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Critical behavior and the limit distribution for long-range oriented percolation. II: Spatial correlation

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Abstract We prove that the Fourier transform of the properly scaled normalized two-point function for sufficiently spread-out long-range oriented percolation with index $\alpha > 0$ converges to $e^{-C|k|^{\alpha\wedge 2}}$ for some $C \in (0, \infty)$ above the upper-critical dimension $d_c \equiv 2(\alpha \wedge 2)$. This answers the open question remained in the previous paper (Chen and Sakai in Probab Theory Relat Fields 142:151–188, 2008). Moreover, we show that the constant C exhibits crossover at $\alpha = 2$, which is a result of interactions among occupied paths. The proof is based on a new method of estimating fractional moments for the spatial variable of the lace-expansion coefficients.

Keywords Long-range oriented percolation · Mean-field critical behavior · Limit theorem · Crossover phenomenon · Lace expansion · Fractional moments

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1 Introduction and the main result

We consider oriented bond percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$, where each time-oriented bond $((x, n), (y, n+1))$ is occupied with probability $pD(y-x)$ and vacant with probability $1-pD(y-x)$, independently of the other bonds. Here, D is a $\mathbb{Z}^d$-symmetric probability distribution on $\mathbb{Z}^d$, hence the parameter $p \in [0, \|D\|_\infty^{-1}]$ can be interpreted as the

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average number of occupied bonds per vertex. We say that a vertex (x, j) is connected to (y, n), and write $(x, j) \to (y, n)$, if either $(x, j) = (y, n)$ or there is a time-oriented path of occupied bonds from (x, j) to (y, n). Let $\mathbb{P}_p$ be the probability distribution of the bond variables, and define the two-point function as

$$\varphi_p(x, n) = \mathbb{P}_p\left((o, 0) \to (x, n)\right),$$

and its Fourier transform as

$$Z_p(k; n) = \sum_{x \in \mathbb{Z}^d} e^{ik\cdot x} \varphi_p(x, n) \qquad (k \in [-\pi, \pi]^d).$$

Notice that $Z_p(0; n) \equiv \sum_{x \in \mathbb{Z}^d} \varphi_p(x, n)$ is the expected number of vertices at time n connected from $(o, 0)$. It has been known ([3] and references therein) that there is a $p_c \geq 1$ such that

$$\chi_p \equiv \sum_{n=0}^{\infty} Z_p(0; n) \begin{cases} < \infty & (p < p_c), \\ = \infty & (p \geq p_c). \end{cases}$$

In the previous paper [1] (often referred to as Part I from now on), we investigated critical behavior of long-range oriented percolation defined by

$$D(x) = \frac{h(x/L)}{\sum_{y \in \mathbb{Z}^d} h(y/L)},$$

where h is a probability density on $\mathbb{R}^d$ satisfying $h(x) \asymp |x|^{-d-\alpha}$ (i.e., $|x|^{d+\alpha} h(x)$ is bounded away from zero and infinity) for large x. Here, $\alpha > 0$ is the characteristic index, and $L \in [1, \infty)$ is the parameter that serves the model to spread out. For example, $\|D\|_\infty = O(\lambda)$, where

$$\lambda = L^{-d}.$$

See [1, Sect. 1.1] for the precise definition and other properties of D. Notice that the variance $\sigma^2 \equiv \sum_{x \in \mathbb{Z}^d} |x|^2 D(x)$ does not exist if $\alpha \leq 2$.

Suppose that there is a positive finite constant v_α $(= \frac{\sigma^2}{2d}$ if $\alpha > 2)$ such that the Fourier transform $\hat{D}(k) \equiv \sum_{x \in \mathbb{Z}^d} D(x) e^{ik\cdot x}$ obeys the asymptotics

$$1 - \hat{D}(k) \underset{|k| \to 0}{\sim} \begin{cases} v_\alpha |k|^{\alpha \wedge 2} & (\alpha \neq 2), \\ v_2 |k|^2 \log \frac{1}{|k|} & (\alpha = 2). \end{cases} \tag{1.1}$$

The assumption (1.1) with $v_\alpha = O(L^{\alpha \wedge 2})$ indeed holds if, e.g., $h(x) \sim c|x|^{-d-\alpha}$ as $|x| \to \infty$ for some constant c (see [7, Sect. 10.5] for the 1-dimensional case). Let

$$k_n = k \times \begin{cases} (v_\alpha n)^{-\frac{1}{\alpha \wedge 2}} & (\alpha \neq 2), \\ (v_2 n \log \sqrt{n})^{-\frac{1}{2}} & (\alpha = 2), \end{cases} \tag{1.2}$$

so that

$$\lim_{n\to\infty} n\big(1-\hat{D}(k_n)\big) = |k|^{\alpha\wedge 2}. \tag{1.3}$$

Among various results, we proved that, for $\alpha > 0$, $d > 2(\alpha \wedge 2)$, $L \gg 1$, $p \in (0, p_{\mathrm{c}}]$ and $k \in \mathbb{R}^d$, there exists $c, c' = 1 + O(\lambda)$ such that the normalized two-point function satisfies

$$e^{-c|k|^{\alpha\wedge 2}} \le \liminf_{n\to\infty} \frac{Z_p(k_n;n)}{Z_p(0;n)} \le \limsup_{n\to\infty} \frac{Z_p(k_n;n)}{Z_p(0;n)} \le e^{-c'|k|^{\alpha\wedge 2}}. \tag{1.4}$$

Here, $d_{\mathrm{c}} \equiv 2(\alpha \wedge 2)$ is the upper-critical dimension of this model. We do not expect that (1.4) holds for $d < d_{\mathrm{c}}$. Compare this result with the behavior of the two-point function for the branching random walk on $\mathbb{Z}^d$ whose mean number of offspring per parent is $p > 0$:

$$Z_p^{\mathrm{BRW}}(k;n) = p^n \hat{D}(k)^n, \qquad \lim_{n\to\infty} \frac{Z_p^{\mathrm{BRW}}(k_n;n)}{Z_p^{\mathrm{BRW}}(0;n)} = e^{-|k|^{\alpha\wedge 2}}. \tag{1.5}$$

The latter is an immediate consequence of the former and (1.3). We note that $e^{-|k|^{\alpha}}$ is the characteristic function of an α-stable random variable (see, e.g., [10]).

The proof in [1] of (1.4) is based on the lace expansion for the two-point function. To derive information of the sequence $Z_p(k;n)$ from its sum (= the Fourier–Laplace transform of the two-point function) and prove (1.4), we established optimal control over fractional moments for the *time* variable of the lace-expansion coefficients. However, due to the long-range nature of our D, we were unable to optimally control fractional moments for the *spatial* variable of the expansion coefficients and squeeze the bounds in (1.4) to identify the limit. We note that, by the standard Taylor-expansion method, the limit has been shown to exist at $p = p_{\mathrm{c}}$ if $\alpha > 2$ [6] and for every $p \in (0, p_{\mathrm{c}}]$ if the model is finite-range [8]. This standard method does not work for $\alpha < 2$ in the current setting.

In this paper, we develop a new method to estimate fractional moments for the spatial variable of the expansion coefficients and achieve the following result on the normalized two-point function:

Theorem 1 *Let $\alpha > 0$, $d > 2(\alpha\wedge 2)$, $L \gg 1$ and $p \in (0, p_{\mathrm{c}}]$. There is a $C = 1+O(\lambda)$ such that, for any $k \in \mathbb{R}^d$,*

$$\lim_{n\to\infty} \frac{Z_p(k_n;n)}{Z_p(0;n)} = e^{-C|k|^{\alpha\wedge 2}},$$

where k_n is defined in (1.2). Moreover,

$$C = \frac{1}{1 + pm_p \sum_{(x,n)} n\,\pi_p(x,n)\,m_p^n} \times \begin{cases} 1 + \frac{pm_p}{\sigma^2} \sum_{(x,n)} |x|^2 \pi_p(x,n)\,m_p^n & (\alpha > 2), \\ 1 & (\alpha \le 2), \end{cases} \tag{1.6}$$

where m_p is the radius of convergence for $\sum_{n=0}^{\infty} Z_p(0;n)\,m^n$, and $\pi_p(x,n)$ is the alternating sum of the lace-expansion coefficients. The sums in (1.6) are absolutely convergent.

See, e.g., [1, Sect. 3.1] for the precise definition of $\pi_p(x,n)$.

The most remarkable observation in the above theorem is that the constant C exhibits crossover at $\alpha = 2$. This phenomenon is observable if π_p, which is model-dependent and contains information about interactions of occupied paths, is nonzero. We recall that, for the branching random walk, occupied paths are independent and $\pi_p \equiv 0$, hence C is always 1 as in (1.5). Therefore, the crossover behavior in (1.6) is a result of interactions among occupied paths.

We should emphasize that our approach developed in this paper and Part I is widely applicable, not only to our long-range oriented percolation, but also to various other (long-range/finite-range) statistical–mechanical models. For example, our methods also apply to show that a similar result to the above limit theorem holds for long-range self-avoiding walk with the characteristic index $\alpha > 0$, studied in [5]. Markus Heydenreich is working in this direction [4]. His work will be a generalization of the results in [2,12], where $D(x)$ is proportional to $|x|^{-2}$ if x is on the coordinate axes, otherwise $D(x) = 0$. Since the coordinate axes are 1-dimensional, we should interpret α for this particular model as 1.

As another nontrivial application of the fractional-moment method of this paper, one of the authors (LCC) will report in his ongoing work that the gyration radius $\xi_p^{(r)}$ of order $r \in (0,\alpha)$ for sufficiently spread-out oriented percolation with $d > 2(\alpha \wedge 2)$ obeys

$$\xi_p^{(r)}(n) \equiv \left(\frac{1}{Z_p(0;n)} \sum_{x \in \mathbb{Z}^d} |x|^r \varphi_p(x,n) \right)^{1/r} \asymp \begin{cases} n^{\frac{1}{\alpha \wedge 2}} & (\alpha \ne 2), \\ (n \ \log n)^{1/2} & (\alpha = 2), \end{cases}$$

for every $p \in (0, p_c]$.

The rest of the paper is organized as follows. In Sect. 2, we summarize the relevant results from Part I. In Sect. 3, we prove Theorem 1 subject to a key proposition on fractional moments for the spatial variable of the lace-expansion coefficients. We prove that proposition in Sect. 4 using a certain integral representation for fractional powers of positive reals.

2 Summary of the relevant results from Part I

In this section, we summarize the results from Part I that will be used in the rest of the paper.

First, we introduce some notation. Let

$$q_p(x,n) = \mathbb{P}_p\left(((o,0),(x,n)) \text{ is occupied}\right) \equiv \begin{cases} pD(x) & (n=1), \\ 0 & (n \neq 1). \end{cases}$$

We denote the space-time convolution of functions f and g on $\mathbb{Z}^d \times \mathbb{Z}_+$ by

$$(f * g)(x,n) = \sum_{(y,t) \in \mathbb{Z}^d \times \mathbb{Z}_+} f(y,t)\, g(x-y, n-t),$$

and the Fourier–Laplace transform of f by

$$\hat{f}(k,z) = \sum_{(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+} f(x,n)\, e^{ik\cdot x} z^n \qquad (k \in [-\pi,\pi]^d,\ z \in \mathbb{C}).$$

Notice that $\frac{-1}{2}\Delta_k \hat{f}(l,z)$, defined as

$$\begin{aligned} \frac{-1}{2}\Delta_k \hat{f}(l,z) &= \hat{f}(l,z) - \frac{\hat{f}(l+k,z) + \hat{f}(l-k,z)}{2} \\ &\equiv \sum_{(x,n) \in \mathbb{Z}^d \times \mathbb{Z}_+} (1 - \cos(k \cdot x))\, f(x,n)\, e^{il\cdot x} z^n, \end{aligned} \tag{2.1}$$

is the Fourier–Laplace transform of $(1 - \cos(k \cdot x)) f(x,n)$.

In [1, Sect. 3.1], we explained the derivation of the lace expansion

$$\varphi_p(x,n) = \pi_p(x,n) + (\pi_p * q_p * \varphi_p)(x,n),$$

where $\pi_p(x,n)$ is the alternating sum of the $\mathbb{Z}^d$-symmetric nonnegative expansion coefficients $\pi_p^{(N)}(x,n)$ for $N = 0,1,2,\dots$:

$$\pi_p(x,n) = \sum_{N=0}^{\infty} (-1)^N \pi_p^{(N)}(x,n). \tag{2.2}$$

The precise definition of $\pi_p^{(N)}$ is unimportant in this paper. However, we will use the following properties of π_p and φ_p:

Proposition 1 *Let* $\alpha > 0$, $d > 2(\alpha \wedge 2)$ *and* $L \gg 1$. *Then,*

$$pm_p \hat{\pi}_p(0, m_p) = 1, \tag{2.3}$$

$$\sum_{(x,n)} n\, |\pi_p(x,n)|\, m^n \le O(\lambda), \tag{2.4}$$

$$\sum_{(x,n)} (1 - \cos(k \cdot x))\, |\pi_p(x,n)|\, m^n \le O(\lambda)\big(1 - \hat{D}(k)\big), \tag{2.5}$$

and

$$|\hat{\varphi}_p(k, me^{i\theta})| \le \frac{O(1)}{pm_p(1 - \frac{m}{m_p}) + |\theta| + 1 - \hat{D}(k)}, \tag{2.6}$$

$$\begin{aligned} |\Delta_k \hat{\varphi}_p(l, me^{i\theta})| \le & \sum_{(j,j')=(0,\pm1),(1,-1)} \frac{1 - \hat{D}(k)}{pm_p(1 - \frac{m}{m_p}) + |\theta| + 1 - \hat{D}(l + jk)} \\ & \times \frac{O(1)}{pm_p(1 - \frac{m}{m_p}) + |\theta| + 1 - \hat{D}(l + j'k)}, \end{aligned} \tag{2.7}$$

uniformly in $p \in (0, p_c]$, $m \in [0, m_p)$, $k, l \in [-\pi, \pi]^d$ *and* $\theta \in [-\pi, \pi]$.

Proof The identity (2.3) for every $p \in (0, p_c]$ was proved in [1, (2.17) and (2.22)]. The bounds (2.4) and (2.6) for $p \in (0, p_c)$ were also proved in Part I, and can be extended up to $p = p_c$, as long as m is strictly less than the radius of convergence, $m_{p_c} = 1$ (cf., [1, Corollary 1.3]).

The same extension applies to the bounds (2.5) and (2.7), if they hold uniformly in $p \in (0, p_c)$ and $m \in [0, m_p)$. In Part I, we showed that

$$\sum_{(x,n)} (1 - \cos(k \cdot x))\, |\pi_p(x,n)|\, m^n \le O(\lambda) \left(1 - \frac{m}{m_p} + 1 - \hat{D}(k)\right),$$

uniformly in $p \in (0, p_c)$, $m \in [0, m_p)$ and $k \in [-\pi, \pi]^d$. However, since the left-hand side is increasing in $m < m_p$, we obtain

$$\begin{aligned} \sum_{(x,n)} (1 - \cos(k \cdot x))\, |\pi_p(x,n)|\, m^n & \le \lim_{m \uparrow m_p} \sum_{(x,n)} (1 - \cos(k \cdot x))\, |\pi_p(x,n)|\, m^n \\ & \le O(\lambda) \lim_{m \uparrow m_p} \left(1 - \frac{m}{m_p} + 1 - \hat{D}(k)\right) \\ & \le O(\lambda)\big(1 - \hat{D}(k)\big), \end{aligned}$$

as required. Using this stronger bound and following the steps in [1, Sect. 4.2], we also obtain (2.7). This completes the proof. ◻

Finally, we summarize the results for the n^{th} coefficient $Z_p(k;n)$ of the series expansion of $\hat{\varphi}_p(k,m)$ in powers of m: $\hat{\varphi}_p(k,m) \equiv \sum_{n=0}^{\infty} Z_p(k;n)\, m^n$. Let (cf., [1, (2.33)–(2.34)])

$$\hat{A}_p^{(1)}(k) = \hat{D}(k) + \frac{m_p \partial_m \hat{\pi}_p(k,m_p)}{p m_p \hat{\pi}_p(k,m_p)^2}, \tag{2.8}$$

$$\hat{B}_p(k) = 1 - \hat{D}(k) + \frac{\hat{\pi}_p(0,m_p) - \hat{\pi}_p(k,m_p)}{\hat{\pi}_p(k,m_p)}, \tag{2.9}$$

where $\partial_m \hat{\pi}_p(k,m_p) = \lim_{m\uparrow m_p} \partial_m \hat{\pi}_p(k,m)$. Notice that, by (2.4)–(2.5),

$$|m_p \partial_m \hat{\pi}_p(k,m_p)| \le \lim_{m\uparrow m_p} \sum_{(x,n)} n\, |\pi_p(x,n)|\, m^n \le O(\lambda), \tag{2.10}$$

$$\begin{aligned} |\hat{\pi}_p(0,m_p) - \hat{\pi}_p(k,m_p)| &\le \sum_{(x,n)} (1-\cos(k\cdot x))\, |\pi_p(x,n)|\, m_p^n \\ &\le O(\lambda)\big(1-\hat{D}(k)\big), \end{aligned} \tag{2.11}$$

where the $O(\lambda)$ terms are uniform in $p \in (0, p_c]$ and $k \in [-\pi,\pi]^d$. Moreover, since $\pi_p(x,0)$ equals the Kronecker delta $\delta_{x,o}$ (cf., [1, (3.2)]), we have $\hat{\pi}_p(k,m_p) = 1 + O(\lambda)$ and thus $\hat{A}_p^{(1)}(k) + \hat{B}_p(k) = 1 + O(\lambda)$ uniformly in $p \in (0, p_c]$ and $k \in [-\pi,\pi]^d$.

In [1, Sect. 2.4], we showed that, for $\alpha > 0$, $d > 2(\alpha\wedge 2)$, $L \gg 1$ and $\varepsilon \in (0, 1 \wedge \frac{d-2(\alpha\wedge 2)}{\alpha\wedge 2})$,

$$m_p^n Z_p(k;n) = \frac{1}{p m_p \left(\hat{A}_p^{(1)}(k) + \hat{B}_p(k)\right)} \left(\frac{\hat{A}_p^{(1)}(k)}{\hat{A}_p^{(1)}(k) + \hat{B}_p(k)}\right)^n + O(n^{-\varepsilon}),$$

hence

$$\frac{Z_p(k;n)}{Z_p(0;n)} = \frac{\hat{A}_p^{(1)}(0)}{\hat{A}_p^{(1)}(k) + \hat{B}_p(k)} \left(\frac{\hat{A}_p^{(1)}(k)}{\hat{A}_p^{(1)}(k) + \hat{B}_p(k)}\right)^n + O(n^{-\varepsilon}), \tag{2.12}$$

uniformly in $p \in (0, p_c]$ and $k \in [-\pi,\pi]^d$. To prove Theorem 1, it thus suffices to investigate the first term in (2.12).

3 Proof of Theorem 1 subject to a key proposition

In this section, we prove Theorem 1 assuming convergence of fractional moments for the spatial variable of π_p, as stated in the following proposition:

Proposition 2 *Let $\alpha > 0$, $d > 2(\alpha \wedge 2)$, $L \gg 1$ and*

$$\delta \begin{cases} \in (0, \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))) & (\alpha \neq 2), \\ = 0 & (\alpha = 2). \end{cases} \tag{3.1}$$

Then, for any $p \in (0, p_c]$,

$$\sum_{(x,n)} |x|^{\alpha\wedge 2+\delta} |\pi_p(x,n)|\, m_p^n < \infty. \tag{3.2}$$

We will roughly explain why δ is chosen as in (3.1), after the proof of Theorem 1 is completed. The proof of Proposition 2 is deferred to Sect. 4.

Proof of Theorem 1 subject to Proposition 2 As explained at the end of Sect. 2, it suffices to investigate the term

$$\left(\frac{\hat{A}_p^{(1)}(k)}{\hat{A}_p^{(1)}(k) + \hat{B}_p(k)}\right)^n \equiv \left(\left(1 + \frac{\hat{B}_p(k)}{\hat{A}_p^{(1)}(k)}\right)^{-\frac{\hat{A}_p^{(1)}(k)}{\hat{B}_p(k)}}\right)^{\frac{n(1-\hat{D}(k))}{\hat{A}_p^{(1)}(k)} \frac{\hat{B}_p(k)}{1-\hat{D}(k)}}.$$

Notice that, by (2.8)–(2.11) and (1.3),

$$\left(1 + \frac{\hat{B}_p(k)}{\hat{A}_p^{(1)}(k)}\right)^{-\frac{\hat{A}_p^{(1)}(k)}{\hat{B}_p(k)}} \underset{|k|\to 0}{\to} e^{-1}, \qquad \frac{n(1-\hat{D}(k_n))}{\hat{A}_p^{(1)}(k_n)} \underset{n\to\infty}{\to} \frac{|k|^{\alpha\wedge 2}}{\hat{A}_p^{(1)}(0)},$$

where

$$\hat{A}_p^{(1)}(0) = 1 + p m_p \sum_{(x,n)} n\, \pi_p(x,n)\, m_p^n.$$

Moreover,

$$\begin{aligned} \frac{\hat{B}_p(k)}{1-\hat{D}(k)} &= 1 + p m_p \frac{\hat{\pi}_p(0, m_p)}{\hat{\pi}_p(k, m_p)} \frac{\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)}{1 - \hat{D}(k)} \\ &\underset{|k|\to 0}{\to} 1 + p m_p \lim_{|k|\to 0} \frac{\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)}{1 - \hat{D}(k)}, \end{aligned}$$

if the limit exists. To complete the proof of Theorem 1, it remains to show

$$\lim_{|k|\to 0} \frac{\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)}{1 - \hat{D}(k)} = \begin{cases} \dfrac{1}{\sigma^2} \displaystyle\sum_{(x,n)} |x|^2 \pi_p(x,n)\, m_p^n & (\alpha > 2), \\ 0 & (\alpha \leq 2). \end{cases} \tag{3.3}$$

Now we choose δ as in (3.1) and use Proposition 2 to prove (3.3) for (i) $\alpha \le 2$ and (ii) $\alpha > 2$, separately.
(i) Let $\alpha \le 2$ and $\alpha + \delta \le 2$. Then, we have

$$0 \le 1 - \cos(k \cdot x) \le O(|k \cdot x|^{\alpha+\delta}).$$

By the spatial symmetry of the model and using (3.2) with δ satisfying $\alpha + \delta \le 2$ and (3.1),

$$\begin{aligned} |\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)| &= \left| \sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+} (1 - \cos(k \cdot x))\, \pi_p(x, n)\, m_p^n \right| \\ &\le O(|k|^{\alpha+\delta}) \sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+} |x|^{\alpha+\delta} |\pi_p(x, n)|\, m_p^n \\ &= O(|k|^{\alpha+\delta}). \end{aligned}$$

By (1.1), we thus obtain that, for small $|k|$,

$$\frac{|\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p)|}{1 - \hat{D}(k)} \le \begin{cases} O(|k|^{\delta}) & (\alpha < 2), \\ O(1/\log \frac{1}{|k|}) & (\alpha = 2). \end{cases}$$

This yields (3.3) for $\alpha \le 2$.
(ii) Let $\alpha > 2$ and $\delta \le 2$. By the Taylor expansion,

$$1 - \cos(k \cdot x) = \frac{(k \cdot x)^2}{2} + O(|k \cdot x|^{2+\delta}).$$

Then, by the spatial symmetry of the model and using (3.2) with δ satisfying (3.1),

$$\hat{\pi}_p(0, m_p) - \hat{\pi}_p(k, m_p) = \frac{|k|^2}{2d} \sum_{(x,n)\in\mathbb{Z}^d\times\mathbb{Z}_+} |x|^2 \pi_p(x, n)\, m_p^n + O(|k|^{2+\delta}). \tag{3.4}$$

The limit (3.3) for $\alpha > 2$ follows from (3.4) and the asymptotics (1.1) with $v_\alpha = \frac{\sigma^2}{2d}$. This completes the proof of Theorem 1 subject to Proposition 2. □

Before closing this section, we roughly explain why $\delta < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2))$ for $\alpha \ne 2$ (the necessity of $\delta = 0$ for $\alpha = 2$ and $\delta > 0$ for $\alpha \ne 2$ is obvious from the above proof of Theorem 1). This is a sort of preview of Sect. 4.

In Sect. 4, we will use diagrammatic bounds on the expansion coefficients $\pi_p^{(N)}$ in (2.2). In each bound (cf., (4.1)–(4.3) below), there are *two* sequences of two-point functions from $(o, 0)$ to (x, n). To bound $\sum_{(x,n)} |x|^{\alpha\wedge 2+\delta} \pi_p^{(N)}(x, n) m^n$, we will split the power $\alpha \wedge 2 + \delta$ into δ_1 and δ_2, and multiply one of the aforementioned two sequences of two-point functions by $|x|^{\delta_1}$ and the other by $|x|^{\delta_2}$. Here, we choose δ_1 and δ_2 both less than $\alpha \wedge 2$, so as to potentially control the weighted two-point functions, like $|y|^{\delta_1} \varphi_p(y, s)$.

Then, $\sum_{(x,n)} |x|^{\alpha\wedge 2+\delta}\pi_p^{(N)}(x,n)m^n$ will be bounded by the product of diagram functions (cf., Lemma 3 below). Those diagram functions are the "triangle" $T_{p,m}$, which is independent of δ_1 and δ_2, its weighted version $T'_{p,m}(\delta_1)$ and the weighted "bubbles" $W'_{p,m}(\delta_2)$ and $W''_{p,m}(\delta_1,\delta_2)$ (cf., (4.4)–(4.7) below). As shown in Sect. 4.3, it is not hard to bound $W'_{p,m}(\delta_2)$ uniformly in p and m for $d > 2(\alpha\wedge 2)$ and $L \gg 1$ as long as $\delta_2 < \alpha\wedge 2$. However, to bound $T'_{p,m}(\delta_1)$ and $W''_{p,m}(\delta_1,\delta_2)$ uniformly in p and m, we will have to choose δ_1 to be small depending on how close d is to the upper-critical dimension $2(\alpha\wedge 2)$. As described in Lemma 5 below, we will choose δ_1 less than $d-2(\alpha\wedge 2)$.

To summarize the above, we have

$$0<\delta_1<\alpha\wedge 2\wedge(d-2(\alpha\wedge 2)), \qquad 0<\delta_2<\alpha\wedge 2, \qquad \delta_1+\delta_2=\alpha\wedge 2+\delta.$$

To satisfy all, it suffices to choose δ_1 "slightly" larger than δ and let $\delta_2=\alpha\wedge 2-(\delta_1-\delta)$. This is why we choose $\delta<\alpha\wedge 2\wedge(d-2(\alpha\wedge 2))$ when $\alpha\neq 2$.

4 Proof of Proposition 2

Finally, in this section, we prove Proposition 2. First, in Sect. 4.1, we bound fractional moments for the spatial variable of the expansion coefficients $\pi_p^{(N)}$ in (2.2) in terms of certain diagram functions. In Sect. 4.2, we use an integral representation of a^δ for $a>0$ and $\delta\in(0,2)$, which is the key to the proof of Proposition 2. In Sect. 4.3, we show that the aforementioned diagram functions are convergent, and complete the proof of Proposition 2.

4.1 Diagrammatic bounds on the expansion coefficients

In this subsection, we bound $\sum_{(x,n)} |x|^r|\pi_p(x,n)|\,m^n$ for $r>0$ in terms of the diagram functions $T_{p,m}$, $T'_{p,m}$, $W'_{p,m}$ and $W''_{p,m}$ defined in Lemma 3 below.

First, we show the following elementary inequality:

Lemma 1 *For any $r>0$ and $m\geq 0$,*

$$\sum_{(x,n)} |x|^r|\pi_p(x,n)|\,m^n \leq d^{\frac{r}{2}+1}\sum_{N=0}^{\infty}\sum_{(x,n)} |x_1|^r\pi_p^{(N)}(x,n)\,m^n,$$

where x_1 is the first coordinate of $x\equiv(x_1,\dots,x_d)$.

Proof For any $r>0$, we have

$$|x|^r=\Bigg(\sum_{j=1}^d |x_j|^2\Bigg)^{r/2}\leq\Bigg(\sum_{j=1}^d \|x\|_r^2\Bigg)^{r/2}=d^{r/2}\|x\|_r^r\equiv d^{r/2}\sum_{j=1}^d |x_j|^r.$$

By this inequality and using the nonnegativity and the spatial symmetry of $\pi_p^{(N)}$, we obtain

$$\begin{aligned}\sum_{(x,n)}|x|^r|\pi_p(x,n)|\,m^n &\le d^{r/2}\sum_{j=1}^{d}\sum_{(x,n)}|x_j|^r\left|\sum_{N=0}^{\infty}(-1)^N\pi_p^{(N)}(x,n)\right|m^n\\ &\le d^{r/2}\sum_{j=1}^{d}\sum_{N=0}^{\infty}\sum_{(x,n)}|x_j|^r\pi_p^{(N)}(x,n)\,m^n\\ &= d^{\frac{r}{2}+1}\sum_{N=0}^{\infty}\sum_{(x,n)}|x_1|^r\pi_p^{(N)}(x,n)\,m^n,\end{aligned}$$

as required. □

Next, we use [9, Lemma 1] to investigate $\sum_{(x,n)}|x_1|^r\pi_p^{(N)}(x,n)\,m^n$. For notational convenience, we denote vertices in $\mathbb{Z}^{d+1}$ by bold letters, e.g., $\mathbf{o}\equiv(o,0)$ and $\mathbf{x}=(x,t_{\mathbf{x}})$, where $t_{\mathbf{x}}$ is the temporal part of $\mathbf{x}$. Let

$$\psi_p(\mathbf{x})=(q_p*\varphi_p)(\mathbf{x}).$$

Given a sequence of vertices $\mathbf{y}_1,\dots,\mathbf{y}_j\in\mathbb{Z}^{d+1}$, we write

$$\vec{\mathbf{y}}_j=\sum_{i=1}^{j}\mathbf{y}_i.$$

For $\mathbf{y}_1,\mathbf{z}_1,\mathbf{y}_2,\mathbf{z}_2,\ldots\in\mathbb{Z}^{d+1}$, we define

$$\Lambda_p(\vec{\mathbf{y}}_{i-1},\vec{\mathbf{z}}_{i-1};\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i)=\psi_p(\mathbf{y}_i)\,\psi_p(\mathbf{z}_i)\,\frac{\varphi_p(\vec{\mathbf{y}}_i-\vec{\mathbf{z}}_i)+\varphi_p(\vec{\mathbf{z}}_i-\vec{\mathbf{y}}_i)}{2^{\delta_{\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i}}},$$

$$\tilde{\Lambda}_p(\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i;\vec{\mathbf{y}}_{i+1},\vec{\mathbf{z}}_{i+1})=\frac{\varphi_p(\vec{\mathbf{y}}_i-\vec{\mathbf{z}}_i)+\varphi_p(\vec{\mathbf{z}}_i-\vec{\mathbf{y}}_i)}{2^{\delta_{\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i}}}\,\psi_p(\mathbf{y}_{i+1})\,\psi_p(\mathbf{z}_{i+1}).$$

Lemma 2 (Equivalent to Lemma 1 [9]) *For $N=0$,*

$$0\le\pi_p^{(0)}(\mathbf{x})-\delta_{\mathbf{x},\mathbf{o}}\le\psi_p(\mathbf{x})^2. \tag{4.1}$$

For $N\ge 1$,

$$\pi_p^{(N)}(\mathbf{x})\le\sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{N+1}\\ \mathbf{z}_1,\dots,\mathbf{z}_{N+1}\\ (\vec{\mathbf{y}}_{N+1}=\vec{\mathbf{z}}_{N+1}=\mathbf{x})\\ (t_{\mathbf{y}_1}\ge t_{\mathbf{z}_1})}}\varphi_p(\mathbf{y}_1)\,\varphi_p(\mathbf{z}_1)\prod_{i=1}^{N}\tilde{\Lambda}_p(\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i;\vec{\mathbf{y}}_{i+1},\vec{\mathbf{z}}_{i+1}), \tag{4.2}$$

and, for any $j\in\{2,\dots,N+1\}$,

$$\pi_p^{(N)}(\mathbf{x}) \le \sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{N+1}\\ \mathbf{z}_1,\dots,\mathbf{z}_{N+1}\\ (\vec{\mathbf{y}}_{N+1}=\vec{\mathbf{z}}_{N+1}=\mathbf{x})}} \varphi_p(\mathbf{y}_1)\,\varphi_p(\mathbf{z}_1)\,\varphi_p(\mathbf{y}_1-\mathbf{z}_1)\left(\prod_{i=2}^{j-1}\Lambda_p(\vec{\mathbf{y}}_{i-1},\vec{\mathbf{z}}_{i-1};\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i)\right)$$

$$\times\,\psi_p(\mathbf{y}_j)\,\psi_p(\mathbf{z}_j)\left(\prod_{i=j}^{N}\tilde{\Lambda}_p(\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i;\vec{\mathbf{y}}_{i+1},\vec{\mathbf{z}}_{i+1})\right), \tag{4.3}$$

where an empty product is regarded as 1 (Fig. 1).

For further notational convenience, we let

$$\begin{aligned}
\varphi_p^{(m)}(\mathbf{x}) &= \varphi_p(\mathbf{x})\,m^{t_{\mathbf{x}}},\\
\psi_p^{(m)}(\mathbf{x}) &= \psi_p(\mathbf{x})\,m^{t_{\mathbf{x}}},\\
\Lambda_p^{(m)}(\vec{\mathbf{y}}_{i-1},\vec{\mathbf{z}}_{i-1};\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i) &= \Lambda_p(\vec{\mathbf{y}}_{i-1},\vec{\mathbf{z}}_{i-1};\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i)\,m^{t_{\mathbf{z}_i}},\\
\tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i;\vec{\mathbf{y}}_{i+1},\vec{\mathbf{z}}_{i+1}) &= \tilde{\Lambda}_p(\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i;\vec{\mathbf{y}}_{i+1},\vec{\mathbf{z}}_{i+1})\,m^{t_{\mathbf{z}_{i+1}}}.
\end{aligned}$$

Given arbitrary $\delta_1,\delta_2>0$, we define $T_{p,m}$, $T'_{p,m}\equiv T'_{p,m}(\delta_1)$, $W'_{p,m}\equiv W'_{p,m}(\delta_2)$ and $W''_{p,m}\equiv W''_{p,m}(\delta_1,\delta_2)$ as

$$T_{p,m} = \sup_{\mathbf{x}}\sum_{\mathbf{y}}\psi_p(\mathbf{y})\left((\psi_p^{(m)}*\varphi_p)(\mathbf{y}-\mathbf{x})+\sum_{\mathbf{z}}\varphi_p(\mathbf{z}-\mathbf{y})\,\psi_p^{(m)}(\mathbf{z}-\mathbf{x})\right), \tag{4.4}$$

$$T'_{p,m} = \sup_{\mathbf{x}}\sum_{\mathbf{y}}|y_1|^{\delta_1}\psi_p(\mathbf{y})\left((\psi_p^{(m)}*\varphi_p)(\mathbf{y}-\mathbf{x})+\sum_{\mathbf{z}}\varphi_p(\mathbf{z}-\mathbf{y})\,\psi_p^{(m)}(\mathbf{z}-\mathbf{x})\right), \tag{4.5}$$

$$W'_{p,m} = \sup_{\mathbf{x}}\sum_{\mathbf{y}}\psi_p(\mathbf{y})\,|y_1-x_1|^{\delta_2}\psi_p^{(m)}(\mathbf{y}-\mathbf{x}), \tag{4.6}$$

$$W''_{p,m} = \sup_{\mathbf{x}}\sum_{\mathbf{y}}|y_1|^{\delta_1}\psi_p(\mathbf{y})\,|y_1-x_1|^{\delta_2}\psi_p^{(m)}(\mathbf{y}-\mathbf{x}). \tag{4.7}$$

Using the above diagram functions and Lemma 2, we obtain the following:

Lemma 3 *For any $N\ge 0$ and $m\ge 0$,*

$$\begin{aligned}
&\sum_{(x,n)}|x_1|^{\delta_1+\delta_2}\pi_p^{(N)}(x,n)\,m^n\\
&\le (N+1)^{\delta_1+\delta_2}(T_{p,m})^{N-2}\Big(\big(N(1+T_{p,m})+T_{p,m}\big)T_{p,m}W''_{p,m}\\
&\quad +N\left((N-1)(1+T_{p,m})+3T_{p,m}\right)T'_{p,m}W'_{p,m}\Big).
\end{aligned} \tag{4.8}$$

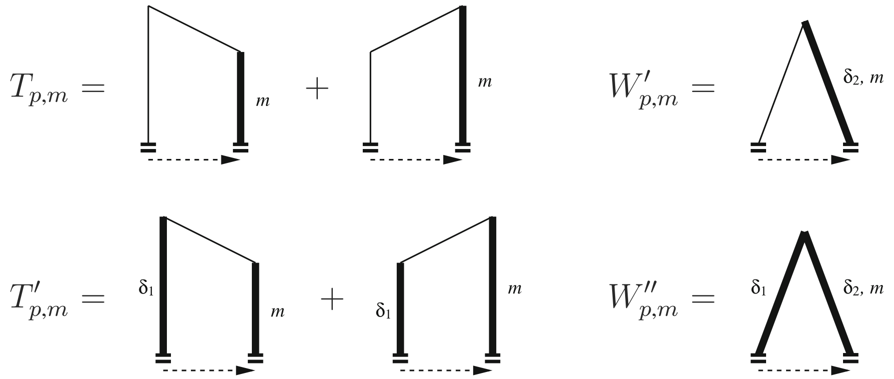

Fig. 1 Schematic representations of the diagram functions. Each pair of horizontal short line segments represents q_p, and the other longer line segments represent φ_p. A bold line segment representing $\varphi_p(x, n)$ is weighted by the factor m^n if the line segment is indexed by m, and by the factor $|x_1|^\delta$ if the line segment is indexed by δ. A *dashed arrow* represents the supremum over its terminal point $\mathbf{x} \in \mathbb{Z}^{d+1}$, with its initial point fixed at the origin $\mathbf{o}$

Proof First of all, by (4.1), we immediately obtain

$$\sum_{(x,n)} |x_1|^{\delta_1+\delta_2} \pi_p^{(0)}(x, n)\, m^n \le \sum_{(x,n)} |x_1|^{\delta_1} \psi_p(x, n)\, |x_1|^{\delta_2} \psi_p^{(m)}(x, n) \le W''_{p,m},$$

as required.

Let $N \ge 1$. We denote the first coordinate of the spatial part of $\mathbf{y}_i$ by $y_{i,1}$: $\mathbf{y}_i = ((y_{i,1}, \ldots, y_{i,d}), t_{\mathbf{y}_i})$. Similarly, we write, e.g., $\vec{\mathbf{y}}_i = ((\vec{y}_{i,1}, \ldots, \vec{y}_{i,d}), t_{\vec{\mathbf{y}}_i})$. Notice that, since

$$|\vec{y}_{N+1,1}|^{\delta_1} = \left| \sum_{j=1}^{N+1} y_{j,1} \right|^{\delta_1} \le (N+1)^{\delta_1} \max_j |y_{j,1}|^{\delta_1} \le (N+1)^{\delta_1} \sum_{j=1}^{N+1} |y_{j,1}|^{\delta_1},$$

we have that, for $\vec{\mathbf{y}}_{N+1} = \vec{\mathbf{z}}_{N+1} = \mathbf{x}$,

$$|x_1|^{\delta_1+\delta_2} = |\vec{y}_{N+1,1}|^{\delta_1} |\vec{z}_{N+1,1}|^{\delta_2} \le (N+1)^{\delta_1+\delta_2} \sum_{j,j'=1}^{N+1} |y_{j,1}|^{\delta_1} |z_{j',1}|^{\delta_2}.$$

By this inequality and using (4.2)–(4.3), we obtain

$$\sum_{(x,n)} |x_1|^{\delta_1+\delta_2} \pi_p^{(N)}(x, n)\, m^n \le (N+1)^{\delta_1+\delta_2} \sum_{j'=1}^{N+1} S_{j'}, \tag{4.9}$$

where

$$S_1 = \sum_{j=1}^{N+1} \sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{N+1} \\ \mathbf{z}_1,\dots,\mathbf{z}_{N+1} \\ (\vec{\mathbf{y}}_{N+1}=\vec{\mathbf{z}}_{N+1}) \\ (t_{\mathbf{y}_1} \ge t_{\mathbf{z}_1})}} |y_{j,1}|^{\delta_1} \varphi_p(\mathbf{y}_1)\, |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \prod_{i=1}^{N} \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}),$$

and, for $j' > 1$,

$$\begin{aligned} S_{j'} = \sum_{j=1}^{N+1} \sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{N+1} \\ \mathbf{z}_1,\dots,\mathbf{z}_{N+1} \\ (\vec{\mathbf{y}}_{N+1}=\vec{\mathbf{z}}_{N+1})}} & |y_{j,1}|^{\delta_1} \varphi_p(\mathbf{y}_1)\, \varphi_p^{(m)}(\mathbf{z}_1)\, \varphi_p(\mathbf{y}_1 - \mathbf{z}_1) \\ & \times \left(\prod_{i=2}^{j'-1} \Lambda_p^{(m)}(\vec{\mathbf{y}}_{i-1}, \vec{\mathbf{z}}_{i-1}; \vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i) \right) \psi_p(\mathbf{y}_{j'})\, |z_{j',1}|^{\delta_2} \psi_p^{(m)}(\mathbf{z}_{j'}) \\ & \times \left(\prod_{i=j'}^{N} \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) \right). \end{aligned}$$

It remains to estimate each $S_{j'}$. To do so, we follow the same line of argument in [9, Sect. 2]. Here, we explain in detail how to estimate S_1. First we note that, by translation-invariance,

$$\sup_{\mathbf{y}} \sum_{\mathbf{w},\mathbf{x}} \tilde{\Lambda}_p^{(m)}(\mathbf{o}, \mathbf{w}; \mathbf{x}, \mathbf{x}+\mathbf{y}) \le T_{p,m}, \tag{4.10}$$

$$\sup_{\mathbf{y}} \sum_{\mathbf{w},\mathbf{x}} |y_1|^{\delta_1} \tilde{\Lambda}_p^{(m)}(\mathbf{o}, \mathbf{w}; \mathbf{x}, \mathbf{x}+\mathbf{y}) \le T'_{p,m}. \tag{4.11}$$

Then, by repeated use of translation-invariance, the contribution to S_1 from $j = 1$ is bounded as

$$\begin{aligned} & \sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{N+1} \\ \mathbf{z}_1,\dots,\mathbf{z}_{N+1} \\ (\vec{\mathbf{y}}_{N+1}=\vec{\mathbf{z}}_{N+1}) \\ (t_{\mathbf{y}_1} \ge t_{\mathbf{z}_1})}} |y_{1,1}|^{\delta_1} \varphi_p(\mathbf{y}_1)\, |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \prod_{i=1}^{N} \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) \\ & = \sum_{\mathbf{w},\mathbf{x}} \tilde{\Lambda}_p^{(m)}(\mathbf{o}, \mathbf{w}; \mathbf{x}, \mathbf{x}) \sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{N} \\ \mathbf{z}_1,\dots,\mathbf{z}_{N} \\ (\vec{\mathbf{z}}_{N}=\vec{\mathbf{y}}_{N}+\mathbf{w}) \\ (t_{\mathbf{y}_1} \ge t_{\mathbf{z}_1})}} |y_{1,1}|^{\delta_1} \varphi_p(\mathbf{y}_1)\, |z_{1,1}|^{\delta_2} \varphi_p^{(m)}(\mathbf{z}_1) \\ & \qquad \times \prod_{i=1}^{N-1} \tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i; \vec{\mathbf{y}}_{i+1}, \vec{\mathbf{z}}_{i+1}) \end{aligned}$$

$$
\begin{aligned}
&\le T_{p,m} \sup_{\mathbf{w}} \sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_N\\ \mathbf{z}_1,\dots,\mathbf{z}_N\\ (\vec{\mathbf{z}}_N=\vec{\mathbf{y}}_N+\mathbf{w})\\ (t_{\mathbf{y}_1}\ge t_{\mathbf{z}_1})}} |y_{1,1}|^{\delta_1}\varphi_p(\mathbf{y}_1)\,|z_{1,1}|^{\delta_2}\varphi_p^{(m)}(\mathbf{z}_1)\prod_{i=1}^{N-1}\tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i;\vec{\mathbf{y}}_{i+1},\vec{\mathbf{z}}_{i+1})\\
&\le (T_{p,m})^2 \sup_{\mathbf{w}} \sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{N-1}\\ \mathbf{z}_1,\dots,\mathbf{z}_{N-1}\\ (\vec{\mathbf{z}}_{N-1}=\vec{\mathbf{y}}_{N-1}+\mathbf{w})\\ (t_{\mathbf{y}_1}\ge t_{\mathbf{z}_1})}} |y_{1,1}|^{\delta_1}\varphi_p(\mathbf{y}_1)\,|z_{1,1}|^{\delta_2}\varphi_p^{(m)}(\mathbf{z}_1)\\
&\qquad\qquad\times\prod_{i=1}^{N-2}\tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i;\vec{\mathbf{y}}_{i+1},\vec{\mathbf{z}}_{i+1})\\
&\ \ \vdots\\
&\le (T_{p,m})^N \sup_{\mathbf{w}:t_{\mathbf{w}}\ge 0} \sum_{\substack{\mathbf{y}_1,\mathbf{z}_1\\ (\mathbf{y}_1=\mathbf{z}_1+\mathbf{w})}} |y_{1,1}|^{\delta_1}\varphi_p(\mathbf{y}_1)\,|z_{1,1}|^{\delta_2}\varphi_p^{(m)}(\mathbf{z}_1)\\
&\le (T_{p,m})^N W''_{p,m},
\end{aligned}
\tag{4.12}
$$

where we have used $\varphi_p(\mathbf{x}) \le \delta_{\mathbf{x},\mathbf{o}} + \psi_p(\mathbf{x})$. Similarly, the contribution to S_1 from $j > 1$ is bounded as

$$
\begin{aligned}
&\sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{N+1}\\ \mathbf{z}_1,\dots,\mathbf{z}_{N+1}\\ (\vec{\mathbf{y}}_{N+1}=\vec{\mathbf{z}}_{N+1})\\ (t_{\mathbf{y}_1}\ge t_{\mathbf{z}_1})}} |y_{j,1}|^{\delta_1}\varphi_p(\mathbf{y}_1)\,|z_{1,1}|^{\delta_2}\varphi_p^{(m)}(\mathbf{z}_1)\prod_{i=1}^{N}\tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i;\vec{\mathbf{y}}_{i+1},\vec{\mathbf{z}}_{i+1})\\
&\le (T_{p,m})^{N-1}T'_{p,m} \sup_{\mathbf{w}:t_{\mathbf{w}\ge 0}} \sum_{\substack{\mathbf{y}_1,\mathbf{z}_1\\ (\mathbf{y}_1=\mathbf{z}_1+\mathbf{w})}} \varphi_p(\mathbf{y}_1)\,|z_{1,1}|^{\delta_2}\varphi_p^{(m)}(\mathbf{z}_1)\\
&\le (T_{p,m})^{N-1}T'_{p,m}W'_{p,m}.
\end{aligned}
$$

Therefore,

$$
S_1 \le N(T_{p,m})^{N-1}T'_{p,m}W'_{p,m} + (T_{p,m})^N W''_{p,m}. \tag{4.13}
$$

To estimate $S_{j'}$ for $j' > 1$, we first use (4.10)–(4.11). For example, the contribution from $j = j'$ is bounded, similarly to (4.12), as

$$
\begin{aligned}
&\sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{N+1}\\ \mathbf{z}_1,\dots,\mathbf{z}_{N+1}\\ (\vec{\mathbf{y}}_{N+1}=\vec{\mathbf{z}}_{N+1})}} \varphi_p(\mathbf{y}_1)\,\varphi_p^{(m)}(\mathbf{z}_1)\,\varphi_p(\mathbf{y}_1-\mathbf{z}_1)\left(\prod_{i=2}^{j'-1}\Lambda_p^{(m)}(\vec{\mathbf{y}}_{i-1},\vec{\mathbf{z}}_{i-1};\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i)\right)\\
&\qquad\times|y_{j',1}|^{\delta_1}\psi_p(\mathbf{y}_{j'})\,|z_{j',1}|^{\delta_2}\psi_p^{(m)}(\mathbf{z}_{j'})\left(\prod_{i=j'}^{N}\tilde{\Lambda}_p^{(m)}(\vec{\mathbf{y}}_i,\vec{\mathbf{z}}_i;\vec{\mathbf{y}}_{i+1},\vec{\mathbf{z}}_{i+1})\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq (T_{p,m})^{N+1-j'} \sup_{\mathbf{x}} \sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{j'} \\ \mathbf{z}_1,\dots,\mathbf{z}_{j'} \\ (\vec{\mathbf{z}}_{j'}=\vec{\mathbf{y}}_{j'}+\mathbf{x})}} \varphi_p(\mathbf{y}_1)\, \varphi_p^{(m)}(\mathbf{z}_1)\, \varphi_p(\mathbf{y}_1-\mathbf{z}_1) \\
&\qquad \times \left(\prod_{i=2}^{j'-1} \Lambda_p^{(m)}(\vec{\mathbf{y}}_{i-1}, \vec{\mathbf{z}}_{i-1}; \vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i) \right) |y_{j',1}|^{\delta_1} \psi_p(\mathbf{y}_{j'})\, |z_{j',1}|^{\delta_2} \psi_p^{(m)}(\mathbf{z}_{j'}). \\
&\leq (T_{p,m})^{N+1-j'} W''_{p,m} \sup_{\mathbf{x}} \sum_{\substack{\mathbf{y}_1,\dots,\mathbf{y}_{j'-1} \\ \mathbf{z}_1,\dots,\mathbf{z}_{j'-1} \\ (\vec{\mathbf{z}}_{j'-1}=\vec{\mathbf{y}}_{j'-1}+\mathbf{x})}} \varphi_p(\mathbf{y}_1)\, \varphi_p^{(m)}(\mathbf{z}_1)\, \varphi_p(\mathbf{y}_1-\mathbf{z}_1) \\
&\qquad \times \left(\prod_{i=2}^{j'-1} \Lambda_p^{(m)}(\vec{\mathbf{y}}_{i-1}, \vec{\mathbf{z}}_{i-1}; \vec{\mathbf{y}}_i, \vec{\mathbf{z}}_i) \right).
\end{aligned}
\tag{4.14}
$$

Notice that

$$
\sup_{\mathbf{x}} \sum_{\mathbf{y},\mathbf{z}} \Lambda_p^{(m)}(\mathbf{o},\mathbf{x};\mathbf{y},\mathbf{z}) \leq T_{p,m}, \qquad \sup_{\mathbf{x}} \sum_{\mathbf{y},\mathbf{z}} |y_1|^{\delta_1} \Lambda_p^{(m)}(\mathbf{o},\mathbf{x};\mathbf{y},\mathbf{z}) \leq T'_{p,m}.
$$

By repeated use of translation-invariance, we obtain

$$
(4.14) \leq (1+T_{p,m})(T_{p,m})^{N-1} W''_{p,m}.
$$

It is not hard to see that the contribution from j not being either j' or 1, which is possible only if $N \geq 2$, is bounded by $(1+T_{p,m})(T_{p,m})^{N-2} T'_{p,m} W'_{p,m}$, and the contribution from $j=1$ is bounded by $2(T_{p,m})^{N-1} T'_{p,m} W'_{p,m}$. Therefore, for $j'>1$,

$$
\begin{aligned}
S_{j'} \leq{} & \big((N-1)(1+T_{p,m}) + 2T_{p,m}\big)\, (T_{p,m})^{N-2} T'_{p,m} W'_{p,m} \\
& + (1+T_{p,m})(T_{p,m})^{N-1} W''_{p,m}.
\end{aligned}
\tag{4.15}
$$

The proof of (4.8) is completed by assembling (4.9), (4.13) and (4.15). □

4.2 Integral representation of fractional-power functions

In this subsection, we use an integral representation of a^δ for $a>0$ and $\delta \in (0,2)$ to bound the diagram functions $T'_{p,m}$, $W'_{p,m}$ and $W''_{p,m}$.

First we note that, for $\delta \in (0,2)$,

$$
K_\delta = \int_0^\infty \frac{1-\cos t}{t^{1+\delta}}\, \mathrm{d}t
$$

is a positive finite constant. Replacing t by $u = t/a$ with $a > 0$, we obtain

$$a^\delta = \frac{1}{K_\delta}\int_0^\infty \frac{1-\cos(ua)}{u^{1+\delta}}\,\mathrm{d}u \le \frac{1}{K_\delta}\left(\frac{2}{\delta} + \int_0^1 \frac{1-\cos(ua)}{u^{1+\delta}}\,\mathrm{d}u\right), \tag{4.16}$$

which is the key inequality.

To describe bounds on $T'_{p,m}$, $W'_{p,m}$ and $W''_{p,m}$ below, we define

$$\hat{Y}_k(l,z) = |\Delta_k \hat{D}(l)|\,|\hat{\varphi}_p(l,z)| + |\Delta_k \hat{\varphi}_p(l,z)|,$$

and, by denoting $\vec{u} = (u, 0, \ldots, 0) \in [-\pi,\pi]^d$,

$$\hat{I}_1(u) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi}\, \hat{Y}_{\vec{u}}(l, e^{i\theta})\,|\hat{\varphi}_p(l,e^{i\theta})|\,|\hat{\varphi}_p(l, me^{i\theta})|, \tag{4.17}$$

$$\hat{I}_2(v) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi}\, |\hat{\varphi}_p(l,e^{i\theta})|\,\hat{Y}_{\vec{v}}(l, me^{i\theta}), \tag{4.18}$$

$$\hat{I}_3(u) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi}\, \hat{Y}_{\vec{u}}(l, e^{i\theta})\,|\hat{\varphi}_p(l, me^{i\theta})|, \tag{4.19}$$

$$\hat{I}_4(u,v) = \int_{[-\pi,\pi]^d} \frac{\mathrm{d}^d l}{(2\pi)^d} \int_{-\pi}^{\pi} \frac{\mathrm{d}\theta}{2\pi}\, \hat{Y}_{\vec{u}}(l, e^{i\theta})\,\hat{Y}_{\vec{v}}(l, me^{i\theta}). \tag{4.20}$$

Taking the Fourier–Laplace transform of (4.5)–(4.7) (also recalling (2.1)) and using (4.16), we obtain the following:

Lemma 4 *For any $p \in (0, p_c)$ and $m \in [0, m_p)$,*

$$T'_{p,m} \le \frac{1}{K_{\delta_1}}\left(\frac{2}{\delta_1}T_{p,m} + 5p^2 m \int_0^1 \frac{\mathrm{d}u}{u^{1+\delta_1}}\,\hat{I}_1(u)\right), \tag{4.21}$$

$$W'_{p,m} \le \frac{1}{K_{\delta_2}}\left(\frac{1}{\delta_2}T_{p,m} + \frac{5p^2 m}{2} \int_0^1 \frac{\mathrm{d}v}{v^{1+\delta_2}}\,\hat{I}_2(v)\right), \tag{4.22}$$

$$\begin{aligned} W''_{p,m} \le \frac{1}{K_{\delta_1}}\Bigg(&\frac{2}{\delta_1}W'_{p,m} + \frac{5p^2 m}{K_{\delta_2}\delta_2}\int_0^1 \frac{\mathrm{d}u}{u^{1+\delta_1}}\,\hat{I}_3(u) \\ &+ \frac{25p^2 m}{4K_{\delta_2}}\int_0^1 \frac{\mathrm{d}u}{u^{1+\delta_1}}\int_0^1 \frac{\mathrm{d}v}{v^{1+\delta_2}}\,\hat{I}_4(u,v)\Bigg). \end{aligned} \tag{4.23}$$

Proof We only prove (4.22), since the other two inequalities can be proved in the same way.

First we use (4.16) to bound $|y_1 - x_1|^{\delta_2}$ in (4.6). The first term in (4.22) is due to the first term in (4.16) and the trivial inequality

$$\sum_{\mathbf{y}} \psi_p(\mathbf{y})\, \psi_p^{(m)}(\mathbf{y}-\mathbf{x}) \le \frac{1}{2} T_{p,m}.$$

To complete the proof of (4.22), it thus remains to show

$$\sum_{\mathbf{y}} \psi_p(\mathbf{y}+\mathbf{x})\ (1-\cos(v y_1))\, \psi_p^{(m)}(\mathbf{y}) \le \frac{5p^2 m}{2} \hat{I}_2(v). \tag{4.24}$$

However, since $1-\cos\sum_{j=1}^J t_j \le (2J+1)\sum_{j=1}^J (1-\cos t_j)$ (cf., [11, (4.50)]), we have

$$\begin{aligned}(1-\cos(v y_1))\, \psi_p(\mathbf{y}) &\equiv (1-\cos(v y_1))\, (q_p * \varphi_p)(\mathbf{y}) \\ &\le 5p \sum_{\mathbf{w}} (1-\cos(v w_1)) \big(D(w)\, \varphi_p(\mathbf{y}-\mathbf{w}) + \varphi_p(\mathbf{w})\, D(y-w)\big).\end{aligned}$$

Applying this to the left-hand side of (4.24), then taking the Fourier–Laplace transform and using $|\hat{\varphi}_p(l, e^{i\theta})| = |\hat{\varphi}_p(l, e^{-i\theta})|$, we obtain (4.24). This completes the proof of (4.22). □

4.3 Bounds on the diagram functions

In this subsection, we complete the proof of Proposition 2 using the following lemma:

Lemma 5 *Let $\alpha > 0$ and $d > 2(\alpha \wedge 2)$, and choose δ as in (3.1) and $\delta_1, \delta_2 \in (0, 2)$ as*

$$\delta < \delta_1 < \alpha \wedge 2 \wedge (d - 2(\alpha \wedge 2)), \quad \delta_2 = \alpha \wedge 2 + \delta - \delta_1.$$

Then,

$$T_{p,m} = O(\lambda), \qquad \left.\begin{matrix} T'_{p,m} \\ W'_{p,m} \\ W''_{p,m} \end{matrix}\right\} = O(1), \tag{4.25}$$

uniformly in $p \in (0, p_c)$ and $m \in [0, m_p)$.

Proof of Proposition 2 First, by Lemmas 1 and 3 with $r = \delta_1 + \delta_2 \equiv \alpha \wedge 2 + \delta < 4$, we obtain that, for any $p \in (0, p_c]$,

$$\sum_{(x,n)} |x|^{\alpha\wedge 2+\delta} |\pi_p(x,n)|\, m_p^n \\ \le d^3 \sum_{N=0}^{\infty} (N+1)^{\delta_1+\delta_2} (T_{p,m_p})^{N-2} \Big(\big(N(1+T_{p,m_p}) + T_{p,m_p}\big) T_{p,m_p} W''_{p,m_p} \\ + N\left((N-1)(1+T_{p,m_p}) + 3T_{p,m_p}\right) T'_{p,m_p} W'_{p,m_p} \Big). \tag{4.26}$$

Since the diagram functions (4.4)–(4.7) are increasing in $m \ge 0$ for every $p \ge 0$ and in $p \ge 0$ for every $m \ge 0$, the uniform bounds in (4.25) imply that these diagram functions at $m = m_p$ obey the same bounds uniformly in $p \in (0, p_c]$. Therefore, the right-hand side of (4.26) is convergent, if λ is sufficiently small. This completes the proof of Proposition 2. □

Proof of Lemma 5 It is not hard to extend [1, Lemma 4.1] to show that $T_{p,m} = O(\lambda)$ uniformly in $p \in (0, p_c)$ and $m \in [0, m_p)$. Recall Lemma 4. To complete the proof of Lemma 5, it thus suffices to show that the integrals in (4.21)–(4.23) of $\hat{I}_1, \ldots, \hat{I}_4$ are bounded uniformly in $p \in (0, p_c)$ and $m \in [0, m_p)$.

The integrals of $\hat{I}_2$ and $\hat{I}_3$ are easy and can be estimated similarly. For example, by (2.6)–(2.7) and $|\Delta_{\vec{v}} \hat{D}(l)| \le 2\big(1 - \hat{D}(\vec{v})\big)$ (cf., (2.1)),

$$\hat{I}_2(v) = \int \frac{d^d l}{(2\pi)^d} \int \frac{d\theta}{2\pi}\, |\hat{\varphi}_p(l, e^{i\theta})| \Big(|\Delta_{\vec{v}} \hat{D}(l)|\, |\hat{\varphi}_p(l, me^{i\theta})| + |\Delta_{\vec{v}} \hat{\varphi}_p(l, me^{i\theta})| \Big) \\ \le O(1 - \hat{D}(\vec{v})) \int \frac{d^d l}{(2\pi)^d} \int \frac{d\theta}{2\pi} \frac{1}{|\theta| + 1 - \hat{D}(l)} \Bigg(\frac{1}{|\theta| + 1 - \hat{D}(l)} \\ + \sum_{(j,j')=(0,\pm1),(1,-1)} \frac{1}{(|\theta| + 1 - \hat{D}(l + j\vec{v}))(|\theta| + 1 - \hat{D}(l + j'\vec{v}))} \Bigg)$$

holds uniformly in $p \in (0, p_c)$ and $m \in [0, m_p)$. Using the Hölder inequality twice and the translation-invariance of D, we have

$$\int \frac{d\theta}{2\pi} \frac{1}{|\theta| + 1 - \hat{D}(l)} \frac{1}{(|\theta| + 1 - \hat{D}(l + j\vec{v}))(|\theta| + 1 - \hat{D}(l + j'\vec{v}))} \\ \le \left(\int \frac{d\theta}{2\pi} \frac{1}{|\theta| + 1 - \hat{D}(l)} \left(\frac{1}{|\theta| + 1 - \hat{D}(l + j\vec{v})} \right)^2 \right)^{1/2} \\ \times \left(\int \frac{d\theta}{2\pi} \frac{1}{|\theta| + 1 - \hat{D}(l)} \left(\frac{1}{|\theta| + 1 - \hat{D}(l + j'\vec{v})} \right)^2 \right)^{1/2}$$

$$
\begin{aligned}
&\le \left(\int \frac{d\theta}{2\pi}\left(\frac{1}{|\theta|+1-\hat{D}(l)}\right)^3\right)^{1/6}\left(\int \frac{d\theta}{2\pi}\left(\frac{1}{|\theta|+1-\hat{D}(l+j\vec{v})}\right)^3\right)^{1/3}\\
&\quad\times\left(\int \frac{d\theta}{2\pi}\left(\frac{1}{|\theta|+1-\hat{D}(l)}\right)^3\right)^{1/6}\left(\int \frac{d\theta}{2\pi}\left(\frac{1}{|\theta|+1-\hat{D}(l+j'\vec{v})}\right)^3\right)^{1/3}\\
&= \int \frac{d\theta}{2\pi}\left(\frac{1}{|\theta|+1-\hat{D}(l)}\right)^3.
\end{aligned}
$$

Since, by (1.1),

$$
\int \frac{d^d l}{(2\pi)^d}\int \frac{d\theta}{2\pi}\left(\frac{1}{|\theta|+1-\hat{D}(l)}\right)^3 \le \int \frac{d^d l}{(2\pi)^d}\,\frac{O(1)}{(1-\hat{D}(l))^2} < \infty
$$

holds for $d > 2(\alpha \wedge 2)$, we conclude that, for $\delta_2 = \alpha \wedge 2 - (\delta_1 - \delta) < \alpha \wedge 2$,

$$
\int_0^1 \frac{dv}{v^{1+\delta_2}}\,\hat{I}_2(v) \le \int_0^1 \frac{dv}{v^{1+\delta_2}}\,O\big(1-\hat{D}(\vec{v})\big) < \infty,
$$

as required.

Next, we consider the integral of $\hat{I}_1$. In fact, we only need consider the contribution from $|\Delta_{\vec{u}}\hat{\varphi}_p(l, e^{i\theta})|$ in $\hat{Y}_{\vec{u}}(l, e^{i\theta})$ of (4.17), because the contribution from the other term in $\hat{Y}_{\vec{u}}(l, e^{i\theta})$ can be estimated similarly to the integral of $\hat{I}_2$, as explained above. Using (2.6)–(2.7) and ignoring some factors of $|\theta|$, we obtain

$$
\begin{aligned}
&\int \frac{d^d l}{(2\pi)^d}\int \frac{d\theta}{2\pi}\,|\Delta_{\vec{u}}\hat{\varphi}_p(l, e^{i\theta})|\,|\hat{\varphi}_p(l, e^{i\theta})|\,|\hat{\varphi}_p(l, me^{i\theta})|\\
&\le \sum_{(j,j')}\int \frac{d^d l}{(2\pi)^d}\,\frac{O(1-\hat{D}(\vec{u}))}{(1-\hat{D}(l+j\vec{u}))(1-\hat{D}(l+j'\vec{u}))}\int \frac{d\theta}{2\pi}\left(\frac{1}{|\theta|+1-\hat{D}(l)}\right)^2\\
&\le O\big(1-\hat{D}(\vec{u})\big)\sum_{(j,j')}\int \frac{d^d l}{(2\pi)^d}\,\frac{1}{(1-\hat{D}(l+j\vec{u}))(1-\hat{D}(l+j'\vec{u}))(1-\hat{D}(l))},
\end{aligned}
\tag{4.27}
$$

where $\sum_{(j,j')}$ is the sum over $(j, j') = (0, \pm 1), (1, -1)$. By the translation-invariance and $\mathbb{Z}^d$-symmetry of D, the integral for $(j, j') = (0, \pm 1)$ equals

$$
\hat{J}(u) = \int_{[-\pi,\pi]^d} \frac{d^d l}{(2\pi)^d}\,\frac{1}{(1-\hat{D}(l))^2(1-\hat{D}(l-\vec{u}))}. \tag{4.28}
$$

Moreover, by the Schwarz inequality, the integral for $(j, j') = (1, -1)$ is bounded by

$$\left(\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(1-\hat{D}(l+\vec{u}))^2(1-\hat{D}(l))}\right)^{1/2} \times \left(\int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(1-\hat{D}(l-\vec{u}))^2(1-\hat{D}(l))}\right)^{1/2} = \hat{J}(u).$$

Therefore,

$$(4.27) \le O\big(1-\hat{D}(\vec{u})\big)\hat{J}(u). \tag{4.29}$$

Now we show

$$\hat{J}(u) \le O\big(u^{(d-3(\alpha\wedge 2))\wedge 0}\big), \tag{4.30}$$

which is sufficient for the integral of $\hat{I}_1$ to be convergent for $\delta_1 < \alpha\wedge 2\wedge(d-2(\alpha\wedge 2))$. Let

$$R_1 = \left\{l \in [-\pi,\pi]^d : |l| \ge \tfrac{3}{2}u\right\}, \tag{4.31}$$

$$R_2 = \left\{l \in [-\pi,\pi]^d : |l| \le \tfrac{3}{2}u,\ |l| \le |l-\vec{u}|\right\}, \tag{4.32}$$

$$R_3 = \left\{l \in [-\pi,\pi]^d : |l| \le \tfrac{3}{2}u,\ |l-\vec{u}| \le |l|\right\}. \tag{4.33}$$

Notice that $1-\hat{D}(l-\vec{u}) \ge O(|l-\vec{u}|^{\alpha\wedge 2})$ for any $l \in [-\pi,\pi]^d$ and $u \in [0,1]$ (cf., [1, Proposition 1.1]). Since $|l-\vec{u}| \ge |l|-u \ge \frac{1}{3}|l|$ for $l \in R_1$, we have

$$\int_{R_1} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(1-\hat{D}(l))^2(1-\hat{D}(l-\vec{u}))} \le O(1)\int_{R_1} \frac{\mathrm{d}^d l}{|l|^{3(\alpha\wedge 2)}} \le O\left(u^{(d-3(\alpha\wedge 2))\wedge 0}\right).$$

Moreover, since $|l-\vec{u}| \ge \frac{u}{2}$ for $l \in R_2$ and $|l| \ge \frac{u}{2}$ for $l \in R_3$, we have

$$\int_{R_2} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(1-\hat{D}(l))^2(1-\hat{D}(l-\vec{u}))} \le O(u^{-\alpha\wedge 2}) \int_{|l|\le\frac{3}{2}u} \frac{\mathrm{d}^d l}{|l|^{2(\alpha\wedge 2)}} \le O\left(u^{d-3(\alpha\wedge 2)}\right),$$

and

$$\int_{R_3} \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(1-\hat{D}(l))^2(1-\hat{D}(l-\vec{u}))} \le O(u^{-2(\alpha\wedge 2)}) \int_{|l|\le\frac{3}{2}u} \frac{\mathrm{d}^d l}{|l|^{\alpha\wedge 2}} \le O\left(u^{d-3(\alpha\wedge 2)}\right).$$

This completes the proof of (4.30), as required.

Finally, we discuss the integral of $\hat{I}_4$. We only need consider the contribution from $|\Delta_{\vec{u}}\hat{\varphi}_p(l,e^{i\theta})||\Delta_{\vec{v}}\hat{\varphi}_p(l,me^{i\theta})|$ in $\hat{Y}_{\vec{u}}(l,e^{i\theta})\hat{Y}_{\vec{v}}(l,me^{i\theta})$ of (4.20), since the contributions from the other combinations are bounded similarly to the integrals of $\hat{I}_1$, $\hat{I}_2$, $\hat{I}_3$ as long as $d > 2(\alpha\wedge 2)$, $\delta_1 < \alpha\wedge 2\wedge(d-2(\alpha\wedge 2))$ and $\delta_2 < \alpha\wedge 2$. Using (2.7) and ignoring some factors of $|\theta|$, we have

$$\begin{aligned}
&\int\frac{\mathrm{d}^d l}{(2\pi)^d}\int\frac{\mathrm{d}\theta}{2\pi}\,|\Delta_{\vec{u}}\hat{\varphi}_p(l,e^{i\theta})|\,|\Delta_{\vec{v}}\hat{\varphi}_p(l,me^{i\theta})|\\
&\le\sum_{(j_1,j_1'),(j_2,j_2')}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{O(1-\hat{D}(\vec{u}))}{(1-\hat{D}(l+j_1\vec{u}))(1-\hat{D}(l+j_1'\vec{u}))}\\
&\qquad\times\int\frac{\mathrm{d}\theta}{2\pi}\,\frac{1-\hat{D}(\vec{v})}{(|\theta|+1-\hat{D}(l+j_2\vec{v}))(|\theta|+1-\hat{D}(l+j_2'\vec{v}))}.
\end{aligned}\tag{4.34}$$

Notice that

$$\begin{aligned}
&\sum_{(j_2,j_2')}\int\frac{\mathrm{d}\theta}{2\pi}\,\frac{1}{(|\theta|+1-\hat{D}(l+j_2\vec{v}))(|\theta|+1-\hat{D}(l+j_2'\vec{v}))}\\
&\le\sum_{(j_2,j_2')}\frac{1}{1-\hat{D}(l+j_2\vec{v})\vee\hat{D}(l+j_2'\vec{v})}\le\sum_{j=0,\pm1}\frac{2}{1-\hat{D}(l+j\vec{v})}.
\end{aligned}$$

The contribution from $j=0$ is bounded, similarly to (4.29), by $O(1-\hat{D}(\vec{u}))\hat{J}(u)(1-\hat{D}(\vec{v}))$, where $(1-\hat{D}(\vec{u}))\hat{J}(u)/u^{1+\delta_1}$ is integrable if $\delta_1<\alpha\wedge 2\wedge(d-2(\alpha\wedge 2))$ and $(1-\hat{D}(\vec{v}))/v^{1+\delta_2}$ is integrable if $\delta_2<\alpha\wedge 2$ (see around (4.30)). On the other hand, the contribution from $j=\pm1$ is bounded, due to the Schwarz inequality and the $\mathbb{Z}^d$-symmetry and translation-invariance of D, by

$$\begin{aligned}
&\sum_{(j_1,j_1')}\int\frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{O(1-\hat{D}(\vec{u}))}{(1-\hat{D}(l+j_1\vec{u}))(1-\hat{D}(l+j_1'\vec{u}))}\,\frac{1-\hat{D}(\vec{v})}{1-\hat{D}(l+j\vec{v})}\\
&\le\sum_{(j_1,j_1')}\left(\int\frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{O(1-\hat{D}(\vec{u}))^2}{(1-\hat{D}(l+j_1\vec{u}))^2(1-\hat{D}(l+j_1'\vec{u}))}\right)^{1/2}\\
&\qquad\times\left(\int\frac{\mathrm{d}^d l}{(2\pi)^d}\,\frac{(1-\hat{D}(\vec{v}))^2}{(1-\hat{D}(l+j_1'\vec{u}))(1-\hat{D}(l+j\vec{v}))^2}\right)^{1/2}\\
&\le O\big(1-\hat{D}(\vec{u})\big)\big(1-\hat{D}(\vec{v})\big)\sum_{(j_1,j_1')}\hat{J}\left((1-j_1j_1')u\right)^{1/2}\hat{J}(|v-jj_1'u|)^{1/2}\\
&=O\big(1-\hat{D}(\vec{u})\big)\big(1-\hat{D}(\vec{v})\big)\left(\left(\hat{J}(u)^{1/2}+\hat{J}(2u)^{1/2}\right)\hat{J}(|v+ju|)^{1/2}\right.\\
&\qquad\left.+\hat{J}(u)^{1/2}\hat{J}(|v-ju|)^{1/2}\right).
\end{aligned}\tag{4.35}$$

It is not hard to show that $\hat{J}(2u)$ and $\hat{J}(v+u)$ obey the same bound as $\hat{J}(u)$ for $u, v \in [0,1]$. Therefore, the contribution to (4.35) from $\hat{J}(v+u)$ is bounded by $O(1-\hat{D}(\vec{u}))(1-\hat{D}(\vec{v}))\hat{J}(u)$, which divided by $u^{1+\delta_1}v^{1+\delta_2}$ is integrable if $d > 2(\alpha\wedge 2)$, $\delta_1 < \alpha\wedge 2\wedge(d-2(\alpha\wedge 2))$ and $\delta_2 < \alpha\wedge 2$, as explained above. Moreover, since

$$\begin{aligned}
&\int_{\frac{u}{2}}^{\frac{3u}{2}} \frac{\mathrm{d}v}{v^{1+\delta_2}} \left(1-\hat{D}(\vec{v})\right) \hat{J}(|v-u|)^{1/2} \\
&\quad \le \underbrace{\int_0^{\frac{u}{2}} \mathrm{d}r\ \hat{J}(r)^{1/2}}_{O\left(u^{\frac{d-3(\alpha\wedge 2)}{2}\wedge 0+1}\right)} \times \begin{cases} O\left(u^{(\alpha\wedge 2-\delta_2-1)\wedge 0}\right) & (\alpha\ne 2)\\ O\left(u^{(1-\delta_2)\wedge 0}\log\frac{1}{u}\right) & (\alpha=2)\end{cases} \\
&\quad \le O\left(u^{\frac{d-3(\alpha\wedge 2)}{2}\wedge 0}\right),
\end{aligned}$$

and

$$\begin{aligned}
\int_{[0,1]\setminus[\frac{u}{2},\frac{3u}{2}]} \frac{\mathrm{d}v}{v^{1+\delta_2}} \left(1-\hat{D}(\vec{v})\right) \hat{J}(|v-u|)^{1/2} &\le O\left(u^{\frac{d-3(\alpha\wedge 2)}{2}\wedge 0}\right) \int_0^1 \frac{\mathrm{d}v}{v^{1+\delta_2}} \left(1-\hat{D}(\vec{v})\right) \\
&\le O\left(u^{\frac{d-3(\alpha\wedge 2)}{2}\wedge 0}\right),
\end{aligned}$$

we have

$$\int_0^1 \frac{\mathrm{d}v}{v^{1+\delta_2}} \left(1-\hat{D}(\vec{v})\right) \hat{J}(|v-u|)^{1/2} \le O\left(u^{\frac{d-3(\alpha\wedge 2)}{2}\wedge 0}\right), \tag{4.36}$$

i.e., the left-hand side of (4.36) obeys the same bound as $\hat{J}(u)^{1/2}$. Therefore, the contribution to (4.35) from $\hat{J}(|v-u|)$, divided by $u^{1+\delta_1}v^{1+\delta_2}$, is also integrable if $d > 2(\alpha\wedge 2)$, $\delta_1 < \alpha\wedge 2\wedge(d-2(\alpha\wedge 2))$ and $\delta_2 < \alpha\wedge 2$, as required. This completes the proof of Lemma 5. $\square$

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