A probabilistic representation of constants in Kesten's renewal theorem

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Abstract The aims of this paper are twofold. Firstly, we derive a probabilistic representation for the constant which appears in the one-dimensional case of Kesten's renewal theorem. Secondly, we estimate the tail of a related random variable which plays an essential role in the description of the stable limit law of one-dimensional transient sub-ballistic random walks in random environment.

Keywords Renewal series · Coupling · Fluctuation theory of random walks

Mathematics Subject Classification (2000) 60H25 · 60K05

1 Introduction

In 1973, Kesten published a famous paper [9] about the tail estimates of renewal series of the form $\sum_{i \geq 1} A_1 \ldots A_{i-1} B_i$, where $(A_i)_{i \geq 0}$ is a sequence of non-negative i.i.d. $d \times d$ random matrices and $(B_i)_{i \geq 1}$ is a sequence of i.i.d. random vectors of $\mathbb{R}^d$.

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His result states that the tail of the projection of this random vector on every direction is equivalent to $Ct^{-\kappa}$, when t tends to infinity, where C and κ are positive constants. The constant κ is defined as the solution of the equation $k(s) = 1$, with $k(s) := \lim_{n \to \infty} \mathbb{E}(\| A_1 \ldots A_n \|^s)^{1/n}$. The proof of his result in the one-dimensional case, even if it is much easier than in dimension $d \geq 2$, is already rather complicated.

Even though we are concerned by the one-dimensional case in this paper, let us mention that a significant generalization of Kesten's result, in the multi-dimensional case, was recently achieved by de Saporta et al. [3], who relaxed the assumption of positivity on A_i.

In 1991, Goldie [7] relaxed, in dimension $d = 1$, the assumption of positivity on the A_i and simplified Kesten's proof. Furthermore, he obtained a formula for the implicit constant C in the special case where A_i is non-negative and κ is an integer.

In 1991, Chamayou and Letac [1] observed that, in dimension $d = 1$, if A_i has the same law as $(1 - X_i)/X_i$, with X_i following a Beta distribution on $(0, 1)$, then the law of the series itself is computable so that the constant C is explicit in this special case also. The following question was then asked. How does one effectively compute the constant C?

In our framework, we consider the case $d = 1$ and we make the following assumptions: $\rho_i = A_i$ is a sequence of i.i.d. positive random variables, $B_i = 1$ and there exists $\kappa > 0$ such that $\mathbb{E}(\rho_1^\kappa) = 1$. Moreover, we assume a weak integrability condition and that the law of $\log \rho_i$, which has a negative expectation by the previous assumptions, is non-arithmetic. In this context we are interested in the random series

$$R = 1 + \sum_{k \geq 1} \rho_1 \cdots \rho_k.$$

The previous assumptions ensure that the tail of the renewal series R is equivalent to $C_K t^{-\kappa}$, when t tends to infinity. We are now aiming at finding a probabilistic representation of the constant C_K.

Besides, this work is motivated by the study of one-dimensional random walks in random environment. In [10], Kesten et al. proved, using the tail estimate derived in [9], that when the RWRE is transient with null asymptotic speed, then the behavior depends on an index $\kappa \leq 1$: the RWRE X_n normalized by $n^{1/\kappa}$ converges in law to $C_\kappa \left(\frac{1}{S_\kappa} \right)^\kappa$ where S_κ is a positive stable random variable with index κ. The computation of the explicit value of C_κ was left open. In [5], the authors derive an explicit expression, either in terms of the Kesten's constant C_K when it is explicit, or in terms of the expectation of a random series when C_K is not explicit. To this end, we need to obtain a tail estimate for a random variable Z, closely related to the random series R, and to relate it to Kesten's constant. This is the other aim of this paper.

The strategy of our proof is based on a coupling argument in the (cf [4], 4.3). We first interpret $\rho_1 \ldots \rho_n$ as the exponential of a random walk $(V_n, n \geq 0)$, which is negatively drifted, since $\mathbb{E}(\log \rho_1) < 0$. We have now to deal with the series $R := \sum_{n \geq 0} e^{V_n}$. One can write

$$R = e^S \sum_{n \geq 0} e^{V_n - S},$$

where S is the maximum of $(V_n, \; n \geq 0)$. The heuristic is that S and $\sum_{n \geq 0} e^{V_n - S}$ are asymptotically independent. The coupling argument is used to derive this asymptotic independence. But, in order to implement this strategy, several difficulties have to be overcome: we first need to condition S to be large. Moreover, we have to couple conditioned processes: this requires us to describe precisely the part of the process $(V_0, \ldots, V_{T_S})$, where T_S is the first hitting time of the level S.

To end this section, let us finally discuss our results and strategy. Let us first remind that Kesten and Goldie's proof were based on a clever use of the renewal theorem but strongly relied on the renewal structure of the series, and also did not lead to satisfying representations of the constant involved in its tail function. Later, Siegmund [11] presented an interesting scheme of proof, inspired by a work on change-point analysis of Pollak and Yakir [12]. He was able to derive formally a representation of the constant, which enables simulation of the constant by Monte Carlo.

We would like to emphasize the flexibility of our proof that allows to study conditioned variables which do not necessarily satisfy a renewal scheme like the variable Z mentioned above, which plays a key role in the analysis of RWRE. This flexibility could hopefully make also possible some generalizations to the d-dimensional case. As explained above, the strength of this method is indeed to prove an asymptotic independence between two different parts of the underlying random walk of step $\log(\rho_n)$, when its maximum is large, namely: the maximum of the random walk and the part of trajectory in the neighbourhood of the absolute maximum. As a consequence, the tail constant of R is expressed as the product of the tail constant of the absolute maximum of the random walk times the expectation of a functional of some meander which comes from the part of the trajectory near its maximum. One of the central interests of this representation is that it is well suited for Monte-Carlo simulation. Compared to Siegmund's formula, our formula is exact and not asymptotic (formula (3.6) of Siegmund [11] must be understood as a limit when j tends to infinity). Our asymptotic independence argument is reminiscent of the argument of Siegmund which remained at a heuristic level, and we want to emphasize that this asymptotic independance is the difficult part of our proof.

On the other hand, let us notice that the analytic expressions found by Goldie when κ is an integer, and Chamayou and Letac in the case of Beta variables are strongly based on the renewal scheme. It is therefore not surprising that the representation found by our method do not recover these results. However, their identification a posteriori leads to explicit formulas for the constants arising in the limit theorems for RWRE in some very interesting special cases see [5].

2 Notation and statement of the results

Let $(\rho_i)_{i \in \mathbb{Z}}$ be a sequence of i.i.d. positive random variables with law $Q = \mu^{\otimes \mathbb{Z}}$. With the sequence $(\rho_i)_{i \in \mathbb{Z}}$ we associate the potential $(V_k)_{k \in \mathbb{Z}}$ defined by

$$V_n := \begin{cases} \sum_{k=1}^{n} \log \rho_k & \text{if } n \geq 1, \\ 0 & \text{if } n = 0, \\ -\sum_{k=n+1}^{0} \log \rho_k & \text{if } n \leq -1. \end{cases}$$

Let ρ have law μ. Suppose now that the law μ is such that there is $\kappa > 0$ satisfying

$$\mathbb{E}^{\mu}(\rho^{\kappa}) = 1 \quad \text{and} \quad \mathbb{E}^{\mu}(\rho^{\kappa} \log^{+} \rho) < \infty. \tag{1}$$

Moreover, we assume that the distribution of $\log \rho$ is non-lattice. Then the law μ is such that $\log \rho$ satisfies

$$\mathbb{E}^{\mu}(\log \rho) < 0, \tag{2}$$

which implies that, Q-almost surely,

$$\lim_{n \to \infty} \frac{V_n}{n} = \int \log \rho \, d\mu < 0.$$

We set

$$S := \max\{V_k, \ k \geq 0\},$$

and

$$H := \max\{V_k, 0 \leq k \leq T_{\mathbb{R}_-}\},$$

where $T_{\mathbb{R}_-}$ is the first positive hitting time of $\mathbb{R}_-$:

$$T_{\mathbb{R}_-} := \inf\{k > 0, \ V_k \leq 0\}.$$

The random variable S is the absolute maximum of the path $(V_k)_{k \geq 0}$ while H is the maximum of the first positive excursion. We also set

$$T_S := \inf\{k \geq 0, \ V_k = S\}, \quad T_H := \inf\{k \geq 0, \ V_k = H\}.$$

We clearly have, Q-almost surely,

$$H \leq S < \infty, \quad T_H \leq T_S < \infty.$$

The following tail estimate for S is a classical consequence of renewal theory, see [6],

$$\mathbb{P}^{Q}(e^S \geq t) \sim C_F t^{-\kappa}, \tag{3}$$

when $t \to \infty$, where

$$C_F = \frac{1 - \mathbb{E}^Q(e^{\kappa V(T_{\mathbb{R}_-})})}{\kappa \mathbb{E}^\mu(\rho^\kappa \log \rho)\mathbb{E}^Q(T_{\mathbb{R}_-})}.$$

The tail estimate of H is derived by Iglehart, in [8],

$$\mathbb{P}^Q(e^H \geq t) \sim C_I t^{-\kappa}, \tag{4}$$

when $t \to \infty$, where

$$C_I = \frac{\left(1 - \mathbb{E}^Q\left(e^{\kappa V(T_{\mathbb{R}_-})}\right)\right)^2}{\kappa \mathbb{E}^\mu(\rho^\kappa \log \rho)\mathbb{E}^Q(T_{\mathbb{R}_-})} = \left(1 - \mathbb{E}^Q\left(e^{\kappa V(T_{\mathbb{R}_-})}\right)\right) C_F.$$

Consider now the random variable

$$R := \sum_{n=0}^{\infty} e^{V_n}.$$

This random variable clearly satisfies the following random affine equation

$$R \stackrel{\text{law}}{=} 1 + \rho R,$$

where ρ is a random variable with law μ independent of R. In [9], Kesten proved (actually his result was more general and concerned by the multidimensional version of this one) that there exists a positive constant C_K such that

$$\mathbb{P}^Q(R \geq t) \sim C_K t^{-\kappa}, \tag{5}$$

when $t \to \infty$. The constant C_K has been made explicit in some particular cases: for κ integer by Goldie, see [7], and when $\rho \stackrel{\text{law}}{=} \frac{W}{1-W}$ where W is a beta variable, by Chamayou and Letac [1]. One aim of this paper is to derive an expression of this constant in terms of the expectation of a functional of the random walk V which is more standard than R.

We need now to introduce some Girsanov transform of Q. Thanks to (1) we can define the law

$$\tilde{\mu} = \rho^\kappa \mu,$$

and the law $\tilde{Q} = \tilde{\mu}^{\otimes \mathbb{Z}}$ which is the law of a sequence of i.i.d. random variables with law $\tilde{\mu}$. The definition of κ implies that

$$\int \log \rho \, \tilde{\mu}(d\rho) > 0,$$

and thus that, $\tilde{Q}$-almost surely,

$$\lim_{n \to \infty} \frac{V_n}{n} = \int \log \rho \, \mathrm{d}\tilde{\mu} > 0.$$

Moreover, $\tilde{Q}$ is a Girsanov transform of Q, i.e. we have for all n

$$\mathbb{E}^{Q}\left(\phi(V_0, \ldots, V_n)\right) = \mathbb{E}^{\tilde{Q}}\left(e^{-\kappa V_n}\phi(V_0, \ldots, V_n)\right),$$

for any bounded test function ϕ. Let us now introduce the random variable M defined by

$$M = \sum_{i<0} e^{-V_i} + \sum_{j\geq 0} e^{-V_j}, \tag{6}$$

where $(V_i)_{i<0}$ is distributed under $Q(\cdot|V_i \geq 0, \forall i < 0)$ and independent of $(V_j)_{j\geq 0}$ which is distributed under $\tilde{Q}(\cdot|V_j > 0, \forall j > 0)$.

Theorem 2.1   (i)   *We have the following tail estimate*

$$\mathbb{P}^{Q}(R \geq t) \sim C_K t^{-\kappa},$$

when $t \to \infty$, where

$$C_K = C_F \mathbb{E}(M^{\kappa}).$$

(ii)   *We have*

$$\mathbb{P}^{Q}(R \geq t; H = S) \sim C_{KI} t^{-\kappa},$$

when $t \to \infty$, where

$$C_{KI} := C_I \mathbb{E}(M^{\kappa}).$$

Remark 2.1 The conditioning $H = S$ means that the path $(V_k)_{k\geq 0}$ never goes above the height of its first excursion.

In [5], we need a tail estimate on a random variable of the type of R but with an extra term. Let us introduce the event

$$\mathcal{I} := \{H = S\} \cap \{V_k \geq 0, \forall k \leq 0\}, \tag{7}$$

and the random variable

$$Z := e^{S} M_1 M_2,$$

where

$$M_1 := \sum_{k=-\infty}^{T_S} e^{-V_k},$$

$$M_2 := \sum_{k=0}^{\infty} e^{V_k-S}.$$

Theorem 2.2 *We have the following tail estimate*

$$\mathbb{P}^Q(Z \geq t|\mathcal{I}) \sim \frac{1}{\mathbb{P}^Q(H = S)} C_U t^{-\kappa},$$

when $t \to \infty$, *where*

$$C_U = C_I \mathbb{E}(M^\kappa)^2 = \frac{C_I}{C_F}(C_K)^2.$$

Remark 2.2 The conditioning event $\mathcal{I}$ gives a nice symmetry property, which is useful to return the path, cf Sect. 3.2.

Let us now discuss the case where the B_i's are not necessarily equal to 1. Let $(B_i)_{i \geq 0}$ be a sequence of positive i.i.d. random variables, which is independent of the sequence $(\rho_i)_{i \geq 0}$, and denote by R^B the random series $R^B := B_0 + \sum_{k \geq 1} B_k \rho_1 \cdots \rho_k$. The result of Theorem 2.1, (i), is then generalized into the following result.

Theorem 2.3 *If there exists* $\varepsilon > 0$ *such that* $\mathbb{E}(|B_1|^{\kappa+\varepsilon}) < \infty$, *then*

$$\mathbb{P}^Q(R^B \geq t) \sim C_{KB} t^{-\kappa},$$

when $t \to \infty$, *where*

$$C_{KB} = C_F \mathbb{E}((M^B)^\kappa)$$

and where M^B *is defined by*

$$M^B = \sum_{k<0} e^{-V_k} \tilde{B}_k + \sum_{k \geq 0} e^{-V_k} \tilde{B}_k,$$

with $(V_k)_{k<0}$ *distributed under* $Q(\cdot|V_k \geq 0, \forall k < 0)$ *and independent of* $(V_k)_{k \geq 0}$ *which is distributed under* $\tilde{Q}(\cdot|V_k > 0, \forall k > 0)$ *while* $(\tilde{B}_k)_{k \in \mathbb{Z}}$ *is a sequence of i.i.d. random variables having the same distribution as* B_1 *and independent of* $(V_k)_{k \in \mathbb{Z}}$.

Sketch of the proof and organization of the paper

The intuition behind Theorems 2.1 and 2.2 is the following. Let us first consider $\mathbb{P}^Q(R \geq t|H = S)$. The law $Q(\cdot|\mathcal{I})$ has a symmetry property which implies that the

variable $R = M_2 e^H$ has the same distribution as $M_1 e^H$ (cf Sect. 3.2). Then, the proof of Theorem 2.1 is based on the following arguments.

Firstly, we prove that the variables M_1 and e^H are asymptotically independent. To this end, we use a delicate coupling argument which works only when H is conditioned to be large. Therefore, we need to restrict ourselves to large values of H. To this end, we need to control the value of R conditioned by H; this is done in Sect. 4. Then, a second difficulty is that we have to couple conditioned processes (namely, the process (V_k) conditioned to have a first high excursion). We overcome this difficulty by using an explicit description of the law of the path $(V_0, \ldots, V_{T_H})$. Namely, the path $(V_0, \ldots, V_{T_H})$ behaves like V under $\tilde{Q}(\cdot | V_k > 0, \forall k > 0)$ stopped at some random time.

Secondly, we observe that the distribution of M_1 is close to the distribution of M as a consequence of the above description of the law of $(V_0, \ldots, V_{T_H})$.

From these two facts, we deduce that $\mathbb{P}^Q(R \geq t \mid \mathcal{I}) \simeq \mathbb{P}^Q(M e^H \geq t \mid \mathcal{I})$, where M and H are roughly independent. Using the tail estimate for H we get the part ii) of Theorem 2.1. For Theorem 2.2, we proceed similarly: the variable Z can be written $M_1 R$ and, for large H, the variables M_1 and R are asymptotically independent and the law of M_1 is close to the law of M. Then the estimate on the tail of R allows us to conclude the proof.

Let us now describe the organization of the proofs. Section 3 contains preliminary results, whose proofs are postponed to the Appendix (see Sect. 6). In Sect. 3.1, we prove that M has finite moments of all orders and we estimate the rest of the series M. Section 3.2 contains some preliminary properties of the law $Q(\cdot | \mathcal{I})$, and Sect. 3.3 presents a representation of the law of the process $(V_0, \ldots, V_S)$ in terms of the law $\tilde{Q}$. Section 4 contains crucial estimates which will allow us to restrict ourselves to large values of H. In Sect. 5, we detail the coupling arguments which roughly give the asymptotic independence of M_1 and $e^H M_2$. Finally, in Sect. 6 we assemble the arguments of the previous sections to prove Theorems 2.2 and 2.1. In the Appendix (see Sect. 6), we give the proof of the claims of Sect. 3 and present a Tauberian version of the tail estimates, which is the version we ultimately use in [5].

Let us finally explain the convention we use concerning constants. We denote by c a positive constant with value changing from place to place, which only depends on κ and the distribution of ρ. The dependence on additional parameters otherwise appears in the notation.

3 Preliminaries

In this section, we give preliminary results, whose proof are postponed to the Appendix (see Sect. 6).

3.1 Moments of M

Here is a series of three lemmas about the moments of the exponential functional of the random walk M. In this section, we denote by $\{V \geq -L\}$ the event $\{V_k \geq -L, \ k \geq 0\}$.

Lemma 3.1 *There exists $c > 0$ such that, for all $L \geq 0$,*

$$\mathbb{E}^{\tilde{Q}}\left(\sum_{k \geq 0} e^{-V_k} \mid V \geq -L\right) \leq c\, e^L.$$

Lemma 3.2 *Under $\tilde{Q}^{\geq 0} := \tilde{Q}(\cdot \mid V_k \geq 0, \forall k \geq 0)$, all the moments of $\sum_{k \geq 0} e^{-V_k}$ are finite.*

We will need further a finer result than Lemma 3.1 as follows.

Lemma 3.3 *For any $\kappa > 0$, there exists $c = c(\kappa) > 0$ such that, for all $L > 0$ and for all $\varepsilon' > 0$, we have*

- *if $\kappa < 1$,*

$$\mathbb{E}^{\tilde{Q}}\left(\sum_{i \geq 0} e^{-V_i} \mid V \geq -L\right) \leq c\, e^{(1-\kappa+\varepsilon')L},$$

- *if $\kappa \geq 1$,*

$$\mathbb{E}^{\tilde{Q}}\left(\sum_{i \geq 0} e^{-V_i} \mid V \geq -L\right) \leq c\, e^{\varepsilon' L}.$$

Remark 3.1 Analogous results as in Lemmas 3.1, 3.2 and 3.3 apply for $\sum_{k \geq 0} e^{V_k}$ under Q and conditionally on the event $\{V_k \leq L, \forall k \geq 0\}$.

3.2 A time reversal

Let us denote by $Q^{\mathcal{I}}$ the conditional law $Q^{\mathcal{I}}(\cdot) := Q(\cdot \mid \mathcal{I})$, where $\mathcal{I}$ is defined in (7). The law $Q^{\mathcal{I}}$ has the following symmetry property.

Lemma 3.4 *Under $Q^{\mathcal{I}}$ we have the following equality in law*

$$(V_k)_{k \in \mathbb{Z}} \stackrel{law}{=} (V_{T_H} - V_{T_H - k})_{k \in \mathbb{Z}}.$$

This implies that under $Q^{\mathcal{I}}$, R has the law of $e^H M_1$. This last formula will be useful since the asymptotic independence of e^H and M_1, in the limit of large H, is more visible than the asymptotic independence of H and M_2 and will be easier to prove.

3.3 The two faces of the mountain

It will be convenient to introduce the following notation: we denote by $Q^{\leq 0}$ the conditional law

$$Q^{\leq 0}(\cdot) = Q(\cdot | V_k \leq 0, \ \forall k \geq 0),$$

and by $\tilde{Q}^{>0}$ the conditional law

$$\tilde{Q}^{>0}(\cdot) = \tilde{Q}(\cdot | V_k > 0, \ \forall k > 0).$$

It will be useful to describe the law of the part of the path $(V_0, \ldots, V_{T_S})$. Let us introduce some notations. If $(Y_k)_{k \geq 0}$ is a random process under the law $\tilde{Q}$, then $Y_k \to +\infty$ a.s. and we can define its strictly increasing ladder times $(e_k)_{k \geq 0}$ by: $e_0 := 0$, and

$$e_{k+1} := \inf\{n > e_k, \ Y_n > Y_{e_k}\}.$$

We define a random variable $((Y_k)_{k \geq 0}, \Theta)$ with values in $\mathbb{R}^{\mathbb{N}} \times \mathbb{N}$ as follows: the random process $(Y_k)_{k \geq 0}$ has a law with density with respect to $\tilde{Q}$ given by

$$\frac{1}{\mathcal{Z}} \left(\sum_{k=0}^{\infty} e^{-\kappa Y_{e_k}} \right) \tilde{Q},$$

where $\mathcal{Z}$ is the normalizing constant given by

$$\mathcal{Z} = \frac{1}{1 - \mathbb{E}^{\tilde{Q}}(e^{-\kappa Y_{e_1}})}.$$

Then, conditionally on $(Y_k)_{k \geq 0}$, Θ takes one of the value of the strictly ladder times with probability

$$\mathbb{P}(\Theta = e_p \mid \sigma((Y_k)_{k \geq 0})) = \frac{e^{-\kappa Y_{e_p}}}{\sum_{k=0}^{\infty} e^{-\kappa Y_{e_k}}}.$$

We denote by $\hat{Q}$ the law of $((Y_k)_{k \geq 0}, \Theta)$. Otherwise stated, it means that, for all test functions ϕ,

$$\mathbb{E}^{\hat{Q}}(\phi(\Theta, (Y_n)_{n \geq 0})) = \frac{1}{\mathcal{Z}} \mathbb{E}^{\tilde{Q}} \left(\sum_{k=0}^{\infty} e^{-\kappa Y_{e_k}} \phi(e_k, (Y_n)_{n \geq 0}) \right).$$

Lemma 3.5 *The processes* $(V_0, \ldots, V_{T_S})$ *and* $(V_{T_S+k} - V_{T_S})_{k \geq 0}$ *are independent and have the following laws:* $(V_{T_S+k} - V_{T_S})_{k \geq 0}$ *has the law* $Q^{\leq 0}$ *and*

$$(V_0, \ldots, V_{T_S}) \overset{law}{=} (Y_0, \ldots, Y_\Theta),$$

where $((Y_k)_{k \geq 0}, \Theta)$ *has the law* $\hat{Q}$.

Denote now by $\hat{Q}^{>0}$ the law

$$\hat{Q}^{>0} = \hat{Q}(\cdot \mid Y_k > 0, \forall k > 0).$$

We will need the following result.

Lemma 3.6 *There exists a positive constant $c > 0$ such that, for all positive test functions ψ,*

$$\mathbb{E}^{Q^{\mathcal{I}}}(\psi(V_0, \ldots, V_{T_H})) \leq c\mathbb{E}^{\hat{Q}^{>0}}(\psi(Y_0, \ldots, Y_\Theta)).$$

4 A preliminary estimate

To derive the tail estimate of R or Z we need to restrict to large values of H: this will be possible, thanks to the following estimate.

Lemma 4.1 *For all $\eta > 0$ there exists a positive constant c_η such that*

$$\mathbb{E}^{Q^{\mathcal{I}}}((M_1)^\eta \mid \lfloor H \rfloor) \leq c_\eta, \quad Q^{\mathcal{I}}\text{- a.s.,}$$

where $\lfloor H \rfloor$ is the integer part of H.

Proof Since $(V_k)_{k \leq 0}$ is independent of H under $Q^{\mathcal{I}}$, we have, for all $p \in \mathbb{N}$,

$$\mathbb{E}^{Q^{\mathcal{I}}}((M_1)^\eta \mid \lfloor H \rfloor = p) \leq 2^\eta \left(\mathbb{E}^{Q^{\leq 0}}\left(\left(\sum_{k=0}^{\infty} e^{V_k} \right)^\eta \right) \right.$$
$$\left. + \mathbb{E}^{Q^{\mathcal{I}}}\left(\left(\sum_{k=0}^{T_H} e^{-V_k} \right)^\eta \mid \lfloor H \rfloor = p \right) \right).$$

The first term on the right-hand side is finite for all $\eta > 0$ as proved in Sect. 3.1. Consider now the last term. Using Lemma 3.6, we get

$$\mathbb{E}^{Q^{\mathcal{I}}}\left(\left(\sum_{k=0}^{T_H} e^{-V_k} \right)^\eta \mid \lfloor H \rfloor = p \right)$$
$$\leq \frac{c}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\hat{Q}^{>0}}\left(\left(\sum_{k=0}^{T_H} e^{-Y_k} \right)^\eta \mathbf{1}_{\lfloor H \rfloor = p} \right)$$
$$\leq \frac{c}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} e^{-\kappa Y_{e_k}} \mathbf{1}_{Y_{e_k} \in [p, p+1[} \right) \left(\sum_{j=0}^{\infty} e^{-V_j} \right)^\eta \right).$$

Now, using the Cauchy–Schwarz inequality in the last expression, we get

$$\mathbb{E}^{Q^{\mathcal{I}}}\left(\left(\sum_{k=0}^{T_H} \mathrm{e}^{-V_k}\right)^{\eta} \mid \lfloor H \rfloor = p\right)$$

$$\leq \frac{c}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} \mathrm{e}^{-\kappa Y_{e_k}} \mathbf{1}_{Y_{e_k} \in [p,p+1[}\right)^2\right)^{\frac{1}{2}} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} \mathrm{e}^{-V_k}\right)^{2\eta}\right)^{\frac{1}{2}}$$

$$\leq \frac{c\,\mathrm{e}^{-\kappa p}}{\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p)} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} \mathbf{1}_{Y_{e_k} \in [p,p+1[}\right)^2\right)^{\frac{1}{2}} \mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} \mathrm{e}^{-V_k}\right)^{2\eta}\right)^{\frac{1}{2}}.$$

But the last term is independent of p and finite by Lemma 3.2. On the other hand, since $\tilde{Q}(V_k > 0, \ \forall k > 0) > 0$ and from the Markov property, we obtain

$$\mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k=0}^{\infty} \mathbf{1}_{Y_{e_k} \in [p,p+1[}\right)^2\right) \leq c\mathbb{E}^{\tilde{Q}}\left(\left(\sum_{k=0}^{\infty} \mathbf{1}_{Y_{e_k} \in [p,p+1[}\right)^2\right)$$

$$\leq c\mathbb{E}^{\tilde{Q}}\left(\left(\sum_{k=0}^{\infty} \mathbf{1}_{Y_{e_k} \in [0,1[}\right)^2\right),$$

which is finite since $(Y_k)_{k\geq 0}$ has a positive drift under $\tilde{Q}$. Finally, using the tail estimate on H, we know that

$$\lim_{p\to\infty} \mathrm{e}^{\kappa p}\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p) = \lim_{p\to\infty} \mathrm{e}^{\kappa p}\left(\mathbb{P}^{Q^{\mathcal{I}}}(H \geq p) - \mathbb{P}^{Q^{\mathcal{I}}}(H \geq p+1)\right)$$

$$= C_I(1 - \mathrm{e}^{-\kappa}). \tag{8}$$

Hence, $(\mathrm{e}^{\kappa p}\mathbb{P}^{Q^{\mathcal{I}}}(\lfloor H \rfloor = p))^{-1}$ is a bounded sequence (we do not have to consider the cases where eventually $\mathbb{P}(\lfloor H \rfloor = p) = 0$ since it is a conditioning by an event of null probability which can be omitted). $\square$

Corollary 4.1 *We have, $Q^{\mathcal{I}}$-almost surely,*

$$\mathbb{E}^{Q^{\mathcal{I}}}(Z \mid \lfloor H \rfloor) \leq \mathrm{e}c_2\mathrm{e}^{\lfloor H \rfloor}.$$

Proof We have $Z = M_1 M_2 \mathrm{e}^H$. Using the Cauchy–Schwarz inequality and Lemma 4.1 we get

$$\mathbb{E}^{Q^{\mathcal{I}}}(Z \mid \lfloor H \rfloor) \leq \mathrm{e}^{\lfloor H \rfloor + 1}\left(\mathbb{E}^{Q^{\mathcal{I}}}((M_1)^2 \mid \lfloor H \rfloor)\mathbb{E}^{Q^{\mathcal{I}}}((M_2)^2 \mid \lfloor H \rfloor)\right)^{\frac{1}{2}}$$

$$\leq \mathrm{e}c_2\mathrm{e}^{\lfloor H \rfloor},$$

since M_1 and M_2 have the same law under $Q^{\mathcal{I}}$. $\square$

Corollary 4.2 *Let $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a function such that*

$$\lim_{t \to \infty} t^{-1} e^{h(t)} = 0.$$

Then, we have

$$\mathbb{P}^{Q^{\mathcal{I}}} (R \geq t, \ H \leq h(t)) = o(t^{-\kappa}),$$
$$\mathbb{P}^{Q^{\mathcal{I}}} (Z \geq t, \ H \leq h(t)) = o(t^{-\kappa}),$$

when t tends to infinity.

Proof Let us do the proof for Z. Let η be a positive real such that

$$\eta > \kappa.$$

We have (all expectations are relative to the measure $Q^{\mathcal{I}}$; so, to simplify the reading, we remove the reference to $Q^{\mathcal{I}}$ in the following)

$$\mathbb{P}^{Q^{\mathcal{I}}} (Z \geq t, \ H \leq h(t)) = \mathbb{E} \left(\mathbb{P} (Z \geq t, \ H \leq h(t) \mid \lfloor H \rfloor) \right)$$

$$\leq \mathbb{E} \left(\mathbf{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor} \mathbb{P} (Z \geq t \mid \lfloor H \rfloor) \right)$$

$$\leq \mathbb{E} \left(\mathbf{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor} \mathbb{P} \left(M_1 M_2 \geq t e^{-(\lfloor H \rfloor + 1)} \mid \lfloor H \rfloor \right) \right)$$

$$\leq e^{\eta} \mathbb{E} \left(\mathbf{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor} t^{-\eta} e^{\eta \lfloor H \rfloor} \mathbb{E} \left((M_1 M_2)^{\eta} \mid \lfloor H \rfloor \right) \right)$$

$$\leq e^{\eta} \mathbb{E} \left(\mathbf{1}_{\lfloor H \rfloor \leq \lfloor h(t) \rfloor} t^{-\eta} e^{\eta \lfloor H \rfloor} \mathbb{E} \left((M_1)^{2\eta} \mid \lfloor H \rfloor \right) \right).$$

In the last formula, we used the Cauchy–Schwarz inequality and the symmetry property of $Q^{\mathcal{I}}$, see Lemma 3.4, to obtain

$$\mathbb{E}((M_2)^{2\eta} \mid \lfloor H \rfloor) = \mathbb{E}((M_1)^{2\eta} \mid \lfloor H \rfloor).$$

We can now use the estimate of Lemma 4.1, which gives

$$\mathbb{P}^{Q^{\mathcal{I}}} (Z \geq t, \ H \leq h(t)) \leq e^{\eta} c_{2\eta} t^{-\eta} \sum_{p=0}^{\lfloor h(t) \rfloor} e^{\eta p} \mathbb{P}(\lfloor H \rfloor = p)$$

$$\leq c t^{-\eta} \sum_{p=0}^{\lfloor h(t) \rfloor} e^{(\eta - \kappa) p}.$$

In the last formula, we used the fact that $\mathbb{P}(\lfloor H \rfloor = p) = O(e^{-\kappa p})$, see (8). Since we chose $\eta > \kappa$ we can bound uniformly

$$\mathbb{P}^{Q^{\mathcal{I}}}\left(Z \geq t,\; H \leq h(t)\right) \leq ct^{-\eta}e^{(\eta-\kappa)h(t)} = ct^{-\kappa}\left(\frac{e^{h(t)}}{t}\right)^{\eta-\kappa}.$$

This gives the result for Z. Since $R \leq Z$, we get the result for R. □

5 The coupling argument

We set

$$I(t) := \mathbb{P}^{Q^{\mathcal{I}}}\left(e^H M_1 M_2 \geq t\right),$$
$$J(t) := \mathbb{P}^{Q^{\mathcal{I}}}\left(e^H M_2 \geq t\right),$$
$$K(t) := \mathbb{P}^{Q^{\mathcal{I}}}\left(e^H \geq t\right).$$

From the estimate of Iglehart, see [8], we know that

$$K(t) \sim \frac{1}{\mathbb{P}^{Q}(H = S)} C_I t^{-\kappa},$$

when $t \to \infty$. Indeed, we have

$$\mathbb{P}^{Q^{\mathcal{I}}}\left(e^H \geq t\right) = \frac{1}{\mathbb{P}^{Q}(H = S)}\left(\mathbb{P}^{Q}(e^H \geq t) - P^{Q}(e^H \geq t,\; S > H)\right).$$

The second term is clearly of order $O(t^{-2\kappa})$, the first term is estimated in [8], cf (4).

We will prove the following key estimates.

Proposition 5.1 *For all $\xi > 0$ there exists a function $\epsilon_\xi(t) > 0$ such that* $\lim_{t\to\infty} \epsilon_\xi(t) = 0$ *and*

$$e^{-3\xi}\mathbb{E}\left(J(e^{3\xi}tM^{-1})\right)(1 - \epsilon_\xi(t)) \leq I(t) \leq e^{3\xi}\mathbb{E}\left(J(e^{-3\xi}tM^{-1})\right)(1 + \epsilon_\xi(t)),$$
$$e^{-2\xi}\mathbb{E}\left(K(e^{2\xi}tM^{-1})\right)(1 - \epsilon_\xi(t)) \leq J(t) \leq e^{2\xi}\mathbb{E}\left(K(e^{-2\xi}tM^{-1})\right)(1 + \epsilon_\xi(t)),$$

where M is the random variable defined in (6).

We see that Theorem 2.1 (ii) is a direct consequence of the second estimate and of the tail estimate for $K(t)$. Theorem 2.2 is a consequence of the estimate (i) and of the estimate for J.

Proof Step 1: We first restrict the expectations to large values of H. Let $h : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be any increasing function such that

$$\lim_{t\to\infty} t^{-1}e^{h(t)} = 0, \tag{9}$$

$$h(t) \geq \frac{9}{10}\log t. \tag{10}$$

From Corollary 4.2, we know that

$$\mathbb{P}^{Q^{\mathcal{I}}} \left(e^H M_1 M_2 \geq t, \ H \leq h(t) \right) = o(t^{-\kappa}) = o(K(t)). \tag{11}$$

Hence, we can restrict ourselves to consider

$$I_h(t) := \mathbb{P}^{Q^{\mathcal{I}}} \left(e^H M_1 M_2 \geq t \mid H \geq h(t) \right),$$
$$J_h(t) := \mathbb{P}^{Q^{\mathcal{I}}} \left(e^H M_2 \geq t \mid H \geq h(t) \right),$$

Step 2: (Truncation of M_1, M_2). We need to truncate the sums M_1 and M_2 so that they do not overlap. Under $Q^{\mathcal{I}}(\cdot | H \geq h(t))$ we consider the random variables

$$\tilde{M}_1 := \sum_{-\infty}^{t_1} e^{-V_k}, \tag{12}$$

$$\tilde{M}_2 := \sum_{t_2}^{\infty} e^{V_k - S}, \tag{13}$$

where

$$t_1 := \inf \left\{ k \geq 0, \ V_k \geq \frac{1}{3} \log t \right\} - 1,$$

$$t_2 := \sup \left\{ k \leq T_H, \ V_k \leq H - \frac{1}{3} \log t \right\} + 1.$$

Since $h(t) \geq \frac{9}{10} \log t$, we have

$$0 \leq t_1 < t_2 \leq T_H.$$

Clearly, by the symmetry property of $Q^{\mathcal{I}}$, $\tilde{M}_1$ and $\tilde{M}_2$ have the same law under $Q^{\mathcal{I}}(\cdot | H \geq h(t))$ (Observe that the random variables $\tilde{M}_1$ and $\tilde{M}_2$ are implicitly defined in terms of the variable t).

Lemma 5.1 *Let ξ be a positive real. There exists a constant $c_\xi > 0$ such that*

$$\mathbb{P}^{Q^{\mathcal{I}}} \left(\tilde{M}_1 \leq e^{-\xi} M_1 \mid H \geq h(t) \right) \leq \begin{cases} c_\xi t^{-\kappa/6} & \text{for } \kappa \leq 1, \\ c_\xi t^{-1/6} & \text{for } \kappa \geq 1. \end{cases}$$

Proof We have, since $M_1 \geq 1$

$$\mathbb{P}^{Q^{\mathcal{I}}} \left(\tilde{M}_1 \leq e^{-\xi} M_1 \mid H \geq h(t) \right)$$
$$\leq \mathbb{P}^{Q^{\mathcal{I}}} \left(M_1 - \tilde{M}_1 \geq 1 - e^{-\xi} \mid H \geq h(t) \right)$$

$$\leq \frac{1}{1-\mathrm{e}^{-\xi}}\mathbb{E}^{Q^{\mathcal{I}}}\left(M_1 - \tilde{M}_1 \mid H \geq h(t)\right)$$

$$\leq c\frac{\mathrm{e}^{-\kappa h(t)}}{\mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t))}\mathbb{E}^{\tilde{Q}>0}\left(\sum_{k=t_1+1}^{\infty}\mathrm{e}^{-Y_k}\left(\sum_{\substack{e_p \geq k,\\ Y_{e_p} \geq h(t)}}\mathrm{e}^{-\kappa(Y_{e_p}-h(t))}\right)\right),$$

where in the last expression we used the result of Lemma 3.6, and the notation of the related section, and where c is a constant depending on ξ and on the parameters of the model. Using the fact that $\mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t)) \sim C\mathrm{e}^{-\kappa h(t)}$, when $t \to \infty$, the Markov property and the fact that

$$\mathbb{E}^{\tilde{Q}>0}\left(\sum_{\substack{e_p \geq k,\\ Y_{e_p} \geq h(t)}}\mathrm{e}^{-\kappa(Y_{e_p}-h(t))}\right) \leq \frac{1}{\mathbb{P}^{\tilde{Q}}(Y_n > 0, \forall n > 0)(1 - \mathbb{E}^{\tilde{Q}}(\mathrm{e}^{-\kappa Y_{e_1}}))},$$

independently of k, we see that

$$\mathbb{P}^{Q^{\mathcal{I}}}\left(\tilde{M}_1 \leq \mathrm{e}^{-\xi}M_1 \mid H \geq h(t)\right) \leq c\mathbb{E}^{\tilde{Q}>0}\left(\sum_{k=t_1+1}^{\infty}\mathrm{e}^{-Y_k}\right)$$

$$\leq c_\xi t^{-\frac{\kappa \wedge 1}{6}},$$

using the estimate of Lemma 3.3.   □

Step 3: (A small modification of the conditioning). We set

$$\mathcal{I}_h^{(t)} := \mathcal{I} \cap \{S \geq h(t)\} = \{V_k \geq 0, \forall k \leq 0\} \cap \{S = H\} \cap \{S \geq h(t)\},$$

the event by which we condition in $I_h(t)$, $J_h(t)$. We set

$$\tilde{\mathcal{I}}_h^{(t)} := \{S \geq h(t)\} \cap \{V_k \geq 0, \forall k \leq 0\} \cap \{V_k > 0, \forall 0 < k < T_{\frac{1}{3}\log t}\},$$

where

$$T_{\frac{1}{3}\log t} := \inf\left\{k \geq 0, \ V_k \geq \frac{1}{3}\log t\right\}.$$

Clearly, we have $\mathcal{I}_h^{(t)} \subset \tilde{\mathcal{I}}_h^{(t)}$ and

$$\mathbb{P}(\tilde{\mathcal{I}}_h^{(t)}\backslash \mathcal{I}_h^{(t)} \mid \tilde{\mathcal{I}}_h^{(t)}) \leq ct^{-\kappa/3},$$

for a constant $c > 0$ depending only on the parameters of the model. We set

$$\tilde{I}_h(t) := \mathbb{P}^Q \left(e^H \tilde{M}_1 \tilde{M}_2 \geq t \mid \tilde{\mathcal{I}}_h^{(t)} \right),$$

$$\tilde{J}_h(t) := \mathbb{P}^Q \left(e^H \tilde{M}_2 \geq t \mid \tilde{\mathcal{I}}_h^{(t)} \right),$$

$$\tilde{K}_h(t) := \mathbb{P}^Q \left(e^H \geq t \mid \tilde{\mathcal{I}}_h^{(t)} \right).$$

From Step 2 (Lemma 5.1) and Step 3, we see that we have, for all $\xi > 0$, the following estimate

$$I_h(e^{2\xi} t) - c_\xi t^{-\frac{\kappa \wedge 1}{6}} \leq \tilde{I}_h(t) \leq I_h(t) + ct^{-\frac{\kappa}{3}}, \tag{14}$$

$$J_h(e^{\xi} t) - c_\xi t^{-\frac{\kappa \wedge 1}{6}} \leq \tilde{J}_h(t) \leq J_h(t) + ct^{-\frac{\kappa}{3}}. \tag{15}$$

Step 4: (The coupling strategy).

Let $(Y'_k)_{k \geq 0}$ and $(Y''_k)_{k \geq 0}$ be two independent processes with law

$$\tilde{Q}(\cdot \mid Y_k > 0, \, 0 < k \leq T_{\frac{1}{3} \log t}).$$

Let us define, for all $u > 0$, the hitting times

$$T'_u := \inf\{k \geq 0, \, Y'_k \geq u\}, \quad T''_u := \inf\{k \geq 0, \, Y''_k \geq u\}.$$

Set

$$N'_0 := T'_{\frac{1}{3} \log t}, \quad N''_0 := T''_{\frac{1}{3} \log t}.$$

We couple the processes $(Y'_{N'_0+k})_{k \geq 0}$ and $(Y''_{N''_0+k})_{k \geq 0}$ as in Durrett (cf [4], (4.3), p. 204): we construct some random times $K' \geq N'_0$ and $K'' \geq N''_0$ such that

$$|Y'_{K'} - Y''_{K''}| \leq \xi,$$

and such that $(Y'_{K'+k} - Y'_{K'})_{k \geq 0}$ and $(Y''_{K''+k} - Y''_{K''})_{k \geq 0}$ are independent of the σ-field generated by $Y'_0, \ldots, Y'_{K'}$ and $Y''_0, \ldots, Y''_{K''}$. The method for this ξ-coupling is the following: we consider some independent Bernoulli random variables $(\eta'_i)_{i \in \mathbb{N}}$ and $(\eta''_i)_{i \in \mathbb{N}}$ (with $\mathbb{P}(\eta'_i = 1) = \mathbb{P}(\eta''_i = 1) = \frac{1}{2}$) and we define

$$(Z'_k) = \left(Y'_{N'_0+\sum_{i=1}^k \eta'_i} \right), \quad (Z''_k) = \left(Y''_{N''_0+\sum_{i=1}^k \eta''_i} \right).$$

This extra randomization ensures that the process $(Z'_k - Z''_k)$ is non arithmetic. Since its expectation is null, there exists a positive random time for which Z'_k and Z''_k are

at a distance at most ξ (cf the proof of Chung–Fuchs theorem (2.7), p. 188 and theorem (2.1), p. 183 in [4]). Then we define

$$Y_k = \begin{cases} Y'_k, & \text{when } k \le K', \\ (Y''_{K''+(k-K')} - Y''_{K''}) + Y'_{K'}, & \text{when } k > K'. \end{cases}$$

Clearly, by construction, since the processes Y' and Y'' are no longer conditioned when they reach the level $\frac{1}{3} \log t$, $(Y_k)_{k \ge 0}$ has the law

$$\tilde{Q} \left(\cdot \mid Y_k > 0, \ \forall 0 < k < T_{\frac{1}{3} \log t} \right).$$

We want that Y' and Y'' to couple before they reach the level $\frac{1}{2} \log t$, so we set

$$\mathcal{A} = \left\{ K' < T'_{\frac{1}{2} \log t} \right\} \cap \left\{ K'' < T''_{\frac{1}{2} \log t} \right\}.$$

Clearly, since the distribution of $Y'_{N'_0} - \frac{1}{3} \log t$ converges (and the same for Y'', cf limit Theorem (4.10), p. 370 in [6]) and since for all starting points $Y'_{N'_0}$ and $Y''_{N''_0}$, Z' and Z'' couple in a finite time almost surely, we have the following result (whose proof is postponed to the end of the section).

Lemma 5.2

$$\lim_{t \to \infty} \mathbb{P}(\mathcal{A}^c) = 0.$$

We set

$$\eta(t) := \mathbb{P}(\mathcal{A}^c),$$

and we choose $h(t)$ in terms of η by

$$h(t) = \left(\log t + \frac{1}{2\kappa} \log \eta(t) \right) \vee \left(\frac{9}{10} \log t \right) \vee \left(\left(1 - \frac{1}{7\kappa} \right) \log t \right), \qquad (16)$$

where $\vee$ stands for the maximum of the three values. Clearly, $h(t)$ satisfies the hypotheses (9), (10).

Consider now two independent processes $(W_k)_{k \ge 0}$ and $(W'_k)_{k \ge 0}$ (and independent of Y', Y'') with the same law $Q^{\le 0}$ (cf Sect. 3.3). Let e be a strictly increasing ladder time of Y and define the process $V(W, W', Y, e) = (V_k)_{k \in \mathbb{Z}}$ by

$$\begin{cases} (V_k)_{k \le 0} = (-W_{-k})_{k \le 0}, \\ \\ (V_k)_{k \ge 0} = (Y_0, \ldots, Y_e, Y_e + W'_1, \ldots, Y_e + W'_k, \ldots). \end{cases}$$

If $Y_e \geq h(t)$ then clearly $(V_k)_{k \in \mathbb{Z}}$ belongs to the event $\tilde{\mathcal{I}}_h^{(t)}$, and the functional $\tilde{M}_1$ defined in (12) depends only on W and Y'; we denote it by $\tilde{M}_1(W, Y')$. The functional $\tilde{M}_2$ depends only on Y, W', e; we denote it by $\tilde{M}_2(Y, W', e)$. Using Lemma 3.5, we see that

$$\tilde{I}_h(t) = \frac{1}{\mathcal{Z}_h(t)} \mathbb{E}\left(\sum_{p=0}^{\infty} e^{-\kappa Y_{e_p}} \mathbf{1}_{Y_{e_p} \geq h(t)} \mathbf{1}_{\tilde{M}_1(W, Y') \tilde{M}_2(Y, W', e_p) e^{Y_{e_p}} \geq t}\right),$$

where $(e_p)_{p \geq 0}$ is the set of strictly increasing ladder times of Y (cf Sect. 3.3) and where $\mathcal{Z}_h(t)$ is the normalizing constant

$$\mathcal{Z}_h(t) = \mathbb{E}\left(\sum_{p=0}^{\infty} e^{-\kappa Y_{e_p}} \mathbf{1}_{Y_{e_p} \geq h(t)}\right).$$

Clearly, $\mathcal{Z}_h(t) \sim_{t \to \infty} c e^{-\kappa h(t)}$. The variable $Y_{T_{h(t)}} - h(t)$ is indeed the residual waiting time of the renewal process defined by the values of the process Y at the successive increasing ladder epochs. Hence, it converges in distribution by the limit Theorem (4.10) in ([6, p. 370]).

On the coupling event $\mathcal{A}$, we have

$$Y''_{e_p - K' + K''} - \xi \leq Y_{e_p} \leq Y''_{e_p - K' + K''} + \xi,$$
$$\tilde{M}_2(Y, W', e_p) = \tilde{M}_2(Y'', W', e_p - K' + K''),$$

for all ladder times e_p such that $Y_{e_p} \geq h(t)$ (indeed $h(t) \geq \frac{9}{10} \log t$) and where $\tilde{M}_2(Y'', W', e_p - K' + K'')$ is the functional obtained from the concatenation of the processes Y'' and W' at time $e_p - K' + K''$, as done for $\tilde{M}_2(Y, W', e_p)$. The first set of inequalities implies that, on the coupling event $\mathcal{A}$, the set $\{e_p - K' + K'', Y_{e_p} \geq h(t)\}$ is included in the set of strictly increasing ladder times of Y'' larger than $h(t) - \xi$. So we have

$$\tilde{I}_h(t) \leq \frac{e^{\kappa \xi}}{\mathcal{Z}_h(t)} \mathbb{E}\left(\mathbf{1}_{\mathcal{A}}\left(\sum_{p=0}^{\infty} e^{-\kappa Y''_{e''_p}} \mathbf{1}_{Y''_{e''_p} \geq h(t) - \xi} \mathbf{1}_{\tilde{M}_1(W, Y') \tilde{M}_2(Y'', W', e''_p) \exp(Y''_{e''_p}) \geq t e^{-\xi}}\right)\right)$$
$$+ \frac{e^{-\kappa h(t)}}{\mathcal{Z}_h(t)} \mathbb{E}\left(\mathbf{1}_{\mathcal{A}^c}\left(\sum_{p=0}^{\infty} e^{-\kappa(Y_{e_p} - h(t))} \mathbf{1}_{Y_{e_p} \geq h(t)}\right)\right),$$

where $(e''_p)_{p \geq 0}$ denote the strictly increasing ladder times for the process Y''. Since the process $\{Y_{e_p}, Y_{e_p} \geq h(t)\}$ depends on the event $\mathcal{A}$ only through the value of $Y_{T_{h(t)}}$, we see that the second term is less than or equal to

$$\frac{1}{1 - \mathbb{E}^{\tilde{Q}}(e^{-\kappa Y_{e_1}})} \frac{e^{-\kappa h(t)}}{\mathcal{Z}_h(t)} \mathbb{P}(\mathcal{A}^c) \leq c \mathbb{P}(\mathcal{A}^c). \tag{17}$$

Now, the first term is lower than

$$
\begin{aligned}
\mathrm{e}^{\kappa\xi} \frac{\mathcal{Z}_{h-\xi}(t)}{\mathcal{Z}_h(t)} & \mathbb{P}\left(\mathrm{e}^{S''}\tilde{M}_1(W, Y')\tilde{M}_2'' \geq t\mathrm{e}^{-\xi}\right) \\
& \leq \mathrm{e}^{3\kappa\xi}\mathbb{P}\left(\mathrm{e}^{S''}\tilde{M}_1(W, Y')\tilde{M}_2'' \geq t\mathrm{e}^{-\xi}\right),
\end{aligned} \tag{18}
$$

for t large enough (using the equivalent of $\mathcal{Z}_h(t)$), where S'' and $\tilde{M}_2''$ are relative to a process V'' independent of W, Y' and with law $Q(\cdot \mid \tilde{\mathcal{I}}_{h-\xi}^{(t)})$. Moreover, let us introduce $M_2'' := \sum_{k=0}^{\infty} \mathrm{e}^{V_k''-S''}$. We need now to replace the truncated sum $\tilde{M}_1$ by M. Using the fact that $\mathbb{P}(\exists k > 0 : Y_k' \leq 0) \leq ct^{-\kappa/3}$, we see that

$$
\begin{aligned}
\mathbb{P}\left(\mathrm{e}^{S''}\tilde{M}_1(W, Y')\tilde{M}_2'' \geq t\mathrm{e}^{-\xi}\right) & \leq \mathbb{P}\left(\mathrm{e}^{S''}M_2''M \geq t\mathrm{e}^{-\xi}\right) + ct^{-\kappa/6} \\
& \leq \mathbb{E}\left(J_{h-\xi}(\mathrm{e}^{-\xi}t/M)\right) + c't^{-\kappa/6},
\end{aligned} \tag{19}
$$

the second inequality being a consequence of $\mathbb{P}(\tilde{\mathcal{I}}_h^{(t)}\backslash\mathcal{I}_h^{(t)} \mid \tilde{\mathcal{I}}_h^{(t)}) \leq ct^{-\kappa/3}$ and M the random variable defined in (6) and independent of V''. Finally, considering the choice made for $h(t)$ (cf (16)), we have

$$
\begin{aligned}
t^{-\frac{\kappa\wedge 1}{6}}\mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t)) &= o(t^{-\kappa}), \\
\mathbb{P}(\mathcal{A}^c)\mathbb{P}^{Q^{\mathcal{I}}}(H \geq h(t)) &\leq ct^{-\kappa}\sqrt{\mathbb{P}(\mathcal{A}^c)} = o(t^{-\kappa}).
\end{aligned}
$$

Putting everything together (i.e., the estimates (11), (14), (17), (18), (19))

$$
\begin{aligned}
I(t) &\leq \mathbb{P}(H \geq h(t))I_h(t) + o(t^{-\kappa}) \\
&\leq \mathbb{P}(H \geq h(t))(\tilde{I}_h(\mathrm{e}^{-2\xi}t) + ct^{-\frac{\kappa\wedge 1}{6}}) + o(t^{-\kappa}) \\
&\leq \mathbb{P}(H \geq h(t))(\mathrm{e}^{3\kappa\xi}\mathbb{E}(J_{h-\xi}(\mathrm{e}^{-3\xi}t/M)) + c\mathbb{P}(\mathcal{A}^c)) + o(t^{-\kappa}) \\
&\leq \mathrm{e}^{3\kappa\xi}\mathbb{P}\left(RM \geq t\mathrm{e}^{-3\xi}, \ H \geq h(t) - \xi\right) + o(t^{-\kappa}),
\end{aligned}
$$

where R and M are independent processes with laws defined in Sect. 2 (indeed, in the last inequality, $\mathbb{P}(H \geq h(t))\mathbb{P}(\mathcal{A}^c) \leq \sqrt{\mathbb{P}(\mathcal{A}^c)}t^{-\kappa} = o(t^{-\kappa})$). Now, proceeding exactly as in Corollary 4.2, we see that

$$
\mathbb{P}(RM \geq t, \ H < h(t) - \xi) = o(t^{-\kappa}),
$$

(indeed, the only difference is that M_1 is replaced by M and that M and R are independent). Finally, we proved that

$$
I(t) \leq \mathrm{e}^{3\kappa\xi}\mathbb{E}(J(\mathrm{e}^{-3\xi}t/M)) + o(t^{-\kappa}).
$$

The lower estimate is similar. We first have, since the set $\{e_p - K' + K'', \ Y_{e_p} \ge h(t)\}$ includes the set of strictly increasing ladder times of Y'' larger than $h(t) + \xi$:

$$\tilde{I}_h(t) \ge \frac{e^{-\kappa\xi}}{\mathcal{Z}_h(t)} \mathbb{E}\left(\mathbf{1}_{\mathcal{A}}\left(\sum_{p=0}^{\infty} e^{-\kappa Y''_{e_p}} \mathbf{1}_{Y''_{e_p} \ge h(t)+\xi} \mathbf{1}_{\tilde{M}_1(W,Y')\tilde{M}_2(Y'',W',e_p'') \exp(Y''_{e_p}) \ge te^{\xi}}\right)\right).$$

Hence, by the same argument as above

$$\tilde{I}_h^{(t)} \ge e^{-3\kappa\xi} \mathbb{P}\left(e^{S''}\tilde{M}_1(W,Y')\tilde{M}_2'' \ge e^{\xi}t\right) + c\mathbb{P}(\mathcal{A}^c),$$

where S'' and $\tilde{M}_2''$ are relative to a process V'' independent of W and Y' and with law $Q(\cdot \mid \tilde{I}_{h+\xi}^{(t)})$. Using, now the fact that $Y_k' > 0$ for all $k > 0$ with probability at least $1 - ct^{-\kappa/3}$ and the fact that $\tilde{M}_2 \ge e^{-\xi} M_2$ with probability at least $1 - ct^{-\kappa/6}$, and the estimate on the tail of the sum $\sum e^{-Y_k'}$ (of Sect. 3.1) we see that

$$\tilde{I}_h^{(t)} \ge e^{-3\kappa\xi} \mathbb{P}\left(Me^{S''}M_2'' \ge e^{3\xi}t\right) + o(t^{-\kappa/6}) + c\mathbb{P}(\mathcal{A}^c),$$

where M is the random variable defined in (6) and independent of V''. Then, we conclude as previously.

To prove the estimate on $J(t)$ and $K(t)$ we proceed exactly in the same way: we first remark that by the property of time reversal (see Lemma 3.4), we have

$$J(t) = \mathbb{P}^{Q^{\mathcal{I}}}(e^H M_1 \ge t).$$

The situation is then even simpler, we just have to decouple M_1 and e^H. $\qquad\square$

Proof of Lemma 5.2 Denote by $F_{y',y''}(u)$ the probability that Z' and Z'' couple before the level $\frac{1}{3}\log t + u$ knowing that $Y'_{N_0'} = \frac{1}{3}\log t + y'$ and $Y''_{N_0''} = \frac{1}{3}\log t + y''$. By the arguments above, $F_{y',y''}(u)$ tends to 1 when u tends to infinity. Let $A > 0$; we first prove that this convergence is uniform in y', y'' on the compact set $y' \le A$, $y'' \le A$. For this we consider the set $\mathcal{S} = (\mathbb{N} \cdot \frac{\xi}{4}) \cap [0, A]$, and for y', y'' in $\mathcal{S} \times \mathcal{S}$ the function $\widehat{F}_{y',y''}(u)$, the probability that Z' and Z'', starting from the points $Y'_{N_0'} = \frac{1}{3}\log t + y'$ and $Y''_{N_0''} = \frac{1}{3}\log t + y''$, couple at a distance $\xi/2$, before the level $\frac{1}{3}\log t + u - \xi$. Let

$$\phi(u) = \inf_{y'\in\mathcal{S},\, y''\in\mathcal{S}} \widehat{F}_{y',y''}(u).$$

Clearly $\phi(u) \to 1$ when $u \to \infty$ and $F_{y',y''}(u) \ge \phi(u)$, whenever y' and y'' are in $[0, A]$. This implies that

$$\liminf_{t\to\infty} \mathbb{P}(\mathcal{A}) \ge \liminf_{A\to\infty} \liminf_{t\to\infty} \left(\mathbb{P}\left(Y'_{N_0'} - \frac{1}{3}\log t \le A\right)\right)^2.$$

Moreover, $\mathbb{P}^{\tilde{Q}}(Y'_k > 0, \, 0 < k \le T_{\frac{1}{3}\log t}) \ge \mathbb{P}^{\tilde{Q}}(Y'_k > 0, \, k \ge 0) > 0$ implies

$$\mathbb{P}\left(Y'_{N'_0} - \frac{1}{3}\log t \ge A\right) = \mathbb{P}^{\tilde{Q}}\left(V_{T_{\frac{1}{3}\log t}} - \frac{1}{3}\log t \ge A \mid V > 0\right)$$

$$\le c\mathbb{P}^{\tilde{Q}}\left(V_{T_{\frac{1}{3}\log t}} - \frac{1}{3}\log t \ge A\right),$$

where here V is the canonical process under $\tilde{Q}$. Therefore, since $V_{T_{\frac{1}{3}\log t}} - \frac{1}{3}\log t$ converges in law (under $\tilde{Q}$) to a finite random variable when t tends to infinity (see limit Theorem (4.10), p. 370 in [6] or Example 4.4 part II, p. 214 in [4]), this yields $\liminf_{t\to\infty}\mathbb{P}(\mathcal{A}) = 1$. $\qquad\square$

6 Proof of Theorem 2.1, Theorem 2.2 and Theorem 2.3

Proof of Theorem 2.1, (ii) *and Theorem* 2.2 Let $\xi > 0$. By Proposition 5.1, we have, for all $A > 0$ and for t large enough,

$$J(t) \le \mathrm{e}^{3\xi}\left(\mathbb{E}(K(\mathrm{e}^{-2\xi}tM^{-1})\mathbf{1}_{M\le A}) + \mathbb{E}(K(\mathrm{e}^{-2\xi}tM^{-1})\mathbf{1}_{M>A})\right).$$

On the first term, for t large enough, we can bound $K\left(\mathrm{e}^{-2\xi}tM^{-1}\right)$ from above by $\left(\frac{C_I}{\mathbb{P}^Q(H=S)} + \xi\right)(t\mathrm{e}^{-2\xi}M^{-1})^{-\kappa}$. For the second term we can use a uniform bound $K(t) \le ct^{-\kappa}$. Thus we get

$$J(t) \le \mathrm{e}^{3(1+\kappa)\xi}\left(\frac{C_I}{\mathbb{P}^Q(H=S)} + \xi\right)t^{-\kappa}(\mathbb{E}(M^{\kappa})\mathbf{1}_{M\le A}) + ct^{-\kappa}\mathbb{E}(M^{\kappa}\mathbf{1}_{M>A})).$$

Since M^{κ} is integrable, letting A tend to ∞, then ξ tend to 0, we get the upper bound

$$\limsup_{t\to\infty} t^{\kappa}J(t) \le \frac{C_{KI}}{\mathbb{P}^Q(H=S)}.$$

For the lower bound it is the same. The proof of Theorem 2.2 is the same: we use the estimate (i) of Proposition 5.1 and the tail estimate for J. $\qquad\square$

Proof of Theorem 2.1 (i) Let us first recall (5) and Theorem 2.1, (ii), which tells us that

$$Q(R > t; \, H = S) = \frac{C_{KI}}{t^{\kappa}} + o(t^{-\kappa}), \qquad t \to \infty, \qquad (20)$$

where $C_{KI} = C_I\mathbb{E}(M^{\kappa})$. Then, introducing

$$KI := \sum_{0\le k\le T_{\mathbb{R}_-}} \mathrm{e}^{V_k}, \qquad O_1 := -V_{T_{\mathbb{R}_-}},$$

Theorem 2.1, (i) is a consequence of Theorem 2.1, (ii) together with the two following lemmas.

Lemma 6.1 *We have*

$$Q(KI > t) = \frac{C_{KI}}{t^\kappa} + o(t^{-\kappa}), \quad t \to \infty. \tag{21}$$

Proof Firstly, observe that $KI \leq R$ implies $Q(KI > t; H = S) \leq Q(R > t; H = S)$. Moreover, Corollary 4.2 implies $Q(KI > t; e^H = e^S \leq t^{2/3}) = o(t^{-\kappa})$, $t \to \infty$, since $KI \leq R$. Furthermore, we have $0 \leq Q(KI > t; e^H > t^{2/3}) - Q(KI > t; e^H = e^S > t^{2/3}) \leq Q(H \neq S; e^H > t^{2/3}) = o(t^{-\kappa})$, $t \to \infty$. Therefore, we obtain, when $t \to \infty$,

$$Q(R > t; H = S) \geq Q(KI > t; e^H > t^{2/3}) + o(t^{-\kappa}). \tag{22}$$

Since, by Corollary 4.2, $Q(KI > t; e^H \leq t^{2/3}) = o(t^{-\kappa})$, $t \to \infty$, we get

$$Q(KI > t; e^H > t^{2/3}) = Q(KI > t) + o(t^{-\kappa}), \tag{23}$$

when $t \to \infty$. Then, assembling (22) and (23) yields

$$Q(R > t; H = S) \geq Q(KI > t) + o(t^{-\kappa}), \quad t \to \infty. \tag{24}$$

On the other hand, observe that Corollary 4.2 implies that $Q(R > t; H = S) = Q(R > t; e^H = e^S > t^{2/3}) + o(t^{-\kappa})$, $t \to \infty$. Moreover, since we have $R = KI + e^{O_1}R'$, with R' a random variable independent of KI and O_1, having the same law as R, we obtain that $Q(R > t; e^H = e^S > t^{2/3}) \leq Q_1 + Q_2$, where

$$Q_1 := Q(KI \leq t - t^{2/3}; R' > t^{2/3}; e^H > t^{2/3}),$$

$$Q_2 := Q(KI > t - t^{2/3}; R > t; e^H = e^S > t^{2/3}).$$

Now, since R' and H are independent, we get $Q_1 \leq Q(e^H > t^{2/3})Q(R' > t^{2/3}) = o(t^{-\kappa})$, $t \to \infty$. Moreover, we easily have $Q_2 \leq Q(KI > t - t^{2/3})$. Therefore

$$Q(R > t; H = S) \leq Q(KI > t - t^{2/3}) + o(t^{-\kappa}), \quad t \to \infty. \tag{25}$$

Recalling (20) and assembling (24) and (25) concludes the proof of Lemma 6.1. □

Lemma 6.2 C_{KI} *satisfies*

$$C_{KI} = (1 - \mathbb{E}^Q(e^{-\kappa O_1}))C_K.$$

Proof First, observe that $Q(R > t) = Q(KI > t) + P_1 + P_2$, where

$$P_1 := Q(KI + e^{-O_1}R' > t; t^{1/2} < KI \leq t),$$

$$P_2 := Q(KI + e^{-O_1} R' > t; \ KI \le t^{1/2}),$$

with R' a random variable independent of KI and O_1, with the same law as R.

Now, let us prove that P_1 is negligible. Observe first that, since $O_1 \ge 0$ by definition, we have $P_1 \le Q(R' > t - KI; \ t^{1/2} < KI \le t)$. Therefore $0 \le P_1 \le P_1' + P_1''$, where

$$P_1' := Q(R' > t - KI; \ t - t^{2/3} < KI \le t),$$
$$P_1'' := Q(R' > t - KI; \ t^{1/2} < KI \le t - t^{2/3}).$$

Since R' and KI are independent, (5) and (21) yield $P_1'' \le Q(R' > t^{2/3}) Q(KI > t^{1/2}) = o(t^\kappa)$, $t \to \infty$. Furthermore, we have

$$\begin{aligned}
P_1' &\le Q(t - t^{2/3} < KI \le t) \\
&\le Q(KI > t - t^{2/3}) - Q(KI > t) \\
&= Q(KI > t) \left(\frac{Q(KI > t - t^{2/3})}{Q(KI > t)} - 1 \right).
\end{aligned}$$

Therefore (21) implies $P_1' = o(t^{-\kappa})$, $t \to \infty$. Then, we obtain $P_1 = o(t^{-\kappa})$, $t \to \infty$.

Now, let us estimate P_2. Observe that $\underline{P}_2 \le P_2 \le \overline{P}_2$, where

$$\underline{P}_2 := Q(e^{-O_1} R' > t; \ KI \le t^{1/2}),$$
$$\overline{P}_2 := Q(e^{-O_1} R' > t - t^{1/2}).$$

Since R' and O_1 are independent (5) yields

$$\overline{P}_2 = \frac{\mathbb{E}^Q(e^{-\kappa O_1}) C_K}{t^\kappa} + o(t^{-\kappa}), \qquad t \to \infty. \tag{26}$$

Therefore, it only remains to estimate $\underline{P}_2$. Since R' is independent of KI and O_1, we obtain for any $\varepsilon > 0$ and t large enough,

$$(1 - \varepsilon) C_K \mathbb{E}^Q \left(\mathbf{1}_{\{KI \le t^{1/2}\}} \frac{e^{-\kappa O_1}}{t^\kappa} \right) \le \underline{P}_2 \le (1 + \varepsilon) C_K \mathbb{E}^Q \left(\mathbf{1}_{\{KI \le t^{1/2}\}} \frac{e^{-\kappa O_1}}{t^\kappa} \right).$$

Moreover,

$$\mathbb{E}^Q \left(\mathbf{1}_{\{KI \le t^{1/2}\}} \frac{e^{-\kappa O_1}}{t^\kappa} \right) = \frac{\mathbb{E}^Q(e^{-\kappa O_1})}{t^\kappa} - \mathbb{E}^Q \left(\mathbf{1}_{\{KI > t^{1/2}\}} \frac{e^{-\kappa O_1}}{t^\kappa} \right),$$

and the second term on the right-hand side is less or equal than $t^{-\kappa} Q(KI > t^{1/2}) = o(t^{-\kappa})$, $t \to \infty$. Thus

$$\underline{P}_2 = \frac{\mathbb{E}^Q(e^{-\kappa O_1}) C_K}{t^\kappa} + o(t^{-\kappa}), \qquad t \to \infty. \tag{27}$$

Assembling (26) and (27) yields $P_2 = \frac{\mathbb{E}^Q(e^{-\kappa O_1})C_K}{t^\kappa} + o(t^{-\kappa})$, $t \to \infty$. Therefore, recalling (5), (21) and $Q(R > t) = Q(KI > t) + P_1 + P_2$, we obtain $C_{KI} = (1 - \mathbb{E}^Q(e^{-\kappa O_1}))C_K$, which concludes the proof of Lemma 6.2. $\qquad\square$

Since Theorem 2.1, (ii) together with Lemma 6.1 and Lemma 6.2 yield $C_{KI} = C_I \mathbb{E}^Q[M^\kappa] = (1 - \mathbb{E}^Q[e^{-\kappa O_1}])C_K$, we get $C_K = C_I \mathbb{E}^Q[M^\kappa](1 - \mathbb{E}^Q[e^{-\kappa O_1}])^{-1}$. Now, recalling that $C_I = (1 - \mathbb{E}^Q[e^{-\kappa O_1}])C_F$, this concludes the proof of Theorem 2.1, (i). $\qquad\square$

Proof of Theorem 2.3 The proof of Theorem 2.3 is based on the same arguments as in the proof of Theorem 2.1, (i). We mainly have to check analogous statements to Lemma 4.1 and Corollary 4.2. Namely, we check that there exists $c > 0$ such that

$$\mathbb{E}^{Q^{\mathcal{I}}}((M_1^B)^{\kappa+\frac{\varepsilon}{2}} \mid \lfloor H \rfloor) \leq c, \quad Q^{\mathcal{I}}\text{- a.s.},$$

where $M_1^B := \sum_{k=-\infty}^{T_S} e^{-V_k} \tilde{B}_k$. Using the Hölder inequality instead of the Cauchy–Schwarz inequality in the proof of Lemma 4.1 we are led to check the integrability of $(M^B)^{\kappa+\varepsilon}$. This is used in the proof of

$$\mathbb{P}^{Q^{\mathcal{I}}}\left(R^B \geq t, \ H \leq h(t)\right) = o(t^{-\kappa}),$$

when t tends to infinity, which is analogous to the proof of Corollary 4.2 (in its R version), choosing $\eta = \kappa + \frac{\varepsilon}{2}$.

Now, it only remains to check the integrability of $(M^B)^{\kappa+\varepsilon}$. To this aim, we prove that $\mathbb{E}^{\tilde{Q}^{>0}}((\sum_{k\geq 0} e^{-V_k} \tilde{B}_k)^{\kappa+\varepsilon}) < \infty$, the case of $\mathbb{E}^{Q^{>0}}((\sum_{k<0} e^{-V_k} \tilde{B}_k)^{\kappa+\varepsilon})$ being similar.

If $\kappa \geq 1$, the Minkowski inequality yields

$$\mathbb{E}^{\tilde{Q}^{>0}}\left(\left(\sum_{k\geq 0} e^{-V_k} \tilde{B}_k\right)^{\kappa+\varepsilon}\right) \leq \left(\sum_{k\geq 0} \mathbb{E}^{\tilde{Q}^{>0}}\left((e^{-V_k} \tilde{B}_k)^{\kappa+\varepsilon}\right)^{\frac{1}{\kappa+\varepsilon}}\right)^{\kappa+\varepsilon}$$

$$\leq c \left(\sum_{k\geq 0} \mathbb{E}^{\tilde{Q}^{>0}}(e^{-(\kappa+\varepsilon)V_k})^{\frac{1}{\kappa+\varepsilon}}\right)^{\kappa+\varepsilon}$$

$$\leq c \left(\sum_{k\geq 0} \mathbb{E}^{\tilde{Q}^{>0}}(e^{-V_k})^{\frac{1}{\kappa+\varepsilon}}\right)^{\kappa+\varepsilon}, \qquad (28)$$

the second inequality being a consequence of the independence between $(\tilde{B}_i)_{i\geq 0}$ and $(V_i)_{i\geq 0}$, while the third inequality is due to the fact that $V_i \geq 0$ for $i \geq 0$ under $\tilde{Q}^{>0}$ together with $\kappa + \varepsilon \geq 1$. Choosing p such that $p/(\kappa + \varepsilon) > 1$, let us write

$$\mathbb{E}^{\tilde{Q}^{>0}}(e^{-V_k}) \leq \frac{1}{k^p} + \mathbb{P}^{\tilde{Q}^{>0}}(e^{-V_k} \geq k^{-p}).$$

Now, as in the proof of Lemma 3.1, since large deviations do occur, we get from Cramer's theory, see [2], that the sequence $(\mathbb{P}^{\tilde{Q}^{>0}}(\mathrm{e}^{-V_k} \geq k^{-p}))_{k \geq 1}$ is exponentially decreasing. This yields that the sum in (28) is finite.

If $\kappa < 1$, observe that we can restrict our attention to the case where $\kappa + \varepsilon < 1$. Then, let us write

$$\mathbb{E}^{\tilde{Q}^{>0}} \left(\left(\sum_{k \geq 0} \mathrm{e}^{-V_k} \tilde{B}_k \right)^{\kappa+\varepsilon} \right) \leq \mathbb{E}^{\tilde{Q}^{>0}} \left(\sum_{k \geq 0} (\mathrm{e}^{-V_k} \tilde{B}_k)^{\kappa+\varepsilon} \right)$$
$$\leq c \sum_{k \geq 0} \mathbb{E}^{\tilde{Q}^{>0}} (\mathrm{e}^{-(\kappa+\varepsilon) V_k}),$$

the second inequality being a consequence of the independence between $(\tilde{B}_i)_{i \geq 0}$ and $(V_i)_{i \geq 0}$. Now, the conclusion is the same as in the case $\kappa \geq 1$. $\qquad \square$

Appendix A

A.1 Preliminaries' proofs

We give here the proofs of the claims from Sect. 3.

Proof of Lemma 3.1 Using the Markov inequality, we get

$$\mathbb{E}^{\tilde{Q}} \left(\sum_{k \geq 0} \mathrm{e}^{-V_k} \mid V \geq -L \right) \leq 1 + \sum_{k \geq 1} \frac{1}{k^2} + \sum_{k \geq 1} \tilde{Q} \left(\mathrm{e}^{-V_k} \geq \frac{1}{k^2} \mid V \geq -L \right) \mathrm{e}^L.$$

Since $\mathbb{P}^{\tilde{Q}}(V \geq -L) \geq \mathbb{P}^{\tilde{Q}}(V \geq 0) > 0$, for all $L \geq 0$,

$$\tilde{Q} \left(\mathrm{e}^{-V_k} \geq \frac{1}{k^2} \mid V \geq -L \right) = \tilde{Q}(V_k \leq 2 \log k \mid V \geq -L) \leq c \, \tilde{Q}(V_k \leq 2 \log k).$$

Now, since large deviations do occur, we get, from Cramer's theory, see [2], that $\mathbb{E}^{\tilde{Q}}(\log \rho_0) > 0$ implies that the sequence $\tilde{Q}(V_k \leq 2 \log k)$ is exponentially decreasing.

The sum $\sum_{k \geq 1} \tilde{Q} \left(\mathrm{e}^{-V_k} \geq \frac{1}{k^2} \mid V \geq -L \right)$ is therefore bounded uniformly in L, and the result follows. $\qquad \square$

Proof of Lemma 3.2 Let us treat, for more readability, the case of the second moment. Observe first that

$$
\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\left(\sum_{i\geq 0}\mathrm{e}^{-V_i}\right)^2\right) \leq 2\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i\geq 0}\mathrm{e}^{-2V_i}\left(\sum_{j\geq i}\mathrm{e}^{-(V_j-V_i)}\right)\right).
$$

Applying the Markov property to the process V under $\tilde{Q}$ at time i, we get

$$
\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\left(\sum_{i\geq 0}\mathrm{e}^{-V_i}\right)^2\right) \leq 2\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i\geq 0}\mathrm{e}^{-2V_i}\mathbb{E}^{\tilde{Q}}\left[\sum_{l\geq 0}\mathrm{e}^{-V_l'}\mid V'\geq -V_i\right]\right),
$$

where V' is a copy of V independent of $(V_k)_{0\leq k\leq i}$. Now, we use Lemma 3.1 to get the upper bound

$$
c\,\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i\geq 0}\mathrm{e}^{-2V_i}\times\mathrm{e}^{V_i}\right) \leq c\,\mathbb{E}^{\tilde{Q}^{\geq 0}}\left(\sum_{i\geq 0}\mathrm{e}^{-V_i}\right),
$$

which is finite, again by applying Lemma 3.1. This scheme is then easily extended to higher moments. □

Proof of Lemma 3.3 Let $\alpha \in [0,1]$ and define $T_{(-\infty,-\alpha L]} := \min\{i\geq 0 : V_i\leq -\alpha L\}$. Let us write

$$
\sum_{i\geq 0}\mathrm{e}^{-V_i} = \left(\sum_{i\geq 0}\mathrm{e}^{-V_i}\right)\mathbf{1}_{\{V>-\alpha L\}}
$$
$$
+ \left(\sum_{i=0}^{T_{(-\infty,-\alpha L]}-1}\mathrm{e}^{-V_i} + \sum_{i=T_{(-\infty,-\alpha L]}}^{\infty}\mathrm{e}^{-V_i}\right)\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}}.
$$

Now, since $\tilde{Q}(V\geq -A)$ is uniformly bounded below, for $A>0$, by $\tilde{Q}(V>0)>0$, we obtain that $\mathbb{E}^{\tilde{Q}}\left(\sum_{i\geq 0}\mathrm{e}^{-V_i}\mid V\geq -L\right)$ is less than or equal to

$$
c\,\mathbb{E}^{\tilde{Q}}\left(\sum_{i\geq 0}\mathrm{e}^{-V_i}\mid V\geq -\alpha L\right)
$$
$$
+c\,\mathbb{E}^{\tilde{Q}}\left(\sum_{i<T_{(-\infty,-\alpha L]}}\mathrm{e}^{-V_i};\ T_{(-\infty,-\alpha L]}<\infty;\ V\geq -L\right)
$$
$$
+c\,\mathbb{E}^{\tilde{Q}}\left(\sum_{i\geq T_{(-\infty,-\alpha L]}}\mathrm{e}^{-V_i};\ T_{(-\infty,-\alpha L]}<\infty;\ V\geq -L\right). \tag{29}
$$

Lemma 3.1 bounds the first term in (29) from above by $ce^{\alpha L}$, for all $L > 0$. Furthermore, $i < T_{(-\infty,-\alpha L]}$ implies $e^{-V_i} \leq e^{\alpha L}$. Therefore, $ce^{\alpha L}\mathbb{E}^{\tilde{Q}}(T_{(-\infty,-\alpha L]} \mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}})$ is an upper bound for the second term in (29), which is treated as follows,

$$
\begin{aligned}
\mathbb{E}^{\tilde{Q}}\left(T_{(-\infty,-\alpha L]}\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}}\right) &\leq \sum_{k\geq 0} k\tilde{Q}\left(T_{(-\infty,-\alpha L]} = k\right) \\
&\leq \sum_{k\geq 0} k\tilde{Q}\left(V_k \leq -\alpha L\right) \\
&\leq \sum_{k\geq 0} k e^{-k\theta \tilde{I}(-\frac{\alpha L}{k})}e^{-k(1-\theta)\tilde{I}(-\frac{\alpha L}{k})},
\end{aligned}
$$

where $0 < \theta < 1$ and $\tilde{I}$ denotes the rate function associated with $\tilde{P}$ which is positive convex and admits a unique minimum on $\mathbb{R}_+$. We can therefore bound below all the terms $\tilde{I}(-\frac{\alpha h}{k})$ by $\tilde{I}(0) > 0$. Moreover, a more sophisticated result yields $\sup_{x\leq 0}\tilde{I}(x)/x \leq -\kappa$ (see definition of κ and formula (2.2.10) in [2, p. 28]). Therefore, we obtain

$$
\mathbb{E}^{\tilde{Q}}\left(T_{(-\infty,-\alpha L]}\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}}\right) \leq e^{-\theta\kappa\alpha L}\sum_{k\geq 0} k e^{-k(1-\theta)\tilde{I}(0)} \leq c\, e^{-\theta\kappa\alpha L}.
$$

As a result, the second term in (29) is bounded by $ce^{(1-\theta\kappa\alpha)L}$, for all $L > 0$.

Finally, concerning the third term in (29), we have that

$$
\begin{aligned}
c\,\mathbb{E}^{\tilde{Q}}&\left(\sum_{i\geq T_{(-\infty,-\alpha L]}} e^{-V_i};\; T_{(-\infty,-\alpha L]} < \infty;\; V \geq -L\right) \\
&\leq c\,\mathbb{E}^{\tilde{Q}}\left(e^{-V_{T_{(-\infty,-\alpha L]}}}\sum_{i\geq T_{(-\infty,-\alpha L]}} e^{-(V_i-V_{T_{(-\infty,-\alpha L]}})};\; T_{(-\infty,-\alpha L]} < \infty\,;\; V \geq -L\right) \\
&\leq c\,\mathbb{E}^{\tilde{Q}}\left(e^{-V_{T_{(-\infty,-\alpha L]}}}\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}}\mathbb{E}^{\tilde{Q}}\left(\sum_{i\geq 0} e^{-V_i'} \mid V' \geq -(L+V_{T_{(-\infty,-\alpha L]}})\right)\right),
\end{aligned}
$$

where $V_i' := V_{T_{(-\infty,-\alpha L]}+i} - V_{T_{(-\infty,-\alpha L]}}$, for $i \geq 0$. The last inequality is a consequence of the strong Markov property applied at $T_{(-\infty,-\alpha L]}$, which implies that $(V_i',\, i \geq 0)$ is a copy of $(V_i,\, i \geq 0)$ independent of $(V_i,\, 0 \leq i \leq T_{(-\infty,-\alpha L]})$. Then, Lemma 3.1 yields that the third term in (29) is less than

$$
\begin{aligned}
c\,\mathbb{E}^{\tilde{Q}}&\left(e^{-V_{T_{(-\infty,-\alpha L]}}}\mathbf{1}_{\{T_{(-\infty,-\alpha L]}<\infty\}}e^{L+V_{T_{(-\infty,-\alpha L]}}}\right) \\
&\leq c\,e^L\,\tilde{Q}(T_{(-\infty,-\alpha L]} < \infty) \leq c\,e^{(1-\kappa\alpha)L}.
\end{aligned}
$$

Since $\theta < 1$ implies $1 - \theta\kappa\alpha \geq 1 - \kappa\alpha$, we optimize the value of α by taking $\alpha = -\alpha\kappa\theta + 1$, i.e. $\alpha = 1/(1 + \kappa\theta)$. As a result, we get already a finer result than Lemma 3.1 with a bound $e^{\frac{L}{1+\kappa\theta}}$ instead of e^L.

Now, the strategy is to use this improved estimation instead of Lemma 3.1 and repeat the same procedure. In that way, we obtain recursively a sequence of bounds, which we denote by $ce^{u_n L}$. The first term in (29) is bounded by $ce^{\alpha u_n L}$ whereas the second and the third term are still bounded respectively by $ce^{(1-\kappa\alpha\theta)L}$ and $ce^{(1-\kappa\alpha)L}$.

Optimizing in α again, one chooses $\alpha u_n = -\alpha\kappa\theta + 1$, i.e. $\alpha = \frac{1}{u_n+\kappa\theta}$. The new exponent is therefore $u_{n+1} = \alpha u_n = \frac{u_n}{u_n+\kappa\theta}$. Thus, the sequence u_n is monotone and converges to a limit satisfying $l = \frac{l}{l+\kappa\theta}$. For $\kappa\theta \leq 1$, the limit is $l = 1 - \kappa\theta$ and for $\kappa\theta \geq 1$, the limit is 0. Since this result holds for any $0 < \theta < 1$, it concludes the proof of Lemma 3.3. $\qquad\square$

Proof of Lemma 3.4 Let ϕ be a positive test function. We have

$$
\mathbb{E}^{Q^{\mathcal{I}}}(\phi((V_{T_H} - V_{T_H-k})_{k\geq 0}))
$$

$$
= \sum_{p=0}^{\infty} \mathbb{E}^{Q^{\mathcal{I}}}(\mathbf{1}_{T_H=p}\phi((V_p - V_{p-k})_{k\geq 0}))
$$

$$
= \frac{1}{\mathbb{P}^Q(\mathcal{I})} \sum_{p=0}^{\infty} \mathbb{E}^Q(\mathbf{1}_{\{V_k\geq 0,\,\forall k\leq 0\}}\mathbf{1}_{\{V_k\leq V_p,\,\forall k\geq p\}}\mathbf{1}_{\{0<V_k<V_p,\,\forall 0<k<p\}}\phi((V_p - V_{p-k})_{k\geq 0})).
$$

By construction we have, for all $p \geq 0$,

$$
(V_p - V_{p-k})_{k\in\mathbb{Z}} \stackrel{\text{law}}{=} (V_k)_{k\in\mathbb{Z}}.
$$

This implies that

$$
\mathbb{E}^{Q^{\mathcal{I}}}(\phi((V_{T_H} - V_{T_H-k})_{k\geq 0}))
$$

$$
= \frac{1}{\mathbb{P}^Q(\mathcal{I})} \sum_{p=0}^{\infty} \mathbb{E}^Q(\mathbf{1}_{\{V_k\geq 0,\,\forall k\leq 0\}}\mathbf{1}_{\{V_k\leq V_p,\,\forall k\geq p\}}\mathbf{1}_{\{0<V_k<V_p,\,\forall 0<k<p\}}\phi((V_k)_{k\geq 0}))
$$

$$
= \mathbb{E}^{Q^{\mathcal{I}}}(\phi((V_k)_{k\in\mathbb{Z}})).
$$

$\qquad\square$

Proof of Lemma 3.5 Let ψ, θ be positive test functions. Let us compute

$$
\mathbb{E}^Q\left(\psi((V_{T_S+k} - V_{T_S})_{k\geq 0})\theta((V_0, \ldots, V_{T_S}))\right)
$$

$$
= \sum_{p=0}^{\infty} \mathbb{E}^Q\left(\mathbf{1}_{T_S=p}\psi((V_{p+k} - V_p)_{k\geq 0})\theta((V_0, \ldots, V_p))\right)
$$

$$= \sum_{p=0}^{\infty} \mathbb{E}^Q \left(\mathbf{1}_{\{V_k < V_p, \forall 0 \leq k < p\}} \mathbf{1}_{\{V_k \leq V_p, \forall k \geq p\}} \psi((V_{p+k} - V_p)_{k \geq 0}) \theta((V_0, \ldots, V_p))\right)$$

$$= \sum_{p=0}^{\infty} \mathbb{E}^Q \left(\mathbf{1}_{\{V_k < V_p, \forall k < p\}} \theta((V_0, \ldots, V_p))\right) \mathbb{E}^Q \left(\mathbf{1}_{\{V_k \leq 0, \forall k \geq 0\}} \psi((V_k)_{k \geq 0})\right),$$

using the Markov property at time p. The second term is equal to

$$\mathbb{P}^Q(V_k \leq 0, \forall k \geq 0) \, \mathbb{E}^{Q^{\leq 0}} \left(\psi((V_k)_{k \geq 0})\right).$$

Let us now consider only the first term. Using the Girsanov property of Q and $\tilde{Q}$ we get

$$\sum_{p=0}^{\infty} \mathbb{E}^Q \left(\mathbf{1}_{\{V_k < V_p, \forall k < p\}} \theta((V_0, \ldots, V_p))\right) = \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}} \left(\mathbf{1}_{\{V_k < V_p, \forall k < p\}} e^{-\kappa V_p} \theta((V_0, \ldots, V_p))\right)$$

$$= \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}} \left(e^{-\kappa V_{e_p}} \theta((V_0, \ldots, V_{e_p}))\right),$$

where $(e_p)_{p \geq 0}$ are the strictly increasing ladder times of $(V_k, \ k \geq 0)$ as defined above. The last formula is exactly the one we need, and also implies that

$$\frac{1}{\mathcal{Z}} = \mathbb{P}^Q(V_k \leq 0, \forall k \geq 0) = 1 - \mathbb{E}^{\tilde{Q}}(e^{-\kappa V_{e_1}}),$$

(which can also be obtained directly). $\qquad \square$

Proof of Lemma 3.6 Let Ψ be a positive test function. Thanks to the previous lemma, we have

$$\mathbb{E}^{Q^{\mathcal{T}}} (\Psi(V_0, \ldots, V_{T_H}))$$

$$= \frac{1}{\mathbb{P}^Q(H = S)} \mathbb{E}^Q (\mathbf{1}_{H=S} \Psi(V_0, \ldots, V_{T_H}))$$

$$= \frac{1}{\mathcal{Z} \mathbb{P}^Q(H = S)} \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}} (\mathbf{1}_{Y_k > 0, \forall 0 < k \leq e_p} e^{-\kappa Y_{e_p}} \Psi(Y_0, \ldots, Y_{e_p}))$$

$$= \frac{1}{\mathcal{Z} \mathbb{P}^Q(H=S)} \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}} (\mathbf{1}_{Y_k > 0, \forall k > 0} \frac{1}{\mathbb{P}^{\tilde{Q}}(V_k > -Y_{e_p}, \forall k > 0)} e^{-\kappa Y_{e_p}} \Psi(Y_0, \ldots, Y_{e_p}))$$

$$\leq \frac{1}{\mathcal{Z} \mathbb{P}^Q(H = S) \mathbb{P}^{\tilde{Q}}(V_k > 0, \forall k > 0)} \sum_{p=0}^{\infty} \mathbb{E}^{\tilde{Q}} (\mathbf{1}_{Y_k > 0, \forall k > 0} e^{-\kappa Y_{e_p}} \Psi(Y_0, \ldots, Y_{e_p}))$$

$$\leq \frac{1}{\mathbb{P}^Q(H = S) \mathbb{P}^{\tilde{Q}}(V_k > 0, \forall k > 0)} \mathbb{E}^{\hat{Q}^{>0}} (\Psi(Y_0, \ldots, Y_{\Theta})),$$

using the Markov property at time e_p in the fourth line. This is exactly what we want.
$\qquad \square$

A.2 A Tauberian result

Corollary A.1 *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be such that*

$$\lim_{\lambda \to 0} \lambda e^{h(\lambda)} = 0, \quad \lim_{\lambda \to 0} h(\lambda) = \infty.$$

Then, for $\kappa < 1$,

$$\mathbb{E}^Q \left(1 - \frac{1}{1 + \lambda Z} \mid \mathcal{I}_h^{(\lambda)} \right) \sim \frac{1}{\mathbb{P}^Q (H \geq h(\lambda))} \frac{\pi \kappa}{\sin(\pi \kappa)} C_U \lambda^\kappa,$$

when $\lambda \to 0$, where $\mathcal{I}_h^{(\lambda)}$ is the event

$$\mathcal{I}_h^{(\lambda)} = \mathcal{I} \cap \{H \geq h(\lambda)\} = \{V_k \geq 0, \, \forall k \leq 0\} \cap \{H = S \geq h(\lambda)\}.$$

Proof Clearly, we have

$$\mathbb{E}^Q \left(1 - \frac{1}{1 + \lambda Z} \mid \mathcal{I}_h^{(\lambda)} \right) = \frac{\mathbb{P}^Q (H = S)}{\mathbb{P}^Q (H = S \geq h(\lambda))} \mathbb{E}^{Q^{\mathcal{I}}} \left(\mathbf{1}_{H \geq h(\lambda)} \left(1 - \frac{1}{1 + \lambda Z} \right) \right).$$

Since $\mathbb{P}^Q (H = S \geq h(\lambda)) \sim \mathbb{P}^Q (H \geq h(\lambda))$ we consider now

$$\mathbb{E}^{Q^{\mathcal{I}}} \left(\mathbf{1}_{H \geq h(\lambda)} \left(1 - \frac{1}{1 + \lambda Z} \right) \right)$$

We will omit in the following the reference to the law $Q^{\mathcal{I}}$, and simply write $\mathbb{E}$ for the expectation with respect to $Q^{\mathcal{I}}$. We have

$$\mathbb{E} \left(\mathbf{1}_{H \geq h(\lambda)} \left(1 - \frac{1}{1 + \lambda Z} \right) \right)$$
$$= \mathbb{E} \left(\mathbf{1}_{Z \geq e^{h(\lambda)}} \left(1 - \frac{1}{1 + \lambda Z} \right) \right) - \mathbb{E} \left(\mathbf{1}_{e^H < e^{h(\lambda)} \leq Z} \left(1 - \frac{1}{1 + \lambda Z} \right) \right). \quad (30)$$

For $\kappa < 1$, the second term can be bounded by

$$\mathbb{E} \left(\mathbf{1}_{e^H < e^{h(\lambda)} \leq Z} \left(1 - \frac{1}{1 + \lambda Z} \right) \right) \leq \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathbb{E} \left(\mathbf{1}_{\lfloor H \rfloor = p} \frac{\lambda Z}{1 + \lambda Z} \right)$$
$$= \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathbb{E} \left(\mathbf{1}_{\lfloor H \rfloor = p} \mathbb{E} \left(\frac{\lambda Z}{1 + \lambda Z} \mid \lfloor H \rfloor = p \right) \right)$$
$$\leq \sum_{p=0}^{\lfloor h(\lambda) \rfloor} \mathbb{E} \left(\mathbf{1}_{\lfloor H \rfloor = p} \frac{c \lambda e^p}{1 + c \lambda e^p} \right),$$

where, in the last inequality, we used the Jensen inequality and Corollary 4.1, and where c denotes a constant independent of λ (which may change from line to line). Now, since $\mathbb{P}(\lfloor H \rfloor = p) \leq c e^{-\kappa p}$ for a positive constant c, we get that

$$\mathbb{E}\left(\mathbf{1}_{e^H < e^{h(\lambda)} \leq Z}\left(1 - \frac{1}{1 + \lambda Z}\right)\right) \leq c\lambda \sum_{p=0}^{\lfloor h(\lambda) \rfloor} e^{(1-\kappa)p} \leq c'\lambda e^{(1-\kappa))h(\lambda)}$$

$$\leq c\lambda^{\kappa}(\lambda e^{h(\lambda)})^{1-\kappa} = o(\lambda^{\kappa}),$$

for $\kappa < 1$, since $\lambda e^{h(\lambda)} \to 0$, $\lambda \to 0$.

By integration by parts, we see that the first term of (30) is equal to

$$\mathbb{E}\left(\mathbf{1}_{Z \geq h(\lambda)}\left(1 - \frac{1}{1 + \lambda Z}\right)\right)$$

$$= \left[\frac{\lambda z}{1 + \lambda z}\mathbb{P}(Z \geq z)\right]_{e^{h(\lambda)}}^{\infty} + \int_{e^{h(\lambda)}}^{\infty} \frac{\lambda}{(1 + \lambda z)^2}\mathbb{P}(Z \geq z)\,\mathrm{d}z.$$

The first term is lower than

$$c\lambda e^{(1-\kappa)h(\lambda)} = c\lambda^{\kappa}(\lambda e^{h(\lambda)})^{1-\kappa} = o(\lambda^{\kappa}),$$

for $\kappa < 1$. For the second term, let us suppose first that

$$h(\lambda) \to \infty.$$

We can estimate $\mathbb{P}(Z \geq z)$ by

$$\left(\frac{C_U}{\mathbb{P}^Q(H = S)} - \eta\right)z^{-\kappa} \leq \mathbb{P}(Z \geq z) \leq \left(\frac{C_U}{\mathbb{P}^Q(H = S)} + \eta\right)z^{-\kappa},$$

for any η, when λ is sufficiently small. Hence we are led to compute the integral

$$\int_{e^{h(\lambda)}}^{\infty} \frac{\lambda}{1 + \lambda z}z^{-\kappa}\,\mathrm{d}z = \lambda^{\kappa} \int_{\frac{\lambda e^{h(\lambda)}}{1 + \lambda e^{h(\lambda)}}}^{1} x^{-\kappa}(1 - x)^{\kappa}\,\mathrm{d}x,$$

(making the change of variables $x = \lambda z/(1 + \lambda z)$). For $\kappa < 1$ this integral converges, when $\lambda \to 0$, to

$$\Gamma(\kappa + 1)\Gamma(-\kappa + 1) = \frac{\pi\kappa}{\sin(\pi\kappa)}.$$

$\square$

Remark A.1 Let us make a final remark useful for [5]. If we truncate the series M_1 on the right and on the left when V_k reaches the level $A > 0$, and if we truncate M_2 when $H - V_k$ reaches the level A then the results of Theorem 2.2 and Corollary A.1 remain valid just by replacing in the tail estimate M by the the random walk M truncated at level A. More precisely, let $A > 0$ and consider

$$\overline{M}_1 = \sum_{k=t_1^-}^{t_1^+} e^{-V_k}, \quad \overline{M}_2 = \sum_{k=t_2^-}^{t_2^+} e^{V_k - H}$$

where

$$t_1^- = \sup\{k \le 0, \ V_k \ge A\}, \quad t_1^+ = \inf\{k \ge 0, \ V_k \ge A\} \wedge T_H$$
$$t_2^- = \sup\{k \le T_H, \ H - V_k \ge A\} \vee 0, \quad t_2^+ = \inf\{k \ge T_H, \ V_k \ge A\}$$

then the results of Theorem 2.2 and Corollary A.1 remain valid when we consider $\overline{Z} = e^H \overline{M}_1 \overline{M}_2$ instead of Z, if we replace in the tail estimate M by $\overline{M} = \sum_{t_-}^{t_+} e^{-V_k}$ where t_- and t_+ are the hitting times of the level A on the left and on the right. Indeed, in the proof of Theorem 2.2 we see that considering the truncated $\overline{M}_1$ and $\overline{M}_2$ only simplifies the proof: we do not need to truncate M_1 and M_2 as we did. In particular, it implies that in Corollary A.1 we can truncate M_1 and M_2 at a level $\underline{h}(\lambda) \le h(\lambda)$: if $\underline{h}(\lambda)$ tends to ∞, we have exactly the same result.

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