Probab. Theory Relat. Fields (2009) 144:79–98
DOI 10.1007/s00440-008-0142-1

Small time two-sided LIL behavior for Lévy processes at zero

Mladen Savov

Received: 1 October 2007 / Revised: 25 January 2008 / Published online: 27 March 2008

Abstract We wish to characterize when a Lévy process X_t crosses boundaries $b(t)$, in a two-sided sense, for small times t, where $b(t)$ satisfies very mild conditions. An integral test is furnished for computing the value of $\limsup_{t\to 0} |X_t|/b(t) = c$. In some cases, we also specify a function $b(t)$ in terms of the Lévy triplet, such that $\limsup_{t\to 0} |X_t|/b(t) = 1$.

Keywords Lévy process · LIL behavior · Norming functions

1 Introduction

Let $X = (X_t, t \geq 0)$ be a Lévy process with characteristic triplet (γ, σ, Π), where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and the Lévy measure Π has the property

$$\int (1 \wedge x^2)\Pi(dx) < \infty.$$

For basic definitions and properties of Lévy processes we refer to [1] and [13].

In this paper we are only interested in the behaviour of X_t, as $t \downarrow 0$. Therefore we can truncate the jumps of the Lévy process, i.e. those with absolute value exceeding 1

M. Savov (✉)
School of Mathematics, University of Manchester, Alan Turing Building,
M13 9PL Manchester, UK
e-mail: Mladen.S.Savov@postgrad.manchester.ac.uk

and deduce that the characteristic exponent has the form

$$\psi(\theta) = i\gamma\theta + \frac{\sigma^2\theta^2}{2} + \int_{-1}^{1} (1 - e^{i\theta x} + i\theta x)\Pi(dx), \tag{1.1}$$

for $\theta \in \mathbb{R}$.

In this paper we are going to study the possible limiting values taken by $|X_t|/b(t)$, as $t \downarrow 0$, where $b(t)$ satisfies some very mild conditions, see (2.1) and (2.2). More precisely, we are interested in deriving an integral criterion, which specifies the quantity $\limsup_{t\to 0} |X_t|/b(t)$ for a given function $b(t)$ and any Lévy process X. An application of our integral criterion allows us, in some cases, to specify a function $b(t)$ in terms of the Lévy triplet, such that $\limsup_{t\to 0} |X_t|/b(t) = 1$ a.s.

A fundamental question, both for random walks and Lévy processes, is to determine their maximal rate of growth, or namely to find a norming function $b(t)$ $(b(n))$, such that

$$\limsup \frac{|X_t|}{b(t)} = 1 \text{ a.s.} \left(\limsup_{n\to\infty} \frac{|S_n|}{b(n)} = 1 \text{ a.s.} \right), \tag{1.2}$$

where for Lévy processes the "lim sup" can be taken both at 0 or infinity. Thus, for example, when $EX_1^2 < \infty$ and $EX_1 = 0$, (1.2) holds at infinity with $b(t) = \sqrt{2EX_1^2 t \ln\ln t} \left(b(n) = \sqrt{2EX_1^2 n \ln\ln n} \right)$, while a result by Khintchine, see [9], gives

$$\limsup_{t\to 0} \frac{X_t}{b(t)} = \limsup_{t\to 0} \frac{-X_t}{b(t)} = \sigma \text{ a.s.}, \tag{1.3}$$

with $b(t) = \sqrt{2t \ln|\ln t|}$, where σ is the coefficient of the Brownian component in (1.1). Since, historically, these were the first results in this vein, we use the terminology "two-sided LIL (law of iterated logarithm) behavior" to refer to the existence of the norming function $b(t)$ $(b(n))$ for a given X_t (S_n), both at zero and infinity.

The main result of this paper extends work by Bertoin, Doney and Maller on two-sided crossings of power law boundaries at small times, see Sect. 2 in [2]. They furnish an integral criterion for $\limsup_{t\to 0} t^{-1/2}|X_t| = c \in [0, \infty]$ and show that even

$$\limsup_{t\to 0} \frac{X_t}{\sqrt{t}} = \limsup_{t\to 0} \frac{-X_t}{\sqrt{t}} = c \text{ a.s.}$$

Thus they extend earlier results by Blumenthal and Getoor [3]. Under some mild conditions on $b(t)$, see (2.1) and (2.2), we provide a similar integral criterion for computing

$$\limsup_{t\to 0} \frac{X_t}{b(t)} = \limsup_{t\to 0} \frac{-X_t}{b(t)} = c \text{ a.s.}, \tag{1.4}$$

where X is a Lévy process with unbounded variation and $\sigma = 0$. The authors of [2] also discuss the possible values of $\limsup_{t\to 0} t^{-\kappa}|X_t|$, for $\kappa > 1/2$, where X is with unbounded variation and $\sigma = 0$. They arrive at the conclusion that $\limsup_{t\to 0} t^{-\kappa}|X_t|$ is either 0 or ∞, according as $\int_{|x|\le 1} |x|^{1/\kappa}\Pi(dx)$ is finite or not. This is achieved by an approach different from the one used to tackle the case $\kappa = 1/2$. Our integral criterion applies to this case and gives the same answer, even when t^{κ} is replaced by any positive, continuous function $b(t)$, satisfying $b(0) = 0$, (2.1) and (2.2) with $\alpha > 1/2$.

In [12], Maller approaches the problem the other way round. He proves necessary and sufficient conditions for the existence of a norming function $b(t)$ such that $\limsup_{t\to 0} |X_t|/b(t) = 1$ a.s. But there is not much information given about the properties of $b(t)$. However, it is important to note that he does show, see Theorem 3 in [12], that if a norming function $b(t)$ exists, then it can be chosen in a way that, $b(t)/t^{(1/2-\varepsilon)}$ is decreasing as $t \downarrow 0$, for each $\varepsilon > 0$: in particular our assumption (2.2) holds.

Ideally, we would like to be able to specify the norming function $b(t)$ ($b(n)$), whenever it exists, by using the basic quantities - the Lévy triplet or the distribution of X_1 in the random walks case. At present, it is only possible, to check for existence of a norming function both at zero, see [12], and infinity, see [8] and specify $b(n)$ in some special cases, see Einmahl and Li [7], and Klass [10,11]. At zero the only known result is an integral criterion, which checks whether $\sqrt{t}$ is a norming function, for any Lévy process X_t, see [2].

In Sect. 3 of this paper we are going to show that in some cases $b(t)$ can be expressed in terms of functions of the basic quantity $\Pi(.)$. We will also provide examples when our approach fails, but taking into account the specific features of the process we are still able to specify $b(t)$.

Recall that a cluster set $C(A)$ of a set $A \in \mathbb{R}$ is the set of all limit points of A. As an immediate corollary of our integral criterion, we show that $C(X_t/b(t), t \le 1) = [-c, c]$ a.s., where $\limsup_{t\to 0} X_t/b(t) = c$ a.s.

Throughout the paper we will try to compare as much as possible the results for random walks at infinity and Lévy processes at zero.

2 Main results

In this section we study two-sided crossings out of regions $\{(t, y) \in [0, \infty) \times \mathbb{R} : y \notin [-b(t), b(t)]\}$, where $b(t)$ is defined as below, see (2.4), with properties (2.1) and (2.2). We wish to find an integral criterion for specifying c in (1.4), given a function $b(t)$.

We proceed to state our main result. Denote by $b(t)$ a positive, continuous function with inverse function $b^{\leftarrow}(t)$ and the following properties

$$\frac{b(t)}{t} \uparrow \infty, \quad \text{as } t \downarrow 0, \tag{2.1}$$

and, for some $\alpha > 1/3$,

$$\frac{b(t)}{t^{\alpha}} \downarrow 0, \quad \text{as } t \downarrow 0. \tag{2.2}$$

Recall that the norming function $b(t)$ can be chosen in a way that condition (2.2) is satisfied for any $\alpha < 1/2$, if (1.4) holds. This follows from Theorem 3 in [12].

Define next

$$V(x) = \int_{-x}^{x} y^2 \Pi(dy); \quad U(x) = V(x) + x^2 \overline{\Pi}(x), \tag{2.3}$$

where $\overline{\Pi}(x) = \int_{|y|>x} \Pi(dy)$. Put

$$b(t) = \sqrt{t} G(t). \tag{2.4}$$

Then the following result holds:

Theorem 2.1 *Let X be a Lévy process with unbounded variation such that $\sigma = 0$. Let $b(t)$ be as above, with properties (2.1), (2.2) and additionally assume that*

$$\int_0^1 \overline{\Pi}(b(t))dt < \infty. \tag{2.5}$$

Then

$$\limsup_{t \to 0} \frac{X_t}{b(t)} = \limsup_{t \to 0} \frac{-X_t}{b(t)} = \lambda \text{ a.s.}, \tag{2.6}$$

where

$$\lambda = \inf\{a : I(a) < \infty\} = \inf\{a : \widetilde{I}(a) < \infty\} \in [0, \infty] \tag{2.7}$$

and

$$\begin{aligned} I(a) &= \int_0^1 \frac{\sqrt{V(b(x))}}{G(x)} e^{-\frac{a^2 G^2(x)}{2V(b(x))}} \frac{dx}{x}, \\ \widetilde{I}(a) &= \int_0^1 \frac{\sqrt{U(b(x))}}{G(x)} e^{-\frac{a^2 G^2(x)}{2U(b(x))}} \frac{dx}{x}. \end{aligned} \tag{2.8}$$

Moreover the cluster set $C(X_t/b(t), t \le 1) = [-\lambda, \lambda]$ a.s.

Remark (i) Denote by

$$J(a)=\int_0^1 e^{-\frac{a^2G^2(x)}{2V(b(x))}}\frac{dx}{x} \quad \text{and} \quad \widetilde{J}(a)=\int_0^1 e^{-\frac{a^2G^2(x)}{2U(b(x))}}\frac{dx}{x}.$$

Note that because $I(a)$ and $J(a)$ are of the type $\int_0^1 y(x)e^{-ay^{-2}(x)}dx/x$ and $\int_0^1 e^{-ay^{-2}(x)} dx/x$, $I(a)<\infty$ implies $J(a+\delta)<\infty$, for each $\delta>0$, and vice versa. Therefore we can specify λ, see (2.7), in terms of $J(a)$ or $\widetilde{J}(a)$.

(ii) Note the similarity between the integral $J(a)$ used for specifying λ and the series used to specify $\alpha_0=\limsup_{n\to\infty}|S_n|/c_n$ in Theorem 3 in [7]:

$$\alpha_0=\sup\left\{\alpha\geq 0:\sum_{n\geq 1} n^{-1}e^{-\frac{\alpha^2c_n^2}{2nH(c_n)}}=\infty\right\},$$

where $H(x)=EX^2 1_{\{|X|\leq x\}}$ and c_n plays the role of $b(t)$. This shows that the similarity between the two-sided LIL behavior at zero and infinity goes beyond the ordinary LIL, see (1.3). A possible explanation of this fact may be the application of Berry-Esseen bounds. At zero we represent $X_t=\sum_{k=1}^n(X_{kt/n}-X_{(k-1)t/n})$ and use Lemma 4.3 in [2]. At infinity, Lemma 3 in [7] applies.

(iii) Define now the iterated logarithms: $l_{k+1}(x)=\ln|l_k(x)|;\ \ l_1(x)=|\ln x|$. Now take for example a Lévy process with

$$\Pi(dx)=\frac{1}{|x|^3|l_1(|x|)|l_2(|x|)l_3^2(|x|)}1_{\{|x|<1\}}dx$$

and hence $V(x)=C\frac{1}{l_3(x)}$. Then it is trivial to check that $b(x)=\frac{\sqrt{2Cxl_2(x)}}{\sqrt{l_3(x)}}$ satisfies conditions (2.1), (2.2) and (2.5). Moreover we get $\lambda=1$, see (2.7), and thus the norming function is specified. The integral criterion can be used to work out the norming function $b(t)$ in many other cases. We will discuss this later.

Next we proceed to discuss what happens if (2.5) fails. We formulate the following result:

Proposition 2.1 *Let X be a Lévy process with unbounded variation and $\sigma=0$, and $b(t)$ be a function satisfying* (2.1) *and* (2.2). *Let*

$$\int_0^1 \overline{\Pi}(b(t))dt=\infty.$$

Then $\limsup_{t\to 0}|X_t|/b(t)=\infty$ a.s.

Remark (i) According to Theorem 3.1 in [2] we can have

$$\limsup_{t\to 0} \frac{X_t}{t^\kappa} < \infty \text{ a.s.}$$

and yet $\int_0^1 \overline{\Pi}(t^\kappa)dt = \infty$. This does not contradict our result, since in this case $\liminf_{t\to 0} X_t/t^\kappa = -\infty$.

We conclude this section with the following application, which extends Theorem 2.1 in [2]. It is a simple corollary of our Theorem 2.1.

Corollary 2.1 *Let X be a Lévy process as defined in Proposition 2.1 and $b(t)$ be a positive, continuous function satisfying $b(0) = 0$, (2.1) and (2.2) with some $\alpha > 1/2$. Then*

$$\lim_{t\to 0} \frac{|X_t|}{b(t)} = 0 \text{ a.s.} \tag{2.9}$$

iff $\int_0^1 \overline{\Pi}(b(t))dt < \infty$. If the latter fails then $\limsup_{t\to 0} \frac{|X_t|}{b(t)} = \infty$ a.s.

Remark (i) Note that this lemma essentially says that the moment we move away from the square root boundary, i.e. the boundary behaves like $t^{1/2+\varepsilon}$ and satisfies (2.1) and (2.2), it is no longer possible to find a Lévy process, whose norming function is the aforementioned boundary.

3 Applications and discussions

In this section we are going to show that in some cases the norming function $b(t)$ can be computed in terms of the basic quantities of the Lévy process. For this purpose define

$$K(t) := \frac{t^2}{U(t)}, \tag{3.1}$$

which is strictly decreasing to zero as $t \downarrow 0$, and its inverse $K^{\leftarrow}(t)$. Then the following result holds:

Lemma 3.1 *Let X have unbounded variation, $\sigma = 0$, $\liminf_{t\to 0} U(t)/t > 0$, and*

$$\int_0^1 \overline{\Pi}(K^{\leftarrow}(2t \ln|\ln t|))dt < \infty.$$

Then with $f(t) = K^{\leftarrow}(2t \ln|\ln t|)$, we have

$$\limsup_{t\to 0} \frac{X_t}{f(t)} = \limsup_{t\to 0} \frac{-X_t}{f(t)} = 1 \text{ a.s.}$$

Remark (i) The form of $f(t)$ does not come in a mysterious way. We simply solve $U(b(x)) = \frac{b^2(x)}{2x \ln|\ln x|}$ in (2.8) to make sure that $\lambda = \inf\{a : \widetilde{I}(a) < \infty\} = 1$.

(ii) Note that $\liminf_{t\to 0} U(t)/t > 0$ is not a stringent restriction. Thus, for example, if X_t is in the domain of attraction of the normal law, i.e. there is a function $c(t)$ such that $\lim_{t\to 0} X_t/c(t) \overset{d}{=} N(0,1)$, then $U(t)$ is slowly varying. Therefore to apply Lemma 3.1 we only need to check that $\int_0^1 \overline{\Pi}(K^{\leftarrow}(2t \ln|\ln t|))dt < \infty$.

Now we proceed to show that $f(t) = K^{\leftarrow}(2t \ln|\ln t|)$ is not always the right norming function, even when X is in the domain of attraction of the normal law. This happens, because (2.5) may fail. Take for example the family of Lévy processes with Lévy measure defined as follows,

$$\Pi_\alpha(dx) = C\frac{1}{|x|^3 |\ln|x||^{1+\alpha}} 1_{\{|x|<1\}}dx; \quad V_\alpha(x) = \frac{1}{|\ln x|^\alpha}, \tag{3.2}$$

where $\alpha > 0$. In (3.2) we choose C in a way that the relation for $V(x)$ holds, as specified above. Then we have the following result:

Proposition 3.1 *For any Lévy process satisfying* (3.2), (1.4) *holds with the norming function specified by*

$$b_\alpha^2(x) = \frac{2xl_2(x)}{|\ln\sqrt{x}|^\alpha}\left(1 + l_2(x)\sin^2(l_3(x))\right). \tag{3.3}$$

Remark (i) Note that, for all $\alpha > 0$, the Lévy processes defined above, are in the domain of attraction of the normal law.

(ii) According to Lemma 3.1 the expected norming function for each process in family (3.2) satisfies $b_\alpha^2(x) = 2xl_2(x)U_\alpha(b(x)) \sim 2xl_2(x)V_\alpha(b(x))$. Then from Remark (i) after Lemma 3.1, we get that $b(x)$ is regularly varying with index $1/2$ and $|\ln b(x)| \sim |\ln x|/2$. Finally $V_\alpha(b(x)) = \frac{1}{|\ln b(x)|^\alpha}$ implies $V_\alpha(b(x)) \sim V_\alpha(\sqrt{x})$ and thus $b^2(x) \sim 2xl_2(x)V_\alpha(\sqrt{x}) = 2xl_2(x)|\ln\sqrt{x}|^{-\alpha}$. Then it is trivial to check that $\int_0^1 \overline{\Pi}(b(x))dx = \infty$ and thus $\limsup_{t\to 0} |X_t|/b(t) = \infty$ a.s.

(iii) The idea for formulating b_α in this specific way comes from [7]. There a particular example of random walks is treated. Here we extend a little bit this idea to study a whole class of Lévy processes and we do not doubt many more examples can be obtained.

(iv) The choice of $\sin^2(x)$ in (3.3) seems to be of no importance. If we replace $\sin^2(x)$ with a positive function $g(x)$ with period T, such that $g(0) = g(T) = 0$ and $g(x) \sim x^2$, as x goes to 0, we will get another plausible norming function.

4 Proof of Theorem 2.1

The proof of Theorem 2.1 relies on a sequence of results, which will be formulated as different lemmas. First let us set up some notation. Write, for $b > 0$,

$$X_t = X_t^b + Z_t^b, \tag{4.1}$$

with

$$X_t^b := \int\limits_{[0,t]\times[-1,1]} 1_{\{|x|\le b\}} x N(ds, dx),$$

$$Z_t^b := \gamma t + \int\limits_{[0,t]\times[-1,1]} 1_{\{|x|>b\}} x N(ds, dx) \tag{4.2}$$

$$= t\left(\gamma - \int\limits_{|x|>b} x\Pi(dx)\right) + \sum_{s\le t} \Delta X_s 1_{\{|\Delta X_s|>b\}},$$

where $N(ds, dx)$ is a Poisson random measure on $[0,\infty)\times[-1,1]$ with intensity $ds\Pi(dx)$, and the stochastic integrals are taken in compensated sense.

Lemma 4.1 *Let X be a Lévy process and $b(t)$ be a function satisfying* (2.1), (2.2) *and* (2.5). *Then*

$$\lim_{t\to 0} \frac{\sup_{0\le s\le t} |Z_s^{b(t)}|}{b(t)} = 0 \text{ a.s.}$$

Proof of Lemma 4.1 The result is standard. From

$$\int\limits_0^1 \overline{\Pi}(b(t))dt < \infty$$

and the monotonicity of $b(t)$ we get $\lim_{t\to 0} t\overline{\Pi}(b(t)) = 0$ and

$$\sum_{n\ge 0} r^n \overline{\Pi}(b(r^n)) < \infty, \quad \forall r \in (0,1).$$

Then denote by $\Delta X_t = X_t - X_{t-}$ and recall that

$$P\left(\sum_{s\le r^n} |\Delta X_s| 1_{\{|\Delta X_s|>b(r^n)\}} > 0\right) = 1 - e^{-r^n\overline{\Pi}(b(r^n))} \sim r^n\overline{\Pi}(b(r^n)),$$

as long as $\lim_{n\to\infty} r^n\overline{\Pi}(b(r^n)) = 0$. Therefore we get by the Borel–Cantelli lemma

$$P\left(\sum_{s\le r^n} |\Delta X_s| 1_{\{|\Delta X_s|>b(r^n)\}} > 0 \text{ i.o.}\right) = 0.$$

Thus, going back to the definition of Z and using (2.1), we get

$$\limsup_{n\to\infty} \frac{\gamma r^n + \sum_{s\le r^n} \Delta X_s 1_{\{|\Delta X_s|>b(r^n)\}} - r^n \int_{|x|>b(r^n)} x\Pi(dx)}{b(r^n)} = \limsup_{n\to\infty} \frac{-r^n \int_{|x|>b(r^n)} x\Pi(dx)}{b(r^n)}.$$

Then (2.1) implies that $b^{\leftarrow}(x)/x \downarrow 0$, which gives, for any $1 > C > 0$,

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{r^n \int_{|x|>b(r^n)} |x|\Pi(dx)}{b(r^n)} \\
&\le \limsup_{n\to\infty} \int\limits_{C\ge|x|>b(r^n)} b^{\leftarrow}(|x|)\Pi(dx) + \limsup_{n\to\infty} \frac{r^n}{b(r^n)} \int\limits_{|x|>C} |x|\Pi(dx) \\
&= \int\limits_{|x|\le C} b^{\leftarrow}(|x|)\Pi(dx).
\end{aligned}$$

Since C is arbitrary, we get $\lim_{n\to\infty} \frac{\sup_{0\le s\le r^n} |Z_s^{b(r^n)}|}{b(r^n)} = 0$. Then from (2.1) and (2.2), it is easy to deduce the statement of Lemma 4.1. □

Lemma 4.1 shows that from now on we can truncate the Lévy process X and study

$$\limsup_{t\to 0} \frac{X_t^{b(t)}}{b(t)}.$$

The next result provides a criterion in terms of series for

$$\limsup_{t\to 0} \frac{X_t}{b(t)} = a \in [0, \infty] \text{ a.s.}$$

Lemma 4.2 *Let X be a Lévy process and $b(t)$ be a function satisfying* (2.1), (2.2) *and* (2.5). *Moreover let, see* (2.4),

$$\lim_{t\to 0} \frac{G^2(t)}{V(b(t))} = \infty. \tag{4.3}$$

Then, if

$$\sum_{n\ge 1} P\left(X_{r^n} > ab(r^n)\right) = \infty, \quad \textit{for some } r \in (0, 1), \tag{4.4}$$

the following holds

$$\limsup_{t\to 0} \frac{X_t}{b(t)} \ge a \text{ a.s.}, \tag{4.5}$$

and if (4.4) *fails for some* $r \in (0, 1)$, *then*

$$\limsup_{t \to 0} \frac{X_t}{b(t)} \le \frac{a}{r} \text{ a.s.} \tag{4.6}$$

Remark (i) We note that, for any $r \in (0, 1)$, $a > 0$ and any $\delta > 0$,

$$\sum_{n \ge 1} P\left(X_{r^n} > ab(r^n)\right) = \infty \Rightarrow \sum_{n \ge 1} P\left(X_{r^n}^{b(r^n)} > (a - \delta)b(r^n)\right) = \infty,$$
$$\sum_{n \ge 1} P\left(X_{r^n} > ab(r^n)\right) < \infty \Rightarrow \sum_{n \ge 1} P\left(X_{r^n}^{b(r^n)} > (a + \delta)b(r^n)\right) < \infty.$$

To show these relations, we observe that with $X_t = X_t^{b(t)} + Z_t^{b(t)}$ and

$$A(t) = \left\{Z^{b(t)} \text{ does not have jumps up to time } t\right\},$$

we have

$$P(X_t \ge ab(t)) \ge P\left(A(t)\right) P(X_t > ab(t)|A(t))$$
$$P(X_t \ge ab(t)) \le P(A^c(t)) + P(X_t \ge ab(t); A(t)).$$

Then from Lemma 4.1 we immediately see that $\lim_{t \to 0} P(A(t)) = 1$, $Z_t^{b(t)} = t(\gamma - \int_{|x| > b(t)} x \Pi(dx)) = o(b(t))$ on $A(t)$, and, for any $r \in (0, 1)$,

$$\sum_{n \ge 1} P(A^c(r^n)) < \infty.$$

Using the independence of $Z^{b(t)}$ and $X^{b(t)}$, and the inequalities above, we verify the relations.

Proof of Lemma 4.2 Assume that (4.4) holds, for some $r \in (0, 1)$ and $a > 0$. Then it is clear that, for each $m > 0$, there is $0 \le k < m$, such that

$$\sum_{n \ge 1} P\left(X_{r^{nm+k}} > ab(r^{nm+k})\right) = \infty. \tag{4.7}$$

Now fix m and k, and set

$$A_n(k) = \left\{X_{\frac{r^{nm+k}}{(1-r^m)}} - X_{\frac{r^{(n+1)m+k}}{(1-r^m)}} > ab(r^{nm+k})\right\}$$

and $B_n(k) = \{|X_{\frac{r^{(n+1)m+k}}{(1-r^m)}}| \le \varepsilon b(r^{nm+k})\}$. Then $\{A_n(k)\}_{n \ge 1}$ are independent and (4.7) implies that $P(A_n(k) \text{ i.o.}) = 1$. Next we write $r(n) = r^{(n+1)m+k}/(1 - r^m)$ and

$d(n) = b(r(n))$, and deduce using Chebyshev's inequality that, for some $K > 0$,

$$\begin{aligned} P(B_n^c(k)) &\le P\left(|X_{r(n)}^{d(n)}| \ge \frac{\varepsilon}{2} b(r^{nm+k})\right) + P\left(|Z_{r(n)}^{d(n)}| > \frac{\varepsilon}{2} b(r^{nm+k})\right) \\ &\le \frac{4E(X_{r(n)}^{d(n)})^2}{\varepsilon^2 b^2(r^{nm+k})} + P\left(|Z_{r(n)}^{d(n)}| > \frac{\varepsilon}{2} b(r^{nm+k})\right) \\ &\le K \frac{V(d(n))}{G^2(d(n))} + P\left(|Z_{r(n)}^{d(n)}| > \frac{\varepsilon}{2} b(r^{nm+k})\right), \end{aligned}$$

where the second term on the right hand side goes to 0 from Lemma 4.1. The first term goes to 0, as n goes to infinity, since (4.3) holds and $G(r^{nm+k}) \asymp G(r(n))$, which follows from (2.1) and (2.2). Thus we get $\lim_{n\to\infty} P(B_n(k)) = 1$. Since, for each $n \ge 0$, $B_n(k)$ is independent of $\{A_1, \dots, A_n\}$, we can apply the Feller-Chung lemma, see ([5] page 69), to deduce

$$P(B_n(k) \cap A_n(k) \text{ i.o.}) = 1.$$

This trivially implies that $P(X_{r^{nm+k}/(1-r^m)} > (a-\varepsilon) b(r^{nm+k}) \text{ i.o.}) = 1$, which using (2.1) entails

$$P(X_{r^{nm+k}/(1-r^m)} > (a-\varepsilon)(1-r^m) b(r^{nm+k}/(1-r^m)) \text{ i.o.}) = 1.$$

This is sufficient to show that

$$\limsup_{t\to 0} \frac{X_t}{b(t)} \ge (a-\varepsilon)(1-r^m) \text{ a.s.}$$

Recall now that m and ε are arbitrary to deduce (4.5).

Assume that (4.4) fails, for some $a > 0$ and $r \in (0,1)$. A standard version of Petrov's inequality yields, see proof of Theorem 2.2 in [2] for more details,

$$P\left(\sup_{s\le r^n} X_s^{b(r^n)} > (a+\varepsilon) b(r^n)\right) \le 2P\left(X_{r^n}^{b(r^n)} > (a+\varepsilon) b(r^n) - \sqrt{2r^n V(b(r^n))}\right),$$

where $\sqrt{2r^n V(b(r^n))}$ is a bound for the median of $X_s^{b(r^n)}$, for each $s \le r^n$. Then (4.3) gives $\sqrt{2r^n V(b(r^n))} = o(b(r^n))$ and thus we conclude that, for any $\varepsilon > 0$,

$$\sum_{n\ge 0} P\left(\sup_{s\le r^n} X_s^{b(r^n)} > (a+\varepsilon) b(r^n)\right) \le 2\sum_{n\ge 0} P\left(X_{r^n}^{b(r^n)} > a b(r^n)\right) < \infty.$$

An application of the Borel-Cantelli lemma implies that

$$\limsup_{n\to\infty} \frac{\sup_{s\le r^n} X_s^{b(r^n)}}{b(r^n)} \le a + \varepsilon \text{ a.s.}$$

Finally, Lemma 4.1 and argument of monotonicity using (2.1) yield

$$\limsup_{t\to 0}\frac{X_t}{b(t)}\le\frac{a+\varepsilon}{r}\ \text{a.s.}$$

To finish the lemma let $\varepsilon\downarrow 0$ and $r\uparrow 1$. □

This result is not very useful, because the criterion requires knowledge of the distribution function of the Lévy process. Nevertheless we shall show that (4.4) is equivalent to the divergence of a series, whose terms are determined as a function of $V(x)$.

Lemma 4.3 *Take X and $b(t)$ as in Lemma 4.2. Then*

$$\sum_{n\ge 1}P\left(X_{r^n}^{b(r^n)}>ab(r^n)\right)=\infty\ \Leftrightarrow\sum_{n\ge 1}\overline{F}\left(\frac{aG(r^n)}{\sqrt{V(b(r^n))}}\right)=\infty, \tag{4.8}$$

where $\overline{F}(x)$ is the tail of a standard, normally distributed random variable.

Proof of Lemma 4.3 A slight modification of Proposition 4.3 in [2] yields immediately

$$|P(X_{r^n}^{b(r^n)}>ab(r^n))-\overline{F}\left(\frac{aG(r^n)}{\sqrt{V(b(r^n))}}\right)|\le\frac{A\rho(b(r^n))}{b(r^n)G^2(r^n)}, \tag{4.9}$$

where $\rho(x)=\int_{|y|\le x}|y|^3\Pi(dy)$. Therefore it will be sufficient to show that

$$\sum_{n=1}^{\infty}\frac{\rho(b(r^n))}{b(r^n)G^2(r^n)}=\sum_{n=1}^{\infty}\frac{r^n\rho(b(r^n))}{b^3(r^n)}<\infty.$$

We write

$$\begin{aligned}\sum_{n=1}^{\infty}r^n\frac{\int_0^{b(r^n)}y^3\Pi(dy)}{b^3(r^n)}&=\sum_{j=1}^{\infty}\int_{b(r^{j+1})}^{b(r^j)}y^3\Pi(dy)\sum_{n=1}^{j}\frac{r^n}{b^3(r^n)}\\&\le\sum_{j=1}^{\infty}\frac{b^3(r^j)}{r^{j+1}}\int_{b(r^{j+1})}^{b(r^j)}b^{\leftarrow}(y)\Pi(dy)\sum_{n=1}^{j}\frac{r^n}{b^3(r^n)}\\&\le\frac{1}{r}\sum_{j=1}^{\infty}\int_{b(r^{j+1})}^{b(r^j)}b^{\leftarrow}(y)\Pi(dy)\sum_{n=1}^{j}r^{n-j+3j\alpha-3n\alpha},\end{aligned}$$

where $b^3(r^j)/b^3(r^n) \le r^{3(j-n)\alpha}$ follows from property (2.2). Then since $\alpha > 1/3$, see (2.2), we get

$$\sum_{n=1}^{\infty} r^n \frac{\int_0^{b(r^n)} y^3 \Pi(dy)}{b^3(r^n)} \le K \int_0^1 b^{\leftarrow}(y)\Pi(dy) < \infty.$$

Similarly we compute $\sum_{n=1}^{\infty} r^n \frac{\int_{-b(r^n)}^0 |y|^3 \Pi(dy)}{b^3(r^n)} < \infty$ and hence the result. □

This lemma is much more useful, since the behaviour of $\overline{F}$ is well known. Nevertheless we shall aim at obtaining an equivalent, integral criterion for (4.4) to hold.

The next result will be the last in the sequence of lemmas.

Lemma 4.4 *Take X and $b(t)$ as in Lemma 4.2. Denote by $S(a, r) = \sum_{n\ge1} \overline{F}\left(\frac{aG(r^n)}{\sqrt{V(b(r^n))}}\right)$. Recall $I(a) = \int_0^1 \frac{\sqrt{V(b(x))}}{G(x)} e^{-\frac{a^2 G^2(x)}{2V(b(x))}} \frac{dx}{x}$. Then, we have*

$$I(a) = \infty \Rightarrow S(a\sqrt{r}, r) = \infty, \quad \forall r \in (0, 1)$$

and

$$I(a) < \infty \Rightarrow S\left(\frac{a}{\sqrt{r}}, r\right) < \infty, \quad \forall r \in (0, 1)$$

Remark Note that since (4.3) holds, we have

$$\overline{F}\left(\frac{aG(x)}{\sqrt{V(b(x))}}\right) \sim \frac{\sqrt{V(b(x))}}{aG(x)} e^{-\frac{a^2}{2}\frac{G^2(x)}{V(b(x))}}.$$

Proof Assume $I(a) = \infty$. Then for each $r \in (0, 1)$, we get

$$\begin{aligned} I(a) &= \sum_{n\ge1} \int_{r^{n+1}}^{r^n} \overline{F}\left(\frac{aG(x)}{\sqrt{V(b(x))}}\right) \frac{dx}{x} \\ &\le \sum_{n\ge1} \int_r^1 \overline{F}\left(\frac{aG(sr^n)}{\sqrt{V(b(r^n))}}\right) \frac{ds}{s} \\ &\le C(r) \sum_{n\ge1} \overline{F}\left(\frac{a\sqrt{r}G(r^n)}{\sqrt{V(b(r^n))}}\right), \end{aligned}$$

where we use $G(sr^n) \ge \sqrt{r}G(r^n)$, for each $s \in (r, 1)$, which is immediate from (2.1).

Let $I(a) < \infty$. Then, similarly, we write

$$I(a) = \sum_{n\geq 1} \int_{r^{n+1}}^{r^n} \overline{F}\left(\frac{aG(x)}{\sqrt{V(b(x))}}\right)\frac{dx}{x} \geq \sum_{n\geq 1} \int_{r}^{1} \overline{F}\left(\frac{aG(sr^n)}{\sqrt{V(b(r^{n+1}))}}\right)\frac{ds}{s}$$
$$\geq C(r)\sum_{n\geq 1} \overline{F}\left(\frac{(a/\sqrt{r})G(r^{n+1})}{\sqrt{V(b(r^{n+1}))}}\right),$$

where $G(sr^n) \leq G(r^{n+1})/\sqrt{r}$ comes from (2.1) as well. □

Now we are ready to prove Theorem 2.1.

Proof of Theorem 2.1 We recall that the function $b(t)$ satisfies (2.1), (2.2) and (2.5). First assume $\lim_{t\to 0} \frac{V(b(t))}{G^2(t)} = 0$. Then from Lemma 4.4 above, we immediately get $I(a) = \infty \Rightarrow S(a\sqrt{r}, r) = \infty$ ($S(.,.)$ is defined in Lemma 4.4), which in turn from Lemma 4.3 implies

$$\sum_{n\geq 1} P\left(X_{r^n}^{b(r^n)} > a\sqrt{r}b(r^n)\right) = \infty, \quad \text{for each } r \in (0, 1).$$

Next Lemma 4.2, see (4.5), and Remark (i) afterwards, give, for any small $\delta > 0$,

$$\limsup_{t\to 0} \frac{X_t}{b(t)} \geq a\sqrt{r} - \delta \text{ a.s.},$$

and this holds for each $r \in (0, 1)$. This shows $\limsup_{t\to 0} \frac{X_t}{b(t)} \geq a$ a.s.

Assume now that $I(a) < \infty$. Therefore from Lemma 4.4 we see that $S(a/\sqrt{r}, r) < \infty$. Then, similarly, we can deduce from Lemma 4.2 and the remark afterwards that, for any small $\delta > 0$,

$$\limsup_{t\to 0} \frac{X_t}{b(t)} \leq \frac{a}{r^{\frac{3}{2}}} + \delta \quad \text{a.s., for each } r \in (0, 1).$$

Thus we get

$$\limsup_{t\to 0} \frac{X_t}{b(t)} = \lambda = \inf\{a : I(a) < \infty\} \in [0, \infty] \text{ a.s.}$$

To show that

$$\limsup_{t\to 0} \frac{-X_t}{b(t)} = \lambda \text{ a.s.},$$

all we need to observe is that $V(x)$ is an even function and perform a sign change.

Let now $\limsup_{t\to 0} \frac{V(b(t))}{G^2(t)} > 0$. Choose $\{t_i\} \downarrow 0$, such that

$$\frac{V(b(t_i))}{G^2(t_i)} > \delta > 0, \quad \text{for each } i \geq 1.$$

Choose a subsequence to ensure that $t_{j+1} \leq rt_j$. Thus we have, with some $K = K(r) > 0$,

$$\frac{V(b(t))}{G^2(t)} \geq K\frac{V(b(t_j))}{G^2(t_j)} \geq K\delta,$$

since $G(t) \asymp G(t_j)$, for every $t \in (t_j, r^{-1}t_j)$. Then

$$\int_{t_j}^{r^{-1}t_j} \frac{\sqrt{V(b(x))}}{G(x)} e^{-\frac{a^2G^2(x)}{2V(b(x))}} \frac{dx}{x} \geq K\delta e^{-K^{-2}a^2\delta^{-2}} |\ln r|.$$

This implies $I(a) = \infty$, for each $a > 0$. From $\frac{V(b(t_j))}{G^2(t_j)} > \delta > 0$ we also obtain

$$\overline{F}\left(\frac{aG(t_j)}{\sqrt{V(b(t_j))}}\right) > h > 0.$$

Therefore from (4.9) and $\lim_{t\to 0} \frac{\rho(b(t))}{b(t)G^2(t)} = 0$, see Lemma 4.1, we get that there is j_0, such that for each $j \geq j_0$

$$P\left(X_{t_j}^{b(t_j)} > ab(t_j)\right) > \frac{h}{2} > 0.$$

Lastly this leads to $P(X_{t_j}^{b(t_j)} > ab(t_j) \text{ i.o.}) > 0$ and from the Blumenthal 0–1 law we get, for each $a > 0$,

$$P\left(X_{t_j}^{b(t_j)} > ab(t_j) \text{ i.o.}\right) = 1.$$

Therefore using Lemma 4.1 we see that $\limsup_{t\to 0} X_t/b(t) = \infty$ a.s. To show that $\limsup_{t\to 0} -X_t/b(t) = \infty$ a.s. simply note that $V(x)$ is an even function and use the same argument for the dual process $-X$.

Now it is easy to show that the same holds if we substitute $U(b(x))$ for $V(b(x))$ in $I(a)$ and work with $\widetilde{I}(a)$. We simply change the truncation and instead of working with $X_t^{b(t)}$, see (4.2), we work with

$$Y_t = X_t^{b(t)} + b(t)\sum_{s\leq t} sign(\Delta X_s)1_{\{|\Delta X_s|\geq b(t)\}}.$$

It is obvious from Lemma 4.1 that $\limsup_{t\to 0} Y_t/b(t) = \limsup_{t\to 0} X_t/b(t)$, while $Var(Y_t) = tU(b(t))$. Then Lemma 4.3 holds with $U(b(t))$ for $V(b(t))$.

It remains to show that $C(X_t/b(t), t \le 1) = [-\lambda, \lambda]$. We argue as follows: From (2.2), we have, for each $c < 1$, $b(ct) \le c^\alpha b(t)$ and hence

$$\int_0^1 \overline{\Pi}\left(c^\alpha b(t)\right) dt < \int_0^1 \overline{\Pi}(b(ct))dt < \infty.$$

This implies that $\{t \le 1 : |\Delta X_t| > c^\alpha b(t)\}$ is a.s. a finite set. This, combined with (2.6), shows that when $X_t/b(t)$ ranges between $[-\lambda, \lambda]$, there will be only finitely many points such that $|\Delta X_t| > c^\alpha b(t)$. Denote by $\mathcal{P}(a)$ a partition of the interval $[-\lambda, \lambda]$ into disjoint intervals of length not exceeding a. Then each member of $\mathcal{P}(c^\alpha)$ is visited i.o. by $X_t/b(t)$, as $t \downarrow 0$. This holds for any $c < 1$ and hence $C(X_t/b(t), t \le 1) = [-\lambda, \lambda]$. $\square$

Next we turn our attention to Proposition 2.1.

Proof of Proposition 2.1 Assume that $\limsup_{t\to 0} |X_t|/b(t) < \infty$. We invoke Proposition 4.2 in [2] to get that there is $C > 1$ such that

$$\int_0^1 \overline{\Pi}(Cb(t))dt < \infty.$$

Then we use (2.2) to get $b(C^{1/\alpha}t) \ge Cb(t)$ and thus $\overline{\Pi}(b(C^{1/\alpha}t)) \le \overline{\Pi}(Cb(t))$. We immediately obtain a contradiction with $\int_0^1 \overline{\Pi}(b(t))dt = \infty$. $\square$

Now we prove Corollary 2.1.

Proof of Corollary 2.1 If $\int_0^1 \overline{\Pi}(b(t))dt = \infty$, then the statement follows immediately from Proposition 2.1. Therefore assume that $\int_0^1 \overline{\Pi}(b(t))dt < \infty$.

Assume, without loss of generality, $b(1) = 1$ and write $|\Pi(dy)| = \Pi(dy) + \Pi(-dy)$. Our first aim is to deduce that

$$\int_{b^{\leftarrow}(y)}^1 \frac{1}{b^2(x)} dx \le D\frac{b^{\leftarrow}(y)}{y^2}, \tag{4.10}$$

for each $y < 1$. First we consider $y = r^k$, for some $k \in \mathbb{N}$ and fixed $r \in (0, 1)$. We estimate

$$\begin{aligned}
\frac{r^{2k}}{b^{\leftarrow}(r^k)} \int_{b^{\leftarrow}(r^k)}^{1} \frac{1}{b^2(x)} dx &= \frac{r^{2k}}{b^{\leftarrow}(r^k)} \sum_{l=1}^{k} \int_{b^{\leftarrow}(r^l)}^{b^{\leftarrow}(r^{l-1})} \frac{1}{b^2(x)} dx \le \sum_{l=1}^{k} \frac{r^{2k}}{r^{2l}} \frac{b^{\leftarrow}(r^{l-1})}{b^{\leftarrow}(r^k)} \\
&\le \frac{1}{r^{\frac{1}{\alpha}}} \sum_{l=1}^{k} \frac{r^{2k}}{r^{2l}} \frac{b^{\leftarrow}(r^l)}{b^{\leftarrow}(r^k)} \le \frac{1}{r^{\frac{1}{\alpha}}} \sum_{l=1}^{k} r^{2k-2l} r^{\frac{(l-k)}{\alpha}} \\
&= \frac{1}{r^{\frac{1}{\alpha}}} \sum_{l=1}^{k} r^{(2-\frac{1}{\alpha})(k-l)} < \frac{1}{r^{\frac{1}{\alpha}}} \sum_{m=0}^{\infty} r^{(2-\frac{1}{\alpha})m} < C,
\end{aligned}$$

where we use that (2.2) implies $b^{\leftarrow}(x)/x^{1/\alpha} \uparrow \infty$, as $x \downarrow 0$, and $2 - 1/\alpha > 0$, which comes from the assumption $\alpha > 1/2$. Therefore we deduce that

$$\int_{b^{\leftarrow}(r^k)}^{1} \frac{1}{b^2(x)} dx \le C \frac{b^{\leftarrow}(r^k)}{r^{2k}},$$

for each $k \in \mathbb{N}$. In order to obtain (4.10) we take $y \in [r^{k+1}, r^k)$ and write

$$\int_{b^{\leftarrow}(y)}^{1} \frac{1}{b^2(x)} dx \le \int_{b^{\leftarrow}(r^{k+1})}^{1} \frac{1}{b^2(x)} dx \le C \frac{b^{\leftarrow}(r^{k+1})}{r^{2k+2}} \le \frac{C}{r^2} \frac{b^{\leftarrow}(y)}{y^2},$$

where r is fixed in $(0, 1)$. Therefore (4.10) holds with $D = C/r^2$. This enables us to derive

$$\begin{aligned}
\int_0^1 \frac{V(b(x))}{b^2(x)} dx &= \int_0^1 y^2 \int_{b^{\leftarrow}(y)}^{1} \frac{1}{b^2(x)} dx |\Pi(dy)| \le D \int_0^1 y^2 \frac{b^{\leftarrow}(y)}{y^2} |\Pi(dy)| \\
&= D \int_0^1 b^{\leftarrow}(y) |\Pi(dy)| < \infty,
\end{aligned}$$

which together with $b(r^{n-1}) \le r^{-1} b(r^n)$ implies that

$$\sum_{n\ge1} \frac{r^n V(b(r^n))}{r^2 b^2(r^n)} \le \frac{1}{r^4} \sum_{n\ge1} \frac{r^n V(b(r^n))}{b^2(r^{n-1})} \le \frac{1}{r^4(1-r)} \sum_{n\ge0} \int_{r^n}^{r^{n-1}} \frac{V(b(x))}{b^2(x)} dx < \infty.$$

Hence $\sum_{n\geq 1}\frac{r^n V(b(r^n))}{b^2(r^n)} < \infty$. Denote by $a_n(r) = \frac{r^n V(b(r^n))}{b^2(r^n)}$ and note that $\sum_{n\geq 1} a_n(r) < \infty$. The latter implies that $\sum_{n\geq 1} e^{-ca_n^{-2}(r)} < \infty$, for each $c > 0$. But the last series is easily comparable to $J(c)$, see Remark (ii), Theorem 2.1. Therefore $J(c) < \infty$, for each $c > 0$, and Theorem 2.1 gives $\lim_{t\to 0}|X_t|/b(t) = 0$ a.s. $\square$

5 Proofs for section 3

Proof of Lemma 3.1 Taking into account the definition of $\widetilde{I}(a)$, see (2.8), we check that $\lambda = 1$, see (2.7), if $U(b(t)) = b^2(t)/2t\ln|\ln t|$. From the definition of $K(t) = t^2/U(t)$, we get $K(b(t)) = 2t\ln|\ln t|$ and hence $b(t) = f(t) = K^{\leftarrow}(2t\ln|\ln t|)$. Now if (2.5) holds, we need to verify that $f(t)$ satisfies (2.1) and (2.2). First we use $K(t)/t^2 = 1/U(t)\uparrow\infty$, to get $K^{\leftarrow}(t)/\sqrt{t}\downarrow 0$ and hence $K^{\leftarrow}(2t\ln|\ln t|)/t^{\alpha}\downarrow 0$, for any $\alpha < 1/2$. Then we check that to have $K^{\leftarrow}(2t\ln|\ln t|)/t = f(t)/t\uparrow\infty$, it is sufficient to have $\liminf_{t\to 0} K^{\leftarrow}(t)/t > 0$. Using the definition of $K(t)$, (3.1), this is equivalent to $\liminf_{t\to 0} U(t)/t > 0$. Therefore we finish the proof. $\square$

Now we proceed to prove Proposition 3.1.

Proof of Proposition 3.1 It is clear that $b_\alpha(t)$ satisfy (2.1) and (2.2). Hence we need to check that (2.5) holds and show that the integral criterion in Theorem 2.1, applied to $b_\alpha(t)$, yields (2.6), with $\lambda = 1$. From (3.2) we note that

$$\overline{\Pi}_\alpha(b_\alpha(t)) \asymp \frac{1}{b_\alpha^2(t)|\ln b_\alpha(t)|^{1+\alpha}} = \frac{|\ln\sqrt{t}|^\alpha}{2tl_2(t)(1+l_2(t)\sin^2(l_3(t)))|\ln b_\alpha(t)|^{1+\alpha}}$$

when $t\downarrow 0$. From the definition of $b_\alpha(t)$, see (3.3), we get $|\ln b_\alpha(t)|\sim|\ln\sqrt{t}|$, as $t\downarrow 0$. Therefore we get

$$\int_0^{1/2}\overline{\Pi}(b_\alpha(x))dx \asymp \int_0^{1/2}\frac{1}{x|\ln x|l_2(x)}\left(\frac{1}{1+l_2(x)\sin^2(l_3(x))}\right)dx$$

and then changing variables we get, for some $C > 0$,

$$\int_0^{1/2}\overline{\Pi}(b_\alpha(t))dt \asymp \int_C^{\infty}\frac{1}{1+e^v\sin^2(v)}dv < \infty.$$

Now to finish the proposition we need to show that $I(a)$ diverges for $a < 1$ and converges for $a > 1$. First we estimate

$$\frac{b_\alpha^2(x)}{xV(b_\alpha(x))} = \frac{b_\alpha^2(x)|\ln b_\alpha(x)|^\alpha}{x} \sim 2l_2(x)\Big(1+l_2(x)\sin^2(l_3(x))\Big), \tag{5.1}$$

and then we investigate, for some $z > 0$,

$$\widehat{J}(\mu) = \int_0^z e^{-\mu^2 l_2(x)\left(1+l_2(x)\sin^2(l_3(x))\right)} \frac{dx}{x}.$$

Two changes of variables give

$$\widehat{J}(\mu) \asymp \int_{C(z)}^{\infty} e^{-\mu^2 \rho(1+\rho\sin^2(\ln\rho))} e^{\rho} d\rho.$$

This integral obviously converges for any $\mu > 1$. For $\mu < 1$, we shall show that $J(\mu) = \infty$. Note that $g(x) = \sin^2(x)$ is periodic with period π and $g(x) \sim x^2$, as $x \downarrow 0$. Write

$$\widehat{J}(\mu) = \sum_{j=1}^{\infty} \int_{e^{j\pi}}^{e^{(j+1)\pi}} e^{-\mu^2 \rho(1+\rho\sin^2(\ln\rho))} e^{\rho} d\rho,$$

and estimate each summand in the following way

$$\begin{aligned}
&\int_{e^{j\pi}}^{e^{(j+1)\pi}} e^{-\mu^2 \rho(1+\rho\sin^2(\ln\rho))} e^{\rho} d\rho \\
&= \int_0^{e^{(j+1)\pi}-e^{j\pi}} e^{-(\mu^2-1)(e^{j\pi}+x)-e^{j\pi}\left((e^{j\pi}+x)\sin^2(x)\right)} dx \\
&\geq \int_0^{e^{-4j\pi}} e^{-(\mu^2-1)e^{j\pi}-e^{2(j+1)\pi}\sin^2(x)} dx \geq e^{-4j\pi} e^{-(\mu^2-1)e^{j\pi}-C},
\end{aligned}$$

where the last inequality comes from $\sin^2 x \leq x^2$ in a neighborhood of zero. This together with $\mu < 1$ implies that $\widehat{J}(\mu) = \infty$. Our statement follows trivially if we note that

$$I(a) < \infty \Rightarrow \int_0^1 e^{-\frac{(a+\delta)^2 b^2(x)}{2xV(b(x))}} \frac{dx}{x} < \infty,$$

for each $\delta > 0$, and

$$I(a) = \infty \Rightarrow \int_0^1 e^{-\frac{(a-\delta)^2 b^2(x)}{2xV(b(x))}} \frac{dx}{x} = \infty,$$

for each $\delta > 0$, and take into account (5.1). $\square$

Acknowledgment I am thankful to my supervisor prof. Ron Doney for his advice throughout the course of this work, which will also be included in my future Ph.D. Thesis, see [14].

References

1. Bertoin, J.: Lévy Processes. Cambridge University Press, London (1996)
2. Bertoin, J., Doney, R.A., Maller, R.A.: Passage of Lévy processes across power law boundaries at small times. Ann. Probab. (to appear) (2007)
3. Blumenthal, R.M., Getoor, R.K.: Sample functions of stochastic processes with stationary independent increments. J. Math. Mech. **10**, 492–516 (1961)
4. Buchmann, B., Maller, R., Szimayer, A.: An alsmost sure functional limit theorem at zero for a Lévy process normed by the square root function and applications (in press, 2006)
5. Chow, Y.S., Teicher, H.: Probability Theory: Independence, Interchangeability, Martingales. Springer, Heidelberg (1978)
6. Doney, R.: Fluctuation Theory for Lévy Processes. Ecole d'été de Probabilités de St-Flour XXXV. Springer, Heidelberg (2007)
7. Einmahl, U., Li, D.: Some results on two-sided LIL behavior. Ann. Probab. **33**(4), 1601–1624 (2005)
8. Kesten, H.: Sums of independent random variables-without moment conditions. Ann. Math. Stat. **43**, 701–732 (1972)
9. Khintchine, A.Ya.: Sur la croissance locale des processes stochastiques homogénes á accroissements indépendants. Izv. Akad. Nauk SSSR **3**, 487–508 (1939)
10. Klass, M.: Toward a universal law of the iterated logarithm. Z. Wahrsch. Verw. Gebiete **36**, 165–178 (1976)
11. Klass, M.: Toward a universal law of the iterated logarithm. Z. Wahrsch. Verw. Gebiete **39**, 151–165 (1977)
12. Maller, R.: Small time versions of Strassen's law for Lévy processes (in press, 2007)
13. Sato, K.: Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge (1999)
14. Savov, M.: Asymptotic behavior of Lévy processes. Ph.D. Thesis, The University of Manchester (in press, 2008)