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Carleson measures and vector-valued BMO martingales

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Abstract We study the relationship between vector-valued BMO martingales and Carleson measures. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $2 \leq q < \infty$. Let X be a Banach space. Given a stopping time τ, let $\widehat{\tau}$ denote the tent over τ:

$$\widehat{\tau} = \{(w, k) \in \Omega \times \mathbb{N} : \tau(w) \leq k, \tau(w) < \infty\}.$$

We prove that there exists a positive constant c such that

$$\sup_{\tau} \frac{1}{P(\tau < \infty)} \int\limits_{\widehat{\tau}} \|df_k\|^q dP \otimes dm \leq c^q \|f\|^q_{BMO(X)}$$

for any finite martingale with values in X iff X admits an equivalent norm which is q-uniformly convex. The validity of the converse inequality is equivalent to the existence of an equivalent p-uniformly smooth norm. And then we also give a characterization of UMD Banach lattices.

Keywords Carleson measures · BMO martingales · Uniformly convex (smooth) spaces

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1 Introduction and preliminaries

This paper deals with vector-valued martingale inequalities. It is well known that the validity of a classical (scalar-valued) result in the vector-valued setting, i.e. for functions or martingales with values in a Banach space X, depends on the geometrical or topological properties of X. For instance, the a.s. convergence of bounded L_p-martingales ($1 < p < \infty$) with values in X amounts to saying that X has the Radon–Nikodym property (see [4]). On the other hand, the validity of a one-sided Burkholder–Gundy inequalities for X-valued martingales is equivalent to the uniform convexity (smoothness) of X (see [12]).

It is also well known that martingale theory is intimately related to harmonic analysis. It was exactly with this in mind that Xu [15] developed the vector-valued Littlewood–Paley theory, which was inspired by Pisier's celebrated work [12] on martingale inequalities in uniformly convex spaces. Very recently, Ouyang and Xu [11] studied the endpoint case of the main results of [10,15] by means of the classical relationship between BMO functions and Carleson measures. Let us recall the main results of [11]. For a cube $I \subset \mathbb{R}^n$ let $\widehat{I}$ denote the tent over I. Let $1 < q < \infty$ and X be a Banach space. Then X has an equivalent norm which is q-uniformly convex iff there exists a positive c such that

$$\sup_{I \text{ cube}} \frac{1}{|I|} \int_{\widehat{I}} (t\|\nabla f(x,t)\|)^q \frac{dxdt}{t} \leq c^q \|f\|^q_{BMO(R^n;X)}, \quad \forall f \in BMO(\mathbb{R}^n; X), \tag{1.1}$$

where f also denotes the Poisson integral of f on $\mathbb{R}^{n+1}_+$, and where

$$\|\nabla f(x,t)\| = \left\| \frac{\partial}{\partial t} f(x,t) \right\| + \sum_{i=1}^{n} \left\| \frac{\partial}{\partial x_i} f(x,t) \right\|.$$

The validity of the converse inequality is equivalent to the existence of an equivalent q-uniformly smooth norm. Inequality (1.1) means that $(t\|\nabla f(x,t)\|)^q \frac{dxdt}{t}$ is a Carleson measure on $\mathbb{R}^{n+1}_+$ for every $f \in BMO(\mathbb{R}^n; X)$.

The main goal of the present paper is to give the martingale version of Ouyang–Xu's results. This can be considered as the endpoint case of Pisier's theorem quoted previously, which we now recall as follows. Let $1 < q < \infty$. Then a Banach space X has an equivalent q-uniformly convex norm iff for one $1 < p < \infty$ (or equivalently, for every $1 < p < \infty$) there exists a positive constant c such that

$$\left\| \left(\|f_1\|^q + \sum_{n\geq 2} \|f_n - f_{n-1}\|^q \right)^{1/q} \right\|_p \leq c \sup_{n\geq 1} \|f_n\|_p \tag{1.2}$$

for all finite L_p-martingales f with values in X. Again, the validity of the converse inequality amounts to saying that X has an equivalent q-uniformly smooth norm.

Ouyang–Xu's arguments heavily rely on Calderon–Zygmund singular integral theory. In fact, the Lusin function S_q in [10,15] can be represented as a singular integral operator with a regular operator-valued kernel. Similarly, Our proofs depend on martingale transform theory. More precisely, we will use operator-valued martingale transform theory as developed by Martinez and Torrea [8,9]. Compared with the function case, the martingale situation is simpler because of stopping time techniques.

In the remainder of this section, we give some preliminaries necessary to the whole paper. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space and X a Banach space. For $1 \leq p \leq \infty$ the usual L_p-space of strongly p-integrable X-valued functions on $(\Omega, \mathcal{F}, P)$ will be denoted by $L_p(\Omega; X)$ or simply by $L_p(X)$. Let $\{\mathcal{F}_n\}_{n\geq 1}$ be an increasing sequence of sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F} = \vee \mathcal{F}_n$. By an X-valued martingale relative to $\{\mathcal{F}_n\}_{n\geq 1}$ we mean a sequence $f = \{f_n\}_{n\geq 1}$ in $L_1(X)$ such that $\mathbb{E}(f_{n+1}|\, \mathcal{F}_n) = f_n$ for every $n \geq 1$. Let $df_n = f_n - f_{n-1}$ with the convention that $f_0 = 0$. $\{df_n\}_{n\geq 1}$ is the martingale difference sequence of f. We will use the following standard notations from martingale theory

$$f^* = \sup_{n\geq 1} \|f_n\| \quad \text{and} \quad S^{(q)}(f) = \left(\sum_{n=1}^{\infty} \|df_n\|^q \right)^{1/q}.$$

To avoid unnecessary (and irrelevant) convergence problem on infinite series we will assume that all martingales considered in the sequel are finite, unless explicitly stated otherwise. Note that f^* is the maximal function of f and $S^{(q)}(f)$ a variant of the usual square function of f. We will adopt the convention that a martingale $f = \{f_n\}$ will be identified with its final value f_∞ whenever the latter exists. Accordingly, if $f \in L_1(X)$ we will denote again by f the associated martingale $\{f_n\}$ with $f_n = \mathbb{E}(f|\, \mathcal{F}_n)$. We refer to [4] for more information on vector-valued martingale theory.

The main object of this paper is the BMO space given in the following

Definition 1.1 Let $1 \leq p < \infty$ and X be a Banach space. The space $BMO_p(X)$ consists of all functions $f \in L_1(X)$ such that

$$\|f\|_{BMO_p(X)} = \sup_{n\geq 1} \left\| \mathbb{E}(\|f - f_{n-1}\|^p|\mathcal{F}_n)^{1/p} \right\|_\infty < \infty.$$

Remark 1.2 The following facts are well known in the scalar-valued case (see [5,6,14]). Their proofs go straightforward over the Banach-valued setting.

(1) The spaces $BMO_p(X)$ are independent of p and all corresponding norms are equivalent. This allows us to denote any of them by $BMO(X)$.

(2) $L_\infty(X) \subset BMO(X) \subset L_p(X)$ for $1 \leq p < \infty$.

(3) We have

$$\|f\|_{BMO(X)} = \sup_{\tau} P(\tau < \infty)^{-1/p} \|f - f_{\tau-1}\|_{L_p(X)}, \quad 1 \leq p < \infty, \tag{1.3}$$

where the supremum is taken over all stopping times τ. On the other hand a function $f \in L_p(X), 1 \leq p < \infty$, belongs to $BMO(X)$ iff there exists an adapted process $(\theta_n)_{n\geq 0}$ such that $\theta_0 = 0$ and

$$C_\theta = \sup_n \left\| \mathbb{E}(\|f - \theta_{n-1}\|^p|\mathcal{F}_n)^{1/p} \right\|_\infty < \infty.$$

In this case, $\|f\|_{BMO(X)} \approx \inf_\theta C_\theta$.

Burkholder's martingale transforms (see [2,3]) are defined by using scalar-valued multiplying sequences. One main tool in our proofs will be martingale transforms with operator-valued multiplying sequences, defined and studied in [8,9].

Definition 1.3 Let X_1 and X_2 be two Banach spaces. Let $L(X_1, X_2)$ denote the space of all bounded linear operators from X_1 to X_2. Let $\upsilon = \{\upsilon_n\}_{n\geq 1}$ be an adapted sequence such that $\upsilon_n \in L_\infty(L(X_1, X_2))$ and $\sup_{n\geq 1} \|\upsilon_n\|_{L_\infty(L(X_1,X_2))} \leq 1$. Then the martingale transform T associated to υ is defined as follows. For any X_1-valued martingale $f = \{f_n\}_{n\geq 1}$

$$(Tf)_n = \sum_{k=1}^n \upsilon_k df_k.$$

We will use the following result from [8,9].

Lemma 1.4 *With the assumptions above the following statements are equivalent*:

(1) *There exists a positive constant c such that*

$$\|Tf\|_{BMO(X_2)} \leq c\,\|f\|_{BMO(X_1)}, \quad \forall\ f \in BMO(X_1).$$

(2) *There exists a positive constant c such that*

$$\|(Tf)^*\|_{BMO(X_2)} \leq c\,\|f\|_{BMO(X_1)}, \quad \forall\ f \in BMO(X_1).$$

(3) *For some* $1 \leq p < \infty$ (*or equivalently, for every* $1 \leq p < \infty$) *there exists a positive constant c such that*

$$\|Tf\|_{L_p(X)} \leq c\,\|f^*\|_{L_p(X)}, \quad \forall\ f \in L_p(X_1).$$

The classical notion of Carleson measures in harmonic analysis has the following martingale analogue.

Definition 1.5 Let μ be a nonnegative measure on $\Omega \times \mathbb{N}$, where $\mathbb{N}$ is equipped with the counting measure dm. μ is called a Carleson measure if

$$\sup \frac{\mu(\widehat{\tau})}{P(\tau < \infty)} < \infty,$$

where the supremum runs over all stopping times τ and where $\widehat{\tau}$ denotes the "tent" over τ:

$$\widehat{\tau} = \{(w, k) \in \Omega \times \mathbb{N} \,:\, \tau(w) \leq k, \tau(w) < \infty\}\,.$$

Throughout the paper, we will use $A \approx B$ to abbreviate $c^{-1}B \leq A \leq cB$ for some positive constant c. The letter c will denote a positive constant, which may depend on p but never on the martingales in consideration, and which may change from line to line.

2 Main results

The following theorem is the main result of this section. Recall that $\widehat{\tau}$ denotes the tent over a stopping time τ.

Theorem 2.1 *Let X be a Banach space and $2 \leq q < \infty$. Then the following statements are equivalent:*

(1) *There exists a positive constant c such that for any finite X-valued martingale*

$$\sup_{\tau} \frac{1}{P(\tau < \infty)} \int\limits_{\widehat{\tau}} \|df_k\|^q dP \otimes dm \leq c^q \|f\|^q_{BMO}\,. \tag{2.1}$$

(2) *X has an equivalent norm which is q-uniformly convex.*

Inequality (3.1) means that $\|df_k\|^q dP \otimes dm$ is a Carleson measure on $\Omega \times \mathbb{N}$ for every $f \in BMO(X)$.

Lemma 2.2 *Let $1 \leq p < \infty$. Then*

$$\|f\|_{BMO(X)} \approx \inf_{\theta} \sup_{\tau} P(\tau < \infty)^{-1/p} \|f - \theta_{\tau-1}\|_p\,,$$

where the supremum runs over all stopping times τ and the infimum over all adapted processes θ such that $\theta_\infty = f$.

Proof Assume $f \in BMO(X)$. Let τ be a stopping time. Then by Remark 1.2, (2)

$$\begin{aligned}\|f - \theta_{\tau-1}\|_p^p &= \mathbb{E}\|f - \theta_{\tau-1}\|^p \chi_{\{\tau<\infty\}}\\ &= \mathbb{E}\left(\mathbb{E}(\|f - \theta_{\tau-1}\|^p \,\middle|\mathcal{F}_\tau)\chi_{\{\tau<\infty\}}\right)\\ &\leq C_\theta^p P(\tau < \infty).\end{aligned}$$

This implies

$$\inf_{\theta} \sup_{\tau} P(\tau < \infty)^{-1/p} \|f - \theta_{\tau-1}\|_p \leq \inf_{\theta} C_\theta \leq c\|f\|_{BMO(X)}.$$

Conversely, assume $\beta = \inf_\theta \sup_\tau P(\tau < \infty)^{-1/p}\|f - \theta_{\tau-1}\|_p < \infty$, τ is any stopping time, $\forall F \in \mathcal{F}_\tau$, $F \subset \{\tau < \infty\}$. By defining $\tau_F = \tau$, if $\omega \in F$; otherwise $\tau_F = \infty$, we get

$$\frac{1}{P(F)}\int\limits_F \|f - \theta_{\tau-1}\|^p dP = P(\tau_F < \infty)^{-1}\int\limits_F \|f - \theta_{\tau_F-1}\|^p dP$$
$$= P(\tau_F < \infty)^{-1}\|f - \theta_{\tau_F-1}\|_p^p,$$

which leads to

$$\sup_\tau \|\mathbb{E}(\|f - \theta_{\tau-1}\|^p|\mathcal{F}_n)^{1/p}\|_\infty \leq P(\tau_F < \infty)^{-1/p}\|f - \theta_{\tau_F-1}\|_p .$$

Thus

$$\|f\|_{BMO(X)} \leq c \inf_\theta C_\theta \leq c \inf_\theta \sup_\tau P(\tau < \infty)^{-1/p}\|f - \theta_{\tau-1}\|_p .$$

□

Proof of Theorem 2.1 (1)$\Longrightarrow$(2). Assume that (1) holds. We first claim that

$$\|S^{(q)}(f)\|_{BMO} \leq c\, \|f\|_{BMO(X)}, \quad \forall f \in BMO(X).$$

Indeed, by Lemma 2.2

$$\begin{aligned}\|S^{(q)}(f)\|_{BMO} &\leq c \sup_\tau P(\tau < \infty)^{-1/q}\|S^{(q)}(f) - S^{(q)}_{\tau-1}(f)\|_q \\ &\leq c \sup_\tau P(\tau < \infty)^{-1/q}\left(\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^q \chi_{\{\tau<\infty\}}\right)^{1/q} \\ &= c \sup_\tau P(\tau < \infty)^{-1/q}\left(\int\limits_{\hat{\tau}} \|df_k\|^q dP \otimes dm\right)^{1/q} \\ &\leq c\, \|f\|_{BMO(X)}.\end{aligned}$$

We now consider a martingale transform operator Q from the family of X-valued martingales to that of $\ell_q(X)$-valued martingales. Let $\upsilon_k \in L(X, \ell_q(X))$ be the operator defined by $\upsilon_k x = \{x_j\}_{j=1}^\infty$ for $x \in X$, where $x_j = x$ if $j = k$ and $x_j = 0$ otherwise. Q is the martingale transform associated to the sequence (υ_k):

$$(Qf)_n = \sum_{k=1}^{n} \upsilon_k df_k = (df_1, df_2, \ldots, df_n, 0, \ldots).$$

Then

$$(Qf)^* = \sup_n \|(Qf)_n\|_{\ell^q(X)} = S^{(q)}(f).$$

It is clear that by the claim above Q satisfies the statement (2) in Lemma 1.4. Therefore, Q is L_q-bounded. Namely,

$$\|S^{(q)}(f)\|_{L_q} = \|(Qf)^*\|_{L_q} \leq c\|f\|_{L_q(X)}.$$

Thus by Pisier' theorem (see 1.2) X has an equivalent q-uniformly convex norm.

(2)$\Longrightarrow$(1). Suppose that X has an equivalent q-uniformly convex norm. By Pisier' theorem, we find for any $1 \leq n \leq m$

$$\mathbb{E}\left(\sum_{i=n}^{m} \|df_i\|^q|\mathcal{F}_n\right) \leq c\mathbb{E}(\|f_m - f_{n-1}\|^q|\mathcal{F}_n) \leq c\mathbb{E}(\|f - f_{n-1}\|^q|\mathcal{F}_n) \leq c\|f\|^q_{BMO(X)}.$$

This implies

$$\mathbb{E}\left(\sum_{i=n}^{\infty} \|df_i\|^q|\mathcal{F}_n\right) \leq c\|f\|^q_{BMO(X)}.$$

Now let τ be a stopping time. We then have

$$\begin{aligned}
&P(\tau < \infty)^{-1/q} \left(\int_{\widehat{\tau}} \|df_k\|^q dP \otimes dm\right)^{1/q}\\
&= P(\tau < \infty)^{-1/q} \left(\mathbb{E}\sum_{k=\tau}^{\infty} \|df_k\|^q \chi_{\{\tau<\infty\}}\right)^{1/q}\\
&= P(\tau < \infty)^{-1/q} \left(\mathbb{E}\left(\mathbb{E}\left(\sum_{k=\tau}^{\infty} \|df_k\|^q|\mathcal{F}_\tau\right)\chi_{\{\tau<\infty\}}\right)\right)^{1/q}\\
&\leq c\, P(\tau < \infty)^{-1/q} \left(\mathbb{E}\|f\|^q_{BMO(X)}\chi_{\{\tau<\infty\}}\right)^{1/q}\\
&\leq c\, \|f\|_{BMO(X)}.
\end{aligned}$$

Taking the supremum over all stopping times τ, we get (2.1). □

Theorem 2.3 *Let X be a Banach space and $1 < p \leq 2$. Then the following statements are equivalent:*

(1) *There exists a positive constant c such that for any finite X-valued martingale*

$$\|f\|^p_{BMO(X)} \leq c^p \sup_\tau P(\tau < \infty)^{-1} \int_{\widehat{\tau}} \|df_k\|^p dP \otimes dm. \tag{2.2}$$

(2) *X has an equivalent norm which is p-uniformly smooth.*

Inequality (2.2) means $f \in BMO(X)$, if $\|df_k\|^p dP \otimes dm$ is a Carleson measure on $\Omega \times \mathbb{N}$.

Proof (1) $\Longrightarrow$ (2). Suppose that (1) holds. Then for any X-valued martingale we have

$$\|f\|_{BMO(X)} \leq c \sup_{\tau} P(\tau < \infty)^{-1/p} \left(\mathbb{E} \sum_{k=\tau}^{\infty} \|df_k\|^p \chi_{\{\tau<\infty\}} \right)^{1/p}. \tag{2.3}$$

Let X^* be the dual space of X. It suffices to prove that X^* has an equivalent q-uniformly smooth norm, where q is the conjugate index of p. By Pisier' theorem, this is equivalent to showing that

$$\|S^{(q)}(g)\|_{L_1} \leq c\|g^*\|_{L_1} = c\|g\|_{H_1(X^*)} \tag{2.4}$$

for any finite L_1-martingale g with values in X^*. To this end, we will use duality. Moreover, by approximation, we can assume that g is an L_2-bounded martingale. Recall that $H_1(X^*)$ is defined by

$$H_1(X^*) = \{g \in L_1(X^*) : g^* \in L_1\}.$$

It is well known that $BMO(X)$ can be identified as a norming subspace of $H_1(X^*)$. Thus for any finite martingale $g \in H_1(X^*)$ and $f \in BMO(X)$

$$|\langle g, f\rangle| = \left| \int_{\Omega} \langle g(w), f(w)\rangle dP \right| \leq c\, \|g\|_{H_1(X^*)} \|f\|_{BMO(X)}.$$

On the other hand, $\|S^{(q)}(g)\|_{L_1}$ is the norm of the difference sequence $\{dg_n\}$ in $L_1\left(\ell_q(X^*)\right)$. Thus

$$\begin{aligned} \|S^{(q)}(g)\|_{L_1} &= \sup \left\{ \left| \sum \langle dg_k, a_k\rangle \right| : \|\{a_k\}\|_{L_\infty(\ell_p(X))} \leq 1 \right\} \\ &= \sup \left\{ \left| \sum \langle dg_k, \mathbb{E}(a_k) - \mathbb{E}_{k-1}(a_k)\rangle \right| : \|\{a_k\}\|_{L_\infty(\ell_p(X))} \leq 1 \right\}. \end{aligned}$$

Set $df_k = \mathbb{E}_k(a_k) - \mathbb{E}_{k-1}(a_k)$ and $f = \sum_k df_k$. Then f is an X-valued martingale. We have

$$\left| \sum \langle dg_k, a_k\rangle \right| = \left| \sum \langle dg_k, df_k\rangle \right| = |\langle g, f\rangle| \leq c\, \|g\|_{H_1(X^*)} \|f\|_{BMO(X)}.$$

It remains to estimate $\|f\|_{BMO(X)}$. This will be done by using (2.3). Indeed,

$$\begin{aligned}\left(\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^p\chi_{\{\tau<\infty\}}\right)^{1/p} &\leq \left(\mathbb{E}\sum_{k=\tau}^{\infty}\|\mathbb{E}_k(a_k)\|^p\chi_{\{\tau<\infty\}}\right)^{1/p}\\ &\quad+\left(\mathbb{E}\sum_{k=\tau}^{\infty}\|\mathbb{E}_{k-1}(a_k)\|^p\chi_{\{\tau<\infty\}}\right)^{1/p}\\ &= \mathrm{I}+\mathrm{II}\end{aligned}$$

Let $\{a_k\} \in L_\infty\big(\ell_p(X)\big)$ and $\|\{a_k\}\|_{L_\infty(l_p(X))} \leq 1$. Then by Doob's stopping time theorem we find

$$\begin{aligned}\mathrm{I}^p &\leq \mathbb{E}\sum_{k=\tau}^{\infty}\mathbb{E}_k\|a_k\|^p\chi_{\{\tau<\infty\}} = \mathbb{E}\mathbb{E}_\tau\left(\sum_{k=1}^{\infty}\mathbb{E}_k\left(\|a_k\|^p\chi_{\{\tau\leq k\}}\right)\right)\\ &= \mathbb{E}\left(\sum_{k=1}^{\infty}\|a_k\|^p\chi_{\{\tau\leq k\}}\right) = \mathbb{E}\left(\sum_{k=\tau}^{\infty}\|a_k\|^p\chi_{\{\tau<\infty\}}\right)\\ &\leq \mathbb{E}\left(\left\|\sum_{k=\tau}^{\infty}\|a_k\|^p\right\|_\infty\chi_{\{\tau<\infty\}}\right)\\ &\leq P(\tau<\infty).\end{aligned}$$

The same argument applies to II:

$$\begin{aligned}\mathrm{II}^p &= \mathbb{E}\|\mathbb{E}_{\tau-1}a_\tau\|^p\chi_{\{\tau<\infty\}} + \mathbb{E}\sum_{k=\tau-1}^{\infty}\|\mathbb{E}_{k-1}(a_k)\|^p\chi_{\{\tau<\infty\}}\\ &\leq 2P(\tau<\infty).\end{aligned}$$

combining the previous estimates with (2.3) we then duduce $\|f\|_{BMO(X)} \leq c$. Therefore, we finally obtain (2.4). Thus X^* has an equivalent q-uniformly norm.

(2) $\Longrightarrow$ (1). Assume that (2) holds. By Remark 1.2, we have

$$\|f\|_{BMO(X)} = \sup_\tau P(\tau<\infty)^{-\frac{1}{p}}\|f-f_{\tau-1}\|_{L_p(X)}.$$

Now we consider the new sequence $\{\mathcal{F}_{k\vee\tau}\}_{k\geq1}$ of σ-fields and the corresponding martingale $\tilde{f}$ generated by $f-f_\tau$. Then by Doob's stopping time theorem, we find

$$\tilde{f}_k = \mathbb{E}(f-f_\tau|\mathcal{F}_{k\vee\tau}) = \mathbb{E}(f|\mathcal{F}_{k\vee\tau}) - f_\tau = f_{k\vee\tau}-f_\tau.$$

By (2), we have $\|\tilde{f}\|_p \le c\|S^{(p)}(\tilde{f})\|_p$. Namely,

$$\mathbb{E}\|f - f_\tau\|^p = \mathbb{E}\|\tilde{f}\|^p \le c^p \mathbb{E}\sum_{k=1}^{\infty}\|d\tilde{f}_k\|^p = c^p \mathbb{E}\sum_{k=1}^{\infty}\|f_{(k+1)\vee\tau} - f_{k\vee\tau}\|^p$$
$$= c^p \mathbb{E}\sum_{k=\tau}^{\infty}\|f_{k+1} - f_k\|^p = c^p \mathbb{E}\sum_{k=\tau+1}^{\infty}\|df_k\|^p \chi_{\{\tau<\infty\}}$$

Therefore,

$$\mathbb{E}\|f - f_{\tau-1}\|^p \le 2^p \left(\mathbb{E}\|f - f_\tau\|^p + \mathbb{E}\|f_\tau - f_{\tau-1}\|^p\right) = c\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^p \chi_{\{\tau<\infty\}}.$$

Then we obtain

$$\|f\|_{BMO(X)} \le c \sup_\tau P(\tau < \infty)^{-1/p} \left(\mathbb{E}\sum_{k=\tau}^{\infty}\|df_k\|^p \chi_{\{\tau<\infty\}}\right)^{1/p}.$$

Thus the theorem is proved. □

Corollary 2.4 *Let X be a Banach space. Then the following statements are equivalent:*

(1) *There exists a positive constant c such that for any finite X-valued martingale*

$$c^{-2} \sup_\tau P(\tau < \infty)^{-1} \int_{\widehat{\tau}} \|df_k\|^2 dP \otimes dm$$
$$\le \|f\|^2_{BMO} \le c^2 \sup_\tau P(\tau < \infty)^{-1} \int_{\widehat{\tau}} \|df_k\|^2 dP \otimes dm.$$

(2) *X is isomorphic to a Hilbert space.*

Proof It is well known that a space which is both 2-uniformly smooth and 2-uniformly convex is isomorphic to a Hilbert space. □

3 UMD Banach lattices

Definition 3.1 A Banach space X is said to satisfy UMD property if there exists a positive constant c such that for $1 < p < \infty$,

$$\|\varepsilon_1 df_1 + \cdots + \varepsilon_n df_n\|_{L_p} \le c\|df_1 + \cdots + df_n\|_{L_p}, \quad \forall n \ge 1$$

for all finite X-valued martingales $\{f_k\}$ and all $\varepsilon_k = \pm 1$.

This notion is due to Burkholder [3]. It is known that the existence of one p_0 satisfying the inequality is enough to assure the existence of the rest of p, $1 < p < \infty$.

We shall be interested in Banach lattice case. Thus X will denote a Banach lattice in this section. Without loss of generality we assume that X is a Banach lattice of measurable functions on some measure space $(\Sigma, d\mu)$. The reader is referred to [7] for information on Banach lattices. It is now nature to consider the following variant of square function $S^{(2)}(f)$.

Definition 3.2 Let X be a Banach lattice and $f = \{f_n\}_{n\geq 1}$ a X-valued martingale. We define the operator

$$\tilde{S}f(\sigma) = \left(\sum_{k=1}^{\infty} |df_k(\sigma)|^2\right)^{1/2}.$$

The following lemma is well known; see [1,13].

Lemma 3.3 *Given a Banach lattice X, the following statements are equivalent:*

(1) *X satisfies the UMD property.*
(2) *There exist $1 < p < \infty$ and a constant c such that*

$$c^{-1}\|f\|_{L_p(X)} \leq \|\tilde{S}f\|_{L_p(X)} \leq c\|f\|_{L_p(X)},$$

for any X-valued martingale.

Now we can prove the following characterization of UMD Banach lattices.

Theorem 3.4 *Given a Banach lattice X, the following statements are equivalent:*

(1) *X satisfies the UMD property.*
(2) *There exists a positive constant c such that for any finite X-value martingale f,*

$$c^{-1}\|f\|_{BMO(X)} \leq \sup_{\tau} P(\tau < \infty)^{-1/2} \times \left(\mathbb{E}\left\|\left(\sum_{k=\tau}^{\infty}|df_k|^2\right)^{1/2}\right\|^2 \chi_{\{\tau<\infty\}}\right)^{1/2} \leq c\|f\|_{BMO(X)}.$$

Proof (2) $\Longrightarrow$ (1). Assume that (2) holds. By

$$\left\|\sum_k df_k\right\|_{BMO(X)} \approx \sup_{\tau} P(\tau < \infty)^{-1/2} \left(\mathbb{E}\left\|\left(\sum_{k=\tau}^{\infty}|df_k|^2\right)^{1/2}\right\|^2 \chi_{\{\tau<\infty\}}\right)^{1/2},$$

we know the martingale difference sequences are unconditional in $BMO(X)$. Namely,

$$\left\|\sum_k \varepsilon_k df_k\right\|_{BMO(X)} \leq c\|f\|_{BMO(X)}, \quad \forall \varepsilon_k = \pm 1, \quad \forall f \in BMO(X).$$

Fix a sequence $\{\varepsilon_k\}$ of signs and we consider the associated martingale transform operator Q:

$$(Qf)_n = \sum_{k=1}^{n} \varepsilon_k df_k.$$

Then the inequality above means that Q is bounded on $BMO(X)$. Thus by Lemma 1.4, we get

$$\|Qf\|_{L_p(X)} \le c\|f\|_{L_p(X)}, \quad \forall p > 1.$$

Therefore, X satisfies the UMD property.

(1) $\Longrightarrow$ (2). This proof is similar to that of (2) $\Longrightarrow$ (1) in Theorem 2.3. We consider the X-valued martingale $\tilde{f}$ defined in the proof of Theorem 2.3. Since X has the UMD property, we have

$$\|\tilde{f}\|_{L_2(X)} \approx \|\tilde{S}\tilde{f}\|_{L_2(X)}.$$

Then

$$\begin{aligned}\mathbb{E}\|f - f_\tau\|^2 = \mathbb{E}\|\tilde{f}\|^2 &\le c\mathbb{E}\left\|\left(\sum_{k=1}^{\infty} |d\tilde{f}_k|^2\right)^{1/2}\right\|^2 \\ &= c\mathbb{E}\left\|\left(\sum_{k=1}^{\infty} |f_{(k+1)\vee\tau} - f_{k\vee\tau}|^2\right)^{1/2}\right\|^2 \\ &= c\mathbb{E}\left\|\left(\sum_{k=\tau+1}^{\infty} |df_k|^2\right)^{1/2}\right\|^2 \chi_{\{\tau<\infty\}}\end{aligned}$$

Then

$$\begin{aligned}\mathbb{E}\|f - f_{\tau-1}\|^2 &\le c\left(\mathbb{E}\|f - f_\tau\|^2 + \mathbb{E}\|f_\tau - f_{\tau-1}\|^2\right) \\ &\le c\mathbb{E}\left\|\left(\sum_{k=\tau}^{\infty} |df_k|^2\right)^{1/2}\right\|^2 \chi_{\{\tau<\infty\}}. \qquad (3.1)\end{aligned}$$

Conversely,

$$\begin{aligned}\mathbb{E}\left\|\left(\sum_{k=\tau+1}^{\infty} |df_k|^2\right)^{1/2}\right\|^2 \chi_{\{\tau<\infty\}} &= \mathbb{E}\left\|\left(\sum_{k=1}^{\infty} |f_{(k+1)\vee\tau} - f_{k\vee\tau}|^2\right)^{1/2}\right\|^2 \\ &\le c\mathbb{E}\|\tilde{f}\|^2 = \mathbb{E}\|f - f_\tau\|^2\end{aligned}$$

On the other hand, since $f_\tau - f_{\tau-1} = \mathbb{E}(f - f_{\tau-1} | \mathcal{F}_\tau)$, we have

$$\mathbb{E}\| f_\tau - f_{\tau-1}\|^2 \le \mathbb{E}\| f - f_{\tau-1}\|^2.$$

Therefore,

$$\begin{aligned}\mathbb{E}\left\|\left(\sum_{k=\tau}^{\infty}|df_k|^2\right)^{1/2}\right\|^2 \chi_{\{\tau<\infty\}} &\le c\left(\mathbb{E}\| f_\tau - f_{\tau-1}\|^2 \chi_{\{\tau<\infty\}}\right.\\ &\quad\left. + \mathbb{E}\left\|\left(\sum_{k=\tau+1}^{\infty}|df_k|^2\right)^{1/2}\right\|^2 \chi_{\{\tau<\infty\}}\right)\\ &\le c\left(\mathbb{E}\| f - f_{\tau-1}\|^2 \chi_{\{\tau<\infty\}} + c\mathbb{E}\| f - f_\tau\|^2 \chi_{\{\tau<\infty\}}\right)\\ &\le c\,\mathbb{E}\| f - f_{\tau-1}\|^2 \chi_{\{\tau<\infty\}}\end{aligned}$$

Combining this inequality with (3.1),

$$\mathbb{E}\| f - f_{\tau-1}\|^2 \approx \mathbb{E}\left\|\left(\sum_{k=\tau}^{\infty}|df_k|^2\right)^{1/2}\right\|^2 \chi_{\{\tau<\infty\}}$$

Thus by (1.3), we obtain the desired inequality. Thus the theorem is proved. □

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