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Laplace approximation for stochastic line integrals

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Abstract We prove a precision of large deviation principle for current-valued processes such as shown in Bolthausen et al. (Ann Probab 23(1):236–267, 1995) for mean empirical measures. The class of processes we consider is determined by the martingale part of stochastic line integrals of 1-forms on a compact Riemannian manifold. For the pair of the current-valued process and mean empirical measures, we give an asymptotic evaluation of a nonlinear Laplace transform under a nondegeneracy assumption on the Hessian of the exponent at equilibrium states. As a direct consequence, our result implies the Laplace approximation for stochastic line integrals or periodic diffusions. In particular, we recover a result in Bolthausen et al. (Ann Probab 23(1):236–267, 1995) in our framework.

Keywords Laplace approximation · Large deviation · Stochastic line integral · Empirical measure

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1 Introduction

Let M be a compact connected Riemannian manifold. Consider a diffusion process $(\{z_t\}_{t\ge 0}, \{\mathbb{P}_x\}_{x\in M})$ associated with the generator $\mathcal{L} = \Delta/2 + b$. Here Δ is the Laplace-Beltrami operator and b a smooth vector field. Our result is an asymptotic evaluation of a functional of $\{z_t\}_{t\ge 0}$ determined by stochastic line integrals. In what follows, we will give a rough sketch of it. For the precise framework, see Sect. 1.1. Let $\mathcal{D}_p$

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be a L^2-Sobolev space of 1-forms on M with the differentiability order p. Then, for sufficiently large $p > 0$, there is a $\mathcal{D}_{-p}$-valued process $\{Y_t\}_{t\geq 0}$ such that $Y_t(\alpha)$ equals the martingale part of the stochastic line integral $\int_{z[0,t]} \alpha$ for each $\alpha \in \mathcal{D}_p$ $\mathbb{P}_x$-almost surely. We are interested in asymptotic behavior of Y_t as $t \to \infty$. Such a process is first introduced in [21] and we know several limit theorems including the law of large numbers [10], the central limit theorem [10,12,21] and the large deviation principle [16–18]. As shown in [16,17,21], the limit theorems for Y_t as stated above involve the corresponding limit theorems for each stochastic line integral or for the empirical measure $l_t = \int_0^t \delta_{z_s} ds$.

Let us consider the pair $\boldsymbol{Y}_t := (Y_t, l_t)$ of Y_t and the empirical measure. We realize l_t as an element of a negatively ordered Sobolev space $H_{-p'}$ of scalar functions. Then $\boldsymbol{Y}_t$ becomes a $\mathcal{D}_{-p} \times H_{-p'}$-valued random functional. Let $\bar{\boldsymbol{Y}}_t := t^{-1}\boldsymbol{Y}_t$. Then $\bar{\boldsymbol{Y}}_t$ satisfies the large deviation principle with the rate function I explicitly given in (1.10). In this paper, we will investigate a precision of the large deviation principle in the following sense. Take a "sufficiently regular" function F on $\mathcal{D}_{-p} \times H_{-p'}$. Then the Varadhan lemma asserts

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\mathrm{e}^{tF(\bar{\boldsymbol{Y}}_t)} \right] = \sup_{\boldsymbol{w} \in \mathcal{D}_{-p} \times H_{-p'}} (F(\boldsymbol{w}) - I(\boldsymbol{w})) =: \kappa_F. \tag{1.1}$$

Our goal is to give an asymptotic evaluation of $\exp(-\kappa_F t) \mathbb{E}_x \left[\exp\left\{ tF(\bar{\boldsymbol{Y}}_t) \right\} \right]$ as $t \to \infty$.

Let us define the set of equilibrium states $\mathcal{K}_F$ by

$$\mathcal{K}_F := \left\{ \boldsymbol{w} \in \mathcal{D}_{-p} \times H_{-p'} \; ; \; F(\boldsymbol{w}) - I(\boldsymbol{w}) = \kappa_F \right\}. \tag{1.2}$$

Then $\mathcal{K}_F$ is nonempty and compact (Lemma 1). Our main theorem will be proved under the condition that, for each $\boldsymbol{w} \in \mathcal{K}_F$, "Hessian of $F - I$" at $\boldsymbol{w}$ is nondegenerate. Under this assumption, $\mathcal{K}_F$ must be a finite set (Lemma 12). In this paper, we will show the following theorem, so-called "Laplace approximation" for $\boldsymbol{Y}_t$.

Theorem 1 *Suppose that "Hessian of $F - I$" is nondegenerate on $\mathcal{K}_F$. Then, for each $\boldsymbol{w} = (w, \mu) \in \mathcal{K}_F$, there is a constant $D_{F,\boldsymbol{w}} > 0$ and $h_{\boldsymbol{w}} \in C_+(M)$ such that*

$$\lim_{t\to\infty} \mathrm{e}^{-t\kappa_F} \mathbb{E}_x \left[\mathrm{e}^{tF(\bar{\boldsymbol{Y}}_t)} \right] = \sum_{\boldsymbol{w}=(w,\mu)\in\mathcal{K}_F} D_{F,\boldsymbol{w}} h_{\boldsymbol{w}}(x) \int_M \frac{1}{h_{\boldsymbol{w}}} d\mu. \tag{1.3}$$

This is a rough version of Theorem 4 and we will describe the precise meaning later.

Note that, by the same argument as given in [2], Theorem 1induces a convergence of path measures associated with $F(\bar{\boldsymbol{Y}}_t)$ (cf.[4]; see Corollary 4 for details). Let $\mathbb{P}_{x,F,T}$ be a probability measure on $C([0,\infty) \to M)$ given by

$$\mathbb{P}_{x,F,T}[dz] = \frac{\exp\left(TF(\bar{\boldsymbol{Y}}_T)\right)}{\mathbb{E}_x\left[\exp\left(TF(\bar{\boldsymbol{Y}}_T)\right)\right]} \mathbb{P}_x[dz] \tag{1.4}$$

for each Borel set $\mathcal{A} \subset C([0,\infty) \to M)$.

Corollary 1 *Suppose the same condition as in Theorem 1. Then we have*

$$\lim_{T\to\infty} \mathbb{P}_{x,F,T} = \frac{1}{Z} \sum_{\boldsymbol{w}=(w,\mu)\in\mathcal{K}_F} \left(D_{F,\boldsymbol{w}}\, h_{\boldsymbol{w}}(x) \int_M \frac{1}{h_{\boldsymbol{w}}} d\mu \right) \mathbb{P}_x^{\boldsymbol{\alpha}_{\boldsymbol{w}}},$$

with respect to the weak convergence on $C([0,\infty) \to M)$. *Here* Z *equals the right-hand side of* (1.3), $\boldsymbol{\alpha}_w = \nabla F(\boldsymbol{w}) \in \mathcal{D}_p \times H_{p'}$ *and* $\mathbb{P}_x^{\boldsymbol{\alpha}}$ *is defined by* (2.6) *for each* $\boldsymbol{\alpha} \in \mathcal{D}_p \times H_{p'}$.

Let $\{X_t\}_{t\geq 0}$ be a $\mathcal{D}_{-p}$-valued process determined by $X_t(\alpha) = \int_{z[0,t]} \alpha$ for each $\alpha \in \mathcal{D}_p$ $\mathbb{P}_x$-almost surely. This is a realization of stochastic line integrals itself as $\mathcal{D}_{-p}$-valued process. We can obtain a result corresponding to Theorem 1 for (X_t, l_t) (Theorem 5). As a result, we can obtain the Laplace approximation for a finite number of stochastic line integrals by taking F which depends only on a finite dimensional subspace of $\mathcal{D}_{-p}$. The key ingredient is the expression of X_t by a sum of a linear functional of Y_t and a remainder term. To see the result and its proof, we can observe an effect of the remainder term. It should be noted that the emergence of the effect is a result of precision of the limit theorem. The remainder term is negligible in considering other known limit theorems including the large deviation for X_t.

Our approach formulated by means of $\boldsymbol{Y}_t$ has an advantage to the analysis of rate function. Actually, if we consider the problem for each individual stochastic line integral or X_t, the rate function corresponding to it is expressed by a variational formula in general (see [1,16,18]). Then computing a perturbation of the rate function seems to be complicated. It should be emphasized that our rate function can be written without any variational expression. This representation enables us to compute an infinitesimal behavior of the rate function near the equilibrium states. An intuition from observations in [16] says that the emergence of variational expression comes from the lack of information about l_t. Actually, the pair (X_t, l_t) has an non-variational expression. But we choose to analyse (Y_t, l_t) since the derivation of the Laplace approximation for Y_t from that for (X_t, l_t) is much more involved than the converse derivation. Note that Y_t itself *has* information about l_t. Indeed, the rate function corresponding to the large deviation for Y_t also needs no variational expression. Thus, in the theoretical point of view, we can derive all the results stated above from the Laplace approximation for Y_t. But it is not easy to identify the influence of l_t in the expression of the rate function for Y_t. Dealing with the pair (Y_t, l_t) helps us to understand the connection between Y_t and l_t. This is a reason why we consider the pair (Y_t, l_t) instead of Y_t only. As another by-product, we can easily obtain the Laplace approximation for l_t by considering the case $F(w, \mu) = F_0(\mu)$ for a functional F_0. In fact, we can derive the same result from the Laplace approximation for Y_t though the calculation is much more involved.

Laplace approximation for mean empirical measures of a (discontinuous) Markov process is intensively studied in connection with some problems in statistical mechanics, for instance, mean field potential or polaron [2,4,14]. In these works, the asymptotic behavior of the same type is obtained for mean empirical measures on a compact metric space in more general situations. The above observation says that our result recovers that in [4] in the case of a diffusion process on a compact manifold. On the other hand, we should mention that more general results [3,15] are known for mean empirical measures even for diffusions on a manifold. These results do not require the nondegeneracy assumption as we impose on the "Hessian of $F - I$" on $\mathcal{K}_F$.

Now we concentrate our attention on the constant $D_{F,\boldsymbol{w}}$ appeared in Theorem 1. Set $\boldsymbol{\alpha_w} := \nabla F(\boldsymbol{w})$. Note that $D_{F,\boldsymbol{w}}$ is expressed by

$$D_{F,\boldsymbol{w}} = \det\left(1 - G^F_{\boldsymbol{w}} \circ S_\mu\right)^{-1/2}. \tag{1.5}$$

Here S_μ is a covariance operator of a Gaussian distribution on $\mathcal{D}_{-p}$. As we will give in (2.19), $G^F_{\boldsymbol{w}}$ is a composite of an operator $\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha_w}}$ from $\mathcal{D}_{-p}$ to $\mathcal{D}_{-p} \times H_{-p'}$ and $\nabla^2 F(\boldsymbol{w})$, the second Fréchet derivative of F at $\boldsymbol{w}$. In the framework of [4], the term corresponding to $G^F_{\boldsymbol{w}}$ also appears but it is given only by the second Fréchet derivative. We will explain below the reason why such a operation appears. We remark that $\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha_w}}$ consists of two operators $\Gamma^*_{\boldsymbol{\alpha_w}}$ and $(d\mathcal{G}_{\boldsymbol{\alpha_w}})^*$; $\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha_w}}(\eta, \nu) = (\Gamma^*_{\boldsymbol{\alpha_w}}\eta, (d\mathcal{G}_{\boldsymbol{\alpha_w}})^*\nu)$. Thus we observe the role of each component respectively. As a matter of fact, the principle behind the derivation of our result is the same as in the case of empirical measures. That is, to evaluate the asymptotic behavior of $\mathbb{E}_x\left[\exp\left\{tF(\bar{\boldsymbol{Y}}_t)\right\}\right]$, we reduce the problem to the estimate of the asymptotic behaviors of $\bar{\boldsymbol{Y}}_t$ near $\boldsymbol{w} = (w, \mu) \in \mathcal{K}_F$ in the scaling of the central limit theorem. Indeed, S_μ is the covariance operator of the Gaussian distribution appearing in the central limit theorem on $\mathcal{D}_{-p}$ near w. For this purpose, we will transform the measure $\mathbb{P}_x$ so that $\bar{\boldsymbol{Y}}_t$ converges to $\boldsymbol{w}$ almost surely. We can naturally establish such a transformation of measure by the Girsanov transform associated with $\boldsymbol{\alpha_w}$. Then the operation of $\Gamma^*_{\boldsymbol{\alpha_w}}$ to $Y_t - tw$ corresponds to the transformation of martingale $Y_t \mapsto Y_t^{\boldsymbol{\alpha_w}}$ under the Girsanov transform (Lemma 7) up to a negligible remainder term. This is the role of the first component $\Gamma^*_{\boldsymbol{\alpha_w}}$ of $\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha_w}}$. To explain the role of the second component, we observe that the central limit theorem for l_t near μ will be shown via the Itô formula for the generator $\mathcal{L}^{\boldsymbol{\alpha_w}}$ obtained as the result of the Girsanov transform:

$$u(z_t) - u(z_0) = Y_t^{\boldsymbol{\alpha_w}}(du) + \int_M \mathcal{L}^{\boldsymbol{\alpha_w}} u \, dl_t.$$

Since the left hand side is negligible in the scaling of the central limit theorem, we can reduce the central limit theorem for l_t to that for Y_t. By using the Green operator $\mathcal{G}_{\boldsymbol{\alpha_w}} = (-\mathcal{L}^{\boldsymbol{\alpha_w}})^{-1}$, we can write $du = -d\mathcal{G}_{\boldsymbol{\alpha_w}}\mathcal{L}^{\boldsymbol{\alpha_w}}u$. Hence the second component $(d\mathcal{G}_{\boldsymbol{\alpha_w}})^*$ of $\boldsymbol{\Gamma}^*_{\alpha_w}$ connects the central limit theorem for l_t with that for $Y_t^{\boldsymbol{\alpha_w}}$. Note that we can avoid to use the operation $(d\mathcal{G}_{\boldsymbol{\alpha_w}})^*$ from $H_{-p'}$ to $\mathcal{D}_{-p}$. Actually, the covariance of the central limit theorem for l_t near μ can be described in the framework of $H_{-p'}$ as we will show in Sect. 6 (cf. [4]). But, if we goes in this direction, we need to use two covariance operators, S_μ and the other one on $H_{-p'}$ to describe the result. To clarify the fact that the central limit theorem for $\boldsymbol{Y}_t$ near (w, μ) is reduced to that only for $Y_t^{\boldsymbol{\alpha_w}}$, we adopt the above expression as (1.5). In this sense, the emergence of the second component of $\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha_w}}$ is not essential for the problem.

Different from the situation on the second component, the emergence of $\Gamma^*_{\boldsymbol{\alpha_w}}$ in our result is essential. It comes from the fact that the definition of Y_t itself depends on $\mathbb{P}_x$. When we consider only the empirical measure l_t, it is invariant under the change of measures. Thus there appears no correction term corresponding to $\Gamma^*_{\boldsymbol{\alpha}_w}$. We should remark that, even in the Laplace approximation for X, some operator like $\Gamma^*_{\boldsymbol{\alpha}_{bw}}$ depending on $\boldsymbol{\alpha}_w$ also appears. The property that $X_t(\alpha)$ is a stochastic line integral of α is independent of the change of measure via Girsanov transform. Thus the transformation of measure causes no change like the operation of $\Gamma^*_{\boldsymbol{\alpha}_w}$. But, as we stated above, we need to reduce the central limit theorem for X_t to that for a linear functional of $Y_t^{\boldsymbol{\alpha}_w}$. This reduction is based on the Itô formula stated above, and therefore it depends on $\mathcal{L}^{\boldsymbol{\alpha}_w}$ or $\boldsymbol{\alpha}_w$ (see Remark 6 for more details). These observations tell us that it is natural and inevitable that some additional operation like $\Gamma^*_{\boldsymbol{\alpha}_w}$ appears when we deal with the long time asymptotic behavior of stochastic line integrals or its martingale parts.

The organization of this paper is as follows. In the rest of this section, we will introduce the precise framework of our results. We review there a series of limit theorems for a current-valued process Y or the pair (Y_t, l_t), including the law of large numbers, the central limit theorem and the large deviation principle. Section 2 is devoted to a functional calculus of a perturbation near an equilibrium state. The argument is based on the analytic perturbation theory of differential operators. Such an approach is available since our state space has a differentiable structure. We should remark that the functional Y_t is not bounded as a random variable while the empirical measure is bounded; $|\int_M f\,dl_t| \le t \sup|f|$. Thus it is not obvious whether a probabilistic perturbation argument based on the boundedness of functionals as used in [4] is also effective or not. In Sect. 3, we will establish the uniform moderate deviation estimate. Such estimates are essentially shown in [17], but we need more precise estimate concerning the uniformity on scale parameters. Our main theorem (Theorem 4) is proved in Sect. 4 with the aid of the estimate obtained in Sects. 2 and 3. The proof goes along the same idea as in [4]. We prove there the Laplace approximation for Y. The Laplace approximation for X will be shown in Sect. 5 as a corollary of our main theorem. In Sect. 6, we consider some special cases of our result. First we show that the Laplace approximation for only l_t. Next we consider the case where F depends only on a finite dimensional subspaces. In particular, we consider the periodic diffusion on $\mathbb{R}^k$.

1.1 Preliminaries

Let $d_0 = \dim M$. For a smooth differential 1-form α and $r \in \mathbb{R}$, we define a norm $\|\alpha\|_r$ by

$$\|\alpha\|_r := \left\{ \int_M |(1-\Delta_1)^{r/2}\alpha|^2 dv \right\}^{1/2},$$

where Δ_1 is the Hodge-Kodaira Laplacian on 1-forms and v is the normalized Riemannian measure. Let $\mathcal{D}_r$ be the completion of the space of smooth 1-forms by $\|\cdot\|_r$. We identify $\mathcal{D}_{-r}$ with $\mathcal{D}_r^*$. In the same way, let us define the L^2-Sobolev space

H_r of scalar functions. That is, H_r is the completion of the space of smooth functions by the L^2-Sobolev norm given by

$$\|f\|_{H_r} := \left\{ \int_M |(1-\Delta)^{r/2} f|^2 dv \right\}^{1/2}.$$

Throughout this paper except for Sect. 5, we take $p > 0$ and $p' > 0$ large enough to satisfy

(D) $p > d_0$ and $p' > \inf\{n \in \mathbb{N} \ ; \ n \geq p-1\}$.

For such p, we can realize a $\mathcal{D}_{-p}$-valued continuous process Y_t introduced in Sect. 1 (see [21]). It means that, for each $\alpha \in \mathcal{D}_p$, $Y_t(\alpha)$ is the martingale part of the stochastic line integral $\int_{z[0,t]} \alpha$ of α along the diffusion path $\{z_s\}_{s\in[0,t]}$ $\mathbb{P}_x$-almost surely. Recall that $\int_{z[0,t]} \alpha$ is defined via the Stratonovich stochastic integral and it becomes a semimartingale. We can realize stochastic line integrals themselves also as $\mathcal{D}_{-p}$-valued process. We deal with it in Sect. 5. Let l_t be the empirical measure defined by $l_t = \int_0^t \delta_{z_s} ds$. By the Sobolev embedding, we can realize l_t as a $H_{-p'}$-valued process. Set $\boldsymbol{Y}_t = (Y_t, l_t)$ and $\boldsymbol{\mathcal{D}}_{p,p'} := \mathcal{D}_p \times H_{p'}$. Then $\boldsymbol{Y}_t$ becomes a $\boldsymbol{\mathcal{D}}^*_{p,p'}$-valued process.

For later use, we review some known facts for Y. For 1-forms α and β, $(\alpha, \beta)(x)$ stands for the inner product at a cotangent space $T^*_x M$. Let $|\alpha|(x) = (\alpha, \alpha)(x)^{1/2}$. Then (α, β) and $|\alpha|$ become a scalar function on M. First, for each $\alpha \in \mathcal{D}_p$, $Y_t(\alpha)$ is a square-integrable martingale and the quadratic variation $\langle Y(\alpha)\rangle_t$ of $Y_t(\alpha)$ is given by

$$\langle Y(\alpha)\rangle_t = \int_0^t |\alpha|^2(z_s) ds.$$

Next we will review some known limit theorems for Y. Set $\bar{Y}_t = t^{-1} Y_t$. The law of large numbers asserts that $\lim_{t\to\infty} \bar{Y}_t = 0$ in $\mathcal{D}_{-p}$ almost surely. To state the central limit theorem, we introduce a Gaussian measure on $\mathcal{D}_{-p}$. We denote the inner product on a Hilbert space H and the dual pairing between H^* and H by $(\cdot,\cdot)_H$ and $\langle \cdot,\cdot\rangle_H$ respectively. For simplicity, we write $(\cdot,\cdot)_{\mathcal{D}_p} =: (\cdot,\cdot)_p$. and $\langle\cdot,\cdot\rangle_{\mathcal{D}_p} = \langle\cdot,\cdot\rangle$. For $\alpha \in H$, we denote the conjugate element by $\alpha^* \in H^*$. We denote by ν_S the centered Gaussian distribution with a covariance operator $S : \mathcal{D}_{-p} \to \mathcal{D}_{-p}$. That is, S is a nonnegative definite, symmetric bounded linear operator satisfying

$$\int_{\mathcal{D}_{-p}} \exp\left(\sqrt{-1}\,\langle w, \alpha\rangle\right) \nu_S(dw) = \exp\left(-\frac{1}{2}\left(\alpha^*, S(\alpha^*)\right)_{-p}\right). \tag{1.6}$$

We denote the normalized invariant measure of $\mathcal{L}$ by m. The central limit theorem (see [10,21]) for Y asserts that the law of $\sqrt{t}\bar{Y}_t$ weakly converges to ν_{S_m} as $t \to \infty$, where

S_m is given by

$$\langle S_m(\alpha^*), \beta \rangle := \int_M (\alpha, \beta)\, dm.$$

We now refer to the large deviation principle and its rate function studied in [16]. Let $\mathcal{M}_1$ be the set of all Borel probability measures on M. For $\mu \in \mathcal{M}_1$, let $L_1^2(d\mu)$ be the space of every measurable 1-form α on M with $|\alpha| \in L^2(d\mu)$. We say $w \in \mathcal{H}'_\mu$ if and only if there is $\hat{w} \in L_1^2(d\mu)$ such that, for all $\alpha \in \mathcal{D}_p$,

$$\langle w, \alpha \rangle = \int_M (\hat{w}, \alpha) d\mu. \tag{1.7}$$

It means that w acts on $\mathcal{D}_p$ as an inner product on $L_1^2(d\mu)$. Let us define a functional I'_μ on $\mathcal{D}_{-p}$ by

$$I'_\mu(w) = \begin{cases} \dfrac{1}{2} \displaystyle\int_M |\hat{w}|^2\, d\mu & \text{if } w \in \mathcal{H}'_\mu, \\ \infty & \text{otherwise.} \end{cases} \tag{1.8}$$

We denote by $\dot{\mathcal{M}}_1$ the totality of $\mu \in \mathcal{M}_1$ which is absolutely continuous with respect to v and $\sqrt{d\mu/dv} \in H_1$. For $\mu \in \mathcal{M}_1$ and $w \in \mathcal{D}_{-p}$, we say $w \in \Omega_\mu$ if and only if

$$\langle w, du \rangle + \int_M \mathcal{L}u\, d\mu = 0 \tag{1.9}$$

holds for all $u \in C^\infty(M)$. Here d means the exterior derivative. We can easily verify that for each $w \in \mathcal{D}_{-p}$, such $\mu \in \mathcal{M}_1$ as $w \in \Omega_\mu$ is unique if it exists. Set $\mathcal{H}_\mu = \mathcal{H}'_\mu \cap \Omega_\mu$. Let us define a functionals $\tilde{I}$ on $\mathcal{D}_{-p}$ by

$$\tilde{I}(w) := \begin{cases} I_\mu(w) & \text{if } w \in \mathcal{H}_\mu \text{ for some } \mu \in \dot{\mathcal{M}}_1, \\ \infty & \text{otherwise.} \end{cases}$$

As shown in [16], $\tilde{I}$ is a convex good rate function and this rate function controls the large deviation principle for $\bar{Y}$. It means that for each Borel subset $\mathcal{A} \subset \mathcal{D}_{-p}$,

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(\sup_{x\in M} \mathbb{P}_x\left[\bar{Y}_t \in \mathcal{A}\right]\right) \le - \inf_{w\in\bar{\mathcal{A}}} \tilde{I}(w),$$
$$\liminf_{t\to\infty} \frac{1}{t} \log\left(\inf_{x\in M} \mathbb{P}_x\left[\bar{Y}_t \in \mathcal{A}\right]\right) \ge - \inf_{w\in\mathcal{A}^\circ} \tilde{I}(w).$$

Set $\bar{l}_t := t^{-1} l_t$ and $\bar{\boldsymbol{Y}}_t := t^{-1}\boldsymbol{Y}_t = (\bar{Y}_t, \bar{l}_t)$. We extend the large deviation for $\bar{Y}_t$ to that for $\bar{\boldsymbol{Y}}_t$ to consider the Laplace approximation for $\bar{\boldsymbol{Y}}_t$.

Definition 1 A functional I on $\mathfrak{D}^*_{p,p'}$ is defined by

$$I(w, \mu) := \begin{cases} I_\mu(w) & \text{if } (w, \mu) \in \mathfrak{H}, \\ \infty & \text{otherwise,} \end{cases} \tag{1.10}$$

where we say $(w, \mu) \in \mathfrak{H}$ if and only if $\mu \in \dot{\mathcal{M}}_1$ and $w \in \mathcal{H}_\mu$.

Theorem 2 *$\bar{Y}_t$ satisfies the Large deviation principle in $\mathfrak{D}^*_{p,p'}$ with the convex good rate function I.*

Before the proof, we give a remark on uniform embeddings. For a bounded measurable function f, set $\|f\|_B := \sup_{x\in M} |f(x)|$. For a bounded 1-form α, we use the same symbol: $\|\alpha\|_B := \sup_{x\in M} |\alpha|(x)$. By (**D**), the Sobolev embedding theorem yields that there is a constant $C_S > 0$ such that

$$\|\alpha\|_B \le C_S \|\alpha\|_p, \quad \|f\|_B \le C_S \min\{\|f\|_{H_{p'}}, \|f\|_{H_{p+1}}\} \tag{1.11}$$

for all $\alpha \in \mathcal{D}_p$ and $f \in H_{p'}$.

Proof The method we will use here is similar to that used in [16] for proving large deviations for stochastic line integrals or empirical measures from that for Y_t. You can refer to it for technical details. For $\varphi \in H_{p'}$, let us consider the differential equation

$$-\mathcal{L}u = \varphi - \int_M \varphi \, dm.$$

Note that this equation has a unique solution up to additive constants (see [9], for example). We denote the solution u with $\int_M u \, dm = 0$ by $\mathcal{G}_0\varphi$. We should remark that $\mathcal{G}_0$ is bounded operator from $H_{p'}$ to H_{p+1} and that $d\mathcal{G}_0\varphi \in \mathcal{D}_p$ follows from it. By the Itô formula, we have

$$\begin{aligned} \mathcal{G}_0\varphi(z_t) - \mathcal{G}_0\varphi(z_0) &= Y_t(d\mathcal{G}_0\varphi) + \int_0^t \mathcal{L}\mathcal{G}_0\varphi(z_s) ds \\ &= Y_t(d\mathcal{G}_0\varphi) - \int_M \varphi \, dl_t + t\int_M \varphi \, dm \\ &= \left\langle (d\mathcal{G}_0)^* Y_t - l_t + tm, \varphi \right\rangle_{H_{p'}}. \end{aligned}$$

This equation connects the large deviation for Y_t with that for $\boldsymbol{Y}_t$. By the contraction principle, $(\bar{Y}_t, (d\mathcal{G}_0)^*\bar{Y}_t + m)$ satisfies the large deviation principle in $\mathfrak{D}^*_{p,p'}$. The Itô formula yields

$$\begin{aligned}\left\| (d\mathcal{G}_0)^* \bar{Y}_t - \bar{l}_t + m \right\|_{H_{-p'}} &= \sup_{\substack{\varphi \in H_{p'} \\ \|\varphi\|_{H_{p'}} \le 1}} \left| \bar{Y}_t(d\mathcal{G}_0 \varphi) - \int_M \varphi \, d\bar{l}_t + \int_M \varphi \, dm \right| \\ &= \frac{1}{t} \sup_{\substack{\varphi \in H_{p'} \\ \|\varphi\|_{H_{p'}} \le 1}} |\mathcal{G}_0 \varphi(z_t) - \mathcal{G}_0 \varphi(z_0)| \le \frac{C_S \|\mathcal{G}_0\|_{H_{p'} \to H_{p+1}}}{t}.\end{aligned}$$

This estimate implies that the approximation of $\bar{\boldsymbol{Y}}_t$ by $(\bar{Y}_t, (d\mathcal{G}_0)^* \bar{Y}_t + m)$ is superexponential. Thus $\bar{\boldsymbol{Y}}_t$ satisfies the large deviation principle with the same rate function as that for $(\bar{Y}_t, (d\mathcal{G}_0)^* \bar{Y}_t + m)$. Let us denote the rate function for $(\bar{Y}_t, (d\mathcal{G}_0)^* \bar{Y}_t + m)$ by J. Note that J is a convex good rate function because so is $\tilde{I}$. The contraction principle involves

$$J(w, \mu) = \inf_{\substack{\eta \in \mathcal{D}_{-p} \\ (\eta, (d\mathcal{G}_0)^* \eta + m) = (w, \mu)}} \tilde{I}(\eta) = \begin{cases} \tilde{I}(w) & \text{if } (d\mathcal{G}_0)^* w = \mu - m, \\ \infty & \text{otherwise.} \end{cases}$$

Suppose $J(w, \mu) < \infty$. Then $w \in \mathcal{H}_\nu$ for some $\nu \in \dot{\mathcal{M}}_1$ and $(d\mathcal{G}_0)^* w = \mu - m$. These two conditions on w implies

$$\begin{aligned}\int_M \varphi \, d\mu &= \left\langle (d\mathcal{G}_0)^* w + m, \varphi \right\rangle_{H_{p'}} = \langle w, d\mathcal{G}_0 \varphi \rangle + \int_M \varphi \, dm \\ &= -\int_M \mathcal{L} \mathcal{G}_0 \varphi \, d\nu + \int_M \varphi \, dm = \int_M \varphi \, d\nu\end{aligned}$$

for any $\varphi \in H_{p'}$. It immediately implies $\mu = \nu$ and therefore $(w, \mu) \in \mathcal{H}$. By a similar argument, we can easily show that $(w, \mu) \in \mathcal{H}$ implies $J(w, \mu) < \infty$ and $J(w, \mu) = I(w, \mu)$. It means $J = I$ and therefore the conclusion follows. □

Remark 1 For $(w, \mu) \in \mathcal{H}$, we have a different expression of (1.9), which will be used later. For a nonnegative function $f \in H_1$, let us denote by P_f the orthogonal projection on $L^2_1(f^2 dv)$ to the closure of the space of smooth exact 1-forms. We denote $\xi_f = \lim_{\varepsilon \to 0} (f + \varepsilon)^{-1} df$. Here we take the limit in $L^2_1(f^2 dv)$. When f is essentially strictly positive, ξ_f equals the logarithmic derivative of f. Let $\hat{b}$ be a 1-form corresponding to the vector field b in the canonical way. Then, the Green formula and (1.7) imply that, for $(w, \mu) \in \mathcal{H}$, (1.9) means

$$P_f \hat{w} + P_f \hat{b} - \xi_f = 0 \tag{1.12}$$

(cf. Lemma 4.3 and the proof of Proposition 5.1 in [16]).

Now we will try to explain an intuitive interpretation of the large deviation for Y_t and $\boldsymbol{Y}_t$. Let $\{B_t\}_{t\ge 0}$ be the stochastic development of the diffusion $\{z_t\}_{t\ge 0}$ and $\{Z_t\}_{t\ge 0}$ the horizontal lift of $\{z_t\}_{t\ge 0}$ to the orthonormal frame bundle $\mathcal{O}(M)$. As shown in [11], stochastic line integral is represented by means of a stochastic integral of B_t. Indeed, for each 1-form α, there exists a scalarization $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_{d_0})$ of α, which is $\mathbb{R}^{d_0}$-valued 1-form on $\mathcal{O}(M)$, such that

$$\int_{z[0,t]} \alpha = \sum_{i=1}^{d_0} \int_0^t \bar{\alpha}_i(Z_s) \circ dB_s^i,$$

$$Y_t(\alpha) = \sum_{i=1}^{d_0} \int_0^t \bar{\alpha}_i(Z_s) dB_s^i$$

hold. As a result, conditioned that the mean empirical measure $\bar{l}_t$ approximately equals $g^2 dv$ for $g > 0$, $Y_t(\alpha)$ behaves like Gaussian with the covariance $\int_M |\alpha|^2 g^2 dv$ *if* we can regard B_t and $\bar{l}_t$ to be nearly independent in long time. This is an intuitive reason why $I(w, \mu)$ *looks like quadratic* with respect to w for fixed μ in viewing only (1.7), (1.8) and (1.10). But this observation is not sufficient because we have one additional condition (1.9), which gives a relation between w and μ. In fact, when α is exact, namely $\alpha = du$ for a scalar function u, $Y_t(\alpha)$ heavily depends on l_t since the Itô formula asserts

$$\frac{1}{t}(u(z_t) - u(z_0)) = \bar{Y}_t(du) + \int_M \mathcal{L}u \, d\bar{l}_t.$$

Letting $t \to \infty$, the left hand sides degenerates to 0 while each term of the right hand side has the large deviations. Comparing it with (1.9), we can interpret that (1.9) inherits this relation. To observe more details, we decompose a 1-form $\hat{w}$ into the exact component and its orthogonal complement. By using (1.12), we can rewrite $I(w, \mu)$ for $(w, \mu) \in \mathcal{H}$ with $d\mu = f^2 dv$ as follows:

$$\begin{aligned} I(w, \mu) &= \frac{1}{2}\left(\int_M |(1 - P_f)\hat{w}|^2 d\mu + \int_M |P_f \hat{w}|^2 d\mu \right) \\ &= \frac{1}{2}\int_M |(1 - P_f)\hat{w}|^2 d\mu + \frac{1}{2}\int_M |P_f \hat{b} - \xi_f|^2 d\mu. \end{aligned} \tag{1.13}$$

The second term of this expression depends only on μ. Indeed, this part is the rate function for $\bar{l}_t$ (see [5]); when $\hat{b} = 0$, this functional is nothing but the L^2-energy functional of $\sqrt{d\mu/dv}$. Since $(1 - P_f)\hat{w}$ is free from (1.9), I is *quadratic* with respect to $(1 - P_f)\hat{w}$. Thus, intuitively saying, under the condition $\bar{l}_t \approx g^2 dv$, $Y_t(\alpha)$ behaves as Gaussian *if and only if* α is orthogonal to exact 1-forms in $L_1^2(g^2 dv)$.

Let $F : \mathfrak{D}^*_{p,p'} \to [-\infty, \infty)$ be a continuous function. We consider the following three conditions on F:

(F1) F is three times Fréchet differentiable at each $\boldsymbol{w} \in \mathfrak{D}^*_{p,p'}$ with $F(\boldsymbol{w}) \in \mathbb{R}$,

(F2)

$$\lim_{R\to\infty} \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp\left(tF(\bar{\boldsymbol{Y}}_t)\right) \; ; \; \|\bar{\boldsymbol{Y}}_t\|_{\mathfrak{D}^*_{p,p'}} \geq R\right] = -\infty,$$

(F3) for each $R > 0$,

$$\sup_{\|\boldsymbol{w}\|_{\mathfrak{D}^*_{p,p'}} \leq R} F(\boldsymbol{w}) < \infty.$$

When F has at most linear growth, **(F2)** and **(F3)** hold by Lemma 8 below. Note that the Varadhan lemma (1.1) holds under (**F2**). Thus we can define κ_F and $\mathcal{K}_F$ by (1.1) and (1.2).

Lemma 1 *Suppose* **(F2)** *and* **(F3)**. *Then $\mathcal{K}_F$ is nonempty and compact.*

Proof Let $\{\boldsymbol{w}_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}^*_{p,p'}$ be a sequence so that $F(\boldsymbol{w}_n) - I(\boldsymbol{w}_n) \to \kappa_F$ as $n \to \infty$. By (**F2**), there is $R > 0$ such that $F(\boldsymbol{w}) - I(\boldsymbol{w}) < \kappa_F - 1$ holds when $\|\boldsymbol{w}\|_{\mathfrak{D}^*_{p,p'}} > R$. Hence $\{\boldsymbol{w}_n\}_{n\in\mathbb{N}}$ is bounded in $\mathfrak{D}^*_{p,p'}$. Then (**F3**) implies that $\{F(\boldsymbol{w}_n)\}_{n\in\mathbb{N}}$ is bounded above and hence $\{I(\boldsymbol{w}_n)\}_{n\in\mathbb{N}}$ is. Since I is good, $\{\boldsymbol{w}_n\}_{n\in\mathbb{N}}$ has a convergent subsequence. By the upper semi-continuity of $F - I$, its subsequential limit of $\{\boldsymbol{w}_n\}_{n\in\mathbb{N}}$ is in $\mathcal{K}_F$ and hence $\mathcal{K}_F \neq \emptyset$. The compactness of $\mathcal{K}_F$ follows from the same argument. □

Remark 2 The assumption **(F3)** is rather technical and it is used only in the proof of Lemma 1. Instead of **(F2)** and **(F3)**, we may assume that

$$\limsup_{R\to\infty} \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp\left(tF(\bar{\boldsymbol{Y}}_t)\right) \; ; \; \|\bar{\boldsymbol{Y}}_t\|_{\mathfrak{D}^*_{p_0,p'}} \geq R\right] = -\infty \tag{1.14}$$

for some $d_0 < p_0 < p$. We show that (1.14) implies **(F2)** and Lemma 1. Since the embedding $\mathfrak{D}^*_{p_0,p'} \to \mathfrak{D}^*_{p,p'}$ is continuous, **(F2)** follows from (1.14). For proving Lemma 1, it suffices to show $\sup_{n\in\mathbb{N}} F(\boldsymbol{w}_n) < \infty$ for each sequence $\{\boldsymbol{w}_n\}_{n\in\mathbb{N}} \subset \mathfrak{D}^*_{p,p'}$ with $\lim_{n\to\infty} F(\boldsymbol{w}_n) - I(\boldsymbol{w}_n) = \kappa_F$. Indeed, (1.14) implies $\sup_{n\in\mathbb{N}} \|\boldsymbol{w}_n\|_{\mathfrak{D}^*_{p_0,p'}} < \infty$. Since the embedding $\mathfrak{D}^*_{p_0,p'} \to \mathfrak{D}^*_{p,p'}$ is compact, the desired result follows.

2 Perturbation

2.1 Regularity of principal eigenfunction

Here we consider the existence and regularity of the principal eigenfunction associated with an operator obtained by lower-order perturbation of the generator $\mathcal{L}$. In the rest of this paper, we abbreviate $L^2(dv)$ as L^2. Take $\alpha \in \mathcal{D}_q$ and $V \in H_{q'}$, where $d_0 < q$

and $d_0 < q'$. We consider a family of operators $\{\tilde{T}_t^{(\alpha,V)}\}_{t>0}$ given as follows:

$$\tilde{T}_t^{(\alpha,V)}\varphi(x) := \mathbb{E}_x\left[\exp\left(Y_t(\alpha) + \int_0^t V(z_s)ds\right)\varphi(z_t)\right] \tag{2.1}$$

for $\varphi \in C(M)$. We can easily verify that

$$\sup_{x\in M}\mathbb{E}_x\left[\exp\left\{a\left(Y_t(\alpha) + \int_0^t V(z_s)ds\right)\right\}\right] < \infty \tag{2.2}$$

for any $a > 0$. Recall that the semigroup $\tilde{T}_t^{(0,0)}$ is ultracontractive and has an extension on L^r as a compact operator for any $r \in [1, \infty]$. Combining it with (2.2), we can show that the same is true for $\tilde{T}_t^{(\alpha,V)}$. In addition, by a standard argument, we can show that $\tilde{T}_t^{(\alpha,V)}$ is a Feller semigroup, i.e., $\tilde{T}_t^{(\alpha,V)} f \in C(M)$ if $f \in C(M)$. The generator $\tilde{\mathfrak{L}}_{(2)}^{(\alpha,V)}$ of $\{\tilde{T}_t^{(\alpha,V)}|_{L^2\to L^2}\}_{t>0}$ is given by

$$\tilde{\mathfrak{L}}_{(2)}^{(\alpha,V)}u := \mathfrak{L}u + (\alpha, du) + \left(\frac{|\alpha|^2}{2} + V\right)u$$

for $u \in C^\infty(M)$. Note that the domain of $\tilde{\mathfrak{L}}_{(2)}^{(\alpha,V)}$ coincides with H_2, the domain of the closure of Δ on L^2. For simplicity, we abbreviate $\tilde{\mathfrak{L}}_{(2)}^{(\alpha,V)}$ as $\tilde{\mathfrak{L}}^{(\alpha,V)}$.

In the following, we discuss the existence and regularity of the principal eigenfunction of $\tilde{T}_t^{(\alpha,V)}$ and its logarithm. Though it may be a kind of well-known argument, we should mention it since α and V are *not* infinitely differentiable.

Since $\tilde{T}_t^{(\alpha,V)}$ is positivity preserving, the Krein-Rutman theorem (see [22] for example) yields that $\tilde{T}_t^{(\alpha,V)}$ has a principal eigenvalue λ and a υ-a.e. nonnegative eigenfunction $h \in L^2$ which is unique up to multiplicative constant. By definition, $h \in H_2$ and

$$\frac{1}{2}\Delta h = \lambda h - bh - (\alpha, dh) - \left(\frac{|\alpha|^2}{2} + V\right)h. \tag{2.3}$$

By virtue of the theory of multipliers (see [19]), if $u \in H_r$ with $r \in \mathbb{N}$, then $(\alpha, du) \in H_{(r-1)\wedge[q]}$ and $(|\alpha|^2/2 + V)u \in H_{r\wedge[q]\wedge[q']}$. Here $[a]$ stands for the integer part of $a \geq 0$. Thus, combining this fact with the hypoellipticity of Δ, we obtain $h \in H_{[q+2]\wedge[q'+2]}$ by a recursive argument using (2.3). By the choice of q, the Sobolev embedding theorem implies $h \in C(M)$. Note that the ultracontractivity of $\tilde{T}_t^{(\alpha,V)}$ implies the L^r-independence of the logarithmic spectral radius:

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\log\|\tilde{T}_t^{(\alpha,V)}\|_{L^2\to L^2} = \lim_{t\to\infty}\frac{1}{t}\log\|\tilde{T}_t^{(\alpha,V)}\|_{L^r\to L^r}$$

for any $1 \leq r \leq \infty$. Thus λ is also a principal eigenvalue of the Feller semigroup $\tilde{T}_t^{(\alpha,V)}|_{C(M)\to C(M)}$. Therefore the Krein-Rutman theorem for $\tilde{T}_t^{(\alpha,V)}|_{C(M)\to C(M)}$ implies that h is strictly positive. The property of the principal eigenfunction h obtained above is summarized in the following proposition.

Proposition 1 *Take $\alpha \in \mathcal{D}_q$ and $V \in H_{q'}$, where $q > d_0$ and $q' > d_0$. Let $\tilde{T}_t^{(\alpha,V)}$ be a semigroup defined by (2.1). Then there is a unique continuous principal eigenfunction $h \in H_{[q+2]\wedge[q'+2]}$ of $\tilde{T}_t^{(\alpha,V)}|_{L^2\to L^2}$ with $\|h\|_{L^2} = 1$ and $h > 0$.*

Since $h > 0$, we can consider $\log h$ and $\sqrt{h}$. We claim $\log h, \sqrt{h} \in H_{[q+2]\wedge[q'+2]}$. Let φ be a bounded smooth function with bounded derivative of each order satisfying $\varphi(r) = \log r$ for $\inf_{x\in M} h(x) \leq r \leq \sup_{x\in M} h(x)$. Then we have $\varphi(h) = \log h$. Thus we can reduce the proof of the claim to the following lemma.

Lemma 2 *Let g be a bounded smooth function on $\mathbb{R}$ with bounded derivative of any order. Then, for each $f \in H_r$, $r \in \mathbb{N}$ with $d/2 < r$, we have $g \circ f \in H_r$.*

By virtue of the equivalence of Sobolev norms [23], we can prove this in the same way as in Chapter 1,§2 of [20].

Corollary 2 *Let h be given in Proposition 1. Then $\log h \in H_{[q+2]}$ and $\sqrt{h} \in H_{[q+2]}$.*

2.2 Hessian at equilibrium states

For each $\boldsymbol{\alpha} = (\alpha, V) \in \mathfrak{D}_{p,p'}$, let $\Lambda(\boldsymbol{\alpha})$ be the principal eigenvalue of $\tilde{\mathfrak{L}}^{\boldsymbol{\alpha}}$, where $\tilde{\mathfrak{L}}^{\boldsymbol{\alpha}}$ is defined in Sect. 1.1. As shown in [16], we have

$$I(\boldsymbol{w}) = \sup_{\boldsymbol{\alpha}\in\mathfrak{D}_{p,p'}} \left(\langle \boldsymbol{w}, \boldsymbol{\alpha}\rangle_{\mathfrak{D}_{p,p'}} - \Lambda(\boldsymbol{\alpha})\right), \tag{2.4}$$

$$\Lambda(\boldsymbol{\alpha}) = \sup_{\boldsymbol{w}\in\mathfrak{D}^*_{p,p'}} \left(\langle \boldsymbol{w}, \boldsymbol{\alpha}\rangle_{\mathfrak{D}_{p,p'}} - I(\boldsymbol{w})\right). \tag{2.5}$$

The L^2-normalized principal eigenfunction of $\tilde{\mathfrak{L}}^{\boldsymbol{\alpha}}$ is denoted by $h^{\boldsymbol{\alpha}}$. By Proposition 1, we can choose $h^{\boldsymbol{\alpha}}$ as a strictly positive continuous function. We give a probability measure $\mathbb{P}_x^{\boldsymbol{\alpha}}$ on $C([0,\infty) \to M)$ by

$$\mathbb{P}_x^{\boldsymbol{\alpha}}[z_t \in E] := \mathrm{e}^{-\Lambda(\boldsymbol{\alpha})t} \frac{1}{h^{\boldsymbol{\alpha}}(x)} \mathbb{E}_x\left[h^{\boldsymbol{\alpha}}(z_t)\exp(Y_t(\alpha) + l_t(V))1_{\{z_t\in E\}}\right]. \tag{2.6}$$

Set $\psi^{\boldsymbol{\alpha}} = -\log h^{\boldsymbol{\alpha}}$. We define a differential operator $\mathfrak{L}^{\boldsymbol{\alpha}}$ as follows:

$$\mathfrak{L}^{\boldsymbol{\alpha}} u := \mathfrak{L} u + (\alpha - d\psi^{\boldsymbol{\alpha}}, du).$$

Note that $d\psi^{\boldsymbol{\alpha}} \in \mathcal{D}_p$ by Corollary 2. Under $\mathbb{P}_x^{\boldsymbol{\alpha}}$, $\{z_t\}_{t\geq 0}$ is a diffusion process associated with the generator $\mathfrak{L}^{\boldsymbol{\alpha}}$. The normalized invariant measure of $\mathfrak{L}^{\boldsymbol{\alpha}}$ is denoted by $m_{\boldsymbol{\alpha}}$.

We define $\Lambda^{\boldsymbol{\alpha}} : \mathfrak{D}_{p,p'} \to \mathbb{R}$ and $I^{\boldsymbol{\alpha}} : \mathfrak{D}^*_{p,p'} \to [0, \infty]$ by

$$\Lambda^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) := \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}^{\boldsymbol{\alpha}}_x \left[\mathrm{e}^{\boldsymbol{Y}_t(\boldsymbol{\beta})} \right],$$
$$I^{\boldsymbol{\alpha}}(w) := \sup_{\boldsymbol{\beta}\in\mathfrak{D}_{p,p'}} \left(\langle \boldsymbol{w}, \boldsymbol{\beta} \rangle_{\mathfrak{D}_{p,p'}} - \Lambda^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) \right).$$

Since

$$\Lambda^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \lim_{t\to\infty} \frac{1}{t} \log \left(\sup_{x\in M} \mathbb{E}^{\boldsymbol{\alpha}}_x \left[\mathrm{e}^{\boldsymbol{Y}_t(\boldsymbol{\beta})} \right] \right) = \lim_{t\to\infty} \frac{1}{t} \log \left(\inf_{x\in M} \mathbb{E}^{\boldsymbol{\alpha}}_x \left[\mathrm{e}^{\boldsymbol{Y}_t(\boldsymbol{\beta})} \right] \right),$$

we can easily verify that $\Lambda^{\boldsymbol{\alpha}}(\boldsymbol{\beta}) = \Lambda(\boldsymbol{\alpha} + \boldsymbol{\beta}) - \Lambda(\boldsymbol{\alpha})$. In addition, (2.4) yields

$$I^{\boldsymbol{\alpha}}(\boldsymbol{w}) = I(\boldsymbol{w}) - \langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle_{\mathfrak{D}_{p,p'}} + \Lambda(\boldsymbol{\alpha}). \tag{2.7}$$

In the rest of this section, we suppose (**F1**) and we set $\boldsymbol{w}_0 = (w_0, \mu_0) \in \mathcal{K}_F$ and $\boldsymbol{\alpha}_0 = (\alpha_0, V_0) := \nabla F(\boldsymbol{w}_0) \in (\mathfrak{D}^*_{p,p'})^* = \mathfrak{D}_{p,p'}$.

Lemma 3 (i) $I^{\boldsymbol{\alpha}_0}(\boldsymbol{w}_0) = 0$.
(ii) *For each* $(\beta, U) \in \mathfrak{D}_{p,p'}$, $\langle \boldsymbol{w}_0, (\beta, U) \rangle = \int_M \left(\left(\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \beta \right) + U \right) dm_{\boldsymbol{\alpha}_0}$. *In particular,* $\mu_0 = m_{\boldsymbol{\alpha}_0}$.

Proof Let $\boldsymbol{\alpha} \in \mathfrak{D}_{p,p'}$. By (2.5), $I^{\boldsymbol{\alpha}} \geq 0$ and $I^{\boldsymbol{\alpha}}(\boldsymbol{w}) = 0$ holds if and only if a functional $\langle \cdot, \boldsymbol{\alpha} \rangle_{\mathfrak{D}_{p,p'}} - I(\cdot)$ attains its supremum at $\boldsymbol{w}$. Thus, to prove the first assertion, it suffices to show that $\boldsymbol{w}_0$ is the maximizer of $\langle \cdot, \boldsymbol{\alpha}_0 \rangle_{\mathfrak{D}_{p,p'}} - I(\cdot)$. The fact $w_0 \in \mathcal{K}_F$ implies that, for each $\boldsymbol{w} \in \mathfrak{D}^*_{p,p'}$ and $0 < \varepsilon < 1$,

$$\begin{aligned} F(\boldsymbol{w}_0) - I(\boldsymbol{w}_0) &\geq F((1-\varepsilon)\boldsymbol{w}_0 + \varepsilon \boldsymbol{w}) - I((1-\varepsilon)\boldsymbol{w}_0 + \varepsilon \boldsymbol{w}) \\ &\geq F((1-\varepsilon)\boldsymbol{w}_0 + \varepsilon \boldsymbol{w}) - (1-\varepsilon) I(\boldsymbol{w}_0) - \varepsilon I(\boldsymbol{w}). \end{aligned}$$

The last inequality follows from the convexity of I. Thus we have

$$F(\boldsymbol{w}_0) - F(\boldsymbol{w}_0 - \varepsilon(\boldsymbol{w}_0 - \boldsymbol{w})) - \varepsilon I(\boldsymbol{w}_0) \geq -\varepsilon I(\boldsymbol{w}).$$

Dividing both sides by ε and $\varepsilon \to 0$, we obtain $\nabla F(\boldsymbol{w}_0)(\boldsymbol{w}_0) - I(\boldsymbol{w}_0) \geq \nabla F(\boldsymbol{w}_0)(\boldsymbol{w}) - I(\boldsymbol{w})$. Since $\boldsymbol{\alpha}_0 = \nabla F(\boldsymbol{w}_0)$, the first assertion holds.

Set $\boldsymbol{\alpha} = (\alpha, V) \in \mathfrak{D}_{p,p'}$. By using (2.5), (1.7) and (1.10), we have

$$\begin{aligned}
\Lambda(\boldsymbol{\alpha}) &= \sup_{\boldsymbol{w} \in \mathfrak{D}^*_{p,p'}} \Big(\langle \boldsymbol{w}, \boldsymbol{\alpha} \rangle_{\mathfrak{D}_{p,p'}} - I(\boldsymbol{w}) \Big) \\
&= \sup_{(w,\mu) \in \mathfrak{H}} \left(\int_M \big((\hat{w}, \alpha) + V \big)\, d\mu - \frac{1}{2} \int_M |\hat{w}|^2 d\mu \right) \\
&= \sup_{(w,\mu) \in \mathfrak{H}} \left(\int_M \left(\frac{|\alpha|^2}{2} + V \right) d\mu - \frac{1}{2} \int_M |\hat{w} - \alpha|^2 d\mu \right).
\end{aligned}$$

When $\boldsymbol{\alpha} = \boldsymbol{\alpha}_0$, $\boldsymbol{w}_0$ is the maximizer of the right hand side. Thus we will consider the condition on the maximizer. By (1.12), for $\mu \in \dot{\mathcal{M}}_1$ with $\sqrt{d\mu/dv} = f$, the functional

$$w \mapsto \int_M |\hat{w} - \alpha|^2 d\mu$$

on $\mathcal{H}_\mu$ attains its minimum only at w with $(1 - P_f)\hat{w} = (1 - P_f)\alpha$. Using (1.12) again, we obtain

$$\Lambda(\boldsymbol{\alpha}) = \sup_{f \in H_1} \left(\int_M \left(\frac{|\alpha|^2}{2} + V \right) f^2 dv - \frac{1}{2} \int_M |P_f \alpha + P_f \hat{b} - \xi_f|^2 f^2 dv \right). \tag{2.8}$$

We now remark that $\psi^{\boldsymbol{\alpha}}$ satisfies

$$\mathfrak{L}^{\boldsymbol{\alpha}} \psi^{\boldsymbol{\alpha}} + \frac{1}{2} \left| d\psi^{\boldsymbol{\alpha}} \right|^2 - \left(\frac{|\alpha|^2}{2} + V \right) = -\Lambda(\boldsymbol{\alpha}). \tag{2.9}$$

Substituting (2.9) to (2.8), we obtain

$$\begin{aligned}
\sup_{f \in H_1} \Bigg(\int_M \Delta \psi^{\boldsymbol{\alpha}} f^2 dv - \int_M \left| d\psi^{\boldsymbol{\alpha}} \right|^2 f^2 dv + 2 \int_M (\hat{b} + \alpha, d\psi^{\boldsymbol{\alpha}}) f^2 dv \\
- \int_M |P_f \alpha + P_f \hat{b} - \xi_f|^2 f^2 dv \Bigg) = 0.
\end{aligned}$$

By the Green formula,

$$\int_M \Delta\psi^{\boldsymbol{\alpha}} f^2 dv - \int_M \left|d\psi^{\boldsymbol{\alpha}}\right|^2 f^2 dv + 2\int_M \left(\hat{b}+\alpha, d\psi^{\boldsymbol{\alpha}}\right) f^2 dv$$
$$- \int_M |P_f\alpha + P_f\hat{b} - \xi_f|^2 f^2 dv$$
$$= -2\int_M (d\psi^{\boldsymbol{\alpha}}, \xi_f) f^2 dv - \int_M |d\psi^{\boldsymbol{\alpha}}|^2 f^2 dv$$
$$+ 2\int_M \left(P_f\hat{b} + P_f\alpha, d\psi^{\boldsymbol{\alpha}}\right) f^2 dv - \int_M |P_f\hat{b} + P_f\alpha - \xi_f|^2 f^2 dv$$
$$= -\int_M |P_f\hat{b} + P_f\alpha - \xi_f - d\psi^{\boldsymbol{\alpha}}|^2 f^2 dv.$$

Thus the maximizer f of the functional must satisfy $P_f\hat{b} + P_f\alpha - \xi_f - d\psi^{\boldsymbol{\alpha}} = 0$. We can see that this condition is equivalent to $f = \sqrt{dm_{\boldsymbol{\alpha}}/dv}$, that is, $f^2 dv = dm_{\boldsymbol{\alpha}}$. Note that $\sqrt{dm_{\boldsymbol{\alpha}}/dv} \in H_1$ holds by Corollary 2. Set $f_0 = \sqrt{d\mu_0/dv}$. These observations conclude that the maximizer $\boldsymbol{w}_0 = (w_0, \mu_0) \in \mathcal{H}$ of $\langle \cdot, \boldsymbol{\alpha}_0 \rangle_{\mathfrak{D}_{p,p'}} - I(\cdot)$ is expressed as follows:

$$\begin{aligned}\langle \boldsymbol{w}_0, (\beta, U)\rangle_{\mathfrak{D}_{p,p'}} &= \int_M (\hat{w}_0, \beta) d\mu_0 + \int_M U\, d\mu_0 \\ &= \int_M ((1-P_{f_0})\hat{w}_0, \beta) d\mu_0 + \int_M (P_{f_0}\hat{w}_0, \beta) d\mu_0 + \int_M U\, d\mu_0 \\ &= \int_M ((1-P_{f_0})\alpha, \beta) dm_{\boldsymbol{\alpha}_0} - \int_M (P_{f_0}\hat{b} - \xi_{f_0}, \beta) dm_{\boldsymbol{\alpha}_0} + \int_M U\, dm_{\boldsymbol{\alpha}_0} \\ &= \int_M \left((\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \beta) + U\right) dm_{\boldsymbol{\alpha}_0}.\end{aligned}$$

Thus the second assertion follows. □

Let $\mathcal{C}$ be a Banach space which consists of continuous 1-forms with the supremum norm $\|\cdot\|_B$. The following proposition provides an infinitesimal behavior of our rate function I near $\boldsymbol{w}_0 \in \mathcal{K}_F$.

Proposition 2 *For $\varepsilon' > 0$, let $\gamma_\cdot : (-\varepsilon', \varepsilon') \to \mathcal{C}$ be a $\mathcal{C}$-valued function with $\gamma_0 = \alpha_0 - d\psi^{\boldsymbol{\alpha}_0}$. We denote by μ_ε the unique probability measure which satisfies the following relation*:

$$\int_M (\mathcal{L}u + (\gamma_\varepsilon, du))\, d\mu_\varepsilon = 0 \tag{2.10}$$

for each $u \in H_2$. *Suppose that* γ_ε *and* μ_ε *depend smoothly on* ε *and that* $\mu_\varepsilon \in \dot{\mathcal{M}}_1$. *Here we regard* μ_ε *as a* $C(M)^*$*-valued functional with weak topology. We denote by* $\gamma_\varepsilon^{(i)}$ *or* $\mu_\varepsilon^{(i)}$ *the* i*-th order derivative of* γ_ε *or* μ_ε *respectively. Let us give a current* $w_\varepsilon \in \mathcal{D}_{-p}$ *as follows*:

$$\langle w_\varepsilon, \beta\rangle = \int_M (\gamma_\varepsilon, \beta)\, d\mu_\varepsilon. \tag{2.11}$$

Then, for $\boldsymbol{w}_\varepsilon = (w_\varepsilon, \mu_\varepsilon)$, *we have*

$$I^{\boldsymbol{\alpha}_0}(\boldsymbol{w}_\varepsilon) = \frac{\varepsilon^2}{2}\int_M |\gamma_0^{(1)}|^2 dm_{\alpha_0} + o(\varepsilon^2).$$

Proof Since $w_\varepsilon \in \mathcal{H}_{\mu_\varepsilon}$ and $\hat{w}_\varepsilon = \gamma_\varepsilon$, we have

$$I(\boldsymbol{w}_\varepsilon) = \frac{1}{2}\int_M |\gamma_\varepsilon|^2\, d\mu_\varepsilon. \tag{2.12}$$

Thus, by the assumption, $I^{\boldsymbol{\alpha}_0}(w_\varepsilon) = I(\boldsymbol{w}_\varepsilon) - \langle \boldsymbol{w}_\varepsilon, \boldsymbol{\alpha}_0\rangle_{\boldsymbol{\mathcal{D}}_{p,p'}} + \Lambda(\boldsymbol{\alpha}_0)$ is smooth at $\varepsilon = 0$. With the aid of Lemma 3, (2.11) and (2.12),

$$\begin{aligned}
\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I^{\boldsymbol{\alpha}_0}(\boldsymbol{w}_\varepsilon) &= \frac{1}{2}\int_M |\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}|^2 d\mu_0^{(1)} - \int_M (\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \alpha_0) d\mu_0^{(1)} - \int_M V_0 d\mu_0^{(1)} \\
&\quad + \int_M (\gamma_0^{(1)}, \alpha_0 - d\psi^{\boldsymbol{\alpha}_0}) d\mu_0 - \int_M (\gamma_0^{(1)}, \alpha_0) d\mu_0 \\
&= \frac{1}{2}\int_M |d\psi^{\boldsymbol{\alpha}_0}|^2 d\mu_0^{(1)} - \int_M \left(\frac{|\alpha_0|^2}{2} + V_0\right) d\mu_0^{(1)} \\
&\quad - \int_M (\gamma_0^{(1)}, d\psi^{\boldsymbol{\alpha}_0}) d\mu_0.
\end{aligned}$$

Differentiating (2.10) once at $\varepsilon = 0$, we obtain

$$\int_M \mathcal{L}^{\boldsymbol{\alpha}_0} u\, d\mu_0^{(1)} + \int_M (\gamma_0^{(1)}, du) d\mu_0 = 0. \tag{2.13}$$

Applying (2.13) to the case $u = \psi^{\boldsymbol{\alpha}_0}$ and substituting (2.9) to it, we obtain

$$\frac{1}{2}\int_M \left|d\psi^{\boldsymbol{\alpha}_0}\right|^2 d\mu_0^{(1)} - \int_M \left(\frac{|\alpha_0|^2}{2} + V_0\right) d\mu_0^{(1)} - \int_M (\gamma_0^{(1)}, d\psi^{\boldsymbol{\alpha}_0}) d\mu_0 = 0$$

since $\mu_0^{(1)}(M) = 0$. It means

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} I^{\boldsymbol{\alpha}_0}(\boldsymbol{w}_\varepsilon) = 0. \tag{2.14}$$

For the second derivative,

$$\begin{aligned}
\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} I^{\boldsymbol{\alpha}_0}(\boldsymbol{w}_\varepsilon) &= \frac{1}{2}\int_M |\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}|^2 d\mu_0^{(2)} - \int_M (\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \alpha_0) d\mu_0^{(2)} - \int_M V_0 d\mu_0^{(2)} \\
&\quad + 2\int_M (\gamma_0^{(1)}, \alpha_0 - d\psi^{\boldsymbol{\alpha}_0}) d\mu_0^{(1)} - 2\int_M (\gamma_0^{(1)}, \alpha_0) d\mu_0^{(1)} \\
&\quad + \int_M (\gamma_0^{(2)}, \alpha_0 - d\psi^{\boldsymbol{\alpha}_0}) d\mu_0 - \int_M (\gamma_0^{(2)}, \alpha_0) d\mu_0 + \int_M |\gamma_0^{(1)}|^2 d\mu_0 \\
&= \frac{1}{2}\int_M \left|d\psi^{\boldsymbol{\alpha}_0}\right|^2 d\mu_0^{(2)} - \int_M \left(\frac{|\alpha_0|^2}{2} + V_0\right) d\mu_0^{(2)} \\
&\quad - 2\int_M (\gamma_0^{(1)}, d\psi^{\boldsymbol{\alpha}_0}) d\mu_0^{(1)} \\
&\quad - \int_M (\gamma_0^{(2)}, d\psi^{\boldsymbol{\alpha}_0}) d\mu_0 + \int_M |\gamma_0^{(1)}|^2 d\mu_0.
\end{aligned} \tag{2.15}$$

Differentiating (2.10) twice at $\varepsilon = 0$, we have

$$\int_M \mathcal{L}^{\boldsymbol{\alpha}_0} u \, d\mu_0^{(2)} + 2\int_M (\gamma_0^{(1)}, du) d\mu_0^{(1)} + \int_M (\gamma_0^{(2)}, du) d\mu_0 = 0. \tag{2.16}$$

Applying (2.16) to the case $u = \psi^{\boldsymbol{\alpha}_0}$ and substituting (2.9) to it, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_M \left|d\psi^{\boldsymbol{\alpha}_0}\right|^2 d\mu_0^{(2)} - \int_M \left(\frac{|\alpha_0|^2}{2} + V_0\right) d\mu_0^{(2)} \\
&\quad = 2\int_M (\gamma_0^{(1)}, d\psi^{\boldsymbol{\alpha}_0}) d\mu_0^{(1)} + \int_M (\gamma_0^{(2)}, d\psi^{\boldsymbol{\alpha}_0}) d\mu_0.
\end{aligned} \tag{2.17}$$

Thus, combining (2.15) with (2.17), we obtain

$$\left.\frac{d^2}{d\varepsilon^2}\right|_{\varepsilon=0} I^{\boldsymbol{\alpha}_0}(\boldsymbol{w}_\varepsilon) = \int_M |\gamma_0^{(1)}|^2 d\mu_0. \tag{2.18}$$

Since $\mu_0 = m_{\boldsymbol{\alpha}_0}$ by Lemma 3, the Taylor expansion of $I^{\boldsymbol{\alpha}_0}(\boldsymbol{w}_\varepsilon)$, (2.14) and (2.18) imply the desired result. □

For $\boldsymbol{\alpha} = (\alpha, V) \in \boldsymbol{\mathcal{D}}_{p,p'}$ and $\varphi \in H_{p'}$, let us denote the solution of the following differential equation

$$-\mathcal{L}^{\boldsymbol{\alpha}} u = \varphi - \int_M \varphi \, dm_{\boldsymbol{\alpha}}$$

with $\int_M u \, dm_{\boldsymbol{\alpha}} = 0$ by $\mathcal{G}_{\boldsymbol{\alpha}}\varphi$. As mentioned in the proof of Theorem 2, the above equation has a unique solution up to additive constants. Thus $\mathcal{G}_{\boldsymbol{\alpha}}\varphi$ is well-defined. In addition, $\mathcal{G}_{\boldsymbol{\alpha}}$ is a bounded linear operator from $H_{p'}$ to H_{p+1}. We define $\Gamma_{\boldsymbol{\alpha}} : \mathcal{D}_p \to \mathcal{D}_p$ by $\Gamma_{\boldsymbol{\alpha}}\beta := \beta + d\mathcal{G}_{\boldsymbol{\alpha}}(\alpha - d\psi^{\boldsymbol{\alpha}}, \beta)$. It should be noted that $\Gamma_{\boldsymbol{\alpha}}$ is a bounded linear operator. We define $\boldsymbol{\Gamma}_{\boldsymbol{\alpha}} : \boldsymbol{\mathcal{D}}_{p,p'} \to \mathcal{D}_p$ by $\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}(\beta, U) = \Gamma_{\boldsymbol{\alpha}}\beta + d\mathcal{G}_{\boldsymbol{\alpha}}U$.

For each $\boldsymbol{w} \in \boldsymbol{\mathcal{D}}^*_{p,p'}$, let $G^F_{\boldsymbol{w}}$ be a bounded symmetric operator on $\mathcal{D}_{-p}$ defined by

$$\left(\eta, G^F_{\boldsymbol{w}}\eta\right)_{-p} := \nabla^2 F(\boldsymbol{w})(\boldsymbol{\Gamma}^*_{\nabla F(\boldsymbol{w})}\eta, \boldsymbol{\Gamma}^*_{\nabla F(\boldsymbol{w})}\eta) \tag{2.19}$$

for each $\eta \in \mathcal{D}_{-p}$.

Proposition 3 *Let γ_ε, μ_ε be as in Proposition 2. We define $\tilde{w}_0 \in \mathcal{D}_{-p}$ by*

$$\langle \tilde{w}_0, \beta \rangle := \int_M (\gamma_0^{(1)}, \beta) dm_{\boldsymbol{\alpha}_0}. \tag{2.20}$$

Then we have

(i) $\boldsymbol{w}_0^{(1)} = \boldsymbol{\Gamma}^*_{\boldsymbol{\alpha}_0}\tilde{w}_0$,

(ii) $\displaystyle\int_M |\gamma_0^{(1)}|^2 d\mu_0 \geq \left(\tilde{w}_0, G^F_{\boldsymbol{w}_0}\tilde{w}_0\right)_{-p}$.

Proof Set $u^{\boldsymbol{\alpha}_0,\beta} = \mathcal{G}_{\boldsymbol{\alpha}_0}(\alpha - d\psi^{\boldsymbol{\alpha}_0}, \beta)$. For $\boldsymbol{\beta} = (\beta, U) \in \boldsymbol{\mathcal{D}}_{p,p'}$, the definition of μ_0, $\mathcal{G}_{\boldsymbol{\alpha}_0}$ and μ_ε implies

$$\begin{aligned}
\langle \boldsymbol{w}_\varepsilon - \boldsymbol{w}_0, \boldsymbol{\beta} \rangle = &\int_M \left(\gamma_\varepsilon - \alpha_0 + d\psi^{\boldsymbol{\alpha}_0}, \beta\right) d\mu_\varepsilon \\
&+ \int_M \left(\left(\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \beta\right) - \int_M \left(\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \beta\right) d\mu_0 \right) d\mu_\varepsilon \\
&+ \int_M \left(U - \int_M U \, d\mu_0 \right) d\mu_\varepsilon
\end{aligned}$$

$$
\begin{aligned}
&= \int_M \left(\gamma_\varepsilon - \alpha_0 + d\psi^{\boldsymbol{\alpha}_0}, \beta\right) d\mu_\varepsilon - \int_M \mathcal{L}^{\boldsymbol{\alpha}_0}(u^{\boldsymbol{\alpha}_0,\beta} + \mathcal{G}_{\boldsymbol{\alpha}_0} U) d\mu_\varepsilon \\
&= \int_M \left(\gamma_\varepsilon - \alpha_0 + d\psi^{\boldsymbol{\alpha}_0}, \beta\right) d\mu_\varepsilon \\
&\quad + \int_M \left(\gamma_\varepsilon - \alpha_0 + d\psi^{\boldsymbol{\alpha}_0}, du^{\boldsymbol{\alpha}_0,\beta} + d\mathcal{G}_{\boldsymbol{\alpha}_0} U\right) d\mu_\varepsilon \\
&= \int_M \left(\gamma_\varepsilon - \gamma_0, \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0} \boldsymbol{\beta}\right) d\mu_\varepsilon .
\end{aligned}
$$

To obtain the third equality, we used (2.10) for $u = u^{\boldsymbol{\alpha}_0,\beta} + \mathcal{G}_{\boldsymbol{\alpha}_0} U$. Dividing by ε and taking $\varepsilon \to 0$, we obtain $\boldsymbol{w}_0^{(1)} = \boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^* \tilde{w}_0$. Thus (i) follows.

Since $\boldsymbol{w}_0 \in \mathcal{K}_F$, we have

$$
F(\boldsymbol{w}_0) - I(\boldsymbol{w}_0) \geq F(\boldsymbol{w}_\varepsilon) - I(\boldsymbol{w}_\varepsilon).
$$

By using (2.7) and Lemma 3 (i), we have

$$
I^{\boldsymbol{\alpha}_0}(\boldsymbol{w}_\varepsilon) \geq F(\boldsymbol{w}_\varepsilon) - F(\boldsymbol{w}_0) - \nabla F(\boldsymbol{w}_0)(\boldsymbol{w}_\varepsilon - \boldsymbol{w}_0).
$$

The Taylor expansion of F yields

$$
\begin{aligned}
&F(\boldsymbol{w}_\varepsilon) - F(\boldsymbol{w}_0) - \nabla F(\boldsymbol{w}_0)(\boldsymbol{w}_\varepsilon - \boldsymbol{w}_0) \\
&\quad = \frac{1}{2} \nabla^2 F(\boldsymbol{w}_0)(\boldsymbol{w}_\varepsilon - \boldsymbol{w}_0, \boldsymbol{w}_\varepsilon - \boldsymbol{w}_0) + o(\|\boldsymbol{w}_\varepsilon - \boldsymbol{w}_0\|_{-p}^2).
\end{aligned}
$$

Thus the conclusion follows from Proposition 2 and (i). □

In the following lemma, we will construct a smooth perturbation explicitly. This construction asserts that, for any $\alpha \in \mathcal{D}_p$, a perturbation infinitesimally toward α-direction is possible.

Lemma 4 *For each $\alpha \in \mathcal{D}_p$, there is $\varepsilon' > 0$ and a smooth perturbation $\gamma_\cdot : (-\varepsilon', \varepsilon') \to \mathcal{C}$ with $\gamma_0^{(1)} = \alpha$ which satisfies the assumption of Proposition 2.*

Proof Fix $\alpha \in \mathcal{D}_p$. For each $\zeta \in \mathbb{C}$, we define a differential operator $\tilde{\mathcal{L}}_\zeta^{\boldsymbol{\alpha}_0}$ by

$$
\tilde{\mathcal{L}}_\zeta^{\boldsymbol{\alpha}_0} u := \mathcal{L}^{\boldsymbol{\alpha}_0} u + \zeta\,(\alpha, du) + \frac{\zeta^2}{2} |\alpha|^2 u.
$$

We can easily show that $\{\mathcal{L}_\zeta^{\boldsymbol{\alpha}_0}\}_{\zeta \in \mathbb{C}}$ is a holomorphic family of type (A) defined in VII §2 of [13]. Thus we can apply the analytic perturbation theory. With the aid of Proposition 1, it yields that there is a small neighborhood U of 0 in $\mathbb{C}$ and a family of functions $\{h_\zeta\}_{\zeta \in U} \subset H_{[p+2]}$ such that h_ζ satisfies the following:

(i) h_ζ is an eigenfunction of $\mathcal{L}_\zeta^{\boldsymbol{\alpha}_0}$,
(ii) h_ζ analytically depends on ζ as $H_{[p+2]}$-valued function,
(iii) $h_0 \equiv 1$ and
(iv) $\inf_{\zeta \in U, x \in M} |h_\zeta(x)| > 0$.

Note that the general analytic perturbation theory only guarantees that h_ζ is analytic as L^2-valued function. In this case, however, it implies (ii) (cf. III Sect. 1 Remark 1.38 of [13]). In the same way as we introduced $\mathcal{L}^{\boldsymbol{\alpha}}$ by using $h^{\boldsymbol{\alpha}}$, we define $\mathcal{L}_\zeta^{\boldsymbol{\alpha}_0}$ by

$$\mathcal{L}_\zeta^{\boldsymbol{\alpha}_0} u := \mathcal{L}^{\boldsymbol{\alpha}_0} u + \left(\zeta\alpha - \frac{dh_\zeta}{h_\zeta}, du \right).$$

By Corollary 2, $h_\zeta^{-1} dh_\zeta$ is analytic as a $H_{[p+1]}$-valued function. By applying the analytic perturbation theory to $(\mathcal{L}_\zeta^{\boldsymbol{\alpha}_0})^*$, the dual operator of $\mathcal{L}_\zeta^{\boldsymbol{\alpha}_0}$ in L^2, we obtain a family of $H_{[p+1]}$-valued function $\{g_\zeta\}_{\zeta \in U'}$ indexed by an open neighborhood $U' \subset \mathbb{C}$ of 0 which satisfies the following:

(i) g_ζ is an eigenfunction of $(\mathcal{L}_\zeta^{\boldsymbol{\alpha}_0})^*$,
(ii) g_ζ is anti-analytically depends on ζ as $H_{[p+1]}$-valued function,
(iii) $g_0 = \dfrac{dm_{\boldsymbol{\alpha}_0}}{dv}$ and
(iv) $\displaystyle\int_M g_\zeta \, dv = 1$ for each $\zeta \in U'$ and g_ζ takes its value in $\mathbb{R}$ if $\zeta \in U' \cap \mathbb{R}$.

Indeed, there exists a family of functions $\{\tilde{g}_\zeta\}_{\zeta \in U'}$ satisfying (i)–(iii) and $\int_M \tilde{g}_\zeta \, dv \neq 0$. Hence $g_\zeta := \tilde{g}_\zeta / \int_M \tilde{g}_\zeta \, dv$ satisfies (i)–(iv). Take $\varepsilon \in U \cap U' \cap \mathbb{R}$. By the continuity argument for the spectra of perturbed operators, we can choose a neighborhood $U_1 \subset U$ of 0 in $\mathbb{R}$ so that h_ε is a principal eigenfunction if $\varepsilon \in U_1$. By the same argument, we obtain a neighborhood $U_1' \subset U'$ so that $g_\varepsilon dv$ is the normalized invariant measure of $\mathcal{L}_\varepsilon^{\boldsymbol{\alpha}_0}$ if $\varepsilon \in U_1'$. It follows from the fact, which is a consequence of the Krein-Rutman theorem, that the principal eigenvalue of $(\mathcal{L}_\varepsilon^{\boldsymbol{\alpha}_0})^*$ is 0.

We take $\varepsilon' > 0$ so that $(-\varepsilon', \varepsilon') \subset U_1 \cap U_1' \cap \mathbb{R}$. Set $\psi_\varepsilon := -\log h_\varepsilon$. We claim that $\gamma_\varepsilon := \varepsilon\alpha - d\psi_\varepsilon$ fulfills all our requirements. By the choice of h_ε, g_ε and ε', $d\psi_\varepsilon = h_\varepsilon^{-1} dh_\varepsilon$ and the invariant measure $g_\varepsilon dv$ of $\mathcal{L}_\varepsilon^{\boldsymbol{\alpha}_0}$ are smooth. In order to complete the proof, it suffices to show that $\psi_0^{(1)}$ is a constant function. Note that ψ_ε satisfies

$$\mathcal{L}^{\boldsymbol{\alpha}_0} \psi_\varepsilon - \frac{1}{2} |\varepsilon\alpha - d\psi_\varepsilon|^2 = \Lambda^{\boldsymbol{\alpha}_0}(\varepsilon\alpha). \tag{2.21}$$

Obviously ψ_0 is a constant function. Differentiating (2.21) at $\varepsilon = 0$, we have

$$\mathcal{L}^{\boldsymbol{\alpha}_0} \psi_0^{(1)} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \Lambda^{\boldsymbol{\alpha}_0}(\varepsilon\alpha).$$

Integrating this equality by $m_{\boldsymbol{\alpha}_0}$, we obtain $\mathcal{L}^{\boldsymbol{\alpha}_0} \psi_0^{(1)} = 0$ and hence $\psi_0^{(1)}$ is also a constant function. ◻

In order to express the result of Proposition 3 in more useful form, we will introduce an operator S_μ and a functional L_μ for $\mu \in \mathcal{M}_1$. We take an orthonormal basis $\{\alpha_n\}_{n\in\mathbb{N}}$ of L^2_1 such that α_n is an n-th eigenform of $-\Delta_1$ corresponding to the n-th eigenvalue λ_n (counting multiplicity) for each $n \in \mathbb{N}$. Obviously $\alpha_n \in \mathcal{D}_p$ for each $n \in \mathbb{N}$. Letting $\bar{\alpha}_n := (1+\lambda_n)^{-p/2}\alpha_n$, we obtain an orthonormal basis $\{\bar{\alpha}_n\}_{n\in\mathbb{N}}$ of $\mathcal{D}_p$. Let $\{\omega_n\}_{n\in\mathbb{N}}$ be the dual basis of $\{\bar{\alpha}_n\}_{n\in\mathbb{N}}$. For each $\mu \in \mathcal{M}_1$, a linear operator $S_\mu : \mathcal{D}_{-p} \to \mathcal{D}_{-p}$ is given by

$$S_\mu \omega_n := \sum_{m\in\mathbb{N}} s_\mu(n,m)\omega_m,$$
$$s_\mu(n,m) := \int_M (\bar{\alpha}_n, \bar{\alpha}_m)\, d\mu.$$

Lemma 5 *Let $\mu \in \mathcal{M}_1$.*

(i) *S_μ is a symmetric, nonnegative definite operator of trace class. In particular, S_μ is positive if* $\mathrm{supp}[\mu] = M$.

(ii) $\langle S_\mu(\alpha^*), \beta\rangle = \int_M (\alpha,\beta)\, d\mu$ *for each $\alpha, \beta \in \mathcal{D}_p$.*

Proof (i): S_μ is obviously symmetric. First we show that S_μ is of trace class. Take $q > 0$ with $d_0/2 < q < p - d_0/2$. Then we have

$$\begin{aligned}\sum_{n\in\mathbb{N}} \left| \left(S_\mu\omega_n, \omega_n\right)_{-p} \right| &\le \sum_{n,m\in\mathbb{N}} \left| s_\mu(n,m)\,(\omega_m,\omega_n)_{-p}\right| \\ &= \sum_{n\in\mathbb{N}} s_\mu(n,n) = \sum_{n\in\mathbb{N}} \int_M |\bar{\alpha}_n|^2 d\mu \le C \sum_{n\in\mathbb{N}} \|\bar{\alpha}_n\|_q^2.\end{aligned}$$

Here $C > 0$ is a constant of the Sobolev embedding from $\mathcal{D}_q$ to $\mathcal{C}$. By the definition of $\bar{\alpha}_n$, $\|\bar{\alpha}_n\|_q^2 = (1+\lambda_n)^{-p+q}$. The asymptotic formula for $\{\lambda_n\}_{n\in\mathbb{N}}$ (see [8]) implies $(1+\lambda_n)^{-p+q} \le C' n^{2(q-p)/d_0}$ for some constant $C' > 0$. It yields $\sum_{n\in\mathbb{N}} \|\bar{\alpha}_n\|_q^2 < \infty$. Hence S_μ is of trace class. Next we show that S_μ is nonnegative definite. For $w = \sum_{n\in\mathbb{N}} a_n\omega_n \in \mathcal{D}_{-p}$,

$$\left(w, S_\mu w\right)_{-p} = \sum_{n,m\in\mathbb{N}} a_n a_m s_\mu(n,m) = \int_M \left|\sum_{n\in\mathbb{N}} a_n\bar{\alpha}_n\right|^2 d\mu \ge 0.$$

Thus the desired result follows. The final part of the assertion is now obvious.

(ii): It suffices to prove the assertion in the case $\alpha = \bar{\alpha}_n$ and $\beta = \bar{\alpha}_m$. Since $\bar{\alpha}_n^* = \omega_n$, we have

$$\left\langle S_\mu\omega_n, \bar{\alpha}_m\right\rangle = \sum_{k\in\mathbb{N}} s_\mu(n,k)\,\langle\omega_k, \bar{\alpha}_m\rangle = s_\mu(n,m) = \int_M (\bar{\alpha}_n, \bar{\alpha}_m)\, d\mu.$$

Thus all assertions are proved. □

Take $\alpha \in \mathcal{D}_p$. Let $\tilde{w}_0$ be defined by (2.20) with γ_ε constructed in Lemma 4 so as to satisfy $\gamma_0^{(1)} = \alpha$. By virtue of Lemma 5 (i), we can easily verify that $\tilde{w}_0 = S_{\mu_0}(\alpha^*)$. Thus, for $\alpha \in \mathcal{D}_p$, Proposition 2 and Lemma 5 (ii) imply

$$\big(\alpha^*, S_{\mu_0}(\alpha^*)\big)_{-p} \geq \Big(S_{\mu_0}(\alpha^*), G^F_{\boldsymbol{w}_0} \circ S_{\mu_0}(\alpha^*)\Big)_{-p}.$$

Since * is bijective, for each $w \in \mathcal{D}_{-p}$,

$$\big(w, S_{\mu_0} w\big)_{-p} \geq \Big(S_{\mu_0} w, G^F_{\boldsymbol{w}_0} \circ S_{\mu_0} w\Big)_{-p}. \tag{2.22}$$

For $\mu \in \mathcal{M}_1$, let us define $L_\mu : \mathcal{D}_{-p} \to [0, \infty]$ by

$$L_\mu(w) := \begin{cases} \dfrac{1}{2} \inf \left\{ \|\eta\|^2_{-p} \ ; \ w = \sqrt{S_\mu}\,\eta \right\} & \text{if } w \in \mathrm{Range}(\sqrt{S_\mu}), \\ \infty & \text{otherwise.} \end{cases} \tag{2.23}$$

By definition, $2L_\mu(S_\mu w) = \big(S_\mu w, w\big)_{-p}$ follows.

Lemma 6 $L_\mu = I'_\mu$. *In particular,* $\mathrm{Range}(\sqrt{S_\mu}) = \mathcal{H}'_\mu$.

Proof Note that $\mathrm{Range}(S_\mu) \subset \mathcal{H}'_\mu$. It follows from Lemma 5 (ii). In addition, for $S_\mu w \in \mathcal{H}'_\mu$, $\widehat{S_\mu w} = w^*$ holds. Thus

$$I'_\mu(S_\mu w) = \frac{1}{2} \int_M |w^*|^2 d\mu = \frac{1}{2} \langle S_\mu w, w^* \rangle = \frac{1}{2} \big(S_\mu w, w\big)_{-p} = L_\mu(S_\mu w).$$

That is, $L_\mu|_{\mathrm{Range}(S_\mu)} = I'_\mu|_{\mathrm{Range}(S_\mu)}$.

We can easily verify that the functional $(I'_\mu)^{1/2}$ defines a norm on $\mathrm{Range}(S_\mu)$, and that its closure by $(I'_\mu)^{1/2}$ coincides with $\mathcal{H}'_\mu$. On the other hand, by using the spectral decomposition of S_μ, we can show that the functional $L_\mu^{1/2}$ defines a norm on $\mathrm{Range}(S_\mu)$ and that its completion by $L_\mu^{1/2}$ coincides with $\mathrm{Range}(\sqrt{S_\mu})$. Through these observations, we obtain the conclusion. □

Remark 3 By the observation in the above proof, $\mathcal{H}'_\mu$ becomes an Hilbert space with norm induced from $\sqrt{2}L_\mu^{1/2}$ when $\mathrm{supp}[\mu] = M$. We can easily show that the triplet $(\mathcal{D}_{-p}, \mathcal{H}'_\mu, \nu_{S_\mu})$ becomes an abstract Wiener space. Here the Gaussian measure ν_{S_μ} is defined by (1.6).

For $\mu \in \mathcal{M}_1$, $w \in \mathrm{Range}(\sqrt{S_\mu})$ and $\beta \in \mathcal{D}_p$, we have

$$|\langle w, \beta \rangle| = \left| \int_M \big(\hat{w}, \beta\big)\, d\mu \right| \leq \left\{ \int_M |\hat{w}|^2 d\mu \right\}^{1/2} \left\{ \int_M |\beta|^2 d\mu \right\}^{1/2} \leq \sqrt{2} \|\beta\|_B L_\mu(w)^{1/2}.$$

Hence we obtain

$$\|w\|_{-p} \leq \sqrt{2} C_S L_\mu(w)^{1/2}. \tag{2.24}$$

Thus, for each $w \in \mathrm{Range}(\sqrt{S_\mu})$, we can take a sequence $\{w_n\}_{n\in\mathbb{N}} \subset \mathrm{Range}(S_\mu)$ so that $\|w_n - w\|_{-p}$ and $L_\mu(w_n - w)$ converge to 0 as $n \to \infty$. Thus, by Lemma 6 and (2.22), we obtain

$$L_{\mu_0}(w) \geq \frac{1}{2} \left(w, G^F_{\boldsymbol{w}_0} w \right)_{-p} \tag{2.25}$$

for each $\boldsymbol{w}_0 \in \mathcal{K}_F$ and $w \in \mathcal{D}_{-p}$.

3 Uniform moderate deviation estimate

Take $\boldsymbol{w}_0 \in \mathcal{K}_F$ and set $\boldsymbol{\alpha}_0 = \nabla F(\boldsymbol{w}_0)$ as in Sect. 2.2. For $\boldsymbol{\alpha} = (\alpha, V) \in \mathfrak{D}_{p,p'}$, let $Y^{\boldsymbol{\alpha}}$ be a $\mathcal{D}_{-p}$-valued process defined by

$$Y^{\boldsymbol{\alpha}}_t(\beta) := Y_t(\beta) - \int_0^t \left(\alpha - d\psi^{\boldsymbol{\alpha}}, \beta\right)(z_s)ds.$$

Also, let $\boldsymbol{R}^{\boldsymbol{\alpha}}_t$ be a $\mathfrak{D}^*_{p,p'}$-valued process defined by

$$\boldsymbol{R}^{\boldsymbol{\alpha}}_t(\beta, U) := \mathcal{G}_{\boldsymbol{\alpha}}((\alpha - d\psi^{\boldsymbol{\alpha}}, \beta) + U)(z_0) - \mathcal{G}_{\boldsymbol{\alpha}}((\alpha - d\psi^{\boldsymbol{\alpha}}, \beta) + U)(z_t).$$

Recall that $\psi^{\boldsymbol{\alpha}}$ is defined in Sect. 2.2. In fact, $\{Y^{\boldsymbol{\alpha}}_t\}_{t\geq 0}$ is a $\mathcal{D}_{-p}$-valued martingale under $\mathbb{P}^{\boldsymbol{\alpha}}_x$. We prove the following auxiliary lemma which explains how $\boldsymbol{Y}_t$ changes under the Girsanov transform driven by $\boldsymbol{w}_0$.

Lemma 7 $\boldsymbol{Y}_t - t\boldsymbol{w}_0 = \boldsymbol{\Gamma}^*_{\boldsymbol{\alpha}_0} Y^{\boldsymbol{\alpha}_0}_t + \boldsymbol{R}^{\boldsymbol{\alpha}_0}_t$.

Proof Note that $\mu_0 = m_{\boldsymbol{\alpha}_0}$ by Lemma 3. The definition of $\mathcal{G}_{\boldsymbol{\alpha}_0}$ and the Itô formula imply

$$\begin{aligned}\langle l_t - t\mu_0, U\rangle &= \int_0^t \left(U(z_s) - \int_M U\, dm_{\boldsymbol{\alpha}_0} \right) ds = -\int_0^t \mathcal{L}^{\boldsymbol{\alpha}_0} \mathcal{G}_{\boldsymbol{\alpha}_0} U(z_s)ds \\ &= \mathcal{G}_{\boldsymbol{\alpha}_0} U(z_0) - \mathcal{G}_{\boldsymbol{\alpha}_0} U(z_t) + Y^{\boldsymbol{\alpha}_0}_t(d\mathcal{G}_{\boldsymbol{\alpha}_0} U). \end{aligned} \tag{3.1}$$

Similarly, Lemma 3 implies

$$
\begin{aligned}
Y_t(\beta) - t\,\langle w_0, \beta\rangle &= Y_t^{\boldsymbol{\alpha}_0}(\beta) + \left(\int_0^t \left(\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \beta\right)(z_s)ds \right. \\
&\qquad \left. - t \int_M \left(\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \beta\right) d\mu_0 \right) \\
&= Y_t^{\boldsymbol{\alpha}_0}(\beta) - \int_0^t \mathcal{L}^{\boldsymbol{\alpha}_0} \mathcal{G}_{\boldsymbol{\alpha}_0}(\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \beta)(z_s)ds \\
&= Y_t^{\boldsymbol{\alpha}_0}(\Gamma_{\boldsymbol{\alpha}_0}\beta) + \left(\mathcal{G}_{\boldsymbol{\alpha}_0}(\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \beta)(z_0) - \mathcal{G}_{\boldsymbol{\alpha}_0}(\alpha_0 - d\psi^{\boldsymbol{\alpha}_0}, \beta)(z_t)\right).
\end{aligned}
\tag{3.2}
$$

Combining (3.1) with (3.2), the conclusion follows. □

Note that there exists a constant $C_{\boldsymbol{R}} = C_{\boldsymbol{R}}(\boldsymbol{\alpha}_0) > 0$ such that

$$
\sup_{\boldsymbol{\beta} \in \boldsymbol{\mathcal{D}}_{p,p'}} |\boldsymbol{R}_t^{\boldsymbol{\alpha}_0}(\boldsymbol{\beta})| \le C_{\boldsymbol{R}} \tag{3.3}
$$

for each $t \ge 0$ by the Sobolev embedding. The following lemma, obtained as a corollary of Lemma 3.3 of [17], plays a crucial role in the proof of Proposition 4 below.

Lemma 8 *For $d_0 < p' \le p$, there is a constant $C_1 = C_1(p')$ so that*

$$
\sup_{x \in M} \mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\left\| \frac{1}{\sqrt{t}} Y_t^{\boldsymbol{\alpha}_0} \right\|_{-p'} \ge r \right] \le C_1 \exp\left(-\frac{r^2}{C_1} \right).
$$

Proposition 4 *For each closed set $\mathcal{A} \subset \mathcal{D}_{-p}$,*

$$
\limsup_{c \to \infty} \left(\sup_{c \le s \le \sqrt{t}/c} \frac{1}{s^2} \log \mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\frac{1}{\sqrt{t}} Y_t^{\boldsymbol{\alpha}_0} \in s\mathcal{A} \right] \right) \le - \inf_{w \in \mathcal{A}} L_{\mu_0}(w).
$$

Proof For simplicity of notation, we assume that $\boldsymbol{\alpha}_0 = 0$. Our proof also works for the case $\boldsymbol{\alpha}_0 \ne 0$ in the same way. We divided the proof into two steps.

Step 1: the case $\mathcal{A}$ is compact. For $\delta > 0$, set $L_m^{(\delta)} := (L_m - \delta) \wedge (1/\delta)$. The functional $\Xi : \mathcal{D}_p \to \mathbb{R}$ given by

$$
\Xi(\beta) := \frac{1}{2} \int_M |\beta|^2 dm
$$

is the Legendre conjugate of L_m. It means

$$L_m(w) = \sup_{\beta \in \mathcal{D}_p} (\langle w, \beta \rangle - \Xi(\beta))$$

Thus, for each $w \in \mathcal{D}_{-p}$, there is $\alpha_w \in \mathcal{D}_p$ such that

$$\langle w, \alpha_w \rangle - \Xi(\alpha_w) \geq L_m^{(\delta)}(w).$$

Since $\alpha_w : \eta \mapsto \langle \eta, \alpha_w \rangle$ is continuous, there is a neighborhood U_w of w in $\mathcal{D}_{-p}$ so that

$$\sup_{\eta \in U_w} (\langle w, \alpha_w \rangle - \langle \eta, \alpha_w \rangle) \leq \delta.$$

For each $\alpha \in \mathcal{D}_p$,

$$\mathbb{P}_x \left[\frac{1}{s\sqrt{t}} Y_t \in U_w \right] \leq \mathbb{E}_x \left[\exp\left(\frac{1}{s\sqrt{t}} Y_t(\alpha) - \langle w, \alpha \rangle \right) \right] \exp\left(\sup_{\eta \in U_w} (\langle w, \alpha \rangle - \langle \eta, \alpha \rangle) \right).$$

Substituting $\alpha = s^2 \alpha_w$,

$$\frac{1}{s^2} \log \mathbb{P}_x \left[\frac{1}{s\sqrt{t}} Y_t \in U_w \right] \leq \frac{1}{s^2} \log \mathbb{E}_x \left[\exp\left(\frac{s}{\sqrt{t}} Y_t(\alpha_w) \right) \right] - \langle w, \alpha_w \rangle + \delta.$$

Choose a finite subcover $\{U_{w_i}\}_{i=1}^n$ of a covering $\{U_w\}_{w \in \mathcal{A}}$ of $\mathcal{A}$. Then

$$\frac{1}{s^2} \log \mathbb{P}_x \left[\frac{1}{s\sqrt{t}} Y_t \in \mathcal{A} \right] \leq \frac{1}{s^2} \log n + \delta - \min_{1 \leq i \leq n} \Bigg(\langle w_i, \alpha_{w_i} \rangle - \frac{1}{s^2} \log \mathbb{E}_x \left[\exp\left(\frac{s}{\sqrt{t}} Y_t(\alpha_{w_i}) \right) \right] \Bigg). \tag{3.4}$$

Now we claim the following:

$$\limsup_{c \to \infty} \sup_{c \leq s \leq \sqrt{t}/c} \left(\frac{1}{s^2} \log \mathbb{E}_x \left[\exp\left(\frac{s}{\sqrt{t}} Y_t(\alpha) \right) \right] \right) \leq \Xi(\alpha). \tag{3.5}$$

If (3.5) holds true, (3.4) implies

$$\begin{aligned} &\limsup_{c \to \infty} \sup_{c \leq s \leq \sqrt{t}/c} \left(\frac{1}{s^2} \log \mathbb{P}_x \left[\frac{1}{s\sqrt{t}} Y_t \in \mathcal{A} \right] \right) \\ &\leq \delta - \min_{1 \leq i \leq n} L_m^{(\delta)}(w_i) \leq \delta - \inf_{w \in \mathcal{A}} L_m^{(\delta)}(w). \end{aligned}$$

Hence, as $\delta \to 0$, the conclusion follows.

Set

$$\mathcal{A}_{\varepsilon,t} := \left\{ \left| \frac{1}{t} \langle Y(\alpha) \rangle_t - \int_M |\alpha|^2 \, dm \right| < \varepsilon \right\}.$$

By the Schwarz inequality,

$$\begin{aligned} \mathbb{E}_x \left[\exp\left(\frac{s}{\sqrt{t}} Y_t(\alpha) \right) ; \ \mathcal{A}^c_{\varepsilon,t} \right] &\le \left\{ \mathbb{E}_x \left[\exp\left(\frac{2s}{\sqrt{t}} Y_t(\alpha) \right) \right] \right\}^{1/2} \mathbb{P}_x \left[\mathcal{A}^c_{\varepsilon,t} \right]^{1/2} \\ &\le \exp\left(s^2 C_S^2 \|\alpha\|_p^2 \right) \mathbb{P}_x \left[\mathcal{A}^c_{\varepsilon,t} \right]^{1/2} \end{aligned} \tag{3.6}$$

holds. Since $\langle Y(\alpha) \rangle_t = \langle l_t, |\alpha|^2 \rangle$, the large deviation for $\bar{l}_t$ yields that there is $r > 0$ such that

$$\mathbb{P}_x \left[\mathcal{A}^c_{\varepsilon,t} \right] \le \exp(-rt) \tag{3.7}$$

for sufficiently large t. On the other hand,

$$\mathbb{E}_x \left[\exp\left(\frac{s}{\sqrt{t}} Y_t(\alpha) \right) ; \ \mathcal{A}_{\varepsilon,t} \right] \le \exp\left(\frac{s^2}{2} \int_M |\alpha|^2 \, dm + \frac{s^2 \varepsilon}{2} \right). \tag{3.8}$$

By (3.6), (3.7) and (3.8), we obtain

$$\begin{aligned} &\sup_{c \le s \le \sqrt{t}/c} \left(\frac{1}{s^2} \log \mathbb{E}_x \left[\exp\left(\frac{s}{\sqrt{t}} Y_t(\alpha) \right) \right] \right) \\ &\le \sup_{c \le s \le \sqrt{t}/c} \left\{ \frac{1}{s^2} \log 2 + \left(\Xi(\alpha) + \frac{\varepsilon}{2} \right) \vee \left(C_S^2 \|\alpha\|_p^2 - \frac{rt}{2s^2} \right) \right\}. \end{aligned}$$

Since

$$C_S^2 \|\alpha\|_p^2 - \frac{rt}{2s^2} \le C_S^2 \|\alpha\|_p^2 - \frac{rc^2}{2} \to -\infty$$

as $c \to \infty$, we have

$$\limsup_{c \to \infty} \sup_{c \le s \le \sqrt{t}/c} \left(\frac{1}{s^2} \log \mathbb{E}_x \left[\exp\left(\frac{s}{\sqrt{t}} Y_t(\alpha) \right) \right] \right) \le \Xi(\alpha) + \frac{\varepsilon}{2}.$$

By letting $\varepsilon \to 0$, (3.5) follows.

Step 2: the case $\mathcal{A}$ is closed. Choose $p_1 > 0$ with $d_0 < p_1 < p$. For $r > 0$, We write $B_r^{p_1} = \{ w \in \mathcal{D}_{-p} \ ; \ \|w\|_{-p_1} \le r \}$. Then we have $\mathcal{A} \subset (B_r^{p_1} \cap \mathcal{A}) \cup (B_r^{p_1})^c$.

By Lemma 8,

$$\frac{1}{s^2}\log\mathbb{P}_x\left[\frac{1}{s\sqrt{t}}Y_t\in\mathcal{A}\right]\le\frac{1}{s^2}\log 2+\left(\frac{1}{s^2}\log C_1-\frac{r^2}{C_1}\right)\vee\left(\frac{1}{s^2}\log\mathbb{P}_x\left[\frac{1}{s\sqrt{t}}Y_t\in\mathcal{A}\cap B_r^{p_1}\right]\right).$$

Since $\mathcal{A}\cap B_r^{p_1}$ is compact, this estimate implies

$$\begin{aligned}\limsup_{c\to\infty}\sup_{c\le s\le\sqrt{t}/c}\left(\frac{1}{s^2}\log\mathbb{P}_x\left[\frac{1}{s\sqrt{t}}Y_t\in\mathcal{A}\right]\right)&\le\left(-\frac{r^2}{C_1}\right)\vee\left(-\inf_{w\in\mathcal{A}\cap B_r^{p_1}}L_m(w)\right)\\&\le\left(-\frac{r^2}{C_1}\right)\vee\left(-\inf_{w\in\mathcal{A}}L_m(w)\right).\end{aligned}$$

As $r\to\infty$, the desired result follows. □

Let $L_{\boldsymbol{\gamma}}^{\boldsymbol{\Gamma}}$ be the rate function determined by the contraction principle for $L_{m_{\boldsymbol{\gamma}}}$ via $\boldsymbol{\Gamma}_{\boldsymbol{\gamma}}^*$:

$$L_{\boldsymbol{\gamma}}^{\boldsymbol{\Gamma}}(\boldsymbol{w}):=\inf_{\boldsymbol{\Gamma}_{\boldsymbol{\gamma}}^*\eta=\boldsymbol{w}}L_{m_{\boldsymbol{\gamma}}}(\eta).\tag{3.9}$$

Corollary 3 *For each closed set $\boldsymbol{\mathcal{A}}\subset\boldsymbol{\mathfrak{D}}_{p,p'}^*$,*

$$\limsup_{c\to\infty}\sup_{c\le s\le\sqrt{t}/c}\left(\frac{1}{s^2}\log\mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\frac{1}{\sqrt{t}}\boldsymbol{Y}_t-\sqrt{t}\boldsymbol{w}_0\in s\boldsymbol{\mathcal{A}}\right]\right)\le-\inf_{\boldsymbol{w}\in\boldsymbol{\mathcal{A}}}L_{\boldsymbol{\alpha}_0}^{\boldsymbol{\Gamma}}(\boldsymbol{w}).$$

Proof Take $\delta>0$. Set $\boldsymbol{\mathcal{A}}^{(\delta)}\subset\boldsymbol{\mathfrak{D}}_{p,p'}^*$ by

$$\boldsymbol{\mathcal{A}}^{(\delta)}:=\left\{\boldsymbol{w}\in\boldsymbol{\mathfrak{D}}_{p,p'}^*\;;\;\inf_{\boldsymbol{\eta}\in\boldsymbol{\mathcal{A}}}\|\boldsymbol{w}-\boldsymbol{\eta}\|_{-p}\le\delta\right\}.$$

Lemma 7 and (3.3) imply that, for $t,s>0$ with $C_{\boldsymbol{R}}\le\delta s\sqrt{t}$, we have

$$\left\{\frac{1}{\sqrt{t}}\boldsymbol{Y}_t-\sqrt{t}\boldsymbol{w}_0\in s\boldsymbol{\mathcal{A}}\right\}\subset\left\{\frac{1}{s\sqrt{t}}Y_t^{\boldsymbol{\alpha}_0}\in(\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^*)^{-1}\boldsymbol{\mathcal{A}}^{(\delta)}\right\}.$$

Hence, by Proposition 4,

$$\begin{aligned}&\limsup_{c\to\infty}\sup_{c\le s\le\sqrt{t}/c}\left(\frac{1}{s^2}\log\mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\frac{1}{\sqrt{t}}\boldsymbol{Y}_t-\sqrt{t}\boldsymbol{w}_0\in s\boldsymbol{\mathcal{A}}\right]\right)\\&\quad\le-\inf_{w\in(\Gamma_{\boldsymbol{\alpha}_0}^*)^{-1}\boldsymbol{\mathcal{A}}^{(\delta)}}L_{\mu_0}(w)=-\inf_{\boldsymbol{w}\in\boldsymbol{\mathcal{A}}^{(\delta)}}L_{\boldsymbol{\alpha}_0}^{\boldsymbol{\Gamma}}(\boldsymbol{w}).\end{aligned}$$

By letting $\delta\downarrow 0$, the conclusion follows. □

4 Proof of main theorems

In addition to (**D**) and (**F1**)-(**F3**), the argument in this section requires the following assumption concerning the nondegeneracy of the Hessian at equilibrium states (cf. (2.25)):

Assumption 1 For each $\boldsymbol{w}_0 = (w_0, \mu_0) \in \mathcal{K}_F$, there exists $\delta_{\boldsymbol{w}_0} > 0$ such that

$$L_{\mu_0}(w) \geq \frac{1}{2}\left(w, G^F_{\boldsymbol{w}_0} w\right)_{-p} + \delta_{\boldsymbol{w}_0} \|w\|^2_{-p} \tag{4.1}$$

holds for all $w \in \mathcal{D}_{-p}$.

Remark 4 In order to verify Assumption 1, it suffices to show (4.1) for $w \in \left(\mathcal{H}'_{\mu_0} \cap \mathrm{Ker}\, G^F_{\boldsymbol{w}_0}\right)^{\perp}$, the orthogonal complement of $\mathcal{H}'_{\mu_0} \cap \mathrm{Ker}\, G^F_{\boldsymbol{w}_0}$ in $\mathcal{H}'_{\mu_0}$ with respect to the inner product induced from L_{μ_0}. To show the assertion, we take $w \in \mathcal{H}'_{\mu_0}$ and write $w = w_K + w_C$, where $w_K \in \mathcal{H}'_{\mu_0} \cap \mathrm{Ker}\, G^F_{\boldsymbol{w}_0}$ and $w_C \in (\mathcal{H}'_{\mu_0} \cap \mathrm{Ker}\, G^F_{\boldsymbol{w}_0})^{\perp}$. Then (2.24) implies

$$\begin{aligned}
&\frac{1}{2}(w\,,\, G^F_{\boldsymbol{w}_0} w)_{-p} + \frac{1}{2}\left\{\delta_{\boldsymbol{w}_0} \wedge \left(\frac{1}{2C_S^2}\right)\right\} \|w\|^2_{-p} \\
&\leq \frac{1}{2C_S^2}\|w_K\|^2_{-p} + \delta_{\boldsymbol{w}_0}\|w_C\|^2_{-p} + \frac{1}{2}(w_C\,,\, G^F_{\boldsymbol{w}_0} w_C)_{-p} \\
&\leq L_{\mu_0}(w_K) + L_{\mu_0}(w_C) = L_{\mu_0}(w).
\end{aligned}$$

Hence Assumption 1 holds with the constant $(\delta_{\boldsymbol{w}_0} \wedge (C_S^{-2}/2))/2$.

First we consider the simplest case, that is, the case $\#\mathcal{K}_F = 1$.

Theorem 3 *Let $\varphi : M \to \mathbb{R}$ be a positive continuous function. Suppose that (**D**), (**F1**)–(**F3**) and Assumption 1 hold. Assume $\#\mathcal{K}_F = 1$. Set $\boldsymbol{w}_0 = (w_0, \mu_0) \in \mathcal{K}_F$. Then*

$$\lim_{t\to\infty} \mathrm{e}^{-t\kappa_F} \mathbb{E}_x\left[\mathrm{e}^{tF(\bar{Y}_t)}\varphi(z_t)\right] = \frac{1}{\det(1 - G^F_{\boldsymbol{w}_0} \circ S_{\mu_0})^{1/2}} h^{\nabla F(\boldsymbol{w}_0)}(x) \int_M \frac{\varphi}{h^{\nabla F(\boldsymbol{w}_0)}} d\mu_0.$$

In order to give a proof, let us decompose the left hand side into three parts. For $c_1, c_2 > 0$,

$$\begin{aligned}
J_1(c_1, t) &:= \mathrm{e}^{-t\kappa_F}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{\boldsymbol{Y}}^t)}\varphi(z_t)\,;\; \|\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0\|_{\mathcal{D}^*_{p,p'}} \leq \frac{c_1}{\sqrt{t}}\right], \\
J_2(c_1, c_2, t) &:= \mathrm{e}^{-t\kappa_F}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{\boldsymbol{Y}}_t)}\varphi(z_t)\,;\; \frac{c_1}{\sqrt{t}} < \|\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0\|_{\mathcal{D}^*_{p,p'}} \leq c_2\right], \\
J_3(c_2, t) &:= \mathrm{e}^{-t\kappa_F}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{\boldsymbol{Y}}_t)}\varphi(z_t)\,;\; c_2 < \|\bar{\boldsymbol{Y}}_t - w_0\|_{\mathcal{D}^*_{p,p'}}\right].
\end{aligned}$$

Associated with this decomposition, the proof is divided into the following three lemmas.

Lemma 9 $\limsup_{t\to\infty} J_3(c_2, t) = 0$ *for all* $c_2 > 0$.

Proof The large deviation principle for $\bar{\boldsymbol{Y}}_t$ implies

$$\limsup_{t\to\infty} \frac{1}{t} \log J_3(c_2, t) \le -\kappa_F + \sup\left\{F(\boldsymbol{w}) - I(\boldsymbol{w}) \; ; \; \|\boldsymbol{w} - \boldsymbol{w}_0\|_{\mathfrak{D}^*_{p,p'}} \ge c_2\right\} < 0.$$

This estimate yields the conclusion. □

For simplicity of notation, we set $\boldsymbol{\alpha}_0 = \nabla F(\boldsymbol{w}_0)$ as in Sects. 2.2 and 3.

Lemma 10 *There exists* $\hat{J}_1(c_1) := \lim_{t\to\infty} J_1(c_1, t)$ *for all but countably many* $c_1 > 0$, *and*

$$\lim_{c_1\to\infty} \hat{J}_1(c_1) = \frac{1}{\det(1 - G^F_{\boldsymbol{w}_0} \circ S_{\mu_0})^{1/2}} h^{\boldsymbol{\alpha}_0}(x) \int_M \frac{\varphi}{h^{\boldsymbol{\alpha}_0}} d\mu_0.$$

Proof Let $\mathcal{A} := \{\|\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0\|_{\mathfrak{D}^*_{p,p'}} \le c_1 t^{-1/2}\}$. The Taylor expansion of F near $\boldsymbol{w}_0$ up to the second order yields

$$\begin{aligned} J_1(c_1, t) &= \exp\left\{t(I(\boldsymbol{w}_0) - \langle \boldsymbol{w}_0, \boldsymbol{\alpha}_0\rangle_{\mathfrak{D}_{p,p'}})\right\} \\ &\quad\times \mathbb{E}_x\left[\varphi(z_t)\exp\left\{\boldsymbol{Y}_t(\boldsymbol{\alpha}_0) + t\left(F(\bar{\boldsymbol{Y}}_t) - F(\boldsymbol{w}_0)\right.\right.\right. \\ &\qquad\qquad \left.\left.\left. -\nabla F(\boldsymbol{w}_0)(\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0)\right)\right\} ; \; \mathcal{A}\right] \\ &= h^{\boldsymbol{\alpha}_0}(x)\mathbb{E}^{\boldsymbol{\alpha}_0}_x\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)} \exp\left\{t\left(\frac{1}{2}\nabla^2 F(\boldsymbol{w}_0)(\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0\,,\; \bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0)\right.\right.\right. \\ &\qquad\qquad \left.\left.\left. + r_F(\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0)\right)\right\} ; \; \mathcal{A}\right]. \end{aligned} \tag{4.2}$$

Here $r_F(\boldsymbol{w})$ is the remainder term satisfying $r_F(\boldsymbol{w}) = o(\|\boldsymbol{w}\|^2_{\mathfrak{D}^*_{p,p'}})$ as $\|\boldsymbol{w}\|_{\mathfrak{D}^*_{p,p'}}$ goes to 0. Take $\delta > 0$. Since $t|r_F(\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0)| \le \delta$ holds on $\mathcal{A}$ for all sufficiently large t, we obtain

$$\begin{aligned} \limsup_{t\to\infty} J_1(c_1, t) \le \mathrm{e}^{\delta} \limsup_{t\to\infty} h^{\boldsymbol{\alpha}_0}(x)\mathbb{E}^{\boldsymbol{\alpha}_0}_x&\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\right. \\ &\left.\times \exp\left(\frac{t}{2}\nabla^2 F(\boldsymbol{w}_0)(\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0\,,\; \bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0)\right) \; ; \; \mathcal{A}\right]. \end{aligned}$$

By Lemma 7 and (3.3), if $t > C^2_{\boldsymbol{R}}\delta^{-2}$,

$$\left\|\frac{1}{\sqrt{t}}\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha}_0} Y^{\boldsymbol{\alpha}_0}_t\right\|_{\mathfrak{D}^*_{p,p'}} \le c_1 + \delta \tag{4.3}$$

holds on $\mathcal{A}$. Let $C_F := \sup_{\|\boldsymbol{w}\|_{\mathfrak{D}^*_{p,p'}} \le 1} |\nabla^2 F(\boldsymbol{w}_0)(\boldsymbol{w}, \boldsymbol{w})|$. Then we have

$$\limsup_{t\to\infty} J_1(c_1, t) \le \exp\left\{\delta + \frac{C_F}{2}\delta(2c_1 + 3\delta)\right\} \times h^{\boldsymbol{\alpha}_0}(x) \limsup_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)} \exp\left(\frac{1}{2t}(Y_t^{\boldsymbol{\alpha}_0}, G^F_{\boldsymbol{w}_0} Y_t^{\boldsymbol{\alpha}_0})_{-p}\right); \mathcal{A}\right]. \tag{4.4}$$

In the same way as we derive the estimate (4.4) from (4.2), we obtain

$$\liminf_{t\to\infty} J_1(c_1, t) \ge \exp\left\{-\delta - \frac{C_F}{2}\delta(2c_1 + 3\delta)\right\} \times h^{\boldsymbol{\alpha}_0}(x) \liminf_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)} \exp\left(\frac{1}{2t}(Y_t^{\boldsymbol{\alpha}_0}, G^F_{\boldsymbol{w}_0} Y_t^{\boldsymbol{\alpha}_0})_{-p}\right); \mathcal{A}\right]. \tag{4.5}$$

Note that the integrand of the expectation in the right hand side of (4.4) or (4.5) is bounded.

We set

$$E_\delta := \left\{\left\|\frac{1}{\sqrt{t}}(Y_t^{\boldsymbol{\alpha}_0} - Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0})\right\|_{-p} \le \delta\right\}.$$

For each random variable W with $|W| \le C$ for some constant $C > 0$, Lemma 8 implies

$$\left|\mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[W\ ;\ E_\delta^c\right]\right| \le C\mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\mathbb{P}_{z_{t-\sqrt{t}}}^{\boldsymbol{\alpha}_0}\left[\|Y_{\sqrt{t}}^{\boldsymbol{\alpha}_0}\|_{-p} > \delta\sqrt{t}\right]\right] \le CC_1 \exp\left(-\frac{\delta^2\sqrt{t}}{C_1}\right).$$

Thus

$$\lim_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[W\ ;\ E_\delta^c\right] = 0. \tag{4.6}$$

We write $\delta' = \delta\|\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha}_0}\|_{\mathfrak{D}_{-p}\to\mathfrak{D}^*_{p,p'}}$. Note that (4.3) implies that, if $t > C_{\boldsymbol{R}}^2\delta^{-2}$,

$$\mathcal{A} \cap E_\delta \subset \left\{\left\|\frac{1}{\sqrt{t}}\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha}_0} Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}\right\|_{\mathfrak{D}^*_{p,p'}} \le c_1 + \delta + \delta'\right\} =: \mathcal{A}_\delta. \tag{4.7}$$

Thus, by virtue of (4.6), we have

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(\frac{1}{2t}\left(Y_t^{\boldsymbol{\alpha}_0}, G_{\boldsymbol{w}_0}^F Y_t^{\boldsymbol{\alpha}_0}\right)_{-p}\right)\ ;\ \mathcal{A}\right]\\
&\quad= \limsup_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(\frac{1}{2t}\left(Y_t^{\boldsymbol{\alpha}_0}, G_{\boldsymbol{w}_0}^F Y_t^{\boldsymbol{\alpha}_0}\right)_{-p}\right)\ ;\ \mathcal{A}\cap E_\delta\right]\\
&\quad\le \exp\left\{\frac{C_F}{2}\delta'(2c_1+2\delta+3\delta')\right\}\\
&\qquad\times \limsup_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(\frac{1}{2t}\left(Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}, G_{\boldsymbol{w}_0}^F Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}\right)_{-p}\right)\ ;\ \mathcal{A}_\delta\right].
\end{aligned}\tag{4.8}$$

Since $\mathcal{A}_{-\delta}\cap E_\delta \subset \mathcal{A}\cap E_\delta$ holds if $t > C_{\boldsymbol{R}}^2\delta^{-2}$, we have

$$\begin{aligned}
&\liminf_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(\frac{1}{2t}\left(Y_t^{\boldsymbol{\alpha}_0}, G_{\boldsymbol{w}_0}^F Y_t^{\boldsymbol{\alpha}_0}\right)_{-p}\right)\ ;\ \mathcal{A}\right]\\
&\quad= \liminf_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(\frac{1}{2t}\left(Y_t^{\boldsymbol{\alpha}_0}, G_{\boldsymbol{w}_0}^F Y_t^{\boldsymbol{\alpha}_0}\right)_{-p}\right)\ ;\ \mathcal{A}\cap E_\delta\right]\\
&\quad\ge \exp\left\{-\frac{C_F}{2}\delta'(2c_1+2\delta+3\delta')\right\}\\
&\qquad\times \liminf_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(\frac{1}{2t}\left(Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}, G_{\boldsymbol{w}_0}^F Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}\right)_{-p}\right)\ ;\ \mathcal{A}_{-\delta}\cap E_\delta\right]\\
&\quad= \exp\left\{-\frac{C_F}{2}\delta'(2c_1+2\delta+3\delta')\right\}\\
&\qquad\times \liminf_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(\frac{1}{2t}\left(Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}, G_{\boldsymbol{w}_0}^F Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}\right)_{-p}\right)\ ;\ \mathcal{A}_{-\delta}\right].
\end{aligned}\tag{4.9}$$

The last equality follows from (4.6).

We choose c_1 and δ so that $\nu_{S_{\mu_0}}(\{w\in\mathcal{D}_{-p}\ ;\ \|\Gamma_{\boldsymbol{\alpha}_0}^* w\|_{\mathcal{D}_{p,p'}^*} = a\}) = 0$ for $a = c_1, c_1+\delta+\delta', c_1-\delta-\delta'$. Recall that $\nu_{S_{\mu_0}}$ is given by (1.6) for $S = S_{\mu_0}$. Since $\{z_t\}_{t\ge 0}$ is strongly mixing under $\mathbb{P}_x^{\boldsymbol{\alpha}_0}$ (see [5,6]), z_t and $Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}$ are asymptotically independent as $t\to\infty$. Therefore, the central limit theorem for $Y^{\boldsymbol{\alpha}_0}$ yields that

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(\frac{1}{2t}\left(Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}, G_{\boldsymbol{w}_0}^F Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}\right)_{-p}\right)\ ;\ \mathcal{A}_\delta\right]\\
&\quad= \int_M \frac{\varphi}{h^{\boldsymbol{\alpha}_0}}d\mu_0 \int_{\|\boldsymbol{\Gamma}_{\boldsymbol{\alpha}_0}^* w\|_{\mathcal{D}_{p,p'}^*}\le c_1+\delta+\delta'} \exp\left(\frac{1}{2}(w, G_{\boldsymbol{w}_0}^F w)_{-p}\right)\nu_{S_{\mu_0}}(dw),\\
&\liminf_{t\to\infty} \mathbb{E}_x^{\boldsymbol{\alpha}_0}\left[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(\frac{1}{2t}\left(Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}, G_{\boldsymbol{w}_0}^F Y_{t-\sqrt{t}}^{\boldsymbol{\alpha}_0}\right)_{-p}\right)\ ;\ \mathcal{A}_{-\delta}\right]
\end{aligned}$$

$$= \int_M \frac{\varphi}{h^{\boldsymbol{\alpha}_0}} d\mu_0 \int_{\|\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha}_0} w\|_{\mathfrak{D}^*_{p,p'}} \le c_1 - \delta - \delta'} \exp\left(\frac{1}{2}(w, G^F_{\boldsymbol{w}_0} w)_{-p}\right) \nu_{S_{\mu_0}}(dw).$$

Combining this fact with (4.4), (4.5), (4.8) and (4.9) and letting $\delta \to 0$, we obtain

$$\hat{J}_1(c_1) = \lim_{t\to\infty} J_1(c_1, t) = h^{\boldsymbol{\alpha}_0}(x) \int_M \frac{\varphi}{h^{\boldsymbol{\alpha}_0}} d\mu_0 \times \int_{\|\boldsymbol{\Gamma}^*_{\boldsymbol{\alpha}_0} w\|_{\mathfrak{D}^*_{p,p'}} \le c_1} \exp\left(\frac{1}{2}(w, G^F_{\boldsymbol{w}_0} w)_{-p}\right) \nu_{S_{\mu_0}}(dw) \tag{4.10}$$

by the choice of c_1. Since we have

$$\int_{\mathfrak{D}_{-p}} \exp\left(\frac{1}{2}(w, G^F_{\boldsymbol{w}_0} w)_{-p}\right) \nu_{S_{\mu_0}}(dw) = \frac{1}{\det\left(1 - G^F_{\boldsymbol{w}_0} \circ S_{\mu_0}\right)^{1/2}}$$

as a result of Lemma 4.4 in [7], the conclusion follows as $c_1 \to \infty$ in (4.10). □

Lemma 11 *For sufficiently small $c_2 > 0$, $\lim_{c_1\to\infty} \sup_{t>1} J_2(c_1, c_2, t) = 0$.*

Proof For $\varepsilon > 0$, let

$$\boldsymbol{\mathcal{A}}_\varepsilon := \left\{ \boldsymbol{w} \in \mathfrak{D}^*_{p,p'} \ ; \ \frac{1}{2}\nabla^2 F(\boldsymbol{w}_0)(\boldsymbol{w}, \boldsymbol{w}) + \frac{\varepsilon}{2}\|\boldsymbol{w}\|^2_{\mathfrak{D}^*_{p,p'}} \ge 1 \right\}.$$

First we will show

$$\inf_{\boldsymbol{w}\in\boldsymbol{\mathcal{A}}_\varepsilon} L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}(\boldsymbol{w}) > 1 \tag{4.11}$$

provided ε is sufficiently small. Recall that $L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}$ is given by (3.9). For $\delta > 0$ we define $\tilde{\boldsymbol{\mathcal{A}}}_\delta$ by

$$\tilde{\boldsymbol{\mathcal{A}}}_\delta := \left\{ \boldsymbol{w} \in \mathfrak{D}^*_{p,p'} \ ; \ \frac{1}{2}\nabla^2 F(\boldsymbol{w}_0)(\boldsymbol{w}, \boldsymbol{w}) \ge 1 - \delta \right\}.$$

Then, for all $r > 0$, we have

$$\inf_{\boldsymbol{w}\in\boldsymbol{\mathcal{A}}_\varepsilon} L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}(\boldsymbol{w}) \ge \left(\inf_{\|\boldsymbol{w}\|_{\mathfrak{D}^*_{p,p'}} \ge r} L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}(\boldsymbol{w}) \right) \wedge \left(\inf_{\boldsymbol{w}\in\tilde{\boldsymbol{\mathcal{A}}}_{r\varepsilon/2}} L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}(\boldsymbol{w}) \right).$$

Since (2.24) involves

$$\lim_{r\to\infty} \inf_{\|\boldsymbol{w}\|_{\mathfrak{D}^*_{p,p'}}\ge r} L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}(\boldsymbol{w}) = \infty,$$

we take $r > 1$ sufficiently large so that $\inf_{\|\boldsymbol{w}\|_{\mathfrak{D}^*_{p,p'}}\ge r} L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}(\boldsymbol{w}) > 1$. Since

$$\inf_{\boldsymbol{w}\in\tilde{\boldsymbol{\mathcal{A}}}_{r\varepsilon/2}} L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}(\boldsymbol{w}) = \left(1 - \frac{r\varepsilon}{2}\right) \inf_{\boldsymbol{w}\in\tilde{\boldsymbol{\mathcal{A}}}_0} L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}(\boldsymbol{w})$$

for $\varepsilon < 2r^{-1}$, it suffices to show that $\inf_{\boldsymbol{w}\in\tilde{\boldsymbol{\mathcal{A}}}_0} L^{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}_0}(\boldsymbol{w}) > 1$. Actually, this assertion follows from Assumption 1. Thus we obtain (4.11) for sufficiently small $\varepsilon > 0$.

Fix such a small $\varepsilon > 0$. If c_2 is small enough, the Taylor expansion implies

$$\begin{aligned} &F(\bar{\boldsymbol{Y}}_t) - F(\boldsymbol{w}_0) - \nabla F(\boldsymbol{w}_0)(\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0) \\ &\le \frac{1}{2}\nabla^2 F(\boldsymbol{w}_0)(\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0\,,\ \bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0) + \frac{\varepsilon}{2}\|\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0\|^2_{\mathfrak{D}^*_{p,p'}} \end{aligned}$$

for $\|\bar{\boldsymbol{Y}}_t - \boldsymbol{w}_0\|_{\mathfrak{D}^*_{p,p'}} \le c_2$. Thus, we have

$$\begin{aligned} J_2(c_1, c_2, t) &\le C_2 \mathbb{E}^{\boldsymbol{\alpha}_0}_x \Bigg[\exp\Bigg\{ \frac{1}{2}\nabla^2 F(\boldsymbol{w}_0)\left(\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0\,,\ \frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0\right) \\ &\qquad + \frac{\varepsilon}{2}\left\|\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0\right\|^2_{\mathfrak{D}^*_{p,p'}} \Bigg\}\ ;\ c_1 \le \left\|\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0\right\|_{\mathfrak{D}^*_{p,p'}} \le c_2\sqrt{t} \Bigg] \\ &= C_2 \int_{\mathbb{R}} ds\ \mathrm{e}^s\ \mathbb{P}^{\boldsymbol{\alpha}_0}_x \Bigg[\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \in \sqrt{s}\boldsymbol{\mathcal{A}}_\varepsilon, \\ &\qquad c_1 \le \left\|\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0\right\|_{\mathfrak{D}^*_{p,p'}} \le c_2\sqrt{t} \Bigg], \end{aligned} \tag{4.12}$$

where $C_2 := \|h^{\boldsymbol{\alpha}_0}\|_B \|\varphi/h^{\boldsymbol{\alpha}_0}\|_B$. By Corollary 3 and (4.11), there exists $r > 1$ and $c > 0$ so that for each $t, s > 0$ with $c \le \sqrt{s} \le \sqrt{t}/c$,

$$\mathbb{P}^{\boldsymbol{\alpha}_0}_x \left[\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \in \sqrt{s}\boldsymbol{\mathcal{A}}_\varepsilon \right] \le \exp(-rs).$$

Note that $\inf_{\boldsymbol{w}\in\boldsymbol{\mathcal{A}}_\varepsilon} \|\boldsymbol{w}\|_{\mathfrak{D}^*_{p,p'}} \ge \sqrt{2/(C_F + \varepsilon)}$ holds. Take c_2 sufficiently small so that $c_2 < c^{-1}\sqrt{2/(C_F + \varepsilon)}$ holds. Then we have

$$\sqrt{s}\boldsymbol{\mathcal{A}}_\varepsilon \cap \left\{\boldsymbol{w}\ ;\ \|\boldsymbol{w}\|_{\mathfrak{D}^*_{p,p'}} \le c_2\sqrt{t}\right\} = \emptyset \quad \text{if } \sqrt{s} \ge \frac{\sqrt{t}}{c}.$$

Take a constant $c_0 > c$ and set

$$J_{2,1} := \int_{-\infty}^{c_0^2} ds\ \mathrm{e}^s\, \mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \in \sqrt{s}\boldsymbol{\mathcal{A}}_\varepsilon, \right. \\ \left. c_1 \le \left\| \frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \right\|_{\mathfrak{D}^*_{p,p'}} \le c_2\sqrt{t} \right],$$

$$J_{2,2} := \int_{c_0^2}^{\infty} ds\ \mathrm{e}^s\, \mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \in \sqrt{s}\boldsymbol{\mathcal{A}}_\varepsilon, \right. \\ \left. c_1 \le \left\| \frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \right\|_{\mathfrak{D}^*_{p,p'}} \le c_2\sqrt{t} \right].$$

By Lemma 7 and Lemma 8,

$$\limsup_{c_1\to\infty} \sup_{t>1} J_{2,1} \le \limsup_{c_1\to\infty} \sup_{t>1} \mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\left\| \frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \right\|_{\mathfrak{D}^*_{p,p'}} \ge c_1 \right] \int_{-\infty}^{c_0^2} \mathrm{e}^s\, ds = 0. \tag{4.13}$$

For $J_{2,2}$,

$$\sup_{t>1} J_{2,2} = \sup_{t>1} \int_{c_0^2}^{t/c^2} ds\ \mathrm{e}^s\, \mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \in \sqrt{s}\boldsymbol{\mathcal{A}}_\varepsilon, \right. \\ \left. c_1 \le \left\| \frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \right\|_{\mathfrak{D}^*_{p,p'}} \le c_2\sqrt{t} \right] \\ \le \sup_{t>1} \int_{c_0^2}^{t/c^2} ds\ \mathrm{e}^s\, \mathbb{P}_x^{\boldsymbol{\alpha}_0}\left[\frac{1}{\sqrt{t}}\boldsymbol{Y}_t - \sqrt{t}\boldsymbol{w}_0 \in \sqrt{s}\boldsymbol{\mathcal{A}}_\varepsilon \right] \\ \le \int_{c_0^2}^{\infty} \mathrm{e}^{(1-r)s} ds = \frac{1}{r-1}\mathrm{e}^{(1-r)c_0^2}. \tag{4.14}$$

Combining (4.13) and (4.14) with (4.12), we obtain

$$\limsup_{c_1\to\infty} \sup_{t>1} J_2(c_1, c_2, t) \le \frac{1}{r-1}\mathrm{e}^{(1-r)c_0^2}.$$

Thus the conclusion follows as $c_0 \to \infty$. □

Before discussing the general case, we will prove the following lemma.

Lemma 12 *Suppose that* (***D***), (***F1***)–(***F3***) *and* Assumption 1 *hold. Then* $\mathcal{K}_F$ *has no accumulation point.*

Proof Take $\boldsymbol{w} = (w, \mu) \in \mathcal{K}_F$ and $\boldsymbol{w}_n = (w_n, \mu_n) \in \mathcal{K}_F$ for $n \in \mathbb{N}$ so that w_n converges to $\boldsymbol{w}$ as $n \to \infty$. We write $\boldsymbol{\alpha}_n = (\alpha_n, V_n) := \nabla F(\boldsymbol{w}_n)$ and $\boldsymbol{\alpha} := (\alpha, V) = \nabla F(\boldsymbol{w})$. By Lemma 3, for $\boldsymbol{\beta} = (\beta, U) \in \mathfrak{D}_{p,p'}$,

$$\langle \boldsymbol{w}_n, \boldsymbol{\beta} \rangle_{\mathfrak{D}^*_{p,p'}} = \int_M \left(\left(\alpha_n - d\psi^{\boldsymbol{\alpha}_n}, \beta \right) + U \right) dm_{\mu_n},$$
$$\langle \boldsymbol{w}, \boldsymbol{\beta} \rangle_{\mathfrak{D}^*_{p,p'}} = \int_M \left(\left(\alpha - d\psi^{\boldsymbol{\alpha}}, \beta \right) + U \right) dm_{\mu}$$

holds. For $\varepsilon \in [0, 1]$, we write $\boldsymbol{w}_{n,\varepsilon} = \varepsilon \boldsymbol{w}_n + (1 - \varepsilon)\boldsymbol{w}$. Let $f_n := d\mu_n/dv$ and $f := d\mu/dv$. Then $(w_{n,\varepsilon}, \mu_{n,\varepsilon}) := \boldsymbol{w}_{n,\varepsilon} \in \mathfrak{H}$ and

$$\mu_{n,\varepsilon} = \varepsilon \mu_n + (1 - \varepsilon)\mu,$$
$$\hat{w}_{n,\varepsilon} = \frac{1}{\varepsilon f_n + (1 - \varepsilon) f} \left\{ \varepsilon f_n (\alpha_n - d\psi^{\boldsymbol{\alpha}_n}) + (1 - \varepsilon) f (\alpha - d\psi^{\boldsymbol{\alpha}}) \right\}.$$

Obviously, $\boldsymbol{w}_{n,\varepsilon}$ satisfies the assumption of Proposition 2. Thus, by the Taylor expansion of $I(\boldsymbol{w}_{n,\varepsilon})$ by ε, there is a constant $0 \le \varepsilon_1 \le 1$ so that

$$I^{\boldsymbol{\alpha}}(\boldsymbol{w}_n) = \frac{1}{2} \int_M |\hat{w}_{n,0}^{(1)}|^2 f \, dv + \frac{1}{6} \left. \frac{d^3}{d\varepsilon^3} \right|_{\varepsilon=\varepsilon_1} I^{\boldsymbol{\alpha}}(\boldsymbol{w}_{n,\varepsilon}). \tag{4.15}$$

Note that we have

$$\hat{w}_{n,0}^{(1)} = \frac{f_n}{f} (\alpha_n - d\psi^{\boldsymbol{\alpha}_n} - \alpha + d\psi^{\boldsymbol{\alpha}}).$$

It yields

$$\int_M |\hat{w}_{n,0}^{(1)}|^2 f dv = \int_M \frac{f_n^2}{f} \left| \alpha - d\psi^{\boldsymbol{\alpha}} - \alpha_n + d\psi^{\boldsymbol{\alpha}_n} \right|^2 dv. \tag{4.16}$$

By the direct calculation, the remainder term in (4.15) is written by

$$\left. \frac{d^3}{d\varepsilon^3} \right|_{\varepsilon=\varepsilon_1} I^{\boldsymbol{\alpha}}(\boldsymbol{w}_{n,\varepsilon}) = 3 \int_M \frac{(f - f_n) f^2 f_n^2}{((1 - \varepsilon_1) f + \varepsilon_1 f_n)^4} \left| \alpha - d\psi^{\boldsymbol{\alpha}} - \alpha_n + d\psi^{\boldsymbol{\alpha}_n} \right|^2 dv. \tag{4.17}$$

Since $\boldsymbol{w}_n$ converges to $\boldsymbol{w}$ in $\mathfrak{D}^*_{p,p'}$ as $n \to \infty$, $\boldsymbol{\alpha}_n$ converges to $\boldsymbol{\alpha}$ in $\mathfrak{D}_{p,p'}$ as $n \to \infty$. Then, by the stability theorem ([13] Chapter IV §3.5), we can show that $d\psi^{\boldsymbol{\alpha}_n}$ converges to $d\psi^{\boldsymbol{\alpha}}$ in $\mathcal{D}_p$ and f_n converges to f in $H_{p'}$ as $n \to \infty$ (cf. Lemma 5.2 of [16]). As a result, we have $C_3 := \sup_{n,x}(f_n(x) \wedge f(x))^{-1} < \infty$. Then, by (4.16) and (4.17),

$$\left| \frac{d^3}{d\varepsilon^3}\bigg|_{\varepsilon=\varepsilon_1} I^{\boldsymbol{\alpha}}(\boldsymbol{w}_{n,\varepsilon}) \right| \leq 3\|f - f_n\|_{H_{p'}} C_3^4 C_S^3 \|f\|^3_{H_{p'}} \int_M |\hat{w}^{(1)}_{n,0}|^2 f\, dv.$$

Take $\delta > 0$ arbitrary. Then, the argument above provides that there is $N_1 \in \mathbb{N}$ such that, for all $n \geq N_1$, we have

$$\left| \frac{d^3}{d\varepsilon^3}\bigg|_{\varepsilon=\varepsilon_1} I(\boldsymbol{w}_{n,\varepsilon}) \right| \leq \frac{\delta}{2} \int_M |\hat{w}^{(1)}_{n,0}|^2 f\, dv. \tag{4.18}$$

Let us turn our attention to the estimate of $F(\boldsymbol{w}_{n,\varepsilon})$. The Taylor expansion of $F(\boldsymbol{w}_{n,\varepsilon})$ by ε yields that there is $0 \leq \varepsilon_2 \leq 1$ such that

$$\begin{aligned} F(\boldsymbol{w}_n) - F(\boldsymbol{w}) &= \nabla F(\boldsymbol{w})(\boldsymbol{w}_n - \boldsymbol{w}) + \frac{1}{2}\nabla^2 F(\boldsymbol{w})(\boldsymbol{w}_n - \boldsymbol{w}, \boldsymbol{w}_n - \boldsymbol{w}) \\ &\quad + \frac{1}{6}\nabla^3 F(\boldsymbol{w})(\boldsymbol{w}_{n,\varepsilon_2} - \boldsymbol{w}, \boldsymbol{w}_{n,\varepsilon_2} - \boldsymbol{w}, \boldsymbol{w}_{n,\varepsilon_2} - \boldsymbol{w}). \end{aligned} \tag{4.19}$$

We define $\tilde{w}_n \in \mathcal{D}_{-p}$ by

$$\langle \tilde{w}_n, \beta \rangle = \int_M (\hat{w}^{(1)}_{n,0}, \beta) d\mu.$$

By Proposition 3 (i), we have $\boldsymbol{w}_n - \boldsymbol{w} = \boldsymbol{\Gamma}^*_{\boldsymbol{\alpha}} \tilde{w}_n$. Thus we have

$$\frac{1}{2}\nabla^2 F(\boldsymbol{w})(\boldsymbol{w}_n - \boldsymbol{w}, \boldsymbol{w}_n - \boldsymbol{w}) = (\tilde{w}_n, G^F_{\boldsymbol{w}} \tilde{w}_n)_{-p}. \tag{4.20}$$

In addition, there is $N_2 \in \mathbb{N}$ such that, for all $n \geq N_2$, we have

$$\frac{1}{6}\left| \nabla^3 F(\boldsymbol{w})(\boldsymbol{w}_{n,\varepsilon_2} - \boldsymbol{w}, \boldsymbol{w}_{n,\varepsilon_2} - \boldsymbol{w}, \boldsymbol{w}_{n,\varepsilon_2} - \boldsymbol{w}) \right| \leq \delta\, \|\tilde{w}_n\|^2_{-p}. \tag{4.21}$$

Take $n \in \mathbb{N}$ with $n \geq N_1 \vee N_2$. Since we have

$$\int_M |\hat{w}^{(1)}_{n,0}|^2 f\, dv = 2L_\mu(\tilde{w}_n),$$

(4.15), (4.18), (4.19), (4.20) and (4.21) yield

$$F(\boldsymbol{w}_n)-F(\boldsymbol{w})-I(\boldsymbol{w}_n)+I(\boldsymbol{w}) \le \frac{1}{2}\left(\tilde{w}_n, G^F_{\boldsymbol{w}}\tilde{w}_n\right)_{-p} - (1-\delta)L_\mu(\tilde{w}_n) + \delta\|\tilde{w}_n\|^2_{-p}. \tag{4.22}$$

By Assumption 1,

$$\frac{1}{2}(\tilde{w}_n, G^F_{\boldsymbol{w}}\tilde{w}_n)_{-p} - L_\mu(\tilde{w}_n) \le -\delta_{\boldsymbol{w}}\,\|\tilde{w}_n\|^2_{-p}\,. \tag{4.23}$$

Then (4.22) and (4.23) yield

$$\begin{aligned} F(\boldsymbol{w}_n) - F(\boldsymbol{w}) - I(\boldsymbol{w}_n) + I(\boldsymbol{w}) &\le \frac{1}{2}\delta(\tilde{w}_n, G^F_{\boldsymbol{w}}\tilde{w}_n)_{-p} + (\delta-(1-\delta)\delta_{\boldsymbol{w}})\|\tilde{w}_n\|^2_{-p} \\ &\le \left\{\left(1+\delta_{\boldsymbol{w}}+\frac{\|G^F_{\boldsymbol{w}}\|}{2}\right)\delta - \delta_{\boldsymbol{w}}\right\}\|\tilde{w}_n\|^2_{-p}. \end{aligned} \tag{4.24}$$

Choose $\delta > 0$ so that $\delta < \delta_{\boldsymbol{w}}(1+\delta_{\boldsymbol{w}}+\|G^F_{\boldsymbol{w}}\|/2)^{-1}$. Then $\tilde{w}_n = 0$ and $\boldsymbol{w}_n = \boldsymbol{w}$ for $n \ge N_1 \vee N_2$ since the left hand side of (4.24) is 0. Thus $\boldsymbol{w}$ is an isolated point in $\mathcal{K}_F$. □

By virtue of Lemma 12, the general case easily follows from Theorem 3.

Theorem 4 *Let $\varphi : M \to \mathbb{R}$ be a positive continuous function. Under (**D**), (**F1**)–(**F3**) and Assumption 1,*

$$\begin{aligned} &\lim_{t\to\infty} \mathrm{e}^{-t\kappa_F}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{Y}_t)}\varphi(z_t)\right] \\ &\quad= \sum_{\boldsymbol{w}=(w,\mu)\in\mathcal{K}_F} \frac{1}{\det(1-G^F_{\boldsymbol{w}}\circ S_\mu)^{1/2}} h^{\nabla F(\boldsymbol{w})}(x)\int_M \frac{\varphi}{h^{\nabla F(\boldsymbol{w})}}d\mu. \end{aligned} \tag{4.25}$$

Obviously, this theorem is a refined version of Theorem 1.

Proof To begin with, we should remark that Lemma 12 together with Lemma 1 implies that $\mathcal{K}_F$ is a finite set. We write $B_r(\boldsymbol{\eta}) := \{\boldsymbol{w} \in \mathfrak{D}^*_{p,p'}\ ;\ \|\boldsymbol{w}-\boldsymbol{\eta}\|_{\mathfrak{D}^*_{p,p'}} < r\}$. Take $\varepsilon > 0$ so that $B_\varepsilon(\boldsymbol{w}) \cap \mathcal{K}_F = \{\boldsymbol{w}\}$ for each $\boldsymbol{w} \in \mathcal{K}_F$. Take a smooth function f on $\mathbb{R}$ which satisfies $f|_{(-1/2,\,1/2)} \equiv 1$, $f|_{(-1,\,1)^c} \equiv 0$ and $0 \le f \le 1$. We define $F_{\boldsymbol{w}} : \mathfrak{D}^*_{p,p'} \to \mathbb{R}$ as follows:

$$F_{\boldsymbol{w}}(\boldsymbol{\eta}) := F(\boldsymbol{\eta}) + \log f\left(\frac{1}{\varepsilon^2}\|\boldsymbol{\eta}-\boldsymbol{w}\|^2_{\mathfrak{D}^*_{p,p'}}\right).$$

Then

$$\begin{aligned}
&\mathrm{e}^{-t\kappa_F}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{Y}_t)}\varphi(z_t)\right]\\
&\quad=\sum_{\boldsymbol{w}\in\mathcal{K}_F}\mathrm{e}^{-t\kappa_F}\mathbb{E}_x\left[\mathrm{e}^{tF_{\boldsymbol{w}}(\bar{Y}_t)}\varphi(z_t)\right]\\
&\qquad+\mathrm{e}^{-t\kappa_F}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{Y}_t)}\varphi(z_t)\left\{1-\sum_{\boldsymbol{w}\in\mathcal{K}_F}f\left(\frac{1}{\varepsilon^2}\left\|\bar{\boldsymbol{Y}}_t-\boldsymbol{w}\right\|^2_{\mathfrak{D}^*_{p,p'}}\right)^t\right\}\right]. \qquad (4.26)
\end{aligned}$$

The large deviation principle for $\bar{\boldsymbol{Y}}_t$ implies

$$\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{Y}_t)}1_{\{\bigcap_{\boldsymbol{w}\in\mathcal{K}_F}B_{\varepsilon/2}(\boldsymbol{w})^c\}}(\bar{\boldsymbol{Y}}_t)\right]\\
&\quad=\sup\left\{F(\boldsymbol{\eta})-I(\boldsymbol{\eta})\ ;\ \boldsymbol{\eta}\in\bigcap_{\boldsymbol{w}\in\mathcal{K}_F}B_{\varepsilon/2}(\boldsymbol{w})^c\right\}<\kappa_F.
\end{aligned}$$

Hence we have

$$\begin{aligned}
0&\le\lim_{t\to\infty}\mathrm{e}^{-t\kappa_F}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{Y}_t)}\varphi(z_t)\left\{1-\sum_{\boldsymbol{w}\in\mathcal{K}_F}f\left(\frac{1}{\varepsilon^2}\left\|\bar{\boldsymbol{Y}}_t-\boldsymbol{w}\right\|^2_{\mathfrak{D}^*_{p,p'}}\right)^t\right\}\right]\\
&\le\lim_{t\to\infty}\mathrm{e}^{-t\kappa_F}\,\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{Y}_t)}1_{\{\bigcap_{\boldsymbol{w}\in\mathcal{K}_F}B_{\varepsilon/2}(\boldsymbol{w})^c\}}(\bar{\boldsymbol{Y}}_t)\right]=0. \qquad (4.27)
\end{aligned}$$

We can easily verify $\kappa_F=\kappa_{F_{\boldsymbol{w}}}$, $\mathcal{K}_{F_{\boldsymbol{w}}}=\{\boldsymbol{w}\}$ and $\nabla^2F(\boldsymbol{w})=\nabla^2F_{\boldsymbol{w}}(\boldsymbol{w})$ for each $\boldsymbol{w}\in\mathcal{K}_F$. Hence (4.26), (4.27) and Theorem 3 imply the conclusion. □

From Theorem 4, we can obtain the convergence of path measure as announced in Sect. 1:

Corollary 4 *Suppose the same condition as in Theorem 4. Let $\mathbb{P}_{x,F,T}$ be given by* (1.4). *Then we have*

$$\begin{aligned}
&\lim_{T\to\infty}\mathbb{P}_{x,F,T}\\
&\quad=\frac{1}{Z}\sum_{\boldsymbol{w}=(w,\mu)\in\mathcal{K}_F}\left(\frac{1}{\det(1-G^F_w\circ S_\mu)^{1/2}}h^{\nabla F(\boldsymbol{w})}(x)\int_M\frac{1}{h^{\nabla F(\boldsymbol{w})}}d\mu\right)\mathbb{P}^{\nabla F(\boldsymbol{w})}_x,
\end{aligned}$$

with respect to the weak convergence on $C([0,\infty)\to M)$. Here Z equals the right-hand side of (4.25).

5 The case for stochastic line integrals

The same idea as used in Sect. 4 also works in dealing with the case for $\mathcal{D}_{-p}$-valued process defined by stochastic line integrals. In this section, we assume

(**D'**) $\quad p > d_0 + 1$ and $p' > \inf\{n \in \mathbb{N} \; ; \; n \geq p - 1\}$

instead of (**D**). Under (**D'**), we can realize a $\mathcal{D}_{-p}$-valued continuous process X_t which is characterized by the following property: $X_t(\alpha) = \int_{z[0,t]} \alpha$ for each $\alpha \in \mathcal{D}_p$ $\mathbb{P}_x$-almost surely (see [21]).

5.1 Preliminaries

We will review some properties for X as we did in Sect. 1.1 for Y. Recall that $\Delta = -d^\dagger d$, where $d^\dagger$ is the adjoint derivative of the exterior derivative d in L^2, that is,

$$\int_M (df, \alpha)\, dv = \int_M f d^\dagger \alpha \, dv.$$

First we see that, for each exact 1-form du, we have

$$X_t(du) = u(z_t) - u(z_0). \tag{5.1}$$

To state the next property, we will introduce an operator $\hat{\Gamma}$ and a functional e. Recall that $\hat{b}$ is a 1-form corresponding to the vector field b. Let us define an operator $\hat{\Gamma}$ by $\hat{\Gamma}\alpha := \alpha + d\mathcal{G}_0((\hat{b}, \alpha) - d^\dagger \alpha/2)$. Note that $\hat{\Gamma}$ becomes a continuous linear idempotent operator on $\mathcal{D}_p$. When $\beta = 0$, $\hat{\Gamma}$ is the orthonormal projection to co-exact 1-forms. Let us define $e : \mathcal{D}_p \to \mathbb{R}$ by

$$e(\alpha) = \int_M \left((\hat{b}, \alpha) - \frac{1}{2} d^\dagger \alpha \right) dm.$$

Then $X_t(\hat{\Gamma}\alpha) - te(\hat{\Gamma}\alpha) = Y_t(\hat{\Gamma}\alpha)$ holds for any $\alpha \in \mathcal{D}_p$. Since $e((1 - \hat{\Gamma})\alpha) = 0$, we have

$$X_t - te = \hat{\Gamma}^* Y_t + (1 - \hat{\Gamma}^*) X_t \tag{5.2}$$

(see [21] for example). Note that

$$\|(1 - \hat{\Gamma}^*) X_t\|_{-p} \leq C' \tag{5.3}$$

holds for some constant $C' > 0$. This estimate comes from (5.1) and the continuity of $\mathcal{G}_0$. These properties mean that X_t equals $\hat{\Gamma}^* Y_t$ plus a remainder term.

Next we see limit theorems for X. Set $\bar{X}_t := t^{-1} X_t - e$. The law of large numbers [10] asserts $\lim_{t \to \infty} \bar{X}_t = 0$ in $\mathcal{D}_{-p}$ $\mathbb{P}_x$-almost surely. The central limit theorem for X asserts that the law of $\sqrt{t}\bar{X}_t$ weakly converges to $\nu_{\hat{\Gamma}^* S_m}$ as $t \to \infty$ respectively. As shown in [21], these limit theorems for X is resulted from that for Y. Indeed, (5.2) and (5.3) imply the desired result. The same idea as stated above involves the large deviation principle for $\bar{\boldsymbol{X}}_t := (\bar{X}_t, \bar{l}_t)$. via the contraction principle (see [16] for more details). Set $\hat{\boldsymbol{\Gamma}} : \mathfrak{D}_{p,p'} \to \mathfrak{D}_{p,p'}$ by $\hat{\boldsymbol{\Gamma}}(\beta, U) = (\hat{\Gamma}\beta, U)$. Then the rate function I_X for $\bar{\boldsymbol{X}}$ is given by $I_X(\boldsymbol{w}) := \inf_{\hat{\boldsymbol{\Gamma}}^*\boldsymbol{\eta}=\boldsymbol{w}} I(\boldsymbol{\eta})$. We should remark that, there appears no influence from the term $(1 - \hat{\Gamma}^*)\bar{X}_t$ in above-mentioned limit theorems for X.

5.2 Laplace approximation for currents of stochastic line integrals

As in the beginning of Sect. 4, we suppose the following assumption concerning the nondegeneracy of the Hessian at equilibrium states. We write $F_1 := F \circ \hat{\boldsymbol{\Gamma}}^*$ and

Assumption 2 For each $\boldsymbol{w}_0 = (w_0, \mu_0) \in \mathcal{K}_{F_1}$, there exists $\delta^{(1)}_{\boldsymbol{w}_0} > 0$ such that

$$L_{\mu_0}(w) \geq \frac{1}{2}\left(w, G^{F_1}_{\boldsymbol{w}_0} w\right)_{-p} + \delta^{(1)}_{\boldsymbol{w}_0} \|w\|^2_{-p}$$

holds for each $w \in \mathcal{D}_{-p}$.

Theorem 5 *Let $\varphi : M \to \mathbb{R}$ be a positive continuous function. Under (**D'**), (**F1**)–(**F3**) and Assumption 2,*

$$\begin{aligned}&\lim_{t\to\infty} \mathrm{e}^{-t\kappa_{F_1}} \mathbb{E}_x\left[\mathrm{e}^{tF(\bar{\boldsymbol{X}}_t)}\varphi(z_t)\right]\\ &= \sum_{\boldsymbol{w}=(w,\mu)\in\mathcal{K}_{F_1}} \frac{1}{\det(1 - G^{F_1}_{\boldsymbol{w}} \circ S_\mu)^{1/2}} h^{\boldsymbol{\alpha}_w}(x)\mathrm{e}^{-u^{\tilde{\alpha}_w}(x)} \int_M \frac{\varphi}{h^{\boldsymbol{\alpha}_w}} \mathrm{e}^{u^{\tilde{\alpha}_w}}\, d\mu,\end{aligned}$$

where $\boldsymbol{\alpha}_w := \nabla F_1(\boldsymbol{w})$, $\tilde{\alpha}_w$ is a projection of $\nabla F(\hat{\boldsymbol{\Gamma}}^*\boldsymbol{w})$ to $\mathcal{D}_p$ and $u^{\tilde{\alpha}_w} = -\mathcal{G}_0((\hat{b}, \tilde{\alpha}_w) - d^\dagger\tilde{\alpha}_w/2)$.

Remark 5 A similar result also holds for the bounded variation part of X_t. But this part is nothing but a different realization of the mean empirical measure l_t. Thus we omit it because it is not important.

Proof In the following, we assume $\mathcal{K}_{F_1} = \{\boldsymbol{w}_0\}$. The extension to the general case is now obvious. Set $\boldsymbol{\eta}_0 = \hat{\Gamma}^*\boldsymbol{w}_0$ and $(w_0, \mu_0) := \boldsymbol{w}_0$. As in the proof of Theorem 3, we decompose the left hand side into three parts. For $c_1, c_2 > 0$,

$$\mathrm{e}^{-t\kappa_{F_1}} \mathbb{E}_x\left[\mathrm{e}^{tF(\bar{\boldsymbol{X}}_t)}\varphi(z_t)\right] = J'_1(c_1, t) + J'_2(c_1, c_2, t) + J'_3(c_2, t),$$

where

$$J_1'(c_1,t) := \mathrm{e}^{-t\kappa_{F_1}}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{X}^t)}\varphi(z_t)\ ;\ \|\bar{\boldsymbol{X}}_t-\boldsymbol{\eta}_0\|_{\mathfrak{D}^*_{p,p'}}\le\frac{c_1}{\sqrt{t}}\right],$$
$$J_2'(c_1,c_2,t) := \mathrm{e}^{-t\kappa_{F_1}}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{X}_t)}\varphi(z_t)\ ;\ \frac{c_1}{\sqrt{t}}<\|\bar{\boldsymbol{X}}_t-\boldsymbol{\eta}_0\|_{\mathfrak{D}^*_{p,p'}}\le c_2\right],$$
$$J_3'(c_2,t) := \mathrm{e}^{-t\kappa_{F_1}}\mathbb{E}_x\left[\mathrm{e}^{tF(\bar{X}_t)}\varphi(z_t)\ ;\ c_2<\|\bar{\boldsymbol{X}}_t-\boldsymbol{\eta}_0\|_{\mathfrak{D}^*_{p,p'}}\right].$$

Since

$$\kappa_{F_1}=\sup_{\boldsymbol{w}\in\mathfrak{D}^*_{p,p'}}\{F_1(\boldsymbol{w})-I(\boldsymbol{w})\}=\sup_{\boldsymbol{\eta}\in\mathfrak{D}^*_{p,p'}}\{F(\boldsymbol{\eta})-I_X(\boldsymbol{\eta})\},$$

the large deviation principle for $\bar{\boldsymbol{X}}_t$ implies $\limsup_{t\to\infty}J_3'(c_2,t)=0$ for each $c_2>0$.

Let us turn to the estimate of $J_1'(c_1,t)$. Let $\boldsymbol{\alpha}_0=(\alpha_0,V_0):=\nabla F_1(\boldsymbol{w}_0)$, $\tilde{\boldsymbol{\alpha}}_0=(\tilde{\alpha}_0,V_0):=\nabla F(\boldsymbol{\eta}_0)$ and $u^{\tilde{\alpha}_0}:=-\mathcal{G}_0((\hat{b},\tilde{\alpha}_0)-d^\dagger\tilde{\alpha}_0/2)$. Note that $\boldsymbol{\alpha}_0=\hat{\boldsymbol{\Gamma}}\tilde{\boldsymbol{\alpha}}_0$ and $(1-\hat{\Gamma})\tilde{\alpha}_0=du^{\tilde{\alpha}_0}$. Then, by (5.1) and (5.2),

$$\begin{aligned}J_1'(c_1,t)&=\exp\left(t\left(I(\boldsymbol{w}_0)-\langle\boldsymbol{w}_0,\boldsymbol{\alpha}_0\rangle_{\mathfrak{D}^*_{p,p'}}\right)\right)\mathbb{E}_x\Bigg[\varphi(z_t)\exp\left(t\bar{\boldsymbol{X}}_t(\tilde{\boldsymbol{\alpha}}_0)\right)\\&\qquad\times\exp\left(t\left\{F(\bar{\boldsymbol{X}}_t)-F(\boldsymbol{\eta}_0)-\nabla F(\boldsymbol{\eta}_0)(\bar{\boldsymbol{X}}_t-\boldsymbol{\eta}_0)\right\}\right);\\&\qquad\left\|\bar{\boldsymbol{X}}_t-\boldsymbol{\eta}_0\right\|_{\mathfrak{D}^*_{p,p'}}\le\frac{c_1}{\sqrt{t}}\Bigg]\\&=h^{\boldsymbol{\alpha}_0}(x)\mathbb{E}_x^{\boldsymbol{\alpha}_0}\Bigg[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(u^{\tilde{\alpha}_0}(z_t)-u^{\tilde{\alpha}_0}(z_0)\right)\\&\qquad\times\exp\left\{t\left(\nabla^2F(\boldsymbol{\eta}_0)(\bar{\boldsymbol{X}}_t-\boldsymbol{\eta}_0,\bar{\boldsymbol{X}}_t-\boldsymbol{\eta}_0)+r_F(\bar{\boldsymbol{X}}_t-\boldsymbol{\eta}_0)\right)\right\};\\&\qquad\left\|\bar{\boldsymbol{X}}_t-\boldsymbol{\eta}_0\right\|_{\mathfrak{D}^*_{p,p'}}\le\frac{c_1}{\sqrt{t}}\Bigg],\end{aligned}$$

where r_F is the remainder term of the Taylor expansion satisfying $r_F(\boldsymbol{w})=o(\|\boldsymbol{w}\|^2_{\mathfrak{D}^*_{p,p'}})$. Thus (5.2) and the remark after (5.2) imply

$$\begin{aligned}\lim_{t\to\infty}J_1'(c_1,t)&=h^{\boldsymbol{\alpha}_0}(x)\exp\left(-u^{\tilde{\alpha}_0}(x)\right)\lim_{t\to\infty}\mathbb{E}_x^{\boldsymbol{\alpha}_0}\Bigg[\frac{\varphi(z_t)}{h^{\boldsymbol{\alpha}_0}(z_t)}\exp\left(u^{\tilde{\alpha}_0}(z_t)\right)\\&\qquad\times\exp\left\{t\left(\nabla^2F_1(\boldsymbol{w}_0)(\bar{\boldsymbol{Y}}_t-\boldsymbol{w}_0,\bar{\boldsymbol{Y}}_t-\boldsymbol{w}_0)\right)\right\};\\&\qquad\left\|\hat{\Gamma}^*(\bar{\boldsymbol{Y}}_t-\boldsymbol{w}_0)\right\|_{\mathfrak{D}^*_{p,p'}}\le\frac{c_1}{\sqrt{t}}\Bigg].\end{aligned}$$

Therefore, as in Lemma 10, the central limit theorem and the mixing property of $\{z_t\}_{t\geq 0}$ induce that there exists $\hat{J}_1'(c_1) := \lim_{t\to\infty} J_1'(c_1, t)$ for all but countably many $c_1 > 0$ and

$$\lim_{c_1\to\infty} \hat{J}_1'(c_1) = \frac{1}{\det(1 - G_{\boldsymbol{w}_0}^{F_1} \circ S_{\mu_0})^{1/2}} h^{\boldsymbol{\alpha}_0}(x) \mathrm{e}^{-u^{\tilde{\alpha}_0}(x)} \int_M \frac{\varphi}{h^{\boldsymbol{\alpha}_0}} \mathrm{e}^{u^{\tilde{\alpha}_0}} d\mu_0.$$

Finally, we will estimate $J_3'(c_1, c_2, t)$. Let $\mathcal{A}_\varepsilon'$ be given by

$$\mathcal{A}_\varepsilon' := \left\{ \boldsymbol{w} \in \mathfrak{D}_{p,p'}^* \ ; \ \frac{1}{2}\nabla^2 F_1(\boldsymbol{w}_0)(\boldsymbol{w}, \boldsymbol{w}) + \frac{\varepsilon}{2} \|\boldsymbol{w}\|_{\mathfrak{D}_{p,p'}^*}^2 \geq 1 \right\}.$$

As in the proof of Lemma 11, we can show

$$\inf_{\boldsymbol{w}\in\mathcal{A}_\varepsilon'} \inf_{\hat{\boldsymbol{\Gamma}}^*\boldsymbol{\eta}=\boldsymbol{w}} L_{\boldsymbol{\alpha}_0}^{\boldsymbol{\Gamma}}(\boldsymbol{\eta}) > 1 \tag{5.4}$$

for sufficiently small $\varepsilon > 0$. With the aid of (5.2), almost the same argument as used in Corollary 3 implies the following uniform moderate deviation estimate

$$\limsup_{c\to\infty} \left(\sup_{c\leq s\leq\sqrt{t}/c} \frac{1}{s^2} \log \mathbb{P}_x^{\boldsymbol{\alpha}_0} \left[\frac{1}{\sqrt{t}} \boldsymbol{X}_t - \sqrt{t}\boldsymbol{\eta}_0 \in s\mathcal{A} \right] \right) \leq - \inf_{\boldsymbol{w}\in\mathcal{A}} \inf_{\hat{\boldsymbol{\Gamma}}^*\boldsymbol{\eta}=\boldsymbol{w}} L_{\boldsymbol{\alpha}_0}^{\boldsymbol{\Gamma}}(\boldsymbol{w}) \tag{5.5}$$

for each closed set $\mathcal{A} \subset \mathfrak{D}_{p,p'}^*$. Then we obtain $\lim_{c_1\to\infty} \sup_{t>1} J_3'(c_1, c_2, t) = 0$ in the same way as in Lemma 11 by using (5.4) and (5.5) instead of (4.11) and Corollary 3. On the basis of these estimates on J_1', J_2' and J_3', the desired result follows. □

For $\boldsymbol{\alpha} = (\alpha, V) \in \mathfrak{D}_{p,p'}$, let us define $\hat{\Gamma}_{\boldsymbol{\alpha}} : \mathcal{D}_p \to \mathcal{D}_p$ by $\hat{\Gamma}_{\boldsymbol{\alpha}}\beta := \beta + d\mathcal{G}_{\boldsymbol{\alpha}}((\hat{b} + \alpha - d\psi^{\boldsymbol{\alpha}}, \beta) - d^\dagger\beta/2)$. Note that $\hat{\Gamma}_0 = \hat{\Gamma}$ holds. The following lemma characterizes the influence of Γ_α in Theorem 5.

Lemma 13 $\Gamma_{\boldsymbol{\alpha}}\hat{\Gamma} = \hat{\Gamma}_{\boldsymbol{\alpha}}$.

Proof By the definition of $\Gamma_{\boldsymbol{\alpha}}$ and $\hat{\Gamma}$,

$$\begin{aligned}
\Gamma_{\boldsymbol{\alpha}}\hat{\Gamma}\beta &= \hat{\Gamma}\beta + d\mathcal{G}_{\boldsymbol{\alpha}}(\alpha - d\psi^{\boldsymbol{\alpha}}, \hat{\Gamma}\beta) \\
&= \hat{\Gamma}\beta + d\mathcal{G}_{\boldsymbol{\alpha}}\left(\alpha - d\psi^{\boldsymbol{\alpha}}, \beta + d\mathcal{G}_0\left((\hat{b}, \beta) - \frac{1}{2}d^{\dagger}\beta\right)\right) \\
&= \hat{\Gamma}\beta + d\mathcal{G}_{\boldsymbol{\alpha}}\left(\alpha - d\psi^{\boldsymbol{\alpha}}, \beta\right) + d\mathcal{G}_{\boldsymbol{\alpha}}\mathcal{L}^{\boldsymbol{\alpha}}\mathcal{G}_0\left((\hat{b}, \beta) - \frac{1}{2}d^{\dagger}\beta\right) \\
&\qquad - d\mathcal{G}_{\boldsymbol{\alpha}}\mathcal{L}\mathcal{G}_0\left((\hat{b}, \beta) - \frac{1}{2}d^{\dagger}\beta\right) \\
&= \beta + d\mathcal{G}_{\boldsymbol{\alpha}}\left(\alpha - d\psi^{\boldsymbol{\alpha}}, \beta\right) + d\mathcal{G}_{\boldsymbol{\alpha}}\left((\hat{b}, \beta) - \frac{1}{2}d^{\dagger}\beta\right) \\
&= \hat{\Gamma}_{\boldsymbol{\alpha}}\beta.
\end{aligned}$$

Hence the desired result follows. □

Remark 6 Let us define a operator $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}} : \boldsymbol{\mathcal{D}}_{p,p'} \to \mathcal{D}_p$ by $\hat{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}}(\beta, U) = \hat{\Gamma}_{\boldsymbol{\alpha}}\beta + d\mathcal{G}_{\boldsymbol{\alpha}}U$. Note that Lemma 13 asserts $\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\hat{\boldsymbol{\Gamma}} = \hat{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}}$. By using this notation, we obtain

$$\left(\boldsymbol{\eta}, G_{\boldsymbol{w}}^{F_1}\boldsymbol{\eta}\right)_{-p} = \nabla^2 F(\hat{\boldsymbol{\Gamma}}^*\boldsymbol{w})(\hat{\boldsymbol{\Gamma}}^*_{\nabla F_1(\boldsymbol{w})}\boldsymbol{\eta}, \hat{\boldsymbol{\Gamma}}^*_{\nabla F_1(\boldsymbol{w})}\boldsymbol{\eta}). \tag{5.6}$$

As we have seen in (5.1), X_t degenerates on exact 1-forms in long time. The equation (5.2) together with (5.3) means that the range of $\hat{\Gamma}$ determines the complementary subspace to degenerate parts. To see the definition of $\hat{\Gamma}_{\boldsymbol{\alpha}}$, we can say that a transform of operator $\hat{\Gamma}_0 \mapsto \hat{\Gamma}_{\boldsymbol{\alpha}}$ corresponds to the transform of generator $\mathcal{L} \mapsto \mathcal{L}^{\boldsymbol{\alpha}}$. Thus $\boldsymbol{\Gamma}_{\boldsymbol{\alpha}}\hat{\boldsymbol{\Gamma}} = \hat{\boldsymbol{\Gamma}}_{\boldsymbol{\alpha}}$ means that the change of generator causes the change of the complementary subspace.

Recall that the emergence of $\boldsymbol{\Gamma}^*_{\cdot}$ in the Laplace approximation for Y_t comes from the transformation of martingales driven by the Girsanov transform. Since the definition of stochastic line integral is invariant under the Grisanov transform, it it natural that $\Gamma^*_{\cdot}$ does not appear in the Laplace approximation for X_t. Indeed, if we rewrites Theorem 5 by using (5.6), $\Gamma^*_{\cdot}$ actually disappears. But as a result, the transform $\hat{\Gamma} \mapsto \hat{\Gamma}_{\boldsymbol{\alpha}_w}$ emerges instead of it.

Remark 7 In our framework, we have considered a (nonsymmetric) diffusion process $\{z_t\}_{t\geq 0}$ with the generator $\Delta/2 + b$. But, when we consider the Laplace approximation for $\boldsymbol{Y}_t$ or $\boldsymbol{X}_t$, it is sufficient to consider the case $b = 0$, that is, $\{z_t\}_{t\geq 0}$ is the Brownian motion. Let $\{\hat{\mathbb{P}}_x\}_{x\in M}$ be the Wiener measure of the Brownian motion. We define a map $\Psi_b : H_{-p'} \to \mathcal{D}_{-p}$ by $\langle\Psi_b(\mu), \beta\rangle = \langle\mu, (\hat{b}, \beta)\rangle_{H_{p'}}$. Then the Girsanov formula implies

$$\begin{aligned}
&\mathbb{E}_x\left[\exp(tF(\boldsymbol{Y}_t))\varphi(z_t)\right] \\
&\quad = \hat{\mathbb{E}}_x\left[\exp\left\{t\left(F(\bar{Y}_t - \Psi_b(\bar{l}_t), \bar{l}_t) + \bar{Y}_t(\hat{b}) - \frac{1}{2}\langle\bar{l}_t, |\hat{b}|^2\rangle_{H_{p'}}\right)\right\}\varphi(z_t)\right].
\end{aligned}$$

Thus letting a functional $\tilde{F}$ on $\mathfrak{D}^*_{p,p'}$ by $\tilde{F}(w,\mu) := F(w - \Psi_b(\mu), \mu) + \langle w, \hat{b} \rangle - \frac{1}{2}\langle \mu, |\hat{b}|^2 \rangle_{H_{p'}}$, we obtain

$$\mathbb{E}_x\left[\exp(tF(\boldsymbol{Y}_t))\varphi(z_t)\right] = \hat{\mathbb{E}}_x\left[\exp(t\tilde{F}(\boldsymbol{Y}_t))\varphi(z_t)\right].$$

Thus we can reduce the Laplace approximation to the case $b = 0$. The same argument also works for $\boldsymbol{X}_t$. We remark that X_t is invariant under the Girsanov transform and therefore Ψ_b does not appear in this case.

6 Some special cases

6.1 Functional of empirical measures

In this section we consider the Laplace approximation only for mean empirical measures $\bar{l}_t$. This problem is nothing but considering Theorem 4 under the following additional assumption:

Assumption 3 there exists $F_0 : H_{-p'} \to [-\infty, \infty)$ such that $F(w,\mu) = F_0(\mu)$ holds for any $(w,\mu) \in \mathfrak{D}^*_{p,p'}$.

Of course, Theorem 4 implies the Laplace approximation for l_t under Assumption 1. The aim in this section is to reveal that our result recovers that in [4], which is obtained as an example of the general Laplace approximation result for mean empirical measures of a Markov process.

Note that all the derivatives in $\mathcal{D}_{-p}$ direction vanishes under Assumption 3. It means that for $\boldsymbol{w} = (w,\mu)$, $\nabla F(\boldsymbol{w}) = (0, \nabla F_0(\mu))$ holds. Thus the Varadhan lemma asserts that

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x\left[\exp\left(tF_0(\bar{l}_t)\right)\right] = \kappa_F$$

holds. Let I_0 be the rate function for $\bar{l}_t$. Set

$$\kappa^0_{F_0} := \sup_{\mu\in H_{-p'}} \left(F_0(\mu) - I_0(\mu)\right),$$
$$\mathcal{K}^0_{F_0} := \{\mu \in H_{-p'} \ ;\ F_0(\mu) - I_0(\mu) = \kappa^0_{F_0}\}.$$

Let $\iota : \mathfrak{D}^*_{p,p'} \to H_{-p'}$ be the canonical projection.

Lemma 14 *In the above framework, $\kappa_F = \kappa^0_{F_0}$ holds. In addition, ι maps $\mathcal{K}_F$ to $\mathcal{K}^0_{F_0}$ as a homeomorphism.*

Proof By the contraction principle, we have $I_0(\mu) = \inf_{w\in\mathcal{D}_{-p}} I(w,\mu)$. Thus $\kappa_F = \kappa^0_{F_0}$ immediately follows. Since ι is continuous and open mapping, it suffices to show that ι is bijective. By (1.12), (1.13) and a remark after that, we have $I(w, \mu) = I_0(\mu)$ if and only if $w \in \mathcal{D}_{-p}$ is given by

$$\langle w, \beta \rangle = -\int_M (P_f \hat{b} - \xi_f, \beta) d\mu, \tag{6.1}$$

where $f = \sqrt{d\mu/d\nu}$. We denote w defined by (6.1) by w_μ. Then the above observation yields that $(w, \mu) \in \mathcal{K}_F$ holds if and only if $w = w_\mu$ and $\mu \in \mathcal{K}_{F_0}$. Thus the bijectivity of ι is now obvious. □

For $V \in H_{p'}$, we simply denote $\mathcal{L}^{(0,V)}$, $h^{(0,V)}$, $m_{(0,V)}$, $\mathcal{G}_{(0,V)}$ and $\Gamma_{(0,V)}$ by $\mathcal{L}^V$, h^V, m_V, $\mathcal{G}_V$ and Γ_V respectively. Set $\bar{\mathcal{G}}_V = \mathcal{G}_V + \mathcal{G}_V^\dagger$, where $\mathcal{G}_V^\dagger$ means the adjoint of $\mathcal{G}_V$ on $L^2(dm_V)$.

Lemma 15 *Under Assumption 3 and (**F1**), (4.1) is equivalent to the following: for any $\mu_0 \in \mathcal{K}_{F_0}^0$, there exists $\delta_{\mu_0} > 0$ such that*

$$\begin{aligned} &\frac{1}{2}\int_M f \bar{\mathcal{G}}_{\nabla F_0(\mu_0)} f \, d\mu_0 \\ &\quad \ge \frac{1}{2}\nabla^2 F_0(\mu_0)(\bar{\mathcal{G}}_{\nabla F_0(\mu_0)} f \mu_0, \bar{\mathcal{G}}_{\nabla F_0(\mu_0)} f \mu_0) + \delta_{\mu_0} \|\bar{\mathcal{G}}_{\nabla F_0(\mu_0)} f \mu_0\|_{H_{-p'}}^2 \end{aligned} \tag{6.2}$$

holds for any $f \in H_{p'}$.

Proof First we derive (6.2) from Assumption 1. Take $\mu_0 \in \mathcal{K}_{F_0}^0$. By Lemma 14, there exists $\boldsymbol{w}_0 \in \mathcal{K}_F$ such that $\iota(\boldsymbol{w}_0) = \mu_0$ holds. Then Lemma 3 asserts $\mu_0 = m_{V_0}$. For $f \in H_{p'}$, let $w \in \mathcal{D}_{-p}$ be given by $\langle w, \beta \rangle = \int_M \left(d\mathcal{G}_{V_0} f, \beta \right) d\mu_0$. Note that H_{p+1} is closed under multiplication and $p + 1 > 2$ because $p > d_0$. Thus we have

$$\begin{aligned} 2L_{\mu_0}(w) &= \int_M |d\mathcal{G}_{V_0} f|^2 d\mu_0 = \int_M \mathcal{L}^{V_0} (\mathcal{G}_{V_0} f)^2 d\mu_0 - 2\int_M \mathcal{G}_{V_0} f \mathcal{L}^{V_0} \mathcal{G}_{V_0} f \, d\mu_0 \\ &= \int_M f \bar{\mathcal{G}}_{V_0} f \, d\mu_0. \end{aligned} \tag{6.3}$$

In a similar way, for $g \in H_{p'}$, we have

$$\left\langle \iota \circ \Gamma_{V_0}^* w, g \right\rangle_{H_{p'}} = \left\langle w, d\mathcal{G}_{V_0} g \right\rangle = \int_M \left(d\mathcal{G}_{V_0} f, d\mathcal{G}_{V_0} g \right) d\mu_0 = \int_M g \bar{\mathcal{G}}_{V_0} f \, d\mu_0. \tag{6.4}$$

Thus we obtain $\iota \circ \Gamma_{V_0}^* w = \bar{\mathcal{G}}_{V_0} f \mu_0$. Hence Assumption 3 yields

$$\left(w, G_{\boldsymbol{w}_0}^F w \right)_{-p} = \nabla^2 F_0(\mu_0)(\iota \circ \Gamma_{V_0}^* w, \iota \circ \Gamma_{V_0}^* w) = \nabla^2 F_0(\mu_0)(\bar{\mathcal{G}}_{V_0} f \mu_0, \bar{\mathcal{G}}_{V_0} f \mu_0). \tag{6.5}$$

Note that there is a constant $C > 0$ such that

$$\begin{aligned}\|w\|_{-p} &= \sup_{\substack{\alpha\in\mathcal{D}_p\\ \|\alpha\|_p\le 1}} |\langle w,\alpha\rangle| \ge \sup_{\substack{g\in H_{p'}\\ \|d\mathcal{G}_{V_0}g\|_p\le 1}} \left|\langle w, d\mathcal{G}_{V_0}g\rangle\right| \\ &\ge C \sup_{\substack{g\in H_{p'}\\ \|g\|_{H_{p'}}\le 1}} \left|\int_M \left(d\mathcal{G}_{V_0}g, d\mathcal{G}_{V_0}f\right) d\mu_0\right| = C\left\|\bar{\mathcal{G}}_{V_0} f\mu_0\right\|_{H_{-p'}}. \end{aligned} \tag{6.6}$$

By substituting (6.3), (6.5) and (6.6) to Assumption 1, we obtain (6.2).

Next we show the converse implication. Take $\boldsymbol{w}_0 \in \mathcal{K}_F$. By Lemma 14, $\mu_0 := \iota(\boldsymbol{w}_0) \in \mathcal{K}^0_{F_0}$ holds. For $\mu_0 \in \mathcal{K}^0_{F_0}$, set $V_0 = \nabla F_0(\mu_0)$. Lemma 3 asserts $\mu_0 = m_{V_0}$. By Remark 4, it suffices to show that there is a constant $\delta'_{\boldsymbol{w}_0} > 0$ such that for any w with $S_{\mu_0} w \in \left(\mathcal{H}'_{\mu_0} \cap \operatorname{Ker} G^F_{\boldsymbol{w}_0}\right)^\perp$,

$$L_{\mu_0}(S_{\mu_0}w) \ge \frac{1}{2}\left(S_{\mu_0}w, G^F_{\boldsymbol{w}_0}\circ S_{\mu_0}w\right)_{-p} + \delta'_{\boldsymbol{w}_0}\left\|S_{\mu_0}w\right\|^2_{-p}. \tag{6.7}$$

Indeed, once we obtain (6.7), we can replace $S_{\mu_0}w$ with $w \in \mathcal{H}'_{\mu_0}$ as we obtained (2.25) from (2.22). Take w as above. Assumption 3 implies $\operatorname{Ker} G^F_{\boldsymbol{w}_0} \supset \operatorname{Ker}(d\mathcal{G}_{V_0})^* = \operatorname{Ker} d^*$. Thus we have $(\mathcal{H}'_{\mu_0} \cap \operatorname{Ker} G^F_{\boldsymbol{w}_0})^\perp \subset (\mathcal{H}'_{\mu_0} \cap \operatorname{Ker} d^*)^\perp$. We set $\alpha = w^*$ and $d\mu_0 = f_0^2 dv$. Then we have $\langle S_{\mu_0}w, \beta\rangle = \int_M (\alpha,\beta)\, d\mu_0$. Since $\eta \in \operatorname{Ker} d^*$ means $\langle \eta, du\rangle = 0$ for any $u \in H_{p'}$, $P_{f_0}\alpha = \alpha$ holds. A regularity argument as we did in Sect. 2.1 implies $f_0 \in H_{p'}$ and $f_0 > 0$. Set $f = (f_0^{-1} df_0, \alpha) - d^\dagger\alpha/2$. Then we can show $P_{f_0}\alpha = d\mathcal{G}_{V_0} f$. Since

$$\langle S_{\mu_0}w, \beta\rangle = \int_M (\alpha,\beta)\, d\mu_0 = \int_M \left(d\mathcal{G}_{V_0} f, \beta\right) d\mu_0,$$

(6.3), (6.4) and (6.5) imply

$$L_{\mu_0}(S_{\mu_0}w) = \frac{1}{2}\int_M f\bar{\mathcal{G}}_{V_0} f\, d\mu_0,$$

$$\left(S_{\mu_0}w, G^F_{\boldsymbol{w}_0}\circ S_{\mu_0}w\right) = \nabla F_0(\mu_0)(\bar{\mathcal{G}}_{V_0} f\mu_0, \bar{\mathcal{G}}_{V_0} f\mu_0).$$

Since there is a constant $C' > 0$ such that $\|g\|_{H_{p'}} \le C'\|d\mathcal{G}_{V_0}g\|_p$ for any $g \in H_{p'}$, there exists a constant $C'' > 0$ such that we have

$$\|\bar{\mathcal{G}}_{V_0} f \mu_0\|_{H_{-p'}} = \sup_{\substack{g \in H_{p'} \\ \|g\|_{H_{p'}} \le 1}} \left| \int_M \left(d\mathcal{G}_{V_0} g, d\mathcal{G}_{V_0} f \right) d\mu_0 \right|$$
$$= \sup_{\substack{g \in H_{p'} \\ \|g\|_{H_{p'}} \le 1}} |\langle S_{\mu_0} w, d\mathcal{G}_{V_0} g \rangle| \ge C'' \|S_{\mu_0} w\|_{-p}.$$

Therefore, substituting these estimates to (6.2) yields Assumption 1. □

For $V \in H_{p'}$, let $S_V^0 : H_{-p'} \to H_{-p'}$ be defined by

$$\left\langle S_V^0(f^*), g \right\rangle_{H_{p'}} = \int_M f \bar{\mathcal{G}}_V g \, dm_V$$

for any $f, g \in H_{p'}$.

Lemma 16 *S_V^0 is a symmetric, nonnegative definite operator of trace class.*

Proof By (6.3), symmetricity and nonnegativity obviously follow. Take $q > 0$ such that $d_0/2 < q < p - d_0/2$. Since $d\mathcal{G}_V : H_{q-1} \to \mathcal{D}_q$ is continuous, the Sobolev embedding from $\mathcal{D}_q$ to $\mathcal{C}$ yields

$$\left(S_V^0(f^*), f^* \right)_{H_{-p'}} = \int_M f \bar{\mathcal{G}}_V f \, dm_V = \int_M |d\mathcal{G}_V f|^2 dm_V \le C \|d\mathcal{G}_V f\|_q^2 \le C' \|f\|_{H_{q-1}}^2$$

for some constants $C, C' > 0$. Since $p' - (q - 1) \ge p - q > d_0/2$ by (**D**), the conclusion follows from a similar argument as in the proof of Lemma 5 by calculating the trace norm with respect to an orthonormal basis of H_{p-1} consisting of normalized eigenfunctions of $-\Delta$. □

Theorem 6 ([4]) *Let $\varphi : M \to \mathbb{R}$ be a positive continuous function. Suppose that* (***D***)*,* (***F1***)*–*(***F3***) *and Assumption 3 hold. Suppose that, for each $\mu_0 \in \mathcal{K}_{F_0}^0$,* (6.2) *holds for any $f \in H_{p'}$. Then $\#\mathcal{K}_{F_0}^0 < \infty$ and*

$$\lim_{t\to\infty} e^{-t\kappa_{F_0}^0} \mathbb{E}_x \left[e^{tF_0(\bar{l}_t)} \varphi(z_t) \right]$$
$$= \sum_{\mu \in \hat{\mathcal{K}}_{\hat{F}}} \frac{1}{\det\left(1 - \nabla^2 F_0(\mu) \circ S_\mu^0\right)^{1/2}} h^{\nabla F_0(\mu)}(x) \int_M \frac{\varphi}{h^{\nabla F_0(\mu)}} d\mu.$$

Here, we identify the quadratic form $\nabla^2 F_0(\mu)$ on $H_{-p'}$ with the operator from $H_{-p'}$ to itself.

Proof By Lemma 15, Assumption 1 holds. In particular $\mathcal{K}_F$ is a finite set by Lemma 12. Thus Lemma 14 asserts $\#\mathcal{K}_{F_0}^0 < \infty$. Moreover, we have $\kappa_F = \kappa_{F_0}^0$ and Theorem 4 yields

$$\begin{aligned}
&\lim_{t\to\infty} \mathrm{e}^{-t\kappa^0_{F_0}} \mathbb{E}_x \left[\mathrm{e}^{tF_0(\bar{l}_t)} \varphi(z_t) \right] \\
&= \lim_{t\to\infty} \mathrm{e}^{-t\kappa_F} \mathbb{E}_x \left[\mathrm{e}^{tF(\bar{Y}_t)} \varphi(z_t) \right] \\
&= \sum_{\boldsymbol{w}_0=(w_0,\mu_0)\in\mathcal{K}_F} h^{\nabla F_0(\mu_0)}(x) \int_M \frac{\varphi}{h^{\nabla F_0(\mu_0)}} d\mu_0 \int_{\mathcal{D}_{-p}} \exp\left(\frac{1}{2} (w, G^F_{\boldsymbol{w}_0} w)_{-p} \right) \nu_{S_{\mu_0}}(dw).
\end{aligned} \tag{6.8}$$

For simplicity we set $V_0 := \nabla F_0(\mu_0)$. By the definition of $\boldsymbol{\Gamma}^*_\cdot$ and $G^F_{\boldsymbol{w}_0}$, we have

$$\left(w, G^F_{\boldsymbol{w}_0} w \right)_{-p} = \nabla^2 F_0(\mu_0)((d\mathcal{G}_{V_0})^* w, (d\mathcal{G}_{V_0})^* w).$$

By (6.3) we have

$$\begin{aligned}
\int_M \exp\left(\sqrt{-1} \left\langle (d\mathcal{G}_{V_0})^* w, g \right\rangle_{H_{p'}} \right) \nu_{S_{V_0}}(dw) &= \int_M \exp\left(\sqrt{-1} \left\langle w, d\mathcal{G}_{V_0} g \right\rangle \right) \nu_{S_{V_0}}(dw) \\
&= \exp\left(-\frac{1}{2} \int_M |d\mathcal{G}_{V_0} g|^2 d\mu_0 \right) \\
&= \exp\left(-\frac{1}{2} \int_M g \bar{\mathcal{G}}_{V_0} g \, d\mu_0 \right) \\
&= \exp\left(-\frac{1}{2} \left(g^*, S^0_{V_0}(g^*) \right)_{H_{-p'}} \right).
\end{aligned}$$

Thus the induced measure $\nu_{S_{V_0}} \circ ((d\mathcal{G}_{V_0})^*)^{-1}$ on $H_{-p'}$ is also a Gaussian measure with the covariance $S^0_{V_0}$. Hence the conclusion follows from (6.8). □

6.2 Finite dimensional case

Let $f : \mathbb{R}^k \to \mathbb{R}$ be a smooth function. Let $F' : \mathcal{D}_{-p} \to \mathbb{R}$ be defined by $F'(w) = f(\langle w, \alpha_1 \rangle, \dots \langle w, \alpha_k \rangle)$ for some $\alpha_i \in \mathcal{D}_p$, $i = 1, \dots, k$. In what follows, we consider the case that F is given by $F(w, \mu) = F'(w)$. In this case, some parts of our argument are reduced to the calculus on a finite dimensional space. Indeed, since we have

$$\nabla F'(w) = \sum_{i=1}^k \frac{\partial f}{\partial x_i}(\langle w, \alpha_1 \rangle, \dots, \langle w, \alpha_k \rangle)\alpha_i,$$

$\{\boldsymbol{\alpha}_{\boldsymbol{w}}\}_{\boldsymbol{w}\in\mathcal{K}_F}$, which appeared in Theorem 4, are all written by linear combinations of $(\alpha_i, 0)_{i=1}^k$. For each $\boldsymbol{w} = (w, \mu) \in \mathcal{K}_F$, Remark 4 says that it suffices to verify Assumption 1 when $w \in \mathcal{D}_{-p}$ acts on $\mathcal{D}_p$ as an inner product on $L^2(d\mu)$ with a linear combination of $\{\Gamma_{\boldsymbol{\alpha}_{\boldsymbol{w}}}\alpha_i\}_{i=1}^k$. Note that, when we consider Theorem 5, all elements in $\{\boldsymbol{\alpha}_{\boldsymbol{w}}\}_{\boldsymbol{w}\in\mathcal{K}_{F_1}}$ are written by linear combinations of $(\hat{\Gamma}\alpha_i, 0)_{i=1}^n$. In this case, we only need to verify Assumption 2 for each $\boldsymbol{w} = (w, \mu) \in \mathcal{K}_{F_1}$ when w acts as an inner product on $L^2(d\mu)$ with a linear combination of $\{\hat{\Gamma}_{\boldsymbol{\alpha}_{\boldsymbol{w}}}\alpha_i\}_{i=1}^k$.

We remark that this framework includes the case of periodic diffusions on $\mathbb{R}^k$. Let z_t be a solution of the following stochastic differential equation on k-dimensional torus $\mathbb{T}^k = [0, 1)^k$:

$$\begin{cases} dz_t^i &= \sum_{j=1}^k \sigma_j^i(z_t) \circ dz_t^j + \theta^i(z_t)dt, \\ z_0 &= x. \end{cases} \tag{6.9}$$

Here $\{\sigma_j^i\}_{i,j=1}^k$ and $\{\theta^i\}_{i=1}^k$ are smooth functions on $\mathbb{T}^k$. Let $g^{ij} = \sum_{l=1}^k \sigma_l^i \sigma_l^j$. Assume that $\{g^{ij}\}_{i,j=1}^k$ is nondegenerate at each point. Then $\{g_{ij}\} = \{g^{ij}\}^{-1}$ induces a Riemannian metric on $\mathbb{T}^k$. With respect to this metric, the generator of z_t is of the form $\Delta/2 + b$ for some smooth vector field b. We take 1-forms $\alpha_i = dx^i$, $i = 1, \dots, k$. Then, the Stokes formula for stochastic line integrals on $\mathbb{R}^k$ implies $X_t(\alpha_i) = \bar{z}_t^i - \bar{z}_0^i$, where $\bar{z}_t$ is a solution of the following stochastic differential equation on $\mathbb{R}^k$, which is a periodic extension of (6.9):

$$\begin{cases} d\bar{z}_t^i &= \sum_{j=1}^k \bar{\sigma}_j^i(\bar{z}_t) \circ d\bar{z}_t^j + \bar{\theta}^i(\bar{z}_t)dt, \\ \bar{z}_0 &= \bar{x}. \end{cases}$$

Here $\bar{\sigma}$ and $\bar{\theta}$ are a periodic extension of σ and θ respectively. Note that, if we denote the canonical projection from $\mathbb{R}^k$ to $\mathbb{T}^k$ by P, $P(\bar{x}) = x$ and $P(\bar{z}_t) = z_t$. These arguments and the remark at the beginning of this subsection tell us that the Laplace approximation for $t^{-1}\bar{z}_t$ follows as a corollary of Theorem 5.

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References

1. Avellaneda, M.: Large deviation estimates and the homological behavior of Brownian motion on manifolds. Ph.D. thesis. University of Minnesota, USA (1985)
2. Bolthausen, E., Deuschel, J.D., Schmock, U.: Convergence of path measures arising from a mean field or polaron type interaction. Probab. Theory Relat. Fields **95**(3), 283–310 (1993)
3. Bolthausen, E., Deuschel, J.D., Tamura, Y.: Laplace approximations for large deviations of non-reversible Markov processes on compact state spaces part II: the degenerate Hessian case (preprint)

4. Bolthausen, E., Deuschel, J.D., Tamura, Y.: Laplace approximations for large deviations of non-reversible Markov processes. The nondegenerate case. Ann. Probab. **23**(1), 236–267 (1995)
5. Deuschel, J.D., Stroock, D.W.: Large deviations. Academic Press, Boston (1989)
6. Doukhan, P.: Mixing. Properties and examples. Lecture Notes in Statistics, vol. 85. Springer, New York (1994)
7. Ellis, R.S., Rosen, J.S.: Asymptotic analysis of Gaussian integrals. I. Isolated minimal points. Trans. Am. Math. Soc. **273**(2), 447–481 (1982)
8. Gaffney, M.P.: Asymptotic distributions associated with the Laplacian for forms. Comm. Pure. Appl. Math. **11**, 535–545 (1958)
9. Hörmander, L.: Linear partial differential operators. Springer, Heidelberg (1963)
10. Ikeda, N.: Limit theorems for a class of random currents, Probabilistic methods in mathematical physics (Katata/Kyoto, 1985). Academic Press, Boston, pp. 181–193 (1987)
11. Ikeda, N., Manabe, S.: Integral of differential forms along the path of diffusion processes. Publ. RIMS, Kyoto Univ. **15**, 827–852 (1979)
12. Ikeda, N., Ochi, Y.: Central limit theorems and random currents. Lect. Notes Contr. Inform. Sci. **78**, 195–205 (1986)
13. Kato, T.: Perturbation theory for linear operators. Springer, Heidelberg (1980)
14. Kusuoka, S., Tamura, Y.: Symmetric Markov processes with mean field potentials. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **34**(2), 371–389 (1987)
15. Kusuoka, S., Tamura, Y.: Precise estimate for large deviation of Donsker-Varadhan type. J. Fac. Sci. Univ. Tokyo Sect. IA Math. **38**(3), 533–565 (1991)
16. Kuwada, K.: On large deviations for random currents induced from stochastic line integrals. Forum Math. **18**, 639–676 (2006)
17. Kuwada, K.: Sample path large deviations for a class of random currents. Stochastic Process. Appl. **108**, 203–228 (2003)
18. Manabe, S.: Large deviation for a class of current-valued processes. Osaka J. Math. **29**, 89–102 (1992)
19. Maz'ya, V.G., Shaposhnikova, T.O.: Theory of multipliers in spaces of differentiable functions. Monographs and Studies in Mathematics, vol. 23. Pitman (Advanced Publishing Program), Boston, MA (1985)
20. Moser, J.: A rapidly convergent iteration method and non-linear partial differential equations. I. Ann. Scuola Norm. Sup. Pisa **20**(3), 265–315 (1966)
21. Ochi, Y.: Limit theorems for a class of diffusion processes. Stoch. **15**, 251–269 (1985)
22. Pinsky, R.G.: Positive harmonic functions and diffusion, Cambridge Studies in Advanced Mathematics, vol. 45. Cambridge University Press, Cambridge (1995)
23. Yoshida, N.: Sobolev spaces on a Riemannian manifold and their equivalence. J. Math. Kyoto Univ. **32**(3), 621–654 (1992)