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Hydrodynamic behavior of 1D subdiffusive exclusion processes with random conductances

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Abstract Consider a system of particles performing nearest neighbor random walks on the lattice $\mathbb{Z}$ under hard-core interaction. The rate for a jump over a given bond is direction-independent and the inverse of the jump rates are i.i.d. random variables belonging to the domain of attraction of an α-stable law, $0 < \alpha < 1$. This exclusion process models conduction in strongly disordered 1D media. We prove that, when varying over the disorder and for a suitable slowly varying function L, under the super-diffusive time scaling $N^{1+1/\alpha}L(N)$, the density profile evolves as the solution of the random equation $\partial_t \rho = \mathfrak{L}_W \rho$, where $\mathfrak{L}_W$ is the generalized second-order differential operator $\frac{d}{du}\frac{d}{dW}$ in which W is a double-sided α-stable subordinator. This result follows from a quenched hydrodynamic limit in the case that the i.i.d. jump rates are replaced by a suitable array $\{\xi_{N,x} : x \in \mathbb{Z}\}$ having same distribution and fulfilling an a.s. invariance principle. We also prove a law of large numbers for a tagged particle.

Keywords Interacting particle system · Hydrodynamic limit · α-stable subordinator · Random environment · Subdiffusion · Quasi-diffusion

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1 Introduction

Fix a sequence of positive numbers $\xi = \{\xi_x : x \in \mathbb{Z}\}$ and consider the random walk $\{X_t : t \geq 0\}$ on $\mathbb{Z}$ which jumps from x to $x+1$ and from $x+1$ to x at rate ξ_x. Assume that

$$\lim_{\ell\to\infty} \frac{1}{\ell} \sum_{x=1}^{\ell} \xi_x^{-1} = \gamma, \quad \lim_{\ell\to\infty} \frac{1}{\ell} \sum_{x=-\ell}^{-1} \xi_x^{-1} = \gamma \tag{1.1}$$

for some $0 < \gamma < \infty$. It is well known that

$$X_{tN}/\sqrt{N} \quad \text{converges in distribution to} \quad \gamma^{-1} B_t, \tag{1.2}$$

as $N \uparrow \infty$, where B_t is a standard Brownian motion (cf. [7,11], for references).

In the particular case where $\{\xi_x : x \in \mathbb{Z}\}$ is an ergodic sequence of positive random variables (e.g. i.i.d. positive random variables) with $E[\xi_0^{-1}] < \infty$, a quenched (i.e. a.s. with respect to the environment ξ) invariance principle follows from the previous result setting $\gamma = E[\xi_0^{-1}]$. Notice that the noise survives in the limit only through the expected value of ξ_0^{-1}. Moreover, in the non trivial case where ξ is not constant, by Jensen's inequality, the diffusion coefficient $E[\xi_0^{-1}]^{-1}$ is strictly smaller than the expected value of the conductance ξ_x.

If the positive i.i.d. random variables $\{\xi_x : x \in \mathbb{Z}\}$ are such that $E[\xi_0^{-1}] = \infty$, the invariance principle (1.2) suggests that the random walk remains freezed in the diffusive scale. As discussed in [1], a natural assumption is to suppose that the distribution of ξ_0^{-1} belongs to the domain of attraction of an α-stable law, $0 < \alpha < 1$. In this case, the partial sums of the sequence $\{\xi_x^{-1}\}$ converge in law, when properly rescaled, to a double-sided α-stable subordinator W: there exists a slowly varying function $L(\cdot)$ such that, for each u in $\mathbb{R}$,

$$\lim_{N\to\infty} \frac{S(\lfloor Nu \rfloor)}{N^{1/\alpha} L(N)} = W(u) \quad \text{in law,} \tag{1.3}$$

where $\lfloor a \rfloor = \max\{k \in \mathbb{Z} : k \leq a\}$ is the integer part of a, and where the function $S : \mathbb{Z} \to \mathbb{R}$ is defined as

$$S(j) = \begin{cases} \sum_{x=0}^{j-1} \xi_x^{-1}, & j > 0, \\ 0, & j = 0, \\ -\sum_{x=j}^{-1} \xi_x^{-1}, & j < 0. \end{cases} \tag{1.4}$$

This case is treated in [11], where the authors prove that varying over the environment ξ and for a suitable slowly varying function $L'(\cdot)$, the process

$X(N^{1+1/\alpha}\ L'(N)t)/N$ converges weakly in the Skorohod space $D([0,\infty),\mathbb{R})$ to a continuous self-similar process.

In order to prove a quenched limiting behavior we need the limit (1.3) to be almost sure as well. To transform the convergence in law into an almost sure convergence we can, for instance, replace the sequence $\{\xi_x : x \in \mathbb{Z}\}$ by an array $\{\xi_{N,x} : x \in \mathbb{Z}\}$, $N \geq 1$, which has the same distribution as $\{\xi_x : x \in \mathbb{Z}\}$ for each $N \geq 1$. The most natural array can be constructed as follows. Define the nonnegative function G on $[0,\infty)$ by $P(W(1) > G(x)) = P(\xi_0^{-1} > x)$. Since $W(1)$ has a continuous strictly increasing distribution, G is well defined, nondecreasing and right continuous. Call G^{-1} the nondecreasing right continuous generalized inverse of G, $G^{-1}(x) = \sup\{y : G(y) \leq x\}$, and set

$$\xi_{N,x}^{-1} = G^{-1}\left(N^{1/\alpha}\{W(x+1/N) - W(x/N)\}\right).$$

Then (cf. [9, Sect. 3]) the array $\{\xi_{N,x} : x \in \mathbb{Z}\}$, defined as function of the subordinator W, has the same distribution of $\{\xi_x : x \in \mathbb{Z}\}$ and

$$\lim_{N\to\infty} \frac{S_N(\lfloor Nu \rfloor)}{N^{1/\alpha}} = W(u), \quad \forall u \in \mathbb{R}, \quad \text{for a.a. } W,$$

where S_N is defined as the function S in (1.4), with ξ_x^{-1} replaced by $\xi_{N,x}^{-1}$.

Fix $N \geq 1$ and consider the random walk $X_N(t)$ which jumps from x to $x+1$ and from $x+1$ to x at rate $\xi_{N,x}$. Kawazu and Kesten proved in [11] that for a.a. realizations of the environment and for a suitable slowly varying function $L'(\cdot)$, $X_N(N^{1+1/\alpha}L'(N)t)/N$ converges weakly to a process $\{Y_t : t \geq 0\}$. The first main result of this article states that $\{Y_t : t \geq 0\}$ is a Markov process with continuous paths which is not strongly Markov. In particular, $\{Y_t : t \geq 0\}$ is not a Feller process. Furthermore, we show in Sect. 3 that the generator $\mathfrak{L}_W$ of $\{Y_t : t \geq 0\}$ is the generalized second-order differential operator

$$\mathfrak{L}_W = \frac{d}{du}\frac{d}{dW}.$$

We point out that in contrast with the i.i.d. case with $E[\xi_0^{-1}] < \infty$, the noise W entirely survives in the limit. In fact, even the generator depends on the realization W of the subordinator.

The second main object of this article is the hydrodynamic behavior of a 1D simple exclusion process with conductances $\xi = \{\xi_x : x \in \mathbb{Z}\}$. This is the Markov process on $\{0,1\}^{\mathbb{Z}}$ which can be informally described as follows. Denote by η the configurations of $\{0,1\}^{\mathbb{Z}}$ so that $\eta(x) = 0$ if site x is vacant and $\eta(x) = 1$ otherwise. We start from a configuration with at most one particle per site. At rate ξ_x the occupation variables $\eta(x)$, $\eta(x+1)$ are exchanged. The generator L of this Markov process acts on local functions f as

$$Lf(\eta) = \sum_{x\in\mathbb{Z}} \xi_x[f(\sigma^{x,x+1}\eta) - f(\eta)], \tag{1.5}$$

where $\sigma^{x,x+1}\eta$ is the configuration obtained from η by exchanging the variables $\eta(x)$, $\eta(x+1)$:

$$(\sigma^{x,x+1}\eta)(y) = \begin{cases} \eta(x+1) & \text{if } y = x, \\ \eta(x) & \text{if } y = x+1, \\ \eta(y) & \text{otherwise.} \end{cases} \tag{1.6}$$

The hydrodynamic behavior of this exclusion process has been derived in [7,10] under the law of large numbers (1.1) for the inverse of the conductances, and previously in [15] under more restrictive assumptions: assume (1.1) for some $0 < \gamma < \infty$. Fix a continuous initial profile $\rho_0 : \mathbb{R} \to [0,1]$ and consider a sequence of probability measures μ^N on $\{0,1\}^{\mathbb{Z}}$ such that

$$\lim_{N\to\infty} \mu^N \left\{ \left| \frac{1}{N} \sum_{x\in\mathbb{Z}} H(x/N)\eta(x) - \int H(u)\rho_0(u)du \right| > \delta \right\} = 0$$

for every $\delta > 0$ and every continuous function H with compact support. As proven in [7,10], if $\mathbb{P}_{\mu^N}$ stands for the probability measure on the path space $D(\mathbb{R}_+, \{0,1\}^{\mathbb{Z}})$ induced by the initial state μ^N and the Markov process speeded up by N^2, then for any $t \geq 0$,

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N} \left\{ \left| \frac{1}{N} \sum_{x\in\mathbb{Z}} H(x/N)\eta_t(x) - \int H(u)\rho(t,u)du \right| > \delta \right\} = 0$$

for every $\delta > 0$ and every continuous function H with compact support. Here ρ is the solution of the heat equation

$$\partial_t \rho = \gamma^{-1} \partial_u^2 \rho$$

with initial condition ρ_0, t stands for the time variable and u for the macroscopic space variable.

In view of the discussion of the first part of this introduction, assume that the environment consists of a sequence of i.i.d. random variables $\{\xi_x : x \in \mathbb{Z}\}$ and that the distribution of ξ_0^{-1} belongs to the domain of attraction of an α-stable law, $0 < \alpha < 1$. Recall the definition of the array $\{\xi_{N,x} : x \in \mathbb{Z}\}$. By extending the methods developed in [7,10], we can prove a quenched hydrodynamic limit for the exclusion process with random conductances given by the array $\{\xi_{N,x} : x \in \mathbb{Z}\}$. Theorem 2.5 below states that, for almost all trajectories W, the density profile of the exclusion process with random conductances $\{\xi_{N,x} : x \in \mathbb{Z}\}$ evolves on the time scale $N^{1+1/\alpha}$ as the solution of

$$\partial_t \rho = \mathfrak{L}_W \rho, \tag{1.7}$$

where $\mathfrak{L}_W$ is the generalized second order differential operator defined above. From this quenched result we deduce in Theorem 2.2 an annealed result for the original exclusion process with random conductances given by $\{\xi_x : x \in \mathbb{Z}\}$.

The asymptotic evolution of a tagged particle is also examined. Under some assumptions on the solution of the hydrodynamic equation (1.7), we show that the asymptotic behavior $u(t)$ of a tagged particle initially at the origin is described by the differential equation

$$\frac{d}{dt+}u(t) = \begin{cases} -\dfrac{1}{\rho_t(u(t))}\dfrac{d\rho_t}{dW}(u(t)) & \text{if } \dfrac{d\rho_t}{dW}(u(t)) < 0 \\ -\dfrac{1}{\rho_t(u(t)-)}\dfrac{d\rho_t}{dW}(u(t)) & \text{if } \dfrac{d\rho_t}{dW}(u(t)) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

In this formula ρ_t is the solution of (1.7), the differential d/dW is defined by (2.2), and $(df/dt+)(t_0) = \lim_{\epsilon\downarrow 0}\epsilon^{-1}[f(t_0+\epsilon) - f(t_0)]$.

The article is conceived as follows. In Sects. 3 and 4, we examine the evolution of a random walk in the quenched environment generated by the subordinator W. We prove in Sect. 5 first the quenched hydrodynamic of the exclusion process. We deduce from this result the annealed behavior as well as the asymptotic behavior of the tagged particle.

2 Notation and results

We state in this section the main results of the article. In what follows, for simplicity of notation, we assume that $\{\xi_x^{-1} : x \in \mathbb{Z}\}$ is a sequence of i.i.d. non-negative α-stable random variables, $0 < \alpha < 1$, defined on some probability state space $(E, \mathcal{E}, \mathfrak{Q})$, i.e., we assume that

$$E_{\mathfrak{Q}}[\exp\{-\lambda\xi_x^{-1}\}] = \exp[-c_0\lambda^\alpha], \quad \lambda > 0, \tag{2.1}$$

for some positive constant c_0. The reader can check that all the results and proofs presented below can be easily extended to the general case where ξ_x^{-1} belongs to the domain of attraction of an α-stable law.

Let us fix some basic notation: given an interval $I \subset \mathbb{R}$ and a metric space $\mathbb{Y}$, we write $D(I, \mathbb{Y})$ for the space of càdlàg functions $f : I \to \mathbb{Y}$, endowed with the Skorohod metric d_S [3,6], and we denote by $\mathbb{D}(f)$ the set of discontinuity points of f. If $\mathbb{Y} = \mathbb{R}$, the generalized inverse of f is defined as

$$f^{-1}(u) = \sup\{v \in I : f(v) \le u\}.$$

Moreover, we denote by $C(\mathbb{Y})$, $C_b(\mathbb{Y})$, $C_c(\mathbb{Y})$, $C_0(\mathbb{Y})$, respectively, the space of continuous real functions on $\mathbb{Y}$, the space of bounded continuous real functions on $\mathbb{Y}$, the space of continuous real functions on $\mathbb{Y}$ with compact support and the space of bounded continuous real functions on $\mathbb{Y}$ vanishing at infinity, i.e. such that for any $\varepsilon > 0$ the function has modulus smaller than ε outside a suitable bounded subset $U \subset \mathbb{Y}$.

In what follows, we will introduce several processes defined in terms of the Brownian motion or the α-stable subordinator. To this aim, let B be the Brownian motion with $\mathbb{E}[B(t)^2] = 2t$, defined on some probability space $(\mathbb{X}, \mathbb{F}, \mathbb{P})$, and let $L(t, y)$ be the local time of B. Then, $\mathbb{P}$-almost surely,

$$\int_a^b L(t, y)dy = \int_0^t \mathbf{1}\{a \leq B(s) \leq b\}ds$$

for all $t \geq 0$, $a \leq b$. In this formula, $\mathbf{1}\{A\}$ stands for the indicator function of the set A.

Let W be a double-sided α-stable subordinator defined on some probability space $(\Omega, \mathcal{F}, P)$ [2, Sect. III.2]. Namely, $W(0) = 0$, W has non-negative independent increments such that for all $s < t$

$$E\left[\exp\{-\lambda[W(t) - W(s)]\}\right] = \exp\{-c_0\lambda^\alpha(t-s)\}$$

for all $\lambda > 0$ and the same positive constant c_0 as in (2.1). The sample paths of W are càdlàg, strictly increasing and of pure jump type, in the sense that

$$W(u) = \sum_{0<v\leq u}\{W(v) - W(v-)\}.$$

The jumps at location $(u, W(u) - W(u-))$ have a Poisson distribution with intensity $c_0^{1/\alpha} w^{-\alpha} du\, dw$ on $\mathbb{R} \times \mathbb{R}_+$. Given a realization of the subordinator W, denote by ν the Radon measure dW^{-1}, so that

$$\int f(u)\nu(du) = \int f(W(u))\, du$$

for all $f \in C_c(\mathbb{R})$. The support of ν is given by

$$\mathrm{supp}(\nu) = \overline{W(\mathbb{R})} = \{W(x), W(x-) : x \in \mathbb{R}\}.$$

Finally, given a Radon measure μ on $\mathbb{R}$ and a Borel function f, we set

$$\int_u^v f(x)\mu(dx) = \begin{cases} \int_{(u,v]} f(x)\mu(dx) & \text{if } u \leq v, \\ -\int_{(v,u]} f(x)\mu(dx) & \text{if } u > v. \end{cases}$$

2.1 Random walk with random conductances

Given $N \geq 1$, $x \in \mathbb{Z}$ and a realization $\{\xi_x : x \in \mathbb{Z}\}$ of the environment, consider the random walk $X_N^\xi(t|x)$ on $\mathbb{Z}$ having starting point x and generator $\mathbb{L}_{\xi,N}$ given by

$$(\mathbb{L}_{\xi,N} f)(x/N) = N^{1+1/\alpha}\xi_x\{f(x+1) - f(x)\} + N^{1+1/\alpha}\xi_{x-1}\{f(x-1) - f(x)\}.$$

To describe the asymptotic behavior of this random walk, fix a realization of the subordinator W and set

$$\psi(t|u) = \int_{\mathbb{R}} L(t, v-u)\nu(dv), \quad \psi^{-1}(t|u) = \sup\{s \geq 0 : \psi(s|u) \leq t\}.$$

It is known [4, V.2.11] that

$$Z(t|u) = u + B\left(\psi^{-1}(t|u)\right), \quad t \geq 0, \quad u \in \overline{W(\mathbb{R})},$$

is a strong Markov process on $\overline{W(\mathbb{R})}$. Let $Y(t|u)$, with $t \geq 0$ and $u \in \mathbb{R}$, be the process defined by

$$Y(t|u) = Y_W(t|u) = W^{-1}\left(Z(t|W(u))\right).$$

For u in $\mathbb{R}$, set $\lceil u \rceil = \min\{k \in \mathbb{Z} : k \geq u\}$. Kawazu and Kesten proved in [11] that the law of $X_N^{\xi}(t|\lceil uN \rceil)/N$ averaged over the environment ξ converges in distribution to the law of $Y(t|u)$ averaged over W.

We examine in Sect. 3 the process $\{Y_t : t \geq 0\}$. We first show that for a.a.realizations of the subordinator W, $\{Y_t : t \geq 0\}$ is Markov process with continuous paths which is not strongly Markov and therefore not Feller.

The definition of the generator $\mathfrak{L}_W = \frac{d}{du}\frac{d}{dW}$ of the process $\{Y_t : t \geq 0\}$ requires some notation. Fix a realization of the subordinator W. Denote by $C_{W,b}(\mathbb{R})$ (resp. $C_{W,0}(\mathbb{R})$) the set of bounded (resp. bounded which vanish at $\pm\infty$) càdlàg functions $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbb{D}(f) \subset \mathbb{D}(W)$. $C_{W,0}(\mathbb{R})$ is provided with the usual sup norm $\|\cdot\|_\infty$. Let $\mathfrak{D}_W$ be the set of functions f in $C_{W,0}(\mathbb{R})$ such that

$$f(x) = a + bW(x) + \int_0^x dW(y) \int_0^y g(z)dz \qquad \forall x \in \mathbb{R},$$

for some function g in $C_{W,0}(\mathbb{R})$ and some a, b in $\mathbb{R}$. One can check that this function g is unique.

Define the linear operator $\mathfrak{L}_W : \mathfrak{D}_W \to C_{W,0}(\mathbb{R})$ by setting $\mathfrak{L}_W f = g$. Formally,

$$\mathfrak{L}_W = \frac{d}{dx}\frac{d}{dW}.$$

Alternatively, one can introduce the generalized derivative $\frac{d}{dW}$ as follows

$$\frac{df}{dW}(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{W(x+\varepsilon) - W(x)}, \tag{2.2}$$

if the above limit exists and is finite. Due to Lemma 0.9 in [5, Appendix], given a right continuous function f and a continuous function h, the following identities are equivalent

$$\frac{df}{dW}(x) = h(x), \quad \forall x \in \mathbb{R}, \tag{2.3}$$

$$f(b) - f(a) = \int_{(a,b]} h(y) dW(y), \quad \forall a < b. \tag{2.4}$$

Hence, a function $f \in C_{W,0}(\mathbb{R})$ belongs to $\mathfrak{D}_W$ if and only if $\frac{df}{dW}(x)$ is well defined and derivable, and $\frac{d}{dx}\left(\frac{df}{dW}\right) \in C_{W,0}(\mathbb{R})$. In this case

$$\mathfrak{L}_W f = \frac{d}{dx}\frac{d}{dW} f = \frac{d}{dx}\left(\frac{df}{dW}\right).$$

We point out that a function f in $\mathfrak{D}_W \cap C(\mathbb{R})$ must be constant, otherwise due to (2.2) $(df/dW)(x)$ would be 0 on $\mathbb{D}(W)$, which is a dense set of $\mathbb{R}$. Since $df/dW(x)$ is differentiable it must be 0 everywhere. The equivalence between (2.3) and (2.4) allows to conclude.

Denote by $\{P_t : t \geq 0\}$ the semigroup of the Markov process Y_t so that for a bounded Borel function $H : \mathbb{R} \to \mathbb{R}$,

$$(P_t H)(u) = E[H(Y(t|u))], \qquad u \in \mathbb{R}.$$

In Sect. 3 we prove

Theorem 2.1 *The space $C_{W,0}(\mathbb{R})$ is P_t-invariant and $\{P_t : t \geq 0\}$ is a strongly continuous contraction semigroup on $C_{W,0}(\mathbb{R})$ with infinitesimal generator $\mathfrak{L}_W = \frac{d}{dx}\frac{d}{dW}$ defined on the domain $\mathfrak{D}_W$.*

In particular, given ρ_0 in $C_0(\mathbb{R})$, the function $\rho_W(t,u) = P_t\rho_0(u)$, $t \geq 0$, u in $\mathbb{R}$, is continuous in t and càdlàg in u, $\mathbb{D}(\rho_W(t,\cdot)) \subset \mathbb{D}(W)$ and $\rho_W(0,u) = \rho_0(u)$. Moreover, we show in [8] that $\rho_W(t,u)$ belongs to $\mathfrak{D}_W$, $\rho_W(t,u)$ is strictly positive and

$$\frac{\partial}{\partial t}\rho_W(t,u) = \frac{d}{du}\frac{d}{dW}\rho_W(t,u) \tag{2.5}$$

for all $t > 0$.

2.2 Annealed hydrodynamic limit

Let $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$ and denote by the Greek letter η the configurations of $\mathcal{X}$ so that $\eta(x) = 0$ if site x is vacant for the configuration η and $\eta(x) = 1$ otherwise.

For each fixed realization $\{\xi_x : x \in \mathbb{Z}\}$, consider the exclusion process on $\mathcal{X}$ with random conductances $\{\xi_x : x \in \mathbb{Z}\}$. This is the Markov process on $\{0,1\}^{\mathbb{Z}}$ with generator L given by (1.5). Given $T > 0$ and a probability measure μ on $\mathcal{X}$, let $\mathbb{P}_\mu^{\xi,N}$

be the law on the path space $D([0,T],\mathcal{X})$ of the exclusion process $\{\eta_t : t \geq 0\}$ with initial distribution μ and generator L *speeded up* by $N^{1+1/\alpha}$. Expectation with respect to $\mathbb{P}^{\xi,N}_{\mu}$ is denoted by $\mathbb{E}^{\xi,N}_{\mu}$.

Denote by $\mathcal{M} = \mathcal{M}(\mathbb{R})$ the space of Radon measures on $\mathbb{R}$ endowed with the vague topology, i.e., $\mathfrak{m}_n \to \mathfrak{m}$ if and only if $\mathfrak{m}_n(f) \to \mathfrak{m}(f)$ for any $f \in C_c(\mathbb{R})$. Let $\pi^N_t \in \mathcal{M}$ be the empirical measure at time t. This is the measure on $\mathbb{R}$ obtained by rescaling space by N and by assigning mass N^{-1} to each particle at time t:

$$\pi^N_t = \frac{1}{N}\sum_{x\in\mathbb{Z}} \eta_t(x)\delta_{x/N}, \tag{2.6}$$

where δ_u is the Dirac measure concentrated on u. For H in $C_c(\mathbb{R})$, $\langle \pi^N_t, H\rangle$ stands for the integral of H with respect to π^N_t:

$$\langle \pi^N_t, H\rangle = \frac{1}{N}\sum_{x\in\mathbb{Z}} H(x/N)\eta_t(x).$$

A sequence of probability measures $\{\mu^N : N \geq 1\}$ on $\mathcal{X}$ is said to be associated to a profile $\rho_0 : \mathbb{R} \to [0,1]$ if π^N_0 converges to $\rho_0(u)du$, as $N \uparrow \infty$:

$$\lim_{N\to\infty} \mu^N \left\{ \left| \langle \pi^N_0, H\rangle - \int_{\mathbb{R}} H(u)\rho_0(u)du \right| > \delta \right\} = 0$$

for all $H \in C_c(\mathbb{R})$ and for all $\delta > 0$.

The following theorem describes the hydrodynamic behavior of the exclusion process with random conductances ξ_x:

Theorem 2.2 *Let $\rho_0 : \mathbb{R} \to [0,1]$ be a uniformly continuous function and let $\{\mu^N : N \geq 1\}$ be a family of probability measures on $\mathcal{X}$ associated to ρ_0. Then, for all $T > 0$, all $\delta > 0$ and all $H \in C_c(\mathbb{R})$,*

$$\lim_{N\to\infty} \int \mathfrak{Q}(d\xi)\mathbb{P}^{\xi,N}_{\mu^N}\left[\sup_{0\leq t\leq T} \left| \langle \pi^N_t, H\rangle - \int_{\mathbb{R}} H(u)(P^{\xi,N}_t \rho_0)([u]_N)du \right| > \delta \right] = 0,$$

where $P^{\xi,N}_t$ is the Markov semigroup associated to the random walk $X^{\xi}_N(t|\cdot)/N$ and $[u]_N = \lceil uN \rceil/N$.

Corollary 2.3 *Given a uniformly continuous function $\rho_0 : \mathbb{R} \to [0,1]$, for each $N \geq 1$ on a common probability space $(\Theta_N, \mathcal{F}_N, Q_N)$ one can define a double-sided α–stable subordinator W and an exclusion process η_t with law $\int \mathfrak{Q}(d\xi)\mathbb{P}^{\xi,N}_{\mu^N}$ such* *that, for all $T > 0$, all $\delta > 0$ and all $H \in C_c(\mathbb{R})$,*

$$\lim_{N\to\infty} Q_N \left(\sup_{0\le t\le T} \left| \langle \pi_t^N, H\rangle - \int_{\mathbb{R}} H(u)\rho_W(t,u)du \right| > \delta \right) = 0,$$

where $\rho_W(t,u) = P_t\rho_0(u)$.

The proof of Theorem 2.2 is presented in Sect. 5. It relies on the quenched hydrodynamic limit stated in Theorem 2.5 below. Corollary 2.3 is a straightforward consequence of Theorem 2.2.

The asymptotic behavior of a tagged particle can be recovered from the hydrodynamic limit of the process. Fix an initial density profile $\rho_0 : \mathbb{R} \to [0,1]$ in $C_c(\mathbb{R})$ and let $\{\mu^N : N \ge 1\}$ be a family of probability measures on $\mathcal{X}$ associated to ρ_0 and with bounded support in the sense that there exists $A > 0$ such that $\mu^N\{\eta(x) = 1\} = 0$ for all $|x/N| > A$, $N \ge 1$. Assume that the origin is occupied at time 0. Tag the particle at the origin and let the process evolve according to the generator (1.5) speeded up by $N^{1+1/\alpha}$. Denote by x_t^N the position of the tagged particle at time t. Since particles cannot jump over other particles and since, according to the previous theorem, the density profile at time t is approximated by $P_t^{\xi,N}\rho_0$, x_t^N/N must be close to $u_t^{\xi,N}$, the unique solution of

$$\int_{-\infty}^{u_t^{\xi,N}} (P_t^{\xi,N}\rho_0)([u]_N)du = \int_{-\infty}^{0} \rho_0(u)du.$$

Note that $u_t^{\xi,N}$ is uniquely determined by this equation because $P_t^{\xi,N}\rho_0$ is strictly positive and, due to Lemma 4.4, is Lebesgue integrable. In Sect. 5, we prove

Theorem 2.4 *Fix an initial density profile $\rho_0 : \mathbb{R} \to [0,1]$ in $C_c(\mathbb{R})$ and let $\{\mu^N : N \ge 1\}$ be a family of probability measures on $\mathcal{X}$ associated to ρ_0, with bounded support and conditioned to have a particle at the origin. Then, for all $t > 0$ and all $\delta > 0$,*

$$\lim_{N\to\infty} \int \mathfrak{Q}(d\xi)\mathbb{P}_{\mu^N}^{\xi,N}\left[\left|x_t^N/N - u_t^{\xi,N}\right| > \delta\right] = 0.$$

2.3 Quenched hydrodynamic limit

For $N \ge 1$, x in $\mathbb{Z}$, set

$$c_x = c_x(W,N) = \frac{1}{\gamma_x}, \quad \gamma_x = \gamma_x(W,N) = N^{1/\alpha}\left\{W\left(\frac{x+1}{N}\right) - W\left(\frac{x}{N}\right)\right\}. \tag{2.7}$$

Note that c_x equals the constant $\xi_{x,N}$ defined in the introduction. Trivially, $\{\gamma_x : x \in \mathbb{Z}\}$ is a family of i.i.d. α-stable random variables such that

$$E\left[\exp\{-\lambda\gamma_x\}\right] = \exp\{-c_0\lambda^\alpha\}$$

for $\lambda > 0$. In particular, for each $N \geq 1$, $\{\gamma_x(W, N) : x \in \mathbb{Z}\}$ has the same distribution as $\{\xi_x^{-1} : x \in \mathbb{Z}\}$.

Consider the exclusion process on $\mathbb{Z}$ in which we exchange the occupation variables $\eta(x)$, $\eta(x+1)$ at rate c_x. This is the Markov process on $\mathcal{X}$ whose generator $\mathcal{L}_N$ acts on local functions $f : \mathcal{X} \to \mathbb{R}$ as

$$(\mathcal{L}_N f)(\eta) = \sum_{x\in\mathbb{Z}} c_x \left\{f(\sigma^{x,x+1}\eta) - f(\eta)\right\}, \tag{2.8}$$

where $\sigma^{x,x+1}\eta$ is the configuration defined by (1.6).

Given $T > 0$ and a probability measure μ on $\mathcal{X}$, let $\mathbb{P}_\mu^{W,N}$ be the law on the path space $D([0, T], \mathcal{X})$ of the exclusion process $\{\eta_t : t \geq 0\}$ with initial distribution μ and generator $\mathcal{L}_N$ speeded up by $N^{1+1/\alpha}$.

Theorem 2.5 *Let $\rho_0 : \mathbb{R} \to [0, 1]$ be a continuous function in $C_0(\mathbb{R})$ and let $\{\mu^N : N \geq 1\}$ be a family of probability measures on $\mathcal{X}$ associated to ρ_0. Then, for almost all W and all $T > 0$ the empirical measure π_t^N converges in probability to the measure $\rho_W(t, u)du$, where $\rho_W(t, u) = (P_t\rho_0)(u)$:*

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N}^{W,N}\left[\sup_{0\leq t\leq T}\left|\langle\pi_t^N, H\rangle - \int_{\mathbb{R}} H(u)\rho_W(t, u)du\right| > \delta\right] = 0$$

for all $H \in C_c(\mathbb{R})$ and for all $\delta > 0$.

As observed above in the annealed case, the asymptotic behavior of a tagged particle can be recovered from the hydrodynamic limit.

Theorem 2.6 *Let $\rho_0 : \mathbb{R} \to [0, 1]$ be a continuous function with compact support. Let $\{\mu^N : N \geq 0\}$ be a family of probability measures on $\mathcal{X}$ associated to ρ_0, with bounded support and conditioned to have a particle at the origin. Then, for almost all W and for all $t > 0$, x_t^N/N converges in $\mathbb{P}_{\mu^N}^{W,N}$-probability, as $N \uparrow \infty$, to $u_W(t)$ given by*

$$\int_{-\infty}^{u_W(t)} \rho_W(t, u)du = \int_{-\infty}^{0} \rho_0(u)du.$$

By Proposition 3.3 and Corollary 3.4 (iv) with $H = \rho_0$ and $\rho = 1$, $\rho_W(t, \cdot)$ is strictly positive and integrable so that $u_W(t)$ is uniquely determined. We prove Theorem 2.6

in Sect. 5.4 and that $u_W(t)$ is continuous. Moreover, we derive in Lemma 5.5, under some extra assumptions on $\rho_W(t,\cdot)$, the following differential equation for u_W:

$$\frac{d}{dt+}u_W(t)=\begin{cases}-\dfrac{1}{\rho_t(u_W(t))}\dfrac{d\rho_t}{dW}(u_W(t)) & \text{if } \dfrac{d\rho_t}{dW}(u_W(t))<0\\ -\dfrac{1}{\rho_t(u_W(t)-)}\dfrac{d\rho_t}{dW}(u_W(t)) & \text{if } \dfrac{d\rho_t}{dW}(u_W(t))>0\\ 0 & \text{otherwise.}\end{cases}$$

In this formula $\rho_t(\cdot)=\rho_W(t,\cdot)$, the differential d/dW is defined by (2.2), and $(df/dt+)(t_0)=\lim_{\epsilon\downarrow 0}\epsilon^{-1}[f(t_0+\epsilon)-f(t_0)]$.

3 Quasi-diffusions

In this section we study the Markov processes Y and Z, defined in Sect. 2.1 in terms of the Brownian motion B and the subordinator W. All the results presented in this section hold for almost all realizations of the subordinator W, although not always explicitly stated.

Fix a realization of the subordinator W. Recall that $\nu=dW^{-1}$ and that the support of ν coincides with $\overline{W(\mathbb{R})}=\{W(x),W(x-):x\in\mathbb{R}\}$.

3.1 The Markov process $Z(t|u)$

We briefly recall some results from [13,14] applied to the Markov process $Z(t|u)=u+B\left(\psi^{-1}(t|u)\right)$ with state space $\overline{W(\mathbb{R})}$ (see in particular Theorems 1.2.1 and 3.3.1 of [13] and Theorem 3.2 of [14]). Denote by $\{Q_t:t\geq 0\}$ the Markov semigroup associated to $Z(t|u)$ acting on the space $C_0(\overline{W(\mathbb{R})})$:

$$Q_tf(u)=\mathbb{E}\left[f(Z(t|u))\right]\tag{3.1}$$

for all f in $C_0(\overline{W(\mathbb{R})})$. By endowing $C_0(\overline{W(\mathbb{R})})$ with the uniform norm $\|\cdot\|_\infty$, $\{Q_t:t\geq 0\}$ is a strongly continuous semigroup of contraction operators: for all $f\in C_0(\overline{W(\mathbb{R})})$, Q_tf belongs to $C_0(\overline{W(\mathbb{R})})$, $\|Q_tf\|_\infty\leq\|f\|_\infty$ and

$$\lim_{s\to 0}\|Q_{t+s}f-Q_tf\|_\infty=0,$$

for all $t>0$. The same statement holds for $t=0$ if s takes only positive values.

Let D_W be the set of functions f in $C_0(\overline{W(\mathbb{R})})$ for which there exists a function h in $C_0(\overline{W(\mathbb{R})})$ and a,b in $\mathbb{R}$ such that

$$f(u)=a+bu+\int_0^u dv\int_0^v h(w)\nu(dw)\tag{3.2}$$

for all u in $\overline{W(\mathbb{R})}$. One can check that h is univocally determined. We denote h by

$$h = \frac{d}{dW^{-1}} \frac{d}{du} f = L_W f.$$

Then, $L_W : D_W \to C_0(\overline{W(\mathbb{R})})$ is the generator of the Markov semigroup $\{Q_t : t \geq 0\}$ on $C_0(\overline{W(\mathbb{R})})$.

We prove in [8] that the process Z admits a strictly positive symmetric transition density function $q_t(x, y)$ w.r.t. ν:

Theorem 3.1 *There exists a strictly positive Borel function q,*

$$q : (0, \infty) \times \overline{W(\mathbb{R})} \times \overline{W(\mathbb{R})} \to (0, \infty),$$

symmetric in x, y, such that

$$\mathbb{E}[f(Z(t|x))] = \int f(y) q_t(x, y) \nu(dy)$$

for all $t > 0$, x in $\overline{W(\mathbb{R})}$ and f in $C_b(\overline{W(\mathbb{R})})$. Moreover,

$$\int q_t(x, y) \nu(dy) = 1$$

for every $t > 0$, x in $\overline{W(\mathbb{R})}$ and $q_t(\cdot, y) \in D_W$,

$$\partial_t q_t(\cdot, y) = L_W q_t(\cdot, y)$$

for every $t > 0$ and y in $\overline{W(\mathbb{R})}$.

3.2 Markovian properties of the process $Y(t|u)$

Denote by $\{x_j : j \geq 1\}$ the jump points of W, which form a countable dense set in $\mathbb{R}$. Since $W^{-1}(W(x_j-)) = W^{-1}(W(x_j))$, the function $W^{-1} : \overline{W(\mathbb{R})} \to \mathbb{R}$ is not one to one and the process $Y(t|u) = W^{-1}(Z(t|W(u)))$ with space state $\mathbb{R}$ could be a non Markov process. The following proposition clarifies the Markovian properties of the process Y

Proposition 3.2 *The stochastic process Y has continuous paths. It is Markov but not strongly Markov. In particular, it is not a Feller Markov process.*

Proof The continuity of paths can be proved by the same arguments used in Lemma 4 of [11]. In what follows we denote, respectively, by $\mathbb{P}^Y_y$, $\mathbb{P}^Z_z$ the law of $Y(\cdot|y)$ and $Z(\cdot|z)$, where $y \in \mathbb{R}$, $z \in \overline{W(\mathbb{R})}$, and by $\mathbb{E}^Y_y$, $\mathbb{E}^Z_z$ the related expectations. Moreover, we write ω for a generic path in $D([0, \infty), \mathbb{R})$ and $\theta_t \omega$ for the time-translated path $\theta_t \omega(s) = \omega(t + s)$.

First we prove that Y is a Markov process. To this aim, we fix $y \in \mathbb{R}$, $t > 0$ and let $\mathcal{A}, \mathcal{B} \subset D([0,\infty),\mathbb{R})$ be of the form

$$\begin{aligned}\mathcal{A} &= \{\omega : \omega(t_i) \in [a_i, b_i] \quad \forall 1 \le i \le n\}\,,\\ \mathcal{B} &= \{\omega : \omega(s_i) \in [c_i, d_i] \quad \forall 1 \le i \le k\}\,,\end{aligned}$$

where $0 \le t_1 < t_2 < \cdots < t_n < t$ and $0 \le s_1 < s_2 < \cdots < s_k$.

Due to the definition of Y,

$$\mathbb{P}^Y_y\,(\omega \in \mathcal{A}, \theta_t\omega \in \mathcal{B}) = \mathbb{P}^Z_{W(y)}\left(W^{-1}\circ\omega \in \mathcal{A}, \theta_t(W^{-1}\circ\omega) \in \mathcal{B}\right).$$

Since Z is a Markov process, the previous expresssion is equal to

$$\mathbb{E}^Z_{W(y)}\left[\mathbf{1}\{W^{-1}\circ\omega \in \mathcal{A}\}\mathbb{P}^Z_{Z(t|W(y))}(W^{-1}\circ\omega \in \mathcal{B})\right].$$

We claim that $Z(t|W(y)) = W(Y(t|y))$ with probability 1. In fact, we know that $Z(t|W(y))$ has value $W(Y(t|y)-)$ or $W(Y(t|y))$. If $W(Y(t|y)-) = W(Y(t|y))$ the conclusion is trivial. Otherwise, it must be $W(Y(t|y)-) = W(x_j-)$ for some j. Since a.s. ν has no atoms, the countable set $\{W(x_j-) : j \ge 1\}$ has zero ν-measure and due to Theorem 3.1

$$\mathbb{P}\left[Z(t|W(y)) \in \{W(x_j-) : j \ge 1\}\right] = 0.$$

This allows to conclude that $Z(t|W(y)) = W(Y(t|y))$ with probability 1. Therefore,

$$\mathbb{P}^Z_{Z(t|W(y))}(W^{-1}\circ\omega \in \mathcal{B}) = \mathbb{P}^Z_{W(Y(t|y))}(W^{-1}\circ\omega \in \mathcal{B})$$

$\mathbb{P}^Z_{W(y)}$-a.s. By definition of Y, putting all previous identities together, we get that

$$\begin{aligned}\mathbb{P}^Y_y\,[\omega \in \mathcal{A}, \theta_t\omega \in \mathcal{B}] &= \mathbb{E}^Z_{W(y)}\left[\mathbf{1}\left\{W^{-1}\circ\omega \in \mathcal{A}\right\}\mathbb{P}^Z_{W(Y(t|y))}\left[W^{-1}\circ\omega \in \mathcal{B}\right]\right]\\ &= \mathbb{E}^Y_y\left[\mathbf{1}\{\omega \in \mathcal{A}\}\mathbb{P}^Y_{Y(t|y)}\,[\omega \in \mathcal{B}]\right].\end{aligned}$$

This proves that Y is a Markov process.

We now show that $Y(t|u)$ is not strongly Markovian with respect to the filtration $\mathcal{F}^Y_t = \sigma(Y_s : s \le t)$. Fix y in $\mathbb{R}$ such that $0 < W(y-) < W(y)$ and consider the sets

$$\begin{aligned}A &= \{\exists\delta > 0 : Y_s \le y \text{ for } 0 \le s \le \delta\}\,,\\ B &= \{\exists\delta > 0 : Z_s \le W(y) \text{ for } 0 \le s \le \delta\}\,.\end{aligned}$$

Let τ be the first time the process Y_t reaches y: $\tau = \inf\{t \ge 0 : Y_t = y\}$ and let σ be the first time the process Z_t reaches $W(y)$ or $W(y-)$: $\sigma = \inf\{t \ge 0 : Z_t = W(y) \text{ or } Z_t = W(y-)\}$. Since Y_t has continuous paths, τ is a stopping time for the filtration $\{\mathcal{F}_t^Y : t \geq 0\}$ and σ is a stopping time for the filtration $\{\mathcal{F}_t^Z : t \geq 0\}$, where $\mathcal{F}_t^Z = \sigma(Z_s : s \leq t)$.

Assume, by contradiction, that Y is strongly Markov. By the strong Markov property,

$$\mathbb{P}_0^Y [\theta_\tau Y \in A] = \mathbb{E}_0^Y \left[\mathbb{P}_{Y_\tau}^Y [A]\right].$$

Since $Y_\tau = y$, the previous expectation is equal to $\mathbb{P}_y^Y[A]$. By definition of the process Y, this probability corresponds to $\mathbb{P}_{W(y)}^Z[B]$. This last probability is equal to 0 in view of the construction of Z through the Brownian motion.

On the other hand, by construction of the Markov process $Y(t|0)$, by definition of the random times τ, σ and since σ is a stopping time,

$$\mathbb{P}_0^Y [\theta_\tau Y \in A] = \mathbb{P}_0^Z [\theta_\sigma Z \in B] = \mathbb{E}_0^Z \left[\mathbb{P}_{Z_\sigma} [B]\right].$$

Since $0 < W(y-)$, $Z_\sigma = W(y-)$. In particular, P_{Z_σ} is equal to $\mathbb{P}_{W(y-)}[B]$ and this probability is equal to 1 by construction of the process Z through the Brownian motion.

Finally, Y cannot be a Feller Markov process since otherwise it would be a strong Markov process. □

3.3 The generator of the process $Y(t|u)$

We obtain in this section the generator of the Markov process Y. To keep notation simple, we denote $Z(t|W(u))$ by Z_t and $Y(t|u)$ by Y_t. Moreover, we write $\{x_j : j \geq 1\}$ for the jump points of W, which form a countable dense set.

Denote by d_W the distance in $\mathbb{R}$ defined by $d_W(x, y) = |W(x) - W(y)|$ and by $\mathbb{R}_W$ the completion of $\mathbb{R}$ with respect to this distance. It is easy to check that $\mathbb{R}_W$ is obtained by dividing in two each jump point of W: $\mathbb{R}_W = \mathbb{R} \cup \{x_j^- : j \geq 1\}$ and

$$d_W(x_j^-, y) = |W(x_j-) - W(y)|, \quad d_W(x_j^-, x_k^-) = |W(x_j-) - W(x_k-)|$$

for every y in $\mathbb{R}$, $j, k \geq 1$.

Recall that $\overline{W(\mathbb{R})} = \{W(x) : x \in \mathbb{R}\} \cup \{W(x_j-) : j \geq 1\}$. Let

$$W_e : \mathbb{R}_W \to \overline{W(\mathbb{R})}$$

be given by $W_e(x) = W(x)$ for x in $\mathbb{R}$, $W_e(x_j^-) = W(x_j-)$ for $j \geq 1$. W_e is an isometry from $(\mathbb{R}_W, d_W)$ to $(\overline{W(\mathbb{R})}, d)$, where d is the usual Euclidean distance in $\mathbb{R}$. Its inverse, $W_e^{-1} : \overline{W(\mathbb{R})} \to \mathbb{R}_W$, is given by $W_e^{-1}(W(x)) = x$, $W_e^{-1}(W(x_j-)) = x_j^-$.

Since W_e is an isometry, all the results concerning the process Z_t with state space $\overline{W(\mathbb{R})}$ can be trivially restated in terms of the pullback process $X_t = W_e^{-1}(Z_t)$ with state space $\mathbb{R}_W$. In particular, X_t is a strong Markov process and, denoting by $\mathcal{Q}_t$ its Markov semigroup, $\{\mathcal{Q}_t : t \geq 0\}$ is a strongly continuous semigroup of contraction operators acting on the space $C_0(\mathbb{R}_W)$ with norm $\|\cdot\|_\infty$. Let us describe its generator $\mathcal{L}_W$.

We have seen that the domain of the generator L_W of the Markov process Z_t is D_W. Hence the domain of the generator $\mathcal{L}_W$ is given by the set $\mathcal{D}_W$ where $\mathcal{D}_W = \{f \circ W_e : f \in D_W\}$. Let $f, h \in C(\overline{W(\mathbb{R})})$ be as in (3.2) for suitable constants a, b. By a change of variables, it simple to check that the functions $F = f \circ W_e$ and $H = h \circ W_e$, belonging to $C(\mathbb{R}_W)$, satisfy the identity

$$F(x) = a + bW(x) + \int_0^x dW(y) \int_0^y H(z)dz, \quad x \in \mathbb{R}, \tag{3.3}$$

and therefore, by continuity,

$$F(x_j^-) = F(x_j-), \qquad \forall j \geq 1.$$

Viceversa, if $F, H \in C(\mathbb{R}_W)$ fulfill (3.3), then $f = F \circ W_e^{-1} \in C(\overline{W(\mathbb{R})})$ and $h = H \circ W_e^{-1} \in C(\overline{W(\mathbb{R})})$ satisfy (3.2). In particular, we get that

$$\mathcal{D}_W = \{F \in C_0(\mathbb{R}_W) : \exists H \in C_0(\mathbb{R}_W), \exists a, b \in \mathbb{R} \text{ satisfying (3.3)}\}.$$

Moreover, due to the above observations, we obtain that $\mathcal{L}_W F = H$ for all $F \in \mathcal{D}_W$ and $H \in C_0(\mathbb{R}_W)$ as in (3.3).

We now turn to the process Y_t. Recall the definition of the spaces $C_{W,0}(\mathbb{R})$ and $\mathfrak{D}_W$ introduced in Sect. 2.1: $C_{W,0}(\mathbb{R})$ is the set of functions $F : \mathbb{R} \to \mathbb{R}$ which are càdlàg, whose discontinuities form a subset of $\{x_j : j \geq 1\}$ and which vanish at $\pm\infty$, while $\mathfrak{D}_W$ denotes the set of functions F in $C_{W,0}(\mathbb{R})$ such that

$$F(x) = a + bW(x) + \int_0^x dW(y) \int_0^y G(z)dz, \quad \forall x \in \mathbb{R},$$

for some function G in $C_{W,0}(\mathbb{R})$ and some a, b in $\mathbb{R}$. G is univocally determined and the linear operator $\mathfrak{L}_W : \mathfrak{D}_W \to C_{W,0}(\mathbb{R})$ is defined by setting $\mathfrak{L}_W F = G$. Formally, $\mathfrak{L}_W = \frac{d}{dx}\frac{d}{dW}$.

Proof of Theorem 2.1 Since $W^{-1}(W(x_j-)) = W^{-1}(W(x_j))$ we can write $Y_t = \Phi(X_t)$ where $\Phi : \mathbb{R}_W \to \mathbb{R}$ is defined as $\Phi(x) = x$ for all $x \in \mathbb{R}$ and $\Phi(x_j^-) = x_j$ for all $j \geq 1$.

Given a function $H : \mathbb{R} \to \mathbb{R}$, define the function $\mathfrak{E}H : \mathbb{R}_W \to \mathbb{R}$ as $\mathfrak{E}H = H \circ \Phi$. Viceversa, given a function $h : \mathbb{R}_W \to \mathbb{R}$, define $\mathfrak{P}h : \mathbb{R} \to \mathbb{R}$ as $\mathfrak{P}h(x) = h(x)$ for all $x \in \mathbb{R}$. One can easily check that $\mathfrak{E}$ maps $C_{W,0}(\mathbb{R})$, $\mathfrak{D}_W$ bijectively onto $C_0(\mathbb{R}_W)$, $\mathcal{D}_W$ with inverse function given by $\mathfrak{P}$. Moreover,

$$\mathfrak{L}_W H = \mathfrak{P}\mathcal{L}_W \mathfrak{E}H, \quad \forall H \in \mathfrak{D}_W. \tag{3.4}$$

Since $Y_t = \Phi(X_t)$, for all $H \in C_{W,0}(\mathbb{R})$ we can write

$$P_t H(u) = \mathbb{E}\left[H(Y(t|u))\right] = \mathbb{E}\left[H \circ \Phi(X(t|u))\right] = \mathcal{Q}_t\left(\mathfrak{C}H\right)(u), \quad \forall u \in \mathbb{R},$$

thus implying that

$$P_t H = \mathfrak{P}\mathcal{Q}_t\mathfrak{C}H, \quad \forall H \in C_{W,0}(\mathbb{R}). \tag{3.5}$$

Due to the above identity and since $C_0(\mathbb{R}_W)$ is $\mathcal{Q}_t$-invariant, the space $C_{W,0}(\mathbb{R})$ is P_t-invariant. Since $\mathfrak{C}$ is an isomorphism between the normed spaces $C_{W,0}(\mathbb{R})$ and $C_0(\mathbb{R}_W)$ (endowed of the uniform norm $\|\cdot\|_\infty$) and since the identities (3.4) and (3.5) hold, the fact that $\{\mathcal{Q}_t : t \geq 0\}$ is a strongly continuous contraction semigroup on $C_0(\mathbb{R}_W)$ having generator $\mathcal{L}_W$ with domain $\mathcal{D}_W$ implies that $\{P_t : t \geq 0\}$ is a strongly continuous contraction semigroup on $C_{W,0}(\mathbb{R})$ with generator $\mathfrak{L}_W$ having domain $\mathfrak{D}_W$. □

We conclude this section with a result which follows easily from Theorem 3.1 by a change of variables. It will be particularly useful in the study of the limiting behavior of the tagged particle in the exclusion process with random conductances.

Proposition 3.3 *The Borel function p defined on $(0,\infty) \times \mathbb{R} \times \mathbb{R}$ as*

$$p_t(x, y) = q_t(W(x), W(y))$$

is the transition density function of the Markov process Y w.r.t. the Lebesgue measure. $p_t(\cdot,\cdot)$ is a strictly positive symmetric function and

$$\int_{\mathbb{R}} p_t(x, y)dy = 1, \quad (P_t f)(x) = \int p_t(x, y) f(y)dy,$$

for all $t > 0$, x in $\mathbb{R}$ and functions f in $C_{W,b}(\mathbb{R})$.

Next result is a consequence of Theorem 2.1 and this proposition.

Corollary 3.4 *Fix a function H in $C_0(\mathbb{R})$.*

(i) *For every $t \geq 0$, $P_t H$ is a càdlàg function vanishing at infinity.*
(ii) *If H has compact support, $P_t H$ belong to $L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} du\, P_t H(u) = \int_{\mathbb{R}} du\, H(u).$$

(iii) *As $t \downarrow 0$, $P_t H$ converges to H pointwisely. If H has compact support this limit takes also place in $L^1(\mathbb{R})$. In this case, $P_{t+s} H$ converges to $P_t H$ in $L^1(\mathbb{R})$ as $s \to 0$.*

(iv) *For any function H in $C_c(\mathbb{R})$ and any function ρ in $C_b(\mathbb{R})$,*

$$\int du (P_t H)(u)\rho(u) = \int du H(u)(P_t \rho)(u).$$

Proof Statement (i) follows from Theorem 2.1 and statement (ii) from Proposition 3.3. The first claim of (iii) follows from Theorem 2.1. To prove the second claim of (iii) assume, without loss of generality, that H is positive and has compact support. By part (ii) of this corollary, $\int P_t H(u)du$ is equal to $\int H(u)du$. In particular, by Scheffé Theorem, $P_t H$ converges to H in $L^1(\mathbb{R})$. It follows from Proposition 3.3 that the semigroup P_t is a contraction in $L^1(\mathbb{R})$. The third claim of (iii) follows from this observation and the convergence of $P_t H$ to H in $L^1(\mathbb{R})$. Claim (iv) follows from the symmetry of the transition density function $p_t(x, y)$ and Fubini's theorem. □

4 Random walk with random conductances

Fix a realization of the subordinator W and $N \geq 1$, and recall the definition of the random variables $\{c_x : x \in \mathbb{Z}\}$ given in (2.7). We examine in this section the limiting behavior of the continuous–time random walk on $\mathbb{Z}$ which jumps from x to $x+1$ at rate c_x and from x to $x-1$ are rate c_{x-1}. We first recall some results due to Stone [17].

Given a Radon measure μ on $\mathbb{R}$ with support, denoted by $\mathrm{supp}(\mu)$, unbounded from below and from above, for each $x \in \mathrm{supp}(\mu)$ and $t \geq 0$ set

$$\psi(t|x, \mu) = \int_{\mathbb{R}} L(t, y-x)\mu(dy), \quad \psi^{-1}(t|x, \mu) = \sup\{s \geq 0 : \psi(s|x, \mu) \leq t\}. \tag{4.1}$$

Then $\psi(\cdot|x, \mu)$ is a continuous function and $\psi^{-1}(\cdot|x, \mu)$ is a nondecreasing càdlàg function. Set

$$Z(t|x, \mu) = B\left(\psi^{-1}(t|x, \mu)\right) + x.$$

$Z = \{Z(t|x, \mu) : t \geq 0\}$, defined on probability space $(\mathbb{X}, \mathbb{F}, \mathbb{P})$ as the Brownian motion B, is a strong Markov process with state space $\mathrm{supp}(\mu)$ and paths in the Skohorod space $D([0, \infty), \mathbb{R})$ endowed of the Skohorod metric d_S [4, V.2.11].

By Theorem 1 and Corollary 1 in [17], we have

Proposition 4.1 *Let $\{\mu_n\}_{n\geq 0}$, μ be Radon measures on $\mathbb{R}$ with support unbounded from below and from above. Suppose that*:

- $\mu_n \to \mu$ *vaguely*,
- *if $y_n \in supp(\mu_n)$ is a converging sequence as $n \uparrow \infty$, then* $\lim_{n\uparrow\infty} y_n \in supp(\mu)$.

Let $x_n \in supp(\mu)$ be a converging sequence with $\lim_{n\uparrow\infty} x_n = x$. Then,

$$\lim_{n\uparrow\infty} d_S\left(Z(\cdot|x_n, \mu_n), Z(\cdot|x, \mu)\right) = 0 \quad \mathbb{P}\ a.s.$$

Let us recall another consequence of the results in [17] (see also Sect. 2 in [11]):

Proposition 4.2 *Let $\{x_k\}_{k\in\mathbb{Z}}$ satisfy $x_k < x_{k+1}$, $\lim_{k\to\pm\infty} x_k = \pm\infty$. Fix positive constants $\{w_k\}_{k\in\mathbb{Z}}$ and set $\mu = \sum_{k\in\mathbb{Z}} w_k \delta_{x_k}$. Then $Z(\cdot|x_j, \mu)$ is the continuous-time random walk on $\{x_k\}_{k\in\mathbb{Z}}$ starting in x_j such that after reaching site x_k it remains in x_k for an exponential time with mean*

$$w_k \frac{(x_{k+1} - x_k)(x_k - x_{k-1})}{x_{k+1} - x_{k-1}}$$

and then it jumps to x_{k-1}, x_{k+1}, respectively, with probability

$$\frac{x_{k+1} - x_k}{x_{k+1} - x_{k-1}} \quad \textit{and} \quad \frac{x_k - x_{k-1}}{x_{k+1} - x_{k-1}}.$$

Given $N \geq 1$, $x \in \mathbb{Z}$ consider the random walk $X_N(t|x)$ on $\mathbb{Z}$ having starting point x and generator

$$\mathbb{L}_N f(x) = c_x N^{1+1/\alpha}\{f(x+1) - f(x)\} + c_{x-1} N^{1+1/\alpha}\{f(x-1) - f(x)\}.$$

Denote the transition probabilities of X_N by p^N:

$$p_t^N(x, y) = P\left[X_N(t|x) = y\right] \tag{4.2}$$

for x, y in $\mathbb{Z}$. By symmetry, $p_t^N(x, y) = p_t^N(y, x)$.

For $N \geq 1$, let ν_N be the discrete measure defined by

$$\nu_N = \frac{1}{N} \sum_{x\in\mathbb{Z}} \delta_{W(x/N)}.$$

As $N \uparrow \infty$, ν_N converges to ν vaguely. By definition of ν_N and by Proposition 4.2 the random walk X_N/N can be expressed as a space–time change of the Brownian motion:

$$N^{-1} X_N(\cdot|x) \sim W^{-1}\left(Z\left(\cdot|W(x/N), \nu_N\right)\right),$$

where "$\sim$" means that the two processes have the same law.

Recall that $[u]_N = \lceil uN \rceil / N$ and define

$$Y_N(t|u) = W^{-1}\left(Z\left(t|W([u]_N), \nu_N\right)\right).$$

It follows from the two previous identities that

$$N^{-1} X_N(\cdot|N[u]_N) \sim Y_N(\cdot|u).$$

Lemma 4.3 *Let $Y(t|u) = W^{-1}(Z(t|W(u), \nu))$. For all $u \in \mathbb{R}$,*

$$\lim_{N\to\infty} d_S\,(Y_N(\cdot|u), Y(\cdot|u)) = 0 \quad \mathbb{P}\, a.s.$$

Moreover, for all u in $\mathbb{R}$ and for all $T > 0$,

$$\lim_{N\uparrow\infty} \sup_{0\le t\le T} |Y_N(t|u) - Y(t|u)| = 0 \quad \mathbb{P}\, a.s.$$

Proof If $y_N \in \mathrm{supp}(\nu_N)$ and $y_N \to y \in \mathbb{R}$ then $y \in \mathrm{supp}(\nu)$ (see the proof of Lemma 2 in [11]). Since $\lim_{N\to\infty} W([u]_N) = W(u)$ for all $u \in \mathbb{R}$, and since ν_N converges vaguely to ν, by Proposition 4.1,

$$\lim_{N\to\infty} d_S\,(Z\,(\cdot|W([u]_N), \nu_N)\,, Z\,(\cdot|W(u), \nu)) = 0, \qquad \mathbb{P}\text{-}a.s.$$

The first claim of the lemma follows by the same arguments used in the proof of Proposition 1 in [11]. On the other hand, since by Proposition 3.2, $Y(\cdot|u)$ has continuous paths $\mathbb{P}$-a.s., the second statement of the lemma follows from the first one. □

Recall that P_t stands for the semigroup of the process $\{Y(t) : t \ge 0\}$ and let $\{P_t^N : t \ge 0\}$ be the semigroup of the process $\{Y_N(t) : t \ge 0\}$. Hence, given a bounded Borel function H,

$$P_t^N H(u) = \mathbb{E}\,[H\,(Y_N(t|u))]\,. \tag{4.3}$$

It follows from Lemma 4.3 and the dominated convergence theorem that $P_t^N H$ converges pointwisely to $P_t H$ for every bounded continuous function H and every $t \ge 0$.

Since W is strictly increasing, W^{-1} is a continuous function. In particular, since $\lim_{x\to\pm\infty} W(x) = \pm\infty$, $H \circ W^{-1}$ belongs to $C_c(\mathbb{R})$, $C_0(\mathbb{R})$ as soon as H belongs. In the next three lemmata, we prove properties of the operators P_t^N and P_t and some convergence results of P_t^N to P_t.

Lemma 4.4 *Fix a continuous function $H : \mathbb{R} \to \mathbb{R}$ with compact support. For every $t \ge 0$, $P_t^N H$ belongs to $L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} du\, P_t^N H(u) = \frac{1}{N} \sum_{x\in\mathbb{Z}} H(x/N).$$

Proof Assume without loss of generality that $H \geq 0$. Since the transition probability $p_t(x, y)$ is symmetric,

$$\begin{aligned}\int_{\mathbb{R}} du\, P_t^N H(u) &= N^{-1} \sum_{x,y\in\mathbb{Z}} p_t^N(x, y) H(y/N) \\ &= N^{-1} \sum_{y\in\mathbb{Z}} H(y/N) \sum_{x\in\mathbb{Z}} p_t^N(y, x) \\ &= N^{-1} \sum_{y\in\mathbb{Z}} H(y/N).\end{aligned}$$

This proves the identity and that $P_t^N H$ belongs to $L^1(\mathbb{R})$. □

Lemma 4.5 *Fix a function H in $C_c(\mathbb{R})$ and $t \geq 0$.*

(i) *$P_t^N H$ converges in $L^1(\mathbb{R})$ to $P_t H$.*
(ii) *If H has bounded variation in $\mathbb{R}$ then, for every $t \geq 0$, $P_t H$ has also bounded variation.*
(iii) *$P_t^N H$ also converges to $P_t H$ with respect to the counting measure*:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{Z}} \Big|(P_t^N H)(x/N) - (P_t H)(x/N)\Big| = 0.$$

(iv) *For any $\varepsilon > 0$ there exists $\Psi \in C_c(\mathbb{R})$ such that*

$$\int_{\mathbb{R}} du |P_t H(u) - \Psi(u)| \leq \varepsilon, \quad \frac{1}{N} \sum_{x\in\mathbb{Z}} |P_t H(x/N) - \Psi(x/N)| \leq \varepsilon$$

for N large enough.

Proof Without loss of generality, fix a positive function H in $C_c(\mathbb{R})$. Since $P_t^N H$ converges pointwisely to $P_t H$, in view of Corollary 3.4 (ii) and Lemma 4.4, $P_t^N H$ converges in $L^1(\mathbb{R})$ to $P_t H$ by Scheffé Theorem.

It is easy to couple two copies of the random walk Y_N in such a way that

$$Y_N(t|u) \leq Y_N(t|v), \qquad \forall t \geq 0, \quad \forall u \leq v.$$

To prove (ii), assume that H is a continuous function of bounded variation in $\mathbb{R}$. Then there exist bounded, continuous, increasing functions, H_- and H_+, such that $H = H_+ - H_-$. By the coupling, $P_t^N H_\pm$ are bounded increasing functions. Taking the pointwise limit as $N \uparrow \infty$ of the identity $P_t^N H = P_t^N H_+ - P_t^N H_-$ we get that $P_t H = P_t H_+ - P_t H_-$ where $P_t H_\pm$ are bounded increasing functions. Therefore, $P_t H$ has bounded variation.

For $N \geq 1$, and a right continuous function $f : \mathbb{R} \to \mathbb{R}$, let $T_N f : \mathbb{R} \to \mathbb{R}$ be given by $(T_N f)(u) = f(\lceil uN \rceil/N)$. We claim that $T_N P_t H$ converges in $L^1(\mathbb{R})$ to $P_t H$. By Corollary 3.4 (i), $P_t H$ is right continuous. In particular, $T_N P_t H$ converges pointwisely to $P_t H$. For x in $\mathbb{Z}$, denote by V_x the total variation of $P_t H$ on $[x, x+1]$. Let $V : \mathbb{R} \to \mathbb{R}_+$ be given by $V(u) = V_{\lceil u \rceil}$. V belongs to $L^1(\mathbb{R})$ because $P_t H$ has bounded variation due to (ii). Moreover, $T_N P_t H \le P_t H + V$ which implies that $T_N P_t H$ belongs to $L^1(\mathbb{R})$. By the dominated convergence theorem, $T_N P_t H$ converges to $P_t H$ in $L^1(\mathbb{R})$ because $T_N P_t H$ converges pointwisely to $P_t H$.

The sum appearing in (iii) can be rewritten as

$$\int_{\mathbb{R}} du \left| (P_t^N H)(u) - (T_N P_t H)(u) \right| .$$

Since $P_t^N H$ and $T_N P_t H$ converge to $P_t H$ in $L^1(\mathbb{R})$, statement (iii) follows.

Fix $\varepsilon > 0$. By Corollary 3.4 (ii), $P_t H$ belongs to $L^1(\mathbb{R})$. In particular, there exists a continuous function with compact support Ψ which approximates $P_t H$ in $L^1(\mathbb{R})$: $\|P_t H - \Psi\|_1 \le \varepsilon$. The sum in (iv) can be estimated by $\|T_N P_t H - P_t H\|_1 + \|P_t H - \Psi\|_1 + \|\Psi - T_N \Psi\|_1$. Since Ψ belongs to $C_c(\mathbb{R})$ and since $T_N P_t H$ converges in $L^1(\mathbb{R})$ to $P_t H$ the first and third term vanish as $N \uparrow \infty$. This proves (iv). □

Denote by $\{R_\lambda^N : \lambda > 0\}$ the resolvent associated to the semigroup $\{P_t^N : t \ge 0\}$:

$$R_\lambda^N H = \int_0^\infty dt e^{-\lambda t} P_t^N H$$

for H in $C_c(\mathbb{R})$.

Lemma 4.6 *Fix a function g in $C_c(\mathbb{R})$. Then,*

$$\lim_{t\to 0} \limsup_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{Z}} \left| P_t^N g(x/N) - g(x/N) \right| = 0,$$
$$\lim_{\lambda\to\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{x\in\mathbb{Z}} \left| \lambda R_\lambda^N g(x/N) - g(x/N) \right| = 0.$$

Moreover, for every $\lambda > 0$,

$$\frac{1}{N} \sum_{x\in\mathbb{Z}} \left| \lambda R_\lambda^N g(x/N) \right| \le \frac{1}{N} \sum_{x\in\mathbb{Z}} |g(x/N)| .$$

Proof The first expression is bounded above by

$$\|P_t^N g - P_t g\|_1 + \|P_t g - g\|_1 + \frac{1}{N} \sum_{x\in\mathbb{Z}} \left| N \int_{x-1/N}^{x/N} g(u)du - g(x/N) \right| ,$$

where $\|\cdot\|_1$ stands for the $L_1(\mathbb{R})$ norm. By Lemma 4.5 (i), the first expression vanishes as $N \uparrow \infty$. By Corollary 3.4 (iii), the second expression vanishes as $t \downarrow 0$. Since the third expression vanishes as $N \uparrow \infty$, the first claim of the lemma is proved.

By definition of the resolvent, the second expression is less than or equal to

$$\int_0^\infty dt\lambda e^{-\lambda t} \frac{1}{N} \sum_{x\in\mathbb{Z}} \left| P_t^N g(x/N) - g(x/N) \right| .$$

By Lemma 4.4 the sum inside the integral is uniformly bounded in t and N. By the first part of this lemma it vanishes as $N \uparrow \infty$, $t \downarrow 0$. This proves the second claim.

The third claim follows from the definition of the resolvent and Lemma 4.4. □

5 Hydrodynamic behavior

We prove in this section the main Theorems of the article. We first examine the convergence of the empirical measure π^N.

Recall that $\mathcal{M}$ stands for the space of Radon measures endowed with the vague topology. Fix a realization of the subordinator W and $T > 0$. For each probability measure μ on $\{0, 1\}^{\mathbb{Z}}$, denote by $\mathbb{Q}_\mu^{W,N}$ the measure on the path space $D([0, T], \mathcal{M})$ induced by the measure μ and the process π_t^N, introduced in (2.6), evolving according to the generator (2.8) speeded up by $N^{1+1/\alpha}$. Fix a continuous profile $\rho_0 : \mathbb{R} \to [0, 1]$ and consider a sequence $\{\mu^N : N \geq 1\}$ of measures on $\{0, 1\}^{\mathbb{Z}}$ associated to ρ_0. Let $\mathbb{Q}_W$ be the probability measure on $D([0, T], \mathcal{M})$ concentrated on the deterministic path $\pi(t, du) = \rho_W(t, u)du$, where $\rho_W(t, u) = P_t\rho_0$.

Proposition 5.1 *The sequence of probability measures $\mathbb{Q}_{\mu^N}^{W,N}$ converges, as $N \uparrow \infty$, to $\mathbb{Q}_W$.*

The proof of this result is divided in two parts. In Sect. 5.1, we show that the sequence $\{\mathbb{Q}_{\mu^N}^{W,N} : N \geq 1\}$ is tight and in Sect. 5.2 we characterize the limit points of this sequence.

5.1 Tightness

Recall that we denote by $\mathcal{M}$ the space of positive Radon measures endowed with the vague topology. In particular, a sequence μ_n in $\mathcal{M}$ converges to μ if

$$\lim_{n\to\infty} \int f\, d\mu_n = \int f\, d\mu$$

for all continuous functions f with compact support.

This topology can be defined through a metric. It is indeed not difficult to find a countable family of functions $\{f_j : j \geq 1\}$ in $C_c(\mathbb{R})$ (even in $C_c^\infty(\mathbb{R})$) such that

(i) For each $\epsilon > 0$, integer $k \geq 1$ and continuous function f with support contained in $[-k, k]$, there exists j such that $\|f_j - f\|_\infty \leq \epsilon$ and such that the support of f_j is contained in $[-k-1, k+1]$.
(ii) For each integer $k \geq 1$, there exists j such that f_j is a non-negative function with support contained in $[-k-1, k+1]$ and greater or equal to 1 in $[-k, k]$.

It is easy to check that μ_n converges to μ vaguely if and only if $\int f_j d\mu_n$ converges to $\int f_j d\mu$ for all j. In particular, μ_n converges to μ if and only if $d(\mu_n, \mu)$ vanishes, where d is the metric defined by

$$d(\mu, \mu') = \sum_{j \geq 1} \frac{1}{2^j} \left\{ 1 \wedge \left| \int f_j d\mu - \int f_j d\mu' \right| \right\}. \tag{5.1}$$

The space $\mathcal{M}$ endowed with this metric is a complete separable metric space.

The closure of a subset M of $\mathcal{M}$ is compact if and only if

$$\sup_{\mu \in M} \mu(K) < \infty$$

for all compact sets K of $\mathbb{R}$. In particular, if g_j is a non-negative smooth function with support contained in $[-j-1, j+1]$ and greater or equal to 1 in $[-j, j]$, a set M such that

$$\sup_{\mu \in M} \sum_{j \geq 1} \frac{1}{2^j} \int g_j d\mu < \infty$$

is compact. We refer to Sect. A.10 of [16] for the proofs of all the previous statements concerning the vague topology in $\mathcal{M}$.

We prove in this section that the sequence of probability measures $\{\mathbb{Q}^{W,N}_{\mu^N} : N \geq 1\}$ is tight in $D([0, T], \mathcal{M})$. The method [10] consists in proving relative compactness of an auxiliary process and then showing that both processes are close.

For $\lambda > 0$, let $\{X^{\lambda,N}_t : t \geq 0\}$ be the $\mathcal{M}$-valued Markov process such that

$$X^{\lambda,N}_t(H) = \langle \pi^N_t, R^N_\lambda H \rangle = \frac{1}{N} \sum_{x \in \mathbb{Z}} \left(R^N_\lambda H \right) (x/N) \eta_t(x)$$

for every H in $C_c(\mathbb{R})$, where $\{R^N_\lambda : \lambda > 0\}$ is the resolvent associated to the semigroup $\{P^N_t : t \geq 0\}$. Here and below, we do not distinguish between a continuous function $H : \mathbb{R} \to \mathbb{R}$ and its restriction to $\mathbb{Z}_N = \{x/N : x \in \mathbb{Z}\}$.

Lemma 5.2 *For each $T > 0$ and $\lambda > 0$, the sequence of processes $\{X^{\lambda,N} : N \geq 1\}$ is tight in $D([0, T], \mathcal{M})$.*

Proof By Proposition IV.1.7 in [12], it is enough to show that $\{X^{\lambda,N}_t(g) : 0 \leq t \leq T\}$ is tight in $D([0, T], \mathbb{R})$ for all functions g in $C^1_c(\mathbb{R})$. Note that the underlying space in [12] is compact, while we are working here on $\mathbb{R}$. However, in both cases the topology is given by a metric of type (5.1) and in both cases the compacts are characterized by integral of functions. It is easy to adapt the proof of Proposition IV.1.7 to the present case.

Fix a function g in $C_c^1(\mathbb{R})$ and let $g_\lambda^N = R_\lambda^N g$. Note that g_λ^N belongs to $\ell^2(\mathbb{Z}_N)$, the space of square summable functions $f : \mathbb{Z}_N \to \mathbb{R}$, because so does g and because R_λ^N is a bounded operator in $\ell^2(\mathbb{Z}_N)$. Clearly, g_λ^N is the solution of

$$\lambda g_\lambda^N - \mathbb{L}_N g_\lambda^N = g. \tag{5.2}$$

There is here a slight abuse of notation. We are using the same symbol $\mathbb{L}_N$ for the generator of $X_N(t)$ and the generator of $N^{-1}X_N(t)$. The context makes clear to which operator we are refering to. Multiplying (5.2) by $N^{-1}g_\lambda^N(\cdot/N)$ and summing over x, we obtain that

$$\frac{\lambda}{N}\sum_{x\in\mathbb{Z}} g_\lambda^N(x/N)^2 + \frac{N^{1/\alpha}}{N^2}\sum_{x\in\mathbb{Z}} c_x(\nabla_N g_\lambda^N)(x/N)^2 = \frac{1}{N}\sum_{x\in\mathbb{Z}} g_\lambda^N(x/N)g(x/N),$$

where ∇_N stands for the discrete gradient: $(\nabla_N h)(x/N) = N\{h(x+1/N) - h(x/N)\}$. By Schwarz inequality, we obtain that

$$\frac{\lambda}{2N}\sum_{x\in\mathbb{Z}} g_\lambda^N(x/N)^2 + \frac{N^{1/\alpha}}{N^2}\sum_{x\in\mathbb{Z}} c_x(\nabla_N g_\lambda^N)(x/N)^2 \le \frac{1}{2\lambda N}\sum_{x\in\mathbb{Z}} g(x/N)^2.$$

Since g is continuous with compact support, $N^{-1}\sum_{x\in\mathbb{Z}} g(x/N)^2$ is bounded uniformly over N. Hence,

$$\sup_N \frac{N^{1/\alpha}}{N^2}\sum_{x\in\mathbb{Z}} c_x(\nabla_N g_\lambda^N)(x/N)^2 \le \frac{C(g)}{\lambda}. \tag{5.3}$$

In this formula and below, $C(g)$ stands for some finite constant depending only on g.

An elementary computation shows that $\mathcal{L}_N\langle \pi_t^N, H\rangle = \langle \pi_t^N, \mathbb{L}_N H\rangle$, where $\mathcal{L}_N$ is the operator defined in (2.8). In particular, the process $M_t^{N,\lambda}$ defined by

$$M_t^{N,\lambda} = X_t^{\lambda,N}(g) - X_0^{\lambda,N}(g) - \int_0^t ds\,\Big\{\lambda\langle \pi_s^N, g_\lambda^N\rangle - \langle \pi_s^N, g\rangle\Big\} \tag{5.4}$$

is a martingale with quadratic variation

$$\langle M^{N,\lambda}\rangle_t = \int_0^t ds\,\frac{N^{1/\alpha}}{N^3}\sum_{x\in\mathbb{Z}} c_x(\nabla_N g_\lambda^N)(x/N)^2\{\eta_s(x+1) - \eta_s(x)\}^2.$$

Due to the previous estimate (5.3), the quadratic variation $\langle M^{N,\lambda}\rangle_t$ satisfies

$$\langle M^{N,\lambda}\rangle_t \le C(g)T/\lambda N, \qquad \forall\, 0 \le t \le T. \tag{5.5}$$

Moreover, by Lemma 4.6,

$$\sup_{N\ge 1} \frac{\lambda}{N} \sum_{x\in\mathbb{Z}} \left| g_\lambda^N(x/N) \right| \le C(g). \tag{5.6}$$

Hence, given $N \ge 1$ and constants $0 \le a < b$,

$$\left| \int_a^b ds \left\{ \lambda \langle \pi_s^N, g_\lambda^N \rangle - \langle \pi_s^N, g \rangle \right\} \right| \le C(g)(b-a). \tag{5.7}$$

Due to the decomposition (5.4), to the estimates (5.5) and (5.7), and to Doob inequality, it is simple to prove (cf. [12], Chap. IV, p. 55) that, given $\varepsilon > 0$,

$$\lim_{\gamma\downarrow 0} \limsup_{N\uparrow\infty} \sup_{\tau,\theta} \mathbb{P}_{\mu^N} \left(\left| X_\tau^{\lambda,N}(g) - X_{\tau+\theta}^{\lambda,N}(g) \right| > \varepsilon \right) = 0, \tag{5.8}$$

where the supremum $\sup_{\tau,\theta}$ is over all stopping times τ bounded by T, and over θ with $0 \le \theta \le \gamma$.

In addition, due to (5.6),

$$\sup_{t\ge 0} \sup_{N\ge 1} \left| X_t^{\lambda,N}(g) \right| \le \frac{C(g)}{\lambda}. \tag{5.9}$$

As discussed in [12], Chap. IV, p. 51, (5.8) and (5.9) allow to apply Prohorov's theorem, thus concluding the proof of the tightness of $X_t^{\lambda,N}(g)$. □

Corollary 5.3 *The sequence of measures* $\{\mathbb{Q}_{\mu^N}^{W,N} : N \ge 1\}$ *is tight.*

Proof It is enough to show that for every function g in $C_c^1(\mathbb{R})$ and every $\epsilon > 0$, there exists $\lambda > 0$ such that

$$\lim_{N\to\infty} \mathbb{P}_{\mu^N}^{W,N} \left[\sup_{0\le t\le T} |X_t^{\lambda,N}(\lambda g) - \langle \pi_t^N, g \rangle| > \epsilon \right] = 0$$

because in this case the tightness of π_t^N follows from the tightness of $X_t^{\lambda,N}$. Since there is at most one particle per site the expression inside the absolute value is less than or equal to

$$\frac{1}{N} \sum_{x\in\mathbb{Z}} \left| \lambda g_\lambda^N(x/N) - g(x/N) \right|.$$

By Lemma 4.6 this expression vanishes as $N \uparrow \infty$, $\lambda \uparrow \infty$. □

5.2 Proof of Theorem 2.5

We start this section with a generalization of [7,15].

Lemma 5.4 *Fix a function H in $C_c(\mathbb{R})$ and a sequence of probability measures $\{\mu^N : N \geq 1\}$ in $\{0,1\}^{\mathbb{Z}}$. For each $t \geq 0$,*

$$\lim_{N\to\infty} \mathbb{E}^{W,N}_{\mu^N}\left[\left\{\frac{1}{N}\sum_{x\in\mathbb{Z}} H(x/N)\eta_t(x) - \frac{1}{N}\sum_{x\in\mathbb{Z}}(P^N_t H)(x/N)\eta_0(x)\right\}^2\right] = 0,$$

where $\mathbb{E}^{W,N}_{\mu^N}$ denotes the expectation w.r.t. $\mathbb{P}^{W,N}_{\mu^N}$.

Proof We assume that the number of particles is finite μ^N-a.s. One can extend the proof to the general case by the same arguments used in [7]. For each x in $\mathbb{Z}$, denote by $J^{x,x+1}_t$ the net current of particles through the bond $\{x, x+1\}$ in the time interval $[0,t]$. This is the total number of particles which jumped from x to $x+1$ minus the total number of particles which jumped from $x+1$ to x in the time interval $[0,t]$. Denote by $M^{x,x+1}_t$ the martingale associated to $J^{x,x+1}_t$: $M^{x,x+1}_t = J^{x,x+1}_t - N^{1+1/\alpha} c_x \int_0^t \{\eta_s(x) - \eta_s(x+1)\}ds$ and let $M^x_t = M^{x-1,x}_t - M^{x,x+1}_t$.

With this notation,

$$\eta_t(x) = \sum_{y\in\mathbb{Z}} p^N_t(x,y)\eta_0(y) + \sum_{y\in\mathbb{Z}}\int_0^t p^N_{t-s}(x,y)dM^y_s,$$

where $p^N_t(\cdot,\cdot)$ is the transition probability defined in (4.2). In particular, since p^N_t is symmetric, the expression inside the square in the statement of the proposition can be rewritten as

$$\Gamma^N_t = \frac{1}{N}\sum_{y\in\mathbb{Z}}\int_0^t (P^N_{t-s}H)(y/N)dM^y_s.$$

To prove the proposition it is therefore enough to show that $\mathbb{E}^{W,N}_{\mu^N}[(\Gamma^N_t)^2]$ vanishes as $N \uparrow \infty$. Since the martingales $M^{z,z+1}_t$ are orthogonal, $\mathbb{E}^{W,N}_{\mu^N}[\Gamma^2_t]$ is equal to

$$\frac{N^{1/\alpha}}{N}\sum_{y\in\mathbb{Z}}\int_0^t ds\left\{(P^N_{t-s}H)((y+1)/N) - (P^N_{t-s}H)(y/N)\right\}^2 c_y \mathbb{E}^{W,N}_{\mu^N}\left[a_{y,y+1}(\eta_s)\right],$$

where $a_{z,z+1}(\eta) = \{\eta(z) - \eta(z+1)\}^2$. Since this function is bounded by 1, the previous expression is less than or equal to

$$\frac{N^{1/\alpha}}{N}\sum_{y\in\mathbb{Z}}\int_0^t dsc_y\left\{(P_s^N H)((y+1)/N)-(P_s^N H)(y/N)\right\}^2.$$

Let $H_t(x/N)=(P_t^N H)(x/N)$ and observe that $\partial_t H_t=\mathbb{L}_N H_t$. Hence the previous expression can be rewritten as

$$\begin{aligned}-\frac{1}{N^2}\int_0^t (H_s,\mathbb{L}_N H_s)_N ds &= -\frac{1}{2N^2}\int_0^t \partial_s(H_s,H_s)_N ds\\ &=\frac{1}{2N^2}\sum_{y\in\mathbb{Z}}H^2(y/N)-\frac{1}{2N^2}\sum_{y\in\mathbb{Z}}\left\{(P_t^N H)(y/N)\right\}^2,\end{aligned}$$

where $(\cdot,\cdot)_N$ denotes the scalar product on $\ell^2(\mathbb{Z}_N)$ w.r.t. the counting measure. This proves the lemma. □

We are now in a position to show that the sequence of probability measures $\mathbb{Q}^{W,N}_{\mu^N}$ converges, as $N\uparrow\infty$, to $\mathbb{Q}_W$.

Proof of Proposition 5.1 By Corollary 5.3, the sequence $\{\mathbb{Q}^{W,N}_{\mu^N}: N\geq 1\}$ is tight. To prove the lemma we only need to characterize the limit points of this sequence.

Fix a function H in $C_c(\mathbb{R})$. We claim that

$$\lim_{N\to\infty}\mathbb{P}^{W,N}_{\mu^N}\left[\left|\langle\pi_t^N,H\rangle-\int_{\mathbb{R}}H(u)\rho_W(t,u)du\right|>\delta\right]=0 \tag{5.10}$$

for all $0\leq t\leq T$, $\delta>0$.

By Lemma 5.4 we only need to prove that

$$\lim_{N\uparrow\infty}\mu^N\left(\left|\frac{1}{N}\sum_{x\in\mathbb{Z}}P_t^N H(x/N)\eta(x)-\int H(u)\rho_W(t,u)du\right|>\delta\right)=0.$$

Since there is at most one particle per site, by Lemma 4.5 (iii), we may replace $P_t^N H$ by $P_t H$. By assumption on μ^N and by Lemma 4.5 (iv), we may also replace $N^{-1}\sum_{x\in\mathbb{Z}}(P_t H)(x/N)\eta(x)$ by $\int(P_t H)(u)\rho_0(u)du$. Since ρ_0 is continuous and bounded, by Corollary 3.4 (iv), this expression is equal to

$$\int H(u)(P_t\rho_0)(u)du=\int H(u)\rho_W(t,u)du.$$

This concludes the proof of (5.10).

By (5.10), the finite dimensional distributions (f.d.d.) of $\mathbb{Q}^{W,N}_{\mu^N}$ converge to the f.d.d. of $\mathbb{Q}_W$. Since the f.d.d. characterize the measure, the proposition is proved. □

Proof of Theorem 2.5 Fix a function H in $C_c(\mathbb{R})$. On the one hand, by Theorem 2.1 and Corollary 3.4 (iv), $\int_{\mathbb{R}} H(u)\rho_W(t,u)du$ is a bounded continuous function of time. On the other hand, for any continuous function $A : [0,T] \to \mathbb{R}$, the map in $D([0,T],\mathcal{M})$

$$\pi \to \sup_{0\le t\le T} |\langle \pi_t, H\rangle - A(t)|$$

is bounded and continuous for the Skorohod topology. In particular, Theorem 2.5 follows from Proposition 5.1. □

5.3 Proof of Theorem 2.2

We first claim that

$$\lim_{N\to\infty} \sup_{0\le t\le T} \left| \int_{\mathbb{R}} H(u) \left\{ P_t \rho_0(u) - P_t^N \rho_0(u) \right\} du \right| = 0 \tag{5.11}$$

for every function H in $C_c(\mathbb{R})$. Fix $\varepsilon > 0$ and recall that we denote by $(\mathbb{X}, \mathbb{F}, \mathbb{P})$ the probability space in which the processes $Y(t|u)$ and $Y_N(t|u)$ are defined. Expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. If K stands for a compact subset of $\mathbb{R}$ which contains the support of H, the previous supremum is bounded above by

$$C_0(H) \sup_{0\le t\le T} \int_K \mathbb{E}\left[|\rho_0(Y(t|u)) - \rho_0(Y_N(t|u))|\right] du.$$

Since ρ_0 is uniformly continuous, there exists $\delta > 0$ for which the previous is less than or equal to

$$C_0(H)\varepsilon + C_0(H,\rho_0) \int_K \mathbb{P}\left[\sup_{0\le t\le T} |Y(t|u) - Y_N(t|u)| > \delta \right] du.$$

By Lemma 4.3 and the dominated convergence theorem, the second expression vanishes as $N \uparrow \infty$ for every $\delta > 0$. This proves Claim (5.11).

It follows from (5.11) and Theorem 2.5 that

$$\lim_{N\to\infty} \mathbb{P}^{W,N}_{\mu^N} \left[\sup_{0\le t\le T} \left| \langle \pi_t^N, H\rangle - \int_{\mathbb{R}} H(u) P_t^N \rho_0(u) du \right| > \delta \right] = 0.$$

Since, for each $N \ge 1$, $\{\gamma_x(W,N): x \in \mathbb{Z}\}$ has the same distribution as $\{\xi_x^{-1}: x \in \mathbb{Z}\}$,

$$\begin{aligned} &\mathbb{P}^{W,N}_{\mu^N} \left[\sup_{0\le t\le T} \left| \langle \pi_t^N, H\rangle - \int_{\mathbb{R}} H(u) P_t^N \rho_0(u) du \right| > \delta \right] \\ &= \mathbb{P}^{\xi,N}_{\mu^N} \left[\sup_{0\le t\le T} \left| \langle \pi_t^N, H\rangle - \int_{\mathbb{R}} H(u) P_t^{N,\xi} \rho_0(u) du \right| > \delta \right] \end{aligned}$$

in distribution. In particular, Theorem 2.2 follows from Theorem 2.5. □

5.4 The tagged particle

We examine in this section the asymptotic behavior of the tagged particle. As mentioned in Sect. 2, the law of large numbers for the tagged particle in the case of compactly supported initial density profiles is a direct consequence of the hydrodynamic limit and the fact that the relative order among particles is preserved by the dynamics.

We first prove that the position of the tagged particle, $u_W(t)$, is uniquely determined.

Recall the notation introduced in Sect. 2 and assume that the initial density profile ρ_0 belongs to $C_c(\mathbb{R})$. It follows from Proposition 3.3 that

$$\rho_W(t,u) = (P_t\rho_0)(u) = \int p_t(u,v)\rho_0(v)dv.$$

Since $p_t(u,v)$ is strictly positive, $\rho_W(t,\cdot)$ is strictly positive as soon as ρ_0 is not identically equal to 0. On the other hand, since P_t is a contraction in $L^1(\mathbb{R})$, $\rho_W(t,\cdot)$, $t \geq 0$, belongs to $L^1(\mathbb{R})$. In particular, for each $s \geq 0$, there exists a unique $u_W(s)$ in $\mathbb{R}$ such that

$$\int_{-\infty}^{u_W(s)} \rho_W(s,v)dv = \int_{-\infty}^{0} \rho_0(v)dv. \tag{5.12}$$

The function $u_W(t)$ is continuous in time. Indeed, on the one hand, it follows from (5.12) with $s = t, t_n$, that

$$\left| \int_{u_W(t)}^{u_W(t_n)} \rho_W(t,v)dv \right| \leq \int_{\mathbb{R}} dv\, |\rho_W(t,v) - \rho_W(t_n,v)|\,.$$

On the other hand, by Corollary 3.4 (iii), $\rho_W(t_n,\cdot)$ converges in $L^1(\mathbb{R})$ to $\rho_W(t,\cdot)$ if $t_n \to t$. Since $\rho_W(t,\cdot)$ is strictly positive, $u_W(t_n)$ must converge to $u_W(t)$.

Proof of Theorem 2.6 Fix a density profile ρ_0 in $C_c(\mathbb{R})$ and let $\rho_W(t,u) = P_t\rho_0$. Observe that $\rho_W(t,\cdot)$ belongs to $L^1(\mathbb{R})$ in virtue of Proposition 3.3. Consider a sequence $\{\mu^N : N \geq 1\}$ of measures associated to ρ_0, conditioned to have a particle at the origin and such that $\mu^N\{\eta(x) = 1\} = 0$ for $|x/N|$ large enough. For a in $\mathbb{R}$, denote by H_a the indicator function of the interval $[a,\infty)$: $H_a(u) = \mathbf{1}\{[a,\infty)\}(u)$. We first claim that Theorem 2.5 can be extended to such test functions:

$$\lim_{N\to\infty} \mathbb{P}^{W,N}_{\mu^N}\left[\sup_{0\leq t\leq T} \left| \langle \pi^N_t, H_a\rangle - \int_a^\infty \rho_W(t,u)du \right| > \delta \right] = 0$$

for all $\delta > 0$, a in $\mathbb{R}$. The same statement holds for $\check{H}_a = 1 - H_a$ in place of H_a.

Indeed, consider a sequence of compactly supported continuous functions G_k (resp. $\check{G}_k$) increasing to H_a (resp. $\check{H}_a$). On the one hand, $\langle \pi_t^N, H_a\rangle \ge \langle \pi_t^N, G_k\rangle$. On the other hand, $\langle \pi_t^N, H_a\rangle \le N^{-1}\sum_x \eta_t(x) - \langle \pi_t^N, \check{G}_k\rangle$. Since the total number of particles is conserved, this expression is equal to $N^{-1}\sum_x \eta_0(x) - \langle \pi_t^N, \check{G}_k\rangle$. To conclude the proof of the claim it remains to let $N \uparrow \infty$ and then $k \uparrow \infty$ and to recall that $\int \rho_0(u)du = \int \rho_W(t,u)du$.

We are now in a position to prove Theorem 2.6. Fix $\delta > 0$ and assume that $x_t^N/N \ge u_W(t) + \delta$. Since the total numbers to the right of the tagged particle doesn't change in time, $\sum_{x\ge 0} \eta_0(x) = \sum_{x\ge x_t^N} \eta_t(x) \le \sum_{x/N \ge u_W(t)+\delta} \eta_t(x)$. Dividing by N and letting $N \uparrow \infty$, by the previous observation, we get that

$$\int_0^\infty \rho_0(u)du \le \int_{u_W(t)+\delta}^\infty \rho_W(t,u)du.$$

This contradicts the definition of $u_W(t)$ because $\rho_W(t,\cdot)$ is strictly positive in view of Proposition 3.3. Similarly, one can prove that the event $x_t^N/N \le u_W(t) - \delta$ has negligible probability as $N \uparrow \infty$. □

Proof of Theorem 2.4 Let $u_t^{W,N}$ be the unique solution of

$$\int_{-\infty}^{u_t^{W,N}} (P_t^N \rho_0)([u]_N)du = \int_{-\infty}^0 \rho_0(u)du,$$

where P_t^N is the semigroup defined in (4.3). Note that $u_t^{W,N}$ is uniquely determined by this equation because $P_t^N\rho_0$ is strictly positive and, due to Lemma 4.4, is Lebesgue integrable.

As in the proof of Theorem 2.2, since $\{\gamma_x(W,N) : x \in \mathbb{Z}\}$ has the same distribution as $\{\xi_x^{-1} : x \in \mathbb{Z}\}$, the random variables

$$\mathbb{P}_{\mu^N}^{\xi,N}\left[\left|x_t^N/N - u^{\xi,N}(t)\right| > \delta\right] \quad \text{and} \quad \mathbb{P}_{\mu^N}^{W,N}\left[\left|x_t^N/N - u^{W,N}(t)\right| > \delta\right]$$

depending on ξ and W, respectively, have the same law.

We claim that for each $t > 0$ and realization W, $u^{W,N}(t)$ converges to $u_W(t)$ as $N \uparrow \infty$. Indeed, since $\int_{-\infty}^{u_W(t)} \rho_W(t,v)dv = \int_{-\infty}^{u^{W,N}(t)} (P_t^N\rho_0)(v)dv$,

$$\left|\int_{u_W(t)}^{u^{W,N}(t)} \rho_W(t,v)dv\right| \le \int_{\mathbb{R}} dv \left|(P_t^N\rho_0)(v) - (P_t\rho_0)(v)\right|.$$

By Lemma 4.5 (i), the right hand side vanishes as $N \uparrow \infty$. Since, by Proposition 3.3, $P_t\rho_0$ is strictly positive, the claim is proved.

In particular, by Theorem 2.5 and the strategy of the proof of Theorem 2.6, for all $t > 0$, $\delta > 0$ and a.a. W,

$$\lim_{N\to\infty} \mathbb{P}^{W,N}_{\mu^N}\left[\left|x_t^N/N - u^{W,N}(t)\right| > \delta\right] = 0.$$

Theorem 2.4 follows from the dominated convergence theorem, from the second observation of the proof and from this last observation. □

We conclude this section deriving a differential equation for the asymptotic position of the tagged particle. Recall the definition of the derivative d/dW introduced in (2.2) and let $\rho_t(\cdot) = \rho_W(t,\cdot) = P_t\rho_0$.

Lemma 5.5 *Fix a profile* $\rho_0 : \mathbb{R} \to [0,1]$. *Assume that* ρ_0 *belongs to* $C_c(\mathbb{R})$ *and that*

$$\lim_{h\to 0}\int_{\mathbb{R}} \left|\frac{P_h\rho_t - \rho_t}{h} - \mathfrak{L}_W\rho_t\right| = 0$$

for all $t > 0$. *Then,* u_W *is differentiable both from the right and from the left for* $t > 0$ *and*

$$\frac{d}{dt+}u_W(t) = \begin{cases} -\dfrac{1}{\rho_t(u_W(t))}\dfrac{d\rho_t}{dW}(u_W(t)) & \text{if } \dfrac{d\rho_t}{dW}(u_W(t)) < 0\\[2mm] -\dfrac{1}{\rho_t(u_W(t)-)}\dfrac{d\rho_t}{dW}(u_W(t)) & \text{if } \dfrac{d\rho_t}{dW}(u_W(t)) > 0\\ 0 & \text{otherwise.}\end{cases}$$

Notice that while $\rho_t = P_t\rho_0$, which belongs to the domain of the generator $\mathfrak{L}_W$ in view of [8], may have discontinuities at $\{x_j : j \geq 1\}$, its derivative $d\rho_t/dW$ is continuous. On the other hand, the definition of the infinitesimal operator $\mathfrak{L}_W$ and the fact that ρ_t belongs to the domain of the generator imply the uniform convergence of $h^{-1}(P_h\rho_t - \rho_t)$ to $\mathfrak{L}_W\rho_t$ on the whole real line. We are requiring here the convergence to take place in $L^1(\mathbb{R})$. In particular, $\mathfrak{L}_W\rho_t$ belongs to $L^1(\mathbb{R})$.

Proof Fix $t > 0$ and take h in $\mathbb{R}$ small. Denote $\mathfrak{L}_W\rho_t$ by λ_t. Since

$$\int_{-\infty}^{u_W(t)} \rho_t(v)dv = \int_{-\infty}^{u_W(t+h)} \rho_{t+h}(v)dv,$$

we have that

$$-\frac{1}{h}\int_{u_W(t)}^{u_W(t+h)} \rho_t(v)dv = \int_{-\infty}^{u_W(t+h)} \frac{\rho_{t+h}(v) - \rho_t(v)}{h}dv.$$

By the main assumption of the lemma, we may replace on the right hand side the ratio $\{\rho_{t+h}(v) - \rho_t(v)\}/h$ by λ_t paying a price of order $o(h)$. Since $u_W(t+h)$ converges to $u_W(t)$ as $h \to 0$, we get that

$$\int_{-\infty}^{u_W(t)} \lambda_t(v)\, dv = -\lim_{h\to 0} \frac{1}{h} \int_{u_W(t)}^{u_W(t+h)} \rho_t(v)\, dv.$$

We assume that $h > 0$ and compute the right derivative of $u_W(t)$. The left derivative is left to the reader. Assume that $\int_{(-\infty, u_W(t)]} \lambda_t(v) dv > 0$ so that $u_W(t+h) < u_W(t)$ for h sufficiently small. Since ρ_t is a càdlàg function and $u_W(t+h) < u_W(t)$ for h sufficiently small, it follows from the previous identity that

$$\lim_{h\downarrow 0} \frac{u_W(t+h) - u_W(t)}{h} = -\frac{1}{\rho_t(u_W(t)-)} \int_{-\infty}^{u_W(t)} \lambda_t(v)\, dv.$$

Similar identities can be obtained if $\int_{(-\infty, u_W(t)]} \lambda_t(v) dv$ vanishes or is less than 0 and for $h \uparrow 0$. Thus, $u_W(\cdot)$ is differentiable both from the right and from the left. To prove the lemma it remains to show that

$$\frac{d\rho_t}{dW}(u_W(t)) = \int_{-\infty}^{u_W(t)} \lambda_t(v)\, dv.$$

Since $\lambda_t = \mathfrak{L}_W \rho_t$, we have that

$$\rho_t(u) = a_t + b_t W(u) + \int_0^u W(dv) \int_0^v \lambda_t(w)\, dw$$

for some finite constants a_t, b_t. In particular, for any $u < v$,

$$\frac{d\rho_t}{dW}(v) = \frac{d\rho_t}{dW}(u) + \int_u^v \lambda_t(w)\, dw.$$

Since λ_t belongs to $L^1(\mathbb{R})$, letting $u \downarrow -\infty$, we find that $d\rho_t/dW(u)$ converges to some constant c_t and that

$$\frac{d\rho_t}{dW}(v) = c_t + \int_{-\infty}^v \lambda_t(w)\, dw.$$

Take $u > 0$. Since $W(0) = 0$, integrating this identity with respect to dW in the interval $(0, u]$ we get that

$$\rho_t(u) - \rho_t(0) = c_t W(u) + \int_{(0,u]} W(dv) \int_{-\infty}^{v} \lambda_t(w)\, dw.$$

Dividing by $W(u)$, since ρ_t is uniformly bounded and since $\lim_{u\to\infty} W(u) = \infty$

$$c_t = \lim_{u\to\infty} \frac{-1}{W(u)} \int_{(0,u]} W(dv) \int_{-\infty}^{v} \lambda_t(w)\, dw.$$

Since the $L^1(\mathbb{R})$ norm of ρ_t is constant in time, $\int_{-\infty}^{\infty} \lambda_t(w)\, dw = 0$ for all $t > 0$. In particular, the previous expression vanishes. This concludes the proof of the lemma.

□

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