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Nonequilibrium fluctuations for a tagged particle in mean-zero one-dimensional zero-range processes

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Abstract We prove a non-equilibrium functional central limit theorem for the position of a tagged particle in mean-zero one-dimensional zero-range process. The asymptotic behavior of the tagged particle is described by a stochastic differential equation governed by the solution of the hydrodynamic equation.

Keywords Hydrodynamic limit · Tagged particle · Scaling limit · Nonequilibrium

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1 Introduction

Informally, the zero-range particle system follows a collection of dependent random walks on the lattice where, from a vertex with k particles, one of the particles displaces

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by j with rate $(g(k)/k)p(j)$. The function on the non-negative integers $g : \mathbb{N}_0 \to \mathbb{R}_+$ is called the process "rate", and $p(\cdot)$ denotes the translation-invariant single particle transition probability. The name "zero-range" derives from the observation that, infinitesimally, the interaction is only with respect to those particles at the particular vertex. The case when $g(k)$ is proportional to k describes the situation of completely independent particles.

The problem of the asymptotics of a distinguished, or tagged, particle interacting with others fits in the general framework of studies of "random walk in random media" and has a long history (cf. [24, Chapters 8.I, 6.II]) given that it was mentioned in Spitzer's seminal paper [23]. The main analytical difficulty is that the tagged particle motion is not in general Markovian due to the interaction with other particles. However, the intuition is that in an adequate scale the tagged particle behaves as a random walk with certain "homogenized" parameters reflecting the system dynamics.

We prove in this article a nonequilibrium invariance principle, with respect to a diffusion process whose coefficients depend on the hydrodynamic density, for the diffusively rescaled position of the tagged particle in one-dimensional zero-range processes on a periodic lattice, or torus, when the transition probability p is finite-range and mean-zero. This invariance principle is the first result which captures the nonequilibrium fluctuations of a single, given particle in a general finite-range interacting particle system.

We remark, however, that in [8] a nonequilibrium central limit theorem was proved for a tagged particle in the nearest-neighbor symmetric one-dimensional simple exclusion model on $\mathbb{Z}$ by completely different methods which rely on the special structure of the nearest-neighbor one-dimensional dynamics. Also, we note, in [17], a "propagation of chaos" type nonequilibrium result was shown for finite-range symmetric $d \geq 1$ dimensional simple exclusion processes on $\mathbb{Z}^d$ which gives the fluctuations for a tagged particle selected at random, or in other words the average tagged particle position; however, this result, which makes key use of the "averaging," does not convey the fluctuations of any fixed, given particle and so is weaker than the one we state in this paper. In addition, with respect to certain interacting Brownian motions in one dimension, the nonequilibrium behavior was found in [7].

Also, previously, in this context with respect to zero-range tagged particles, we mention works on laws of large numbers, in equilibrium [19, 20] and non-equilibrium [18], and equilibrium central limit theorems when the jump probability p is mean-zero, $\sum jp(j) = 0$ [19, 20], and also when p is totally asymmetric and nearest-neighbor in $d = 1$ [22], and also some diffusive variance results when p has a drift $\sum jp(j) \neq 0$ in $d = 1$ and $d \geq 3$ [22].

We also note, mostly with respect to simple exclusion processes in equilibrium, some reviews, and more references on the tagged particle problem can be found in [4], [14, Section VIII.4], [15, Chapter III.4], [10, Sections IV.3, VIII.4 and XI.5], and [21, 16].

Denote by $\xi \in \mathbb{N}_0^{\mathbb{Z}}$, $\mathbb{N}_0 = \{0, 1, \ldots\}$, the states of the zero-range process, so that $\xi(x)$, $x \in \mathbb{Z}$, stands for the total number of particles at site x for the configuration ξ.

Fix an integer $N \geq 1$, scale space by N^{-1} and assume that the zero-range process rescaled diffusively (i.e. speeded up by N^2), $\{\xi_t^N : t \geq 0\}$, starts from a local equilibrium state with density profile $\rho_0 : \mathbb{R} \to \mathbb{R}_+$ (defined before Theorem 2.1). Denote by $\{\pi_t^{N,0} : t \geq 0\}$ its empirical density. This is the measure obtained by assigning mass N^{-1} to each particle. The celebrated "hydrodynamics" result (Theorem 2.1), proved in [3], and [10, Chapter V] by the method of [6], is that $\pi_t^{N,0}$ converges in probability to an absolutely continuous measure $\rho(t,u)du$, where $\rho(t,u)$ is the solution of a non-linear parabolic equation with initial condition ρ_0.

Tag a particle initially at the origin and denote by X_t^N its position at time t. It is relatively simple to show that the rescaled trajectory $\{X_t^N/N : 0 \leq t \leq T\}$ is tight for the uniform topology. In particular, to prove convergence, one needs only to characterize the limit points.

In contrast with other models, such as simple exclusion processes where infinitesimal interaction is longer range, it turns out that in mean-zero, zero-range processes X_t^N is a square integrable martingale with a bounded quadratic variation $\langle X^N\rangle_t$ given by the time integral of a local function of the process as seen from the tagged particle:

$$\langle X^N\rangle_t = \sigma^2 N^2 \int_0^t \frac{g(\eta_s^N(0))}{\eta_s^N(0)}\, ds,$$

where σ^2 is the variance of the transition probability $p(\cdot)$, $g(\cdot)$ is the jump rate mentioned before, and $\eta_s^N = \tau_{X_s^N}\xi_s^N$ is the state of the process as seen from the tagged particle. Here $\{\tau_x : x \in \mathbb{Z}\}$ stands for the group of translations. In particular, if the rescaled position of the tagged particle $x_t^N = X_t^N/N$ converges to some path x_t, this process x_t inherits the martingale property from X_t^N. If in addition x_t is continuous, to complete the characterization, one needs to examine the asymptotic behavior of its quadratic variation.

Denote by $\{\nu_\rho : \rho \geq 0\}$ the one-parameter family, indexed by the density, of invariant states for the process as seen from the tagged particle. Let π_t^N be the empirical density associated to this process: $\pi_t^N = \tau_{X_t^N}\pi_t^{N,0}$ and suppose that one can replace the local function $g(\eta_s^N(0))/\eta_s^N(0)$ by a function of the empirical density. If we assume "conservation of local equilibrium" for the process as seen from the tagged particle, i.e., that the empirical density around the tagged particle at a macroscopic time converges to a certain density (cf. [10, Chapters I, III, VIII]), this function should be $h(\lambda(s,0))$, where $h(\rho)$ is the expected value of $g(\eta(0))/\eta(0)$ under the invariant state ν_ρ and $\lambda(s,0)$ is the density of particles around the tagged particle, i.e., the density of particles around the origin for the system as seen from the tagged particle.

As we are assuming that X_t^N/N converges to x_t, since $\pi_t^N = \tau_{X_t^N}\pi_t^{N,0}$ and $\pi_t^{N,0}$ converges to $\rho(t,u)du$, we must have $\lambda(s,0) = \rho(s,x_s)$. Therefore, if the quadratic variation of X_t^N/N converges to the quadratic variation of x_t, $\langle x\rangle_t = \sigma^2\int_0^t h(\rho(s,x_s))ds$. In particular, by the characterization of continuous martingales, x_t satisfies the stochastic differential equation

$$dx_t = \sigma\sqrt{h(\rho(s,x_s))}\, dB_s,$$

where ρ is the solution of the hydrodynamic equation, h is defined above and B is a Brownian motion.

We see from this sketch that the main difficulty consists in proving the conservation of local equilibrium around the tagged particle, without assuming any type of "attractiveness", a certain monotonicity of the infinitesimal rates which allows the useful technical device of "basic coupling" (cf. [14, Chapter II]), which is relied upon in [11]. The absence of a space average creates a major obstacle in this step. In contrast with the proof of the hydrodynamic limit, we need to replace a local function instead of a space average of translations of a local function. We may, therefore, only use the bonds close to the origin of the Dirichlet form to perform the replacement and we may not exclude large densities of particles close to the origin. In particular, all estimates (equivalence of ensembles and local central limit theorems) need to be uniform over the density. This lack of translation invariance confines us to one-dimension.

The method presented here may apply to other one-dimensional mean-zero interacting particle systems. However, instead of replacing a local function by a function of the empirical density, one needs to replace the mean-zero "current" or "drift" function of the tagged particle (which vanishes for mean-zero zero-range processes) multiplied by N by a function of the empirical density. This is the main step in the proof of the hydrodynamic behavior of nongradient systems (cf. [10, Chapter VII]). For the tagged particle problem, however, there is an additional difficulty since no space average is available.

2 Notation and results

Since the seminal work [6], the hydrodynamic limit of particle systems is usually obtained in finite volume with periodic boundary conditions. Passing from finite volume to infinite volume is not trivial, and requires extra arguments, like a control of entropy flux [12] (which assumes sublinear growth of the rate g), [5] (for Ginzburg–Landau systems), or attractiveness [2] (which for zero-range processes means the rate g is increasing, a strong condition). Our approach might be carried out in infinite volume by modifying and extending these arguments for a restricted set of rates g, but this is not pursued here. Rather, in the following, we concentrate on the classic setting of finite volume where our results hold for a general class of rates g with linear growth [cf. Assumptions (LG), (M) below].

We consider a one-dimensional zero-range process with periodic boundary conditions. This process is a system of random walks on the discrete torus $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ where particles interact infinitesimally only when they are at the same site. Fix a rate function $g : \mathbb{N}_0 = \{0, 1, \ldots\} \to \mathbb{R}_+$ with $g(0) = 0$, $g(k) > 0$, $k \geq 1$, and a finite range probability measure $p(\cdot)$ on $\mathbb{Z}$ with $p(0) = 0$. The particle dynamics is described as follows. If there are k particles at a site x, one of these particles jumps to site y with an exponential rate $(g(k)/k)p(y - x)$. In the following, the scaling parameter N is always taken larger than the support of $p(\cdot)$, and the argument $y - x$ of $p(\cdot)$ is taken to be in $[-N/2, N/2]$.

For simplicity, we assume that $p(\cdot)$ is symmetric, but our results remain true, with straightforward modifications, for any irreducible, finite-range, mean-zero transition probability $p(\cdot)$ because mean-zero zero-range processes are gradient processes.

For the rate function g, we assume the next conditions, first given in [13], which allow some spectral gap estimates, among other properties:

$$\text{(LG)}\ \exists a_1 > 0 \text{ such that } |g(n+1) - g(n)| \le a_1 \quad \text{for } n \ge 0,$$
$$\text{(M)}\ \exists a_0 > 0, b \ge 1, \text{ such that } g(n+b) - g(n) > a_0 \quad \text{for } n \ge 0.$$

Under (LG) and (M), g is bounded between two linear slopes: There is a constant $0 < a < \infty$ such that $a^{-1}n \le g(n) \le an$ for all $n \ge 0$. Clearly, condition (FEM) introduced in [10, Section V.1] follows from (LG) and (M).

Denote by $\Omega_N = \mathbb{N}_0^{\mathbb{T}_N}$ the state space and by ξ the configurations of Ω_N so that $\xi(x)$, $x \in \mathbb{T}_N$, stands for the number of particles in site x for the configuration ξ. In this setting, the zero-range process is a continuous-time countable state Markov chain ξ_t generated by

$$(\mathcal{L}_N f)(\xi) \ = \ \sum_{x\in\mathbb{T}_N} \sum_{z\in\mathbb{Z}} p(z)\, g(\xi(x)) \left[f(\xi^{x,x+z}) - f(\xi) \right], \tag{2.1}$$

where we assume N to be larger than the support of p. In this formula the sums are performed modulo N and $\xi^{x,y}$ represents the configuration obtained from ξ by displacing a particle from x to y:

$$\xi^{x,y}(z) = \begin{cases} \xi(x) - 1 & \text{for } z = x \\ \xi(y) + 1 & \text{for } z = y \\ \xi(z) & \text{for } z \ne x, y. \end{cases}$$

Now consider an initial configuration ξ such that $\xi(0) \ge 1$. Tag one of the particles initially at the origin, and follow its trajectory X_t jointly with the evolution of the process ξ_t. Specially convenient for our purposes is to consider the process as seen by the tagged particle defined by $\eta_t(x) = \xi_t(x + X_t)$. This process is again Markovian, now on the set $\Omega_N^* = \{\eta \in \Omega_N; \eta(0) \ge 1\}$ and generated by the operator $L_N = L_N^{env} + L_N^{tp}$, where L_N^{env}, L_N^{tp} are defined by

$$\begin{aligned} (L_N^{env} f)(\eta) &= \sum_{x\ne 0} \sum_{z\in\mathbb{Z}} p(z)\, g(\eta(x))\, [f(\eta^{x,x+z}) - f(\eta)] \\ &\quad + \sum_{z\in\mathbb{Z}} p(z)\, g(\eta(0))\, \frac{\eta(0) - 1}{\eta(0)}\, [f(\eta^{0,z}) - f(\eta)], \\ (L_N^{tp} f)(\eta) &= \sum_{z\in\mathbb{Z}} p(z)\, \frac{g(\eta(0))}{\eta(0)}\, [f(\theta_z \eta) - f(\eta)]. \end{aligned}$$

In this formula, the translation θ_z is defined by

$$(\theta_z \eta)(x) = \begin{cases} \eta(x + z) & \text{for } x \ne 0, -z \\ \eta(z) + 1 & \text{for } x = 0 \\ \eta(0) - 1 & \text{for } x = -z. \end{cases}$$

The operator L_N^{tp} corresponds to jumps of the tagged particle, while L_N^{env} corresponds to jumps of the other particles, called the environment.

In order to recover the position of the tagged particle from the evolution of the process η_t, let N_t^z be the number of translations of length z up to time t: $N_t^z = N_{t-}^z + 1 \iff \eta_t = \theta_z \eta_{t-}$. In this case, $X_t = \sum_z z N_t^z$. From computing $L_N N^z$ and $L_N (N^z)^2 - 2N^z L_N N^z$, the processes

$$M_t^z \;=\; N_t^z - \int_0^t p(z) \frac{g(\eta_s(0))}{\eta_s(0)} ds$$

are martingales with quadratic variation $\langle M^z \rangle_t = \int_0^t p(z) g(\eta_s(0))/\eta_s(0) ds$. As jumps are not simultaneous, using the strong Markov property, $\{M_t^z\}$ are orthogonal, and as $\sum z p(z) = 0$ we see that X_t is itself a martingale, with respect to the natural sigma-fields of the process η_t, with quadratic variation

$$\langle X \rangle_t \;=\; \sigma^2 \int_0^t \frac{g(\eta_s(0))}{\eta_s(0)}\, ds,$$

where $\sigma^2 = \sum_z |z|^2 p(z)$.

We now discuss the invariant measures. For each $\varphi \geq 0$, consider the product probability measures $\bar{\mu}_\varphi = \bar{\mu}_\varphi^{N,g}$ in Ω_N defined by

$$\bar{\mu}_\varphi(\xi(x) = k) = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g(k)!},$$

where $g(k)! = g(1) \cdots g(k)$ for $k \geq 1$, $g(0)! = 1$ and $Z(\varphi)$ is the normalization constant. For all $\varphi \geq 0$, $Z(\varphi)$ and $\bar{\mu}_\varphi$ are well defined as g is assumed to have linear growth due to conditions (LG), (M). Let $\rho = \rho(\varphi) = \int \eta(0) d\bar{\mu}_\varphi$. Then, also, by the linear growth consequences of (LG), (M), $\varphi \mapsto \rho$ is a diffeomorphism from $[0, \infty)$ into itself. Define then $\mu_\rho = \bar{\mu}_{\varphi(\rho)}$, since ρ corresponds to the density of particles at each site. The measures $\{\mu_\rho : \rho \geq 0\}$ are invariant for the process ξ_t (cf. [1]).

Due to the inhomogeneity introduced at the origin by the tagged particle, μ_ρ is no longer invariant for the process η_t. However, the size biased, or Palm measures ν_ρ defined by $d\nu_\rho / d\mu_\rho \;=\; \eta(0)/\rho$ are invariant for the process as seen by the tagged particle, reversible when $p(\cdot)$ is symmetric by computing on test functions $\int L_N f d\nu_\rho = 0$ and

$$\begin{aligned} &-2 \int f\, (L_N h)\, d\nu_\rho \\ &\quad = \sum_{x \neq 0} \sum_z \int s(z)\, g(\eta(x))\, [f(\eta^{x,x+z}) - f(\eta)]\, [h(\eta^{x,x+z}) - h(\eta)]\, \nu_\rho(d\eta) \end{aligned}$$

$$+ \sum_z s(z) \int g(\eta(0)) \frac{\eta(0)-1}{\eta(0)} [f(\eta^{0,z}) - f(\eta)] [h(\eta^{0,z}) - h(\eta)] \, \nu_\rho(d\eta)$$

$$+ \sum_z s(z) \int \frac{g(\eta(0))}{\eta(0)} [f(\theta_z \eta) - f(\eta)] [h(\theta_z \eta) - h(\eta)] \, \nu_\rho(d\eta),$$

where $s(z) = (p(z) + p(-z))/2$ (cf. [19]). Here, we take $\nu_0 = \delta_{\mathfrak{d}_0}$, the Dirac measure concentrated on the configuration $\mathfrak{d}_0$ with exactly one particle at the origin, and note $\nu_\rho \Rightarrow \delta_{\mathfrak{d}_0}$ as $\rho \downarrow 0$.

From now on we define every process in a finite time interval $[0, T]$, where $T < \infty$ is fixed. Let $\mathbb{T}$ be the unit torus and let $\mathcal{M}_+(\mathbb{T})$ be the set of positive Radon measures in $\mathbb{T}$.

Consider the process $\xi_t^N =: \xi_{tN^2}$, generated by $N^2 \mathcal{L}_N$. Define the process $\pi_t^{N,0}$ in $\mathcal{D}([0, T], \mathcal{M}_+(\mathbb{T}))$, the path space of càdlàg trajectories on $\mathcal{M}_+(\mathbb{T})$ endowed with the Skorohod topology, as

$$\pi_t^{N,0}(du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \xi_t^N(x) \delta_{x/N}(du) \ ,$$

where δ_u is the Dirac distribution at point u.

For a continuous function $\rho_0 : \mathbb{T} \to \mathbb{R}_+$, define $\mu^N_{\rho_0(\cdot)}$ as the "local equilibrium" product measure with "density profile" ρ_0 in Ω_N given by $\mu^N_{\rho_0(\cdot)}(\eta(x) = k) = \mu_{\rho_0(x/N)}(\eta(x) = k)$. The next result is well known (cf. Chapter V [10]; see also [3, 6]).

Theorem 2.1 *For each $0 \le t \le T$, $\pi_t^{N,0}$ converges in probability to the deterministic measure $\rho(t,u)du$, where $\rho(t,u)$ is the solution of the hydrodynamic equation*

$$\begin{cases} \partial_t \rho = \sigma^2 \partial_x^2 \varphi(\rho) \\ \rho(0,u) = \rho_0(u), \end{cases} \tag{2.2}$$

and $\varphi(\rho) = \int g(\xi(0)) d\mu_\rho$.

Define now the product measure $\nu^N = \nu^N_{\rho_0(\cdot)}$ in Ω_N^* given by $\nu^N_{\rho_0(\cdot)}(\eta(x) = k) = \nu_{\rho_0(x/N)}(\eta(x) = k)$, and let $\eta_t^N =: \eta_{tN^2}$ be the process generated by $N^2 L_N$ and starting from the initial measure ν^N. Define the empirical density π_t^N in $\mathcal{D}([0, T], \mathcal{M}_+(\mathbb{T}))$ by

$$\pi_t^N(du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \eta_t^N(x) \delta_{x/N}(du).$$

Define also the continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\psi(\rho) = \int (g(\eta(0))/\eta(0)) \, d\nu_\rho = \begin{cases} \varphi(\rho)/\rho & \text{for } \rho > 0 \\ g(1) & \text{for } \rho = 0. \end{cases}$$

The next theorems are the main results of this article. We first identify the scaling limit of the tagged particle as a diffusion process:

Theorem 2.2 *Let $x_t^N = X_t^N/N$ be the rescaled position of the tagged particle for the process ξ_t^N. Then, $\{x_t^N : t \in [0, T]\}$ converges in distribution in the uniform topology to the diffusion $\{x_t : t \in [0, T]\}$ defined by the stochastic differential equation*

$$dx_t = \sigma \sqrt{\psi(\rho(t, x_t))}\, dB_t, \tag{2.3}$$

where $\rho(t, u)$ is the solution of the hydrodynamic equation (2.2), and B_t is a standard Brownian motion on $\mathbb{T}$.

Through this characterization we can describe the evolution of the empirical density as seen from the tagged particle:

Theorem 2.3 *$\{\pi_t^N : t \in [0, T]\}$ converges in distribution on $\mathcal{D}([0, T], \mathcal{M}_+(\mathbb{T}))$ to the measure-valued process $\{\rho(t, u+x_t)du : t \in [0, T]\}$, where $\rho(t, u)$ is the solution of the hydrodynamic equation (2.2) and x_t is given by Eq. 2.3.*

Recall that η_0^N is distributed according to $\nu_{\rho_0(\cdot)}^N$. Denote by $\mathbb{P}^N$ the probability measure in $\mathcal{D}([0, T], \Omega_N^*)$ induced by the process η_t^N, and by $\mathbb{E}^N$ the expectation with respect to this process. Denote also by $E_\mu[h]$ and $\langle h\rangle_\mu$ the expectation of a function $h : \Omega_N \to \mathbb{R}$ with respect to the measure μ; when $\mu = \nu_\rho$, let $E_\rho[h]$, $\langle h\rangle_\rho$ stand for $E_{\nu_\rho}[h]$, $\langle h\rangle_{\nu_\rho}$. Finally, since in the next sections we consider only the speeded-up process η_t^N we omit hereafter the superscript N.

The plan of the paper is now the following. After some tightness estimates in Sect. 3, certain limits are established in Theorem 4.1 in Sect. 4—with the aid of "global" and "local" hydrodynamics results in Sects. 5 and 6—which give the main Theorems 2.2 and 2.3.

3 Tightness

Denote by $\mathcal{C}(\mathbb{T})$ the space of real continuous functions on $\mathbb{T}$ and by $\mathcal{C}^2(\mathbb{T})$ the space of twice continuously differentiable functions on $\mathbb{T}$. For a function G in $\mathcal{C}(\mathbb{T})$, let $\pi_t^N(G)$ be the integral of G with respect to π_t^N:

$$\pi_t^N(G) \;=\; \int G(u)\pi_t^N(du) \;=\; \frac{1}{N}\sum_{x\in\mathbb{T}_N} G(x/N)\eta_t^N(x).$$

For $T > 0$, denote by $\mathcal{D}_T = \mathcal{D}([0, T], \mathcal{M}_+(\mathbb{T})\times\mathcal{M}_+(\mathbb{T})\times\mathbb{T}\times\mathbb{R}_+)$ the path space of càdlàg trajectories on $\mathcal{M}_+(\mathbb{T}) \times \mathcal{M}_+(\mathbb{T}) \times \mathbb{T} \times \mathbb{R}_+$ endowed with the Skorohod topology. For $N \geq 1$, let Q_N be the probability measure on $\mathcal{D}_T$ induced by the process $(\pi_t^{N,0}, \pi_t^N, x_t^N, \langle x^N\rangle_t)$, where $\langle x^N\rangle_t$ stands for the quadratic variation of the martingale x_t^N. We prove in this section that the sequence $\{Q_N : N \geq 1\}$ is tight, which follows from the tightness of each component of $(\pi_t^{N,0}, \pi_t^N, x_t^N, \langle x^N\rangle_t)$.

Let Q_N^0 be the probability measure in $\mathcal{D}([0,T],\mathcal{M}_+(\mathbb{T}))$ corresponding to the process $\pi_t^{N,0}$. As mentioned in Theorem 2.1, Q_N^0 converges to the Dirac-δ measure concentrated on the path $\rho(t,u)du$, where ρ is the solution of Eq. 2.2. Hence, the sequence $\{Q_N^0 : N \geq 1\}$ is tight.

On the other hand, as $\mathcal{M}_+(\mathbb{T})$ is a metrizable space under the dual topology of $\mathcal{C}(\mathbb{T})$, to show that $\{\pi_\cdot^N : N \geq 1\}$ is tight, it is enough to prove tightness of the projections $\{\pi_\cdot^N(G) : N \geq 1\}$ for a suitable set of functions G, dense in $\mathcal{C}(\mathbb{T})$. For G in $\mathcal{C}(\mathbb{T})$, let Q_N^G be the measure in $\mathcal{D}([0,T],\mathbb{R})$ corresponding to the process $\{\pi_t^N(G) : 0 \leq t \leq T\}$. Tightness of the sequence $\{Q_N^G : N \geq 1\}$ follows from Aldous's criterion in the next lemma.

Lemma 3.1 *The sequence $\{Q_N^G : N \geq 1\}$ is tight if*

(i) *For every $t \in [0,T]$ and every $\epsilon > 0$, there exists $M > 0$ such that*

$$\sup_N \mathbb{P}^N \Big[|\pi_t^N(G)| > M \Big] < \epsilon \ .$$

(ii) *Let $\mathcal{T}_T$ be the set of stopping times bounded by T. Then, for every $\epsilon > 0$,*

$$\lim_{\gamma\to 0} \limsup_{N\to\infty} \sup_{\tau\in\mathcal{T}_T} \sup_{\theta\leq\gamma} \mathbb{P}^N \Big[|\pi_{\tau+\theta}^N(G) - \pi_\tau^N(G)| > \epsilon \Big] = 0.$$

Lemma 3.2 *The sequence $\{Q_N^G : N \geq 1\}$, G in $\mathcal{C}^2(\mathbb{T})$, is tight.*

Proof A computation of $N^2 L_N \pi_t^N(G)$ and $N^2 L_N(\pi_t^N(G)) - 2N^2\pi_t^N(G) L_N(\pi_t^N(G))$ shows that for each G in $\mathcal{C}(\mathbb{T})$,

$$\begin{aligned} M_t^{N,G} &= \pi_t^N(G) - \pi_0^N(G) - \int_0^t \frac{1}{N} \sum_{x\in\mathbb{T}_N} (\Delta_N^+ G)(x/N)\, g(\eta_s(x))\, ds \\ &\quad - \int_0^t \frac{g(\eta_s(0))}{\eta_s(0)} \pi_s^N(\Delta_N^- G)\, ds + \int_0^t \frac{1}{N} \frac{g(\eta_s(0))}{\eta_s(0)} [(\Delta_N^+ G)(0) + (\Delta_N^- G)(0)]\, ds \end{aligned} \tag{3.1}$$

is a martingale of quadratic variation $\langle M^{N,G}\rangle_t$ given by

$$\langle M^{N,G}\rangle_t = \frac{1}{N^2}\int_0^t \sum_{\substack{x\in\mathbb{T}_N\setminus\{0\}\\ z\in\mathbb{Z}}} p(z)\, g(\eta_s(x))\, [\nabla_{N,z}G(x/N)]^2\, ds$$
$$+ \frac{1}{N^2}\int_0^t \sum_{z\in\mathbb{Z}} p(z)\, g(\eta_s(0))\, \frac{\eta_s(0)-1}{\eta_s(0)}\, [(\nabla_{N,z}G)(0)]^2\, ds + \int_0^t \sum_{z\in\mathbb{Z}} p(z)$$
$$\times \frac{g(\eta_s(0))}{\eta_s(0)} \left(\frac{1}{N}\sum_{x\in\mathbb{T}_N} (\nabla_{N,-z}G)(x/N)\eta_s(x) - \frac{1}{N}(\nabla_{N,-z}G)(0)\right)^2 ds.$$

In these formulas, $\nabla_{N,z}G$, $\Delta_N^{\pm}G$ correspond to the discrete first and second derivatives of G:

$$(\nabla_{N,z}G)(u) = N[G(u+z/N) - G(u)],$$
$$(\Delta_N^{+}G)(u) = N^2 \sum_{z\in\mathbb{Z}} p(z)\, [G(u+z/N) - G(u)],$$
$$(\Delta_N^{-}G)(u) = N^2 \sum_{z\in\mathbb{Z}} p(-z)\, [G(u+z/N) - G(u)].$$

Since the rate function g grows at most linearly and since the total number of particles is preserved by the dynamics,

$$\langle M^{N,G}\rangle_t = \int_0^t \sum_{z\in\mathbb{Z}} p(z)\, \frac{g(\eta_s(0))}{\eta_s(0)} \left(\frac{1}{N}\sum_{x\in\mathbb{T}_N} \nabla_{N,-z}G(x/N)\eta_s(x)\right)^2 ds + R_t^{N,G},$$

where $|R_t^{N,G}| \le C_0 N^{-2}\sum_{x\in\mathbb{T}_N}\eta_0(x)$ and C_0 is a finite constant which depends only on G, g, p and T. In particular, $\mathbb{E}^N\Big[|R_t^{N,G}|\Big] \le C_1 N^{-1}$. Hereafter, we use the convention that C_0, C_1 stand for finite constants whose value may change from line to line.

Note that in contrast with the martingale associated to empirical density $\pi_t^{N,0}$ (cf. [10, Section V.1]), due to the jumps of the tagged particle, the martingale $M^{N,G}$ *does not* vanish in $L^2(\mathbb{P}^N)$. In particular, we should not expect the convergence of the empirical density π_t^N to a deterministic trajectory, but to one which is randomly shifted in terms of the tagged particle which is the content of the first part of Theorem 2.3.

We are now in a position to prove the lemma. Condition (i) of Lemma 3.1 is a direct consequence of the conservation of the total number of particles. In order to prove condition (ii), recall the decomposition (Eq. 3.1) of $\pi_t^N(G)$ as an integral term plus a martingale. The martingale term can be estimated by Chebychev's inequality and the explicit form of its quadratic variation:

$$\begin{aligned}
\mathbb{P}^N\left[\,|M^{N,G}_{\tau+\theta}-M^{N,G}_{\tau}|>\epsilon\,\right] &\le \frac{1}{\epsilon^2}\mathbb{E}^N\left[(M^{N,G}_{\tau+\theta})^2-(M^{N,G}_{\tau})^2\right]\\
&\le \frac{C_1}{\epsilon^2}\,\|G'\|^2_\infty\,\mathbb{E}^N\left[\int_\tau^{\tau+\theta}\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N}\eta_s(x)\right)^2 ds\right]+\frac{C_1}{\epsilon^2 N}\\
&\le \frac{C_1\|G'\|^2_\infty\,\theta}{\epsilon^2}E_{\nu^N_{\rho_0(\cdot)}}\left[\left(\frac{1}{N}\sum_{x\in\mathbb{T}_N}\eta(x)\right)^2\right]+\frac{C_1}{\epsilon^2 N}
\end{aligned}$$

which converges to 0 as $N\uparrow\infty$ and $\gamma\downarrow 0$. The integral term can be estimated in the same way, using again the conservation of the total number of particles. This proves the lemma. □

It remains to consider the scaled position of the tagged particle x^N_t and its quadratic variation. We recall that x^N_t is a martingale with quadratic variation

$$\langle x^N\rangle_t=\sigma^2\int_0^t\frac{g(\eta^N_s(0))}{\eta^N_s(0)}\,ds\;. \tag{3.2}$$

where we recall $\sigma^2=\sum_z z^2p(z)$.

Lemma 3.3 *The process* $\{(x^N,\langle x^N\rangle):N\ge 1\}$ *is tight for the uniform topology.*

Proof We need to show that

$$\lim_{\epsilon\to 0}\limsup_{N\to\infty}Q_N\left[\sup_{|t-s|\le\epsilon}|x^N_t-x^N_s|>\delta\right]\;=\;0 \tag{3.3}$$

for all $\delta>0$ and a similar statement for the quadratic variation $\langle x^N\rangle_t$. Recall that $\sup_{k\ge1}g(k)/k\le a<\infty$ and consider a mean-zero random walk Z^N_t on the discrete torus $\mathbb{T}_N$ with jump rate a and transition probability $p(\cdot)$. We may couple Z^N_t and X^N_t in such a way that the skeleton chains are equal, i.e., that the sequence of sites visited by both processes are the same, and the holding times of Z^N are always less than or equal to the holding times of X^N. In particular,

$$\sup_{|t-s|\le\epsilon}|x^N_t-x^N_s|\;\le\;\sup_{|t-s|\le\epsilon}|z^N_t-z^N_s|$$

if $z^N_t=Z^N_t/N$. Therefore, Eq. 3.3 follows from the tightness in the uniform topology of a rescaled mean-zero random walk.

Tightness of the quadratic variation $\langle x^N\rangle_t$ in the uniform topology is an elementary consequence of its explicit expression (Eq. 3.2) and the boundedness of $g(k)/k$. □

4 Limit points and proof of Theorems 2.2, and 2.3

The following, which characterizes certain limit points, is the main result of this section, which yields Theorems 2.2 and 2.3.

Theorem 4.1 *The sequence Q_N converges in the Skorohod topology to the law Q concentrated on trajectories $\{(\pi_t^0, \pi_t, x_t, A_t) : 0 \le t \le T\}$ such that $\pi_t^0(du) = \rho(t,u)du$, where ρ is the unique weak solution of Eq. 2.2; x_t is the solution of the stochastic differential equation (2.3); $\pi_t(du)=\rho(t, x_t + u)du$ and $A_t=\sigma^2 \int_0^t \psi(\rho(s, x_s))\, ds$.*

The proof of Theorem 4.1 is divided in a sequence of lemmas. Denote by $\{\tau_u : u \in \mathbb{T}\}$ the group of translations in $\mathbb{T}$ acting on points, functions and measures.

Lemma 4.2 *All limit points Q of the sequence $\{Q_N : N \ge 1\}$ are concentrated on trajectories $\{(\pi_t^0, \pi_t, x_t, A_t) : 0 \le t \le T\}$ in which x_t is a continuous square integrable martingale.*

Proof Assume, without loss of generality, that Q_N converges to Q. Since, by Lemma 3.3, $\{x^N : N \ge 1\}$ is tight for the uniform topology, Q is concentrated on continuous paths x_t. In particular, x_t^N converges in law to x_t for all $0 \le t \le T$.

The martingale property is inherited by x_t because x_t^N converges in law to x_t and

$$\mathbb{E}^N\left[(x_t^N)^2\right] = \mathbb{E}^N\left[\sigma^2 \int_0^t \frac{g(\eta_s(0))}{\eta_s(0)}\, ds\right] \le a\sigma^2 t$$

uniformly in N. Therefore, x_t is a square integrable martingale relative to the filtration it generates (cf. Proposition IX.1.12 [9]). □

Lemma 4.3 *All limit points Q of the sequence $\{Q_N : N \ge 1\}$ are concentrated on trajectories $\{(\pi_t^0, \pi_t, x_t, A_t) : 0 \le t \le T\}$ in which $\pi_t^0(du) = \rho(t,u)du$, where ρ is the unique weak solution of Eq. 2.2, and $\pi_t(du) = \tau_{x_t}\pi_t^0(du) = \rho(t, x_t + u)du$.*

Proof Assume, without loss of generality, that Q_N converges to Q. The first statement follows from Theorem 2.1. On the other hand, by Lemma 4.2 and since ρ is continuous, Q is concentrated on continuous trajectories $\{(\pi_t^0, x_t) : 0 \le t \le T\}$. Hence, as $\pi_t^{N,0}$ in fact converges in probability to π_t^0, as π_t^0 is deterministic, all finite dimensional distributions (f.d.d.) of $(\pi_t^{0,N}, x_t^N)$ (and therefore of $\tau_{x_t^N}\pi_t^{N,0}$) converge to the f.d.d. of (π_t^0, x_t) $(\tau_{x_t}\pi_t^0 = \rho(t, u + x_t)du)$. Since $\pi_t^N = \tau_{x_t^N}\pi_t^{N,0}$ and since the f.d.d. characterize a measure on $\mathcal{D}([0,T], \mathcal{M}_+(\mathbb{T}) \times \mathcal{M}_+(\mathbb{T}) \times \mathbb{T})$, the lemma is proved. □

For $\varepsilon > 0$, denote $\iota_\varepsilon = \varepsilon^{-1}\mathbf{1}\{(0,\varepsilon]\}(u)$ and $\alpha_\varepsilon = (2\varepsilon)^{-1}\mathbf{1}\{(-\varepsilon,\varepsilon)\}(u)$. For $l \ge 1$ and $x \in \mathbb{Z}$, denote by $\eta_s^l(x)$ the mean number of particles in a cube of length $2l+1$ centered at $x \in \mathbb{T}_N$ at time $s \ge 0$:

$$\eta_s^l(x) = \frac{1}{2l+1} \sum_{|y-x| \le l} \eta_s(y).$$

When $s = 0$, we drop the suffix "s" for simplicity.

A function $h : \Omega_N \to \mathbb{R}$ is said to be local if it depends only on a finite number of sites. For a local, bounded function $h : \Omega_N \to \mathbb{R}$, denote by $H(\rho)$ and $\bar{h}(\rho)$ its expectations with respect to ν_ρ and μ_ρ respectively. Thus, $H, \bar{h} : \mathbb{R}_+ \to \mathbb{R}$ are the functions defined by

$$H(\rho) \;=\; E_{\nu_\rho}[h(\eta)]\,, \quad \text{and} \quad \bar{h}(\rho) \;=\; E_{\mu_\rho}[h(\xi)]. \tag{4.1}$$

Also, define for $l \geq 1$ the local function $H_l : \Omega_N \to \mathbb{R}$ given by

$$H_l(\eta) \;=\; H(\eta^l(0)).$$

Then, $\bar{H}_l : \mathbb{R}_+ \to \mathbb{R}$ is the function $\bar{H}_l(\rho) = E_{\mu_\rho}[H_l]$.

A local function $h : \Omega_N \to \mathbb{R}$, supported on coordinates $A \subset \mathbb{Z}$, is said to be Lipschitz if there exists a finite constant C_0 such that

$$\left|h(\xi) - h(\xi')\right| \;\leq\; C_0 \sum_{x\in A} \left|\xi(x) - \xi'(x)\right| \tag{4.2}$$

for all configurations ξ, ξ' of Ω_N. We assume here that N is larger than the support of h.

Consider in particular the local function $h_0(\eta(0)) = g(\eta(0))/\eta(0)$. It follows from assumptions (LG), (M) that $h_0(\cdot)$ is a Lipschitz function, bounded above by a finite constant and below by a strictly positive constant.

We now characterize the quadratic variation of x_t.

Lemma 4.4 *All limit points Q of the sequence $\{Q_N : N \geq 1\}$ are concentrated on trajectories $\{(\pi_t^0, \pi_t, x_t, A_t) : 0 \leq t \leq T\}$ such that*

$$A_t \;=\; \sigma^2 \int_0^t \psi(\rho(s, x_s))\, ds$$

for all $0 \leq t \leq T$. Moreover, A_t is the quadratic variation of the martingale x_t.

Proof Assume, without loss of generality, that Q_N converges to Q. Since $\langle x^N \rangle_t$ is tight for the uniform topology by Lemma 3.3, $\langle x^N \rangle_t$ converges to a limit A_t for all $0 \leq t \leq T$. By Eq. 3.2 and Proposition 6.1, with respect to $h(\eta(0)) = g(\eta(0))/\eta(0)$ and $H(\rho) = \psi(\rho)$, and since for each $0 \leq t \leq T$ the map $\pi_\cdot \to \int_0^t ds \int \iota_\epsilon(x)\bar{\psi}_l(\pi_s(\tau_x\alpha_\varepsilon))\, dx$ is continuous for the Skorohod topology,

$$\lim_{l\to\infty} \lim_{\epsilon\to 0} \lim_{\varepsilon\to 0} Q\left[\left|A_t - \sigma^2 \int_0^t ds \int \iota_\epsilon(x)\bar{\psi}_l(\pi_s(\tau_x\alpha_\varepsilon))\, dx\right| > \delta\right] \;=\; 0$$

for all $0 \leq t \leq T$ and $\delta > 0$. By Lemma 4.3, $\pi_t(du) = \rho(t, x_t + u)du$. Also, $\rho(s, \cdot)$ is continuous for $0 \leq s \leq T$, and $\bar{\psi}_l(a) \to \psi(a)$ as $l \uparrow \infty$ by bounded convergence.

Then, as $\varepsilon \downarrow 0$, $\epsilon \downarrow 0$, and $l \uparrow \infty$, we have a.s.

$$\int \iota_\epsilon(x) \int_0^t \bar{\psi}_l(\pi_s(\tau_x \alpha_\varepsilon))\, ds dx \;\to\; \int_0^t \psi(\rho(s, x_s))\, ds.$$

Since A_t is continuous, this identifies $A_t = \sigma^2 \int_0^t \psi(\rho(s, x_s))ds$ for all $0 \le t \le T$.

It remains to show that A_t corresponds to the quadratic variation of the square integrable martingale x_t. By [9, Corollary VI.6.29], $\{(x_t^N, \langle x^N \rangle_t) : 0 \le t \le T\}$ converges in law to $\{(x_t, \langle x \rangle_t) : 0 \le t \le T\}$. Since by the first part of the lemma, $\{(x_t^N, \langle x^N \rangle_t) : 0 \le t \le T\}$ converges to $\{(x_t, A_t) : 0 \le t \le T\}$, $\langle x \rangle_t = A_t$. This concludes the proof of the lemma. □

Alternatively, recall that the quadratic variation $\langle x \rangle_t$ of a martingale x_t is equal to $x_t^2 - x_0^2 - 2\int_0^t x_s\, dx_s$ and that $\langle x \rangle_t$ can be approximated in L^2 by the sequence of Riemannian sums $\sum_j (x_{t_{j+1}} - x_{t_j})^2$, as the mesh of a partition $\{t_j : 1 \le j \le M\}$ of the interval $[0, t]$ vanishes. In particular, one can prove directly in our context the identity between A_t and the quadratic variation $\langle x \rangle_t$.

Corollary 4.5 *The rescaled position of the tagged particle* $\{x_t^N : 0 \le t \le T\}$ *converges in law to the solution of the stochastic differential equation*

$$dx_t \;=\; \sigma \sqrt{\psi(\rho(t, x_t))}\, dB_t,$$

where B_t *is a Brownian motion and* ρ *is the solution of the differential equation* (2.2).

Proof From Lemma 4.4, and Lévy's characterization of continuous martingales (cf. Theorem II.4.4 [9]), all limit points x_t are such that $x_{A_t^{-1}}$ is a standard Brownian motion. This shows that x_t^N converges in law to the diffusion given in the display. □

Proof of Theorem 4.1 By Sect. 3, the sequence Q^N is tight. On the other hand, by Lemma 4.3 and Corollary 4.5, the law of the first and the third components of the vector $(\pi_t^0, \pi_t, x_t, A_t)$ are uniquely determined. Since, by Lemmas 4.3, 4.4, the distribution of the second and fourth components are characterized by the distribution of x_t, and $\rho(t, x)$, the theorem is proved. □

Proof of Theorems 2.2 *and* 2.3 As the limit x_t is concentrated on continuous paths, Theorem 4.1 straightforwardly implies Theorems 2.2 and 2.3. □

5 Global replacement lemma

In this section, we replace the full empirical average of a local, bounded and Lipschitz function in terms of the density field. The proof involves only a few modifications of the standard hydrodynamics proof of [10, Lemma V.1.10, Lemma V.5.5].

Proposition 5.1 (Global replacement) *Let* $r : \Omega_N \to \mathbb{R}$ *be a local, bounded and Lipschitz function. Then, for every* $\delta > 0$,

$$\limsup_{\varepsilon\to\infty}\limsup_{N\to\infty}\mathbb{P}^N\left[\int_0^T \frac{1}{N}\sum_{x\in\mathbb{T}_N}\tau_x\mathcal{V}_{\varepsilon N}(\eta_s)ds \geq \delta\right]=0,$$

where

$$\mathcal{V}_l(\eta)=\left|\frac{1}{2l+1}\sum_{|y|\leq l}\tau_y r(\eta)-\bar{r}(\eta^l(0))\right|,\quad \text{and}\quad \bar{r}(a)=E_{\mu_a}[r].$$

For two measures μ, ν defined on Ω_N (or Ω_N^*), denote by $\mathcal{H}(\mu|\nu)$ the entropy of μ with respect to ν:

$$\mathcal{H}(\mu|\nu) \;=\; \sup_f\left\{\int f d\mu \;-\; \log\int e^f d\nu\right\},$$

where the supremum is carried over all bounded continuous functions f.

A computation, with respect to the two product measures $\nu^N_{\rho_0(\cdot)}$ and ν_ρ, shows that the initial entropy $\mathcal{H}(\nu^N_{\rho_0(\cdot)}|\nu_\rho)$ is bounded by $C_0 N$ for some finite constant C_0 depending only on $\rho_0(\cdot)$ and g. Let $f_t^N(\eta)$ be the density of η_t under $\mathbb{P}^N$ with respect to a reference measure ν_ρ for $\rho>0$, and let $\hat{f}_t^N(\eta)=t^{-1}\int_0^t f_s^N(\eta)ds$. By standard arguments (cf. Section V.2 [10]),

$$\begin{aligned}\mathcal{H}_N(\hat{f}_t^N) &:= \mathcal{H}(\hat{f}_t^N d\nu_\rho|\nu_\rho)\leq C_0 N \quad\text{and}\quad \mathcal{D}_N(\hat{f}_t^N)\\ &:= \left\langle\sqrt{\hat{f}_t^N}\left(-L_N\sqrt{\hat{f}_t^N}\right)\right\rangle_\rho \leq \frac{C_0}{N}.\end{aligned}$$

Consequently, by Chebyshev inequality, to prove Proposition 5.1 it is enough to show, for any finite constant C, that

$$\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\sup_{\substack{\mathcal{H}_N(f)\leq CN\\ \mathcal{D}_N(f)\leq C/N}}\int\frac{1}{N}\sum_{x\in\mathbb{T}_N}\tau_x\mathcal{V}_{\varepsilon N}(\eta)f(\eta)d\nu_\rho=0 \tag{5.1}$$

where the supremum is with respect to ν_ρ-densities f.

The proof of this limit follows the strategy of [10, Lemma V.1.10] where the reference measure μ_ρ is homogeneous. First, we observe that we may remove from the sum in Eq. 5.1 the integers x close to the origin, say $|x|\leq 2\varepsilon N$, because $\mathcal{V}_{\varepsilon N}$ is bounded. Proposition 5.1 now follows from the two standard lemmas below as given in [10, Section V.3] for the proof of [10, Lemma V.1.10].

Lemma 5.2 (Global 1-block estimate)

$$\limsup_{k\to\infty}\limsup_{N\to\infty}\sup_{\substack{\mathcal{H}_N(f)\leq CN\\ \mathcal{D}_N(f)\leq C/N}}\int\frac{1}{N}\sum_{|x|>2\varepsilon N}\tau_x\mathcal{V}_k(\eta)f(\eta)d\nu_\rho=0.$$

Lemma 5.3 (Global 2-block estimate)

$$\limsup_{k\to\infty}\limsup_{\varepsilon\to 0}\limsup_{N\to\infty}\sup_{\substack{\mathcal{H}_N(f)\le CN\\ \mathcal{D}_N(f)\le C/N}} \frac{1}{2N\varepsilon+1}\sum_{|y|\le N\varepsilon}\int\frac{1}{N}\sum_{|x|>2\varepsilon N}|\eta^k(x+y)-\eta^k(x)|f(\eta)d\nu_\rho=0.$$

We now indicate the proofs of Lemmas 5.2 and 5.3 in relation to [10, Sections V.4, V.5].

Proofs of Lemmas 5.2 and 5.3 To be brief, we discuss only the proof of Lemma 5.2 through some modifications of the argument in [10, Section V.4], as the proof of Lemma 5.3, using the modifications for Lemma 5.2 given below, is on similar lines to that in [10, Section V.5].

In the first step of the 1-block estimate we cut-off high densities. We claim that

$$\limsup_{A\to\infty}\limsup_{k\to\infty}\limsup_{N\to\infty}\sup_{\mathcal{H}_N(f)\le CN}\int\frac{1}{N}\sum_{|x|>2\varepsilon N}\tau_x\mathcal{V}_k(\eta)\mathbf{1}\{\eta^k(x)>A\}f(\eta)d\nu_\rho=0.$$

Since $\mathcal{V}_k$ is bounded, and $\mathbf{1}\{\eta^k(x)>A\}\le A^{-1}\eta^k(x)$, we may bound the integral, using the sum in the definition of $\mathcal{V}_k$ to perform a summation by parts, by

$$C_0\int\frac{1}{AN}\sum_{|x|>2\varepsilon N}\eta^k(x)f(\eta)d\nu_\rho \;\le\; C_0\int\frac{1}{AN}\sum_{x\ne 0}\eta(x)f(\eta)d\nu_\rho,$$

for some finite constant C_0. The proof now is almost the same as for [10, Lemma V.4.1], keeping in mind that in applying the entropy inequality with respect to ν_ρ, the marginals of μ_ρ and ν_ρ coincide on sites $x\ne 0$.

Define now $\mathcal{V}_{k,A}(\eta)=\mathcal{V}_k(\eta)1\{\eta^k(0)\le A\}$. By the previous argument, it is enough to show that for every $A>0$,

$$\limsup_{k\to\infty}\limsup_{N\to\infty}\sup_{\mathcal{D}_N(f)\le C/N}\int\frac{1}{N}\sum_{|x|\ge 2\varepsilon N}\tau_x\mathcal{V}_{k,A}(\eta)f(\eta)d\nu_\rho=0. \tag{5.2}$$

The proof, except for notation, is the same as to show [10, Equation (V.4.1)]. In words, since the origin does not appear, both the Dirichlet form $\mathcal{D}_N$ and the measure ν_ρ coincide with the Dirichlet form of the space homogeneous zero-range process and the stationary state μ_ρ. In particular, all estimates needed involve only functionals of the space-homogeneous process already considered in [10, Section V.4]. □

6 Local replacement lemma

In this section, we replace a bounded, Lipschitz function supported at the origin by a function of the empirical density.

Proposition 6.1 (Local replacement) *For any bounded, Lipschitz function $h : \mathbb{N}_0 \to \mathbb{R}$, and any $t > 0$,*

$$\limsup_{l\to\infty} \limsup_{\epsilon\to 0} \limsup_{\varepsilon\to 0} \limsup_{N\to\infty} \mathbb{E}^N \left[\left| \int_0^t h(\eta_s(0)) - \frac{1}{\epsilon N} \sum_{x=1}^{\epsilon N} \bar{H}_l(\eta_s^{\varepsilon N}(x))\, ds \right| \right] = 0,$$

where $H(\rho) = E_{\nu_\rho}[h]$, $H_l(\eta) = H(\eta^l(0))$, and $\bar{H}_l(\rho) = E_{\mu_\rho}[H_l]$.

In the proof of this lemma, there are two difficulties. The first and the most important one is the absence of a spatial average, a crucial point in the standard one and two blocks estimates since it allows a cut-off of large densities and a reduction to translation-invariant densities in the estimation of the largest eigenvalue of a local perturbation of the generator of the process. Without the density cut-off, the equivalence of ensembles, and therefore the local central limit theorem, has to be proved uniformly over all densities. Moreover, this absence of space average confines us to one-dimension because in the proof of the replacement lemma (cf. Lemmata V.3.1 and V.3.2 in [10]), the supremum is restricted to density functions f with Dirichlet form bounded by N^{d-2}. Thus in dimension 1 and only in dimension 1, the Dirichlet form vanishes in the limit. In all other dimensions, one needs to use the translations to show that the we may restrict the supremum to density functions with Dirichlet form vanishing in the limit.

A second obstacle is the lack of translation invariance of the stationary state, turning the origin into a special site. Functions $h(\eta(0))$ and $h(\eta(x))$, for instance, have different distributions. In particular, in contrast with the original zero-range process, the integral $\int \{g(\eta(0)) - g(\eta(x))\} f\, d\nu_\rho$ cannot be estimated by the Dirichlet form of f.

The proof of Proposition 6.1 is divided in several steps. We start with a spectral gap for the evolution of the environment restricted to a finite cube. For $l \geq 1$, denote by Λ_l a cube of length $2l + 1$ around the origin: $\Lambda_l = \{-l, \ldots, l\}$ and by $L^{env}_{\Lambda_l}$ the restriction of the environment part of the generator to the cube Λ_l:

$$\begin{aligned}(L^{env}_{\Lambda_l} f)(\eta) = {} & \sum_{\substack{x\in\Lambda_l\\ x\neq 0}} \sum_{y\in\Lambda_l} p(y-x)\, g(\eta(x))\, [f(\eta^{x,y}) - f(\eta)] \\ & + \sum_{z\in\Lambda_l} p(z)\, g(\eta(0))\, \frac{\eta(0)-1}{\eta(0)}\, [f(\eta^{0,z}) - f(\eta)].\end{aligned}$$

We assume above, without loss of generality, that l is larger than the range of $p(\cdot)$.

Let $\nu_\rho^{\Lambda_l}$ be the measure ν_ρ restricted to the set Λ_l. For $j \geq 1$, denote by $\Sigma_{\Lambda_l, j}$ the set of all configurations in Λ_l with at least one particle at the origin and j particles in Λ_l, and by $\nu_{\Lambda_l, j}$ the measure $\nu_\rho^{\Lambda_l}$ conditioned to $\Sigma_{\Lambda_l, j}$:

$$\Sigma_{\Lambda_l, j} = \left\{ \eta \in \mathbb{N}_0^{\Lambda_l} : \eta(0) \geq 1, \sum_{x\in\Lambda_l} \eta(x) = j \right\}, \quad \nu_{\Lambda_l, j}(\cdot) = \nu_\rho^{\Lambda_l}\left(\cdot \middle| \Sigma_{\Lambda_l, j}\right). \tag{6.1}$$

Note that $\nu_{\Lambda_l, j}$ does not depend on the parameter ρ.

Lemma 6.2 *There exists a finite constant C_0 such that*

$$\langle f; f\rangle_{\nu_{\Lambda_l,j}} \;\le\; C_0\, l^2\, \langle f\,(-L^{env}_{\Lambda_l} f)\rangle_{\nu_{\Lambda_l,j}}$$

for all $j \ge 1$, all $l \ge 1$ and all functions f in $L^2(\nu_{\Lambda_l,j})$. In this formula, $\langle f; f\rangle_{\nu_{\Lambda_l,j}}$ stands for the variance of f with respect to $\nu_{\Lambda_l,j}$.

Proof This result follows from the spectral gap of the zero-range process proved in [13]. Note that $0 < a^{-1} \le g(k)/k \le a$ is bounded, and the computation $E_{\nu_\rho}[g(\eta(0)) f(\eta)] = \varphi(\rho) E_{\mu_\rho}[f'(\eta)]$ with $f'(\eta) = f(\eta + \mathfrak{d}_0)$, where $\mathfrak{d}_0$ is the configuration with exactly one particle at the origin and summation of configurations is performed componentwise. Let also $\mathcal{L}_{\Lambda_l}$ be the generator of the zero-range process (Eq. 2.1) restricted to the set Λ_l, and $\mu_{\Lambda_l,k}$ be the measure μ_ρ conditioned on the hyperplane $\Sigma^0_{\Lambda_l,k} = \{\xi \in \mathbb{N}_0^{\Lambda_l} : \sum_{x\in\Lambda_l} \xi(x) = k\}$. Then, after a calculation,

$$\langle f; f\rangle_{\nu_{\Lambda_l,j}} \;=\; \inf_c \langle (f-c)^2\rangle_{\nu_{\Lambda_l,j}} \le\; a^2 \inf_c \langle (f'-c)^2\rangle_{\mu_{\Lambda_l,j-1}} \;=\; a^2 \langle f'; f'\rangle_{\mu_{\Lambda_l,j-1}}$$

$$\text{and} \quad \langle f'(-\mathcal{L}_{\Lambda_l} f')\rangle_{\mu_{\Lambda_l,j-1}} \;\le\; a^2 \langle f(-L^{env}_{\Lambda_l} f)\rangle_{\nu_{\Lambda_l,j}}.$$

The Poincaré inequality $\langle f'; f'\rangle_{\mu_{\Lambda_l,j-1}} \le C_0' l^2 \langle f'(-\mathcal{L}_{\Lambda_l} f')\rangle_{\mu_{\Lambda_l,j-1}}$, proved in [13], now applies. □

6.1 Local one-block estimate

For $l \ge 1$, define the function $V_l(\eta)$ by

$$V_l(\eta) = h(\eta(0)) - H(\eta^l(0))$$

where we recall h is a bounded, Lipschitz function, and $H(a) = E_{\nu_a}[h(\eta(0))]$. In this subsection we give the second step for the proof of Proposition 6.1:

Lemma 6.3 (One-block estimate) *For every $0 \le t \le T$,*

$$\limsup_{l\to\infty} \limsup_{N\to\infty} \mathbb{E}^N \left[\left| \int_0^t V_l(\eta_s)\, ds \right| \right] = 0.$$

Proof Since the initial entropy $\mathcal{H}(\nu^N_{\rho_0(\cdot)} | \nu_\rho)$ is bounded by $C_0 N$, by the entropy inequality,

$$\mathbb{E}^N \left[\left| \int_0^t V_l(\eta_s)\, ds \right| \right] \;\le\; \frac{C_0}{\gamma} + \frac{1}{\gamma N} \log \mathbb{E}_\rho \left[\exp\left\{ \gamma N \left| \int_0^t V_l(\eta_s)\, ds \right| \right\} \right],$$

where $\mathbb{E}_\rho$ denotes expectation with respect to the process starting from the invariant measure ν_ρ. Using the elementary inequality $e^{|x|} \le e^x + e^{-x}$, we can get rid of the absolute value in the previous integral, considering h and $-h$. In this case, by Feynman–Kac formula, the second term on the right hand side is bounded by $(\gamma N)^{-1} T \lambda_{N,l}$, where $\lambda_{N,l}$ is the largest eigenvalue of $N^2 L_N + \gamma N V_l$. Therefore, to prove the lemma, it is enough to show that $(\gamma N)^{-1}\lambda_{N,l}$ vanishes, as $N \uparrow \infty$ and then $l \uparrow \infty$, for every $\gamma > 0$.

By the variational formula for $\lambda_{N,l}$,

$$(\gamma N)^{-1}\lambda_{N,l} \;=\; \sup_f \left\{ \langle V_l\, f\rangle_\rho - \gamma^{-1} N \left\langle \sqrt{f}\left(-L_N\sqrt{f}\right)\right\rangle_\rho \right\}, \tag{6.2}$$

where the supremum is carried over all densities f with respect to ν_ρ. Recall that we denote by $L^{env}_{\Lambda_l}$ the restriction of the environment part of the generator to the cube Λ_l. By a computation (cf. [22, Equation (3.1)]), the Dirichlet forms satisfy $\left\langle \sqrt{f}\left(-L^{env}_{\Lambda_l}\sqrt{f}\right)\right\rangle_\rho \le \left\langle \sqrt{f}\left(-L_N\sqrt{f}\right)\right\rangle_\rho$. Then, we may bound the previous expression by a similar one where L_N is replaced by $L^{env}_{\Lambda_l}$.

Denote by $\hat{f}_l$ the conditional expectation of f given $\{\eta(z) : z \in \Lambda_l\}$. Since V_l depends on the configuration η only through $\{\eta(z) : z \in \Lambda_l\}$ and since the Dirichlet form is convex, the expression inside braces in Eq. 6.2 is less than or equal to

$$\int V_l\, \hat{f}_l \, d\nu_\rho^{\Lambda_l} \;-\; \gamma^{-1} N \int \sqrt{\hat{f}_l}\left(-L^{env}_{\Lambda_l}\sqrt{\hat{f}_l}\right) d\nu_\rho^{\Lambda_l}, \tag{6.3}$$

where, as in Eq. 6.1, $\nu_\rho^{\Lambda_l}$ stands for the restriction of the product measure ν_ρ to $\mathbb{N}_0^{\Lambda_l}$.

The linear term in this formula is equal to

$$\sum_{j\ge 1} c_{l,j}(f) \int V_l\, \hat{f}_{l,j}\, d\nu_{\Lambda_l,j},$$

where $\nu_{\Lambda_l,j}$ is the canonical measure defined in Eq. 6.1 and

$$c_{l,j}(f) \;=\; \int_{\Sigma_{\Lambda_l,j}} \hat{f}_l\, d\nu_\rho^{\Lambda_l}, \quad \hat{f}_{l,j}(\eta) \;=\; c_{l,j}(f)^{-1}\, \nu_\rho^{\Lambda_l}(\Sigma_{\Lambda_l,j})\, \hat{f}_l(\eta)\,.$$

The sum starts at $j = 1$ because there is always a particle at the origin. Note also that $\sum_{j\ge 1} c_{l,j}(f) = 1$ and that $\hat{f}_{l,j}(\cdot)$ is a density with respect to $\nu_{\Lambda_l,j}$.

By the same reasons, the quadratic term of Eq. 6.3 can be written as

$$\gamma^{-1} N \sum_{j\ge 1} c_{l,j}(f) \int \sqrt{\hat{f}_{l,j}}\left(-L^{env}_{\Lambda_l}\sqrt{\hat{f}_{l,j}}\right) d\nu_{\Lambda_l,j}.$$

In view of this decomposition, Eq. 6.2 is bounded above by

$$\sup_{j\geq 1}\sup_{f}\left\{\int V_l\, f\, d\nu_{\Lambda_l,j} \;-\; \gamma^{-1}N\int \sqrt{f}\,\left(-L^{env}_{\Lambda_l}\sqrt{f}\right)\, d\nu_{\Lambda_l,j}\right\}\,,$$

where the second supremum is carried over all densities with respect to $\nu_{\Lambda_l,j}$.

Recall that $V_l(\eta) = h(\eta(0)) - H(\eta^l(0))$. Let $\tilde{H}_l(j/2l+1) = \int h(\eta(0))\, d\nu_{\Lambda_l,j}$. By Lemma 6.4 below, we can replace $H(\eta^l(0))$ by $\tilde{H}_l(\eta^l(0))$ in the previous expression. Let $V_{l,j}(\eta) = h(\eta(0)) - \tilde{H}_l(j/2l+1)$ and notice that $V_{l,j}$ has mean zero with respect to $\nu_{\Lambda_l,j}$ for all $j \geq 1$. By Lemma 6.2, $L^{env}_{\Lambda_l}$ has a spectral gap on $\Sigma_{\Lambda_l,j}$ of order $C_0 l^{-2}$, uniformly in j. Then, as h is bounded, by Rayleigh expansion [10, Theorem A3.1.1], for sufficiently large N,

$$\begin{aligned}
&\int V_{l,j}\, f\, d\nu_{\Lambda_l,j} \;-\; \gamma^{-1}N\int \sqrt{f}\,\left(-L^{env}_{\Lambda_l}\sqrt{f}\right)\, d\nu_{\Lambda_l,j}\\
&\quad\leq\; \frac{\gamma N^{-1}}{1-2\|V_l\|_{L^\infty}C_0 l^2\gamma N^{-1}}\int V_{l,j}(-L^{env}_{\Lambda_l})^{-1}V_{l,j}\, d\nu_{\Lambda_l,j}\\
&\quad\leq\; 2\gamma N^{-1}\int V_{l,j}(-L^{env}_{\Lambda_l})^{-1}V_{l,j}\, d\nu_{\Lambda_l,j}
\end{aligned}$$

uniformly in $j \geq 1$. By the spectral gap estimate of $L^{env}_{\Lambda_l}$, the L^2 norm on mean-zero functions of the operator $L^{env}_{\Lambda_l}$ satisfies $\|L^{env}_{\Lambda_l}\|^2 \leq C_0 l^2$, and so the above expression is less than or equal to

$$C_0 l^2\gamma N^{-1}\int V^2_{l,j}\, d\nu_{\Lambda_l,j} \;\leq\; C_0(h) l^2\gamma N^{-1}$$

because h is bounded. This proves that Eq. 6.2 vanishes as $N\uparrow\infty$ and then $l\uparrow\infty$, and therefore the lemma. □

Lemma 6.4 *For any bounded, Lipschitz function $h : \mathbb{N}_0 \to \mathbb{R}$,*

$$\limsup_{l\to\infty}\sup_{k\geq 0}\left|E_{\nu_{\Lambda_l,k}}[h(\eta(0))] - E_{\nu_{k/|\Lambda_l|}}[h(\eta(0))]\right| \;=\; 0.$$

Proof Fix $\epsilon > 0$ and consider (l,k) such that $k/|\Lambda_l| \leq \epsilon$. We may subtract $h(1)$ to both expectations. Since h is Lipschitz, the absolute value appearing in the statement of the lemma is bounded by

$$C(h)\left\{\int\{\eta(0)-1\}\, d\nu_{\Lambda_l,k} \;+\; \int\{\eta(0)-1\}\, d\nu_{k/|\Lambda_l|}\right\}\,. \tag{6.4}$$

Note that both terms are positive because both measures are concentrated on configurations with at least one particle at the origin. We claim that each term is bounded by $a^2\epsilon$.

On the one hand, by the explicit formula for $\nu_{\Lambda_l,k}$, the first term in Eq. 6.4 is equal to

$$\sum \eta(0)\{\eta(0)-1\} \prod_{x\in\Lambda_l} \frac{1}{g(\eta(x))!} \Big/ \sum \eta(0) \prod_{x\in\Lambda_l} \frac{1}{g(\eta(x))!}\,,$$

where both sums are performed over $\Sigma_{\Lambda_l,k}$. Replacing $\eta(0)$ by $a^{\pm 1}g(\eta(0))$ in the numerator and in the denominator, we obtain that the previous expression is less than or equal to $a^2 E_{\mu_{\Lambda_l,k-1}}[\eta(0)] \le a^2 k/|\Lambda_l|$. In last formula, $\mu_{\Lambda_l,k}$ is the measure given in the proof of Lemma 6.2.

On the other hand, since $\nu_\rho(d\eta) = \{\eta(0)/\rho\}\mu_\rho(d\eta)$, the second term inside braces is equal to $\rho^{-1}\int \eta(0)\{\eta(0)-1\}\,d\mu_{k/|\Lambda_l|}$, where $\rho = k/|\Lambda_l|$. since $k \le ag(k)$, we may replace $\eta(0)$ by $ag(\eta(0))$ and perform a change of variables $\eta' = \eta - \mathfrak{d}_0$ to bound the second term in Eq. 6.4 by $a\varphi(\rho) \le a^2\rho$.

For (l,k) such that $k/|\Lambda_l| \ge \epsilon$, write

$$E_{\nu_{\Lambda_l,k}}[h(\eta(0))] - E_{\nu_{k/|\Lambda_l|}}[h(\eta(0))] \;=\; \frac{1}{\rho}\left\{E_{\mu_{\Lambda_l,k}}[h'(\eta(0))] - E_{\mu_{k/|\Lambda_l|}}[h'(\eta(0))]\right\},$$

where $\rho = k/|\Lambda_l|$ and $h'(j) = h(j)j$. By Corollary 6.1 (parts a,b) [13] the last difference in absolute value is bounded by $C(h)\epsilon^{-1}l^{-1}$. □

6.2 Local two-blocks estimate

In this subsection we show how to go from a box of size l to a box of size ϵN:

Lemma 6.5 (Two-blocks estimate) *Let $H : \mathbb{R}_+ \to \mathbb{R}$ be a bounded, Lipschitz function. For every $t > 0$,*

$$\limsup_{l\to\infty}\limsup_{\epsilon\to 0}\limsup_{N\to\infty}\mathbb{E}^N\left[\left|\int_0^t \left\{H(\eta_s^l(0)) - \frac{1}{\epsilon N}\sum_{x=1}^{\epsilon N} H(\eta_s^l(x))\right\} ds\right|\right] = 0. \tag{6.5}$$

The proof of this lemma is very similar to the proof of Lemma 6.3. As H is bounded, the expectation in Eq. 6.5 is bounded by

$$\frac{1}{\epsilon N}\sum_{x=2l+1}^{\epsilon N}\mathbb{E}^N\left[\left|\int_0^t \left\{H(\eta_s^l(0)) - H(\eta_s^l(x))\right\} ds\right|\right] + \frac{C(H)(2l+1)}{\epsilon N}\,.$$

Following the proof of the one-block estimate, we see that it is enough to estimate, uniformly in $2l+1 \le x \le \epsilon N$, the quantity

$$\sup_f \left\{\langle V_{l,x} f\rangle_\rho - N\gamma^{-1}\left\langle\sqrt{f}\left(-L_N\sqrt{f}\right)\right\rangle_\rho\right\},$$

where the supremum, as before, is over all density functions f with $\int f d\nu_\rho = 1$ and $V_{l,x}$ is defined by

$$V_{l,x}(\eta) = H(\eta^l(0)) - H(\eta^l(x)).$$

Notice that the blocks Λ_l and $\Lambda_l(x) =: \{-l + x, \dots, l + x\}$ are disjoint. Let $L^{env}_{\Lambda_{l,x}}$ be the restriction of L^{env}_N to the set $\Lambda_{l,x} = \Lambda_l \cup \Lambda_l(x)$ and define the operator $L_{l,x}$ by

$$L_{l,x} f(\eta) = L^{env}_{\Lambda_{l,x}} f(\eta) + g(\eta(l))[f(\eta^{l,x-l}) - f(\eta)] + g(\eta(x-l))[f(\eta^{x-l,l}) - f(\eta)].$$

The operator $L_{l,x}$ corresponds to the environment generator of a zero-range dynamics on which particles can jump between adjacent sites on each box, and between endpoints l and $x - l$. Since $x \le \epsilon N$, we see, by adding and subtracting at most ϵN terms (cf. [10, pp. 94–95], [22, Equation (3.1)]), that

$$\langle f(-L_{l,x} f)\rangle_\rho \le (1 + \epsilon N)\langle f(-L_N f)\rangle_\rho.$$

Then, it is enough to prove that

$$\sup_f \left\{ \left\langle \left\{ H(\eta^l(0)) - H(\zeta^l(0)) \right\} f \right\rangle_{\nu_\rho^{\Lambda_l^*}} - \frac{1}{2\epsilon\gamma} \left\langle \sqrt{f} \left(-L_{l,l}\sqrt{f} \right) \right\rangle_{\nu_\rho^{\Lambda_l^*}} \right\}$$

vanishes as $\epsilon \downarrow 0$ and then $l \uparrow \infty$. In this formula, the state space is $\mathbb{N}_0^{\Lambda_l^*}$, where $\Lambda_l^* = \{-l, \dots, 3l + 1\}$, the configurations of this space are denoted by the pair $\beta = (\eta, \zeta)$, where η belongs to $\mathbb{N}_0^{\Lambda_l}$ and ζ belongs to $\mathbb{N}_0^{\{l+1,\dots,3l+1\}}$, expectation is taken with respect to the measure $\nu_\rho^{\Lambda_l^*}$, the projection of ν_ρ on Λ_l^*, $L_{l,l}$ is the generator of the environment restricted to the set Λ_l^*:

$$\begin{aligned}(L_{l,l} f)(\beta) = &\sum_{\substack{x \ne 0, x \in \Lambda_l^* \\ y \in \Lambda_l^*}} p(y - x)\, g(\beta(x))\, [f(\beta^{x,y}) - f(\beta)] \\ &+ \sum_{z \in \mathbb{Z}} p(z)\, g(\beta(0))\, \frac{\beta(0) - 1}{\beta(0)}\, [f(\beta^{0,z}) - f(\beta)]\,;\end{aligned}$$

and the supremum is carried over all densities f with respect to $\nu_\rho^{\Lambda_l^*}$.

Following the proof of the one-block Lemma 6.3, we need only to prove that

$$\sup_{k \ge 1} \sup_f \left\{ \int \left\{ H(\eta^l(0)) - H(\zeta^l(0)) \right\} f\, d\nu_{\Lambda_l^*,k} - \frac{1}{2\gamma\epsilon} \int \sqrt{f} \left(-L_{l,l}\sqrt{f} \right) d\nu_{\Lambda_l^*,k} \right\}$$

vanishes with limits on ϵ and l where the supremum is on densities f with respect to the canonical measure $\nu_{\Lambda_l^*,k}(\cdot) = \nu_\rho^{\Lambda_l^*}(\cdot|\Sigma_{\Lambda_l^*,k})$, defined similarly as in Eq. 6.1, where $\Sigma_{\Lambda_l^*,k} = \left\{ \beta \in \mathbb{N}_0^{\Lambda_l^*} : \beta(0) \ge 1, \sum_{y \in \Lambda_l^*} \beta(y) = k \right\}$.

Let $W_l(\beta) = H(\eta^l(0)) - H(\zeta^l(0))$, and note that its expectation with respect to $\nu_{\Lambda_l^*,k}$ does not vanish. Let $\hat{W}_l(\beta) = W_l(\beta) - E_{\nu_{\Lambda_l^*,k}}[W_l]$. Then, by the Rayleigh expansion [10, Theorem A3.1.1], spectral gap estimate Lemma 6.2 applied to $L_{l,l}$ (which can be thought of as the environment generator on a block of length $2|\Lambda_l|$) and boundedness of H, for small ϵ, adding and subtracting $E_{\nu_{\Lambda_l^*,k}}[W_l]$, we obtain that

$$\begin{aligned}
&\int W_l\, f\, d\nu_{\Lambda_l^*,k} - \frac{1}{2\gamma\epsilon}\int \sqrt{f}\left(-L_{l,l}\sqrt{f}\right) d\nu_{\Lambda_l^*,k}\\
&\quad\le \int W_l\, d\nu_{\Lambda_l^*,k} + \frac{2\gamma\epsilon}{1-C(H)l^2\gamma\epsilon}\int \hat{W}_l\left\{(-L_{l,l})^{-1}\hat{W}_l\right\} d\nu_{\Lambda_l^*,k}\\
&\quad\le \int W_l\, d\nu_{\Lambda_l^*,k} + C(H)l^2\gamma\epsilon
\end{aligned}$$

for some finite constant $C(H)$ depending on H. The last term vanishes as $\epsilon \downarrow 0$, while the first term vanishes uniformly in k as $l \uparrow \infty$ by Lemma 6.6 below. □

Lemma 6.6 *For a bounded, Lipschitz function $H : \mathbb{R}_+ \to \mathbb{R}$, we have that*

$$\limsup_{l\to\infty}\sup_{k\ge 0}\left| E_{\nu_{\Lambda_l^*,k}}\left[H(\eta^l(0)) - H(\zeta^l(0))\right]\right| = 0.$$

Proof Fix $\epsilon > 0$. Using that H is Lipschitz, we have that $|H(\eta^l(0)) - H(\zeta^l(0))| \le C(H)\{\eta^l(0) + \zeta^l(0)\}$, and so the expectation appearing in the statement of the lemma is less than or equal to $C(H)E_{\nu_{\Lambda_l^*,k}}[\eta^l(0) + \zeta^l(0)]$. A computation, similar to the one presented in the proof of Lemma 6.4, shows that

$$E_{\nu_{\Lambda_l^*,k}}[\beta(0)] \le a^2\left\{1 + E_{\mu_{\Lambda_l^*,k-1}}[\xi(0)]\right\}, \quad E_{\nu_{\Lambda_l^*,k}}[\beta(y)] \le a^2 E_{\mu_{\Lambda_l^*,k-1}}[\xi(0)]$$

for all $y \ne 0$. In this formula, $\mu_{\Lambda_l^*,k}$ stands for the canonical measure defined by $\mu_{\Lambda_l^*,k}(\cdot) = \mu_\rho^{\Lambda_l^*}\left(\cdot \,|\, \sum_{x\in\Lambda_l^*}\beta(x) = k\right)$, where $\mu_\rho^{\Lambda_l^*}$ is the product measure μ_ρ restricted to the set Λ_l^*. In particular, the expectation appearing in the statement of the lemma is less than or equal to $C(H)a^2\{1+k\}/2|\Lambda_l|$. This concludes the proof of the lemma for the supremum restricted to $k/2|\Lambda_l| \le \epsilon$.

Assume now that $k/2|\Lambda_l| \ge \epsilon$. By definition of the canonical measure $\nu_{\Lambda_l^*,k}$ and the grand-canonical measure $\nu_\rho^{\Lambda_l^*}$, the expectation appearing in the statement of the lemma is equal to

$$\frac{1}{E_{\mu_{\Lambda_l^*,k}}[\beta(0)]}\, E_{\mu_{\Lambda_l^*,k}}\left[\beta(0)\left\{H(\eta^l(0)) - H(\zeta^l(0))\right\}\right]. \tag{6.6}$$

Since the measure is space homogeneous, the denominator is equal to $\rho_{l,k} = k/2|\Lambda_l|$, while in the numerator we may replace $\beta(0)$ by $\eta^l(0)$. The numerator can therefore be rewritten as

$$E_{\mu_{\Lambda_l^*,k}}\left[\left\{\eta^l(0)-\rho_{l,k}\right\}\left\{H(\eta^l(0))-H(\zeta^l(0))\right\}\right]$$
$$+\rho_{l,k}E_{\mu_{\Lambda_l^*,k}}\left[H(\eta^l(0))-H(\zeta^l(0))\right].$$

The second term vanishes because the measure $\mu_{\Lambda_l^*,k}$ is space homogeneous, while the first one, as H is bounded, is absolutely dominated by $C(H)E_{\mu_{\Lambda_l^*,k}}[\,|\eta^l(0)-\rho_{l,k}|\,]$. By [13, Corollary 6.1 (C)], this expression is less than or equal to

$$C'(H)E_{\mu_{\rho_{l,k}}^{\Lambda_l^*}}\left[\left|\xi^l(0)-\rho_{l,k}\right|\right]\;\le\;C'(H)\,\sigma(\rho_{l,k})\,l^{-1/2},$$

where $\sigma(\rho)$ stands for the variance of $\xi(0)$ under μ_ρ. By [13, (5.2)], which applies under our assumptions (LG), (M), $\sigma(\rho_{l,k})^2\le C\rho_{l,k}$. Therefore, if we recall the denominator in Eq. 6.6, we obtain that

$$E_{\nu_{\Lambda_l^*,k}}\left[H(\eta^l(0))-H(\zeta^l(0))\right]\;\le\;\frac{C(H)}{\sqrt{l\,\rho_{l,k}}},$$

which concludes the proof of the lemma since we assumed the density to be bounded below by ϵ. $\square$

6.3 Proof of Proposition 6.1

Recall $H(\rho)=E_{\nu_\rho}[h]$, $H_l(\eta)=H(\eta^l(0))$, and $\bar H_l(\rho)=E_{\mu_\rho}[H_l]$. Then, we have that

$$\mathbb{E}^N\left[\left|\int_0^t\left\{h(\eta_s)-\frac{1}{\epsilon N}\sum_{x=1}^{\epsilon N}\bar H_l(\eta_s^{\varepsilon N}(x))\right\}ds\right|\right]$$
$$\le\;\mathbb{E}^N\left[\left|\int_0^t\left\{h(\eta_s)-H(\eta_s^l(0))\right\}ds\right|\right]$$
$$+\,\mathbb{E}^N\left[\left|\int_0^t\left\{H(\eta_s^l(0))-\frac{1}{\epsilon N}\sum_{x=1}^{\epsilon N}H(\eta_s^l(x))\right\}ds\right|\right]$$
$$+\,\mathbb{E}^N\left[\left|\int_0^t\left\{\frac{1}{\epsilon N}\sum_{x=1}^{\epsilon N}\left(H(\eta_s^l(x))-\bar H_l(\eta_s^{\varepsilon N}(x)\right)\right\}ds\right|\right].$$

As h is bounded, Lipschitz, we have that H is bounded, Lipschitz by Lemma 6.7 below, and so the first and second terms vanish by Lemmas 6.3 and 6.5. For the third term, we can rewrite it as

$$\mathbb{E}^N \left[\left| \int_0^t \left\{ \frac{1}{N} \sum_{x \in \mathbb{T}_N} \iota_\epsilon(x/N) \Big(H(\eta_s^l(x)) - \bar{H}_l(\eta_s^{\varepsilon N}(x)) \Big) \right\} ds \right| \right]$$

where $\iota_\epsilon(\cdot) = \epsilon^{-1} 1\{(0, \epsilon]\}$. Now, as h is bounded (and so H is bounded), we can replace ι_ϵ in the last expression by a smooth approximation J_ϵ, which allows us to replace the term in braces by

$$\frac{1}{N} \sum_{x \in \mathbb{T}_N} J_\epsilon(x/N) \left[\frac{1}{2N\varepsilon + 1} \sum_{|y-x| \le N\varepsilon} \Big(H(\eta_s^l(y)) - \bar{H}_l(\eta_s^{\varepsilon N}(x)) \Big) \right].$$

Then, for fixed $\epsilon > 0$ and $l \ge 1$, treating $H_l(\eta) = H(\eta^l(0))$ as a local function, which is also bounded, Lipschitz as H is bounded, Lipschitz, the third term vanishes using Proposition 5.1 by taking $N \uparrow \infty$, and $\varepsilon \downarrow 0$. □

Lemma 6.7 *Let $h : \Omega_N \to \mathbb{R}$ be a local, Lipschitz function. Then, $H : \mathbb{R}_+ \to \mathbb{R}$ given by $H(\rho) = E_{\nu_\rho}[h]$ is also Lipschitz.*

Proof The proof is similar to that of Corollary II.3.7 [10] which shows $\bar{h}(\rho) = E_{\mu_\rho}[h]$ is Lipschitz. Following the proof of Corollary II.3.7 [10], it is not difficult to show $\{\nu_\rho : \rho \ge 0\}$ is a stochastically increasing family, and for $\rho_1 < \rho_2$ that

$$|H(\rho_1) - H(\rho_2)| \le C_h \sum_{x \in A} |E_{\nu_{\rho_1}}[\eta(x)] - E_{\nu_{\rho_2}}[\eta(x)]|$$

where C_h is the Lipschitz constant of h, and $A \subset \mathbb{Z}$ corresponds to the support of h. If A does not contain the origin, the proof is the same as for Corollary II.3.7 [10].

Otherwise, it is enough to estimate the difference $|E_{\nu_{\rho_1}}[\eta(0)] - E_{\nu_{\rho_2}}[\eta(0)]|$. When $0 = \rho_2 < \rho_1$, the difference equals $E_{\mu_{\rho_1}}[\eta(0)(\eta(0) - 1)]/\rho_1 \le a\varphi(\rho_1) \le a^2 \rho_1$ as $a^{-1} \le g(k)/k, \varphi(\rho)/\rho \le a$ through (LG), (M), and $E_{\mu_\rho}[g(\eta(0)) f(\eta)] = \varphi(\rho) E_{\mu_\rho}[f(\eta + \mathfrak{d}_0)]$ where $\mathfrak{d}_0$ is the configuration with exactly one particle at the origin. When $0 < \rho_2 < \rho_1$, the difference equals

$$|\rho_1^{-1} E_{\mu_{\rho_1}}[\eta(0)^2] - \rho_2^{-1} E_{\mu_{\rho_2}}[\eta(0)^2]| \le |\sigma^2(\rho_1)/\rho_1 - \sigma^2(\rho_2)/\rho_1| + |\rho_1 - \rho_2|$$

where $\sigma^2(\rho) = E_{\mu_\rho}[(\eta(0) - \rho)^2]$. The Lipschitz estimate now follows by calculating a uniform bound on the derivative

$$\partial_\rho \frac{\sigma^2(\rho)}{\rho} = \frac{m_3(\rho)}{\rho \sigma^2(\rho)} - \frac{\sigma^2(\rho)}{\rho^2}$$

where $m_3(\rho) = E_{\mu_\rho}[(\eta(0) - \rho)^3]$. For ρ large, under assumptions (LG), (M), this is on order $O(\rho^{-1/2})$ from Lemma 5.2 [13] and bound $a^{-1} \le \varphi(\rho)/\rho \le a$; on the other hand, as $\rho \downarrow 0$, the derivative is also bounded. □

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