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The logarithmic Sobolev inequality for the Wasserstein diffusion

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Abstract We prove that the Dirichlet form associated with the Wasserstein diffusion on the set of all probability measures on the unit interval, introduced in von Renesse and Sturm (Entropic measure and Wasserstein diffusion. Ann Probab, 2008) satisfies a logarithmic Sobolev inequality. This implies hypercontractivity of the associated transition semigroup. We also study functional inequalities for related diffusion processes.

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1 Introduction and main results

Let $\mathcal{P}_0$ denote the set of all probability measures on the measurable space ([0, 1], $\mathcal{B}([0, 1])$). Since [0, 1] is compact, it follows that $\mathcal{P}_0$, endowed with the weak topology, is a compact space too. A compatible metric is given by the Wasserstein distance

$$d_W(\mu, \nu) := \inf_{\gamma} \left(\int_0^1 \int_0^1 |x - y|^2 \, \gamma(dx, dy) \right)^{\frac{1}{2}},$$

where the infimum is taken over all probability measures γ on $[0, 1]^2$ having marginals μ and ν.

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The space $\mathcal{P}_0$ can be identified with the space

$$\mathcal{G}_0 := \{g : [0, 1[\to [0, 1] \text{ right continuous , nondecreasing}\}$$

endowed with the L^2-distance

$$\|g_2 - g_1\| := \left(\int_0^1 (g_2(t) - g_1(t))^2 \, dt \right)^{\frac{1}{2}}$$

via the transformation

$$\chi : \mathcal{G}_0 \to \mathcal{P}_0, \quad g \to \mu_g,$$

where the probability measure μ_g is defined by

$$\int_0^1 f \, d\mu_g := \int_0^1 f(g(t)) \, dt. \tag{1}$$

Note that this identification is not the usual identification of μ_g with its distribution function $t \mapsto \mu_g([0, t])$, but with its right-continuous inverse.

Given $\beta > 0$, there exists a unique probability measure $\mathbb{Q}_0^\beta$ on $\mathcal{G}_0$ whose finite dimensional distributions are given by Dirichlet distributions as follows: for any $n \in \mathbb{N}$, $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$

$$\begin{aligned} &\mathbb{Q}_0^\beta \left(g_{t_1} \in dx_1, \ldots, g_{t_n} \in dx_n \right) \\ &\quad = \nu_{\beta(t_1, t_2 - t_1, \ldots, t_n - t_{n-1}, 1 - t_n)} (dx_1, \ldots, dx_n) \end{aligned}$$

where for $q \in \mathbb{R}_+^{n+1}$

$$\begin{aligned} &\nu_q (dx_1, \ldots, dx_n) \\ &\quad = \frac{\Gamma(|q|)}{\prod_{i=0}^n \Gamma(q_i)} \prod_{i=0}^n (x_{i+1} - x_i)^{q_i - 1} \, dx_1, \ldots, dx_n \end{aligned}$$

on the space

$$\Sigma_n = \left\{ (x_1, \ldots, x_n) \in [0, 1]^n : 0 < x_1 < \cdots < x_n < 1 \right\}.$$

Here, $x_0 := 0$ and $x_{n+1} := 1$ and $|q| := q_1 + \cdots + q_{n+1}$.

Remark 1.1 Recall that a Dirichlet process Π_ν with intensity measure ν on a compact separable metric space S is a probability measure on the set $\mathcal{P}(S)$ of probability measures on S, whose finite dimensional distributions are defined as follows:

given any finite measurable partition $A_1, \ldots, A_{n+1}$ of S the joint distribution of $\mu(A_1), \ldots, \mu(A_n)$ on the n-dimensional simplex

$$\Delta_n := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n, x_i \geq 0 \text{ and } |x| := \sum_{i=1}^{n} x_i \leq 1 \right\}$$

is given by the Dirichlet distribution $\pi_{(\nu(A_1),\ldots,\nu(A_{n+1}))}$. Here, for given $q \in \mathbb{R}_+^{n+1}$, $q > 0$,

$$\pi_q (dx_1, \ldots, dx_n) := \frac{\Gamma(|q|)}{\prod_{i=1}^{n+1} \Gamma(q_i)} \prod_{i=1}^{n} x_i^{q_i - 1} (1 - |x|)^{q_{n+1} - 1} \, dx_1, \ldots, dx_n$$

(see [1], Sect. 3.7). Hence, identifying μ with its distribution function $g(t) := \mu([0, t])$, $t \in [0, 1]$, [and not with its right-continuous inverse as in (1)] we can identify the measure $\mathbb{Q}_0^\beta$ with the Dirichlet process $\Pi_{\beta \cdot dt}$.

1.1 The Wasserstein diffusion

In [1], the authors construct a time-reversible diffusion process $\mathbb{M} = ((X_t)_{t \geq 0}, (P_g)_{g \in \mathcal{G}_0})$ on the space $\mathcal{G}_0$ which they call the Wasserstein diffusion, because its intrinsic metric is exactly the L^2-Wasserstein distance d_W. The existence of this process is shown, using the theory of symmetric Dirichlet forms (see [2]). Indeed, the Dirichlet form associated with the Wasserstein diffusion is given as the closure $(\mathbb{E}, D(\mathbb{E}))$ of the quadratic form

$$\mathbb{E}(F) := \int |\mathbb{D}F(g)|^2_{L^2(0,1)} \, d\mathbb{Q}_0^\beta (g)$$

with domain

$$\begin{aligned} \mathcal{F}C_b^1 := \{ & F(g) = \varphi(\langle f_1, g \rangle, \ldots, \langle f_m, g \rangle) : \\ & m \geq 1, \varphi \in C_b^1 \left(\mathbb{R}^m \right), \ f_1, \ldots, f_m \in L^2(0, 1) \right\}, \end{aligned}$$

in $L^2(\mathbb{Q}_0^\beta)$. Here, $\langle f, g \rangle := \int_0^1 f(x) g(x) \, dx$ denotes integration w.r.t. the Lebesgue measure and $\mathbb{D}F(g)$ is the L^2-Frechet derivative of F at g. Note that for F with representation $F(g) = \varphi(\langle f_1, g \rangle, \ldots, \langle f_m, g \rangle)$ we have that

$$\mathbb{D}F(g)(x) = \sum_{i=1}^{m} (\partial_i \varphi)(\langle f_1, g \rangle, \ldots, \langle f_m, g \rangle) \, f_i(x).$$

By the general theory of symmetric Dirichlet forms, $\mathbb{E}$ uniquely determines a negative semi-definite self-adjoint linear operator $(\mathbb{L}, D(\mathbb{L}))$ on $L^2(\mathbb{Q}_0^\beta)$ that generates a

Markovian C_0-semigroup $(e^{t\mathbb{L}})_{t\geq 0}$ of contractions. This semigroup gives the transition probabilities of the Wasserstein diffusion $\mathbb{M}$ in the sense that for all $F \in \mathcal{B}_b(\mathcal{G}_0)$, $E.[F(X_t)]$ is a $\mathbb{Q}_0^\beta$-version of $e^{t\mathbb{L}}F$.

The main results of this paper are the following two functional inequalities satisfied by $\mathbb{E}$:

Theorem 1.2 *$\mathbb{E}$ satisfies a Poincaré inequality with constant less than $\frac{1}{\beta}$, i.e.,*

$$Var_{\mathbb{Q}_0^\beta}(F) \leq \frac{1}{\beta}\mathbb{E}(F); \quad F \in D(\mathbb{E}). \tag{2}$$

Remark 1.3 (a) Using the Rothaus Simon mass gap theorem, the Poincaré inequality for $\mathbb{E}$ would also follow from our next theorem concerning the logarithmic Sobolev inequality for $\mathcal{E}$. Note that the constant would be different from the constant obtained in the last Theorem, which we think is almost optimal.

(b) It is possible to find explicit lower bounds on the optimal constant for the Poincaré inequality satisfied by $\mathbb{E}$. Indeed, the explicit formula for the finite dimensional distributions of g_t, $t \in [0, 1]$, allows to calculate explicitly certain moments of $\mathbb{Q}_0^\beta$. More precisely, let $f \in C([0, 1])$, then

$$\begin{aligned}\int \langle f, g\rangle \, d\mathbb{Q}_0^\beta(g) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f\left(\frac{i}{n}\right)\int g\left(\frac{i}{n}\right) d\mathbb{Q}_0^\beta(g)\\ &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^n f\left(\frac{i}{n}\right)\frac{i}{n} = \int_0^1 f(t)t\,dt\,.\end{aligned}$$

Similarly, it can be shown that

$$\int \langle f, g\rangle^2 \, d\mathbb{Q}_0^\beta(g) = \frac{1}{\beta+1}\int_0^1\int_0^1 f(s)f(t)s\wedge t\,ds\,dt + \frac{\beta}{\beta+1}\left(\int_0^1 f(t)t\,dt\right)^2.$$

In particular,

$$\mathrm{Var}_{\mathbb{Q}_0^\beta}(\langle f, \cdot\rangle) = \frac{1}{\beta+1}\left(\int_0^1\int_0^1 f(s)f(t)s\wedge t\,ds\,dt - \left(\int_0^1 f(t)t\,dt\right)^2\right).$$

If we choose $f(t) = 1$, hence $\mathbb{E}(\langle f, \cdot\rangle) = 1$, we obtain that

$$\mathrm{Var}_{\mathbb{Q}_0^\beta}(\langle f, \cdot\rangle) = \frac{1}{12(\beta+1)} = \frac{1}{12(\beta+1)}\mathbb{E}(\langle f, \cdot\rangle),$$

so that the optimal constant for the Poincaré inequality satisfied by $\mathbb{E}$ must be greater or equal to $\frac{1}{12(\beta+1)}$.

Theorem 1.4 *There exists a finite universal constant c such that* $\mathbb{E}$ *satisfies a logarithmic Sobolev inequality with constant less than* $\frac{c}{\beta}$, *i.e.,*

$$\int F^2 \log\left(\frac{F^2}{\|F\|^2_{L^2(\mathbb{Q}_0^\beta)}}\right) d\mathbb{Q}_0^\beta \le \frac{c}{\beta}\mathbb{E}(F), \quad F \in D(\mathbb{E}). \tag{3}$$

The proofs of Theorems 1.2 and 1.4 are given in Sect. 2 below.

Well-known consequences of these functional inequalities are:

(a) (2) implies a mass gap of size greater than β in the spectrum of the generator $\mathbb{L}$ below the eigenvalue 0. In particular,

$$\left\| e^{t\mathbb{L}} F - \int F\, d\mathbb{Q}_0^\beta \right\|_{L^2(\mathbb{Q}_0^\beta)} \le e^{-t\beta} \|F\|_{L^2(\mathbb{Q}_0^\beta)},$$

so that the transition semigroup of the Wasserstein diffusion converges to equilibrium in $L^2(\mathbb{Q}_0^\beta)$ with exponential rate β.

(b) (3) implies that $(e^{t\mathbb{L}})_{t\ge 0}$ is hypercontractive, i.e., $\left\|e^{t\mathbb{L}}\right\|_{2,4} \le 1$ for some $t > 0$.

(c) (3) implies that for all $R > 0$, the set

$$\left\{ F^2 | F \in D(\mathbb{E}), \|F\|^2_{L^2(\mathbb{Q}_0^\beta)} + \mathbb{E}(F) \le R \right\}$$

is uniformly integrable.

Proofs and further implications of the two functional inequalities (2) and (3) can be found in the survey articles [3,4].

Remark 1.5 It is worth to compare the Wasserstein Dirichlet form with the Dirichlet form associated with a Fleming–Viot process studied in population genetics (see [5–7] and references therein), which is defined as follows: Let S be a compact separable metric space, ν be a finite positive measure on S and Π_ν be the associated Dirichlet process. Let

$$\begin{aligned} \mathcal{D} := \{ & F(\mu) = \varphi(\langle f_1, \mu\rangle, \ldots, \langle f_m, \mu\rangle) : m \ge 1, \\ & \varphi \in C_b^1(\mathbb{R}^m), f_1, \ldots, f_m \in \mathcal{B}_b(S)\}, \end{aligned}$$

where $\langle f, \mu\rangle := \int_S f\, d\mu$ this time denotes integration w.r.t. the (probability) measure μ on S and let

$$\frac{\partial F}{\partial \delta_x}(\mu) := \frac{dF}{d\varepsilon}(\mu + \varepsilon\delta_x)_{|\varepsilon=0}$$

be the Gateaux derivative of F at μ in direction δ_x. Note that for F with representation $F(\mu) = \varphi(\langle f_1, \mu\rangle, \ldots, \langle f_m, \mu\rangle)$ it follows that

$$\frac{\partial F}{\partial \delta_x}(\mu) = \sum_{i=1}^{m} (\partial_i \varphi)(\langle f_1, \mu\rangle, \ldots, \langle f_m, \mu\rangle) f_i(x).$$

The Dirichlet form $\mathbb{E}^{FV}$ associated with the Fleming–Viot process with parent independent mutation

$$Af(x) = \int_S f(y) - f(x)\,\nu(dy)\,, \quad f \in \mathcal{B}_b(S)$$

is then given by the closure of the quadratic form

$$\mathbb{E}^{FV}(F) := \int \mathrm{Var}_\mu \left(\frac{\partial F}{\partial \delta_x} \right) \Pi_\nu(d\mu)\,, \quad F \in \mathcal{D} \tag{4}$$

in $L^2(\Pi_\nu)$.

To compare the two forms $\mathbb{E}$ and $\mathbb{E}^{FV}$ in the particular case $S = [0, 1]$ with $\nu = \beta\, dt$, fix a cylindrical function

$$F(g) = \varphi(\langle f_1, g\rangle, \ldots, \langle f_m, g\rangle) \in \mathcal{F}C_b^1,$$

and define $F_i(t) := \int_t^1 f_i(s)\, ds$, $1 \le i \le m$. If we the denote the probability measure with distribution function g by $\tilde{\mu}_g$, we can write

$$\langle f_i, g\rangle = \int_0^1 F_i\, dg = \langle F_i, \tilde{\mu}_g\rangle,$$

hence

$$\mathbb{D}F(g)(x) = -\sum_{i=1}^{m} \partial_i \varphi(\langle F_1, \tilde{\mu}_g\rangle, \ldots, \langle F_m, \tilde{\mu}_g\rangle) \frac{d}{dx} F_i(x) = -\frac{d}{dx} \frac{\partial \tilde{F}}{\partial \delta_\cdot}(\tilde{\mu}_g),$$

where

$$\tilde{F}(\mu) := \varphi(\langle F_1, \mu\rangle, \ldots, \langle F_m, \mu\rangle)\,.$$

It follows that

$$\mathbb{E}(F) = \int |\mathbb{D}F(g)|^2_{L^2([0,1])}\, \mathbb{Q}_0^\beta(dg) = \int \left| \frac{d}{dx} \frac{\partial \tilde{F}}{\partial \delta_\cdot} \right|^2_{L^2([0,1])} \Pi_\nu(d\tilde{\mu}_g)\,.$$

Hence, in the Wasserstein Dirichlet form, the carré du champ $\mathrm{Var}_\mu(\frac{\partial F}{\partial \delta_.}(\mu))$ is replaced by the stronger carré du champ $\left|\frac{d}{dx}\frac{\partial \tilde{F}}{\partial \delta_.}\right|^2_{L^2([0,1])}$.

1.2 Functional inequalities for related Dirichlet forms

Von Renesse and Sturm also studied a second family of gradient forms, based on a different gradient defined as follows: let

$$\begin{aligned}\tilde{\mathcal{F}}C_b^1 := \{&F(g) = f(g(t_1), \dots, g(t_n)) :\\ &n \geq 1,\ 0 < t_1 < \cdots < t_n < 1,\ f \in C_b^1(\mathbb{R}^n)\Big\}.\end{aligned}$$

For a function $\varphi : [0,1] \to \mathbb{R}$ and for given $F \in \tilde{\mathcal{F}}C_b^1$ with representation $F(g) = f(g(t_1), \dots, g(t_n))$, the directional gradient $\tilde{\mathbb{D}}_\varphi F(g)$ of F at the point $g \in \mathcal{G}_0$ is defined by

$$\tilde{\mathbb{D}}_\varphi F(g) := \sum_{i=1}^n \partial_i f(g(t_1), \dots, g(t_n))\, \varphi(g(t_i)).$$

In the following let $H_0^s([0,1])$ be the Sobolev space of order s with Dirichlet boundary condition on $[0,1]$. Denote by $\varphi_k(x) := \sqrt{2}\sin(\pi k x)$, $k \geq 1$, the orthonormal basis of $L^2([0,1])$, consisting of eigenfunctions of the Laplacian Δ_D on $[0,1]$ with Dirichlet boundary conditions. Then

$$\|f\|_{H^s} := \left(\sum_{k=1}^\infty (\pi k)^{2s} \langle f, \varphi_k\rangle^2_{L^2([0,1])}\right)^{\frac{1}{2}}$$

defines a norm on $H_0^s([0,1])$ which is equivalent to the usual norm on $H_0^s([0,1])$. A corresponding orthonormal basis of $H_0^s([0,1])$ is given by the functions $\varphi_k^{(s)}(x) := (\pi k)^{-s}\varphi_k(x)$.

For given s one can then define the bilinear form

$$\tilde{\mathbb{E}}_0^{(s)}(F) := \sum_{k=1}^\infty \int_{\mathcal{G}_0} \left|\tilde{\mathbb{D}}_{\varphi_k^{(s)}} F(g)\right|^2 \mathbb{Q}_0^\beta(dg), \quad F \in \tilde{\mathcal{F}}C_b^1$$

on the space $L^2(\mathbb{Q}_0^\beta)$.

It has been shown in Corollary 6.11 of [1] that for $s > \frac{1}{2}$, the above bilinear form is closable (in $L^2(\mathbb{Q}_0^\beta)$) and its closure $(\tilde{\mathbb{E}}_0^{(s)}, D(\tilde{\mathbb{E}}_0^{(s)}))$ defines a regular, strongly local, recurrent Dirichlet form.

Remark 1.6 Monotonicity in s and general consequences

Note that for fixed $F(g) = f(g(t_1), \ldots, g(t_N)) \in \tilde{\mathcal{F}}C_b^1$ the energy $\tilde{\mathbb{E}}_0^{(s)}(F)$ decreases with s. Indeed, since $\varphi_k^{(s)} = (\pi k)^{-s}\varphi_k$, we conclude that

$$\left|\tilde{\mathbb{D}}_{\varphi_k^{(s)}} F(g)\right|^2 = (\pi k)^{-2s} \left(\sum_{i=1}^{N} (\partial_i f)(g(t_1), \ldots, g(t_N))\, \varphi_k(g(t_i))\right)^2$$

is decreasing in s for all k, so that

$$\tilde{\mathbb{E}}_0^{(s)}(F) = \sum_{k=1}^{\infty} \int \left|\tilde{D}_{\varphi_k^{(s)}} F(g)\right|^2 d\mathbb{Q}_0^{\beta}$$

decreases with s, too.

This monotonicity in s has the following two consequences:

(a) Let $M : \mathbb{R}_+ \to \mathbb{R}$ be nondecreasing on R_+ and nonnegative on $[m, \infty)$ for some $m \in \mathbb{R}_+$, and suppose that $\tilde{\mathbb{E}}_0^{(s_0)}$ satisfies a functional inequality of the type

$$\int F^2 M\left(\frac{F^2}{\|F\|^2_{L^2\left(\mathbb{Q}_0^{\beta}\right)}}\right) d\mathbb{Q}_0^{\beta} \le c\, \tilde{\mathbb{E}}_0^{(s_0)}(F), \quad F \in \tilde{\mathcal{F}}C_b^1, \tag{5}$$

then $\tilde{\mathbb{E}}_0^{(s)}$, $s < s_0$, satisfies the same functional inequality.

(b) Let $M : \mathbb{R}_+ \to \mathbb{R}$ be as in (a) and suppose that

$$\lim_{t \to \infty} M(t) = +\infty. \tag{6}$$

It follows from Theorem 1.2 in [8] that the set

$$B_1\left(\tilde{\mathbb{E}}_0^{(s_0)}\right) := \left\{F^2 : F \in \tilde{\mathcal{F}}C_b^1, \tilde{\mathbb{E}}_0^{(s_0)}(F) + \|F\|^2_{L^2\left(\mathbb{Q}_0^{\beta}\right)} \le 1\right\}$$

is uniformly integrable if and only if $\tilde{\mathbb{E}}_0^{(s_0)}$ satisfies an inequality of type (5) with some M satisfying (6). Remark (a) implies that in this case $\tilde{\mathbb{E}}_0^{(s)}$, too, cannot satisfy an inequality of type (5) with some M satisfying (6) for $s \ge s_0$.

Similar to the case of the Wasserstein diffusion of Sect. 1.1, we will study the Poincaré inequality and the logarithmic Sobolev inequality (and possible generalizations) for $\tilde{\mathbb{E}}_0^{(s)}$.

Theorem 1.7 $(\tilde{\mathbb{E}}_0^{(1)}, D(\tilde{\mathbb{E}}_0^{(1)}))$ *satisfies a Poincaré inequality with constant* $\frac{1}{\beta}$. *Consequently,* $(\tilde{\mathbb{E}}_0^{(s)}, D(\tilde{\mathbb{E}}_0^{(s)}))$ *satisfies a Poincaré inequality with constant* $\frac{1}{\beta}$ *for* $s < 1$, *too.*

Our next result concerns the question, whether $\tilde{\mathbb{E}}_0^{(s)}$ satisfies any functional inequality of hypercontractive type, i.e., any inequality of the type (5) with M increasing to infinity. Note that the logarithmic Sobolev inequality is a particular example for such an inequality.

Theorem 1.8 *Let $s > \frac{1}{2}$. Then the set*

$$\left\{ F^2 : \|F\|^2_{L^2\left(\mathbb{Q}_0^\beta\right)} + \tilde{\mathbb{E}}_0^{(s)}(F) \leq 1 \right\}$$

is not uniformly integrable. In particular, $\tilde{\mathbb{E}}_0^{(s)}$ does not satisfy a functional inequality of hypercontractive type.

The proofs of the last two theorems are given in Sect. 3.

2 Proofs of Theorem 1.2 and 1.4

2.1 Proof of Theorem 1.2

The basic idea for the proof will be an approximation of $\mathbb{E}(F)$ using the quadratic forms

$$\mathbb{E}^n(F) := \sum_{i,j=1}^m \int (\partial_i \varphi \partial_j \varphi)(s_n(f_1, g), \ldots, s_n(f_m, g)) \times s_n(f_i, f_j)\, d\mathbb{Q}_0^\beta(g) \tag{7}$$

for $F(g) = \varphi(\langle f_1, g\rangle, \ldots, \langle f_m, g\rangle) \in \mathcal{F}C_b^1$. Here

$$s_n(f, g) := \frac{1}{n}\sum_{l=1}^{n-1} f\left(\frac{l}{n}\right) g\left(\frac{l}{n}\right).$$

Assume for the moment that $f_1, \ldots, f_m$ are continuous, so that

$$\lim_{n\to\infty} s_n(f_i, g) = \langle f_i, g\rangle, \quad 1 \leq i \leq m,$$

for all $g \in \mathcal{G}_0$. This implies, $\lim_{n\to\infty} \mathbb{E}^n(F) = \mathbb{E}(F)$ and similarly, $\lim_{n\to\infty} F^n = F$ in $L^2(\mathbb{Q}_0^\beta)$, where

$$F^n(g) := \varphi(s_n(f_1, g), \ldots, s_n(f_m, g)). \tag{8}$$

In the following denote by $\mathcal{F}^n C_b^1$ the set of all functions F^n that can be represented in this way.

Proposition 2.1 $\left(\mathbb{E}^n, \mathcal{F}^n C_b^1\right)$ *satisfies a Poincaré inequality with constant less than* $\frac{1}{\beta}$.

The proof of Proposition 2.1 requires the following lemma concerning the finite dimensional projections of $\mathbb{E}^n$. To this end define the quadratic form

$$\mathcal{E}^n(\varphi) := n \sum_{i=1}^{n-1} \int_{\Sigma_{n-1}} (\partial_i \varphi)^2 \, d\nu_{q_n}, \quad \varphi \in C^1\left(\mathbb{R}^{n-1}\right),$$

where $q_n := \beta(\frac{1}{n}, \ldots, \frac{1}{n})$, $x_0 := 0$, $x_n := 1$. It follows for F with representation $F(g) = \varphi(\langle f_1, g\rangle, \ldots, \langle f_m, g\rangle)$, that

$$\mathbb{E}^n(F) = \mathcal{E}^n(\varphi_n),$$

where $\varphi_n(x) = \varphi(\tilde{s}_n(f_1, x), \ldots, \tilde{s}_n(f_m, x))$, and $\tilde{s}_n(f_i, x) = \frac{1}{n} \sum_{k=1}^{n-1} f_i\left(\frac{k}{n}\right) x_k$. Indeed, note that

$$\partial_{x_i} \varphi_n(x) = \frac{1}{n} \sum_{r=1}^{m} (\partial_r \varphi)(\tilde{s}_n(f_1, x), \ldots, \tilde{s}_n(f_m, x)) \, f_r\left(\frac{i}{n}\right),$$

so that

$$\begin{aligned}
&\mathcal{E}^n(\varphi_n) \\
&\quad = \frac{1}{n} \sum_{i=1}^{n-1} \int_{\Sigma_{n-1}} \left(\sum_{r=1}^{m} (\partial_r \varphi)(\tilde{s}_n(f_1, x), \ldots, \tilde{s}_n(f_m, x)) \, f_r\left(\frac{i}{n}\right) \right)^2 d\nu_{q_n}(x) \\
&\quad = \sum_{r,s=1}^{m} \int_{\Sigma_{n-1}} (\partial_r \varphi \partial_s \varphi)(\tilde{s}_n(f_1, x), \ldots, \tilde{s}_n(f_m, x)) \\
&\qquad \times \left(\frac{1}{n} \sum_{i=1}^{n-1} f_r\left(\frac{i}{n}\right) f_s\left(\frac{i}{n}\right) \right) d\nu_{q_n}(x) \\
&\quad = \sum_{r,s=1}^{m} s_n(f_r, f_s) d\mathbb{Q}_0^{\beta}(g) \int_{\mathcal{G}_0} (\partial_r \varphi \partial_s \varphi)(s_n(f_1, g), \ldots, s_n(f_m, g)) \\
&\quad = \mathbb{E}^n(F).
\end{aligned}$$

Lemma 2.2 $\left(\mathcal{E}^n, C_b^1\left(\mathbb{R}^{n-1}\right)\right)$ *satisfies a Poincaré inequality on* $L^2(\nu_{q_n})$ *with constant less than* $\frac{1}{\beta}$.

Proof We will show in Sect. 3 below [see estimate (16)] that the quadratic form

$$\tilde{A}_{q_n}(\varphi) = \sum_{i,j=1}^{n-1} \int_{\Sigma_{n-1}} \left(\partial_i \varphi \partial_j \varphi\right)(x) \left(x_i \wedge x_j - x_i x_j\right) d\nu_{q_n}(dx)$$

satisfies a Poincaré inequality with constant $\frac{1}{|q_n|} = \frac{1}{\beta}$. Moreover,

$$\tilde{A}_{q_n}(\varphi) \leq n \sum_{i=1}^{n-1} \int_{\Sigma_{n-1}} (\partial_i \varphi)^2 \, d\nu_{q_n} = \mathcal{E}^n(\varphi),$$

hence $\mathcal{E}^n$, too, satisfies a Poincaré inequality with constant less than $\frac{1}{\beta}$. □

Proof of Proposition 2.1 Fix $F(g) = \varphi(\langle f_1, g\rangle, \ldots, \langle f_m, g\rangle) \in \mathcal{F}C_b^1$, and let

$$\begin{aligned} F^n(g) &:= \varphi(s_n(f_1, g), \ldots, s_n(f_m, g)) \\ \varphi_n(x) &:= \varphi(\tilde{s}_n(f_1, x), \ldots, \tilde{s}_n(f_m, x)) \end{aligned}$$

Then Lemma 2.2 implies that

$$\mathrm{Var}_{\mathbb{Q}_0^\beta}\left(F^n\right) = \mathrm{Var}_{\nu_n}(\varphi_n) \leq \frac{1}{\beta}\mathcal{E}^n(\varphi_n) = \frac{1}{\beta}\mathbb{E}^n(F). \tag{9}$$

Hence, Proposition 2.1 is proven. □

Proof of Theorem 1.2 It is sufficient to prove inequality (2) for all functions $F \in \mathcal{F}C_b^1$ with $F(g) = \varphi(\langle f_1, g\rangle, \ldots, \langle f_m, g\rangle)$ with continuous $f_1, \ldots, f_m$, since the subspace of these functions is dense in the domain of the Wasserstein Dirichlet form $\mathbb{E}$. But for F of this type it follows that

$$F^n(g) = \varphi(s_n(f_1, g), \ldots, s_n(f_m, g)) \to F(g) \text{ in } L^2(\mathbb{Q}_0^\beta)$$

and

$$\lim_{n\to\infty} \mathbb{E}^n(F) = \mathbb{E}(F)$$

so that Proposition 2.1 now implies

$$\mathrm{Var}_{\mathbb{Q}_0^\beta}(F) = \lim_{n\to\infty} \mathrm{Var}_{\mathbb{Q}_0^\beta}\left(F^n\right) \leq \frac{1}{\beta} \lim_{n\to\infty} \mathbb{E}^n(F) = \frac{1}{\beta}\mathbb{E}(F).$$

□

2.2 Proof of Theorem 1.4

In the following, let C_1 be any constant such that for all $q \in \mathring{\mathbb{R}}^2_+$, the bilinear form

$$\tilde{\mathcal{E}}_q(f) := \int_0^1 t(1-t)\dot{f}^2(t)\, d\nu_q(t)\,,\ f \in C^1([0,1])$$

satisfies a logarithmic Sobolev inequality with constant less than $\frac{C_1}{q_1 \wedge q_1}$, i.e.,

$$\int_0^1 f^2 \log\left(\frac{f^2}{\|f\|^2_{L^2(\nu_q)}}\right) d\nu_q \leq \frac{C_1}{q_1 \wedge q_2}\tilde{\mathcal{E}}_q(f)\,,\ f \in C^1([0,1])\,. \tag{10}$$

According to Lemma 2.7 in [6] we can choose $C_1 = 160$.

Proposition 2.3 *Let $q \in \mathring{\mathbb{R}}^{n+1}_+$ and $q_* := \min_{1 \leq i \leq n+1} q_i$. Then*

$$A_q(f) := \sum_{i=1}^n \int_{\Sigma_n} x_i \left(\partial_{x_i} f\right)^2 d\nu_q, \quad f \in C^1_b\left(\mathbb{R}^n_+\right)$$

satisfies a logarithmic Sobolev inequality with constant less than $4\frac{C_1}{q_*}$.

Proposition 2.3 will be proven by induction on n. The case $n = 1$ follows from (10), since

$$A_q(f) = \int_0^1 t\dot{f}^2(t)\, d\nu_q(t) \geq \tilde{E}_q(f).$$

For the induction step we need the following

Proposition 2.4 *Let $p \in \mathring{\mathbb{R}}^{m+1}_+$, $q \in \mathring{\mathbb{R}}^{n+1}_+$,*

$$|p| := \sum_{i=1}^{m+1} p_i\,, \qquad |q| := \sum_{i=1}^{n+1} q_i$$
$$T : (0,1) \times \Sigma_m \times \Sigma_n \to \Sigma_{m+n+1}$$
$$(t, x, y) \mapsto (tx, t, t + (1-t)\, y) =: z\,,$$

and $r := (|p|, |q|) \in \mathring{\mathbb{R}}^2_+$. Then:

(i) $T\left(\nu_r \otimes \nu_p \otimes \nu_q\right) = \nu_{(p,q)}$.

(ii)

$$\frac{1}{|p| \wedge |q|} \int_{\Sigma_m} \int_{\Sigma_n} \tilde{\mathcal{E}}_r \left(f \circ T(\cdot, x, y)\right) \nu_p(dx)\, \nu_q(dy)$$

$$+ \frac{4}{p_*} \int_0^1 \int_{\Sigma_n} A_p \left(f \circ T(t, \cdot, y)\right) \nu_r(dt)\, \nu_q(dy)$$

$$+ \frac{4}{q_*} \int_0^1 \int_{\Sigma_m} A_q \left(f \circ T(t, x, \cdot)\right) \nu_r(dt)\, \nu_p(dx)$$

$$\leq \left(\frac{m+n+1}{m \wedge n} \vee 4\right) \frac{1}{(p,q)_*} A_{(p,q)}(f).$$

Proof (i) For the proof of (i) note that

$$DT(t, x, y) = \begin{pmatrix} x & tE_m & 0 \\ 1 & 0 & 0 \\ 1-y & 0 & (1-t)E_n \end{pmatrix}, \quad \text{where } E_m \text{ (resp. } E_n\text{)}$$

denotes the identity matrix with dimension m (resp. n). Clearly, $|\det DT(t, x, y)| = t^m (1-t)^n$, so that

$$\begin{aligned} &\nu_r(dt)\, \nu_p(dx)\, \nu_q(dy) \\ &= |\det DT(t,x,y)| \; \frac{\Gamma(|(p,q)|)}{\Gamma(|p|)\,\Gamma(|q|)} \; \frac{\Gamma(|p|)}{\prod_{i=1}^{m+1} \Gamma(p_i)} \; \frac{\Gamma(|q|)}{\prod_{i=1}^{n+1} \Gamma(q_i)} \\ &\quad \prod_{i=1}^{m+1} \left(t\,(x_i - x_{i-1})\right)^{p_i - 1} \prod_{i=1}^{n+1} \left((1-t)(y_i - y_{i-1})\right)^{q_i - 1} dt\, dx\, dy, \end{aligned}$$

which implies the first assertion by the change of variables formula.

(ii) To simplify notations, let

$$d_i f(t, x, y) := \partial_{x_i} f(T(t, x, y)).$$

Clearly

$$\begin{aligned} &t(1-t)\, \partial_t (f \circ T)(t, x, y)^2 \\ &= t(1-t) \left(\sum_{i=1}^{m+1} x_i d_i f(t, x, y) + \sum_{j=1}^{n} \left(1 - y_j\right) d_{j+m+1} f(t, x, y) \right)^2 \end{aligned}$$

$$\leq (m+n+1)\, t\,(1-t) \times \left(\sum_{i=1}^{m+1} x_i^2 (d_i f)^2 (t,x,y) + \sum_{j=1}^{n} \left(1-y_j\right)^2 (d_{j+m+1} f)^2 (t,x,y) \right), \tag{11}$$

$$\sum_{i=1}^{m} x_i \left(\partial_{x_i} (f \circ T)\right)^2 (t,x,y) = \sum_{i=1}^{m} t^2 x_i \,(d_i f)^2 (t,x,y) \tag{12}$$

and

$$\sum_{j=1}^{n} y_j \left(\partial_{y_j} (f \circ T)\right)^2 (t,x,y) = \sum_{j=1}^{n} (1-t)^2\, y_j \left(d_{j+m+1} f\right)^2 (t,x,y). \tag{13}$$

Using the inequalities

$$\begin{aligned} t\,(1-t)\, x_i^2 + t^2 x_i &\leq t\,(1-t)\, x_i + t^2 x_i = t x_i = z_i, \quad 1 \leq i \leq m+1 \\ t\,(1-t)\,(1-y_i)^2 + (1-t)^2\, y_i &\leq t + (1-t)\, y_i = z_{i+m+1}, \quad 1 \leq i \leq n, \end{aligned}$$

and combining with inequality (11) and equalities (12) and (13), we conclude that the left hand side of (ii) can be estimated from above by

$$\begin{aligned} &\int_0^1 \int_{\Sigma_m} \int_{\Sigma_n} \left(\frac{m+n+1}{|p| \wedge |q|} \vee \frac{4}{p_*} \right) \sum_{i=1}^{m+1} z_i \,(d_i f)^2 (t,x,y) \\ &\quad + \left(\frac{m+n+1}{|p| \wedge |q|} \vee \frac{4}{q_*} \right) \sum_{j=m+2}^{m+n+1} z_j \left(d_j f\right)^2 (t,x,y)\, \nu_r(dt)\ \nu_p(dx)\ \nu_q(dy) \\ &\leq \left(\frac{m+n+1}{m \wedge n} \vee 4 \right) \frac{1}{(p,q)_*} A_{(p,q)}(f). \end{aligned}$$

□

Proof of Proposition 2.3 As mentioned earlier, we proceed by induction on n. The case $n = 1$ being contained in [6].

The case $n = 2$ also has to be checked separately. In this case we decompose $q = (q_1, q_2, q_3)$ into $p = (q_1, q_2)$ and $r = (|p|, q_3)$. Similar to Proposition 2.4, one can show that if

$$T : (0,1) \times \Sigma_1 \to \Sigma_2, \qquad (t,x) \mapsto (tx, t),$$

then $T\left(\nu_r \otimes \nu_p\right) = \nu_q$ and

$$\frac{1}{|p| \wedge q_3} \int_{\Sigma_1} \tilde{\mathcal{E}}_r \left(f \circ T(\cdot, x)\right) \nu_p (dx) + \frac{1}{p_*} \int_0^1 A_p \left(f \circ T(t, \cdot)\right) \nu_r (dt) \leq \frac{4}{q_*} A_q (f) ,$$

so that by the assertion in the case $n = 1$ and Faris additivity theorem (see [4]), the direct sum of $\frac{1}{|p| \wedge q_3} \tilde{\mathcal{E}}_r$ and $\frac{4}{p_*} A_p$ satisfies a logarithmic Sobolev inequality with constant less than C_1, hence A_q satisfies a logarithmic Sobolev inequality with constant less than $4\frac{C_1}{q_*}$,

Suppose now that the assertion has been proven for all A_p, $p \in \mathring{\mathbb{R}}^m_+$, $m \leq N$ for $N \geq 2$, and let $Q \in \mathring{\mathbb{R}}^{N+1}_+$. Let $m = \left\lfloor \frac{N+1}{2} \right\rfloor$, $n = N + 1 - m$, $p := (Q_1, \ldots, Q_{m+1})$, $q := (Q_{m+2}, \ldots, Q_{N+1})$ and $r = (|p|, |q|)$. By assumption A_p resp. A_q satisfies a logarithmic Sobolev inequality with constant less than $4\frac{C_1}{p_*}$ resp. $4\frac{C_1}{q_*}$. (10) implies that $\tilde{\mathcal{E}}_r$ satisfies a logarithmic Sobolev inequality with constant less than $\frac{C_1}{|p| \wedge |q|}$. It follows from Faris additivity theorem (see [4]) that the direct sum of $\frac{1}{|p| \wedge |q|} \tilde{\mathcal{E}}_r$, $\frac{4}{p_*} A_p$ and $\frac{4}{q_*} A_q$ satisfies a logarithmic Sobolev inequality with constant less than C_1. Consequently, by Proposition 2.4, $A_{(p.q)}$ satisfies a logarithmic Sobolev inequality with constant less than $C_1 \left(\frac{m+n+1}{m \wedge n} \vee 4\right) \frac{1}{Q_*}$. Due to our choice of m and n we can estimate $\frac{m+n+1}{m \wedge n} \leq 4$, hence the assertion follows. □

Proof of Theorem 1.4 Recall the notations from the proof of Theorem 1.2. Proposition 2.4 implies that $\mathcal{E}^n(\varphi)$ satisfies a logarithmic Sobolev inequality with constant less than $\frac{C_1}{\beta}$ where C_1 is independent of n. Consequently,

$$\int F_n^2 \log \left(\frac{F_n^2}{\|F_n\|^2_{L^2(\mathbb{Q}_0^\beta)}} \right) d\mathbb{Q}_0^\beta = \int \varphi_n^2 \log \left(\frac{\varphi_n^2}{\|\varphi_n\|^2_{L^2(\nu_{q_n})}} \right) d\nu_{q_n} \leq \frac{C_1}{\beta} \mathcal{E}^n (\varphi_n) = \frac{C_1}{\beta} \mathbb{E}^n (F) . \tag{14}$$

Since

$$\lim_{n \to \infty} \int F_n^2 \log \left(\frac{F_n^2}{\|F_n\|^2_{L^2(\mathbb{Q}_0^\beta)}} \right) d\mathbb{Q}_0^\beta = \int F^2 \log \left(\frac{F^2}{\|F\|^2_{L^2(\mathbb{Q}_0^\beta)}} \right) d\mathbb{Q}_0^\beta$$

and $\lim_{n \to \infty} \mathbb{E}^n (F) = \mathbb{E}(F)$, we conclude that (14) holds with F_n replaced by F and $\mathbb{E}^n(F)$ replaced by $\mathbb{E}(F)$. □

3 Proofs of Theorems 1.7 and 1.8

3.1 Proof of Theorem 1.7

Let us first derive an alternative representation for $\tilde{\mathbb{E}}_0^{(s)}$. Note that for $F(g) = f(g(t_1), \ldots, g(t_n)) \in \mathcal{F}C_b^1$ we can write

$$\tilde{\mathbb{E}}_0^{(s)}(F) = \sum_{i,j=1}^{n} \int \left(\partial_i f \partial_j f\right)(g(t_1), \ldots, g(t_n))\, K^{(s)}\left(g(t_i), g\left(t_j\right)\right) \mathbb{Q}_0^{\beta}(dg)$$

with

$$K^{(s)}(x, y) := \sum_{k=1}^{\infty} \varphi_k^{(s)}(x)\, \varphi_k^{(s)}(y); \quad x, y \in [0, 1].$$

Note that $s > \frac{1}{2}$ implies that the sum converges.

Lemma 3.1

$$K^{(1)}(x, y) = x \wedge y - xy$$

Proof *Let $h \in \mathbb{R}^n$, $h = (h_1, \ldots, h_n)$ and define*

$$f(x) := \sum_{i=1}^{n} h_i\,(x \wedge x_i - x x_i)$$

Then $f \in H_0^1([0, 1])$, $\dot{f}(x) = \sum_{i=1}^{n} h_i \left(1_{[0,x_i]}(x) - x_i\right)$, $\int_0^1 \dot{f}\, dx = 0$, and thus

$$\sum_{i,j=1}^{n} h_i h_j \left(x_i \wedge x_j - x_i x_j\right) = \int_0^1 \left(\sum_{i=1}^{n} h_i \left(1_{[0,x_i]}(x) - x_i\right)\right)^2 dx$$

$$= \int_0^1 \dot{f}^2(x)\, dx = \sum_{k=1}^{\infty} \langle \dot{f}, \sqrt{2}\cos(\pi k x)\rangle^2 = \sum_{k=1}^{\infty} \left(\int_0^1 \dot{f}(x)\, \dot{\varphi}_k^{(1)}(x)\, dx\right)^2$$

$$= \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} h_i \varphi_k^{(1)}(x_i)\right)^2 = \sum_{i,j=1}^{n} h_i h_j K^{(1)}\left(x_i, x_j\right).$$

□

As a consequence, we conclude that the finite dimensional projections of $\tilde{\mathbb{E}}_0^{(1)}$ are given by

$$\tilde{A}_q(f) := \sum_{i,j=1}^{n} \int_{\Sigma_n} \left(\partial_i f \partial_j f\right)(x) \left(x_i \wedge x_j - x_i x_j\right) \nu_q(dx)$$

with $q = \beta(t_1, t_2 - t_1, \ldots, 1 - t_n)$ in the following sense: let $F(g) = f(g(t_1), \ldots, g(t_n))$, $f \in C_b^1(\mathbb{R}^n)$. Then

$$\tilde{\mathbb{E}}_0^{(1)}(F) = \tilde{A}_q(f).$$

We also define the bilinear form

$$\tilde{\mathcal{E}}_q(f) := \sum_{i,j=1}^{n} \int_{\Delta_n} x_i \left(\delta_{ij} - x_j\right) \partial_i f \partial_j f \, d\pi_q(x), \quad f \in C_b^1(\mathbb{R}^n).$$

In the following let

$$\Gamma_n\left(\tilde{A}\right)(f)(x) := \sum_{i,j=1}^{n} \left(x_i \wedge x_j - x_i x_j\right)\left(\partial_i f \partial_j f\right)(x)$$

be the carré du champ associated with $\tilde{A}_q$ and

$$\Gamma_n\left(\tilde{\mathcal{E}}\right)(f)(x) := \sum_{i,j=1}^{n} x_i \left(\delta_{ij} - x_j\right)\left(\partial_i f \partial_j f\right)(x)$$

be the carré du champ associated with $\tilde{\mathcal{E}}_q$. Finally, let us define the transformation

$$\begin{aligned} T_n : \Sigma_n &\to \Delta_n, \\ (x_1, \ldots, x_n) &\mapsto (x_1, x_2 - x_1, \ldots, x_n - x_{n-1}) = (z_1, \ldots, z_n). \end{aligned}$$

Lemma 3.2

$$\Gamma_n\left(\tilde{A}\right)(f \circ T_n)(x) = \Gamma_n\left(\tilde{\mathcal{E}}\right)(f)(z), \ f \in C_b^1\left(\mathbb{R}^n\right).$$

Proof Fix $f \in C_b^1(\mathbb{R}^n)$. To simplify notations, let

$$d_i f(x) := \partial_i f(T_n(x)), \quad 1 \le i \le n.$$

Then

$$\partial_{x_i}(f \circ T_n)(x) = \begin{cases} d_i f(x) - d_{i+1} f(x) & \text{if } 1 \le i < n \\ d_n f(x) & \text{if } i = n, \end{cases}$$

and

$$\begin{aligned}
\Gamma_n\left(\tilde{A}\right)(f\circ T_n)(x) &= \sum_{i,j=1}^{n}\left(x_i\wedge x_j - x_i x_j\right)\partial_{x_i}(f\circ T_n)(x)\partial_{x_j}(f\circ T_n)(x)\\
&= \sum_{i,j=1}^{n}\sum_{k=1}^{i\wedge j} z_k\,\partial_{x_i}(f\circ T_n)(x)\partial_{x_j}(f\circ T_n)(x)\\
&\quad -\left(\sum_{i=1}^{n}\sum_{k=1}^{i} z_k\,\partial_{x_i}(f\circ T_n)(x)\right)^2\\
&= \sum_{k=1}^{n} z_k\left(\sum_{i=k}^{n}\partial_{x_i}(f\circ T_n)(x)\right)\left(\sum_{j=k}^{n}\partial_{x_j}(f\circ T_n)(x)\right)\\
&\quad -\left(\sum_{k=1}^{n} z_k\sum_{i=k}^{n}\partial_{x_i}(f\circ T_n)(x)\right)^2\\
&= \sum_{k=1}^{n} z_k\left(d_k f(x)\right)^2 - \left(\sum_{k=1}^{n} z_k d_k f(x)\right)^2\\
&= \sum_{k,l=1}^{n} z_k(\delta_{kl}-z_l)d_k f(x) d_l f(x)\,.
\end{aligned}$$

□

The previous Lemma in particular implies that for $q \in \mathring{\mathbb{R}}_+^{n+1}$ and $f \in C_b^1(\mathbb{R}^n)$ we have that

$$\tilde{A}_q(f\circ T_n) = \tilde{\mathcal{E}}_q(f)\,. \tag{15}$$

Proof of Theorem 1.7 First let $s = 1$ and consider $F(g) = f(g(t_1), \ldots, g(t_n)) \in \tilde{\mathcal{F}}C_b^1$. Then Remark 2.2 of [6] implies that

$$\mathrm{Var}_{\pi_q}\left(f\circ T_n^{-1}\right) \le \frac{1}{\beta}\tilde{\mathcal{E}}_q\left(f\circ T_n^{-1}\right),$$

where $q = \beta(t_1, t_2 - t_1, \ldots, t_n - t_{n-1}, 1 - t_n) \in \mathring{\mathbb{R}}_+^{n+1}$. Clearly

$$\mathrm{Var}_{\nu_q}(f) = \mathrm{Var}_{\pi_q}\left(f\circ T_n^{-1}\right),$$

so that (15) implies

$$\mathrm{Var}_{\nu_q}(f) \le \frac{1}{\beta}\tilde{A}_q(f)\,. \tag{16}$$

Since on the other hand $\mathrm{Var}_{\mathbb{Q}_0^\beta}(F) = \mathrm{Var}_{\nu_q}(f)$ and $\tilde{\mathbb{E}}_0^{(1)}(F) = \tilde{A}_q(f)$, inequality (16) now implies that

$$\mathrm{Var}_{\mathbb{Q}_0^\beta}(F) \le \frac{1}{\beta}\tilde{\mathbb{E}}_0^{(1)}(F).$$

Since $\tilde{\mathcal{F}}C_b^1 \subset D\left(\tilde{\mathbb{E}}_0^{(1)}\right)$ dense, the last inequality also extends to the closure, which proves the assertion for the case $s = 1$. The case $s < 1$ clearly follows from Remark 1.6 (a). □

3.2 Proof of Theorem 1.8

We first need the following lemma.

Lemma 3.3 *Let $s > \frac{1}{2}$ and $\delta \in \left(0, s - \frac{1}{2}\right)$. Then there exists a finite positive constant $C_{\delta,s}$ such that*

$$\left|K^{(s)}(x,x)\right| \le C_{\delta,s}x^{2\delta}, \quad x \in [0,1].$$

Proof Using the inequality

$$|\sin(t)| \le |t|, \quad t \in \mathbb{R}$$

we conclude that

$$|\varphi_k(x)| \le \sqrt{2}\pi kx, \quad x \in [0,1].$$

Consequently,

$$\begin{aligned} K^{(s)}(x,x) &= \sum_{k=1}^{\infty}(\pi k)^{-2s}\varphi_k^2(x) \le \sum_{k=1}^{\infty}(\pi k)^{-2s}\varphi_k^{2-2\delta}(x)\left(\sqrt{2}\pi kx\right)^{2\delta} \\ &\le 2\pi^{2(\delta-s)}\left(\sum_{k=1}^{\infty}k^{-2(s-\delta)}\right)x^{2\delta}, \end{aligned}$$

which implies the assertion for $C_{\delta,s} := 2\pi^{2(\delta-2)}\sum_{k=1}^{\infty}k^{-2(s-\delta)}$, since $s-\delta > \frac{1}{2}$ by assumption. □

Proof of Theorem 1.8 Choose a sequence $(t_n) \subset (0,1)$ such that $\lim_{n\to\infty} t_n = 0$ and define

$$\tilde{F}_n(g); = \frac{1}{\sqrt{t_n}}g(t_n).$$

Let $\delta \in \left(0, s - \frac{1}{2}\right)$. Then

$$\int \tilde{F}_n^2 d\mathbb{Q}_0^\beta = \frac{1}{t_n} \int_0^1 t^2 d\pi_{(\beta t_n, \beta(1-t_n))}(t)$$
$$= \frac{1}{t_n} \frac{\Gamma(\beta t_n + 2)}{\Gamma(\beta t_n)} \frac{\Gamma(\beta)}{\Gamma(\beta+2)} = \frac{\beta t_n + 1}{\beta + 1} \leq 1$$

and

$$\tilde{\mathbb{E}}_0^{(s)}\left(\tilde{F}_n\right) = \frac{1}{t_n} \int_0^1 K^{(s)}(t,t)\, d\pi_{(\beta t_n, \beta(1-t_n))}(t) \leq C_{\delta,s} \frac{1}{t_n} \int_0^1 t^{2\delta} d\pi_{(\beta t_n, \beta(1-t_n))}(t)$$
$$= C_{\delta,s} \frac{1}{t_n} \frac{\Gamma(\beta t_n + 2\delta)}{\Gamma(\beta t_n)} \frac{\Gamma(\beta)}{\Gamma(\beta + 2\delta)} = C_{\delta,s} \beta \frac{\Gamma(\beta t_n + 2\delta)}{\Gamma(\beta t_n + 1)} \frac{\Gamma(\beta)}{\Gamma(\beta + 2\delta)}.$$

It follows that

$$c_n := \left\| \tilde{F}_n \right\|^2_{L^2\left(\mathbb{Q}_0^\beta\right)} + \tilde{\mathbb{E}}_0^{(s)}\left(\tilde{F}_n\right)$$

is bounded, hence

$$F_n := \frac{1}{\sqrt{c_n}} \tilde{F}_n \in B_1\left(\tilde{\mathcal{E}}_0^{(s)}\right).$$

We will show next that $(F_n^2)_{n \geq 1}$ is not uniformly integrable, which then implies that $B_1(\tilde{\mathbb{E}}_0^{(s)})$ is neither uniformly integrable. To this end note that for any $c > 0$

$$\int_{\{F_n^2 \geq c\}} F_n^2 d\mathbb{Q}_0^\beta = \frac{1}{c_n t_n} \int_{\left\{g : g(t_n) \geq \sqrt{c c_n t_n}\right\}} g(t_n)^2 \, d\mathbb{Q}_0^\beta(g)$$
$$= \frac{1}{c_n t_n} \int_{\sqrt{c c_n t_n}}^1 t^2 d\pi_{(\beta t_n, \beta(1-t_n))}(t)$$
$$= \frac{1}{c_n t_n} \frac{\Gamma(\beta)}{\Gamma(\beta t_n)\, \Gamma(\beta(1-t_n))} \int_{\sqrt{c c_n t_n}}^1 t^{\beta t_n + 1} (1-t)^{\beta(1-t_n) - 1} dt$$
$$= \frac{1}{c_n} \frac{\Gamma(\beta+1)}{\Gamma(\beta t_n + 1)\, \Gamma(\beta(1-t_n))} \int_{\sqrt{c c_n t_n}}^1 t^{\beta t_n + 1} (1-t)^{\beta(1-t_n) - 1} dt.$$

Consequently,

$$\sup_{n\geq 1} \int_{\{F_n^2 \geq c\}} F_n^2 d\mathbb{Q}_0^\beta \geq \frac{1}{\sup_{n\geq 1} c_n} \frac{\Gamma(\beta+1)}{\Gamma(1)\Gamma(\beta)} \int_0^1 t(1-t)^{\beta-1} dt$$
$$= \frac{1}{\sup_{n\geq 1} c_n} \frac{\Gamma(\beta+1)}{\Gamma(\beta+2)}$$

which is a lower bound independent of c. Consequently,

$$\lim_{c\to\infty} \int_{\{F_n^2 \geq c\}} F_n^2 d\mathbb{Q}_0^\beta \geq \frac{1}{\sup_{n\geq 1} c_n} \frac{1}{\beta+1} > 0,$$

which implies that $\left(F_n^2\right)_{n\geq 1}$ is not uniformly integrable. □

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