

Convergence in distribution of random metric measure spaces (Λ -coalescent measure trees)

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Abstract We consider the space of complete and separable metric spaces which are equipped with a probability measure. A notion of convergence is given based on the philosophy that a sequence of metric measure spaces converges if and only if all finite subspaces sampled from these spaces converge. This topology is metrized following Gromov’s idea of embedding two metric spaces isometrically into a common metric space combined with the Prohorov metric between probability measures on a fixed metric space. We show that for this topology convergence in distribution follows—provided the sequence is tight—from convergence of all randomly sampled finite subspaces. We give a characterization of tightness based on quantities which are reasonably easy to calculate. Subspaces of particular interest are the space of real trees and of ultra-metric spaces equipped with a probability measure. As an example we characterize convergence in distribution for the (ultra-)metric measure spaces given by the random genealogies of the Λ -coalescents. We show that the Λ -coalescent defines an infinite (random) metric measure space if and only if the so-called “dust-free”-property holds.

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1 Introduction and motivation

In this paper we study random metric measure spaces which appear frequently in the form of random trees in probability theory. Prominent examples are random binary search trees as a special case of random recursive trees [9], ultra-metric structures in spin-glasses (see, for example, [8, 22]), and coalescent processes in population genetics (for example, [12, 19]). Of special interest is the continuum random tree, introduced in [2], which is related to several objects, for example, Galton–Watson trees, spanning trees and Brownian excursions.

Moreover, examples for Markov chains with values in finite trees are the Aldous–Broder Markov chain which is related to spanning trees ([1]), growing Galton–Watson trees, and tree-bisection and reconnection which is a method to search through tree space in phylogenetic reconstruction (see e.g., [14]).

Because of the exponential growth of the state space with an increasing number of vertices tree-valued Markov chains are—even so easy to construct by standard theory—hard to analyze for their qualitative properties. It therefore seems to be reasonable to pass to a continuum limit and to construct certain limit dynamics and study them with methods from stochastic analysis.

We will apply this approach in the forthcoming paper [16] to trees encoding genealogical relationships in exchangeable models of populations of constant size. The result will be the tree-valued Fleming–Viot dynamics. For this purpose it is necessary to develop systematically the topological properties of the state space and the corresponding convergence in distribution. The present paper focuses on these topological properties.

As one passes from finite trees to “infinite” trees the necessity arises to equip the tree with a probability measure which allows to sample typical finite subtrees. In [2], Aldous discusses a notion of convergence in distribution of a “consistent” family of finite random trees towards a certain limit: the continuum random tree. In order to define convergence Aldous codes trees as separable and complete metric spaces satisfying some special properties for the metric characterizing them as trees which are embedded into ℓ_1^+ and equipped with a probability measure. In this setting finite trees, i.e., trees with finitely many leaves, are always equipped with the uniform distribution on the set of leaves. The idea of convergence in distribution of a “consistent” family of finite random trees follows Kolmogorov’s theorem which gives the characterization of convergence of \mathbb{R} -indexed stochastic processes with regular paths. That is, a sequence has a unique limit provided a tightness condition holds on path space and assuming that the “finite-dimensional distributions” converge. The analogs of finite-dimensional

distributions are “subtrees spanned by finitely many randomly chosen leaves” and the tightness criterion is built on characterizations of compact sets in ℓ_1^+ .

Aldous’s notion of convergence has been successful for the purpose of rescaling a given family of trees and showing convergence in distribution towards a specific limit random tree. For example, Aldous shows that suitably rescaled families of critical finite variance offspring distribution Galton–Watson trees conditioned to have total population size N converge as $N \rightarrow \infty$ to the Brownian continuum random tree, i.e., the \mathbb{R} -tree associated with a Brownian excursion. Furthermore, Aldous constructs the genealogical tree of a resampling population as a metric measure space associated with the Kingman coalescent, as the limit of N -coalescent trees with weight $1/N$ on each of their leaves.

However, Aldous’s ansatz to view trees as closed subsets of ℓ_1^+ , and thereby using a very particular embedding for the construction of the topology, seemed not quite easy and elegant to work with once one wants to construct tree-valued limit dynamics (see, for example, [11, 13, 16]). More recently, isometry classes of \mathbb{R} -trees, i.e., a particular class of metric spaces, were introduced, and a means of measuring the distance between two (isometry classes of) metric spaces were provided based on an “optimal matching” of the two spaces yielding the Gromov–Hausdorff metric (see, for example, [4, Chap. 7]).

The main emphasis of the present paper is to exploit Aldous’s philosophy of convergence without using Aldous’s particular embedding. That is, we equip the space of separable and complete real trees which are equipped with a probability measure with the following topology:

- A sequence of trees (equipped with a probability measure) converges to a limit tree (equipped with a probability measure) if and only if all randomly sampled finite subtrees converge to the corresponding limit subtrees. The resulting topology is referred to as the *Gromov-weak topology* (compare Definition 2.8).

Since the construction of the topology works not only for tree-like metric spaces, but also for the space of (measure preserving isometry classes of) metric measure spaces we formulate everything within this framework.

- We will see that the Gromov-weak topology on the space of metric measure spaces is Polish (Theorem 1).

In fact, we metrize the space of metric measure spaces equipped with the Gromov-weak topology by the *Gromov–Prohorov metric* which combines the two concepts of metrizing the space of metric spaces and the space of probability measures on a given metric space in a straightforward way. Moreover, we present a number of equivalent metrics which might be useful in different contexts.

This then allows to discuss convergence of random variables taking values in that space.

- We next characterize compact sets (Theorem 2 combined with Theorem 5) and tightness (Theorem 3 combined with Theorem 5) via quantities which are reasonably easy to compute.
- We then illustrate with the example of the Λ -coalescent tree (Theorem 4) how the tightness characterization can be applied.

We remark that topologies on metric measure spaces are considered in detail in Sect. 3 $\frac{1}{2}$ of [17]. We are aware that several of our results (in particular, Theorems 1, 2 and 5) are stated in [17] in a different set-up. While Gromov focuses on geometric aspects, we provide the tools necessary to do probability theory on the space of metric measure spaces. See Remark 5.3 for more details on the connection to Gromov's work.

Further related topologies on particular subspaces of isometry classes of complete and separable metric spaces have already been considered in [13,27]. Convergence in these two topologies implies convergence in the Gromov-weak topology but not vice versa.

Outline

The rest of the paper is organized as follows. In the next two sections we formulate the main results. In Sect. 2 we introduce the space of metric measure spaces equipped with the Gromov-weak topology and characterize their compact sets. In Sect. 3 we discuss convergence in distribution and characterize tightness. We then illustrate the main results introduced so far with the example of the metric measure tree associated with genealogies generated by the infinite Λ -coalescent in Sect. 4.

Sections 5–9 are devoted to the proofs of the theorems. In Sect. 5 we introduce the Gromov–Prohorov metric as a candidate for a complete metric which generates the Gromov-weak topology and show that the generated topology is separable. As a technical preparation we collect results on the modulus of mass distribution and the distance distribution (see Definition 2.9) in Sect. 6. In Sects. 7 and 8 we give characterizations on pre-compactness and tightness for the topology generated by the Gromov–Prohorov metric. In Sect. 9 we prove that the topology generated by the Gromov–Prohorov metric coincides with the Gromov-weak topology.

Finally, in Sect. 10 we provide several other metrics that generate the Gromov-weak topology.

2 Metric measure spaces

As usual, given a topological space (X, \mathcal{O}) , we denote by $\mathcal{M}_1(X)$ the space of all probability measures on X equipped with the Borel- σ -algebra $\mathcal{B}(X)$. Recall that the support of μ , $\text{supp}(\mu)$, is the smallest closed set $X_0 \subseteq X$ such that $\mu(X \setminus X_0) = 0$. The push forward of μ under a measurable map φ from X into another metric space (Z, r_Z) is the probability measure $\varphi_*\mu \in \mathcal{M}_1(Z)$ defined by

$$\varphi_*\mu(A) := \mu\left(\varphi^{-1}(A)\right), \quad (2.1)$$

for all $A \in \mathcal{B}(Z)$. Weak convergence in $\mathcal{M}_1(X)$ is denoted by \implies .

In the following we focus on complete and separable metric spaces.

Definition 2.1 (Metric measure space) A *metric measure space* is a complete and separable metric space (X, r) which is equipped with a probability measure $\mu \in \mathcal{M}_1(X)$. We write \mathbb{M} for the space of measure-preserving isometry classes of complete and separable metric measure spaces, where we say that (X, r, μ) and (X', r', μ') are measure-preserving isometric if there exists an isometry φ between the supports of μ on (X, r) and of μ' on (X', r') such that $\mu' = \varphi_*\mu$. It is clear that the property of being measure-preserving isometric is an equivalence relation.

We abbreviate $\mathcal{X} = (X, r, \mu)$ for a whole isometry class of metric spaces whenever no confusion seems to be possible.

Remark 2.2

- (i) Metric measure spaces, or short *mm-spaces*, are discussed in [17] in detail. Therefore they are sometimes also referred to as *Gromov metric triples* (see, for example, [28]).
- (ii) We have to be careful to deal with *sets* in the sense of the Zermelo-Fraenkel axioms. The reason is that we will show in Theorem 1 that \mathbb{M} can be metrized, say by d , such that (\mathbb{M}, d) is complete and separable. Hence if $\mathbb{P} \in \mathcal{M}_1(\mathbb{M})$ then the measure preserving isometry class represented by $(\mathbb{M}, d, \mathbb{P})$ yields an element in \mathbb{M} . The way out is to define \mathbb{M} as the space of measure preserving isometry classes of those metric spaces equipped with a probability measure whose elements are not themselves metric spaces. Using this restriction we avoid the usual pitfalls which lead to Russell’s antinomy.

To be in a position to formalize that for a sequence of metric measure spaces all finite subspaces sampled by the measures sitting on the corresponding metric spaces converge we next introduce the algebra of polynomials on \mathbb{M} .

Definition 2.3 (Polynomials) A function $\Phi = \Phi^{n,\phi} : \mathbb{M} \rightarrow \mathbb{R}$ is called a *polynomial (of degree n with respect to the test function ϕ)* on \mathbb{M} if and only if $n \in \mathbb{N}$ is the minimal number such that there exists a bounded continuous function $\phi : [0, \infty)^{\binom{n}{2}} \rightarrow \mathbb{R}$ such that

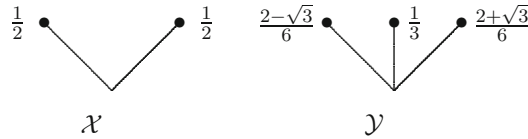
$$\Phi((X, r, \mu)) = \int \mu^{\otimes n}(d(x_1, \dots, x_n)) \phi((r(x_i, x_j))_{1 \leq i < j \leq n}), \tag{2.2}$$

where $\mu^{\otimes n}$ is the n -fold product measure of μ . Denote by Π the algebra of all polynomials on \mathbb{M} .

Example 2.4 In future work, we are particularly interested in tree-like metric spaces, i.e., ultra-metric spaces and \mathbb{R} -trees. In this setting, functions of the form (2.2) can be, for example, the mean total length or the averaged diameter of the sub-tree spanned by n points sampled independently according to μ from the underlying tree.

The next example illustrates that one can, of course, not separate metric measure spaces by polynomials of degree 2 only.

Example 2.5 Consider the following two metric measure spaces.



Assume that in both spaces the mutual distances between different points are 1. In both cases, the empirical distribution of the distances between two points equals $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, and hence all polynomials of degree $n = 2$ agree. But obviously, \mathcal{X} and \mathcal{Y} are not measure preserving isometric.

The first key observation is that the algebra of polynomials is a rich enough subclass to determine a metric measure space.

Proposition 2.6 (Polynomials separate points) *The algebra Π of polynomials separates points in \mathbb{M} .*

We need the useful notion of the distance matrix distribution.

Definition 2.7 (Distance matrix distribution) Let $\mathcal{X} = (X, r, \mu) \in \mathbb{M}$ and the space of infinite (pseudo-)distance matrices

$$\mathbb{R}^{\text{met}} := \{(r_{ij})_{1 \leq i < j < \infty} : r_{ij} + r_{jk} \geq r_{ik}, \forall 1 \leq i < j < k < \infty\}. \tag{2.3}$$

Define the map $\iota^{\mathcal{X}} : X^{\mathbb{N}} \rightarrow \mathbb{R}^{\text{met}}$ by

$$\iota^{\mathcal{X}}(x_1, x_2, \dots) := (r(x_i, x_j))_{1 \leq i < j < \infty}, \tag{2.4}$$

and the *distance matrix distribution* of \mathcal{X} by

$$\nu^{\mathcal{X}} := (\iota^{\mathcal{X}})_* \mu^{\otimes \mathbb{N}}. \tag{2.5}$$

Note that for $\mathcal{X} \in \mathbb{M}$ and Φ of the form (2.2), we have that

$$\Phi(\mathcal{X}) = \int \nu^{\mathcal{X}}(\mathbf{d}(r_{ij})_{1 \leq i < j}) \phi((r_{ij})_{1 \leq i < j \leq n}). \tag{2.6}$$

Proof of Proposition 2.6 Let $\mathcal{X}_\ell = (X_\ell, r_\ell, \mu_\ell) \in \mathbb{M}$, $\ell = 1, 2$, and assume that $\Phi(\mathcal{X}_1) = \Phi(\mathcal{X}_2)$, for all $\Phi \in \Pi$. The algebra $\{\phi \in \mathcal{C}_b(\mathbb{R}^{\binom{\mathbb{N}}{n}}); n \in \mathbb{N}\}$ is separating in $\mathcal{M}_1(\mathbb{R}^{\text{met}})$ and so $\nu^{\mathcal{X}_1} = \nu^{\mathcal{X}_2}$ by (2.6). Applying Gromov’s Reconstruction theorem for mm-spaces (see Paragraph 3 $\frac{1}{2}$.5 in [17]), we find that $\mathcal{X}_1 = \mathcal{X}_2$. \square

We are now in a position to define the Gromov-weak topology.

Definition 2.8 (Gromov-weak topology) A sequence $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is said to converge Gromov-weakly to \mathcal{X} in \mathbb{M} if and only if $\Phi(\mathcal{X}_n)$ converges to $\Phi(\mathcal{X})$ in \mathbb{R} , for all polynomials $\Phi \in \Pi$. We call the corresponding topology $\mathcal{O}_{\mathbb{M}}$ on \mathbb{M} the Gromov-weak topology.

The following result ensures that the state space is suitable to do probability theory on it.

Theorem 1 *The space $(\mathbb{M}, \mathcal{O}_{\mathbb{M}})$ is Polish.*

In order to obtain later tightness criteria for laws of random elements in \mathbb{M} we need a characterization of the compact sets of $(\mathbb{M}, \mathcal{O}_{\mathbb{M}})$. Informally, a subset of \mathbb{M} will turn out to be pre-compact iff the corresponding sequence of probability measures put most of their mass on subspaces of a uniformly bounded diameter, and if the contribution of points which do not carry much mass in their vicinity is small.

These two criteria lead to the following definitions.

Definition 2.9 (Distance distribution and Modulus of mass distribution) Let $\mathcal{X} = (X, r, \mu) \in \mathbb{M}$.

- (i) The *distance distribution*, which is an element in $\mathcal{M}_1([0, \infty))$, is given by $w_{\mathcal{X}} := r_* \mu^{\otimes 2}$, i.e.,

$$w_{\mathcal{X}}(\cdot) := \mu^{\otimes 2} \{ (x, x') : r(x, x') \in \cdot \}. \tag{2.7}$$

- (ii) For $\delta > 0$, define the *modulus of mass distribution* as

$$v_{\delta}(\mathcal{X}) := \inf \{ \varepsilon > 0 : \mu \{ x \in X : \mu(B_{\varepsilon}(x)) \leq \delta \} \leq \varepsilon \} \tag{2.8}$$

where $B_{\varepsilon}(x)$ is the open ball with radius ε and center x .

Remark 2.10 Observe that $w_{\mathcal{X}}$ and v_{δ} are well-defined because they are constant on isometry classes of a given metric measure space.

The next result characterizes pre-compactness in $(\mathbb{M}, \mathcal{O}_{\mathbb{M}})$.

Theorem 2 (Characterization of pre-compactness) *A set $\Gamma \subseteq \mathbb{M}$ is pre-compact in the Gromov-weak topology if and only if the following hold.*

- (i) *The family $\{w_{\mathcal{X}} : \mathcal{X} \in \Gamma\}$ is tight.*
- (ii) *For all $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ such that*

$$\sup_{\mathcal{X} \in \Gamma} v_{\delta}(\mathcal{X}) < \varepsilon. \tag{2.9}$$

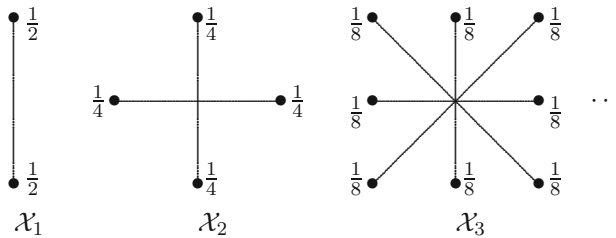
Remark 2.11 If $\Gamma = \{\mathcal{X}_1, \mathcal{X}_2, \dots\}$ then we can replace sup by lim sup in (2.9).

Example 2.12 In the following we illustrate the two requirements for a family in \mathbb{M} to be pre-compact which are given in Theorem 2 by two counter-examples.

- (i) Consider the isometry classes of the metric measure spaces $\mathcal{X}_n := (\{1, 2\}, r_n(1, 2) = n, \mu_n\{1\} = \mu_n\{2\} = \frac{1}{2})$. A potential limit object would be a metric space with masses $\frac{1}{2}$ within distance infinity. This clearly does not exist. Indeed, the family $\{w_{\mathcal{X}_n} = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_n; n \in \mathbb{N}\}$ is not tight, and hence $\{\mathcal{X}_n; n \in \mathbb{N}\}$ is not pre-compact in \mathbb{M} by Condition (i) of Theorem 2.
- (ii) Consider the isometry classes of the metric measure spaces $\mathcal{X}_n = (X_n, r_n, \mu_n)$ given for $n \in \mathbb{N}$ by

$$X_n := \{1, \dots, 2^n\}, \quad r_n(x, y) := \mathbf{1}\{x \neq y\}, \quad \mu_n := 2^{-n} \sum_{i=1}^{2^n} \delta_i, \quad (2.10)$$

i.e., \mathcal{X}_n consists of 2^n points of mutual distance 1 and is equipped with a uniform measure on all points.



A potential limit object would consist of infinitely many points of mutual distance 1 with a uniform measure. Such a space does not exist. Indeed, notice that for $\delta > 0$,

$$v_\delta(\mathcal{X}_n) = \begin{cases} 0, & \delta < 2^{-n}, \\ 1, & \delta \geq 2^{-n}, \end{cases} \quad (2.11)$$

so $\sup_{n \in \mathbb{N}} v_\delta(\mathcal{X}_n) = 1$, for all $\delta > 0$. Hence $\{\mathcal{X}_n; n \in \mathbb{N}\}$ does not fulfil Condition (ii) of Theorem 2, and is therefore not pre-compact.

3 Distributions of random metric measure spaces

From Theorem 1 and Definition 2.8 we immediately conclude the characterization of weak convergence for a sequence of probability measures on \mathbb{M} .

Corollary 3.1 (Characterization of weak convergence) *A sequence $(\mathbb{P}_n)_{n \in \mathbb{N}}$ in $\mathcal{M}_1(\mathbb{M})$ converges weakly w.r.t. the Gromov-weak topology if and only if*

- (i) *the family $\{\mathbb{P}_n; n \in \mathbb{N}\}$ is relatively compact in $\mathcal{M}_1(\mathbb{M})$, and*
- (ii) *for all polynomials $\Phi \in \Pi$, $(\mathbb{P}_n[\Phi])_{n \in \mathbb{N}}$ converges in \mathbb{R} .*

Proof The “only if” direction is clear, as polynomials are bounded and continuous functions by definition. To see the converse, recall from Lemma 3.4.3 in [10] that given a relative compact sequence of probability measures, each separating family of bounded continuous functions is convergence determining. \square

While Condition (ii) of the characterization of convergence given in Corollary ii can be checked in particular examples, we still need a manageable characterization of tightness on $\mathcal{M}_1(\mathbb{M})$ which we can conclude from Theorem 2. It will be given in terms of the distance distribution and the modulus of mass distribution.

Theorem 3 (Characterization of tightness) *A set $\mathbf{A} \subseteq \mathcal{M}_1(\mathbb{M})$ is tight if and only if the following holds:*

- (i) *The family $\{\mathbb{P}[w_{\mathcal{X}}] : \mathbb{P} \in \mathbf{A}\}$ is tight in $\mathcal{M}_1(\mathbb{R})$.*
- (ii) *For all $\varepsilon > 0$ there exist a $\delta = \delta(\varepsilon) > 0$ such that*

$$\sup_{\mathbb{P} \in \mathbf{A}} \mathbb{P}[v_{\delta}(\mathcal{X})] < \varepsilon. \tag{3.1}$$

Remark 3.2 (i) Using the properties of v_{δ} from Lemmata 6.4 and 6.5 it can be seen that (3.1) can be replaced either by

$$\sup_{\mathbb{P} \in \mathbf{A}} \mathbb{P}\{v_{\delta}(\mathcal{X}) \geq \varepsilon\} < \varepsilon \tag{3.2}$$

or

$$\sup_{\mathbb{P} \in \mathbf{A}} \mathbb{P}\{\mu\{x : \mu(B_{\varepsilon}(x)) \leq \delta\}\} < \varepsilon. \tag{3.3}$$

- (ii) If $\mathbf{A} = \{\mathbb{P}_1, \mathbb{P}_2, \dots\}$ then we can replace sup by lim sup in (3.1), (3.2) and (3.3).

The usage of Theorem 3 will be illustrated with the example of the Λ -coalescent measure tree constructed in the next section, and with examples of trees corresponding to spatially structured coalescents [15] and of evolving coalescents [16] in forthcoming work.

Remark 3.3 Starting with Theorem 3 one characterizes easily tightness for the stronger topology given in [27] based on certain L^2 -Wasserstein metrics if one requires in addition to (i) and (ii) uniform integrability of sampled mutual distance.

Similarly, with Theorem 3 one characterizes tightness in the space of measure preserving isometry classes of metric spaces equipped with a finite measure (rather than a probability measure) if one requires in addition tightness of the family of total masses (compare, also with Remark 7.2(ii)).

4 Example: Λ -coalescent measure trees

In this section we apply the theory of metric measure spaces to a class of genealogies which arise in population models. Often such genealogies are represented by coalescent processes and we focus on Λ -coalescents introduced in [24] (see also [26]). The family of Λ -coalescents appears in the description of the genealogies of population models with evolution based on resampling and branching. Such coalescent processes have since been the subject of many papers (see, for example, [3,5,6,21,23]).

In resampling models where the offspring variance of an individual during a reproduction event is finite, the Kingman coalescent appears as a special Λ -coalescent. The fact that general Λ -coalescents allow for multiple collisions is reflected in an infinite variance of the offspring distribution. Furthermore a Λ -coalescent is up to time change dual to the process of relative frequencies of families of a Galton–Watson process with possibly infinite variance offspring distribution (compare [3]). Our goal here is to decide for which Λ -coalescents the genealogies are described by a metric measure space.

We start with a quick description of Λ -coalescents. Recall that a partition of a set S is a collection $\{A_\lambda\}$ of pairwise disjoint subsets of S , also called *blocks*, such that $S = \cup_\lambda A_\lambda$. Denote by \mathbb{S}_∞ the collection of partitions of $\mathbb{N} := \{1, 2, 3, \dots\}$, and for all $n \in \mathbb{N}$, by \mathbb{S}_n the collection of partitions of $\{1, 2, 3, \dots, n\}$. Each partition $\mathcal{P} \in \mathbb{S}_\infty$ defines an equivalence relation $\sim_{\mathcal{P}}$ by $i \sim_{\mathcal{P}} j$ if and only if there exists a partition element $\pi \in \mathcal{P}$ with $i, j \in \pi$. Write ρ_n for the restriction map from \mathbb{S}_∞ to \mathbb{S}_n . We say that a sequence $(\mathcal{P}_k)_{k \in \mathbb{N}}$ converges in \mathbb{S}_∞ if for all $n \in \mathbb{N}$, the sequence $(\rho_n \mathcal{P}_k)_{k \in \mathbb{N}}$ converges in \mathbb{S}_n equipped with the discrete topology.

We are looking for a strong Markov process ξ starting in $\mathcal{P}_0 \in \mathbb{S}_\infty$ such that for all $n \in \mathbb{N}$, the restricted process $\xi_n := \rho_n \circ \xi$ is an \mathbb{S}_n -valued Markov chain which starts in $\rho_n \mathcal{P}_0 \in \mathbb{S}_n$, and given that $\xi_n(t)$ has b blocks, each k -tuple of blocks of \mathbb{S}_n is merging to form a single block at rate $\lambda_{b,k}$. Pitman [24] showed that such a process exists and is unique (in law) if and only if

$$\lambda_{b,k} := \int_0^1 \Lambda(dx) x^{k-2} (1-x)^{b-k} \tag{4.1}$$

for some non-negative and finite measure Λ on the Borel subsets of $[0, 1]$.

Let therefore Λ be a non-negative finite measure on $\mathcal{B}([0, 1])$ and $\mathcal{P} \in \mathbb{S}_\infty$. We denote by $\mathbb{P}^{\Lambda, \mathcal{P}}$ the probability distribution governing ξ with $\xi(0) = \mathcal{P}$ on the space of cadlag paths with the Skorohod topology.

Example 4.1 If we choose

$$\mathcal{P}^0 := \{\{1\}, \{2\}, \dots\}, \tag{4.2}$$

$\Lambda = \delta_0$, or $\Lambda(dx) = dx$, then $\mathbb{P}^{\Lambda, \mathcal{P}^0}$ is the *Kingman* and the *Bolthausen-Sznitman* coalescent, respectively.

For each non-negative and finite measure Λ , all initial partitions $\mathcal{P} \in \mathbb{S}_\infty$ and $\mathbb{P}^{\Lambda, \mathcal{P}}$ -almost all ξ , there is a (random) metric r^ξ on \mathbb{N} defined by

$$r^\xi(i, j) := \inf \{t \geq 0 : i \sim_{\xi(t)} j\}. \tag{4.3}$$

That is, for a realization ξ of the Λ coalescent, $r^\xi(i, j)$ is the time it needs i and j to coalesce. Notice that r^ξ is an *ultra-metric* on \mathbb{N} , almost surely, i.e., for all $i, j, k \in \mathbb{N}$,

$$r^\xi(i, j) \leq r^\xi(i, k) \vee r^\xi(k, j). \tag{4.4}$$

Let (L^ξ, r^ξ) denote the completion of (\mathbb{N}, r^ξ) . Clearly, the extension of r^ξ to L^ξ is also an ultra-metric. Recall that ultra-metric spaces are associated with tree-like structures.

The main goal of this section is to introduce the Λ -coalescent measure tree as the metric space (L^ξ, r^ξ) equipped with the “uniform distribution”. Notice that since the Kingman coalescent is known to “come down immediately to finitely many partition elements” the corresponding metric space is almost surely compact [12]. Even though there is no abstract concept of the “uniform distribution” on compact spaces, the reader may find it not surprising that in particular examples one can easily make sense out of this notion by approximation. We will see, that for Λ -coalescents, under an additional assumption on Λ , one can extend the uniform distribution to locally compact metric spaces. Within this class falls, for example, the Bolthausen–Sznitman coalescent which is known to have infinitely many partition elements for all times, and whose corresponding metric space is therefore not compact.

Define H_n to be the map which takes a realization of the \mathbb{S}_∞ -valued coalescent and maps it to (an isometry class of) a metric measure space as follows:

$$H_n : \xi \mapsto \left(L^\xi, r^\xi, \mu_n^\xi := \frac{1}{n} \sum_{i=1}^n \delta_i \right). \tag{4.5}$$

Put then for given $\mathcal{P}_0 \in \mathbb{S}_\infty$,

$$\mathbb{Q}^{\Lambda, n} := (H_n)_* \mathbb{P}^{\Lambda, \mathcal{P}_0}. \tag{4.6}$$

Next we give the characterization of existence and uniqueness of the Λ -coalescent measure tree.

Theorem 4 (The Λ -coalescent measure tree) *The family $\{\mathbb{Q}^{\Lambda, n}; n \in \mathbb{N}\}$ converges in the weak topology with respect to the Gromov-weak topology if and only if*

$$\int_0^1 \Lambda(dx) x^{-1} = \infty. \tag{4.7}$$

Remark 4.2 (“Dust-free” property) Notice first that Condition (4.7) is equivalent to the total coalescence rate of a given $\{i\} \in \mathcal{P}_0$ being infinite (compare with the proof of Lemma 25 in [24]).

By exchangeability and the de Finetti Theorem, the family $\{\tilde{f}(\pi); \pi \in \xi(t)\}$ of frequencies

$$\tilde{f}(\pi) := \lim_{n \rightarrow \infty} \frac{1}{n} \#\{j \in \{1, \dots, n\} : j \in \pi\} \tag{4.8}$$

exists for $\mathbb{P}^{\Lambda, \mathcal{P}_0}$ almost all $\pi \in \xi(t)$ and all $t > 0$. Define $f := (f(\pi); \pi \in \xi(t))$ to be the ranked rearrangements of $\{\tilde{f}(\pi); \pi \in \xi(t)\}$ meaning that the entrees of the vector f are non-increasing. Let $\mathbf{P}^{\Lambda, \mathcal{P}_0}$ denote the probability distribution of f . Call the frequencies f proper if $\sum_{i \geq 1} f(\pi_i) = 1$. By Theorem 8 in [24], the Λ -coalescent has in the limit $n \rightarrow \infty$ proper frequencies if and only if Condition (4.7) holds.

According to Kingman’s correspondence (see, for example, Theorem 14 in [24]), the distribution $\mathbb{P}^{\Lambda, \mathcal{P}_0}$ and $\mathbf{P}^{\Lambda, \mathcal{P}_0}$ determine each other uniquely. For $\mathcal{P} \in \mathbb{S}_\infty$ and $i \in \mathbb{N}$, let $\mathcal{P}^i := \{j \in \mathbb{N} : i \sim_{\mathcal{P}} j\}$ denote the partition element in \mathcal{P} which contains i . Then Condition (4.7) holds if and only if for all $t > 0$,

$$\mathbb{P}^{\Lambda, \mathcal{P}_0} \left\{ \tilde{f} \left((\xi(t))^1 \right) = 0 \right\} = 0. \tag{4.9}$$

The latter is often referred to as the “dust”-free property.

Proof of Theorem 4 For existence we will apply the characterization of tightness as given in Theorem 3, and verify the two conditions.

(i) By definition, for all $n \in \mathbb{N}$, $\mathbb{Q}^{\Lambda, n}[w_{\mathcal{X}}]$ is exponentially distributed with parameter $\lambda_{2,2}$. Hence the family $\{\mathbb{Q}^{\Lambda, n}[w_{\mathcal{X}}]; n \in \mathbb{N}\}$ is tight.

(ii) Fix $t \in (0, 1)$. Then for all $\delta > 0$, by the uniform distribution and exchangeability,

$$\begin{aligned} & \mathbb{Q}^{\Lambda, n} [\mu\{x : B_\varepsilon(x) \leq \delta\}] \\ &= \mathbb{P}^{\Lambda, \mathcal{P}_0} [\mu_n^\xi \{x \in L^\xi : \mu_n^\xi(B_t(x)) \leq \delta \mid x = 1\}] \\ &= \mathbb{P}^{\Lambda, \mathcal{P}_0} \{ \mu_n^\xi(B_t(1)) \leq \delta \}. \end{aligned} \tag{4.10}$$

By the de Finetti theorem, $\mu_n^\xi(B_t(1)) \xrightarrow{n \rightarrow \infty} \tilde{f}((\xi(t))^1)$, $\mathbb{P}^{\Lambda, \mathcal{P}_0}$ -almost surely. Hence, dominated convergence yields

$$\begin{aligned} \lim_{\delta \rightarrow 0} \lim_{n \rightarrow \infty} \mathbb{Q}^{\Lambda, n} [\mu\{x : B_\varepsilon(x) \leq \delta\}] &= \lim_{\delta \rightarrow 0} \mathbb{P}^{\Lambda, \mathcal{P}_0} \left\{ \tilde{f}((\xi(t))^1) \leq \delta \right\} \\ &= \mathbb{P}^{\Lambda, \mathcal{P}_0} \left\{ \tilde{f}((\xi(t))^1) = 0 \right\}. \end{aligned} \tag{4.11}$$

We have shown that Condition (4.7) is equivalent to (4.9), and therefore, using (3.3), a limit of $\mathbb{Q}^{\Lambda, n}$ exists if and only if the “dust-free”-property holds.

Uniqueness of the limit points follows from the projective property, i.e. restricting the observation to a tagged subset of initial individuals is the same as starting in this restricted initial state. □

5 A complete metric: the Gromov–Prohorov metric

In this section we introduce the Gromov–Prohorov metric d_{GPr} on \mathbb{M} and prove that the metric space $(\mathbb{M}, d_{\text{GPr}})$ is complete and separable. In Sect. 9 we will see that the Gromov–Prohorov metric generates the Gromov-weak topology.

Notice that the first naive approach to metrize the Gromov-weak topology could be to fix a countably dense subset $\{\Phi_n; n \in \mathbb{N}\}$ in the algebra of all polynomials, and to put for $\mathcal{X}, \mathcal{Y} \in \mathbb{M}$,

$$d_{\text{naive}}(\mathcal{X}, \mathcal{Y}) := \sum_{n \in \mathbb{N}} 2^{-n} |\Phi_n(\mathcal{X}) - \Phi_n(\mathcal{Y})|. \tag{5.1}$$

However, such a metric is not complete. Indeed one can check that the sequence $\{\mathcal{X}_n; n \in \mathbb{N}\}$ given in Example 2.12(ii) is a Cauchy sequence w.r.t d_{naive} which does not converge.

Recall that metrics on the space of probability measures on a fixed complete and separable metric space are well-studied (see, for example, [18, 25]). Some of them, like the Prohorov metric and the Wasserstein metric (on compact spaces) generate the weak topology. On the other hand the space of all (isometry classes of compact) metric spaces, not carrying a measure, is complete and separable once equipped with the Gromov–Hausdorff metric (see, [11]). We recall the notion of the Prohorov and Gromov–Hausdorff metric below.

Metrics on metric measure spaces should take both components into account and compare the spaces and the measures simultaneously. This was, for example, done in [13, 27]. We will follow along similar lines as in [27], but replace the Wasserstein metric with the Prohorov metric.

Recall that the *Prohorov metric* between two probability measures μ_1 and μ_2 on a common metric space (Z, r_Z) is defined by

$$d_{\text{Pr}}^{(Z, r_Z)}(\mu_1, \mu_2) := \inf \{ \varepsilon > 0 : \mu_1(F) \leq \mu_2(F^\varepsilon) + \varepsilon, \forall F \text{ closed} \} \tag{5.2}$$

where

$$F^\varepsilon := \{ z \in Z : r_Z(z, z') < \varepsilon, \text{ for some } z' \in F \}. \tag{5.3}$$

Sometimes it is easier to work with the equivalent formulation based on *couplings* of the measures μ_1 and μ_2 , i.e., measures $\tilde{\mu}$ on $X \times Y$ with $\tilde{\mu}(\cdot \times Y) = \mu_1(\cdot)$ and $\tilde{\mu}(X \times \cdot) = \mu_2(\cdot)$. Notice that the product measure $\mu_1 \otimes \mu_2$ is a coupling, and so the set of all couplings of two measures is not empty. By Theorem 3.1.2 in [10],

$$d_{Pr}^{(Z,r_Z)}(\mu_1, \mu_2) = \inf_{\tilde{\mu}} \inf \{ \varepsilon > 0 : \tilde{\mu} \{ (z, z') \in Z \times Z : r_Z(z, z') \geq \varepsilon \} \leq \varepsilon \}, \tag{5.4}$$

where the infimum is taken over all couplings $\tilde{\mu}$ of μ_1 and μ_2 . The metric $d_{Pr}^{(Z,r_Z)}$ is complete and separable if (Z, r_Z) is complete and separable ([10], Theorem 3.1.7).

The Gromov–Hausdorff metric is a metric on the space \mathbb{X}_c of (isometry classes of) compact metric spaces. For (X, r_X) and (Y, r_Y) in \mathbb{X}_c the *Gromov–Hausdorff metric* is given by

$$d_{GH}((X, r_X), (Y, r_Y)) := \inf_{(\varphi_X, \varphi_Y, Z)} d_H^Z(\varphi_X(X), \varphi_Y(Y)), \tag{5.5}$$

where the infimum is taken over isometric embeddings φ_X and φ_Y from X and Y , respectively, into some common metric space (Z, r_Z) , and the Hausdorff metric $d_H^{(Z,r_Z)}$ for closed subsets of a metric space (Z, r_Z) is given by

$$d_H^{(Z,r_Z)}(X, Y) := \inf \{ \varepsilon > 0 : X \subseteq Y^\varepsilon, Y \subseteq X^\varepsilon \}, \tag{5.6}$$

where X^ε and Y^ε are given by (5.3) (compare [4, 7, 17]).

Sometimes, it is handy to use an equivalent formulation of the Gromov–Hausdorff metric based on *correspondences*. Recall that a *relation* R between two compact metric spaces (X, r_X) and (Y, r_Y) is any subset of $X \times Y$. A relation $R \subseteq X \times Y$ is called a *correspondence* iff for each $x \in X$ there exists at least one $y \in Y$ such that $(x, y) \in R$, and for each $y' \in Y$ there exists at least one $x' \in X$ such that $(x', y') \in R$. Define the *distortion* of a (non-empty) relation as

$$\text{dis}(R) := \sup \{ |r_X(x, x') - r_Y(y, y')| : (x, y), (x', y') \in R \}. \tag{5.7}$$

Then by Theorem 7.3.25 in [4], the Gromov–Hausdorff metric can be given in terms of a minimal distortion of all correspondences, i.e.,

$$d_{GH}((X, r_X), (Y, r_Y)) = \frac{1}{2} \inf_R \text{dis}(R), \tag{5.8}$$

where the infimum is over all correspondences R between X and Y .

To define a metric between two metric measure spaces $\mathcal{X} = (X, r_X, \mu_X)$ and $\mathcal{Y} = (Y, r_Y, \mu_Y)$ in \mathbb{M} , we can neither use the Prohorov metric nor the Gromov–Hausdorff metric directly. However, we can use the idea due to Gromov and embed (X, r_X) and (Y, r_Y) isometrically into a common metric space and measure the distance of the image measures.

Definition 5.1 (Gromov–Prohorov metric) The Gromov–Prohorov distance between two metric measure spaces $\mathcal{X} = (X, r_X, \mu_X)$ and $\mathcal{Y} = (Y, r_Y, \mu_Y)$ in \mathbb{M} is defined by

$$d_{GPr}(\mathcal{X}, \mathcal{Y}) := \inf_{(\varphi_X, \varphi_Y, Z)} d_{Pr}^{(Z,r_Z)}((\varphi_X)_*\mu_X, (\varphi_Y)_*\mu_Y), \tag{5.9}$$

where the infimum is taken over all isometric embeddings φ_X and φ_Y from X and Y , respectively, into some common metric space (Z, r_Z) .

- Remark 5.2* (i) To see that the Gromov–Prohorov metric is well-defined we have to check that the right hand side of (5.9) does not depend on the element of the isometry class of (X, r_X, μ_X) and (Y, r_Y, μ_Y) . We leave out the straightforward details.
- (ii) Notice that w.l.o.g. the common metric space (Z, r_Z) and the isometric embeddings φ_X and φ_Y from X and Y can be chosen to be $X \sqcup Y$ and the canonical embeddings φ_X and φ_Y from X and Y to $X \sqcup Y$, respectively (compare, for example, Remark 3.3(iii) in [27]). We can therefore also write

$$d_{\text{GPr}}(\mathcal{X}, \mathcal{Y}) := \inf_{r^{X,Y}} d_{\text{Pr}}^{(X \sqcup Y, r^{X,Y})}((\varphi_X)_* \mu_X, (\varphi_Y)_* \mu_Y), \tag{5.10}$$

where the infimum is here taken over all complete and separable metrics $r^{X,Y}$ which extend the metrics r_X on X and r_Y on Y to $X \sqcup Y$.

Remark 5.3 (Gromov’s \square_1 -metric) Even though the material presented in this paper was developed independently of Gromov’s work, some of the most important ideas are already contained in Chap. 3 $\frac{1}{2}$ in [17].

More detailed, one can also start with a Polish space (X, \mathcal{O}) which is equipped with an probability measure $\mu \in \mathcal{M}_1(X)$ on $\mathcal{B}(X)$, and then introduce a metric $r : X \times X \rightarrow \mathbb{R}_+$ as a measurable function satisfying the metric axioms. Polish measure spaces (X, μ) can be parameterized by the segment $[0, 1)$ where the parametrization refers to a measure preserving map $\varphi : [0, 1) \rightarrow X$. If r is a metric on X then r can be pulled back to a metric $(\varphi^{-1})_* r$ on $[0, 1)$ by letting

$$(\varphi^{-1})_* r(t, t') := r(\varphi(t), \varphi(t')). \tag{5.11}$$

Notice that such a measure-preserving parametrization is far from unique and Gromov introduces his \square_1 -distance between (X, r, μ) and (X', r', μ') as the infimum of distances \square_1 between the two metric spaces $([0, 1), (\varphi^{-1})_* r)$ and $([0, 1), (\psi^{-1})_* r')$ defined as

$$\begin{aligned} \square_1(d, d') &:= \sup \left\{ \varepsilon > 0 : \exists X_\varepsilon \in \mathcal{B}([0, 1)) : \lambda(X_\varepsilon) \leq \varepsilon, \text{ s.t.} \right. \\ &\quad \left. |d(t_1, t_2) - d'(t_1, t_2)| \leq \varepsilon, \forall t_1, t_2 \in X \setminus X_\varepsilon \right\}, \end{aligned} \tag{5.12}$$

where the infimum is taken all possible measure preserving parameterizations and λ denotes the Lebesgue measure.

The interchange of first embedding in a measure preserving way and then taking the distance between the pulled back metric spaces versus first embedding isometrically and then taking the distance between the pushed forward measures explains the similarities between Gromov’s ε -partition lemma (Sect. 3 $\frac{1}{2}$.8 in [17]), his union lemma

(Sect. 3 $\frac{1}{2}$.12 in [17]) and his pre-compactness criterion (Sect. 3 $\frac{1}{2}$.D in [17]) on the one hand and our Lemma 6.9, Lemma 5.8 and Proposition 7.1, respectively, on the other.

We strongly conjecture that the Gromov-weak topology agrees with the topology generated by Gromov’s \square_1 -metric but a (straightforward) proof is not obvious to us.

We first show that the Gromov–Prohorov distance is indeed a metric.

Lemma 5.4 d_{GPr} defines a metric on \mathbb{M} .

In the following we refer to the topology generated by the Gromov–Prohorov metric as the *Gromov–Prohorov topology*. In Theorem 5 of Sect.9 we will prove that the Gromov–Prohorov topology and the Gromov-weak topology coincide.

Remark 5.5 (Extension of metrics via relations) The proof of the lemma and some of the following results is based on the extension of two metric spaces (X_1, r_{X_1}) and (X_2, r_{X_2}) if a non-empty relation $R \subseteq X_1 \times X_2$ is known. The result is a metric on $X_1 \sqcup X_2$ where \sqcup is the disjoint union. Recall the distortion of a relation from (5.7) Define the metric space $(X_1 \sqcup X_2, r_{X_1 \sqcup X_2}^R)$ by letting $r_{X_1 \sqcup X_2}^R(x, x') := r_{X_i}(x, x')$ if $x, x' \in X_i, i = 1, 2$ and for $x_1 \in X_1$ and $x_2 \in X_2$,

$$r_{X_1 \sqcup X_2}^R(x_1, x_2) := \inf \{ r_{X_1}(x_1, x'_1) + \frac{1}{2} \text{dis}(R) + r_{X_2}(x_2, x'_2) : (x'_1, x'_2) \in R \}. \tag{5.13}$$

It is then easy to check that $r_{X_1 \sqcup X_2}^R$ defines a (pseudo-)metric on $X_1 \sqcup X_2$ which extends the metrics on X_1 and X_2 . In particular, $r_{X_1 \sqcup X_2}^R(x_1, x_2) = \frac{1}{2} \text{dis}(R)$, for any pair $(x_1, x_2) \in R$, and

$$d_H^{(X_1 \sqcup X_2, r_{X_1 \sqcup X_2}^R)}(\pi_1 R, \pi_2 R) = \frac{1}{2} \text{dis}(R), \tag{5.14}$$

where π_1 and π_2 are the projection operators on X_1 and X_2 , respectively.

Proof of Lemma 5.4 *Symmetry* is obvious and *positive definiteness* can be shown by standard arguments. To see the *triangle inequality*, let $\varepsilon, \delta > 0$ and $\mathcal{X}_i := (X_i, r_{X_i}, \mu_{X_i}) \in \mathbb{M}, i = 1, 2, 3$, be such that $d_{\text{GPr}}(\mathcal{X}_1, \mathcal{X}_2) < \varepsilon$ and $d_{\text{GPr}}(\mathcal{X}_2, \mathcal{X}_3) < \delta$. Then, by the definition (5.9) together with Remark 5.2(ii), we can find metrics $r^{1,2}$ and $r^{2,3}$ on $X_1 \sqcup X_2$ and $X_2 \sqcup X_3$, respectively, such that

$$d_{\text{Pr}}^{(X_1 \sqcup X_2, r^{1,2})}((\tilde{\varphi}_1)_* \mu_{X_1}, (\tilde{\varphi}_2)_* \mu_{X_2}) < \varepsilon, \tag{5.15}$$

and

$$d_{\text{Pr}}^{(X_2 \sqcup X_3, r^{2,3})}((\tilde{\varphi}'_2)_* \mu_{X_2}, (\tilde{\varphi}_3)_* \mu_{X_3}) < \delta, \tag{5.16}$$

where $\tilde{\varphi}_1, \tilde{\varphi}_2$ and $\tilde{\varphi}'_2, \tilde{\varphi}_3$ are canonical embeddings from X_1, X_2 to $X_1 \sqcup X_2$ and X_2, X_3 to $X_2 \sqcup X_3$, respectively. Setting $Z := (X_1 \sqcup X_2) \sqcup (X_2 \sqcup X_3)$ we define the metric

r_Z^R on Z using the relation

$$R := \{(\tilde{\varphi}_2(x), \tilde{\varphi}'_2(x)) : x \in X_2\} \subseteq (X_1 \sqcup X_2) \times (X_2 \sqcup X_3) \tag{5.17}$$

and Remark 5.5. Denote the canonical embeddings from X_1 , the two copies of X_2 and X_3 to Z by $\varphi_1, \varphi_2, \varphi'_2$ and φ_3 , respectively. Since $\text{dis}(R) = 0$ and

$$d_{\text{Pr}}^{(Z, r_Z^R)}((\varphi_2)_*\mu_2, (\varphi'_2)_*\mu_2) = 0, \tag{5.18}$$

by the triangle inequality of the Prohorov metric,

$$\begin{aligned} d_{\text{GPr}}(\mathcal{X}_1, \mathcal{X}_3) &\leq d_{\text{Pr}}^{(Z, r_Z^R)}((\varphi_1)_*\mu_1, (\varphi_3)_*\mu_3) \\ &\leq d_{\text{Pr}}^{(Z, r_Z^R)}((\varphi_1)_*\mu_1, (\varphi_2)_*\mu_2) + d_{\text{Pr}}^{(Z, r_Z^R)}((\varphi_2)_*\mu_2, (\varphi'_2)_*\mu_2) \\ &\quad + d_{\text{Pr}}^{(Z, r_Z^R)}((\varphi'_2)_*\mu_2, (\varphi_3)_*\mu_3) \\ &< \varepsilon + \delta. \end{aligned} \tag{5.19}$$

Hence the triangle inequality follows by taking the infimum over all ε and δ . □

Proposition 5.6 *The metric space $(\mathbb{M}, d_{\text{GPr}})$ is complete and separable.*

We prepare the proof with a lemma.

Lemma 5.7 *Fix $(\varepsilon_n)_{n \in \mathbb{N}}$ in $(0, 1)$. A sequence $(\mathcal{X}_n := (X_n, r_n, \mu_n))_{n \in \mathbb{N}}$ in \mathbb{M} satisfies*

$$d_{\text{GPr}}(\mathcal{X}_n, \mathcal{X}_{n+1}) < \varepsilon_n \tag{5.20}$$

if and only if there exist a complete and separable metric space (Z, r_Z) and isometric embeddings $\varphi_1, \varphi_2, \dots$ from X_1, X_2, \dots , respectively, into (Z, r_Z) , such that

$$d_{\text{Pr}}^{(Z, r_Z)}((\varphi_n)_*\mu_n, (\varphi_{n+1})_*\mu_{n+1}) < \varepsilon_n. \tag{5.21}$$

Proof The “if” direction is clear. For the “only if” direction, take sequences $(\mathcal{X}_n := (X_n, r_n, \mu_n))_{n \in \mathbb{N}}$ and $(\varepsilon_n)_{n \in \mathbb{N}}$ which satisfy (5.20). By Remark 5.2, for $Y_n := X_n \sqcup X_{n+1}$ and all $n \in \mathbb{N}$, there is a metric r_{Y_n} on Y_n such that

$$d_{\text{Pr}}^{(Y_n, r_{Y_n})}((\varphi_n)_*\mu_n, (\varphi_{n+1})_*\mu_{n+1}) < \varepsilon_n \tag{5.22}$$

where φ_n and φ_{n+1} are the canonical embeddings from X_n and X_{n+1} to Y_n . Put

$$R_n := \{(x, x') \in X_n \times X_{n+1} : r_{Y_n}(\varphi_n(x), \varphi_{n+1}(x')) < \varepsilon_n\}. \tag{5.23}$$

Recall from (5.4) that (5.22) implies the existence of a coupling $\tilde{\mu}_n$ of $(\varphi_n)_*\mu_n$ and $(\varphi_{n+1})_*\mu_{n+1}$ such that

$$\tilde{\mu}_n \{(x, x') : r_{Y_n}(y, y') < \varepsilon_n\} > 1 - \varepsilon_n. \tag{5.24}$$

This implies that R_n is not empty and

$$d_{Pr}^{(Y_n, r_{Y_n}^{R_n})} ((\varphi_n)_* \mu_n, (\varphi_{n+1})_* \mu_{n+1}) \leq \varepsilon_n. \tag{5.25}$$

Using the metric spaces $(Y_n, r_{Y_n}^{R_n})$ we define recursively metrics r_{Z_n} on $Z_n := \bigsqcup_{k=1}^n X_k$. Starting with $n = 1$, we set $(Z_1, r_{Z_1}) := (X_1, r_1)$. Next, assume we are given a metric r_{Z_n} on Z_n . Consider the isometric embeddings ψ_k^n from X_k to Z_n , for $k = 1, \dots, n$ which arise from the canonical embedding of X_k in Z_n . Define for all $n \in \mathbb{N}$,

$$\tilde{R}_n := \left\{ (z, x) \in Z_n \times X_{n+1} : ((\psi_n^n)^{-1}(z), x) \in R_n \right\} \tag{5.26}$$

which defines metrics $r_{Z_{n+1}}^{\tilde{R}_n}$ on Z_{n+1} via (5.13).

By this procedure we obtain in the limit a separable metric space $(Z' := \bigsqcup_{n=1}^\infty X_n, r_{Z'})$. Denote its completion by (Z, r_Z) and isometric embeddings from X_n to Z which arise by the canonical embedding by $\psi_n, n \in \mathbb{N}$. Observe that the restriction of r_Z to $X_n \sqcup X_{n+1}$ is isometric to $(Y_n, r_{Y_n}^{R_n})$ and thus

$$d_{Pr}^{(Z, r_Z)} ((\psi_n)_* \mu_{X_n}, (\psi_{n+1})_* \mu_{X_{n+1}}) \leq \varepsilon_n \tag{5.27}$$

by (5.25). So the claim follows. □

Proof of Proposition 5.6 To get *separability*, we partly follow the proof of Theorem 3.2.2 in [10]. Given $\mathcal{X} := (X, r, \mu) \in \mathbb{M}$ and $\varepsilon > 0$, we can find $\mathcal{X}^\varepsilon := (X, r, \mu^\varepsilon) \in \mathbb{M}$ such that μ^ε is a finitely supported atomic measure on X and $d_{Pr}(\mu^\varepsilon, \mu) < \varepsilon$. Now $d_{GPr}(\mathcal{X}^\varepsilon, \mathcal{X}) < \varepsilon$, while X^ε is just a “finite metric space” and can clearly be approximated arbitrarily closely in the Gromov–Prohorov metric by finite metric spaces with rational mutual distances and weights. The set of isometry classes of finite metric spaces with rational edge-lengths is countable, and so (\mathbb{M}, d_{GPr}) is separable.

To get *completeness*, it suffices to show that every Cauchy sequence has a convergent subsequence. Take therefore a Cauchy sequence $(\mathcal{X}_n)_{n \in \mathbb{N}}$ in (\mathbb{M}, d_{GPr}) and a subsequence $(\mathcal{Y}_n)_{n \in \mathbb{N}}, \mathcal{Y}_n = (Y_n, r_n, \mu_n)$ with $d_{GPr}(\mathcal{Y}_n, \mathcal{Y}_{n+1}) \leq 2^{-n}$. By Lemma 5.7 we can choose a complete and separable metric space (Z, r_Z) and, for each $n \in \mathbb{N}$, an isometric embedding φ_n from Y_n into (Z, r_Z) such that $((\varphi_n)_* \mu_n)_{n \in \mathbb{N}}$ is a Cauchy sequence on $\mathcal{M}_1(Z)$ equipped with the weak topology. By the completeness of $\mathcal{M}_1(Z)$, $((\varphi_n)_* \mu_n)_{n \in \mathbb{N}}$ converges to some $\bar{\mu} \in \mathcal{M}_1(Z)$.

Putting the arguments together yields that with $\mathcal{Z} := (Z, r_Z, \bar{\mu})$,

$$d_{GPr}(\mathcal{Y}_n, \mathcal{Z}) \xrightarrow{n \rightarrow \infty} 0, \tag{5.28}$$

so that \mathcal{Z} is the desired limit object, which finishes the proof. □

We conclude this section by another Lemma.

Lemma 5.8 *Let $\mathcal{X} = (X, r, \mu)$, $\mathcal{X}_1 = (X_1, r_1, \mu_1)$, $\mathcal{X}_2 = (X_2, r_2, \mu_2), \dots$ be in \mathbb{M} . Then,*

$$d_{\text{GPr}}(\mathcal{X}_n, \mathcal{X}) \xrightarrow{n \rightarrow \infty} 0 \tag{5.29}$$

if and only if there exists a complete and separable metric space (Z, r_Z) and isometric embeddings $\varphi, \varphi_1, \varphi_2, \dots$ from X, X_1, X_2 into (Z, r_Z) , respectively, such that

$$d_{\text{Pr}}^{(Z, r_Z)}((\varphi_n)_* \mu_n, \varphi_* \mu) \xrightarrow{n \rightarrow \infty} 0. \tag{5.30}$$

Proof Again the “if” direction is clear by definition. For the “only if” direction, assume that (5.29) holds. To conclude (5.30) we can follow the same line of argument as in the proof of Lemma 5.7 but with a metric r extending the metrics r, r_1, r_2, \dots built on correspondences between X and X_n (rather than X_n and X_{n+1}). We leave out the details. □

6 Distance distribution and Modulus of mass distribution

In this section we provide results on the distance distribution and on the modulus of mass distribution. These will be heavily used in the following sections, where we present metrics which are equivalent to the Gromov–Prohorov metric and which are very helpful in proving the characterizations of compactness and tightness in the Gromov–Prohorov topology.

We start by introducing the *random distance distribution* of a given metric measure space.

Definition 6.1 (Random distance distribution) Let $\mathcal{X} = (X, r, \mu) \in \mathbb{M}$. For each $x \in X$, define the map $r_x : X \rightarrow [0, \infty)$ by $r_x(x') := r(x, x')$, and put $\mu^x := (r_x)_* \mu \in \mathcal{M}_1([0, \infty))$, i.e., μ^x defines the distribution of distances to the point $x \in X$. Moreover, define the map $\hat{r} : X \rightarrow \mathcal{M}_1([0, \infty))$ by $\hat{r}(x) := \mu^x$, and let

$$\hat{\mu}_{\mathcal{X}} := \hat{r}_* \mu \in \mathcal{M}_1(\mathcal{M}_1([0, \infty))) \tag{6.1}$$

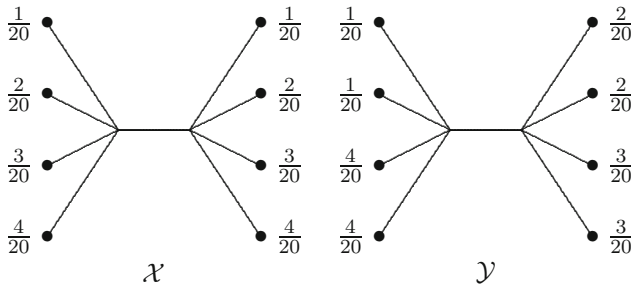
be the *random distance distribution* of \mathcal{X} .

Notice first that the random distance distribution does not characterizes the metric measure space uniquely. We will illustrate this with an example.

Example 6.2 Consider the following two metric measure spaces:

That is, both spaces consist of 8 points. The distance between two points equals the minimal number of edges one has to cross to come from one point to the other. The measures μ_X and μ_Y are given by the numbers in the figure. We find that

$$\begin{aligned} \hat{\mu}_X = \hat{\mu}_Y &= \frac{1}{10} \delta_{\frac{1}{20} \delta_0 + \frac{9}{20} \delta_2 + \frac{1}{2} \delta_3} + \frac{1}{5} \delta_{\frac{1}{10} \delta_0 + \frac{2}{5} \delta_2 + \frac{1}{2} \delta_3} \\ &+ \frac{3}{10} \delta_{\frac{3}{20} \delta_0 + \frac{7}{20} \delta_2 + \frac{1}{2} \delta_3} + \frac{2}{5} \delta_{\frac{1}{5} \delta_0 + \frac{3}{10} \delta_2 + \frac{1}{2} \delta_3}. \end{aligned} \tag{6.2}$$



Hence, the random distance distributions agree. But obviously, \mathcal{X} and \mathcal{Y} are not measure preserving isometric.

Recall the distance distribution w , and the modulus of mass distribution $v_\delta(\cdot)$ from Definition 2.9. Both can be expressed through the random distance distribution $\hat{\mu}(\cdot)$. These facts follow directly from the definitions, so we omit the proof.

Lemma 6.3 (Reformulation of w , and $v_\delta(\cdot)$ in terms of $\hat{\mu}(\cdot)$) *Let $\mathcal{X} \in \mathbb{M}$.*

(i) *The distance distribution $w_{\mathcal{X}}$ satisfies*

$$w_{\mathcal{X}} = \int_{\mathcal{M}_1([0, \infty))} \hat{\mu}_{\mathcal{X}}(dv) v. \tag{6.3}$$

(ii) *For all $\delta > 0$, the modulus of mass distribution $v_\delta(\mathcal{X})$ satisfies*

$$v_\delta(\mathcal{X}) = \inf \{ \varepsilon > 0 : \hat{\mu}_{\mathcal{X}}\{v \in \mathcal{M}_1([0, \infty)) : v([0, \varepsilon)) \leq \delta\} \leq \varepsilon \}. \tag{6.4}$$

The next result will be used frequently.

Lemma 6.4 *Let $\mathcal{X} = (X, r, \mu) \in \mathbb{M}$ and $\delta > 0$. If $v_\delta(\mathcal{X}) < \varepsilon$, for some $\varepsilon > 0$, then*

$$\mu \{x \in X : \mu(B_\varepsilon(x)) \leq \delta\} < \varepsilon. \tag{6.5}$$

Proof By definition of $v_\delta(\cdot)$, there exists $\varepsilon' < \varepsilon$ for which $\mu \{x \in X : \mu(B_{\varepsilon'}(x)) \leq \delta\} \leq \varepsilon'$. Consequently, since $\{x : \mu(B_\varepsilon(x)) \leq \delta\} \subseteq \{x : \mu(B_{\varepsilon'}(x)) \leq \delta\}$,

$$\mu \{x : \mu(B_\varepsilon(x)) \leq \delta\} \leq \mu \{x : \mu(B_{\varepsilon'}(x)) \leq \delta\} \leq \varepsilon' < \varepsilon, \tag{6.6}$$

and we are done. □

The next result states basic properties of the map $\delta \mapsto v_\delta$.

Lemma 6.5 (Properties of $v_\delta(\cdot)$) *Fix $\mathcal{X} \in \mathbb{M}$. The map which sends $\delta \geq 0$ to $v_\delta(\mathcal{X})$ is non-decreasing, right-continuous and bounded by 1. Moreover, $v_\delta(\mathcal{X}) \xrightarrow{\delta \rightarrow 0} 0$.*

Proof The first three properties are trivial. For the fourth, fix $\varepsilon > 0$, and let $\mathcal{X} = (X, r, \mu) \in \mathbb{M}$. Since X is complete and separable there exists a compact set $K_\varepsilon \subseteq X$ with $\mu(K_\varepsilon) > 1 - \varepsilon$ (see [10], Lemma 3.2.1). In particular, K_ε can be covered by finitely many balls $A_1, \dots, A_{N_\varepsilon}$ of radius $\varepsilon/2$ and positive μ -mass. Choose δ such that

$$0 < \delta < \min \{ \mu(A_i) : 1 \leq i \leq N_\varepsilon \}. \tag{6.7}$$

Then

$$\begin{aligned} \mu \{ x \in X : \mu(B_\varepsilon(x)) > \delta \} &\geq \mu \left(\bigcup_{i=1}^{N_\varepsilon} A_i \right) \\ &\geq \mu(K_\varepsilon) \\ &> 1 - \varepsilon. \end{aligned} \tag{6.8}$$

Therefore, by definition, $v_\delta(\mathcal{X}) \leq \varepsilon$, and since ε was chosen arbitrary, the assertion follows. \square

The following proposition states continuity properties of $\hat{\mu}(\cdot)$, w , and $v_\delta(\cdot)$. The reader should have in mind that we finally prove with Theorem 5 in Sect. 9 that the Gromov-weak and the Gromov-Prohorov topology are the same.

Proposition 6.6 (Continuity properties of $\hat{\mu}(\cdot)$, w , and $v_\delta(\cdot)$)

- (i) *The map $\mathcal{X} \mapsto \hat{\mu}_\mathcal{X}$ is continuous with respect to the Gromov-weak topology on \mathbb{M} and the weak topology on $\mathcal{M}_1(\mathcal{M}_1([0, \infty)))$.*
- (ii) *The map $\mathcal{X} \mapsto \hat{\mu}_\mathcal{X}$ is continuous with respect to the Gromov-Prohorov topology on \mathbb{M} and the weak topology on $\mathcal{M}_1(\mathcal{M}_1([0, \infty)))$.*
- (iii) *The map $\mathcal{X} \mapsto w_\mathcal{X}$ is continuous with respect to both the Gromov-weak and the Gromov-Prohorov topology on \mathbb{M} and the weak topology on $\mathcal{M}_1([0, \infty))$.*
- (iv) *Let $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \dots$ in \mathbb{M} such that $\hat{\mu}_{\mathcal{X}_n} \xrightarrow{n \rightarrow \infty} \hat{\mu}_\mathcal{X}$ and $\delta > 0$. Then*

$$\limsup_{n \rightarrow \infty} v_\delta(\mathcal{X}_n) \leq v_\delta(\mathcal{X}). \tag{6.9}$$

The proof of Parts (i) and (ii) of Proposition 6.6 are based on the notion of moment measures.

Definition 6.7 (Moment measures of $\hat{\mu}_\mathcal{X}$) For $\mathcal{X} = (X, r, \mu) \in \mathbb{M}$ and $k \in \mathbb{N}$, define the k^{th} moment measure $\hat{\mu}_\mathcal{X}^k \in \mathcal{M}_1([0, \infty)^k)$ of $\hat{\mu}_\mathcal{X}$ by

$$\hat{\mu}_\mathcal{X}^k(d(r_1, \dots, r_k)) := \int \hat{\mu}_\mathcal{X}(dv) v^{\otimes k}(d(r_1, \dots, r_k)). \tag{6.10}$$

Remark 6.8 (Moment measures determine $\hat{\mu}_\mathcal{X}$) Observe that for all $k \in \mathbb{N}$,

$$\begin{aligned} &\hat{\mu}_\mathcal{X}^k(A_1 \times \dots \times A_k) \\ &= \mu^{\otimes k+1} \{ (u_0, u_1, \dots, u_k) : r(u_0, u_1) \in A_1, \dots, r(u_0, u_k) \in A_k \}. \end{aligned} \tag{6.11}$$

By Theorem 16.16 of [20], the moment measures $\hat{\mu}_{\mathcal{X}}^k, k = 1, 2, \dots$ determine $\hat{\mu}_{\mathcal{X}}$ uniquely. Moreover, weak convergence of random measures is equivalent to convergence of all moment measures.

Proof of Proposition 6.6 (i) Take $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \dots$ in \mathbb{M} such that

$$\Phi(\mathcal{X}_n) \xrightarrow{n \rightarrow \infty} \Phi(\mathcal{X}), \tag{6.12}$$

for all $\Phi \in \Pi$. For $k \in \mathbb{N}$, consider all $\phi \in C_b([0, \infty)^{\binom{k+1}{2}})$ which depend on $(r_{ij})_{0 \leq i < j \leq k}$ only through $(r_{0,1}, \dots, r_{0,k})$, i.e., there exists $\tilde{\phi} \in C_b([0, \infty)^k)$ with $\phi((r_{ij})_{0 \leq i < j \leq k}) = \tilde{\phi}((r_{0,j})_{1 \leq j \leq k})$. Since for any $\mathcal{Y} = (Y, r, \mu) \in \mathbb{M}$,

$$\begin{aligned} & \int \hat{\mu}_{\mathcal{Y}}^k(d(r_1, \dots, r_k)) \tilde{\phi}(r_1, \dots, r_k) \\ &= \int \mu^{\otimes k+1}(d(u_0, u_1, \dots, u_k)) \tilde{\phi}(r(u_0, u_1), \dots, r(u_0, u_k)) \\ &= \int \mu^{\otimes k+1}(d(u_0, u_1, \dots, u_k)) \phi((r(u_i, u_j))_{0 \leq i < j \leq k}) \end{aligned} \tag{6.13}$$

it follows from (6.12) that $\hat{\mu}_{\mathcal{X}_n}^k \xrightarrow{n \rightarrow \infty} \hat{\mu}_{\mathcal{X}}^k$ in the topology of weak convergence. Since k was arbitrary the convergence $\hat{\mu}_{\mathcal{X}_n} \xrightarrow{n \rightarrow \infty} \hat{\mu}_{\mathcal{X}}$ follows by Remark 6.8.

(ii) Once more it suffices to prove that all moment measures converge.

Let $\mathcal{X} = (X, r_X, \mu_X) \in \mathbb{M}$ and $\varepsilon > 0$ be given. Now consider a metric measure space $\mathcal{Y} = (Y, r_Y, \mu_Y) \in \mathbb{M}$ with $d_{\text{GPr}}(\mathcal{X}, \mathcal{Y}) < \varepsilon$.

We know that there exists a metric space (Z, r_Z) , isometric embeddings φ_X and φ_Y of $\text{supp}(\mu_X)$ and $\text{supp}(\mu_Y)$ into Z , respectively, and a coupling $\tilde{\mu}$ of $(\varphi_X)_* \mu_X$ and $(\varphi_Y)_* \mu_Y$ such that

$$\tilde{\mu} \{ (z, z') : r_Z(z, z') \geq \varepsilon \} \leq \varepsilon. \tag{6.14}$$

Given $k \in \mathbb{N}$, define a coupling $\tilde{\mu}^k$ of $\hat{\mu}_{\mathcal{X}}^k$ and $\hat{\mu}_{\mathcal{Y}}^k$ by

$$\begin{aligned} & \tilde{\mu}^k(A_1 \times \dots \times A_k \times B_1 \times \dots \times B_k) \\ &:= \tilde{\mu}^{\otimes(k+1)} \{ (z_0, z'_0), \dots, (z_k, z'_k) : r_Z(z_0, z_i) \in A_i, r_Z(z'_0, z'_i) \in B_i, i = 1, \dots, k \} \end{aligned} \tag{6.15}$$

for all $A_1 \times \dots \times A_k \times B_1 \times \dots \times B_k \in \mathcal{B}(\mathbb{R}_+^{2k})$. Then

$$\begin{aligned} & \tilde{\mu}^k \{ (r_1, \dots, r_k, r'_1, \dots, r'_k) : |r_i - r'_i| \geq 2\varepsilon \text{ for at least one } i \} \\ & \leq k \cdot \tilde{\mu}^1 \{ (r_1, r'_1) : |r_1 - r'_1| \geq 2\varepsilon \} \\ & = k \cdot \tilde{\mu}^{\otimes 2} \{ (z, z'), (\tilde{z}, \tilde{z}') : |r_Z(z, \tilde{z}) - r_Z(z', \tilde{z}')| \geq 2\varepsilon \} \\ & \leq k \cdot \tilde{\mu}^{\otimes 2} \{ (z, z'), (\tilde{z}, \tilde{z}') : r_Z(z, z') \geq \varepsilon \text{ or } r_Z(\tilde{z}, \tilde{z}') \geq \varepsilon \} \\ & \leq 2k\varepsilon, \end{aligned} \tag{6.16}$$

which implies that $d_{\text{Pr}}^{\mathbb{R}_+^k}(\tilde{\mu}_{\mathcal{X}}^k, \tilde{\mu}_{\mathcal{Y}}^k) \leq 2k\varepsilon$, and the claim follows.

(iii) By Part (i) of Lemma 6.3, for $\mathcal{X} \in \mathbb{M}$, $w_{\mathcal{X}}$ equals the first moment measure of $\hat{\mu}_{\mathcal{X}}$. The continuity properties of $\mathcal{X} \mapsto w_{\mathcal{X}}$ are therefore a direct consequence of (i) and (ii).

(iv) Let $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \dots$ in \mathbb{M} such that $\hat{\mu}_{\mathcal{X}_n} \xrightarrow{n \rightarrow \infty} \hat{\mu}_{\mathcal{X}}$ and $\delta > 0$. Assume that $\varepsilon > 0$ is such that $\varepsilon > v_{\delta}(\mathcal{X})$. Then by Lemmata 6.3(ii) and 6.4,

$$\hat{\mu}_{\mathcal{X}} \{v \in \mathcal{M}_1([0, \infty)) : v([0, \varepsilon]) \leq \delta\} < \varepsilon. \tag{6.17}$$

The set $\{v \in \mathcal{M}_1([0, \infty)) : v([0, \varepsilon]) \leq \delta\}$ is closed in $\mathcal{M}_1([0, \infty))$. Hence by the Portmanteau Theorem (see, for example, Theorem 3.3.1 in [10]),

$$\begin{aligned} \limsup_{n \rightarrow \infty} \hat{\mu}_{\mathcal{X}_n} \{v \in \mathcal{M}_1([0, \infty)) : v([0, \varepsilon]) \leq \delta\} \\ \leq \hat{\mu}_{\mathcal{X}} \{v \in \mathcal{M}_1([0, \infty)) : v([0, \varepsilon]) \leq \delta\} < \varepsilon. \end{aligned} \tag{6.18}$$

That is, we have $v_{\delta}(\mathcal{X}_n) < \varepsilon$, for all but finitely many n , by (6.9). Therefore we find that $\limsup_{n \rightarrow \infty} v_{\delta}(\mathcal{X}_n) < \varepsilon$. This holds for every $\varepsilon > v_{\delta}(\mathcal{X})$, and we are done. \square

The following estimate will be used in the proofs of the pre-compactness characterization given in Proposition 7.1 and of Part (i) of Lemma 10.3.

Lemma 6.9 *Let $\delta > 0$, $\varepsilon \geq 0$, and $\mathcal{X} = (X, r, \mu) \in \mathbb{M}$. If $v_{\delta}(\mathcal{X}) < \varepsilon$, then there exists $N \leq \lfloor \frac{1}{\delta} \rfloor$ and points $x_1, \dots, x_N \in X$ such that the following hold.*

- For $i = 1, \dots, N$, $\mu(B_{\varepsilon}(x_i)) > \delta$, and $\mu\left(\bigcup_{i=1}^N B_{2\varepsilon}(x_i)\right) > 1 - \varepsilon$.
- For all $i, j = 1, \dots, N$ with $i \neq j$, $r(x_i, x_j) > \varepsilon$.

Proof Consider the set $D := \{x \in X : \mu(B_{\varepsilon}(x)) > \delta\}$. Since $v_{\delta}(\mathcal{X}) < \varepsilon$, Lemma 6.4 implies that $\mu(D) > 1 - \varepsilon$. Take a maximal 2ε separated net $\{x_i : i \in I\} \subseteq D$, i.e.,

$$D \subseteq \bigcup_{i \in I} B_{2\varepsilon}(x_i), \tag{6.19}$$

and for all $i \neq j$,

$$r(x_i, x_j) > 2\varepsilon, \tag{6.20}$$

while adding a further point to D would destroy (6.20). Such a net exists in every metric space (see, for example, in [4], p. 278). Since

$$1 \geq \mu\left(\bigcup_{i \in I} B_{\varepsilon}(x_i)\right) = \sum_{i \in I} \mu(B_{\varepsilon}(x_i)) \geq |I|\delta, \tag{6.21}$$

$|I| \leq \lfloor \frac{1}{\delta} \rfloor$ follows. \square

7 Compact sets

By Prohorov’s Theorem, in a complete and separable metric space, a set of probability measures is relatively compact iff it is tight. This implies that compact sets in \mathbb{M} play a special role for convergence results. In this section we characterize the (pre-)compact sets in the Gromov–Prohorov topology.

Recall the distance measure $w_{\mathcal{X}}$ from (2.7) and the modulus of mass distribution $v_{\delta}(\mathcal{X})$ from (2.8). Denote by (\mathbb{X}_c, d_{GH}) the space of all isometry classes of compact metric spaces equipped with the Gromov–Hausdorff metric (see Sect. 5 for basic definitions).

The following characterizations together with Theorem 5 stated in Sect. 9 which states the equivalence of the Gromov–Prohorov and the Gromov-weak topology imply the result stated in Theorem 2.

Proposition 7.1 (Pre-compactness characterization) *Let Γ be a family in \mathbb{M} . The following four conditions are equivalent.*

- (a) *The family Γ is pre-compact in the Gromov–Prohorov topology.*
- (b) *The family $\{w(\mathcal{X}); \mathcal{X} \in \Gamma\}$ is tight, and*

$$\sup_{\mathcal{X} \in \Gamma} v_{\delta}(\mathcal{X}) \xrightarrow{\delta \rightarrow 0} 0. \tag{7.1}$$

- (c) *For all $\varepsilon > 0$ there exists $N_{\varepsilon} \in \mathbb{N}$ such that for all $\mathcal{X} = (X, r, \mu) \in \Gamma$ there is a subset $X_{\varepsilon, \mathcal{X}} \subseteq X$ with*
 - $\mu(X_{\varepsilon, \mathcal{X}}) \geq 1 - \varepsilon,$
 - $X_{\varepsilon, \mathcal{X}}$ *can be covered by at most N_{ε} balls of radius $\varepsilon,$ and*
 - $X_{\varepsilon, \mathcal{X}}$ *has diameter at most $N_{\varepsilon}.$*
- (d) *For all $\varepsilon > 0$ and $\mathcal{X} = (X, r, \mu) \in \Gamma$ there exists a compact subset $K_{\varepsilon, \mathcal{X}} \subseteq X$ with*
 - $\mu(K_{\varepsilon, \mathcal{X}}) \geq 1 - \varepsilon,$ and
 - *the family $K_{\varepsilon} := \{K_{\varepsilon, \mathcal{X}}; \mathcal{X} \in \Gamma\}$ is pre-compact in $(\mathbb{X}_c, d_{GH}).$*

Remark 7.2

- (i) In the space of compact metric spaces equipped with a probability measure with full support, Proposition 2.4 in [13] states that Condition (d) is sufficient for pre-compactness.
- (ii) Proposition 7.1(b) characterizes tightness for the stronger topology given in [27] based on certain L^2 -Wasserstein metrics if one requires in addition uniform integrability of sampled mutual distance. Similarly, (b) characterizes tightness in the space of measure preserving isometry classes of metric spaces equipped with a finite measure (rather than a probability measure) if one requires in addition tightness of the family of total masses. □

Proof of Proposition 7.1 As before, we abbreviate $\mathcal{X} = (X, r_X, \mu_X)$. We prove four implications giving the statement.

(a) \Rightarrow (b). Assume that $\Gamma \in \mathbb{M}$ is pre-compact in the Gromov–Prohorov topology.

To show that $\{w(\mathcal{X}); \mathcal{X} \in \Gamma\}$ is tight, consider a sequence $\mathcal{X}_1, \mathcal{X}_2, \dots$ in Γ . Since Γ is relatively compact by assumption, there is a converging subsequence, i.e., we find $\mathcal{X} \in \mathbb{M}$ such that $d_{\text{GPr}}(\mathcal{X}_{n_k}, \mathcal{X}) \xrightarrow{k \rightarrow \infty} 0$ along a suitable subsequence $(n_k)_{k \in \mathbb{N}}$. By Part (iii) of Proposition 6.6, $w_{\mathcal{X}_{n_k}} \xrightarrow{k \rightarrow \infty} w_{\mathcal{X}}$. As the sequence was chosen arbitrary it follows that $\{w(\mathcal{X}); \mathcal{X} \in \Gamma\}$ is tight.

The second part of the assertion in (b) is by contradiction. Assume that $v_{\delta}(\mathcal{X})$ does not converge to 0 uniformly in $\mathcal{X} \in \Gamma$, as $\delta \rightarrow 0$. Then we find an $\varepsilon > 0$ such that for all $n \in \mathbb{N}$ there exist sequences $(\delta_n)_{n \in \mathbb{N}}$ converging to 0 and $\mathcal{X}_n \in \Gamma$ with

$$v_{\delta_n}(\mathcal{X}_n) \geq \varepsilon. \tag{7.2}$$

By assumption, there is a subsequence $\{\mathcal{X}_{n_k}; k \in \mathbb{N}\}$, and a metric measure space $\mathcal{X} \in \Gamma$ such that $d_{\text{GPr}}(\mathcal{X}_{n_k}, \mathcal{X}) \xrightarrow{k \rightarrow \infty} 0$. By Parts (ii) and (iv) of Proposition 6.6, we find that $\limsup_{k \rightarrow \infty} v_{\delta_{n_k}}(\mathcal{X}_{n_k}) = 0$ which contradicts (7.2).

(b) \Rightarrow (c). By assumption, for all $\varepsilon > 0$ there are $C(\varepsilon)$ with

$$\sup_{\mathcal{X} \in \Gamma} w_{\mathcal{X}}([C(\varepsilon), \infty)) < \varepsilon, \tag{7.3}$$

and $\delta(\varepsilon)$ such that

$$\sup_{\mathcal{X} \in \Gamma} v_{\delta(\varepsilon)}(\mathcal{X}) < \varepsilon. \tag{7.4}$$

Set

$$X'_{\varepsilon, \mathcal{X}} := \left\{ x \in X : \mu_X \left(B_{C(\frac{\varepsilon^2}{4})}(x) \right) > 1 - \varepsilon/2 \right\}. \tag{7.5}$$

We claim that $\mu_X(X'_{\varepsilon, \mathcal{X}}) > 1 - \varepsilon/2$. If this were not the case, there would be $\mathcal{X} \in \Gamma$ with

$$\begin{aligned} w_{\mathcal{X}} \left([C(\frac{1}{4}\varepsilon^2); \infty) \right) &= \mu_X^{\otimes 2} \left\{ (x, x') \in X \times X : r_X(x, x') \geq C(\frac{1}{4}\varepsilon^2) \right\} \\ &\geq \mu_X^{\otimes 2} \left\{ (x, x') : x \notin X'_{\varepsilon, \mathcal{X}}, x' \notin B_{C(\frac{\varepsilon^2}{4})}(x) \right\} \\ &\geq \frac{\varepsilon}{2} \mu_X(\mathbb{G}X'_{\varepsilon, \mathcal{X}}) \\ &\geq \frac{\varepsilon^2}{4}, \end{aligned} \tag{7.6}$$

which contradicts (7.3). Furthermore, the diameter of $X'_{\varepsilon, \mathcal{X}}$ is bounded by $4C(\frac{\varepsilon^2}{4})$. Indeed, otherwise we would find points $x, x' \in X'_{\varepsilon, \mathcal{X}}$ with $B_{C(\frac{\varepsilon^2}{4})}(x) \cap B_{C(\frac{\varepsilon^2}{4})}(x') = \emptyset$, which contradicts that

$$\begin{aligned} \mu_X \left(B_{C(\frac{\varepsilon^2}{4})}(x) \cap B_{C(\frac{\varepsilon^2}{4})}(x') \right) &\geq 1 - \mu_X \left(\complement B_{C(\frac{\varepsilon^2}{4})}(x) \right) - \mu_X \left(\complement B_{C(\frac{\varepsilon^2}{4})}(x') \right) \\ &\geq 1 - \varepsilon. \end{aligned} \tag{7.7}$$

By Lemma 6.9, for all $\mathcal{X} = (X, r_X, \mu_X) \in \Gamma$, we can choose points $x_1, \dots, x_{N_\varepsilon \mathcal{X}} \in X$ with $N_\varepsilon \mathcal{X} \leq N(\varepsilon) := \lfloor \frac{1}{\delta(\varepsilon/2)} \rfloor$, $r_X(x_i, x_j) > \varepsilon/2$, $1 \leq i < j \leq N_\varepsilon \mathcal{X}$, and with $\mu_X \left(\bigcup_{i=1}^{N_\varepsilon \mathcal{X}} B_\varepsilon(x_i) \right) > 1 - \varepsilon/2$.

Set

$$X_{\varepsilon, \mathcal{X}} := X'_{\varepsilon, \mathcal{X}} \cap \bigcup_{i=1}^{N_\varepsilon \mathcal{X}} B_\varepsilon(x_i). \tag{7.8}$$

Then $\mu_X(X_{\varepsilon, \mathcal{X}}) > 1 - \varepsilon$. In addition, $X_{\varepsilon, \mathcal{X}}$ can be covered by at most $N(\varepsilon)$ balls of radius ε and $X'_{\varepsilon, \mathcal{X}}$ has diameter at most $4C(\frac{\varepsilon^2}{4})$, so the same is true for $X_{\varepsilon, \mathcal{X}}$.

(c) \Rightarrow (d). Fix $\varepsilon > 0$, and set $\varepsilon_n := \varepsilon 2^{-(n+1)}$, for all $n \in \mathbb{N}$. By assumption we may choose for each $n \in \mathbb{N}$, $N_{\varepsilon_n} \in \mathbb{N}$ such that for all $\mathcal{X} \in \Gamma$ there is a subset $X_{\varepsilon_n, \mathcal{X}} \subseteq X$ of diameter at most N_{ε_n} with $\mu(X_{\varepsilon_n, \mathcal{X}}) \geq 1 - \varepsilon_n$, and such that $X_{\varepsilon_n, \mathcal{X}}$ can be covered by at most N_{ε_n} balls of radius ε_n . Without loss of generality we may assume that all $\{X_{\varepsilon_n, \mathcal{X}}; n \in \mathbb{N}, \mathcal{X} \in \Gamma\}$ are closed. Otherwise we just take their closure. For every $\mathcal{X} \in \Gamma$ take compact sets $K_{\varepsilon_n, \mathcal{X}} \subseteq X$ with $\mu_X(K_{\varepsilon_n, \mathcal{X}}) > 1 - \varepsilon_n$. Then the set

$$K_{\varepsilon, \mathcal{X}} := \bigcap_{n=1}^{\infty} (X_{\varepsilon_n, \mathcal{X}} \cap K_{\varepsilon_n, \mathcal{X}}) \tag{7.9}$$

is compact since it is the intersection of a compact set with closed sets, and

$$\mu_X(K_{\varepsilon, \mathcal{X}}) \geq 1 - \sum_{n=1}^{\infty} (\mu_X(\complement X_{\varepsilon_n, \mathcal{X}}) + \mu_X(\complement K_{\varepsilon_n, \mathcal{X}})) > 1 - \varepsilon. \tag{7.10}$$

Consider

$$\mathcal{K}_\varepsilon := \{K_{\varepsilon, \mathcal{X}}; \mathcal{X} \in \Gamma\}. \tag{7.11}$$

To show that \mathcal{K}_ε is pre-compact we use the pre-compactness criterion given in Theorem 7.4.15 in [4], i.e., we have to show that \mathcal{K}_ε is uniformly totally bounded. This means that the elements of \mathcal{K}_ε have bounded diameter and for all $\varepsilon' > 0$ there is a number $N_{\varepsilon'}$ such that all elements of \mathcal{K}_ε can be covered by $N_{\varepsilon'}$ balls of radius ε' . By definition, $K_{\varepsilon, \mathcal{X}} \subseteq X_{\varepsilon_1, \mathcal{X}}$ and so, $K_{\varepsilon, \mathcal{X}}$ has diameter at most N_{ε_1} . So, take $\varepsilon' < \varepsilon$ and n large enough for $\varepsilon_n < \varepsilon'$. Then $X_{\varepsilon_n, \mathcal{X}}$ as well as $K_{\varepsilon_n, \mathcal{X}}$ can be covered by N_{ε_n} balls of radius ε' . So \mathcal{K}_ε is pre-compact in (\mathbb{X}_c, d_{GH}) .

(d) \Rightarrow (a). The proof is in two steps. Assume first that all metric spaces (X, r_X) with $(X, r_X, \mu_X) \in \Gamma$ are compact, and that the family $\{(X, r_X) : (X, r_X, \mu_X) \in \Gamma\}$ is pre-compact in the Gromov–Hausdorff topology.

Under these assumptions we can choose for every sequence in Γ a subsequence $(\mathcal{X}_m)_{m \in \mathbb{N}}$, $\mathcal{X}_m = (X_m, r_{X_m}, \mu_{X_m})$, and a metric space (X, r_X) , such that

$$d_{\text{GH}}(X, X_m) \xrightarrow{m \rightarrow \infty} 0. \tag{7.12}$$

By Lemma A.1, there are a compact metric space (Z, r_Z) and isometric embeddings $\varphi_X, \varphi_{X_1}, \varphi_{X_2}, \dots$ from X, X_1, X_2, \dots , respectively, to Z , such that $d_{\text{H}}(\varphi_X(X), \varphi_{X_m}(X_m)) \xrightarrow{m \rightarrow \infty} 0$. Since Z is compact, the set $\{(\varphi_{X_m})_* \mu_{X_m} : m \in \mathbb{N}\}$ is pre-compact in $\mathcal{M}_1(Z)$ equipped with the weak topology. Therefore $(\varphi_{X_m})_* \mu_{X_m}$ has a converging subsequence, and (a) follows in this case.

In the second step we consider the general case. Let $\varepsilon_n := 2^{-n}$, fix for every $\mathcal{X} \in \Gamma$ and every $n \in \mathbb{N}$, $x \in K_{\varepsilon_n, \mathcal{X}}$. Put

$$\mu_{X,n}(\cdot) := \mu_X(\cdot \cap K_{\varepsilon_n, \mathcal{X}}) + (1 - \mu_X(K_{\varepsilon_n, \mathcal{X}}))\delta_x(\cdot) \tag{7.13}$$

and let $\mathcal{X}^n := (X, r_X, \mu_{X,n})$. By construction, for all $\mathcal{X} \in \Gamma$,

$$d_{\text{GPr}}(\mathcal{X}^n, \mathcal{X}) \leq \varepsilon_n, \tag{7.14}$$

and $\mu_{X,n}$ is supported by $K_{\varepsilon_n, \mathcal{X}}$. Hence, $\Gamma_n := \{\mathcal{X}^n; \mathcal{X} \in \Gamma\}$ is pre-compact in \mathbb{X}_{c} equipped with the Gromov–Hausdorff topology, for all $n \in \mathbb{N}$. We can therefore find a converging subsequence in Γ_n , for all n , by the first step.

By a diagonal argument we find a subsequence $(\mathcal{X}_m)_{m \in \mathbb{N}}$ with $\mathcal{X}_m = (X_m, r_{X_m}, \mu_{X_m})$ such that $(\mathcal{X}_m^n)_{m \in \mathbb{N}}$ converges for every $n \in \mathbb{N}$ to some metric measure space \mathcal{Z}_n . Pick a subsequence such that for all $n \in \mathbb{N}$ and $m \geq n$,

$$d_{\text{GPr}}(\mathcal{X}_m^n, \mathcal{Z}_n) \leq \varepsilon_m. \tag{7.15}$$

Then

$$d_{\text{GPr}}(\mathcal{X}_m^n, \mathcal{X}_{m'}^n) \leq 2\varepsilon_n, \tag{7.16}$$

for all $m, m' \geq n$. We conclude that $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $(\mathbb{M}, d_{\text{GPr}})$ since $\sum_{n \geq 1} \varepsilon_n < \infty$. Indeed,

$$\begin{aligned} d_{\text{GPr}}(\mathcal{X}_n, \mathcal{X}_{n+1}) &\leq d_{\text{GPr}}(\mathcal{X}_n, \mathcal{X}_n^n) + d_{\text{GPr}}(\mathcal{X}_n^n, \mathcal{X}_{n+1}^n) + d_{\text{GPr}}(\mathcal{X}_{n+1}^n, \mathcal{X}_{n+1}) \\ &\leq 4\varepsilon_n. \end{aligned} \tag{7.17}$$

Since $(\mathbb{M}, d_{\text{GPr}})$ is complete, this sequence converges and we are done.

8 Tightness

In Proposition 7.1 we have given a characterization for relative compactness in \mathbb{M} with respect to the Gromov–Prohorov topology. This characterization extends to the following tightness characterization in $\mathcal{M}_1(\mathbb{M})$ which is equivalent to Theorem 3, once we have shown the equivalence of the Gromov–Prohorov and the Gromov-weak topology in Theorem 5 in Sect. 10.

Proposition 8.1 (Tightness with respect to the Gromov–Prohorov topology) *A set $\mathbf{A} \subseteq \mathcal{M}_1(\mathbb{M})$ is tight with respect to the Gromov–Prohorov topology on \mathbb{M} if and only if for all $\varepsilon > 0$ there exist $\delta > 0$ and $C > 0$ such that*

$$\sup_{\mathbb{P} \in \mathbf{A}} \mathbb{P} [v_\delta(\mathcal{X}) + w_{\mathcal{X}}([C; \infty))] < \varepsilon. \tag{8.1}$$

Proof of Proposition 8.1 For the “only if” direction assume that \mathbf{A} is tight and fix $\varepsilon > 0$. By definition, we find a compact set Γ_ε in $(\mathbb{M}, d_{\text{GPr}})$ such that $\inf_{\mathbb{P} \in \mathbf{A}} \mathbb{P}(\Gamma_\varepsilon) > 1 - \varepsilon/4$. Since Γ_ε is compact there are, by part (b) of Proposition 7.1, $\delta = \delta(\varepsilon) > 0$ and $C = C(\varepsilon) > 0$ such that $v_\delta(\mathcal{X}) < \varepsilon/4$ and $w_{\mathcal{X}}([C, \infty)) < \varepsilon/4$, for all $\mathcal{X} \in \Gamma_\varepsilon$. Furthermore both $v_\delta(\cdot)$ and $w_\cdot([C, \infty))$ are bounded above by 1. Hence for all $\mathbb{P} \in \mathbf{A}$,

$$\begin{aligned} &\mathbb{P} [v_\delta(\mathcal{X}) + w_{\mathcal{X}}([C, \infty))] \\ &= \mathbb{P} [v_\delta(\mathcal{X}) + w_{\mathcal{X}}([C, \infty)); \Gamma_\varepsilon] + \mathbb{P} [v_\delta(\mathcal{X}) + w_{\mathcal{X}}([C, \infty)); \mathbb{C} \setminus \Gamma_\varepsilon] \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned} \tag{8.2}$$

Therefore (8.1) holds.

For the “if” direction assume (8.1) is true and fix $\varepsilon > 0$. For all $n \in \mathbb{N}$, there are $\delta_n > 0$ and $C_n > 0$ such that

$$\sup_{\mathbb{P} \in \mathbf{A}} \mathbb{P} [v_{\delta_n}(\mathcal{X}) + w_{\mathcal{X}}([C_n, \infty))] < 2^{-2n} \varepsilon^2. \tag{8.3}$$

By Tschebychev’s inequality, we conclude that for all $n \in \mathbb{N}$,

$$\sup_{\mathbb{P} \in \mathbf{A}} \mathbb{P} \{ \mathcal{X} : v_{\delta_n}(\mathcal{X}) + w_{\mathcal{X}}([C_n, \infty)) > 2^{-n} \varepsilon \} < 2^{-n} \varepsilon. \tag{8.4}$$

By the equivalence of (a) and (b) in Proposition 7.1 the closure of

$$\Gamma_\varepsilon := \bigcap_{n=1}^\infty \{ \mathcal{X} : v_{\delta_n}(\mathcal{X}) + w_{\mathcal{X}}([C_n, \infty)) \leq 2^{-n} \varepsilon \} \tag{8.5}$$

is compact. We conclude

$$\begin{aligned}
 \mathbb{P}(\overline{\Gamma_\varepsilon}) &\geq \mathbb{P}(\Gamma_\varepsilon) \\
 &\geq 1 - \sum_{n=1}^\infty \mathbb{P}\{\mathcal{X} : v_{\delta_n}(\mathcal{X}) + w_{\mathcal{X}}([C_n, \infty)) > \frac{\varepsilon}{2^n}\} \\
 &> 1 - \varepsilon.
 \end{aligned}
 \tag{8.6}$$

Since ε was arbitrary, \mathbf{A} is tight. □

9 Gromov–Prohorov and Gromov-weak topology coincide

In this section we show that the topologies induced by convergence of polynomials and convergence in the Gromov–Prohorov metric coincide. This implies that the characterizations of compact subsets of \mathbb{M} and tight families in $\mathcal{M}_1(\mathbb{M})$ in Gromov-weak topology stated in Theorems 2 and 3 are covered by the corresponding characterizations with respect to the Gromov–Prohorov topology given in Propositions 7.1 and 8.1, respectively. Recall the distance matrix distribution from Definition 2.7

Theorem 5 *Let $\mathcal{X}, \mathcal{X}_1, \mathcal{X}_2, \dots \in \mathbb{M}$. The following are equivalent:*

(a) *The Gromov–Prohorov metric converges, i.e.,*

$$d_{\text{GPr}}(\mathcal{X}_n, \mathcal{X}) \xrightarrow{n \rightarrow \infty} 0. \tag{9.1}$$

(b) *Distance matrix distributions converge, i.e.,*

$$v^{\mathcal{X}_n} \implies v^{\mathcal{X}} \text{ as } n \rightarrow \infty. \tag{9.2}$$

(c) *All polynomials $\Phi \in \Pi$ converge, i.e.,*

$$\Phi(\mathcal{X}_n) \xrightarrow{n \rightarrow \infty} \Phi(\mathcal{X}). \tag{9.3}$$

Proof (a) \implies (b). Let $\mathcal{X} = (X, r_X, \mu_X)$, $\mathcal{X}_1 = (X_1, r_1, \mu_1)$, $\mathcal{X}_2 = (X_2, r_2, \mu_2)$. By Lemma 5.8 there are a complete and separable metric space (Z, r_Z) and isometric embeddings $\varphi, \varphi_1, \varphi_2, \dots$ from (X, r_X) , (X_1, r_1) , (X_2, r_2) , ..., respectively, to (Z, r_Z) such that $(\varphi_n)_* \mu_n$ converges weakly to $\varphi_* \mu_X$ on (Z, r_Z) . Consequently, using (2.4),

$$v^{\mathcal{X}_n} = (t^{\mathcal{X}_n})_* \mu_n^{\mathbb{N}} = (t^{\mathcal{Z}})_* \left(((\varphi_n)_* \mu_n)^{\mathbb{N}} \right) \implies (t^{\mathcal{Z}})_* \left((\varphi_* \mu)^{\mathbb{N}} \right) = (t^{\mathcal{X}})_* \mu^{\mathbb{N}} = v^{\mathcal{X}} \tag{9.4}$$

(b) \implies (c). This is a consequence of the two different representation of polynomials from (2.2) and (2.6).

(c) \implies (a). Assume that for all $\Phi \in \Pi$, $\Phi(\mathcal{X}_n) \xrightarrow{n \rightarrow \infty} \Phi(\mathcal{X})$. It is enough to show that the sequence $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is pre-compact with respect to the Gromov–Prohorov topology, since by Proposition 2.6, this would imply that all limit points coincide and

equal \mathcal{X} . We need to check the two conditions guaranteeing pre-compactness given by Part (b) of Proposition 7.1.

By Part (iii) of Proposition 6.6, the map $\mathcal{X} \mapsto w_{\mathcal{X}}$ is continuous with respect to the Gromov-weak topology. Hence, the family $\{w_{\mathcal{X}_n}; n \in \mathbb{N}\}$ is tight.

In addition, by Parts (i) and (iv) of Proposition 6.6, $\limsup_{n \rightarrow \infty} v_{\delta}(\mathcal{X}_n) \leq v_{\delta}(\mathcal{X}) \xrightarrow{\delta \rightarrow 0} 0$. By Remark 2.11, the latter implies (7.1), and we are done. □

10 Equivalent metrics

In Sect. 5 we have seen that \mathbb{M} equipped with the Gromov–Prohorov metric is separable and complete. In this section we conclude the paper by presenting further metrics (not necessarily complete) which are all equivalent to the Gromov–Prohorov metric and which may be in some situations easier to work with.

The Eurandom metric¹ Recall from Definition 2.3 the algebra of polynomials, i.e., functions which evaluate distances of finitely many points sampled from a metric measure space. By Proposition 2.6, polynomials separate points in \mathbb{M} . Consequently, two metric measure spaces are different if and only if the distributions of sampled finite subspaces are different.

We therefore define

$$d_{\text{Eur}}(\mathcal{X}, \mathcal{Y}) := \inf_{\tilde{\mu}} \inf \left\{ \varepsilon > 0 : \tilde{\mu}^{\otimes 2} \{(x, y), (x', y') \in (X \times Y)^2 : |r_X(x, x') - r_Y(y, y')| \geq \varepsilon\} < \varepsilon \right\}, \tag{10.1}$$

where the infimum is over all couplings $\tilde{\mu}$ of μ_X and μ_Y . We will refer to d_{Eur} as the *Eurandom metric*.

Not only is d_{Eur} a metric on \mathbb{M} , it also generates the Gromov–Prohorov topology.

Proposition 10.1 (Equivalent metrics) *The distance d_{Eur} is a metric on \mathbb{M} . It is equivalent to d_{GPr} , i.e., the generated topology is the Gromov-weak topology.*

Before we prove the proposition we give an example to show that the Eurandom metric is not complete.

Example 10.2 (Eurandom metric is not complete) Let for all $n \in \mathbb{N}$, $\mathcal{X}_n := (X_n, r_n, \mu_n)$ as in Example 2.12(ii). For all $n \in \mathbb{N}$,

$$\begin{aligned} & d_{\text{Eur}}(\mathcal{X}_n, \mathcal{X}_{n+1}) \\ & \leq \inf \left\{ \varepsilon > 0 : (\mu_n \otimes \mu_{n+1})^{\otimes 2} \{|\mathbf{1}\{x = x'\} - \mathbf{1}\{y = y'\}| \geq \varepsilon\} \leq \varepsilon \right\} \\ & \leq 2^{-(n-1)}, \end{aligned} \tag{10.2}$$

¹ When we first discussed how to metrize the Gromov-weak topology the Eurandom metric came up. Since the discussion took place during a meeting at Eurandom, we decided to name the metric accordingly.

i.e., $(\mathcal{X}_n)_{n \in \mathbb{N}}$ is a Cauchy sequence for d_{Eur} which does not converge. Hence $(\mathbb{M}, d_{\text{Eur}})$ is not complete. The Gromov–Prohorov metric was shown to be complete, and hence the above sequence is not Cauchy in this metric. Indeed,

$$d_{\text{GPr}}(\mathcal{X}_n, \mathcal{X}_{n+1}) = 2^{-1} \not\xrightarrow{n \rightarrow \infty} 0. \tag{10.3}$$

To prepare the proof of Proposition 10.1, we provide bounds on the introduced “distances”.

Lemma 10.3 (Equivalence)] *Let $\mathcal{X}, \mathcal{Y} \in \mathbb{M}$, and $\delta \in (0, \frac{1}{2})$.*

(i) *If $d_{\text{Eur}}(\mathcal{X}, \mathcal{Y}) < \delta^4$ then $d_{\text{GPr}}(\mathcal{X}, \mathcal{Y}) < 12(2v_\delta(\mathcal{X}) + \delta)$.*

(ii)
$$d_{\text{Eur}}(\mathcal{X}, \mathcal{Y}) \leq 2d_{\text{GPr}}(\mathcal{X}, \mathcal{Y}). \tag{10.4}$$

Proof (i) The Gromov–Prohorov metric relies on the Prohorov metric of embeddings of μ_X and μ_Y in $\mathcal{M}_1(Z)$ in a metric space (Z, r_Z) . This is in contrast to the Eurandom metric which is based on an optimal coupling of the two measures μ_X and μ_Y without referring to a space of measures over a third metric space. Since we want to bound the Gromov–Prohorov metric in terms of the Eurandom metric the main goal of the proof is to construct a suitable metric space (Z, r_Z) .

The construction proceeds in three steps. We start in Step 1 with finding a suitable ε -net $\{x_1, \dots, x_N\}$ in (X, r_X) , and show that this net has a suitable corresponding net $\{y_1, \dots, y_N\}$ in (Y, r_Y) . In Step 2 we then verify that these nets have the property that $r_X(x_i, x_j) \approx r_Y(y_i, y_j)$ (where the ‘ \approx ’ is made precise below) and δ -balls around these nets carry almost all μ_X - and μ_Y -mass. Finally, in Step 3 we will use these nets to define a metric space (Z, r_Z) containing both (X, r_X) and (Y, r_Y) , and bound the Prohorov metric of the images of μ_X and μ_Y .

Step 1 (Construction of suitable ε -nets in X and Y)

Fix $\delta \in (0, \frac{1}{2})$. Assume that $\mathcal{X}, \mathcal{Y} \in \mathbb{M}$ are such that $d_{\text{Eur}}(\mathcal{X}, \mathcal{Y}) < \delta^4$. By definition, we find a coupling $\tilde{\mu}$ of μ_X and μ_Y such that

$$\tilde{\mu}^{\otimes 2} \{(x_1, y_1), (x_2, y_2) : |r_X(x_1, x_2) - r_Y(y_1, y_2)| > 2\delta\} < \delta^4. \tag{10.5}$$

Set $\varepsilon := 4v_\delta(\mathcal{X}) \geq 0$. By Lemma 6.9, there are $N \leq \lfloor \frac{1}{\delta} \rfloor$ points $x_1, \dots, x_N \in X$ with pairwise distances at least ε ,

$$\mu(B_\varepsilon(x_i)) > \delta, \tag{10.6}$$

for all $i = 1, \dots, N$, and

$$\mu\left(\bigcup_{i=1}^N B_\varepsilon(x_i)\right) \geq 1 - \varepsilon. \tag{10.7}$$

Put $D := \bigcup_{i=1}^N B_\varepsilon(x_i)$. We claim that for every $i = 1, \dots, N$ there is $y_i \in Y$ with

$$\tilde{\mu} \left(B_\varepsilon(x_i) \times B_{2(\varepsilon+\delta)}(y_i) \right) \geq (1 - \delta^2)\mu_X \left(B_\varepsilon(x_i) \right). \tag{10.8}$$

Indeed, assume the assertion is not true for some $1 \leq i \leq N$. Then, for all $y \in Y$,

$$\tilde{\mu} \left(B_\varepsilon(x_i) \times \complement B_{2(\varepsilon+\delta)}(y) \right) \geq \delta^2\mu_X \left(B_\varepsilon(x_i) \right). \tag{10.9}$$

which implies that

$$\begin{aligned} & \tilde{\mu}^{\otimes 2} \left\{ (x', y'), (x'', y'') : |r_X(x', x'') - r_Y(y', y'')| > 2\delta \right\} \\ & \geq \tilde{\mu}^{\otimes 2} \left\{ (x', y'), (x'', y'') : x', x'' \in B_\varepsilon(x_i), y'' \notin B_{2(\varepsilon+\delta)}(y') \right\} \\ & \geq \mu_X(B_\varepsilon(x_i))^2 \delta^2 \\ & > \delta^4, \end{aligned} \tag{10.10}$$

by (6.19) and (10.9) which contradicts (10.5).

Step 2 (Distortion of $\{x_1, \dots, x_N\}$ and $\{y_1, \dots, y_n\}$)

Assume that $\{x_1, \dots, x_N\}$ and $\{y_1, \dots, y_n\}$ are such that (10.6) through (10.8) hold. We claim that then

$$\left| r_X(x_i, x_j) - r_Y(y_i, y_j) \right| \leq 6(\varepsilon + \delta), \tag{10.11}$$

for all $i, j = 1, \dots, N$. Assume that (10.11) is not true for some pair (i, j) . Then for all $x' \in B_\varepsilon(x_i), x'' \in B_\varepsilon(x_j), y' \in B_{2(\varepsilon+\delta)}(y_i)$, and $y'' \in B_{2(\varepsilon+\delta)}(y_j)$,

$$\left| r_X(x', x'') - r_Y(y', y'') \right| > 6(\varepsilon + \delta) - 2\varepsilon - 4(\varepsilon + \delta) = 2\delta. \tag{10.12}$$

Then

$$\begin{aligned} & \tilde{\mu}^{\otimes 2} \left\{ (x', y'), (x'', y'') : |r_X(x', x'') - r_Y(y', y'')| > 2\delta \right\} \\ & \geq \tilde{\mu}^{\otimes 2} \left\{ (x', y'), (x'', y'') : \right. \\ & \quad \left. x' \in B_\varepsilon(x_i), x'' \in B_\varepsilon(x_j), y' \in B_{2(\varepsilon+\delta)}(y_i), y'' \in B_{2(\varepsilon+\delta)}(y_j) \right\} \\ & = \tilde{\mu} \left(B_\varepsilon(x_i) \times B_{2(\varepsilon+\delta)}(y_i) \right) \tilde{\mu} \left(B_\varepsilon(x_j) \times B_{2(\varepsilon+\delta)}(y_j) \right) \\ & \quad \delta^2(1 - \delta)^2 \\ & > \delta^4, \end{aligned} \tag{10.13}$$

where we used (10.8), (6.19) and $\delta < \frac{1}{2}$. Since (10.13) contradicts (10.5), we are done.

Step 3 (Definition of a suitable metric space (Z, r_Z))

Define the relation $R := \{(x_i, y_i) : i = 1, \dots, N\}$ between X and Y and consider the metric space (Z, r_Z) defined by $Z := X \sqcup Y$ and $r_Z := r_{X \sqcup Y}^R$, given as in Remark 5.5. Choose isometric embeddings φ_X and φ_Y from (X, r_X) and (Y, r_Y) , respectively, into (Z, r_Z) . As $\text{dis}(R) \leq 6(\varepsilon + \delta)$ (see (5.7) for definition), by Remark 5.5, $r_Z(\varphi_X(x_i), \varphi_Y(y_i)) \leq 3(\varepsilon + \delta)$, for all $i = 1, \dots, N$.

If $x \in X$ and $y \in Y$ are such that $r_Z(\varphi_X(x), \varphi_Y(y)) \geq 6(\varepsilon + \delta)$ and $r_X(x, x_i) < \varepsilon$ then

$$\begin{aligned} r_Y(y, y_i) &\geq r_Z(\varphi_X(x), \varphi_Y(y)) - r_X(x, x_i) - r_Z(\varphi_X(x_i), \varphi_Y(y_i)) \\ &\geq 6(\varepsilon + \delta) - \varepsilon - 3(\varepsilon + \delta) \\ &\geq 2(\varepsilon + \delta) \end{aligned} \tag{10.14}$$

and so for all $x \in B_\varepsilon(x_i)$,

$$\{y \in Y : r_Z(\varphi_X(x), \varphi_Y(y)) \geq 6(\varepsilon + \delta)\} \subseteq \mathbb{C}B_{2(\varepsilon + \delta)}(y_i). \tag{10.15}$$

Let $\tilde{\mu}$ be the probability measure on $Z \times Z$ defined by $\hat{\mu}(A \times B) := \tilde{\mu}(\varphi_X^{-1}(A) \times \varphi_Y^{-1}(B))$, for all $A, B \in \mathcal{B}(Z)$. Therefore, by (10.11), (10.15), (10.8) and as $N \leq \lfloor 1/\delta \rfloor$,

$$\begin{aligned} &\hat{\mu} \{(z, z') : r_Z(z, z') \geq 6(\varepsilon + \delta)\} \\ &\leq \hat{\mu} (\varphi_X(\mathbb{C}D) \times \varphi_Y(Y)) + \hat{\mu} \left(\bigcup_{i=1}^N B_\varepsilon(\varphi_X(x_i)) \times \mathbb{C}B_{2(\varepsilon + \delta)}(\varphi_Y(y_i)) \right) \\ &\leq \varepsilon + \sum_{i=1}^N \mu_X(B_\varepsilon(x_i))\delta^2 \\ &\leq \varepsilon + \delta. \end{aligned} \tag{10.16}$$

Hence, using (5.4) and $\varepsilon = 4v_\delta(\mathcal{X})$,

$$d_{\text{Pr}}^{(Z, r_Z)}((\varphi_X)_*\mu_X, (\varphi_Y)_*\mu_Y) \leq 6(4v_\delta(\mathcal{X}) + 2\delta), \tag{10.17}$$

and so $d_{\text{GPr}}(\mathcal{X}, \mathcal{Y}) \leq 12(2v_\delta(\mathcal{X}) + \delta)$, as claimed.

(ii) Assume that $d_{\text{GPr}}(\mathcal{X}, \mathcal{Y}) < \delta$. Then, by definition, there exists a metric space (Z, r_Z) , isometric embeddings φ_X and φ_Y between $\text{supp}(\mu_X)$ and $\text{supp}(\mu_Y)$ and Z , respectively, and a coupling $\hat{\mu}$ of $(\varphi_X)_*\mu_X$ and $(\varphi_Y)_*\mu_Y$ such that

$$\hat{\mu} \{(z, z') : r_Z(z, z') \geq \delta\} < \delta. \tag{10.18}$$

Hence with the special choice of a coupling $\tilde{\mu}$ of μ_X and μ_Y defined by $\tilde{\mu}(A \times B) = \hat{\mu}(\varphi_X(A) \times \varphi_Y(B))$, for all $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$,

$$\begin{aligned} & \tilde{\mu}^{\otimes 2} \{ (x, y), (x', y') \in X \times Y : |r_X(x, x') - r_Y(y, y')| \geq 2\delta \} \\ & \leq \tilde{\mu}^{\otimes 2} \{ (x, y), (x', y') \in X \times Y \\ & \quad : r_Z(\varphi_X(x), \varphi_Y(y)) \geq \delta \text{ or } r_Z(\varphi_X(x'), \varphi_Y(y')) \geq \delta \} \\ & < 2\delta. \end{aligned} \tag{10.19}$$

This implies that $d_{\text{Eur}}(\mathcal{X}, \mathcal{Y}) < 2\delta$. □

Proof of Proposition 10.1 Observe that by Lemma 6.5, $v_\delta(\mathcal{X}) \xrightarrow{\delta \rightarrow 0} 0$. So Lemma 10.3 implies the equivalence of d_{GPr} and d_{Eur} once we have shown that d_{Eur} is indeed a metric.

The symmetry is clear. If $\mathcal{X}, \mathcal{Y} \in \mathbb{M}$ are such that $d_{\text{Eur}}(\mathcal{X}, \mathcal{Y}) = 0$, by equivalence, $d_{\text{GPr}}(\mathcal{X}, \mathcal{Y}) = 0$ and hence $\mathcal{X} = \mathcal{Y}$.

For the triangle inequality, let $\mathcal{X}_i = (X_i, r_i, \mu_i) \in \mathbb{M}, i = 1, 2, 3$, be such that $d_{\text{Eur}}(\mathcal{X}_1, \mathcal{X}_2) < \varepsilon$ and $d_{\text{Eur}}(\mathcal{X}_2, \mathcal{X}_3) < \delta$ for some $\varepsilon, \delta > 0$. Then there exist couplings $\tilde{\mu}_{1,2}$ of μ_1 and μ_2 and $\tilde{\mu}_{2,3}$ of μ_2 and μ_3 with

$$\tilde{\mu}_{1,2}^{\otimes 2} \{ (x_1, x_2), (x'_1, x'_2) : |r_1(x_1, x'_1) - r_2(x_2, x'_2)| \geq \varepsilon \} < \varepsilon \tag{10.20}$$

and

$$\tilde{\mu}_{2,3}^{\otimes 2} \{ (x_2, x_3), (x'_2, x'_3) : |r_2(x_2, x'_2) - r_3(x_3, x'_3)| \geq \delta \} < \delta. \tag{10.21}$$

Introduce the transition kernel $K_{2,3}$ from X_2 to X_3 defined by

$$\tilde{\mu}_{2,3}(d(x_2, x_3)) = \mu_2(dx_2)K_{2,3}(x_2, dx_3). \tag{10.22}$$

which exists since X_2 and X_3 are Polish.

Using this kernel, define a coupling $\tilde{\mu}_{1,3}$ of μ_1 and μ_3 by

$$\tilde{\mu}_{1,3}(d(x_1, x_3)) := \int_{X_2} \tilde{\mu}_{1,2}(d(x_1, x_2))K_{2,3}(x_2, dx_3). \tag{10.23}$$

Then

$$\begin{aligned} & \tilde{\mu}_{1,3}^{\otimes 2} \{ (x_1, x_3), (x'_1, x'_3) : |r_1(x_1, x'_1) - r_3(x_3, x'_3)| \geq \varepsilon + \delta \} \\ & = \int_{X_1^2 \times X_2^2 \times X_3^2} \tilde{\mu}_{1,2}(d(x_1, x_2))\tilde{\mu}_{1,2}(d(x'_1, x'_2))K_{2,3}(x_2, dx_3)K_{2,3}(x'_2, dx'_3) \\ & \quad \times \mathbf{1} \{ |r_1(x_1, x'_1) - r_3(x_3, x'_3)| \geq \varepsilon + \delta \} \end{aligned}$$

$$\begin{aligned}
 &\leq \int_{X_1^2 \times X_2^2 \times X_3^2} \tilde{\mu}_{1,2}(d(x_1, x_2)) \tilde{\mu}_{1,2}(d(x'_1, x'_2)) K_{2,3}(x_2, dx_3) K_{2,3}(x'_2, dx'_3) \\
 &\quad \times (\mathbf{1}\{|r_1(x_1, x'_1) - r_2(x_2, x'_2)| \geq \varepsilon\} + \mathbf{1}\{|r_2(x_2, x'_2) - r_3(x_3, x'_3)| \geq \delta\}) \\
 &= \tilde{\mu}_{1,2}^{\otimes 2} \{(x_1, x_2), (x_1, x'_2) : |r_1(x_1, x'_1) - r_2(x_2, x'_2)| \geq \varepsilon\} \\
 &\quad + \tilde{\mu}_{2,3}^{\otimes 2} \{(x_2, x_3), (x_2, x'_3) : |r_2(x_2, x'_2) - r_3(x_3, x'_3)| \geq \delta\} \\
 &< \varepsilon + \delta
 \end{aligned} \tag{10.24}$$

which yields $d_{\text{Eur}}(\mathcal{X}_1, \mathcal{X}_3) < \varepsilon + \delta$. □

The Gromov-Wasserstein and the modified Eurandom metric

The topology of weak convergence for probability measures on a fixed metric space (Z, r) is generated not only by the Prohorov metric, but also by

$$d_W^{(Z,rZ)}(\mu_1, \mu_2) := \inf_{\tilde{\mu}} \int_{Z \times Z} \tilde{\mu}(d(x, x')) (r(x, x') \wedge 1), \tag{10.25}$$

where the infimum is over all couplings $\tilde{\mu}$ of μ_1 and μ_2 . This is a version of the *Wasserstein metric* (see, for example, [25]). If we rely on the Wasserstein rather than the Prohorov metric, this results in two further metrics: in the Gromov-Wasserstein metric, i.e.,

$$d_{\text{GW}}(\mathcal{X}, \mathcal{Y}) := \inf_{(\varphi_X, \varphi_Y, Z)} d_W^{(Z,rZ)}((\varphi_X)_* \mu_X, (\varphi_Y)_* \mu_Y), \tag{10.26}$$

where the infimum is over all isometric embeddings from $\text{supp}(\mu_X)$ and $\text{supp}(\mu_Y)$ into a common metric Z and in the modified Eurandom metric

$$\begin{aligned}
 &d'_{\text{Eur}}(\mathcal{X}, \mathcal{Y}) \\
 &:= \inf_{\tilde{\mu}} \int \tilde{\mu}(d(x, y)) \tilde{\mu}(d(x', y')) (|r_X(x, x') - r_Y(y, y')| \wedge 1),
 \end{aligned} \tag{10.27}$$

where the infimum is over all couplings of μ_X and μ_Y .

Remark 10.4 An L^2 -version of d_{GW} on the set of compact metric measure spaces is already used in [27]. It turned out that the metric is complete and the generated topology is separable.

Altogether, we might ask if we could achieve similar bounds to those given in Lemma 10.3 by exchanging the Gromov-Prohorov with the Gromov-Wasserstein metric and the Eurandom with the modified Eurandom metric.

Proposition 10.5 *The distances d_{GW} and d'_{Eur} define metrics on \mathbb{M} . They all generate the Gromov-Prohorov topology. Bounds that relate these two metrics with d_{GPr} and*

d_{Eur} are for $\mathcal{X}, \mathcal{Y} \in \mathbb{M}$,

$$(d_{\text{GPr}}(\mathcal{X}, \mathcal{Y}))^2 \leq d_{\text{GW}}(\mathcal{X}, \mathcal{Y}) \leq d_{\text{GPr}}(\mathcal{X}, \mathcal{Y}) \tag{10.28}$$

and

$$(d_{\text{Eur}}(\mathcal{X}, \mathcal{Y}))^2 \leq d'_{\text{Eur}}(\mathcal{X}, \mathcal{Y}) \leq d_{\text{Eur}}(\mathcal{X}, \mathcal{Y}) \tag{10.29}$$

Consequently, the Gromov-Wasserstein metric is complete.

Proof The fact that d_{GW} and d'_{Eur} define metrics on \mathbb{M} is proved analogously as for the Gromov-Prohorov and the Eurandom metric. The Prohorov and the version of the Wasserstein metric used in (10.26) and (10.27) on fixed metric spaces can be bounded uniformly (see, for example, Theorem 3 in [18]). This immediately carries over to the present case. \square

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Appendix A. Additional facts on Gromov-Hausdorff convergence

Recall the notion of the Gromov-Hausdorff distance on the space \mathbb{X}_c of isometry classes of compact metric spaces given in (5.5). We give a statement concerning convergence in the Gromov-Hausdorff metric which is analogous to Lemma 5.8 for Gromov-Prohorov convergence.

Lemma A.1 *Let $(X, r_X), (X_1, r_{X_1}), (X_2, r_{X_2}), \dots$ be in \mathbb{X}_c . Then*

$$d_{\text{GH}}(X_n, X) \xrightarrow{n \rightarrow \infty} 0$$

if and only if there is a compact metric space (Z, r_Z) and isometric embeddings $\varphi, \varphi_1, \varphi_2, \dots$ of $(X, r), (X_1, r_{X_1}), (X_2, r_{X_2}), \dots$, respectively, into (Z, r_Z) such that

$$d_{\text{H}}^{(Z, r_Z)}(\varphi_n(X_n), \varphi(X)) \xrightarrow{n \rightarrow \infty} 0. \tag{A.1}$$

Proof The “if”-direction is clear. So we come immediately to the “only if” direction. If $d_{\text{GH}}(X_n, X) \xrightarrow{n \rightarrow \infty} 0$, then by (5.8) we find correspondences R_n between X and X_n such that $\text{dis}(R_n) \xrightarrow{n \rightarrow \infty} 0$. Using these and $X_0 := X$, we define recursively metrics r_{X_n} on $Z_n := \bigsqcup_{k=0}^n X_k$. First, set $Z_1 := X_0 \sqcup X_1$ and $r_{Z_1} := r_{Z_1}^{R_1}$ (recall Remark 5.5). In the n th step, we are given a metric on Z_n . Consider the canonical isometric embedding φ from X to Z_n and define the relation $\tilde{R}_n \subseteq Z_n \times X_{n+1}$ by

$$\tilde{R}_{n+1} := \left\{ (z, x) \in Z_n \times X_{n+1} : (\varphi^{-1}(z), x) \in R_{n+1} \right\}, \tag{A.2}$$

and set $r_{Z_{n+1}} := r_{Z_{n+1}}^{\tilde{R}_{n+1}}$. By this procedure we end up with a metric r_Z on $Z := \bigsqcup_{n=0}^{\infty} X_n$ and isometric embeddings $\varphi_0, \varphi_1, \dots$ between X_0, X_1, \dots and Z , respectively, such that

$$d_H^{(Z, r_Z)}(\varphi_n(X_n), \varphi(X)) = \frac{1}{2} \text{dis}(R_n) \xrightarrow{n \rightarrow \infty} 0. \quad (\text{A.3})$$

W.l.o.g. we can assume that Z is complete. Otherwise we just embed everything into the completion of Z . To verify *compactness* of (Z, r_Z) it is therefore sufficient to show that Z is totally bounded (see, for example, Theorem 1.6.5 in [4]). For that purpose fix $\varepsilon > 0$, and let $n \in \mathbb{N}$. Since X is compact, we can choose a finite $\varepsilon/2$ -net S in X . Then for all $x \in Z$ with $r_Z(x, X) < \varepsilon/2$ there exists $x' \in S$ such that $r_Z(x, x') < \varepsilon$. Moreover, $d_H(\varphi_n(X_n), \varphi(X)) < \varepsilon$, for all but finitely many $n \in \mathbb{N}$. For the remaining $\varphi_n(X_n)$ choose finite ε -nets and denote their union by \tilde{S} . In this way, $S \cup \tilde{S}$ is a finite set, and $\{B_\varepsilon(s) : s \in S \cup \tilde{S}\}$ is a covering of Z . \square

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