(\Omega_n) \leq d^{-\theta n}$, *hence* $\nu(\Omega(n)) \leq c_\theta\, d^{-\theta n}$.
3. *if* $\tilde{\alpha} \notin \Omega(n)$, *then* $d(z_{m-1}(\tilde{\alpha}), z_m(\tilde{\alpha})) \leq \tilde{c}\, d^{-\epsilon m}$ *for any* $m \geq n$.

Proof We have $\mathcal{B}_n = \{C_n(\tilde{\alpha}),\ \mathsf{diam}\,\gamma_n(\tilde{\alpha}) > \tilde{c}\, d^{-\epsilon n}\}$. For every $\alpha \in \mathcal{A}$, we set $\mathcal{B}_n(\alpha) \subset \mathcal{B}_n$ to be the collection of $(n+1)$-cylinders whose last coordinate is equal to α. The Lemma 4.1 implies that $\mathsf{Card}\,(\mathcal{B}_n(\alpha)) \leq d_t^n d^{-\theta n}$ and thus $\mathsf{Card}\,(\mathcal{B}_n) = \sum_{\alpha \in \mathcal{A}} \mathsf{Card}\,(\mathcal{B}_n(\alpha)) \leq d_t^{n+1} d^{-\theta n}$, which is the point 1. The point 2 follows:

$$\nu(\Omega_n) = \nu(\mathcal{B}_n^*) = \mathsf{Card}\,(\mathcal{B}_n)/d_t^{n+1} \leq d^{-\theta n}.$$

For the point 3, observe that $d(z_{m-1}(\tilde{\alpha}), z_m(\tilde{\alpha})) \leq \mathsf{diam}\,\gamma_m(\tilde{\alpha})$. If $\tilde{\alpha} \notin \Omega(n)$, then $\tilde{\alpha} \notin \Omega_m$ for any $m \geq n$, hence $\mathsf{diam}\,\gamma_m(\tilde{\alpha}) \leq \tilde{c}\, d^{-\epsilon m}$. □

Let $\Omega := \bigcap_{n \geq n_z} \Omega(n) = \limsup_{n \geq n_z} \Omega_n$. The set $\Sigma'' := \Sigma \backslash \Omega$ has full ν-measure since $\nu(\Omega) \leq \nu(\Omega(n)) \leq c_\theta d^{-\theta n}$ for any $n \geq n_z$. For every $\tilde{\alpha} \in \Sigma''$, we define $\tilde{n}(\tilde{\alpha})$ to be the least integer $n \geq n_z$ satisfying $\tilde{\alpha} \notin \Omega(n)$. Let $\Theta_q := \{\tilde{n} \leq q\}$.

Lemma 4.3 1. $\omega(\tilde{\alpha}) = \lim_{n \to \infty} z_n(\tilde{\alpha})$ *is well defined for every* $\tilde{\alpha} \in \Sigma''$.

2. $d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon\, d^{-\epsilon n}$ *for every* $n \geq \tilde{n}(\tilde{\alpha})$.
3. $\omega : \Sigma'' \to \mathbb{P}^k$ *satisfies* $\omega_* \nu = \mu$.
4. $\nu(\Theta_q) \geq 1 - c_\theta\, d^{-\theta q}$ *for any* $q \geq n_z$.

Proof The points 1, 2 and 4 come from Lemma 4.2(3,2) and the definition of $\tilde{n}(\tilde{\alpha})$. Now we prove the point 3. Let us consider the surjective map $z_n : \Sigma'' \to f^{-(n+1)}(z)$. Since $z_n(\tilde{\alpha})$ depends only on $\underline{\alpha} := (\alpha_0, \ldots, \alpha_n) \in \mathcal{A}^{n+1}$, the measure $z_{n*}\nu$ is equal to:

$$z_{n*}\nu = \sum_{\underline{\alpha} \in \mathcal{A}^{n+1}} \nu\left(\Sigma'' \cap C_n(\underline{\alpha})\right)\, \delta_{z_n(\underline{\alpha})} = \frac{1}{d_t^{n+1}} \sum_{f^{n+1}(y) = z} \delta_y = \mu_{n+1,z}.$$

Since $z \notin \mathcal{S}$ and $\mathcal{E} \subset \mathcal{V} \subset \mathcal{S}$, the sequence of probability measures $(\mu_{n,z})_n$ converges to μ (see Sect. 2.1). Hence it remains to prove $z_{n*}\nu \to \omega_*\nu$, meaning that $\int_{\Sigma''} \varphi \circ z_n\, d\nu \to \int_{\Sigma''} \varphi \circ \omega\, d\nu$ for every test function $\varphi : \mathbb{P}^k \to \mathbb{R}$. But this follows from point 1 and Lebesgue convergence theorem. □

It remains to define Σ', $\mathcal{J}'$ and to verify the relation $f \circ \omega = \omega \circ s$ on Σ'. The Lemma 4.3(3) implies that $\Sigma_* := \omega(\Sigma'')$ satisfies $\mu(\Sigma_*) = \nu(\omega^{-1}\Sigma_*) \geq \nu(\Sigma'') = 1$. We define $\mathcal{J}' := \bigcap_{n \in \mathbb{Z}} f^n(\mathcal{J} \cap \Sigma_*)$ and $\Sigma' := \bigcap_{n \in \mathbb{Z}} s^n(\Sigma'' \cap \omega^{-1}\mathcal{J}')$. These are invariant subsets of full measure. We obtain $f \circ \omega = \omega \circ s$ on Σ' by taking limits in $f \circ z_n(\tilde{\alpha}) = z_{n-1} \circ s(\tilde{\alpha})$. That completes the proof of Theorem A.

5 Proof of Theorem B

Let us recall the statement.

Theorem B *Let* $\psi \in \mathcal{U}$ *be a* μ*-centered observable and* ω *be a coding map provided by Theorem A. Let* $\chi := \psi \circ \omega$ *and* $1 \leq p < +\infty$.

1. *there exist* $\hat{c}_p, \lambda_p > 0$ *such that* $\| \chi - \mathbb{E}(\chi | \mathcal{C}_n) \|_p \leq \hat{c}_p \, e^{-n\lambda_p}$ *for every* $n \geq 0$.
2. $R_j(\chi) := \int_\Sigma \chi \cdot \chi \circ s^j \, d\nu$ *satisfies* $|R_j(\chi)| \leq 2 \| \chi \|_2 \, \hat{c}_2 \, e^{-(j-1)\lambda_2}$ *for every* $j \geq 1$.

5.1 Proof of Theorem B(1)

We set $\chi_B := \chi.1_B$ for any $B \subset \Sigma$ and use the following estimates provided by Theorem A. We recall that $\Theta_n = \{\tilde{n}(\tilde{\alpha}) \leq n\}$.

($\star$) $d(z_n(\tilde{\alpha}), \omega(\tilde{\alpha})) \leq \tilde{c}_\epsilon \, d^{-\epsilon n}$ for every $\tilde{\alpha} \in \Theta_n$,
($\star\star$) $\nu(\Theta_n) \geq 1 - c_\theta \, d^{-n\theta}$ for every $n \geq n_z$.

We will need the following lemma, which is a direct consequence of ($\star$).

Lemma 5.1 *Let* $\tilde{\alpha} \in \Theta_n$ *and* $\tilde{\beta} \in C_n(\tilde{\alpha}) \cap \Theta_n$. *Then* $d(\omega(\tilde{\alpha}), \omega(\tilde{\beta})) \leq 2 \, \tilde{c}_\epsilon \, d^{-\epsilon n}$.

5.1.1 The Hölder case

Let ψ be an h-Hölder and μ-centered observable on $\mathbb{P}^k$. We set $\chi := \psi \circ \omega$. The Theorem B(1) is a consequence of the following estimates, which hold for every $n \geq n_z$.

Lemma 5.2 $\left\| \chi_{\Theta_n^c} - \mathbb{E}(\chi_{\Theta_n^c} | \mathcal{C}_n) \right\|_p \leq 2 \| \chi \|_\infty \, \left(c_\theta \, d^{-n\theta}\right)^{1/p}$.

Proof The left-hand side is less than $2 \left\| \chi_{\Theta_n^c} \right\|_p$ by Jensen inequality. Then the conclusion follows from ($\star\star$). □

Lemma 5.3 $\left\| \chi_{\Theta_n} - \mathbb{E}(\chi_{\Theta_n} | \mathcal{C}_n) \right\|_p \leq c \, d^{-n\tau}$ *for some* $c, \tau > 0$.

Proof We denote $\varphi := \chi_{\Theta_n} - \mathbb{E}(\chi_{\Theta_n} | \mathcal{C}_n)$ and estimate $\left\| \varphi_{\Theta_n^c} \right\|_p$, $\left\| \varphi_{\Theta_n} \right\|_p$. Since $\varphi_{\Theta_n^c} = -\mathbb{E}(\chi_{\Theta_n} | \mathcal{C}_n) \cdot 1_{\Theta_n^c}$, we have:

$$\left\| \varphi_{\Theta_n^c} \right\|_p \leq \left\| \mathbb{E}(\chi_{\Theta_n} | \mathcal{C}_n) \right\|_{2p} \cdot \nu(\Theta_n^c)^{1/2p} \leq \| \chi \|_{2p} \cdot \left(c_\theta \, d^{-n\theta}\right)^{1/2p}.$$

We now deal with $\left\| \varphi_{\Theta_n} \right\|_p$. For every $\tilde{\alpha} \in \Theta_n$, let $\nu_{\tilde{\alpha}}$ be the conditional measure of ν on the cylinder $C_n(\tilde{\alpha})$. We have for every $\tilde{\alpha} \in \Theta_n$:

$$\varphi_{\Theta_n}(\tilde{\alpha}) = \int_{C_n(\tilde{\alpha}) \cap \Theta_n} \left(\chi(\tilde{\alpha}) - \chi(\tilde{\beta})\right) d\nu_{\tilde{\alpha}}(\tilde{\beta}) + \chi(\tilde{\alpha}) \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Theta_n^c). \tag{9}$$

We deduce from $\chi = \psi \circ \omega$, Lemma 5.1 and the fact that ψ is h-Hölder:

$$\forall \tilde{\alpha} \in \Theta_n, \quad |\varphi_{\Theta_n}(\tilde{\alpha})| \leq \left(2 \, \tilde{c}_\epsilon \, d^{-n\epsilon}\right)^h + \left\| \chi_{\Theta_n} \right\|_\infty \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Theta_n^c).$$

Hence we get for every $p \geq 1$ up to a multiplicative constant:

$$\forall \tilde{\alpha} \in \Theta_n, \quad |\varphi_{\Theta_n}(\tilde{\alpha})|^p \leq d^{-nhp\epsilon} + \left\| \chi_{\Theta_n} \right\|_\infty^p \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Theta_n^c).$$

By integrating over Θ_n and using $(\star\star)$, we deduce:

$$\| \varphi_{\Theta_n} \|_p^p \leq d^{-nhp\epsilon} + \| \chi \|_\infty^p \cdot c_\theta d^{-n\theta}.$$

That completes the proof of the lemma. □

5.1.2 The general case $\psi \in \mathcal{U}$

Let $\psi : \mathbb{P}^k \to \mathbb{R} \cup \{-\infty\}$ be a μ-centered observable in $\mathcal{U}$: the function e^ψ is h-Hölder and satisfies $\psi \geq \log d(\cdot, \mathcal{N}_\psi)^\rho$ on $\mathbb{P}^k$ (see Definition 2.1). Observe in particular that ψ is bounded from above. We recall that $\mathcal{N}_\psi[r]$ is the r-neighbourhood of $\{\psi = -\infty\}$ and that $\chi = \psi \circ \omega$. We consider the following subsets of Σ:

$$\Gamma_n := \Theta_n^c \backslash \mathcal{N}_n, \quad \Gamma_n = \Theta_n \backslash \mathcal{N}_n, \ \mathcal{N}_n := \omega^{-1}\left(\mathcal{N}_\psi[d^{-n(h\epsilon/2\rho)}]\right).$$

We shall need the following observations. First, we have $\nu(\mathcal{N}_n) = \mu(\mathcal{N}_\psi[d^{-n(h\epsilon/2\rho)}]) \leq d^{-n\gamma(h\epsilon/2\rho)}$ up to a multiplicative constant (see Sect. 2.1). We deduce from $(\star\star)$:

$$\nu(\Gamma_n^c) = \nu(\Theta_n^c \cup \mathcal{N}_n) \leq c_\theta d^{-n\theta} + d^{-n\gamma(h\epsilon/2\rho)} \leq c\, d^{-n\eta} \tag{10}$$

for some $c, \eta > 0$. Second, for every $\tilde{\alpha} \in \mathcal{N}_n^c = \mathcal{S}_n \cup \Gamma_n$, we have $\chi(\tilde{\alpha}) \geq \log d(\omega(\tilde{\alpha}), \mathcal{N}_\psi)^\rho \geq \log d^{-\rho n(h\epsilon/2\rho)}$, hence:

$$\| \chi_{\mathcal{S}_n \cup \Gamma_n} \|_\infty \leq n\,(h\epsilon \log d)/2. \tag{11}$$

The Theorem B(1) is now a consequence of the following estimates.

Lemma 5.4 $\| \chi_{\mathcal{N}_n} - \mathbb{E}(\chi_{\mathcal{N}_n} | \mathcal{C}_n) \|_p \leq \left(\kappa\, d^{-n(h\epsilon/2\rho)\cdot(\gamma/2)}\right)^{1/p}$.

Proof The left-hand side is less than $2 \| \chi_{\mathcal{N}_n} \|_p$. Proposition 2.2 yields $\| \chi_{\mathcal{N}_n} \|_p = \| \psi \circ \omega \cdot 1_{\mathcal{N}_n} \|_p \leq \left(\kappa\, d^{-n(h\epsilon/2\rho)\cdot(\gamma/2)}\right)^{1/p}$ for every n such that $d^{-n(h\epsilon/2\rho)} < 1/2$. □

Lemma 5.5 $\| \chi_{\mathcal{S}_n} - \mathbb{E}(\chi_{\mathcal{S}_n} | \mathcal{C}_n) \|_p \leq n\,(h\epsilon \log d) \cdot \left(c\, d^{-n\eta}\right)^{1/p}$.

Proof The left-hand side is less than $2 \| \chi_{\mathcal{S}_n} \|_p$. We conclude by using (10) and (11) (observe that $\mathcal{S}_n \subset \Gamma_n^c$). □

Lemma 5.6 $\| \chi_{\Gamma_n} - \mathbb{E}(\chi_{\Gamma_n} | \mathcal{C}_n) \|_p \leq c\, d^{-n\tau}$ *for some* $c, \tau > 0$.

Proof We follow the proof of Lemma 5.3: we set $\varphi := \chi_{\Gamma_n} - \mathbb{E}(\chi_{\Gamma_n} | \mathcal{C}_n)$ and estimate $\| \varphi_{\Gamma_n^c} \|_p$, $\| \varphi_{\Gamma_n} \|_p$. The line (10) yields:

$$\| \varphi_{\Gamma_n^c} \|_p \leq \| \mathbb{E}(\chi_{\Gamma_n} | \mathcal{C}_n) \|_{2p} \cdot \nu(\Gamma_n^c)^{1/2p} \leq \| \chi \|_{2p} \cdot \left(c\, d^{-n\eta}\right)^{1/2p}.$$

Now we deal with $\| \varphi_{\Gamma_n} \|_p$. We can write as in (9):

$$\forall \tilde{\alpha} \in \Gamma_n, \quad \varphi(\tilde{\alpha}) = \int_{C_n(\tilde{\alpha}) \cap \Gamma_n} \Big(\chi(\tilde{\alpha}) - \chi(\tilde{\beta})\Big)\, d\nu_{\tilde{\alpha}}(\tilde{\beta}) + \chi(\tilde{\alpha}) \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Gamma_n^c). \tag{12}$$

Let $\tilde{\alpha} \in \Gamma_n$ and $\tilde{\beta} \in C_n(\tilde{\alpha}) \cap \Gamma_n$. We deduce from $(\tilde{\alpha}, \tilde{\beta}) \notin \mathcal{N}_n$ that $e^{\psi} \circ \omega(\tilde{\alpha})$ and $e^{\psi} \circ \omega(\tilde{\beta})$ are larger than $d^{-nh\epsilon/2}$. This implies:

$$|\chi(\tilde{\alpha}) - \chi(\tilde{\beta})| = |\log e^{\psi} \circ \omega(\tilde{\alpha}) - \log e^{\psi} \circ \omega(\tilde{\beta})| \leq d^{nh\epsilon/2} |e^{\psi} \circ \omega(\tilde{\alpha}) - e^{\psi} \circ \omega(\tilde{\beta})|.$$

Using Lemma 5.1 and the fact that e^{ψ} is h-Hölder, the last term is less than $d^{nh\epsilon/2} \cdot \big(2\, \tilde{c}_{\epsilon}\, d^{-n\epsilon}\big)^h$. Then we deduce from (12), up to a multiplicative constant:

$$\forall \tilde{\alpha} \in \Gamma_n, \quad |\varphi(\tilde{\alpha})| \leq d^{-nh\epsilon/2} + \| \chi_{\Gamma_n} \|_{\infty} \cdot \nu_{\tilde{\alpha}}(C_n(\tilde{\alpha}) \cap \Gamma_n^c).$$

Taking the pth power, integrating over Γ_n and using (10), (11), we obtain up to a multiplicative constant:

$$\| \varphi_{\Gamma_n} \|_p^p \leq d^{-nhp\epsilon/2} + (n(h\epsilon \log d)/2)^p \cdot c\, d^{-n\eta}.$$

That completes the proof of the lemma. □

5.2 Proof of Theorem B(2)

Let $\psi \in \mathcal{U}$ be a μ-centered observable and $\chi = \psi \circ \omega$. Let $j \geq 1$ and $n \geq 0$ to be specified later. We set $\chi_n := \mathbb{E}(\chi | \mathcal{C}_n)$ and write:

$$\chi \cdot \chi \circ s^j = (\chi - \chi_n) \cdot \chi \circ s^j + \chi_n \cdot (\chi \circ s^j - \chi_n \circ s^j) + \chi_n \cdot \chi_n \circ s^j.$$

By using the s-invariance of ν and Jensen inequality $\| \chi_n \|_2 \leq \| \chi \|_2$, we deduce:

$$|R_j(\chi)| = \left| \int_{\Sigma} \chi \cdot \chi \circ s^j \, d\nu \right| \leq 2 \| \chi \|_2 \| \chi - \chi_n \|_2 + \left| \int_{\Sigma} \chi_n \cdot \chi_n \circ s^j \, d\nu \right|. \tag{13}$$

The variables χ_n and $\chi_n \circ s^j$ respectively depend on $(\xi_0, \ldots, \xi_n)$ and $(\xi_j, \ldots, \xi_{n+j})$, where $\xi_n : \Sigma \to \mathcal{A}$ is the projection $\xi_n(\tilde{\alpha}) = \alpha_n$. These are independent variables when $n = j - 1$, hence $\int_{\Sigma} \chi_n \cdot \chi_n \circ s^j \, d\nu = \int_{\Sigma} \chi_n \, d\nu \int_{\Sigma} \chi_n \circ s^j \, d\nu$ in that case. But this product is zero since χ is ν-centered. The conclusion then follows from (13) with $n = j - 1$ and Theorem B(1).

6 Proof of Theorem C

Let us recall the statement.

Theorem C *For every μ-centered observable $\psi \in \mathcal{U}$, we have:*

1. $\sigma := \lim_{n\to\infty} \frac{1}{\sqrt{n}} \| S_n(\psi) \|_2$ *exists, and* $\sigma^2 = \int_{\mathbb{P}^k} \psi^2\, d\mu + 2\sum_{j\geq 1} \int_{\mathbb{P}^k} \psi \cdot \psi \circ f^j\, d\mu$.
2. *If* $\sigma = 0$, *then* $\psi = u - u \circ f$ μ*-a.e. for some* $u \in L^2(\mu)$.
3. *If* $\sigma > 0$, *then* ψ *satisfies the* σ*-ASIP.*

The points 1 and 2 are consequences of classical Lemma 6.1 below, whose condition $\sum_{j\geq 1} j|R_j(\varphi)| < \infty$ is fulfilled by Theorem B(2). The point 3 follows from Proposition 2.2, Theorem B(1) and Philipp–Stout's theorem (see Sect. 2.4).

Lemma 6.1 *Let* (X, g, m) *be a dynamical system and* $\varphi \in L^2(m)$ *be a* m*-centered observable. We denote* $S_n(\varphi) = \sum_{j=0}^{n-1} \varphi \circ g^j$ *and* $R_j(\varphi) = \int_X \varphi \cdot \varphi \circ g^j\, dm$. *Let* $\sigma^2 := R_0(\varphi) + 2\sum_{j\geq 1} R_j(\varphi)$. *If* $\sum_{j\geq 1} j|R_j(\varphi)| < \infty$, *then* σ^2 *is finite and we have:*

1. $\| S_n(\varphi) \|_2^2 = n\sigma^2 + O(1)$. *In particular,* $\lim_{n\to\infty} \frac{1}{n} \| S_n(\varphi) \|_2^2 = \sigma^2$.
2. $\sigma^2 = 0$ *if and only if* $\varphi = u - u \circ g$ m*-a.e. for some* $u \in L^2(m)$.

Proof Let $S_n := S_n(\varphi)$ and $R_j := R_j(\varphi)$. Since m is g-invariant, we have $\| S_n \|_2^2 = nR_0 + 2\sum_{j=1}^{n-1} (n-j)\, R_j$. We deduce for every $n \geq 1$:

$$\| S_n \|_2^2 = n\left(R_0 + 2\sum_{j=1}^{\infty} R_j\right) + (-2)\left(\sum_{j=1}^{n-1} jR_j + \sum_{j=n}^{\infty} nR_j\right) = n\sigma^2 + A_n, \tag{14}$$

where $|A_n| \leq 2\sum_{j\geq 1} j|R_j|$. That proves the point 1. Let us show the point 2. Suppose $\sigma^2 = 0$. In view of (14), the function $u_p := \frac{1}{p}\sum_{n=1}^{p} S_n$ satisfies $\| u_p \|_2 \leq (2\sum_{j\geq 1} j|R_j|)^{1/2}$ for every $p \geq 1$. Let $u := \lim_{j\to\infty} u_{p_j}$ be a weak cluster point in $L^2(m)$ and observe that:

$$\forall j \geq 1, \quad u_{p_j} - u_{p_j} \circ g = \frac{1}{p_j} \sum_{n=0}^{p_j - 1} \left(\varphi - \varphi \circ g^n\right) = \varphi - \frac{1}{p_j} S_{p_j}.$$

We deduce $\varphi = u - u \circ g$ m-a.e. by taking limits in $L^2(m)$: $\lim_{j\to\infty} u_{p_j} \circ g = u \circ g$ since m is g-invariant, and $\lim_{j\to\infty} \frac{1}{p_j} S_{p_j} = \int_X \varphi\, dm = 0$ by Von Neumann theorem. The reverse implication of the point 2 comes from $\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \| S_n(\varphi) \|_2^2 = \lim_{n\to\infty} \frac{1}{n} \| u - u \circ g^n \|_2^2 = 0$. □

7 Proof of Theorem D

We recall that $J := \log \mathsf{Jac} f - \int_{\mathbb{P}^k} \log \mathsf{Jac} f \, d\mu$, this is an unbounded μ-centered observable in $\mathcal{U}$. We set $\sigma_J := \lim_n \frac{1}{\sqrt{n}} \| S_n(J) \|_2$, which is well defined by Theorem C. We denote by $\chi_1 \leq \cdots \leq \chi_k$ the Lyapunov exponent of μ, they are larger than or equal to $\log d^{1/2}$.

Theorem D *If the Lyapunov exponents of μ are minimal equal to $\log d^{1/2}$, then $\sigma_J = 0$ and μ is absolutely continuous with respect to Lebesgue measure.*

The first part $\sigma_J = 0$ will be proved in Sect. 7.2. The second part is a consequence of Theorem 7.1 below (that theorem will be proved in Sect. 7.3 by using $\sigma_J = 0$). In the sequel, the maps f^n and $d_x f^n$ are implicitely written in some fixed charts of $\mathbb{P}^k$.

Theorem 7.1 *Assume that the Lyapunov exponents are minimal. Then for μ almost every $x \in \mathbb{P}^k$, there exists $\rho(x) > 0$ and a subsequence $(n_j(x))_{j\geq 1}$ such that $f^{n_j} \circ \left(x + d^{-n_j/2} \cdot \mathrm{Id}_{\mathbb{C}^k}\right) : B(\rho(x)) \to \mathbb{P}^k$ is injective.*

Proof of the second part of Theorem D (abolute continuity) We use the notations of Theorem 7.1. Let $x \in \mathbb{P}^k$ be a μ-generic point and set $n_j := n_j(x)$. Since f^{n_j} is injective on the ball $B_j := B_x(\rho(x) d^{-n_j/2})$ and μ has constant jacobian d^k (see Sect. 2.1), we obtain $\mu(B_j) = \mu(f^{n_j}(B_j)) d^{-kn_j}$. Observe also that $\mathsf{Leb}(B_j) = \rho(x)^{2k} \left(d^{-n_j/2}\right)^{2k} = \rho(x)^{2k} d^{-kn_j}$ up to a multiplicative constant. We obtain therefore for μ-a.e. $x \in \mathbb{P}^k$:

$$\liminf_{r\to 0} \frac{\mu(B_x(r))}{\mathsf{Leb}(B_x(r))} \leq \liminf_{j\to\infty} \frac{\mu(B_j)}{\mathsf{Leb}(B_j)} = \liminf_{j\to\infty} \frac{\mu(f^{n_j}(B_j))}{\rho(x)^{2k}} \leq \frac{1}{\rho(x)^{2k}} < \infty.$$

That proves the absolute continuity of μ (see [31], theorem 2.12). ◻

7.1 Preliminaries

Observe that $J = \log \mathsf{Jac} f - \log d^k$ when the Lyapunov exponents are equal to $\log d^{1/2}$. Since the jacobian is a multiplicative function, we have in that case:

$$S_n(J) = \sum_{i=0}^{n-1} J \circ f^i(x) = \log \mathsf{Jac} f^n - \log d^{kn}. \tag{15}$$

The *singular values* $\delta_1 \leq \cdots \leq \delta_k$ of the linear map $A := d_x f^n$ are defined as the eigenvalues of $\sqrt{AA^*}$. In particular, there exist unitary matrices (U, V) such that $d_x f^n = U \, \mathsf{Diag}(\delta_1, \ldots, \delta_k) \, V$. We have therefore:

$$\delta_1 = \left\| (d_x f^n)^{-1} \right\|^{-1} \quad \text{and} \quad \prod_{i=1}^{k} {\delta_i}^2 = \mathsf{Jac} f^n(x) \geq \delta_1^{2k}. \tag{16}$$

For any $\rho, \tau > 0$ and $n \geq 1$, we define:

$$\mathcal{B}_n(\rho) := \left\{ x \in \mathbb{P}^k,\ f^n \circ (x + d_x f^n)^{-1} : B(\rho) \to \mathbb{P}^k \text{ is an injective map} \right\},$$
$$\mathcal{R}_n(\tau) := \left\{ x \in \mathbb{P}^k,\ \| (d_x f^n)^{-1} \|^{-1} \geq d^{n/2}/\tau \right\}.$$

The following estimates were proved by Berteloot and Dupont [2]. They hold for every system $(\mathbb{P}^k, f, \mu)$ whose Lyapunov exponents satisfy $\chi_k < 2\chi_1$.

Theorem 7.2 *There exists* $\alpha :]0, 1] \to \mathbb{R}_+^*$ *satisfying* $\lim_{\rho \to 0} \alpha(\rho) = 1$ *and for* $n \geq 1$*:*

1. $\mu(\mathcal{B}_n(\rho)) \geq \alpha(\rho)$,
2. $\mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(\tau)^c) \leq (\rho\, \tau)^{-2}$.

That result implies the following lemma.

Lemma 7.3 *Let* $\rho \in]0, 1]$. *There exists* $\mathcal{H} \subset \mathbb{P}^k$ *satisfying* $\mu(\mathcal{H}) = 1$ *and:*

$$\forall x \in \mathcal{H}, \quad \exists n(x) \geq 1, \quad \forall n \geq n(x), \quad x \notin \mathcal{B}_n(\rho) \text{ or } \mathsf{Jac}\, f^n(x) \geq d^{kn}/n^{2k}.$$

Proof We apply Proposition 7.2(2) with $\tau = n$ to get $\mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(n)^c) \leq (\rho\, n)^{-2}$. Since $\sum_{n \geq 1} \mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(n)^c) < \infty$, there exists by Borel–Cantelli lemma a subset $\mathcal{H}$ of full μ-measure satisfying:

$$\forall x \in \mathcal{H}, \quad \exists n(x) \geq 1, \quad \forall n \geq n(x), \quad x \notin \mathcal{B}_n(\rho) \text{ or } x \in \mathcal{R}_n(n).$$

But $x \in \mathcal{R}_n(n)$ implies by (16): $\mathsf{Jac}\, f^n(x) \geq \left(d^{n/2}/n\right)^{2k} = d^{kn}/n^{2k}$. □

7.2 Proof of the first part of Theorem D ($\sigma_J = 0$)

Suppose that the exponents are minimal and that $\sigma_J = \lim_n \frac{1}{\sqrt{n}} \| S_n(J) \|_2 > 0$. Then J satisfies the CLT: if $V := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-1} e^{-u^2/2}\, du$, we get $\mu\left(\mathcal{G}_n := \{\frac{S_n(J)}{\sqrt{n}} \leq -\sigma_J\}\right) \geq V/2$ for n larger than some N (see Sect. 2.4).

Let $\rho > 0$ be such that $\mu(\mathcal{B}_n(\rho)) > 1 - V/4$ for every $n \geq 1$. If we set $\mathcal{F}_n := \mathcal{B}_n(\rho) \cap \mathcal{G}_n$, then $\mathcal{F} := \limsup_{n \geq N} \mathcal{F}_n$ satisfies $\mu(\mathcal{F}) \geq V/4$. Let $x \in \mathcal{F} \cap \mathcal{H}$, where $\mathcal{H}$ is provided by Lemma 7.3. Let $(n_j(x))_{j \geq 1}$ be such that $x \in \mathcal{F}_{n_j}$ for every $j \geq 1$. The inclusion $\mathcal{F}_{n_j} \subset \mathcal{G}_{n_j}$ yields $S_{n_j}(J)(x) \leq -\sigma_J \sqrt{n_j}$ for every $j \geq 1$. Since $S_{n_j}(J) = \log \mathsf{Jac}\, f^{n_j} - \log d^{kn_j}$ (the exponents are indeed minimal, see (15)), we deduce:

$$\forall j \geq 1, \quad \mathsf{Jac}\, f^{n_j}(x) \leq d^{kn_j} e^{-\sigma_J \sqrt{n_j}}. \tag{17}$$

But $\mathsf{Jac}\, f^{n_j}(x) \geq d^{kn_j}/n_j^{2k}$ for every $n_j \geq n(x)$, following from $x \in \mathcal{B}_{n_j}(\rho) \cap \mathcal{H}$ and lemma 7.3. That contradicts (17) when j tends to infinity.

7.3 Proof of Theorem 7.1

We proved in Sect. 7.2 that $\sigma_J = 0$. Hence $J = u - u \circ f$ μ-a.e. for some $u \in L^2(\mu)$ by Theorem C. We obtain therefore:

$$u - u \circ f^n(x) = \sum_{i=0}^{n-1} J \circ f^i(x) = \log \mathsf{Jac}\, f^n(x) - \log d^{kn}. \tag{18}$$

Let $\epsilon > 0$ and $m \geq 1$ such that $\mathcal{M} := \{|u| \leq \log m\}$ satisfies $\mu(\mathcal{M}) \geq (1-\epsilon)^{1/2}$. Since μ is mixing, $\mathcal{M}_n := \mathcal{M} \cap f^{-n}\mathcal{M}$ satisfies $\mu(\mathcal{M}_n) \geq \mu(\mathcal{M})^2 - \epsilon \geq 1 - 2\epsilon$ for n larger than some N'. Let ρ be small and τ be large enough such that $\mu(\mathcal{B}_n(\rho) \cap \mathcal{R}_n(\tau)) \geq 1 - 2\epsilon$ for every $n \geq 1$. We define $\mathcal{T}_n := \mathcal{B}_n(\rho) \cap \mathcal{R}_n(\tau) \cap \mathcal{M}_n$ and $\mathcal{T} := \limsup_{n \geq N'} \mathcal{T}_n$. Observe that $\mu(\mathcal{T}) \geq 1 - 4\epsilon$. Let $x \in \mathcal{T}$ and $(n_j)_j$ (depending on x) such that $x \in \mathcal{T}_{n_j}$ for every $j \geq 1$. Since $x \in \mathcal{T}_{n_j} \subset \mathcal{B}_{n_j}(\rho)$, the map $f^{n_j} \circ (x + (d_x f^{n_j})^{-1}) : B(\rho) \to \mathbb{P}^k$ is injective.

Let $\Lambda_n = d^{-n/2} \cdot \mathrm{Id}_{\mathbb{C}^k}$. It is enough to prove that $d_x f^{n_j} = (U_j\, P_j\, V_j)\, \Lambda_{n_j}^{-1}$, where (U_j, V_j) are unitary matrices and P_j is a diagonal matrix with entries in $[a, b] \subset \mathbb{R}_+^*$ ((a, b) being independent of j). Indeed, this implies that $f^{n_j} \circ (x + \Lambda_{n_j})$ is injective on $B(\rho/b)$, completing the proof of Theorem 7.1. We shall omit the subscript j for simplification, and denote by $\delta_1 \leq \cdots \leq \delta_k$ the singular values of $d_x f^n$. Let (U, V) be unitary matrices such that $d_x f^n = U\, \mathsf{Diag}(\delta_1, \ldots, \delta_k)\, V$ (see Sect. 7.1). The fact that $x \in \mathcal{R}_n(\tau)$ yields:

$$\delta_1 = \left\| (d_x f^n)^{-1} \right\|^{-1} \geq d^{n/2}/\tau. \tag{19}$$

Now we give an upper bound for δ_k. Since $x \in \mathcal{T}_n \subset \mathcal{M}_n$, we have $(x, f^n(x)) \in \mathcal{M} = \{|u| \leq \log m\}$. This implies by (18):

$$d^{kn/2}/m \leq \prod_{i=1}^{k} \delta_i = \mathsf{Jac}\, f^n(x)^{1/2} \leq d^{kn/2} m.$$

We deduce from (19):

$$\delta_k \leq \frac{\delta_1 \ldots \delta_{k-1}}{\delta_1^{k-1}} \delta_k = \frac{\mathsf{Jac}\, f^n(x)^{1/2}}{\delta_1^{k-1}} \leq \frac{d^{kn/2} m}{(d^{n/2}/\tau)^{k-1}} = d^{n/2} \tau^{k-1} m.$$

Thus $\mathsf{Diag}(\delta_1, \ldots, \delta_k) = \Lambda_n^{-1} P$, where P is diagonal with entries in $[1/\tau, \tau^{k-1} m]$. We obtain finally $d_x f^n = U\, \Lambda_n^{-1}\, P\, V = (U\, P\, V)\, \Lambda_n^{-1}$, as desired.

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